Testing the Master Constraint Programme for Loop Quantum Gravity III. SL(2,R) Models

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Abstract

This is the third paper in our series of five in which we test the Master Constraint Programme for solving the Hamiltonian constraint in Loop Quantum Gravity. In this work we analyze models which, despite the fact that the phase space is finite dimensional, are much more complicated than in the second paper: These are systems with an $SL(2,\mathbb{R})$ gauge symmetry and the complications arise because non – compact semisimple Lie groups are not amenable (have no finite translation invariant measure). This leads to severe obstacles in the refined algebraic quantization programme (group averaging) and we see a trace of that in the fact that the spectrum of the Master Constraint does not contain the point zero. However, the minimum of the spectrum is of order $\hbar^2$ which can be interpreted as a normal ordering constant arising from first class constraints (while second class systems lead to $\hbar$ normal ordering constants). The physical Hilbert space can then be be obtained after subtracting this normal ordering correction.

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Contents

1 Introduction

We continue our test of the Master Constraint Programme [1] for Loop Quantum Gravity (LQG) [6, 7, 8] which we started in the companion papers [2, 3] and will continue in [4, 5]. The Master Constraint Programme is a new idea to improve on the current situation with the Hamiltonian constraint operator for LQG [9]. In short, progress on the solution of the Hamiltonian constraint has been slow because of a technical reason: the Hamiltonian constraints themselves are not spatially diffeomorphism invariant. This means that one cannot first solve the spatial diffeomorphism constraints and then the Hamiltonian constraints because the latter do not preserve the space of solutions to the spatial diffeomorphism constraint [10]. On the other hand, the space of solutions to the spatial diffeomorphism constraint [10] is relatively easy to construct starting from the spatially diffeomorphism invariant representations on which LQG is based [11] which are therefore very natural to use and, moreover, essentially unique. Therefore one would really like to keep these structures. The Master Constraint Programme removes that technical obstacle by replacing the Hamiltonian constraints by a single Master Constraint which is a spatially diffeomorphism invariant integral of squares of the individual Hamiltonian constraints which encodes all the necessary information about the constraint surface and the associated invariants. See e.g. [1, 2] for a full discussion of these issues. Notice that the idea of squaring constraints is not new, see e.g. [13], however, our concrete implementation is new and also the Direct Integral Decomposition (DID) method for solving them, see [1, 2] for all the details.

The Master Constraint for four dimensional General Relativity will appear in [14] but before we test its semiclassical limit, e.g. using the methods of [15, 16] and try to solve it by DID methods we want to test the programme in the series of papers [2, 3, 4, 5]. In the previous papers we focussed on finite dimensional systems of various degrees of complexity. In this article we will apply the Master Constraint Progamme to constraint algebras which generate a non-abelean and non-compact gauge group. In the first example we are concerned with the gauge group $SO(2,1)$ and in the second example with the gauge group $SL(2,\mathbb{R})$, which is the double cover of $SO(2,1)$.

We will see that both examples share the same problem – the spectrum of the Master Constraint Operator does not include the value zero. The reason for this is the following: The value zero in the spectrum of the Master Constraint Operator corresponds to the appearence of the trivial representation in a Hilbert space decomposition of the given unitary representation of the gauge group on the kinematical Hilbert space.

Now, the groups $SO(2,1)$ and $SL(2,\mathbb{R})$ (and all groups which have these two groups as subgroups, e.g. symplectic groups and $SO(p,q)$ with $p,q>1$ and $p+q>2$) are non-amenable groups, see [18]. One characteristic of non-amenable groups is, that the trivial representation does not appear in a Hilbert space decomposition of the regular representation into irreducible unitary subrepresentations. Since the decomposition of the regular decomposition is often used to decompose tensor products, it will often happen, that a given representation of a non-amenable group does not include the trivial representation in its Hilbert space decomposition.

Since the value zero is not included in the spectrum of the Master Constraint Operator $\widehat{\mathbf{M}}$ we will use a redefined operator $\widehat{\mathbf{M}}' = \widehat{\mathbf{M}} - \lambda_{\text{min}}$ as proposed in [2], where λ_{min} is the minimum of the spectrum. One can interprete this procedure as a quantum correction (which is proportional to $\hbar^2$). Nevertheless the redefinition of the Master Constraint Operator has to be treated carefully, since it is not guaranteed that all relations between the observables implied by the constraints are realized on the resulting physical Hilbert space. This phenomenon will occur in the second example. However we will show that it is possible to alter the Master Constraint Operator again and to obtain a physical quantum theory which has the correct classical limit.

In both examples we will use the representation theory of $SL(2,\mathbb{R})$ and its covering groups to find the spectra and the direct integral decompositions with respect to the Master Constraint Operator. As we will see, the diagonalization of the Master Constraint Operator is equivalent to the diagonalization of the Casimir Operator of the given gauge group representation, which in turn is equivalent to the decomposition of the given gauge group representation into a direct sum and/or direct integral of irreducible unitary representations.

Furthermore, we will see that our examples exhibit the structure of a dual pair, see [21]. These are defined to be two subgroups in a larger group, where one subgroup is the maximal commutant of the other and vice versa. In our examples one subgroup is the group generated by the observables and the other is the group generated by the constraints. Now, given this structure of dual pairs, one can show that the decomposition of the representation of the gauge group is equivalent to the reduction of the representation of the observable algebra (on the kinematical Hilbert space). This is explained in further detail in appendix A.2. In our examples this fact will help us to determine the induced representation of the observable algebra on the physical Hilbert space. Moreover we can determine in this way the induced inner product on the physical Hilbert space, so that it will not be necessary to perform all the steps of the direct Hilbert space decomposition as explained in [2] to find the physical inner product.

We summarized the representation theory of the $sl(2,\mathbb{R})$ algebra in appendix A.1. Appendix A.2 explains the theory of oscillator representations for $sl(2,\mathbb{R})$, which is havily used in the two examples. It also contains a discussion how the representation theory of dual pairs can be applied to our and similar examples.

2 $SL(2,\mathbb{R})$ Model with Non – Compact Gauge Orbits

Here we consider the configuration space $\mathbb{R}^3$ with the three $so(2,1)$-generators as constraints:

$$L_i = \epsilon^k_{\ ij} x^j p_k, \qquad \{L_i, L_j\} = \epsilon^k_{\ ij} L_k \tag{2.1}$$

where $\epsilon^k_{\ ij} = g^{km}\epsilon_{mij}$, ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$ and g^{ik} is the inverse of the metric $g_{ik} = \mathrm{diag}(+,+,-)$. Indices are raised and lowered with g^{ik} resp. g_{ik} and we sum over repeated indices.

The gauge group $SO(n,1)$ was previously discussed in [29], where group averaging was used to construct the physical Hilbert space. We will compare the results of [29] and the results obtained here at the end of the section.

The observable algebra of the system above is generated by

$$d = x^i p_i \qquad e^+ = x^i x_i \qquad e^- = p^i p_i \quad . \tag{2.2}$$

This set of observables exhibits the commutation relations of the generators of the $sl(2,\mathbb{R})$-algebra (which coincides with $so(2,1)$):

$$\{d, e^\pm\} = \mp 2e^\pm \qquad \{e^+, e^-\} = 4d \quad . \tag{2.3}$$

We have the identity

$$d^2 - e^+ e^- = L_i L^i \tag{2.4}$$

between the Casimirs of the constraint and observable algebra.

2.1 Quantization

We start with the auxilary Hilbert space $\mathfrak{L}^2(\mathbb{R}^3)$ of square integrable functions of the coordinates. The momentum operators are $\hat{p}_j = -i(\hbar)\partial_j$ and the $\hat{x}^j$ act as multiplication operators. There arises no factor ordering ambiguity for the quantization of the constraints, but to ensure a closed observable algebra, we have to choose:

$$\begin{aligned} \hat{d} &= \tfrac{1}{2}(\hat{x}^i\hat{p}_i + \hat{p}_i\hat{x}^i) = \hat{x}^i\hat{p}_i - \tfrac{3}{2}i\hbar \\ \hat{e}^+ &= \hat{x}^i\hat{x}_i \qquad \hat{e}^- = \hat{p}^i\hat{p}_i \end{aligned} \tag{2.5}$$

The commutators between constraints and between observables are then obtained by replacing the Poisson bracket with $\frac{1}{i\hbar}[\cdot,\cdot]$.

The identity (2.4) is altered to be:

$$\hat{d}^2 - \tfrac{1}{2}(\hat{e}^+\hat{e}^- + \hat{e}^-\hat{e}^+) - \tfrac{3}{4}\hbar^2 = \hat{L}_i\hat{L}^i \quad . \tag{2.6}$$

(From now on we will skip the hats and set $\hbar$ to 1.)

For the implementation of the Master Constraint Programme we have to construct the spectral resolution of the $\widehat{\mathbf{M}}$

$$\widehat{\mathbf{M}} := L_1^2 + L_2^2 + L_3^2 = L^i L_i + 2L_3^2 = d^2 - \tfrac{1}{2}(e^+e^- + e^-e^+) - \tfrac{3}{4} + 2L_3^2 \quad . \tag{2.7}$$

To this end we will use the following strategy: The operators $\widehat{\mathbf{M}}$ and L_3 commute, so we can diagonalize them simultaneously. The diagonalization of L_3 is easy to achieve, its spectrum being purely discrete, namely $\mathrm{spec}(L_3) = \mathbb{Z}$. Now we can diagonalize $\widehat{\mathbf{M}}$ on each eigenspace of L_3 seperately. On these eigenspaces the diagonalization of $\widehat{\mathbf{M}}$ is equivalent to the diagonalization of the $so(2,1)$-Casimir L_iL^i and because of identity (2.6) equivalent to the diagonalization of the $sl(2,\mathbb{R})$-Casimir $\mathfrak{C} = -\frac{1}{4}(d^2 - \frac{1}{2}(e^+e^- + e^-e^+))$. As we will show below the $sl(2,\mathbb{R})$-representation given by (2.5) is a tensor product of three representations, which are known as oscillator and contragredient oscillator representations. To obtain the spectral resolution of the Casimir $\mathfrak{C}$, we will reduce this tensor product into its irreducible components.

2.2 The Oscillator Representations and its Reduction

For the reduction process it will be very convenient to work with the following basis of the $sl(2,\mathbb{R})$-algebra:

$$\begin{aligned} h &= \tfrac{1}{4}(e^+ + e^-) &&= \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3 + \tfrac{1}{2}) \\ n^+ &= -\tfrac{1}{2}(i\,d - \tfrac{1}{2}(e^+ - e^-)) &&= \tfrac{1}{2}(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger - a_3 a_3) \\ n^- &= \tfrac{1}{2}(i\,d + \tfrac{1}{2}(e^+ - e^-)) &&= \tfrac{1}{2}(a_1 a_1 + a_2 a_2 - a_3^\dagger a_3^\dagger) \\ \mathfrak{C} &= \tfrac{1}{4}(d^2 - \tfrac{1}{2}(e^+e^- + e^-e^+)) &&= -h^2 + \tfrac{1}{2}(n^+n^- + n^-n^+) \quad , \end{aligned} \tag{2.8}$$

with commutation and adjointness relations

$$[h, n^\pm] = \pm n^\pm \qquad [n^+, n^-] = -2h \qquad (n^+)^\dagger = n^- \quad . \tag{2.9}$$

Here we introduced the anihilation and creation operators

$$a_i = \tfrac{1}{\sqrt{2}}(x^i + ip_i) \quad \text{and} \quad a_i^\dagger = \tfrac{1}{\sqrt{2}}(x^i - ip_i) \quad . \tag{2.10}$$

Now it is easy to see that this representation is a tensor product of the following three $sl(2,\mathbb{R})$-representations:

$$\begin{aligned} h_i &= \tfrac{1}{2}(a_i^\dagger a_i + \tfrac{1}{2}) \quad \text{for } i = 1,2 \quad \text{and} & h_3 &= -\tfrac{1}{2}(a_3^\dagger a_3 + \tfrac{1}{2}) \\ n_i^+ &= \tfrac{1}{2}a_i^\dagger a_i^\dagger & n_3^+ &= -\tfrac{1}{2}a_3 a_3 \\ n_i^- &= \tfrac{1}{2}a_i a_i & n_3^- &= -\tfrac{1}{2}a_3^\dagger a_3^\dagger \end{aligned} \tag{2.11}$$

These representations are known as oscillator representation ω (for $i = 1,2$) and contragredient oscillator representation ω^* (for $i = 3$), see [19] and A.2, where these representations are explained.

The oscillator representation is the sum of two irreducible representations, which are the representations $D(1/2)$ and $D(3/2)$ from the positive discrete series of the double cover of $Sl(2,\mathbb{R})$ (corresponding to even and odd number Fock states). Similarly $\omega^* \simeq D^*(1/2) \oplus D^*(3/2)$, where $D^*(1/2)$ and $D^*(3/2)$ are from the negative discrete series.

As mentioned before we will reduce this tensor product to its irreducible subrepresentations in order to obtain the spectrum of the Casimir (and with it the spectrum of the Master Constraint Operator). In appendix A.2 one can find the general strategy and some formulas to reduce such tensor products. Furthermore appendix A.1 reviews the $sl(2,\mathbb{R})$-representations, which will appear below.

To begin the reduction of $\omega \otimes \omega \otimes \omega^*$ we will reduce $\omega \otimes \omega$. This we can achieve by utilizising the observable L_3. It commutes with the $sl(2,\mathbb{R})$-algebra (2.8), therefore according to Schur's Lemma its eigenspaces are left invariant by the $sl(2,\mathbb{R})$-algebra, i.e. its eigenspaces are subrepresentations of $sl(2,\mathbb{R})$.

To diagonalize L_3 and reduce the tensor product $\omega \otimes \omega$ we will employ the "polarized" anihilation and creation operators

$$A_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2) \qquad A_\pm^\dagger = \frac{1}{\sqrt{2}}(a_1 \pm ia_2). \tag{2.12}$$

With the help of these, we can write

$$\begin{aligned} h &= \tfrac{1}{2}(A_+^\dagger A_+ + A_-^\dagger A_- - a_3^\dagger a_3 + \tfrac{1}{2}) \\ n^+ &= A_+^\dagger A_-^\dagger - \tfrac{1}{2}a_3 a_3 \\ n^- &= A_+ A_- - \tfrac{1}{2}a_3^\dagger a_3^\dagger \end{aligned} \tag{2.13}$$

$$L_3 = A_+^\dagger A_+ - A_-^\dagger A_- \quad . \tag{2.14}$$

In the following we will denote by $|k_+, k_-, k_3 >$ Fock states with respect to $A_+^\dagger$, $A_-^\dagger$ and $a_3^\dagger$. The operator L_3 acts on them diagonally. Its eigenspaces $V(\pm j)$ corresponding to the eigenvalue $\pm j$, $j \in \mathbb{N}$ are generated by $\{|j, 0, k_3 >, k_3 \in \mathbb{N}\}$ and $\{|0, j, k_3 >, k_3 \in \mathbb{N}\}$ respectively, i.e. $V(j)$ is (the closure of) the linear span of the $|j, 0, k_3 >$'s resp. $|0, j, k_3 >$'s and all vectors are obtained by applying repeatedly n^+ to them. These eigenspaces are invariant subspaces of the representation (2.13). This representation restricted to $V(j)$ is still a tensor product representation, namely the representation $D(|j|+1) \otimes \omega^*$. Its factors are given by

$$\begin{array}{lll} h_{12}|_{V(j)} = \frac{1}{2}(A_+^\dagger A_+ + A_-^\dagger A_- + 1)|_{V(j)} & \text{and} & h_3|_{V(j)} = -\frac{1}{2}(a_3^\dagger a_3 + \frac{1}{2})|_{V(j)} \\ n_{12}^+|_{V(j)} = A_+^\dagger A_-^\dagger|_{V(j)} & \text{and} & n_3^+|_{V(j)} = -\frac{1}{2}a_3 a_3 \\ n_{12}^-|_{V(j)} = A_+ A_-|_{V(j)} & \text{and} & n_3^-|_{V(j)} = -\frac{1}{2}a_3^\dagger a_3^\dagger \quad . \end{array} \tag{2.15}$$

Since h_{12} has a smallest eigenvalue $\frac{1}{2}(|j|+1)$ on the subspace $V(j)$, this subspace carries a $D(|j|+1)$-representation from the positive discrete series (of $SL(2,\mathbb{R})$) with lowest weight $\frac{1}{2}(|j|+1)$ (see A.1).

So far we have achieved the reduction $\omega \otimes \omega \otimes \omega^* \simeq (D(1) \oplus \sum_{j=0}^{\infty} 2D(j+1)) \otimes \omega^*$. To reduce the representation (2.13) completely, we have to consider tensor products of the form $D(|j|+1) \otimes D^*(1/2)$ and $D(|j|+1) \otimes D^*(3/2)$. We take this reduction from [19], see also A.2 and A.1 for a description of the $sl(2,\mathbb{R})$-representations, appearing below:

For j even, we have

$$\begin{array}{c} D(|j|+1) \otimes D^*(1/2) \simeq \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} P(t, 1/4) d\mu(t) \oplus \sum_l D(|j| + 1/2 - 2l) \\ \text{with } 0 \leq 2l < |j| - 1/2,\ l \in \mathbb{N} \end{array} \tag{2.16}$$

and for j odd we get

$$\begin{array}{c} D(|j|+1) \otimes D^*(1/2) \simeq \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} P(t, -1/4) d\mu(t) \oplus \sum_l D(|j| + 1/2 - 2l) \\ \text{with } 0 \leq 2l < |j| - 1/2,\ l \in \mathbb{N} \quad . \end{array} \tag{2.17}$$

In particular, we have for $j = 0$

$$D(1) \otimes D^*(1/2) \simeq \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} P(t, 1/4) d\mu(t) \quad . \tag{2.18}$$

The remainig tensor products are

$$\begin{array}{rcl} D(1) \otimes D^*(3/2) & \simeq & \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} P(t, -1/4) d\mu(t) \\ D(|j|+1) \otimes D^*(3/2) & \simeq & D(|j|) \otimes D^*(1/2) \quad \text{for } j > 0 \quad . \end{array} \tag{2.19}$$

$P(t, \epsilon), \epsilon = \frac{1}{4}, -\frac{1}{4}$ is the principal series (of the metaplectic group, i.e. the double cover of $SL(2,\mathbb{R})$) characterized by an h-spectrum $\text{spec}(h) = \{\epsilon + z, z \in \mathbb{Z}\}$ and a Casimir $\mathfrak{C}(P(t,\epsilon)) = t(1-t)\text{Id}$. The measure $d\mu(t)$ is the Plancherel measure on the unitary dual of the metaplectic goup.

The representations $D(l+1/2), l \in \mathbb{N}-\{0\}$ are positive discrete series representations of the metaplectic group. The h-spectrum in these representations is given by $\{\frac{1}{2}(l+1/2)+n, n \in \mathbb{N}\}$ and the Casimir by $\mathfrak{C}(D(l+1/2)) = -\frac{1}{4}(l+1/2)^2 + \frac{1}{2}(l+1/2)$.

The spectrum of the Casimir $\mathfrak{C}$ is non-degenerate on each tensor product $D(k) \otimes D^*(l)$, $l = \frac{1}{2}, \frac{3}{2}$, i.e. the Casimir discriminates the irreducible representations, which appear in this tensor product and the irreducible representations have multiplicity one. The spectrum of h is non-degenerate in each irreducible representation of the metaplectic group . This implies that we can find a (generalized) basis $|j, \epsilon, \mathfrak{c}, \mathfrak{h}>$, which is labeled by the L_3-eigenvalue j, the values $\epsilon = \frac{1}{4}, -\frac{1}{4}$, the Casimir eigenvalue $\mathfrak{c}$ and the h-eigenvalue $\mathfrak{h}$.

Summarizing, we have for the (highly degenerate) spectrum of the Casimir $\mathfrak{C} = -h^2 + \frac{1}{2}(n^+n^- + n^-n^+)$ on $\mathfrak{L}^2(\mathbb{R}^3)$:

$$\mathrm{spec}(\mathfrak{C}) = \{\tfrac{1}{4} + x^2, x \in \mathbb{R}, x \geq 0\} \cup \{-\tfrac{1}{4}(l+1/2)^2 + \tfrac{1}{2}(l+1/2), l \in \mathbb{N}-\{0\}\} \quad . \tag{2.20}$$

The continuous part of the spectrum originates from the principal series $P(t, 1/4)$ and $P(t, -1/4)$ and the discrete part from those positive discrete series representations $D(l)$, which appear in the decompositions above. This results in the following expression for the spectrum of the $so(2,1)$-Casimir $L^iL_i = (4\mathfrak{C} - \frac{3}{4})$:

$$\mathrm{spec}(L^iL_i) = \{\tfrac{1}{4} + s^2, s \in \mathbb{R}\, s \geq 0\} \cup \{-q^2 + q, q \in \mathbb{N}-\{0\}\} \quad . \tag{2.21}$$

As explained in appendix A.1 these values correspond to the principal series $P(\frac{1}{2}+s, 0)$ of $SO(2,1)$ and the positive or negative discrete series $D(2q)$ resp. $D^*(2q)$ for $q \in \mathbb{N}-\{0\}$. In the $SO(2,1)$-principal series the spectrum of L_3 is given by $\mathrm{spec}(L_3) = \{\mathbb{Z}\}$. In the discrete series $D(2q)$ we have $\mathrm{spec}(L_3) = \{q+n, n \in \mathbb{N}\}$ and in $D^*(q)$ $\mathrm{spec}(L_3) = \{-q-n, n \in \mathbb{N}\}$.

2.3 The Physical Hilbert space

Now we can determine the spectrum of the Master Constraint Operator $\widehat{\mathbf{M}} = L_iL^i + 2L_3^2$ on the j-eigenspaces $V(j)$ of L_3. From the principal series we get the continous part of the spectrum

$$\mathrm{spec}_{\mathrm{cont}}(\widehat{\mathbf{M}}\,|_{V(j)}) = \{\tfrac{1}{4} + s^2 + 2j^2, s \in \mathbb{R}, s \geq 0\} \tag{2.22}$$

and from the discrete series the discrete part (for $j \geq 1$, since there is no discrete part for $j = 0$)

$$\mathrm{spec}_{\mathrm{discr}}(\widehat{\mathbf{M}}\,|_{V(j)}) = \{(-q^2+q) + 2j^2, q \in \mathbb{N}, q \leq |j|\} \geq 2\ . \tag{2.23}$$

(The inequality $q \leq |j|$ follows from the fact, that in a representation $D(2q)$ or $D^*(2q)$ we have $|j| \geq q$ for the L_3-eigenvalues j.)

One can see immedeatily, that zero is not included in the spectrum of the Master Constraint, the lowest generalized eigenvalue being $\frac{1}{4}$. Therefore we alter the Master Constraint to $\widehat{\mathbf{M}}' = \widehat{\mathbf{M}} - \frac{1}{4}(\hbar^2)$ where appropriate powers of $\hbar$ have been restored. The generalized null eigenspace of $\widehat{\mathbf{M}}'$ is given by the linear span of all states $|j = 0, \epsilon, \mathfrak{c} = \frac{1}{4}, \mathfrak{h}>$. The spectral measure of $\widehat{\mathbf{M}}'$ induces a scalar product on this space, which can then be completed to a Hilbert space. In particular with this scalar product one can normalize the states $|j = 0, \epsilon, \mathfrak{c} = \frac{1}{4}, \mathfrak{h}>$, obtaining an ortho-normal basis $||\epsilon, \mathfrak{h}>>$.

This Hilbert space has to carry a unitary representation of the metaplectic group. Actually, it carries a sum of two irreducible representations $P(t = 1/2, 1/4)$ and $P(t = 1/2, -1/4)$, corresponding to the labels $\epsilon = 1/4$ and $\epsilon = -1/4$ of the basis $\{||\epsilon, \mathfrak{h}>>\}$. States in these representations are distinguished by the transformation under the reflection $R_3 : x_3 \mapsto -x_3$, which is a group element of $O(2,1)$. As an operator on $\mathfrak{L}^2(\mathbb{R}^3)$ it acts as:

$$\hat{R}_3 : \psi(x_1, x_2, x_3) \mapsto \psi(x_1, x_2, -x_3) \quad . \tag{2.24}$$

$\hat{R}_3$ acts on states with $\epsilon = \frac{1}{4}$ as the identity operator (since these states are linear combinations of even number Fock states with respect to $a_3^\dagger$) and on states with $\epsilon = -\frac{1}{4}$ by multiplying them with (-1) (since these states are linear combinations of odd number Fock states). It seems natural, to exclude the states with nontrivial behaviour under this reflection. This leaves us with the unitary irreducible representation $P(t = 1/2, 1/4)$. As explained in A.1 the action of the observable algebra $sl(2, \mathbb{R})$ on the states $||\mathfrak{h} >>:= ||1/4, \mathfrak{h} >>$ is determined (up to a phase, which can be fixed by adjusting the phases of the states $||\mathfrak{h} >>$) by this representation to be:

$$\begin{aligned} h||\mathfrak{h} >> &= \mathfrak{h}||\mathfrak{h} >> \qquad (\mathfrak{h} \in \{\tfrac{1}{4} + \mathbb{Z}\}) \\ n^+||\mathfrak{h} >> &= (\mathfrak{h} + \tfrac{1}{2})||\mathfrak{h} + 1 >> \\ n^-||\mathfrak{h} >> &= (\mathfrak{h} - \tfrac{1}{2})||\mathfrak{h} - 1 >> \quad . \end{aligned} \tag{2.25}$$

This gives for matrix elements of the observables d and $e^\pm$ (see 2.5)

$$<< \mathfrak{h}'|d|\mathfrak{h} >> = i(\mathfrak{h} + \tfrac{1}{2})\delta_{\mathfrak{h}',\mathfrak{h}+1} - i(\mathfrak{h} - \tfrac{1}{2})\delta_{\mathfrak{h}',\mathfrak{h}-1} \tag{2.26}$$

$$<< \mathfrak{h}'|e^\pm|\mathfrak{h} >> = 2\mathfrak{h}\,\delta_{\mathfrak{h}',\mathfrak{h}} \pm (\mathfrak{h} + \tfrac{1}{2})\delta_{\mathfrak{h}',\mathfrak{h}+1} \pm (\mathfrak{h} - \tfrac{1}{2})\delta_{\mathfrak{h}',\mathfrak{h}-1} \quad . \tag{2.27}$$

The operators $e^+ = \hat{x}^i\hat{x}_i$ and $e^- = \hat{p}^i\hat{p}_i$ are indefinite operators, i.e. their spectra include positive and negative numbers.

To sum up, we obtained a physical Hilbert space, which carries an irreducible unitary representation of the observable algebra. In contrast to these results the group averaging procedure in [29] leads to (two) superselection sectors and therefore to a reducible representation of the observables. These sectors are functions with compact support inside the light cone and functions with compact support outside the light cone. Hence the observable $e^+ = \hat{x}^i\hat{x}_i$ is either strictly positive or strictly negative definite on theses superselection sectors. From that point of view our physical Hilbert space is preferred because physically e^+ should be indefinite. However, as mentioned in [29] the appearence of superselection sectors may depend on the choice of the domain Φ, on which the group averaging procedure has to be defined and thus other choices of Φ may not suffer from this superselection problem. We see, at least in this example, that the DID method outlined in [2] with the prescription given there gives a more natural and unique result. However, as the given system lacks a realistic interpretation anyway, this difference may just be an artefact of a pathological model.

3 Model with Two Hamiltonian Constraints and Non – Compact Gauge Orbits

3.1 Introduction of the Model

Here we consider a reparametrization invariant model introduced by Montesinos, Rovelli and T.T. in [23]. It has an $Sl(2, \mathbb{R})$ gauge symmetry and a global $O(2, 2)$ symmetry and has attracted interest because its constraint structure is in some sense similar to the constraint structure found in general relativity. Further work on this model has appeared in [24, 25, 26, 27] and references therein.

We will shortly summarize the classical (canonical) theory (see [23] for an extended discussion). The configuration space is $\mathbb{R}^4$ parametrized by coordinates (u_1, u_2) and (v_1, v_2) and the canonically conjugated momenta are (p_1, p_2) and (π_1, π_2). The system is a totally constrained (first class) system. The constraints form a realization of an $sl(2, \mathbb{R})$-algebra:

$$H_1 = \tfrac{1}{2}(\vec{p}^2 - \vec{v}^2) \qquad H_2 = \tfrac{1}{2}(\vec{\pi}^2 - \vec{u}^2) \qquad D = \vec{u} \cdot \vec{p} - \vec{v} \cdot \vec{\pi} \tag{3.1}$$

$$\{H_1, H_2\} = D \qquad \{H_1, D\} = -2H_1 \qquad \{H_2, D\} = 2H_2 \tag{3.2}$$

The canonical Hamiltonian governing the time evolution (which is pure gauge) is $H = N\,H_1 + M\,H_2 + \lambda\,D$ where N, M and λ are Lagrange multipliers. Since H_1 and H_2 are quadratic in the momenta and their Poisson bracket gives a constraint which is linear in the momenta, one could say that this model has an analogy with general relativity. There, one has Hamiltonian constraints $H(x)$ quadratic in the momenta and diffeomorphism constraints $D(x)$ linear in the momenta which have the Poisson structure $\{H(x), H(y)\} \sim \delta(x-y)D(x)$ and $\{H(x), D(y)\} \sim \delta(x-y)H(x)$.

However, one can make the following canonical transformation to new canonical coordinates (U_i, V_i, P_i, Π_i), $i = 1, 2$ that transforms the constraint into phase space functions which are linear in the momenta:

$$\begin{aligned} u_i &= \tfrac{1}{\sqrt{2}}(U_i + \Pi_i) & v_i &= \tfrac{1}{\sqrt{2}}(V_i + P_i) \\ p_i &= \tfrac{1}{\sqrt{2}}(-V_i + P_i) & \pi_i &= \tfrac{1}{\sqrt{2}}(-U_i + \Pi_i) \end{aligned} \tag{3.3}$$

$$H_1 = -P_1V_1 - P_2V_2 \qquad H_2 = -U_1\Pi_1 - U_2\Pi_2 \qquad D = P_1U_1 + P_2U_2 - V_1\Pi_1 - V_2\Pi_2 \quad . \tag{3.4}$$

These coordinates have the advantage, that the constraints act on the configuration variables (U_1, V_1) and (U_2, V_2) in the defining two-dimensional representation of $sl(2, \mathbb{R})$ (i.e. by matrix multiplication).

For reasons that will become clear later, it is easier for us to stick to the old coordinates (u_i, v_i, p_i, π_i).

Now we will list the Dirac observables of this system. They reflect the global $O(2, 2)$-symmetry of this model and are given by (see [23])

$$\begin{aligned} O_{12} &= u_1p_2 - p_1u_2 & O_{23} &= u_2v_1 - p_2\pi_1 \\ O_{13} &= u_1v_1 - p_1\pi_1 & O_{24} &= u_2v_2 - p_2\pi_2 \\ O_{14} &= u_1v_2 - p_1\pi_2 & O_{34} &= \pi_1v_2 - v_1\pi_2 \end{aligned} \tag{3.5}$$

They constitute the Lie algebra $so(2, 2)$ which is isomorphic to $so(2, 1) \times so(2, 1)$. A basis adapted to the $so(2, 1) \times so(2, 1)$-structure is (see [26])

$$\begin{aligned} Q_1 &= \tfrac{1}{2}(O_{23} + O_{14}) & P_1 &= \tfrac{1}{2}(O_{23} - O_{14}) \\ Q_2 &= \tfrac{1}{2}(-O_{13} + O_{24}) & P_2 &= \tfrac{1}{2}(-O_{13} - O_{24}) \\ Q_3 &= \tfrac{1}{2}(O_{12} - O_{34}) & P_3 &= \tfrac{1}{2}(O_{12} + O_{34}) \end{aligned} \tag{3.6}$$

The Poisson brackets between these observables are

$$\{Q_i, Q_j\} = \epsilon_{ij}{}^k Q_k \qquad \{P_i, P_j\} = \epsilon_{ij}{}^k P_k \qquad \{Q_i, P_j\} = 0 \tag{3.7}$$

where $\epsilon_{ij}{}^k = g^{lk}\epsilon_{ijk}$, with g^{lk} being the inverse of the metric $g_{lk} = \mathrm{diag}(+1, +1, -1)$. The Levi-Civita symbol ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$ and we sum over repeated indices. Lateron the (ladder) operators $Q_\pm := \frac{1}{\sqrt{2}}(Q_1 \pm iQ_2)$ and $P_\pm := \frac{1}{\sqrt{2}}(P_1 \pm iP_2)$ will be usefull.

One can find the following identities between observables and constraints (see [26]):

$$Q_1^2 + Q_2^2 - Q_3^2 = P_1^2 + P_2^2 - P_3^2 = \tfrac{1}{4}(D^2 + 4H_1H_2) \tag{3.8}$$
$$4Q_3P_3 = (\vec{u}^2 - \vec{v}^2)(H_1 + H_2) - (\vec{u}\cdot\vec{p} + \vec{v}\cdot\vec{\pi})D + (\vec{u}^2 + \vec{v}^2)(H_1 - H_2) \tag{3.9}$$

They imply that on the constraint hypersurface we have $Q_i = 0\ \forall i$ or $P_i = 0\ \forall i$. certain submanifold of the phase space $\mathbb{R}^8$. Notice that the constraint hypersurface consists of the disjoint union of the following five varieties: $\{Q_j = P_j = 0,\ j = 1, 2, 3\}$, $\{\pm Q_3 > 0,\ P_3 = 0\}$, $\{Q_3 = 0,\ \pm P_3 > 0\}$.

3.2 Quantization

For the quantization we will follow [23] and choose the coordinate representation where the momentum operators act as derivative operators and the configuration operators as multiplication operators on the Hilbert space $\mathfrak{L}^2(\mathbb{R}^4)$ of square integrable functions $\psi(\vec{u},\vec{v})$:

$$\hat{\vec{p}}\psi(\vec{u},\vec{v}) = -i\hbar\vec{\nabla}_u\psi(\vec{u},\vec{v}) \qquad \hat{\vec{\pi}}\psi(\vec{u},\vec{v}) = -i\hbar\vec{\nabla}_v\psi(\vec{u},\vec{v})$$
$$\hat{u}_i\psi(\vec{u},\vec{v}) = u_i\psi(\vec{u},\vec{v}) \qquad \hat{v}_i\psi(\vec{u},\vec{v}) = v_i\psi(\vec{u},\vec{v}) \quad . \tag{3.10}$$

In the following we will skip the hats and set $\hbar = 1$.

For the constraint algebra to close we have to quantize the constraints in the following way:

$$H_1 = -\tfrac{1}{2}(\Delta_u + \vec{v}^2) \qquad H_2 = -\tfrac{1}{2}(\Delta_v + \vec{u}^2) \qquad D = -i(\vec{u}\cdot\vec{\nabla}_u - \vec{v}\cdot\vec{\nabla}_v)$$
$$\big[H_1, H_2\big] = iD \qquad \big[D, H_1\big] = 2iH_1 \qquad \big[D, H_2\big] = -2iH_2 \quad . \tag{3.11}$$

There arises no factor ordering ambiguity for the quantization of the observable algebra. The algebraic properties are preserved in the quantization process, i.e. Poisson brackets between observables O_{ij} are simply replaced by $-i\big[\cdot,\cdot\big]$.

We introduce a more convenient basis for the constraints:

$$H_+ = H_1 + H_2 \qquad H_- = H_1 - H_2 \qquad D = D \quad . \tag{3.12}$$

H_- is just the sum and difference of Hamiltonians for one-dimensional harmonic oscillators. (It is the generator of the compact subgroup $SO(2)$ of $Sl(2,\mathbb{R})$ and has discrete spectrum in $\mathbb{Z}$). The commutation relations are now:

$$\big[H_-, D\big] = -2iH_+ \qquad \big[H_+, D\big] = -2iH_- \qquad \big[H_+, H_-\big] = -2iD \quad . \tag{3.13}$$

The operator

$$\mathfrak{C} = \tfrac{1}{4}(D^2 + H_+^2 - H_-^2) \tag{3.14}$$

commutes with all three constraints (3.12), since it is the (quadratic) Casimir operator for $sl(2,r)$ (see Appendix A.1). According to Schur's lemma, it acts as a constant on the irreducible subspaces of the $sl(2,\mathbb{R})$ representation given by (3.12).

The quantum analogs of the classical identities (3.8) are

$$Q_1^2 + Q_2^2 - Q_3^2 = P_1^2 + P_2^2 - P_3^2 = \tfrac{1}{4}(D^2 + H_+^2 - H_-^2) = \mathfrak{C} \tag{3.15}$$
$$4Q_3P_3 = (\vec{u}^2 - \vec{v}^2)(H_+) - (\vec{u}\cdot\vec{p} + \vec{v}\cdot\vec{\pi})D + (\vec{u}^2 + \vec{v}^2)(H_-) \quad . \tag{3.16}$$

3.3 The Oscillator Representation

We are interested in the spectral decomposition of the Master Constraint Operator, which we define as

$$\widehat{\mathbf{M}} = D^2 + H_+^2 + H_-^2 = 4\mathfrak{C} + 2H_-^2. \tag{3.17}$$

The Master Constraint Operator is the sum of (a multiple of) the Casimir operator and H_-, which commutes with the Casimir. Therefore we can diagonalize these two operators simultaneously, obtaining a diagonalization of $\widehat{\mathbf{M}}$. We can achieve a diagonalization of the Casimir by looking for the irreducible subspaces of the $sl(2,\mathbb{R})$-representation given by (3.12), since the Casimir acts as a multiple of the identity operator on these subspaces. Hence we will attempt do determine the representation given by (3.12).

By introducing creation and annihilation operators

$$a_i = \frac{1}{\sqrt{2}}(u_i + \partial_{u_i}) \qquad a_i^\dagger = \frac{1}{\sqrt{2}}(u_i - \partial_{u_i}) \tag{3.18}$$

$$b_i = \frac{1}{\sqrt{2}}(v_i + \partial_{v_i}) \qquad b_i^\dagger = \frac{1}{\sqrt{2}}(v_i - \partial_{v_i}) \tag{3.19}$$

we can rewrite the constraints as

$$H_- = \sum_{i=1,2} (a_i^\dagger a_i - b_i^\dagger b_i) \tag{3.20}$$

$$H_+ = -\frac{1}{2}\sum_{i=1,2} (a_i^2 + (a_i^\dagger)^2 + b_i^2 + (b_i^\dagger)^2) \tag{3.21}$$

$$D = \frac{i}{2}\sum_{i=1,2} (-a_i^2 + (a_i^\dagger)^2 + b_i^2 - (b_i^\dagger)^2) \quad . \tag{3.22}$$

This $sl(2,\mathbb{R})$ representation is a tensor product of the following four representations (with $i \in \{1,2\}$):

$$\begin{aligned} (h_-)_{u_i} &= a_i^\dagger a_i + \frac{1}{2} & (h_-)_{v_i} &= -b_i^\dagger b_i - \frac{1}{2} \\ (h_+)_{u_i} &= -\frac{1}{2}((a_i^\dagger)^2 + a_i^2) & (h_+)_{v_i} &= -\frac{1}{2}((b_i^\dagger)^2 + b_i^2) \\ d_{u_i} &= \frac{i}{2}((a_i^\dagger)^2 - a_i^2) & d_{v_i} &= \frac{i}{2}(-(b_i^\dagger)^2 + b_i^2) \end{aligned} \tag{3.23}$$

The u_i-representations are known as oscillator representations ω and the v_i-representations as contragredient oscilalator representations ω^*, see appendix A.2 for a discussion of these representations. As is also explained there these representation are reducible into two irreducible representations $D(1/2)$ and $D(3/2)$ for the oscillator representation ω and $D^*(1/2)$ and $D^*(3/2)$ for the contragredient oscillator representation ω^*. The representation $D(1/2)$ respectively $D^*(1/2)$ acts on the space of even number Fock states, whereas $D(3/2)$ respectively $D^*(3/2)$ acts on the space of uneven Fock states. The representations $D(1/2)$ and $D(3/2)$ are members of the positive discrete series (of the two-fold covering group of $Sl(2,\mathbb{R})$), $D^*(1/2)$ and $D^*(3/2)$ are members of the negative discrete series. (We have listed all $sl(2,\mathbb{R})$-representations in appendix A.1.)

Our aim is to reduce the tensor product $\omega \otimes \omega \otimes \omega^* \otimes \omega^*$ into its irreducible components. The isotypical component with respect to the trivial representation would correspond to the physical Hilbert space. To begin with we consider the tensor product $\omega \otimes \omega$. The discussion for $\omega^* \otimes \omega^*$ is analogous.

To this end we utilize the observable O_{12} (and O_{34} for the tensor product $\omega^* \otimes \omega^*$). Since O_{12} commutes with the $sl(2,\mathbb{R})$-generators the eigenspaces of O_{12} are $sl(2,\mathbb{R})$-invariant. The observable O_{12} is diagonal in the "polarized" Fock basis, which is defined as the Fock basis with respect to the new creation and annihilation operators

$$A_\pm = \frac{1}{\sqrt{2}}(a_1 \mp i a_2) \qquad A_\pm^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger \pm i a_2^\dagger). \tag{3.24}$$

The "polarized" creation and annihilation operators for the v-coordiantes are

$$B_\pm = \frac{1}{\sqrt{2}}(b_1 \mp i b_2) \qquad B_\pm^\dagger = \frac{1}{\sqrt{2}}(b_1^\dagger \pm i b_2^\dagger). \tag{3.25}$$

With help of these operators we can write the $sl(2,\mathbb{R})$-generators for the $\omega \otimes \omega$ representation and for the $\omega^* \otimes \omega^*$ representation as

$$\begin{aligned} h_{-A} &= A_+^\dagger A_+ + A_-^\dagger A_- + 1 & h_{-B} &= -B_+^\dagger B_+ - B_-^\dagger B_- - 1 \\ h_A+ &= -(A_+ A_- + A_+^\dagger A_-^\dagger) & h_{+B} &= -(B_+ B_- + B_+^\dagger B_-^\dagger) \\ d_A &= i(A_+^\dagger A_-^\dagger - A_+ A_-) & d_B &= i(B_+ B_- - B_+^\dagger B_-^\dagger) \end{aligned} \tag{3.26}$$

and the observables O_{12} and O_{34} as

$$\begin{aligned} O_{12} &= u_1 p_2 - p_1 u_2 = A_+^\dagger A_+ - A_-^\dagger A_- \\ O_{34} &= \pi_1 v_2 - v_1 \pi_2 = -B_+^\dagger B_+ + B_-^\dagger B_- \quad . \end{aligned} \tag{3.27}$$

The (common) eigenspaces (corresponding to the eigenvalues $j, j' \in \mathbb{Z}$) for these observables are spanned by $\{|k_+, k_-, k'_+, k'_- >; k_+ - k_- = j \text{ and } k'_- - k'_+ = j'; k_+, k_-, k'_+, k'_- \in \mathbb{N}\}$, where $|k_+, k_-, k'_+, k'_- >$ denotes a Fock state with respect to the annihilation operators A_+, A_-, B_+, B_-. A closer inspection reveals that these eigenspaces are indeed invariant under the $sl(2,\mathbb{R})$-algebra.

The action of the $sl(2,\mathbb{R})$-algebra on each of the above subspaces is a realization of the tensor product representation $D(|j|+1) \otimes D^*(|j'|+1)$ (see A.1). That can be verified by considering the h_{-A}- and the h_{-B}-spectrum on these subspaces. The h_{-A}-spectrum is bounded from below by $(|j|+1)$, whereas the h_{-B}-spectrum is bounded from above by $-(|j'|+1)$. This characterizes $D(|j|+1)$- and $D^*(|j'|+1)$-representations respectively.

Up to now we have achieved

$$\omega \otimes \omega \otimes \omega^* \otimes \omega^* = \left[D(1) \oplus \sum_{k=2}^{\infty} 2D(k) \right] \otimes \left[D^*(1) \oplus \sum_{k=2}^{\infty} 2D^*(k) \right] \quad . \tag{3.28}$$

For a complete reduction of $\omega \otimes \omega \otimes \omega^* \otimes \omega^*$ we have to reduce the tensor products $D(|j|+1) \otimes D^*(|j'|+1)$.

3.4 The Spectrum of the Master Constraint Operator

In [22] the decomposition of all possible tensor products between unitary irreducible representations of $SL(2,R)$ was achieved.

(Actually [22] considers only representations of $SL(2,\mathbb{R})/\pm \text{Id}$, i.e. representations with uneven j and j'. However the results generalize to representations with even j or j'. See [28] for a reduction of all tensor products of $SL(2,\mathbb{R})$, using different methods.)

The strategy in this article is to calculate the spectral decomposition of the Casimir operator. Since the Casimir commutes with H_- one can consider the Casimir operator on each eigenspace of H_-.

Since the Master Constraint Operator is the sum $\widehat{\mathbf{M}} = 4\mathfrak{C} + 2H_-$ we can easily adapt the results of [22] for the spectral decomposition of the Master Constraint Operator. In the following we will shortly summarize the results for the spectrum of the Master Constraint Operator. The explicit eigenfunctions are constructed in appendix A.3.

To this end we define the subspaces $V(k,j,j')$, $k \in \mathbb{Z}$, $|j| \in \mathbb{N}$ by

$$H_-|_{V(k,j,j')} = k \qquad \text{and} \qquad O_{12}|_{V(k,j,j')} = j \qquad \text{and} \qquad O_{34}|_{V(k,j,j')} = j' \ . \tag{3.29}$$

$V(k,j,j')$ is the H_--eigenspace corresponding to the eigenvalue k of the tensor product representation $D(|j|+1) \otimes D^*(|j'|+1)$. Since the H_--spectrum is even for $(j-j')$ even and uneven for $(j-j')$ uneven these subspaces are vacuous for $k+j-j'$ uneven. One result of [22] is, that the spectrum of the Casimir operator is non-degenerate on these subspaces, which means that there exists a generalized eigenbasis in $\mathfrak{L}^2(\mathbb{R}^4)$ labeled by (k, j, j') and the eigenvalue $\lambda_{\mathfrak{C}}$ of the Casimir.

The spectrum of the Casimir $\mathfrak{C}$ on the subspace $V(k, j, j')$ has a discrete part only if $k > 0$ for $|j| - |j'| \geq 2$ or $k < 0$ for $|j| - |j'| \leq 2$. There is no discrete part if $||j| - |j'|| < 2$. The discrete part is for $(j - j')$ and k even

$$\begin{aligned}\lambda_{\mathfrak{C}} &= t(1-t) \quad \text{with} \quad t = 1, 2, \ldots, \tfrac{1}{2}\text{min}(|k|, ||j| - |j'||) \\ &= 0, -2, -6, \ldots \ . \end{aligned} \tag{3.30}$$

For $(j - j')$ and k odd we have

$$\begin{aligned}\lambda_{\mathfrak{C}} &= t(1-t) \quad \text{with} \quad t = \tfrac{3}{2}, \tfrac{5}{2}, \ldots, \tfrac{1}{2}\text{min}(|k|, ||j| - |j'||) \\ &= -\tfrac{3}{4}, -\tfrac{15}{4}, -\tfrac{35}{4} \ldots \ . \end{aligned} \tag{3.31}$$

The continuous part is in all cases the same and given by:

$$\lambda_{\mathfrak{C}} = \tfrac{1}{4} + x^2 \quad \text{with} \quad x \in \big[0, \infty\big) \quad . \tag{3.32}$$

The discrete part corresponds to unitary irreducible representations from the positive and negative discrete series of $SL(2, \mathbb{R})$, the continous part corresponds to the (two) principal series of $SL(2, \mathbb{R})$ (see appendix A.1).

For the spectrum of the Master Constraint Operator we have to multiply with 4 and add $2k^2$:

$$\begin{gathered}\lambda_{\widehat{\mathbf{M}}} = 4t(1-t) + 2k^2 \geq 2k^2 - k^2 + 2|k| \\ \text{with} \quad t = 1, 2, \ldots, \tfrac{1}{2}\text{min}(|k|, ||j| - |j'||) \text{ for even } k \\ \text{with} \quad t = \tfrac{3}{2}, \tfrac{5}{2} \ldots, \tfrac{1}{2}\text{min}(|k|, ||j| - |j'||) \text{ for odd } k \\ \text{and} \end{gathered} \tag{3.33}$$

$$\lambda_{\widehat{\mathbf{M}}} = 1 + x^2 + 2k^2 > 0 \tag{3.34}$$

As one can immediately see, the spectrum does not include zero. Since we have no discrete spectrum for $k = 0$ the lowest generalized eigenvalue for the master constraint is 1 from the continuous part.

We will attempt to overcome this problem by introducing a quantum correction to the Master Constraint Operator. Since 1 is the minimum of the spectrum we substract 1 ($\hbar^2$ if units are restored) from the Master Constraint Operator.

For the modified Master Constraint Operator we get one solution appearing in the spectral decomposition for each value of j and j'. We call this solution $|\lambda_{\mathfrak{C}} = \frac{1}{4}, k = 0, j, j' >$ and the linear span of these soltutions SOL'.(The above results show, that these quantum numbers are sufficient to label uniquely vectors in the kinematical Hilbert space.)

At the classical level we have several relations between observables, which are valid on the constraint hypersurface. For a physical meaningful quantization we have to check, whether these relations are valid or modified by quantum corrections.

For our modified Master Constraint Operator this seems not to be the case: At the classical level we have $Q_3 = 0$ or $P_3 = 0$. But on SOL', these observables evaluate to:

$$\begin{aligned} Q_3\,|\lambda_{\mathfrak{C}} = \tfrac{1}{4}, k = 0, j, j' > &= \tfrac{1}{2}(j - j')\,|\lambda_{\mathfrak{C}} = \tfrac{1}{4}, k = 0, j, j' > \\ P_3\,|\lambda_{\mathfrak{C}} = \tfrac{1}{4}, k = 0, j, j' > &= \tfrac{1}{2}(j + j')\,|\lambda_{\mathfrak{C}} = \tfrac{1}{4}, k = 0, j, j' > \quad . \end{aligned} \tag{3.35}$$

Since j and j' are arbitrary whole numbers, both Q_3 and P_3 can have arbitrary large eigenvalues (on the same eigenvector) in SOL'.

To solve this problem, we will modify the Master Constraint Operator again, by adding a constraint, which implements the condition $Q_3 = 0$ or $P_3 = 0$. Together with the identities (3.15) this would ensure that $Q_i = 0\,\forall i$ or $P_i = 0\,\forall i$ modulo quantum corrections.

One possibility for the modified constraint is $\widehat{\mathbf{M}}'' = \widehat{\mathbf{M}} - 1 + (Q_3P_3)^2$.(This operator is hermitian, since Q_3 and P_3 commute.) Because of the last relation of (3.15) this modification can be seen as adding the square (of one quarter) of the right hand side of this relation, i.e. the added part is the square of a linear combination of the constraints.

We already know the spectral resolution of $\widehat{\mathbf{M}}''$, since we used Q_3 and P_3 (or O_{12} and O_{34}) in the reduction process for the Master Constraint Operator. Solutions to the Master Constraint Operator $\widehat{\mathbf{M}}''$ are the states $|\lambda_{\mathfrak{C}} = \frac{1}{4}, k = 0, j, j' >$ with $|j| = |j'|$. We call this solution space SOL''. (Up to now this spase is just the linear span of states $|\lambda_{\mathfrak{C}} = \frac{1}{4}, k = 0, j, j' >$ with $|j| = |j'|$. Later we will specify a topology for this space.)

Now the observable algebra (3.6) does not leave this solution space invariant, since not all observables commute with the added constraint Q_3P_3. However, the observables (3.6) are redundant on SOL'', since they obey the relations (3.15). So the question is, whether one can find enough observables, which commute with Q_3P_3 (and with the constraints, we started with) to carry all relevant physical information. Apart from Q_3 and P_3 the operators $Q_1^2 + Q_2^2$ and $P_1^2 + P_2^2$ commute with the added constraint. But the latter do not carry additional information about physical states, because of the first relation in (3.15). Operators of the form $p_1(Q)Q_3 + p_2(P)P_3$, where $p_1(Q)$ (resp. $p_2(P)$) represents a polynomial in the Q-observables (P-observables), commute with Q_3P_3 on the subspace defined by $Q_3P_3 = 0$. Likewise operators of the form $p_1(Q)|\mathrm{sgn}(Q_3)| + p_2(P)|\mathrm{sgn}(P_3)|$, where sgn has values $1, 0$ and -1 (and is defined by the spectral theorem) leave SOL'' (formally) invariant.

In the following we will take as observable algebra the algebra generated by the elementary operators $|\mathrm{sgn}(Q_3)|Q_i|\mathrm{sgn}(Q_3)|$ and $|\mathrm{sgn}(P_3)|P_i|\mathrm{sgn}(P_3)|$. This algebra is closed under taking adjoints. Notice, however, that we may add operators such as $|\mathrm{sgn}(Q_3)|Q_+Q_+|\mathrm{sgn}(Q_3)|$ which does not leave the sectors invariant and thus destroy the superselection structure which is a physical difference from the results of [24]. The next section shows that the latter operator transforms states from the sector $\{\mathrm{sgn}(Q_3) = -1, \mathrm{sgn}(P_3) = 0\}$ to the sector $\{\mathrm{sgn}(Q_3) = +1, \mathrm{sgn}(P_3) = 0\}$ (since $Q_\pm, P_\pm$ are ladder operators, which raise or lower the Q_3, P_3 eigenvalues by 1 respectively). Likewise one can construct operators wich transform from the sector $\{\mathrm{sgn}(P_3) = 0\}$ to $\{\mathrm{sgn}(Q_3) = 0\}$ and vice versa: For instance

$$|\mathrm{sgn}(P_3)|\, P_+ \cdots P_+\, (1 - |\mathrm{sgn}(Q_3)|)(1 - |\mathrm{sgn}(P_3)|)\, Q_+ \cdots Q_+\, |\mathrm{sgn}(Q_3)| \tag{3.36}$$

has this property and leaves the solution space to the modified Master Constraint Operator invariant. Its adjoint is of the same form, transforming from the sector $\{\mathrm{sgn}(Q_3) = 0\}$ to the sector $\{\mathrm{sgn}(P_3) = 0\}$. Thus we may map between all five sectors mentioned before except for the origin. There seems to be no natural exclusion principle for these operators from the point of view of DID and thus we should take them seriously.

3.5 The Physical Hilbert space

Now one can use the spectral measure for the Master Constraint Operator and construct a scalar product in SOL' and SOL'' and then complete them into Hilbert spaces $\mathfrak{H}'$ and $\mathfrak{H}''$. This is done explicitly in Appendix A.3, here we only need that this can be done in principle.

The so achieved Hilbert space $\mathfrak{H}'$ has to carry a unitary representation of the observable algebra $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ (since these observables commute with the constraints). In particular we already know the spectra of Q_3 and P_3 to be the integers $\mathbb{Z}$, since we diagonalized them simultaneously with the Master Constraint Operator. These spectra are discrete, which means that in the constructed scalar product the states $|\lambda_{\mathfrak{C}} = \frac{1}{4}, k = 0, j, j' >$ (which are eigenstates for Q_3 and P_3) have a finite norm. So we can normalize them to states $||j,j'>>$ and in this way obtain a basis of $\mathfrak{H}'$.

Now, because of the identity (3.15) we also know the value of the $sl(2,\mathbb{R})$ Casimirs $Q_1^2+Q_2^2-Q_3^2$ and $P_1^2+P_2^2-P_3^2$ on $\mathfrak{H}'$ to be $\frac{1}{4}$. Together with the fact that $\mathfrak{H}'$ has a normalized eigenbasis $\{||j,j'>>, j,j' \in \mathbb{Z}\}$ (with respect to Q_3 and P_3) we can determine the unitary representation of $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ to be $P(t=1/2,\epsilon=0)_Q \otimes P(t=1/2,\epsilon=0)_P$ (see Appendix A.1). This fixes the action of the (primary) observable algebra to be (modulo phase factors, which can be made to unity by adjusting phases of the basis vectors):

$$\begin{aligned}
Q_+||j,j'>> &= \tfrac{1}{2\sqrt{2}}((j-j')+1)\,||j+1,j'-1>> \\
Q_-||j,j'>> &= \tfrac{1}{2\sqrt{2}}((j-j')-1)\,||j-1,j'+1>> \\
Q_3||j,j'>> &= \tfrac{1}{2}(j-j')\,||j,j'>> \\
P_+||j,j'>> &= \frac{1}{2\sqrt{2}}((j+j')+1)\,||j+1,j'+1>> \\
P_-||j,j'>> &= \frac{1}{2\sqrt{2}}((j+j')-1)\,||j-1,j'-1>> \\
P_3||j,j'>> &= \tfrac{1}{2}(j+j')|\,|j,j'>> \qquad (3.37)
\end{aligned}$$

From these results we can derive the action of the altered observable algebra on $\mathfrak{H}''$, i.e. on states $||j,\epsilon j>>$ with $\epsilon=\pm 1$:

$$\begin{aligned}
\Theta(Q_3)Q_+\Theta(Q_3)\,||j,\epsilon j>> &= \delta_{-1,\epsilon}\,(1-\delta_{-1,j})\,\tfrac{1}{\sqrt{2}}(j+\tfrac{1}{2})\,||j+1,\epsilon(j+1)>> \\
\Theta(Q_3)Q_-\Theta(Q_3)\,||j,j'>> &= \delta_{-1,\epsilon}\,(1-\delta_{+1,j})\,\tfrac{1}{\sqrt{2}}(j-\tfrac{1}{2})\,||j-1,\epsilon(j-1)>> \\
Q_3\,||j,\epsilon j>> &= \delta_{-1,\epsilon}\,j\,||j,\epsilon j>> \\
\Theta(P_3)P_+\Theta(P_3)\,||j,j'>> &= \delta_{1,\epsilon}\,(1-\delta_{-1,j})\,\tfrac{1}{\sqrt{2}}\,(j+\tfrac{1}{2})\,||j+1,\epsilon(j+1)>> \\
\Theta(P_3)P_-\Theta(P_3)\,||j,j'>> &= \delta_{1,\epsilon}\,(1-\delta_{+1,j})\,\tfrac{1}{\sqrt{2}}(j-\tfrac{1}{2})\,||j-1,\epsilon(j-1)>> \\
P_3\,||j,\epsilon j>> &= \delta_{1,\epsilon}\,j\,||j,\epsilon j>> \qquad (3.38)
\end{aligned}$$

where we abbreviated $|\mathrm{sgn}(\mathbf{O})|$ by $\Theta(\mathbf{O})$. The state $|0,0>$ is annihilated by all (altered) observables.

3.6 Algebraic Quantization

In [24] the $SL(2,\mathbb{R})$-model has been quantized in the Algebraic and Refined Algebraic Quantization framework. We will shortly review the results of the Algebraic Quantization scheme in order to compare them with the Master Constaint Programme.

In this scheme one starts with the auxilary Hilbert space $L^2(\mathbb{R}^4)$, the constraints (3.11) and a *- algebra of observables $\mathcal{A}^*$. One looks for a solution space for the constraints which carries an irreducible representation of $\mathcal{A}^*$ and for a scalar product on this space in which the star-operation becomes the adjoint operation.

The solution space $\tilde{V}$, which was found in [24] is the linear span of states $|j,\epsilon j>$ where j is in $\mathbb{Z}$ and $\epsilon \in \{-1,+1\}$. These states are expressible as smooth functions on the $(\vec{u},\vec{v})$ configuration space $\mathbb{R}^4$ and they solve the constraints (3.11).

The solution states can be expressed in our "polarized" Fock basis as follows

$$\begin{aligned}
|j,\epsilon j>= \sum_{m=0} (-1)^m |m+\tfrac{1}{2}(j+|j|), m+\tfrac{1}{2}(-j+|j|)> \otimes \\
|m+\tfrac{1}{2}(-\epsilon j+|j|), m+\tfrac{1}{2}(\epsilon j+|j|)> \ . \qquad (3.39)
\end{aligned}$$

(These states are the solutions $f(t=1;k=0,j,j'=\epsilon j)$, see (A.3).) Clearly, the states $|j,\epsilon j>$ solve the Master Constraint Operator $\widehat{\mathbf{M}}$. However there are much more solutions to the Master Constraint Operator (which do not necessarily solve the three constraints (3.11)).

The algebra $\mathcal{A}^*$ used in [24] is the algebra generated by the observables (3.6). The star-operation is defined by $Q_i^* = Q_i,\ P_i^* = P_i$ and extended to the full algebra by complex anti-linearity. This algebra is supplemented to the algebra $\mathcal{A}^*_{ext}$ by the operators $R_{\epsilon_1,\epsilon_2} = R^*_{\epsilon_1,\epsilon_2}$, which permute between the four different sectors of the classical constraint phase space:

$$R_{\epsilon_1,\epsilon_2} : (u_1,u_2,v_1,v_2,p_1,p_2,\pi_1,\pi_2) \mapsto (u_1,\epsilon_1 u_2,v_1,\epsilon_1\epsilon_2 v_2,p_1,\epsilon_1 p_2,\pi_1,\epsilon_1\epsilon_2\pi_2) \tag{3.40}$$

The algebra $\mathcal{A}^*$ has the following representation on $\tilde{V}$:

$$\begin{aligned} Q_3|j,\epsilon j> &= \delta_{-1,\epsilon}\, j\,|j,\epsilon j> \\ Q_\pm|j,\epsilon j> &= \delta_{-1,\epsilon}\,(\tfrac{\mp i}{\sqrt{2}}|j|)\,|(j\pm 1),\epsilon\,(j\pm 1)> \\ P_3|j,\epsilon j> &= \delta_{+1,\epsilon}\, j\,|j,\epsilon j> \\ P_\pm|j,\epsilon j> &= \delta_{+1,\epsilon}\,(\tfrac{\mp i}{\sqrt{2}}|j|)\,|(j\pm 1),\epsilon\,(j\pm 1)> \end{aligned} \tag{3.41}$$

The state $|j=0,j'=0>$ is annihilated by all operators in $\mathcal{A}^*$, in particular, it generates an invariant subspace for $\mathcal{A}^*$. Now, it is not possible to introduce an inner product on $\tilde{V}$, in which the star-operation becomes the adjoint operation (because the $SO(2,2)$-representation defined by (3.41) is non-unitary). However, since $|0,0>$ generates an invariant subspace one can take the quotient $\tilde{V}/\{c|0,0>, c\in\mathbb{C}\}$, consisting of equivalence classes $\big[v\big] = \{v + c\,|0,0>,\ c\in\mathbb{C}\}$ where $v\in\tilde{V}$. (In particular $\big[|0,0>\ \big]$ is the null vector $\big[0\big]$.) The $so(2,2)$-representation on $\tilde{V}$ then defines a representation on this quotient space by $O([v]) = [O(v)]$, where O is an $so(2,2)$-operator. (This representation is well defined because we are quotienting out an invariant subspace.) A basis in this quotient space is $\{\big[|j,\epsilon j>\ \big], j\in\mathbb{N}-\{0\}\}$. In the following we will drop the equivalence class brackets $[\cdot]$.

The quotient representation is the direct sum of four (unitary) irreducible representations of $so(2,2)$, labeled by $\text{sgn}(j)=\pm 1$ and $\epsilon=\pm 1$. The inner product, which makes these representations unitary is

$$< j_1,\epsilon_1 j_1|j_2,\epsilon_2 j_2 >= c(\text{sgn}(j),\epsilon)\,\delta_{j_1,j_2}\,\delta_{\epsilon_1,\epsilon_2}\,|j| \tag{3.42}$$

where $c(\text{sgn}(j),\epsilon)$ are four independent positiv constants.

By taking the reflections $R_{\epsilon_1,\epsilon_2}\in O(2,2)$ into account, we can partially fix these constants. Their action on states in $\mathfrak{L}^2(\mathbb{R}^4)$ is

$$(R_{\epsilon_1,\epsilon_2}\psi)(u_1,u_2,v_1,v_2) = \psi(u_1,\epsilon_1 u_2,v_1,\epsilon_1\epsilon_2 v_2) \quad . \tag{3.43}$$

States with angular momenta j and j' are mapped to states with angular momenta $\epsilon_1 j$ and $\epsilon_1\epsilon_2 j'$. Therefore the $R_{\epsilon_1,\epsilon_2}$'s effect, that the quotient representation of the observable algebra becomes an irreducible one. Since $R_{\epsilon_1,\epsilon_2}$ is in $O(2,2)$, it is a natural requirement for them to act by unitary operators. This fixes the four constants $c(\text{sgn}(j),\epsilon)$ to be equal (and in the following we will set them to 1).

This gives for the action of the algebra $\mathcal{A}^*$ on the normalized basis vectors $|j,\epsilon j>_N := \frac{1}{\sqrt{|j|}}|j,\epsilon\, j>,\ j\in\mathbb{Z}-\{0\}$:

$$\begin{aligned} Q_3|j,\epsilon j>_N &= \delta_{-1,\epsilon}\, j\ |j,\epsilon j>_N \\ Q_\pm|j,\epsilon j>_N &= \delta_{-1,\epsilon}\,(\tfrac{\mp i}{\sqrt{2}}\sqrt{|j(j\pm 1)|})\ |(j\pm 1),\epsilon\,(j\pm 1)>_N \\ P_3|j,\epsilon j>_N &= \delta_{+1,\epsilon}\, j\ |j,\epsilon j>_N \\ P_\pm|j,\epsilon j>_N &= \delta_{+1,\epsilon}\,(\tfrac{\mp i}{\sqrt{2}}\sqrt{|j(j\pm 1)|})\ |(j\pm 1),\epsilon\,(j\pm 1)>_N \quad . \end{aligned} \tag{3.44}$$

In the limit of large j the right hand sides of (3.38) and (3.44), ie. the matrix elements of the observables in the two qunatizations, coincide except for phase factors. These can be made equal by adjusting the phase factors of the respective basic vectors. Therefore both quantization programs lead to the same semiclassical limit.

A first crucial difference in the results of the two quantization approaches is that in the Master Constraint Programme the vector $||j = 0, j' = 0 >>$ is included in the physical Hilbert space whereas it is excluded during the Algebraic Quantization process. If we exclude the sector changing operators mentioned above by hand, then $||j = 0, j' = 0 >>$ is annihilated by the altered observable algebra and likewise cannot be reached by applying observables to other states in the physical Hilbert space. If we include the sector changing operators then $|j = 0, j' = 0 >$ is still not in the range of any observable because the observables are sandwiched beween operators of the form $|\text{sgn}(Q_3)|$, $|\text{sgn}(Q_3)|$. However, one can map between all the remaining sectors which thus provides a second difference with [24].

4 Conclusions

What we learnt in this paper is that the Master Constraint Programme can also successfully be applied to the difficult of constraint algebras generating $non-amenable$, $non-compact$ gauge groups. As was observed for instance in [29] this is a complication which affects the group averaging proposal [12] for solving the quantum constraints quite drastically in the sense that the physical Hilbert space depends critically on the choice of a dense subspace of the Hilbert space. The Master Constraint Programme also faces complications: The spectrum is supported on a genuine subset of the positive real line not containing zero. Our proposal to subract the zero point of the spectrum from the Master Constraint, which can be considered as a quantum correction[1] because it is proportional to $\hbar^2$ worked and produced an acceptable physical Hilbert space.

Of course, it is unclear whether that physical Hilbert space is in a sense the only correct choice because the models discussed are themselves not very physical and therefore we have only mathematical consistency as a selection criterion at our disposal, such as the fact that the algebraic approach reaches the same semiclassical limit by an independent method. Nevertheless, it is important to notice that DID produces somewhat different results than algebraic and RAQ methods, in particular, the superselection theory is typically trivial in contrast to those programmes. It would be good to know the deeper or intuitive reason behind this and other differences. Obviously, further work on $non-amenable$ groups is necessary, preferrably in an example which has a physical interpretation, in order to settle these interesting questions.

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[1]The fact that the correction is quadratic in $\hbar$ rather than linear in contrast to the normal ordering correction of the harmonic osciallor can be traced back to the fact that harmonic oscillator Hamiltonian can be considered the Master Constraint for the second class pair of constraints $p = q = 0$ while the $sl(2R)$ constraints are first class.

A Review of the Representation Theory of $SL(2,\mathbb{R})$ and its various Covering Groups

A.1 $sl(2,\mathbb{R})$ Representations

In this section we will review unitary representations of $sl(2,\mathbb{R})$, see [31, 30].

In the defining two-dimensional representation the $sl(2,\mathbb{R})$-algebra is spanned by

$$h = \frac{-1}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad n_1 = \frac{-1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad n_2 = \frac{-1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

with commutation relations

$$\big[h, n_1\big] = in_2 \qquad \big[n_2, h\big] = in_1 \qquad \big[n_1, n_2\big] = -ih \quad . \tag{A.2}$$

We introduce raising and lowering operators $n^\pm$ as complex linear combinations $n^\pm = n_1 \pm in_2$ of n_1 and n_2, which fulfill the algebra

$$\big[h, n^\pm\big] = \pm n^\pm \qquad \big[n^+, n^-\big] = -2h \quad . \tag{A.3}$$

The Casimir operator, which commutes with all $sl(2,\mathbb{R})$-algebra operators, is

$$\mathfrak{C} = -h^2 + \tfrac{1}{2}(n^+n^- + n^-n^+) = -h^2 + n_1^2 + n_2^2 \quad . \tag{A.4}$$

We are interested in unitary irreducible representations of $sl(2,\mathbb{R})$, i.e. representations where h, n_1 and n_2 act by self-adjoint operators on a Hilbert space, which does not have non-trivial subspaces, that are left invariant by the $sl(2,\mathbb{R})$-operators. According to Schur's Lemma the Casimir operator acts on an irreducible space as a multiple of the identity operator $\mathfrak{C} = \mathfrak{c}\,\mathrm{Id}$. Since n_1 and n_2 are self-adjoint operators, the raising operator n^+ is the adjoint of the lowering operator n^- and vice versa. (For notational convenience we often do not discriminate between elements of the algebra and the operators representing them.)

In general the $sl(2,\mathbb{R})$-representations do not exponentiate to a representation of the group $SL(2,\mathbb{R})$ but to the universal covering group $\widetilde{SL(2,\mathbb{R})}$. Since h is the generator of the compact subgroup of $\widetilde{SL(2,\mathbb{R})}$ it will have discrete spectrum (and therefore normalizable eigenvectors) in a unitary representation. If the $sl(2,\mathbb{R})$-representation exponentiates to an $SL(2,\mathbb{R})$ representation, h has spectrum in $\{\frac{1}{2}n, n \in \mathbb{Z}\}$. If in this group representation the center $\pm\mathrm{Id}$ acts trivially, it is also an $SO(2,1)$ representation, since $SO(2,1)$ is isomorphic to the quotient group $SL(2,\mathbb{R})/\{\pm\mathrm{Id}\}$. In this case h has spectrum in $\mathbb{Z}$.

Now, assume that $|\mathfrak{h}>$ is an eigenvector of h with eigenvalue $\mathfrak{h}$. Using the commutation relations (A.3) one can see, that $n^\pm|\mathfrak{h}>$ is either zero or an eigenvector of h with eigenvalue $\mathfrak{h}\pm 1$. By repeated application of n^+ or n^- to $|\mathfrak{h}>$ one therefore obtains a set of eigenvectors $\{|\mathfrak{h}+n>\}$ and corresponding eigenvalues $\{\mathfrak{h}+n\}$, where n is an integer. This set of eigenvalues may or may not be bounded from above or below.

Similarly, one can deduce from the commutation relations that $n^+n^-|\mathfrak{h}>$ and $n^-n^+|\mathfrak{h}>$ are both eigenvectors of h with eigenvalue $\mathfrak{h}$ (or zero). Apriori these eigenvectors do not have to be a multiple of $|\mathfrak{h}>$, since it may be, that h has degenerate spectrum. But this is excluded by the relations

$$n^+n^- = h^2 - h + \mathfrak{C} \qquad n^-n^+ = h^2 + h + \mathfrak{C} \quad , \tag{A.5}$$

obtained by using (A.3,A.4). (Remember, that h and $\mathfrak{C}$ act as multiples of the identity on $|\mathfrak{h}>$.) From this one can conclude that the set $\{|\mathfrak{h}+n>\}$ is invariant under the $sl(2,\mathbb{R})$-algebra (modulo multiples) and hence can be taken as a complete basis of the representation space.

One can use the relations (A.5) to set constraints on possible eigenvalues of h and $\mathfrak{C}$. Consider the scalar products

$$\begin{aligned} <\mathfrak{h}\pm 1|\mathfrak{h}\pm 1> &= <\mathfrak{h}|(n^{\pm})^{\dagger}n^{\pm}|\mathfrak{h}>=<\mathfrak{h}|n^{\mp}n^{\pm}|\mathfrak{h}> \\ &= <\mathfrak{h}|h^2 \pm h + \mathfrak{C}|\mathfrak{h}>=(\mathfrak{h}^2 \pm \mathfrak{h} + \mathfrak{c}) <\mathfrak{h}|\mathfrak{h}> \ . \end{aligned} \tag{A.6}$$

Since the norm of a vector has to be positive one obtains the inequalities

$$\mathfrak{h}^2 \pm \mathfrak{h} + \mathfrak{c} \geq 0 \tag{A.7}$$

for the spectrum of h and the value of the Casimir $\mathfrak{C} = \mathfrak{c}\,\mathrm{Id}$.

To summarize what we have said so far, we can specify a unitary irreducible representation with the help of the spectrum of h and the eigenvalue of the Casimir $\mathfrak{c}$. The spectrum of h is non-degenerate and may be unbounded or bounded from below or from above. Together with $\mathfrak{c}$ the spectrum has to fulfill the inequalities (A.7). In this way one can find the following irreducible representations of $sl(2,\mathbb{R})$ (For an explicit description, how one can find the allowed representation parameters, see [31, 30]):

(a) The principal series $P(t,\epsilon)$ where $t \in \{\frac{1}{2} + ix, x \in \mathbb{R} \wedge x \geq 0\}$ and $\epsilon = \mathfrak{h}(\mathrm{mod}\, 1) \in (-\frac{1}{2}, \frac{1}{2}]$.
The spectrum of h is unbounded and given by $\{\epsilon + n, n \in \mathbb{Z}\}$. The Casimir eigenvalue is $\mathfrak{c} = t\,(1-t) \geq \frac{1}{4}$. (For $t = \frac{1}{2}, \epsilon = \frac{1}{2}$ the representation $P(t,\epsilon)$ is reducible into $D(1)$ and $D^*(1)$ see below.)

(b) The complementary series $P_c(t,\epsilon)$ where $\frac{1}{2} < t < 1$ and $|\epsilon| < 1 - t$.
The spectrum of h is unbounded and given by $\{\epsilon + n, n \in \mathbb{Z}\}$. The Casimir eigenvalue is $0 < \mathfrak{c} = t\,(1-t) < \frac{1}{4}$

(c) The positive discrete series $D(k)$ where $k > 0$.
Here, the spectrum of h is bounded from below by $\frac{1}{2}k$ and we have $\mathrm{spec}(h) = \{\frac{1}{2}k + n, n \in \mathbb{N}\}$. The value of the Casimir is $\mathfrak{c} = \frac{1}{2}k - \frac{1}{4}k^2 \leq \frac{1}{4}$.

(d) The negative discrete series $D^*(k)$ where $k > 0$.
In this case the spectrum of h is bounded from above by $-\frac{1}{2}k$ and we have $\mathrm{spec}(h) = \{-\frac{1}{2}k - n, n \in \mathbb{N}\}$. The value of the Casimir is $\mathfrak{c} = \frac{1}{2}k - \frac{1}{4}k^2 \leq \frac{1}{4}$.

(e) The trivial representation.

As mentioned above representations with an integral h-spectrum can be exponentiated to representations of the group $SO(2,1)$, if the spectrum includes half integers one obtains representations of $SL(2,\mathbb{R})$ and for $\mathrm{spec}(h) \in \{\frac{1}{4}n, n \in \mathbb{Z}\}$ representations of the double cover of $SL(2,\mathbb{R})$ (the metaplectic group).

Finally we want to show, how one can uniquely determine the action of the $sl(2,\mathbb{R})$-algebra in a representation from the principal series. (The other cases are analogous, but we need this case in section 3.5.) To this end we assume that the vectors $||\mathfrak{h}>>$ are normalized eigenvectors of h with eigenvalue $\mathfrak{h}$. Applying $n^{\pm}$ gives a multiple of $||\mathfrak{h} \pm 1>>$:

$$n^{\pm}||\mathfrak{h}>> = A_{\pm}(\mathfrak{h})||\mathfrak{h} \pm 1>> \quad . \tag{A.8}$$

Using relation (A.5) one obtains for the coefficients $A_{\pm}(\mathfrak{h})$

$$n^{\mp}n^{\pm}||\mathfrak{h}>> = A_{\mp}(\mathfrak{h} \pm 1)A_{\pm}(\mathfrak{h})|\mathfrak{h}>> = (\mathfrak{h}^2 \pm \mathfrak{h} + \mathfrak{c})||\mathfrak{h}>> \quad . \tag{A.9}$$

Furthermore

$$A_+(\mathfrak{h}) = << \mathfrak{h}+1||n^+||\mathfrak{h} >> = \overline{<< \mathfrak{h}||n^-||\mathfrak{h}+1 >>} = \overline{A_-(\mathfrak{h}+1)} \quad . \tag{A.10}$$

The solution to these equations is

$$A_+ = c_+(\mathfrak{h})\,(\mathfrak{h}+t) \qquad A_-(\mathfrak{h}) = c_-(\mathfrak{h})\,(\mathfrak{h}-t) \tag{A.11}$$

where $|c_\pm| = 1$ and $c_+(\mathfrak{h})c_-(\mathfrak{h}+1) = 1$. Solutions with different $c_\pm$ are related by a phase change for the states $||\mathfrak{h} >>$.

A.2 Oscillator Representations

A.2.1 The Oscillator Representation

Here we will summarize some facts about oscillator representations, following [19, 20].

The oscillator representation is a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ the double cover of $SL(2,\mathbb{R})$ (the so-called metaplectic group) and is also known under the names Weil representation, Segal-Shale-Weil representation or harmonic representation.

The associated representation ω of the Lie algebra $sl(2,\mathbb{R})$ on $\mathfrak{L}^2(\mathbb{R})$ is given by

$$\begin{aligned} h = \tfrac{1}{2}(a^\dagger a + \tfrac{1}{2}) \qquad & n_1 = \tfrac{1}{4}(a^\dagger a^\dagger + aa) \qquad && n_2 = -i\tfrac{1}{4}(a^\dagger a^\dagger - aa) \\ & n^+ = n_1 + in_2 = \tfrac{1}{2}a^\dagger a^\dagger \qquad && n^- = n_1 - in_2 = \tfrac{1}{2}aa \end{aligned} \tag{A.12}$$

where we introduced annihilation and creation operators

$$a = \tfrac{1}{\sqrt{2}}(x+ip) \quad \text{and} \quad a^\dagger = \tfrac{1}{\sqrt{2}}(x-ip) \quad \text{with} \quad p = -i\tfrac{d}{dx} \quad . \tag{A.13}$$

The operator h is (half of) the harmonic oscillator Hamiltonian and represents the infinitesimal generator of the two-fold covering group of $SO(2)$. It has discrete spectrum spec$(h) = \{\frac{1}{4} + \frac{1}{2}n, n \in \mathbb{N}\}$ and its eigenstates are the Fock states $|n >= (n!)^{-1/2}(a^\dagger)^n|0>$; $a|0 >= 0$, which form an orthonormal basis of $\mathfrak{L}^2(\mathbb{R})$. As can be easily seen, the representation (A.12) leaves the spaces of even and odd number Fock states invariant, therefore the representation is reducible into two subspaces. These subspaces are irreducibel since one can reach each (un-)even Fock state $|n>$ by applying powers of n^+ or n^- to an arbitrary (un-)even Fock state $|n'>$.

Since we have an h-spectrum which is bounded from below by $\frac{1}{4}$ for the even number states and $\frac{3}{4}$ for the uneven number states, the corresponding representations are $D(1/2)$ and $D(3/2)$ from the positive discrete series (of the metaplectic group).

The Casimir of the oscillator representation is a constant:

$$\mathfrak{C}(\omega) = -h^2 + \tfrac{1}{2}(n^+n^- + n^-n^+) = \tfrac{3}{16} \quad . \tag{A.14}$$

This confirms the finding $\omega \simeq D(1/2) \oplus D(3/2)$, since we have

$$\mathfrak{C}(D(1/2)) = \tfrac{1}{4}\tfrac{1}{2}(2-\tfrac{1}{2}) = \tfrac{3}{16} = \tfrac{1}{4}\tfrac{3}{2}(2-\tfrac{3}{2}) = \mathfrak{C}(D(3/2)) \quad . \tag{A.15}$$

In chapter 3 we use an $sl(2,\mathbb{R})$-basis $\{h_{-u_i} = 2h,\ h_{+u_i} = 2n_1,\ d = 2n_2\}$. If one would rewrite the (u_i)-representation (3.23) in terms of $\{h, n_1, n_2\}$ it would differ from the representation (A.12) by minus signs in n_1 and n_2. Nevertheless the (u_i)-representation is unitarily equivalent to the representation (A.12), where the unitary map is given by the Fourier transform $\mathfrak{F}$. This can be easily seen by using the transformation properties of annihilation and creation operators under Fourier transformation: $\mathfrak{F}a\mathfrak{F}^{-1} = ia$ and $\mathfrak{F}a^\dagger\mathfrak{F}^{-1} = -ia^\dagger$.

A.2.2 Contragredient Oscillator Representations

The contragredient oscillator representation ω^* on $\mathfrak{L}^2(\mathbb{R})$ is given by

$$h^* = -\tfrac{1}{2}(a^\dagger a + \tfrac{1}{2}) \qquad n_1^* = -\tfrac{1}{4}(a^\dagger a^\dagger + aa) \qquad n_2^* = -i\tfrac{1}{4}(a^\dagger a^\dagger - aa)$$
$$n^{+*} = n_1^* + in_2^* = -\tfrac{1}{2}aa \qquad n^{-*} = n_1^* - in_2^* = -\tfrac{1}{2}a^\dagger a^\dagger \tag{A.16}$$

Here, h^* has strictly negativ spectrum $\text{spec}(h^*) = \{-\frac{1}{4} - \frac{1}{2}n, n \in \mathbb{N}\}$. An analogous discussion to the one above reveals that the contragredient oscillator representation is the direct sum of the representations $D^*(1/2)$ and $D^*(3/2)$ from the negative discrete series (of the metaplectic group). The Casimir evaluates to the same constant as above $\mathfrak{C}(\omega^*) = \frac{3}{16} = \mathfrak{C}(D^*(1/2) = \mathfrak{C}(D^*(3/2))$.

A.2.3 Tensor Products of Oscillator Representations

Now one can consider the tensor product of p oscillator and q contragredient oscillator representations. The tensor product $(\otimes_p\omega) \otimes (\otimes_q\omega^*)$ will be abbreviated by $\omega^{(p,q)}$. The representation space is $(\otimes_p\mathfrak{L}^2(\mathbb{R})) \otimes (\otimes_q\mathfrak{L}^2(\mathbb{R}))$ wich can be identified with $\mathfrak{L}^2(\mathbb{R}^{p+q})$. The tensor product representation (of the $sl(2,\mathbb{R})$-algebra) is given by

$$\begin{aligned}
h^{(p,q)} &= \tfrac{1}{2}\sum_{j=1}^{p}(a_j^\dagger a_j + \tfrac{1}{2}) - \tfrac{1}{2}\sum_{j=p+1}^{p+q}(a_j^\dagger a_j + \tfrac{1}{2}) \\
n_1^{(p,q)} &= \tfrac{1}{4}\sum_{j=1}^{p}(a_j^\dagger a_j^\dagger + a_j a_j) - \tfrac{1}{4}\sum_{j=p+1}^{p+q}(a_j^\dagger a_j^\dagger + a_j a_j) \\
n_2^{(p,q)} &= -\tfrac{i}{4}\sum_{j=1}^{p+q}(a_j^\dagger a_j^\dagger - a_j a_j)
\end{aligned} \tag{A.17}$$

where a_j and $a_j^\dagger$ denote annihilation and creation operators for the j-th coordinate in $\mathbb{R}^{p+q}$:

$$a_j = \tfrac{1}{\sqrt{2}}(x^j + ip_j) \quad \text{and} \quad a^\dagger = \tfrac{1}{\sqrt{2}}(x^j - ip_j) \quad \text{with} \quad p_j = -i\partial_j \quad . \tag{A.18}$$

On $\mathfrak{L}^2(\mathbb{R}^{p+q})$ we also have a natural action of the generalized orthogonal group $O(p,q)$ given by $g \cdot f(\vec{x}) = f(g^{-1}(\vec{x}))$, where $g \in O(p,q)$ and $\vec{x} \in \mathbb{R}^{p+q}$. This action commutes with the $sl(2,\mathbb{R})$ action, which can rapidly be seen, if we calculate the following $sl(2,\mathbb{R})$ basis:

$$\begin{aligned}
e^+ &= 2h^{(p,q)} + 2n_1^{(p,q)} = \sum_{j=1}^{p}(x^j)^2 - \sum_{j=p+1}^{p+q}(x^j)^2 = g_{ij}x^i x^j \\
e^- &= 2h^{(p,q)} - 2n_1^{(p,q)} = \sum_{j=1}^{p}(p_j)^2 - \sum_{j=p+1}^{p+q}(p_j)^2 = g^{ij}p_i p_j \\
d &= -2n_2^{(p,q)} = \tfrac{1}{2}\sum_{j=1}^{p+q}(x^j p_j + p_j x^j) = \tfrac{1}{2}g_j^i(x^j p_i + p_i x^j) \quad .
\end{aligned} \tag{A.19}$$

Here g^{ij} is inverse to the metric $g_{ij} = \text{diag}(+1,\ldots,+1,-1,\ldots,-1)$ (with p positive and q negative entries) so that $g_j^i := g^{ik}g_{kj} = \delta_j^i$, where in the last formula and in the right hand sides of (A.19) we summed over repeated indices. Since $O(p,q)$ leaves by definition the metric g_{ij} invariant, the $sl(2,\mathbb{R})$-operators (A.19) (and all their linear combinations) are left invariant by the $O(p,q)$-action: $\rho(g^{-1})s\hat{\rho}(g) = s$, where s is an element from the $sl(2,\mathbb{R})$-algebra representation and $\rho(g)$ denotes the action of $g \in O(p,q)$ on states in $\mathfrak{L}^2(\mathbb{R}^{p+q})$.

The action of $O(p,q)$ induces a unitary representation ρ of $O(p,q)$ on $\mathfrak{L}^2(\mathbb{R}^{p+q})$ (defined by $\rho(g)f(\vec{x}) = f(g^{-1}(\vec{x}))$ for $f(\vec{x}) \in \mathfrak{L}^2(\mathbb{R}^{p+q})$). The derived representation of the Lie algebra $so(p,q)$ is given by:

$$A_{jk} = x^j p_k - x^k p_j, \quad j,k = 1,\ldots,p \tag{A.20}$$
$$B_{jk} = x^j p_k - x^k p_j, \quad j,k = p+1,\ldots,p+q \tag{A.21}$$
$$C_{jk} = x^j p_k + x^k p_j, \quad j = 1,\ldots p,\ k = p+1,\ldots,p+q \quad . \tag{A.22}$$

The operators A_{jk} and B_{jk} span the Lie algebra $so(p) \times so(q)$ of the maximal compact group $O(p) \times O(q)$. (From this one can conclude that A_{jk} and B_{jk} have discrete spectra.)

For the representations $\rho(O(p,q)$ and $\omega^{(p,q)}(\widetilde{SL(2,\mathbb{R})})$ there is a remarkable theorem, which we will cite from [20]:

The groups of operators $\rho(O(p,q))$ and $\omega^{(p,q)}(SL(\tilde{2},\mathbb{R}))$ generate each other commutants in the sense of von Neumann algebras. Thus there is a direct integral decomposition

$$\mathfrak{L}^2(\mathbb{R}^{p+q}) \simeq \int \sigma_s \otimes \tau_s \, ds \tag{A.23}$$

where ds is a Borel measure on the unitary dual of $\widetilde{SL(2,\mathbb{R})}$, and σ_s and τ_s are irreducible representations of $O(p,q)$ and $\widetilde{SL(2,\mathbb{R})}$, respectively. Moreover σ_s and τ_s determine each other almost everywhere with respect to ds.

This means, that if we are interested in the decomposition of the $\rho(O(p,q))$-representation, we can equally well decompose $\widetilde{SL(2,\mathbb{R})}$, which may be an easier task. (We used this in example 2.)

Furthermore, this theorem is very helpful if one of the two group algebras represents the constraints (say $so(p,q)$) and the other coincides with the algebra of observables (as is the case in examples 2 and 3). The constraints would then impose that the physical Hilbert space has to carry the trivial representation of $so(p,q)$. Now if the trivial representation is included in the decomposition (A.23) we can adopt as a physical Hilbert space the isotypical component of the trivial representation (i.e. the direct sum of all trivial representations which appear in the decomposition of $\mathfrak{L}^2(\mathbb{R}^{p+q})$ with respect to the group $O(p,q)$)). The above cited theorem ensures that this space carries a unitary irreducible[2] representation of the observable algebra. The scalar product on this Hilbert space is determined by this representation.

The same holds if we have the $sl(2,\mathbb{R})$ algebra as constraints and $so(p,q)$ as the algebra of observables.

To determine the representation of the observable algebra on the physical Hilbert space the following relation between the (quadratic) Casimirs of the two algebras involved is administrable (see [19]):

$$4(-(h^{(p,q)})^2 + (n_1^{(p,q)})^2 + (n_2^{(p,q)})^2) = -\sum_{j<k\leq p} A_{jk}^2 - \sum_{p<j<k\leq p+q} B_{jk}^2 + \sum_{j\leq p,\, k>p} C_{jk}^2 + 1 - (\tfrac{p+q}{2} - 1)^2 \tag{A.24}$$

One can check this relation by direct computation.

Now, what was said above works perfectly well in the case of compact gauge groups as for the case of $SO(3)$ in [3] but in the examples 2 and 3 the trivial representation of the corresponding constraint algebra does not appear in the decomposition (A.23). (In fact it never appears, if the constraint algebra is $sl(2,\mathbb{R})$.)

[2]In general the decomposition with respect to $SO(p,q)$ differs from the decomposition of $O(p,q)$, i.e. if one considers in addition to rotations reflections. One has to take the transformation behavior of vectors in $\mathfrak{L}^2(\mathbb{R}^{p+q})$ under reflections in $O(p,q)$ into account to get uniqueness and irreducibility.

To elaborate on this, we will sketch (following [20]) how one can achieve the decomposition (A.23) using $sl(2,\mathbb{R})$-representation theory. This decomposition is used in examples 2 and 3.

To begin with, we consider the p-fold tensor product $\otimes_p \omega \simeq \otimes_p (D(1/2) \oplus D(3/2))$. One can reduce this tensor product by using repeatedly

$$D(l_1) \otimes D(l_2) \simeq \sum_{j=0}^{\infty} D(l_1 + l_2 + 2j) \tag{A.25}$$

from [19]. Or one uses the above theorem with $q = 0$ and reduces rather the regular representation of $O(p)$ on $\mathfrak{L}^2(\mathbb{R}^p)$. This reduction is known to be given by (generalized) spherical harmonics. Via the identity (A.24) one can determine the Casimir of the corresponding $sl(2,\mathbb{R})$ representation. This determines uniquely the representation for $p \geq 2$, since we know from (A.25) that there can only appear representations $D(j)$ with $j \geq 1$. (For the case $p = 1$ we already have the decomposition $\omega = D(1/2) \oplus D(3/2)$, where $D(1/2)$ and $D(3/2)$ are not being distinguished by the Casimir. But the vectors in these two representations are being distinguished by their transformation behavior under $O(1)$, where $O(1)$ consists just of the reflection $x \mapsto -x$ and the identity.) In this way on gets the explicit form of (A.23) for the case $q = 0$ (see [20]):

$$\omega^{(p,0)} \simeq \sum_{j=0}^{\infty} \mathfrak{H}_{p,j} \otimes D(j + p/2) \tag{A.26}$$

where $\mathfrak{H}_{p,j}$ is the representation of $O(p)$ defined by the spherical harmonics (for $S^{(p-1)}$) of degree j. The dimension of $\mathfrak{H}_{p,j}$, in the following denoted by $C_{p,j}$, is finite and is equal to the multiplicity of $D(j + p/2)$ in $\omega^{(p,0)}$. So, if one is just interested in the $sl(2,\mathbb{R})$ structure, one would have

$$\omega^{(p,0)} \simeq \sum_{j=0}^{\infty} C_{p,j}\, D(j + p/2) \quad . \tag{A.27}$$

The discussion for the tensor product $\omega^{(0,q)}$ is analogous, all representations $D(l)$ are just replaced by $D^*(l)$:

$$\omega^{(0,q)} \simeq \sum_{j=0}^{\infty} C_{q,j}\, D^*(j + q/2) \quad . \tag{A.28}$$

Therefore, for the complete reduction of $\omega^{(p,q)}$ we have to tackle

$$\omega^{(p,q)} \simeq \sum_{j,j'=0}^{\infty} (C_{p,j}\, C_{q,j'})\, D(j + p/2) \otimes D^*(j + q/2) \quad , \tag{A.29}$$

i.e. tensor products of the form $D(l_1) \otimes D^*(l_2)$ (for l_1, l_2 positive half integers). We will take these from [20]: Suppose $l_2 \geq l_1$. Then $D(l_1) \otimes D^*(l_2)$ decomposes as

$$D(l_1) \otimes D^*(l_2) \simeq \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} P(t,\epsilon) d\mu(t) \oplus \sum_{\substack{0 \leq 2l < (l_1 - l_2 - 1) \\ l \in \mathbb{N}}} D(l_1 - l_2 - 2l) \tag{A.30}$$

where

$$\begin{aligned}
\epsilon &= \tfrac{1}{4} && \text{for} \quad (l_1 + l_2) \in \{\tfrac{1}{2} + 2n, n \in \mathbb{N}\} \\
\epsilon &= -\tfrac{1}{4} && \text{for} \quad (l_1 + l_2) \in \{\tfrac{3}{2} + 2n, n \in \mathbb{N}\} \\
\epsilon &= 0 && \text{for} \quad (l_1 + l_2) \in \{0 + 2n, n \in \mathbb{N}\} \\
\epsilon &= \tfrac{1}{2} && \text{for} \quad (l_1 + l_2) \in \{1 + 2n, n \in \mathbb{N}\} \quad ,
\end{aligned} \tag{A.31}$$

and the meausre $d\mu(t)$ is the Plancherel measure on the unitary dual of the double cover of $SL(2,\mathbb{R})$. The reduction for $l_1 \le l_2$ is obtained by using $D(l_1)\otimes D^*(l_2) \simeq (D^*(l_1)\otimes D(l_2))^*$.

In these decompositions the trivial representation never appears, therefore the Master Constraint Operator $\widehat{\mathbf{M}} = h^2 + n_1^2 + n_2^2$ never includes zero in its spectrum.

On the other hand, in the Refined Algebraic Quantization approach one can find trivial representations (for $p,q \ge 2$ and $p+q$ even), see [25]. But these trivial representations do not appear in the decomposition of $\omega^{(p,q)}$ as a (continuous) sum of Hilbert spaces. One can find trivial representations if one looks at the algebraic dual Φ^* of the dense subspace Φ in $\mathfrak{L}^2(\mathbb{R}^{p+q})$, where Φ is the linear span of all Fock states.

A.3 Explicit Calculations for Example 3

Here we will elaborate on example 3 and construct the explicit solutions to the Master Constraint Operator using the results of [22]. This may provide some hints how to tackle examples, which do not carry such an amount of group structure, as the present one.

We will start with equation 3.26, where we achieved the reduction of $\omega^{(2,0)}$ and $\omega^{(0,2)}$. We managed to write the constraints as

$$\begin{aligned} H_- &= A_+^\dagger A_+ + A_-^\dagger A_- - B_+^\dagger B_+ - B_-^\dagger B_- \\ H_+ &= -(A_+A_- + A_+^\dagger A_-^\dagger + B_+B_- + B_+^\dagger B_-^\dagger) \\ D &= i(A_+^\dagger A_-^\dagger - A_+A_- + B_+B_- - B_+^\dagger B_-^\dagger) \quad . \end{aligned} \tag{A.32}$$

The generators of the maximal compact subgroup $O(2)\times O(2)$ of $O(2,2)$ can be written as

$$\begin{aligned} O_{12} &= u_1p_2 - p_1u_2 = A_+^\dagger A_+ - A_-^\dagger A_- \\ O_{34} &= \pi_1 v_2 - v_1\pi_2 = -B_+^\dagger B_+ + B_-^\dagger B_- \quad . \end{aligned} \tag{A.33}$$

A convenient (ortho-normal) basis in the kinematical Hilbert space $\mathfrak{L}^2(\mathbb{R}^4)$ is given by the Fock states with respect to A_+, A_-, B_+ and B_- given by

$$|k_+,k_-,k'_+,k'_- >= \frac{1}{\sqrt{k_+k_-k'_+k'_-}}(A_+^\dagger)^{k_+}(A_-^\dagger)^{k_-}(B_+^\dagger)^{k'_+}(B_-^\dagger)^{k'_-}|0,0,0,0> \tag{A.34}$$

where $|0,0,0,0>$ is the state which is annihilated by all four annihilation operators and $k_+,k_-,k'_+,k'_- \in \mathbb{N}$. These states are eigenstates of H_- , O_{12} and O_{34} with eigenvalues

$$\begin{aligned} k &:= \text{eigenval}(H_-) = k_+ + k_- - k'_+ - k'_- \\ j &:= \text{eigenval}(O_{12}) = k_+ - k_- \\ j' &:= \text{eigenval}(O_{34}) = -k'_+ + k'_- \quad . \end{aligned} \tag{A.35}$$

The common eigenspaces $V(j,j')$ of the operators O_{12} and O_{34} are left invariant by the $sl(2,\mathbb{R})$-algebra (A.32), since there only appear combinations of A_+A_- , B_+B_-, their adjoints and number operators, which leave the difference between particels in the plus polarization and particels in the minus polarization invariant. Moreover the kinematical Hilbert space is a direct sum of all the (Hilbert) subspaces $V(j,j')$ (since these $V(j,j')$ constitute the spectral decomposition of the self adjoint operators O_{12} and O_{34}):

$$\mathfrak{L}^2(\mathbb{R}^4) = \sum_{j,j'\in\mathbb{Z}} V(j,j') \quad . \tag{A.36}$$

The scalar product on $V(j,j')$ is simply gained by restriction of the $\mathfrak{L}^2$-scalar product to $V(j,j')$.

The space $V(j,j')$ still carries an $sl(2,\mathbb{R})$- representation, which can be written as a tensor product, where the two factor representations are

$$
\begin{aligned}
H_-^{(A)} &= A_+^\dagger A_+ + A_-^\dagger A_- + 1 & H_-^{(B)} &= -(B_+^\dagger B_+ + B_-^\dagger B_- + 1)\\
H_+^{(A)} &= -(A_+ A_- + A_+^\dagger A_-^\dagger) & H_+^{(B)} &= -(B_+ B_- + B_+^\dagger B_-^\dagger)\\
D^{(A)} &= i(A_+^\dagger A_-^\dagger - A_+ A_-) & D^{(B)} &= i(B_+ B_- - B_+^\dagger B_-^\dagger) \quad . \qquad \text{(A.37)}
\end{aligned}
$$

Each $V(j,j')$ has a basis $\{|k_+,k_-,k'_+,k'_- >, k_+ - k_- = j \wedge -k'_+ + k'_- = j'\}$, which is also an eigenbasis for $H_-^{(A)}$ and $H_-^{(B)}$. Therefore it is easy to check that $H_-^{(A)}$ has on $V(j,j')$ a lowest eigenvalue given by $(|j|+1)$, more generally the spectrum of $H_-^{(A)}$ is non-degenerate and given by $\text{spec}(H_-^{(A)}) = \{|j|+1+2n, n \in \mathbb{N}\}$. Similarly, $H_-^{(B)}$ has a highest eigenvalue $-(|j|+1)$ on $V(j,j')$ and the spectrum is $\text{spec}(H_-^{(B)}) = \{-(|j|+1+2n), n \in \mathbb{N}\}$. From this one can deduce that the representation given on $V(j,j')$ is isomorphic to $D(|j|+1) \otimes D(|j'|+1)$, i.e. a tensor product of a positive discrete series representation and a representation fom the negative discrete series.

Now, what we want achieve is a spectral composition of the Master Constraint Operator $\widehat{\mathbf{M}}$ on each of the subspaces $V(j,j')$. (Clearly, the Master Constraint Operator leaves these spaces invariant.) The Master Constraint Operator is the sum of a multiple of the $sl(2,\mathbb{R})$-Casimir and $2H_-^2$. The latter two operators commute, so we can diagonalize them simultanously. This problem was solved in [22]. There, another realization of the representation $D(|j|+1)\otimes D(|j'|+1)$ was used, hence to use the results of [22], we have to construct a (unitary) map, which intertwines between our realization and the realization in [22].

To this end, we will depict the realization used in [22], at first for representations from the positive and negative discrete series. The Hilbert spaces for these realizations are function spaces on the open unit disc in $\mathbb{C}$. For the positive discrete series $D(l), l \in \mathbb{N} - 0$ the Hilbert space, which we will denote by $\mathfrak{H}_l$, consists of holomorphic functions and for the negativ discrete series $D^*(l), l \in \mathbb{N}_0$ the Hilbert space ($\mathfrak{H}_{*l}$) is composed of anti-holomorphic functions. The scalar product is in both cases

$$
< f,h >_l = \frac{l-1}{\pi} \int_D f(z)\overline{h(z)}(1-|z|^2)^{l-2} dx\, dy \qquad \text{(A.38)}
$$

where D is the unit disc and $dx\,dy$ is the Lebesgue measure on $\mathbb{C}$. (For $l=1$ one has to take the limit $l \to 1$ of the above expression.) An ortho-normal basis is given by

$$
f_n^{(l)} := (\mu_l(n))^{-\frac{1}{2}} z^n \quad (n \in \mathbb{N}) \quad \text{with } \mu_l(n) = \frac{\Gamma(n+1)\Gamma(l)}{\Gamma(l+n)} \qquad \text{(A.39)}
$$

for the positiv discrete series; for the negative series an ortho-normal basis is

$$
f_n^{(*l)} := (\mu_l(n))^{-\frac{1}{2}} \overline{z}^n \quad (n \in \mathbb{N}) \quad . \qquad \text{(A.40)}
$$

In this realizations the $sl(2,\mathbb{R})$-algebra acts as follows: for the positive discrete series $D(l)$

$$
\begin{aligned}
H_-^{(l)} &= l + 2z\frac{d}{dz}\\
H_+^{(l)} &= -lz - (z+z^{-1})z\frac{d}{dz}\\
D^{(l)} &= ilz + i(z-z^{-1})z\frac{d}{dz} \qquad \text{(A.41)}
\end{aligned}
$$

and for the negative discrete series $D^*(l)$

$$\begin{aligned} H_-^{(*l)} &= -l - 2\overline{z}\frac{d}{d\overline{z}} \\ H_+^{(*l)} &= -l\overline{z} + (\overline{z} + \overline{z}^{-1})\overline{z}\frac{d}{d\overline{z}} \\ D^{(*l)} &= -il\overline{z} + i(\overline{z} - \overline{z}^{-1})\overline{z}\frac{d}{d\overline{z}} \quad . \end{aligned} \tag{A.42}$$

The aforementioned bases $\{f_n^{(l)}\}$ and $\{f_n^{(*l)}\}$ are eigen-bases for $H_-^{(l)}$ resp. $H_-^{(*l)}$ with eigenvalues $\{l + 2n\}$ resp. $\{-l - 2n\}$ (where always $n \in \mathbb{N}$).

The representation space of the tensor product $D(l) \otimes D^*(l')$ is the tensor product $\mathfrak{H}_l \otimes \mathfrak{H}_{*l'}$, which has as an ortho-normal basis $\{f_n^{(l)} \otimes f_{n'}^{(*l')},\ n, n' \in \mathbb{N}\}$. The tensor product representation is obtained by adding the corresponding $sl(2,\mathbb{R})$-representatives from (A.41) and (A.42).

Now, considering the properties of the bases $\{|k_+, k_-, k'_+, k'_- >, k_+ - k_- = j \wedge -k'_+ + k'_- = j'\}$ for $V(j, j')$ and $\{f_n^{(|j|+1)} \otimes f_{n'}^{(*(|j'|+1))}\}$ for $\mathfrak{H}_{|j|+1} \otimes \mathfrak{H}_{*(|j'|+1)}$ it is very suggestive to construct a unitary map between these two Hilbert spaces by simply matching the bases:

$$\begin{aligned} U: \quad \mathfrak{H}_{|j|+1} \otimes \mathfrak{H}_{*(|j'|+1)} &\rightarrow V(j, j') \\ f_n^{(|j|+1)} \otimes f_{n'}^{(*(|j'|+1))} &\mapsto (-1)^{n'} |k_+, k_-, k'_+, k'_- > \quad \text{where} \\ & \quad 2n = k_+ + k_- - |j|\,, \quad j = k_+ - k_-\,, \\ & \quad 2n' = k'_+ + k'_- - |j'|\,, \quad j = -k'_+ + k'_- \quad . \end{aligned} \tag{A.43}$$

One can check that this map intertwines the $sl(2,\mathbb{R})$-representations. (For this to be the case the factor $(-1)^{n'}$ in (A.43) is needed.) Since this map maps an orthonormal basis to an orthonormal basis it is an (invertible) isometry and can be continued to the whole Hilbert space (which justifies the notation in (A.43)). We will later use this map to adopt the results of [22] to our situation.

In the following we will sketch how the spectral decomposition of the Casimir operator in the $D(l) \otimes D^*(l')$-representation is achieved in [22]. The Casimir operator is

$$\begin{aligned} \mathfrak{C} &= \tfrac{1}{4}(-(H_-^{(l)} + H_-^{(*l')})^2 + (H_+^{(l)} + H_+^{(*l')})^2 + (D^{(l)} + D^{(*l')})^2) \\ &= -(1 - z_1\overline{z}_2)^2 \partial_{z_1}\partial_{\overline{z}_2} + l'(1 - z_1\overline{z}_2) z_1 \partial_{z_1} + l(1 - z_1\overline{z}_2)\overline{z}_2\partial_{\overline{z}_2} - \tfrac{1}{4}(l - l')^2 + \tfrac{1}{2}(l + l') - l\,l' z_1\overline{z}_2\,. \end{aligned} \tag{A.44}$$

This operator commutes with all $sl(2,\mathbb{R})$-generators and in particular with $H_-^{ll'} = (H_-^{(l)} + H_-^{(*l')})$, i.e. it leaves the eigenspaces of $H_-^{ll'}$ invariant. To take advantage of this fact one introduces new coordinates $z = z_1\overline{z}_2\,, w = z_1$ and rewrites functions in $\mathfrak{H}_l \otimes \mathfrak{H}_{*l'}$ as a Laurent series in w (where the coefficients are functions of z). Since functions of the form $f(z)w^{\frac{1}{2}(k-l+l')}$ are eigenfunctions of $H_-^{ll'}$ with eigenvalue k one has effectively achieved the spectral decomposition of $H_-^{ll'}$. (The number $\frac{1}{2}(k - l + l')$ is always a whole number, since k is (un)even iff $(l - l')$ is (un)even.) The linear span of all these functions (with fixed k) completed with respect to the subspace-topology coming from $\mathfrak{H}_l \otimes \mathfrak{H}_{*l')}$ is a Hilbert space, abbreviated by $\mathfrak{H}(k, l, l')$. Since the power of w is fixed, this Hilbert space is a space of functions of z. The scalar product in this Hilbert space is characterized by the fact that

$$\{(\mu_l(n + \tfrac{1}{2}(k - l + l'))\mu_{l'}(n) z^n \quad | \quad \max(0, \tfrac{1}{2}(-k + l - l') \leq n < \infty\} \tag{A.45}$$

is an orthonormal basis.

One can restrict the Casimir (A.44) to this Hilbert space (since it leaves the $H_-^{ll'}$-eigenspaces invariant) obtaining

$$\mathfrak{C}_k = (1-z)\Big(-z(1-z)\frac{d^2}{dz^2} - \tfrac{1}{2}(k-l+l'+2-(k+l+3l'+2)z)\frac{d}{dz} + \tfrac{1}{2}l'\,(k+l+l') + \tfrac{1}{4}(l+l')(2-l-l')(1-z)^{-1}\Big) \quad . \tag{A.46}$$

Likewise the Master Constraint Operator restricts to $\mathfrak{H}(k,l,l')$ and can be written as

$$M_k = 4\mathfrak{C}_k + 2k^2 \quad . \tag{A.47}$$

These operators are ordinary second order differenential operators and their spectral decomposition is effected in [22] by using (modifications of) the Rellich-Titchmarsh-Kodaira-theory. We will not explain this procedure but merely cite the results.

The eigenvalue equation for the master constraint $(M_k - \lambda)f = 0$ on $\mathfrak{H}(k, l = |j|+1, l' = |j'|+1)$ has two linearly independent solutions (since it is a second order differential operator), a near $z = 0$ regular solution being

$$f_{k,j,j'}(z,t) = (1-z)^{1-t-\frac{1}{2}(|j|+|j'|+2)} F(1-t+\tfrac{1}{2}(-|j|+|j'|), 1-t+\tfrac{1}{2}k, 1+\tfrac{1}{2}(k-|j|+|j'|) \tag{A.48}$$

for $k - |j| + |j'| \geq 0$ and

$$\begin{aligned} f_{k,j,j'}(z,t) &= (1-z)^{1-t-\frac{1}{2}(|j|+|j'|+2)} z^{\frac{1}{2}(-k+|j|-|j'|)} \times \\ &\times F(1-t-\tfrac{1}{2}k, 1-t+\tfrac{1}{2}(|j|-|j'|), 1+\tfrac{1}{2}(-k+|j|-|j'|); z) \end{aligned} \tag{A.49}$$

for $k - |j| + |j'| \leq 0$, where $t = \frac{1}{2}(1+\sqrt{1-\lambda+2k^2})$, $\mathrm{Re}(t) \geq \frac{1}{2}$ and $F(a,b,c;z)$ is the hypergeometric function. For $\lambda(k,|j|,|j'|)$ in the spectrum of the Master Constraint Operator these solutions are generalized eigenvectors of the Master Constraint Operator.

The spectrum has a continuous part and a discrete part. There is a discrete part only if $k > 0$ for $|j| - |j'| \geq 2$ or $k < 0$ for $|j| - |j'| \leq 2$.:

$$\begin{gathered} \lambda_{\mathrm{discr}} = 4t(1-t) + 2k^2 \geq 2k^2 - k^2 + 2|k| \\ \text{with} \quad t = 1, 2, \ldots, \tfrac{1}{2}\min(|k|, ||j|-|j'||) \text{ for even } k \\ \text{with} \quad t = \tfrac{3}{2}, \tfrac{5}{2} \ldots, \tfrac{1}{2}\min(|k|, ||j|-|j'||) \text{ for odd } k \\ \text{and} \end{gathered} \tag{A.50}$$

$$\lambda_{\mathrm{cont}} = 1 + x^2 + 2k^2 > 0 \quad x \in \big[0, \infty\big) \quad . \tag{A.51}$$

The spectral resolution of a function $f(z)$ in $\mathfrak{H}(k, l = |j|+1, l' = |j|+1)$ is for $k - |j| + |j'| \geq 0$

$$\begin{aligned} f(z) &= \sum_{\lambda_{\mathrm{discr}}} A(\lambda_{\mathrm{discr}}) + \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} (2t-1)\mu(j,j',k,t) < f, f_{k,j,j'}(\cdot,t) > f_{k,j,j'}(z,t)\, dt \\ \mu(j,j',k,t) &= \frac{1}{i\pi^2\Gamma(|j|+1)\Gamma(|j'|+1)\Gamma^2(\frac{1}{2}(k-|j|+|j'|+2))} \sin\pi t\, \cos\pi t \;\times \\ &\quad \times |\Gamma(t+\tfrac{1}{2}k)\Gamma(t-\tfrac{1}{2}(2-|j|-|j'|))|^2 |\Gamma(t-\tfrac{1}{2}(|j|-|j'|))|^2 \end{aligned} \tag{A.52}$$

and for $k - |j| + |j'| \leq 0$

$$\begin{aligned} f(z) &= \sum_{\lambda_{\mathrm{discr}}} B(\lambda_{\mathrm{discr}}) + \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} (2t-1)\mu(j,j',k,t) < f, f_{k,j,j'}(\cdot,t) > f_{k,j,j'}(z,t)\, dt \\ \mu(j,j',k,t) &= \frac{1}{i\pi^2\Gamma(|j|+1)\Gamma(|j'|+1)\Gamma^2(\frac{1}{2}(-k+|j|-|j'|+2))} \sin\pi t\, \cos\pi t \;\times \\ &\quad \times |\Gamma(t-\tfrac{1}{2}k)\Gamma(t-\tfrac{1}{2}(2-|j|-|j'|))|^2 |\Gamma(t+\tfrac{1}{2}(|j|-|j'|))|^2 \end{aligned} \tag{A.53}$$

where in the following we do not need $A(\lambda_{\rm discr})$ and $B(\lambda_{\rm discr})$ in explicit form. This gives the following resolution of a function $f(z_1,\overline{z}_2)$ in $\mathfrak{H}_{|j|+1}\otimes\mathfrak{H}_{*(|j'|+1))}$:

$$f(z_1,\overline{z}_2) = \text{discr. part} + \sum_k \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} (2t-1)\mu(j,j',k,t) < f, f_{k,j,j'}(\cdot,\cdot,t) > f_{k,j,j'}(z_1,\overline{z}_2,t)\, dt \quad \text{(A.54)}$$

where

$$f_{k,j,j'}(z_1,\overline{z}_2,t) = f_{k,j,j'}(z_1\overline{z}_2,t)\, z_1^{\frac{1}{2}(k-|j|+|j'|)}, \quad \text{(A.55)}$$

and the sum is over all whole numbers k with the same parity as $(j-j')$.

Now we can use the map U in (A.43) to transfer these results to the subspaces $V(j,j')$ of the kinematical Hilbert space $\mathfrak{L}^2(\mathbb{R}^4)$. To this end we rewrite (A.55) into a power series in z_1 and $\overline{z}_2$ using the definition of the hypergeometric function

$$F(a,b,c\,;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} z^n \quad \text{(A.56)}$$

and

$$(1-z)^{1-d} = \sum_{n=0} \frac{\Gamma(d+k-1)}{\Gamma(d-1)\Gamma(k+1)} z^k \quad . \quad \text{(A.57)}$$

For $k-|j|+|j'|\geq 0$ we obtain

$$f(t;k,j,j') = U(f_{k,j,j'}(z_1,\overline{z}_2)) = \sum_{m=0} a_m\, |k_+(m),k_-(m),k'_+(m),k'_-(m)> \quad \text{(A.58)}$$

where

$$\begin{array}{ll} k_+ = m+\frac{1}{2}(k+j+|j'|) & k_- = m+\frac{1}{2}(k-j+|j'|) \\ k'_+ = m+\frac{1}{2}(|j'|-j') & k'_- = m+\frac{1}{2}(|j'|+j') \end{array} \quad \text{(A.59)}$$

and

$$\begin{aligned} a_m \;=\; & (-1)^m(\mu_{(|j|+1)}(m+\frac{1}{2}(k-|j|+|j'|)))^{\frac{1}{2}}(\mu_{(|j'|+1)}(m))^{\frac{1}{2}} \;\times \\ & \times \frac{\Gamma(1+\frac{1}{2}(k-|j|+|j'|))}{\Gamma(1-t+\frac{1}{2}(-|j|+|j'|))\Gamma(1-t+\frac{1}{2}k)} \;\times \\ & \times \sum_{l=0}^{m} \frac{\Gamma(1-t+\frac{1}{2}(-|j|+|j'|)+l)\Gamma(1-t+\frac{1}{2}k+l)}{\Gamma(1+\frac{1}{2}(k-|j|+|j'|)+l)\Gamma(1+l)} \frac{\Gamma(t+\frac{1}{2}(|j|+|j'|)+(m-l))}{\Gamma(m-l+1)\Gamma(t+\frac{1}{2}(|j|+|j'|))} \quad . \end{aligned} \quad \text{(A.60)}$$

For $k-|j|+|j'|\leq 0$ the coefficient a_m in (A.58) is obtained from (A.60) by replacing k with $-k$, switching $|j|$ and $|j'|$ and multiplying with $(-1)^{\frac{1}{2}(-k+|j|-|j'|)}$.

We could use the vectors $f(t;k,j,j')$ to construct the spectral decomposition of $\mathfrak{L}^2(\mathbb{R}^4)$. However, we want to achieve a spectral measure, which is independent of k,j and j'. For this purpose we normalize the solutions (A.58) to

$$|t,k,j,j'> = \left(i\,\frac{\mu(j,j',k,t)}{\sin\pi t\,\cos\pi t}\right)^{\frac{1}{2}} f(t;k,j,j') \quad . \quad \text{(A.61)}$$

Now we can decompose a vector $|f>\in \mathfrak{L}^2(\mathbb{R}^4)$ as follows

$$\begin{aligned} |f> &= \text{discrete part} + \\ & \sum_{k,j,j'} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} i\,(1-2t)\sin\pi t\,\cos\pi t\, < f|t,k,j,j'> |t,k,j,j'> dt \end{aligned} \tag{A.62}$$

where the sum is over all whole numbers k, j, j' with $(-1)^k = (-1)^{j-j'}$.

From this it follows, that $\mathfrak{L}^2(\mathbb{R}^4)$ decomposes into a direct sum (for the discrete part) and direct integral of Hilbert spaces $\mathfrak{H}(t)$, where in each $\mathfrak{H}(t)$ an ortho-normal basis is given by the vectors $|t,k,j,j'>$. As explained in section 3 our physical Hilbert space $\mathfrak{H}''$ consists of vectors with $t=\frac{1}{2}, k=0$ and $|j|=|j'|$. In this case these vectors are given by

$$\begin{aligned} |j,j'> &= |t=\tfrac{1}{2}, k=0, j, j'> \quad = \sum b_m\, |k_+(m), k_-(m), k'_+(m), k'_-(m)> \\ b_m &= (-1)^m \frac{\Gamma(m+1)}{\Gamma(|j|+1+m)} \sum_{l=0}^{m} \frac{(\Gamma(\frac{1}{2}+l))^2}{(\Gamma(1+l))^2} \frac{\Gamma(|j|+\frac{1}{2}+(m-l))}{\Gamma(m-l+1)} \end{aligned} \tag{A.63}$$

with $k_+(m), k_-(m), k'_+(m), k'_-(m)$ given by (A.59).

Now we want to check our results by calculating the action of the Master Constraint Operator and of the observables on the states (A.63).

The Master Constraint Operator rewritten in terms of annihilation and creation operators is

$$\begin{aligned} \widehat{\mathbf{M}} &= 2\big(2N(A_+)N(A_-) + N(A_+) + N(A_-) + 2N(B_+)N(B_-) + \\ &\quad N(B_+) + N(B_-) + 2 + 2A_+^\dagger A_-^\dagger B_+^\dagger B_-^\dagger + 2A_+A_-B_+B_-\big) + \\ &\quad \big(N(A_+) + N(A_-) - N(B_+) - N(B_-)\big)^2 \end{aligned} \tag{A.64}$$

where $N(i)$ stands for the number operator for quanta of type i.

The eigenvalue equation $(M-\lambda)|j,j'>= 0$ for the states (A.63) can be written as an equation for the coefficients b_m:

$$0 = (8(m+|j|+1)m + 4|j| + 4 - \lambda)\, b_m + 4m(m+|j|)\, b_{m-1} + 4(m+1)(m+|j|+1)\, b_{m+1} \tag{A.65}$$

(The coeffecient b_{-1} is defined to be zero.) One can check, that the coefficients (A.63) fulfill this equation for $\lambda = 1$: For this purpose one introduces

$$\tilde{b}_m = (-1)^m \frac{\Gamma(|j|+1+m)}{\Gamma(m+1)} b_m = \sum_{l=0}^{m} \frac{(\Gamma(\frac{1}{2}+l))^2}{(\Gamma(1+l))^2} \frac{\Gamma(|j|+\frac{1}{2}+(m-l))}{\Gamma(m-l+1)} \tag{A.66}$$

and realizes that the $\tilde{b}_m$'s are the coefficients in the power expansion of the function $\Gamma(|j|+1/2)\, f_{0,j,j'}(z, t=\frac{1}{2})$ (with $|j|=|j'|$) from (A.48). This function fullfills the differential equation $M_0 \cdot f = f$ where $M_{k=0}$ is the differential operator from (A.47). One can rewrite the differential equation for f into a equation for the coefficients $\tilde{b}_m$ in a power expansion for f. If one replaces $\tilde{b}_m$ with b_m according to the first part of equation (A.66) one will get equation (A.65). Therefore the coefficients b_m fulfill this equation.

The observables can be written as

$$
\begin{aligned}
Q_1 &= \tfrac{i}{2}(A_+B_+ - A_-B_- - A_+^\dagger B_+^\dagger + A_-^\dagger B_-^\dagger) \\
Q_2 &= \tfrac{-1}{2}(A_+B_+ + A_-B_- + A_+^\dagger B_+^\dagger + A_-^\dagger B_-^\dagger) \\
Q_3 &= \tfrac{1}{2}(N(A_+) - N(A_-) + N(B_+) - N(B_-)) \\
P_1 &= \tfrac{i}{2}(A_+B_- - A_+^\dagger B_-^\dagger - A_-B_+ + A_+^\dagger B_+^\dagger) \\
P_2 &= \tfrac{-1}{2}(A_+B_- + A_+^\dagger B_-^\dagger + A_-B_+ + A_+^\dagger B_+^\dagger) \\
P_3 &= \tfrac{1}{2}(N(A_+) - N(A_-) - N(B_+) + N(B_-))
\end{aligned}
\tag{A.67}
$$

$$
\begin{aligned}
Q_+ &= \tfrac{1}{\sqrt{2}}(Q_1 + iQ_2) = \tfrac{-i}{\sqrt{2}}(A_+^\dagger B_+^\dagger + A_-B_-) \\
Q_- &= \tfrac{1}{\sqrt{2}}(Q_1 - iQ_2) = \tfrac{+i}{\sqrt{2}}(A_-^\dagger B_-^\dagger + A_+B_+) \\
P_+ &= \tfrac{1}{\sqrt{2}}(P_1 + iP_2) = \tfrac{-i}{\sqrt{2}}(A_+^\dagger B_-^\dagger + A_-B_+) \\
P_- &= \tfrac{1}{\sqrt{2}}(P_1 - iP_2) = \tfrac{+i}{\sqrt{2}}(A_-^\dagger B_+^\dagger + A_+B_-) \quad .
\end{aligned}
\tag{A.68}
$$

In section 3 we concluded that on the physical Hilbert space $\mathfrak{H}''$ the observable algebra is generated by operators of the form $\Theta(Q_3)Q_i\Theta(Q_3)$ and $\Theta(P_3)P_i\Theta(P_3)$. Therefore we will just depict the action of $Q_\pm$ on states with zero P_3-eigenvalue and of $P_\pm$ on states with zero Q_3-eigenvalue. One can determine from this the action of the observable algebra on $\mathfrak{H}''$.

To begin with, we consider the action of Q_+ on states $|j,-j>$, $j \geq 0$, i.e. states with P_3-eigenvalue zero and nonnegative Q_3-eigenvalue:

$$
\begin{aligned}
|j,-j> &= \sum_{m=0} b_m(j)|m+j,m,m+j,m> \\
b_m(j) &= (-1)^m \frac{\Gamma(m+1)}{\Gamma(|j|+1+m)} \sum_{l=0}^{m} \frac{(\Gamma(\frac{1}{2}+l))^2}{(\Gamma(1+l))^2} \frac{\Gamma(|j|+\frac{1}{2}+(m-l))}{\Gamma(m-l+1)} \quad .
\end{aligned}
\tag{A.69}
$$

On these states Q_+ acts as

$$
\begin{aligned}
Q_+|j,-j> &= \tfrac{-i}{\sqrt{2}} \sum_{m=0} ((m+1)b_{m+1}(j) + (m+j+1)b_m(j)) \; |m+j+1,m,m+j+1,m> \\
&\underset{(*)}{=} \tfrac{-i}{\sqrt{2}}(j+\tfrac{1}{2}) \; |j+1,-(j+1)> \quad .
\end{aligned}
\tag{A.70}
$$

For $j<0$ we have

$$
\begin{aligned}
Q_+|j,-j> &= \tfrac{-i}{\sqrt{2}} \sum_{m=0} (m\, b_{m-1}(j) + (m+|j|)b_m(j)) \; |m,m+|j|-1,m,m+|j|-1> \\
&\underset{(*)}{=} \tfrac{i}{\sqrt{2}}(j+\tfrac{1}{2}) \; |j+1,-(j+1)> \quad .
\end{aligned}
\tag{A.71}
$$

For the equalities marked with a star $(*)$ we have to check the relations

$$
\begin{aligned}
(m+1)b_{m+1}(j) + (m+j+1)b_m(j) &= (j+\tfrac{1}{2})b_m(j+1) \quad \text{for} \quad j \geq 0 \\
m\, b_{m-1}(|j|) + (m+|j|)b_m(j) &= (|j|-\tfrac{1}{2})b_m(|j|-1) \quad \text{for} \quad j<0 \quad .
\end{aligned}
\tag{A.72}
$$

The last equation is verified by using the $\tilde{b}_m(j)$'s defined in (A.66), which are the coefficients of

$$
f_{|j|}(z) := \Gamma(|j|+1/2)\, f_{0,j,\pm j}(z, t=\tfrac{1}{2}) = \Gamma(|j|+1/2)(1-z)^{-|j|-1/2}F(1/2,1/2,1;z) \tag{A.73}
$$

in a power expansion in z. Then, rewriting of the identity $(1-z)f_{|j|}(z) = (|j|-1/2)\, f_{|j|-1}(z)$ into an equation for the $\tilde{b}_m(j)$ and furthermore for the $b_m(j)$ results in the last equation of (A.72). For the first equation one starts with the differential equation $M_0 \cdot f_{|j|}(z) = f_{|j|}(z)$ and replaces there $(1-z)^{-1} f_{|j|}(z)$ with $(|j|+1/2)^{-1} f_{|j|+1}(z)$. This then translates into the first equation of (A.72) for the coefficients $b_m(j)$.

The relations (A.72) will also ensure the following equalities:

$$\begin{aligned} Q_-|j,-j> &= \tfrac{i}{\sqrt{2}}(j-\tfrac{1}{2})|j-1,-(j-1)> \\ P_+|j,j> &= \tfrac{-i}{\sqrt{2}}(j+\tfrac{1}{2})|j+1,j+1> \\ P_-|j,j> &= \tfrac{i}{\sqrt{2}}(j-\tfrac{1}{2})|j-1,j-1> \quad . \end{aligned} \tag{A.74}$$

These formulas differ by phase factors from the formulas in 3.5. One can adjust these phase factors to one by choosing a new basis $|j,\epsilon j>' = (-i)^j|j,\epsilon j>$. Therefore the results of this section and section 3 are consistent.

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