On $\mathbb{Z}$-gradations of twisted loop Lie algebras of complex simple Lie algebras

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Abstract

We define the twisted loop Lie algebra of a finite dimensional Lie algebra $\mathfrak{g}$ as the Fréchet space of all twisted periodic smooth mappings from $\mathbb{R}$ to $\mathfrak{g}$. Here the Lie algebra operation is continuous. We call such Lie algebras Fréchet Lie algebras. We introduce the notion of an integrable $\mathbb{Z}$-gradation of a Fréchet Lie algebra, and find all inequivalent integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces of twisted loop Lie algebras of complex simple Lie algebras.

1 Introduction

The theory of loop groups and loop Lie algebras has a lot of applications to mathematical and physical problems. In particular, it is a necessary tool for formulation of many integrable systems and construction of appropriate integration methods. Here one or another version of factorization problem for the underlying group arises (see, for example, [1]). For the so-called Toda systems associated with loop groups the required factorization is induced by a $\mathbb{Z}$-gradation of the corresponding loop Lie algebra, and, at least from this point of view, the classification of $\mathbb{Z}$-gradations of loop Lie algebras is quite important. The definition and general integration procedure for the Toda systems can be found in [2, 3, 4]. The classification of $\mathbb{Z}$-gradations of complex semisimple finite dimensional Lie algebras is well known (see, for example, [5]). The corresponding classification of the Toda systems associated with compex classical Lie groups was given in papers [6, 7].

There are two main definitions of the loop Lie algebras. In accordance with the first definition used, for example, by Kac in his famous monograph [8], a loop Lie algebra is the set of finite Laurent polynomials with coefficients in a finite dimensional Lie algebra. It is rather difficult to associate a Lie group with such a Lie algebra. Actually this is connected to the fact that the exponential of a finite polynomial is usually not a finite polynomial. However, it

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should be noted that with this definition in the case when the underlying Lie algebra is complex and simple one can classify all $\mathbb{Z}$-gradations of the loop Lie algebras with finite dimensional grading subspaces [8][1].

In accordance with the second definition, used in the monograph by Pressley and Segal [9], a loop Lie algebra is the set of smooth mappings from the circle S^1 to a finite dimensional Lie algebra. This set is endowed with the structure of a Fréchet space. Here the Lie algebra operation defined pointwise is continuous. The definition given in [9] is more convenient for applications to the theory of integrable systems, because in this case we always have an appropriate Lie group. Therefore, it would be interesting and useful to obtain a classification of $\mathbb{Z}$-gradations for loop Lie algebras defined as in [9]. In the present paper we introduce the concept of an integrable $\mathbb{Z}$-gradation and classify all integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces of loop Lie algebras and twisted loop Lie algebras of finite dimensional complex simple Lie algebras. The result of the classification is actually the same as for loop Lie algebras and twisted loop Lie algebras defined as in [8]. Namely, to classify all integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces of the Lie algebras under consideration one has to classify all $\mathbb{Z}_K$-gradations of the underlying Lie algebras or, equivalently, all their automorphisms of finite order.

2 Loop Lie algebras and loop Lie groups

Consider the vector space $C^\infty(S^1, V)$ of smooth mappings from the circle S^1 to a finite dimensional vector space V. It is convenient to treat the circle S^1 as the set of complex numbers of modulus one. There is a natural mapping from the set $\mathbb{R}$ of real numbers to S^1 which takes $\sigma \in \mathbb{R}$ to $\mathrm{e}^{\mathrm{i}\sigma} \in S^1$. Given an element $\xi \in C^\infty(S^1, V)$, one defines a mapping $\tilde{\xi}$ from $\mathbb{R}$ to V by the equality

$$\tilde{\xi}(\sigma) = \xi(\mathrm{e}^{\mathrm{i}\sigma}).$$

The mapping $\tilde{\xi}$ is smooth and satisfies the relation $\tilde{\xi}(\sigma + 2\pi) = \tilde{\xi}(\sigma)$. Conversely, any smooth periodic mapping from $\mathbb{R}$ to V induces a smooth mapping from S^1 to V. Introduce the notation

$$\tilde{\xi}^{(k)} = \mathrm{d}^k \tilde{\xi} / \mathrm{d}s^k,$$

where s is the standard coordinate function on $\mathbb{R}$. It is customary to assume that $\tilde{\xi}^{(0)} = \tilde{\xi}$. Given an element $\xi \in C^\infty(S^1, V)$, we denote by $\xi^{(k)}$ the element of $C^\infty(S^1, V)$ induced by $\tilde{\xi}^{(k)}$.

Endow $C^\infty(S^1, V)$ with the structure of a topological vector space in the following way. Let $\|\cdot\|$ be a norm on V. Define a countable collection of norms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on $C^\infty(S^1, V)$ by

$$\|\xi\|_m = \max_{0 \leqslant k < m} \max_{p \in S^1} \|\xi^{(k)}(p)\|,$$

or via the corresponding mapping $\tilde{\xi} : \mathbb{R} \to V$ by

$$\|\tilde{\xi}\|_m = \max_{0 \leqslant k < m} \max_{\sigma \in [0, 2\pi]} \|\tilde{\xi}^{(k)}(\sigma)\|.$$

Note that for any $\xi \in C^\infty(S^1, V)$, if $m_1 < m_2$, then $\|\xi\|_{m_1} \leqslant \|\xi\|_{m_2}$.

Given a positive integer m, denote

$$U_m = \{\xi \in C^\infty(S^1, V) \mid \|\xi\|_m < 1/m\}.$$

[1] Actually in [8] one can find the classification of $\mathbb{Z}$-gradations of the affine Kac–Moody algebras. The classification of $\mathbb{Z}$-gradations of the corresponding loop Lie algebras immediately follows from that classification.

The collection formed by the sets U_m is a local base of a topology on $C^\infty(S^1, V)$. As a base of the topology we can take the collection of subsets of the form

$$U_{\xi,m} = \xi + U_m, \qquad \xi \in C^\infty(S^1, V).$$

A sequence (ξ_n) in $C^\infty(S^1, V)$ converges to $\xi \in C^\infty(S^1, V)$ relative to this topology, if and only if for each nonnegative integer k the sequence $\left(\tilde{\xi}_n^{(k)}\right)$ converges uniformly to $\tilde{\xi}^{(k)}$. One can show that actually we have a Fréchet space. We define a Fréchet space as a complete topological vector space whose topology is induced by a countable family of semi-norms.[2]

Let now $\mathfrak{g}$ be a real or complex finite dimensional Lie algebra. Supply the Fréchet space $C^\infty(S^1, \mathfrak{g})$ with the Lie algebra structure defining the Lie algebra operation pointwise. The obtained Lie algebra is called the *loop Lie algebra* of $\mathfrak{g}$ and denoted $\mathcal{L}(\mathfrak{g})$. It is clear that constant mappings form a subalgebra of $\mathcal{L}(\mathfrak{g})$ which is isomorphic to the initial Lie algebra $\mathfrak{g}$.

Let again V be a finite dimensional vector space, and let a be an automorphism of V. Consider the quotient space E of the direct product $\mathbb{R} \times V$ by the equivalence relation which identifies (σ, v) with $(\sigma + 2\pi, a(v))$. Define the projection $\pi : E \to S^1$ by the relation

$$\pi([(\sigma, v)]) = \mathrm{e}^{\mathrm{i}\sigma}.$$

It is not difficult to show that in such a way one obtains a smooth vector bundle $E \xrightarrow{\pi} S^1$ with fiber V.

Let ξ be a smooth section of E. For any $\sigma \in \mathbb{R}$ there exists a unique element $\tilde{\xi}(\sigma) \in V$ such that

$$[(\sigma, \tilde{\xi}(\sigma))] = \xi(\mathrm{e}^{\mathrm{i}\sigma}).$$

This relation defines a smooth mapping $\tilde{\xi}$ from $\mathbb{R}$ to V which satisfies the relation

$$\tilde{\xi}(\sigma + 2\pi) = a(\tilde{\xi}(\sigma)),$$

called *twisted periodicity*. Conversely, given a mapping $\tilde{\xi} : \mathbb{R} \to V$ which is twisted periodic, the equality

$$\xi(p) = [(\sigma, \tilde{\xi}(\sigma))], \qquad p = \mathrm{e}^{\mathrm{i}\sigma},$$

defines a smooth section of E. One can make the space $C^\infty(S^1 \xleftarrow{\pi} E)$ of smooth sections of $E \xrightarrow{\pi} S^1$ a Fréchet space in the same way as it was done above for the space $C^\infty(S^1, V)$. Here it is natural and useful to assume that the corresponding norm on V is invariant with respect to the automorphism a.

If the vector space V is a Lie algebra $\mathfrak{g}$ and a is an automorphism of $\mathfrak{g}$, one can supply the vector space $C^\infty(S^1 \xleftarrow{\pi} E)$, or equivalently the vector space of twisted periodic mappings from $\mathbb{R}$ to $\mathfrak{g}$, with the structure of a Lie algebra defining Lie algebra operation pointwise. We denote this Lie algebra by $\mathcal{L}_a(\mathfrak{g})$ and call a *twisted loop Lie algebra*. The loop Lie algebra $\mathcal{L}(\mathfrak{g})$ can be considered as the twisted loop Lie algebra $\mathcal{L}_a(\mathfrak{g})$ with $a = \mathrm{id}_\mathfrak{g}$.

Let G be a Lie group whose Lie algebra coincides with $\mathfrak{g}$ and Ad be the adjoint representation of G in $\mathfrak{g}$. For any $g \in G$ the linear operator $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$. Such automorphisms are called *inner automorphisms*. They form a normal subgroup $\mathrm{Int}\,\mathfrak{g}$ of the group $\mathrm{Aut}\,\mathfrak{g}$ of automorphisms of $\mathfrak{g}$. One can show that if the automorphisms a and b of $\mathfrak{g}$ differ by an inner automorphism of $\mathfrak{g}$ then the twisted loop Lie algebras $\mathcal{L}_a(\mathfrak{g})$ and $\mathcal{L}_b(\mathfrak{g})$ are naturally isomorphic. This means, in particular, that if the Lie algebra $\mathfrak{g}$ is semisimple one can consider only the twisted loop Lie algebras $\mathcal{L}_a(\mathfrak{g})$ with a belonging to the finite subgroup of

[2] Sometimes a more general definition of a Fréchet space is used (see, for example, [10]).

$\operatorname{Aut}\mathfrak{g}$ identified with the automorphism group $\operatorname{Aut}\mathit{\Pi}$ of some simple root system $\mathit{\Pi}$ of $\mathfrak{g}$. In particular, one can assume that $a^K = \mathrm{id}_\mathfrak{g}$ for some positive integer K. It is convenient for our purposes to assume that K does not necessarily coincide with the order of a.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Consider an arbitrary element η of $\mathcal{L}_a(\mathfrak{g})$ and the corresponding mapping $\tilde{\eta}$ from $\mathbb{R}$ to $\mathfrak{g}$. It is clear that the mapping $\tilde{\xi}$ defined as

$$\tilde{\xi}(\sigma) = \tilde{\eta}(K\sigma),$$

is a periodic mapping from $\mathbb{R}$ to $\mathfrak{g}$. Therefore, it induces an element ξ of $\mathcal{L}(\mathfrak{g})$. It is clear that in this way we obtain an injective homomorphism from $\mathcal{L}_a(\mathfrak{g})$ to $\mathcal{L}(\mathfrak{g})$. The image of this homomorphism is formed by the elements ξ satisfying the condition

$$\xi(\varepsilon_K p) = a(\xi(p)),$$

where $\varepsilon_K = \exp(2\pi\mathrm{i}/K)$ is the Kth principal root of unity. We will denote this image as $\mathcal{L}_{a,K}(\mathfrak{g})$. For the corresponding mapping $\tilde{\xi}$ from $\mathbb{R}$ to $\mathfrak{g}$ one has

$$\tilde{\xi}(\sigma + 2\pi/K) = a(\tilde{\xi}(\sigma)).$$

Thus, when $\mathfrak{g}$ is a semisimple Lie algebra, the twisted loop Lie algebra $\mathcal{L}_a(\mathfrak{g})$ can be identified with a subalgebra of the loop Lie algebra $\mathcal{L}(\mathfrak{g})$.

We call a Lie algebra $\mathfrak{G}$ a *Fréchet Lie algebra* if $\mathfrak{G}$ is a Fréchet space and the Lie algebra operation in $\mathfrak{G}$, considered as a mapping from $\mathfrak{G} \times \mathfrak{G}$ to $\mathfrak{G}$, is continuous. Actually one can consider a Fréchet Lie algebra as a smooth manifold modelled on itself. Here the Lie algebra operation is a smooth mapping.

To prove that $\mathcal{L}_a(\mathfrak{g})$ is a Fréchet Lie algebra we start with the following simple lemmas.

Lemma 2.1 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\|\cdot\|$ be a norm on $\mathfrak{g}$. There exists a positive real number C such that*

$$\|[x, y]\| \leqslant C\,\|x\|\,\|y\|$$

for all $x, y \in \mathfrak{g}$.

Proof. First prove the statement of the proposition for a special choice of the norm $\|\cdot\|$. Let (e_i) be a basis of $\mathfrak{g}$. Expand an arbitrary element x of $\mathfrak{g}$ over the basis (e_i), $x = \sum_i e_i x^i$, and define

$$\|x\| = \max_i\{|x^i|\}.$$

In this case for any i one has $|x^i| \leqslant \|x\|$. It is also evident that $\|e_i\| = 1$. For arbitrary elements $x, y \in \mathfrak{g}$ one has

$$[x, y] = \sum_{i,j} [e_i x^i, e_j y^j] = \sum_{i,j,k} e_k \, c^k{}_{ij} \, x^i y^j,$$

where $c^k{}_{ij}$ are structure constants of the Lie algebra $\mathfrak{g}$,

$$[e_i, e_j] = \sum_k e_k \, c^k{}_{ij}.$$

Therefore,

$$\|[x, y]\| \leqslant \sum_{i,j,k} \|e_k\| |c^k{}_{ij}| |x^i| |y^j| \leqslant \left(\sum_{i,j,k} |c^k{}_{ij}|\right) \|x\| \|y\|.$$

Thus, the statement of the proposition is valid for the norm chosen and for

$$C = \sum_{i,j,k} |c^k{}_{ij}|.$$

Since in the finite dimensional case all norms are equivalent, the statement of the proposition is valid for an arbitrary norm $\|\cdot\|$. □

Lemma 2.2 *There are positive real numbers C_m, $m = 1, 2, \ldots$, such that*

$$\|[\xi, \eta]\|_m \leqslant C_m \|\xi\|_m \|\eta\|_m$$

for all $\xi, \eta \in \mathcal{L}_a(\mathfrak{g})$.

Proof. For $m = 1$, using Lemma 2.1, one easily obtains

$$\|[\xi, \eta]\|_1 = \max_{p \in S^1} \|[\xi(p), \eta(p)]\| \leqslant \max_{p \in S^1} C \|\xi(p)\| \|\eta(p)\| \leqslant C \|\xi\|_1 \|\eta\|_1.$$

For $m = 2$ one has

$$\|[\xi, \eta]\|_2 = \max \left\{ \max_{p \in S^1} \|[\xi(p), \eta(p)]\|, \max_{p \in S^1} \|[\xi, \eta]^{(1)}(p)\| \right\}.$$

It is clear that

$$\begin{aligned}\max_{p \in S^1} \|[\xi, \eta]^{(1)}(p)\| &= \max_{p \in S^1} \|[\xi^{(1)}, \eta](p) + [\xi, \eta^{(1)}](p)\| \\ &\leqslant C \|\xi^{(1)}\|_1 \|\eta\|_1 + C \|\xi\|_1 \|\eta^{(1)}\|_1 \leqslant 2C \|\xi\|_2 \|\eta\|_2.\end{aligned}$$

Taking into account that $\|\cdot\|_1 \leqslant \|\cdot\|_2$, we conclude that

$$\|[\xi, \eta]\|_2 \leqslant 2C \|\eta\|_2 \|\xi\|_2.$$

Similarly one obtains

$$\|[\xi, \eta]\|_m \leqslant 2^{m-1} C \|\xi\|_m \|\eta\|_m.$$

Thus, the statement of the proposition is valid for $C_m = 2^{m-1} C$. □

Now we are able to prove the desired result.

Proposition 2.1 *The twisted loop Lie algebra $\mathcal{L}_a(\mathfrak{g})$ is a Fréchet Lie algebra.*

Proof. It suffices to show that for any fixed elements $\xi_1, \xi_2 \in \mathcal{L}_a(\mathfrak{g})$ and any positive integer m, there are positive integers m_1 and m_2 such that for any $\xi'_1 \in U_{\xi_1, m_1}$ and $\xi'_2 \in U_{\xi_2, m_2}$ one has $[\xi'_1, \xi'_2] \in U_{[\xi_1, \xi_2], m}$. It is clear that one can assume that $m_1 \geqslant m$ and $m_2 \geqslant m$.

Let m be a fixed positive integer, ξ_1 and ξ_2 be arbitrary elements of $\mathcal{L}(\mathfrak{g})$, m_1, m_2 be arbitrary positive integers greater than m. For any $\xi'_1 \in U_{\xi_1, m_1}$ and $\xi'_2 \in U_{\xi_2, m_2}$ write the equalities

$$\begin{aligned}[\xi'_1, \xi'_2] - [\xi_1, \xi_2] &= [(\xi'_1 - \xi_1) + \xi_1, (\xi'_2 - \xi_2) + \xi_2] - [\xi_1, \xi_2] \\ &= [\xi'_1 - \xi_1, \xi'_2 - \xi_2] + [\xi_1, \xi'_2 - \xi_2] + [\xi'_1 - \xi_1, \xi_2].\end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned}\|[\xi_1', \xi_2'] - [\xi_1, \xi_2]\|_m &\leqslant C_m(\|\xi_1' - \xi_1\|_m \|\xi_2' - \xi_2\|_m \\ &\qquad + \|\xi_1' - \xi_1\|_m \|\xi_2\|_m + \|\xi_1\|_m \|\xi_2' - \xi_2\|_m) \\ &\leqslant C_m(\|\xi_1' - \xi_1\|_{m_1} \|\xi_2' - \xi_2\|_{m_2} \\ &\qquad + \|\xi_1' - \xi_1\|_{m_1} \|\xi_2\|_m + \|\xi_1\|_m \|\xi_2' - \xi_2\|_{m_2}).\end{aligned}$$

Thus, we have

$$\|[\xi_1', \xi_2'] - [\xi_1, \xi_2]\|_m < C_m \left(\frac{1}{m_1} \frac{1}{m_2} + \frac{1}{m_1} \|\xi_2\|_m + \|\xi_1\|_m \frac{1}{m_2} \right).$$

It is clear that for sufficiently large m_1 and m_2 one has

$$\|[\xi_1', \xi_2'] - [\xi_1, \xi_2]\|_m < 1/m,$$

that means that $[\xi_1', \xi_2'] \in U_{[\xi_1, \xi_2], m}$. □

Let G be a finite dimensional Lie group with the Lie algebra $\mathfrak{g}$. The loop group $\mathcal{L}(G)$ is defined as the set of all smooth mappings from the circle S^1 to G with the group law being pointwise composition in G. Here, as for the case of loop Lie algebras, for any element γ of $\mathcal{L}(G)$ one can define a smooth mapping $\tilde{\gamma}$ from $\mathbb{R}$ to G connected with γ by the equality

$$\tilde{\gamma}(\sigma) = \gamma(\mathrm{e}^{\mathrm{i}\sigma}),$$

and satisfying the relation $\tilde{\gamma}(\sigma + 2\pi) = \tilde{\gamma}(\sigma)$. Conversly, any periodic smooth mapping from $\mathbb{R}$ to G induces an element of $\mathcal{L}(G)$.

One can endow the loop group $\mathcal{L}(G)$ with the structure of an infinite dimensional manifold and a Lie group in the following way.

Recall that the exponential mapping $\exp : \mathfrak{g} \to G$ is a local diffeomorphism near the identity. Let $\breve{U}_e$ be an open neighbourhood of the identity of G diffeomorphic to some open neighbourhood of the zero element of $\mathfrak{g}$, and $\breve{\varphi}$ be the restriction of the inverse of the exponential mapping to $\breve{U}_e$. Denote $U_e = C^\infty(S^1, \breve{U}_e)$ and define a mapping $\varphi : U_e \to C^\infty(S^1, \breve{\varphi}(\breve{U}_e))$ by

$$\varphi(\gamma) = \breve{\varphi} \circ \gamma.$$

Note that the set $C^\infty(S^1, \breve{\varphi}(\breve{U}_e))$ is open in $\mathcal{L}(\mathfrak{g})$ and we can consider the pair (U_e, φ) as a chart on $\mathcal{L}(G)$.

For an arbitrary element $\gamma \in \mathcal{L}(G)$ denote $U_\gamma = \gamma U_e$, and define the mapping $\varphi_\gamma : U_\gamma \to C^\infty(S^1, \breve{\varphi}(\breve{U}_e))$ by

$$\varphi_\gamma(\gamma') = \breve{\varphi} \circ (\gamma^{-1} \gamma').$$

In this way we obtain an atlas which makes $\mathcal{L}(G)$ into a smooth manifold modelled on the Fréchet space $\mathcal{L}(\mathfrak{g})$. Actually in this way $\mathcal{L}(G)$ becomes a Lie group. The Lie algebra of $\mathcal{L}(G)$ can be naturally identified with $\mathcal{L}(\mathfrak{g})$.

We say that the set $U \subset \mathcal{L}(G)$ is open if for any $\gamma \in \mathcal{L}(G)$ the set $\varphi_\gamma(U \cap U_\gamma)$ is open. This definition supplies $\mathcal{L}(G)$ with the structure of a topological space. As any Lie group the loop Lie group $\mathcal{L}(G)$ is a Hausdorff topological space.

Twisted loop groups are defined in full analogy with twisted loop Lie algebras. Let a be an automorphism of a Lie group G and E be the quotient space of the direct product $\mathbb{R} \times G$

by the equivalence relation which identifies (σ, g) with $(\sigma + 2\pi, a(g))$. Defining the projection $\pi : E \to S^1$ by the relation

$$\pi([(\sigma, g)]) = e^{i\sigma},$$

we obtain a smooth fiber bundle $E \xrightarrow{\pi} S^1$ with fiber G. Endow the space of smooth sections of this bundle with the structure of a group defining the group composition pointwise. This group is called the *twisted loop group* of G and denoted $\mathcal{L}_a(G)$. Similarly, as for the case of $\mathcal{L}(\mathfrak{g})$, one endows $\mathcal{L}_a(G)$ with the structure of an infinite dimensional manifold modelled on the Fréchet space $\mathcal{L}_a(\mathfrak{g})$, where we denote the automorphism of $\mathfrak{g}$ induced by the automorphism of G by the same letter a. One can verify that in such a way $\mathcal{L}_a(G)$ becomes a Lie group with the Lie algebra $\mathcal{L}_a(\mathfrak{g})$.

Recall that for any $g \in G$ the mapping $\mathrm{Int}(g) : h \in G \mapsto ghg^{-1} \in G$ is an automorphism of G. Such automorphisms are called *inner* and form a normal subgroup of the group $\mathrm{Aut}\, G$. Similarly, as for the twisted loop Lie algebras, if the automorphisms a and b of G differ by an inner automorphism of G, then the twisted loop Lie groups $\mathcal{L}_a(G)$ and $\mathcal{L}_b(G)$ are naturally isomorphic. Therefore, for the case of a semisimple Lie group G one can consider only twisted loop groups $\mathcal{L}_a(G)$ where $a^K = \mathrm{id}_G$ for some positive integer K.

One can show that there is a bijective correspondence between elements of $\mathcal{L}_a(G)$ and twisted periodic mappings from $\mathbb{R}$ to G. We denote by $\tilde{\gamma}$ the twisted periodic mapping from $\mathbb{R}$ to G corresponding to the element $\gamma \in \mathcal{L}_a(G)$. Let G be a semisimple Lie group, and a be an automorphism of G such that $a^K = \mathrm{id}_G$ for some positive integer K. The transformation $\sigma \to K\sigma$ induces an injective homomorphism from $\mathcal{L}_a(G)$ to $\mathcal{L}(G)$ whose image is formed by the elements γ satisfying the condition

$$\gamma(\varepsilon_K p) = a(\gamma(p)),$$

and will be denoted by $\mathcal{L}_{a,K}(G)$. For the corresponding mapping $\tilde{\gamma}$ from $\mathbb{R}$ to G the above condition becomes

$$\tilde{\gamma}(\sigma + 2\pi/K) = a(\tilde{\gamma}(\sigma)).$$

Thus, when G is a semisimple Lie group the twisted loop group $\mathcal{L}_a(G)$ can be identified with a subgroup of $\mathcal{L}(G)$.

3 Automorphisms of twisted loop Lie algebras

In this section $\mathfrak{g}$ is always a complex simple Lie algebra and a is an automorphism of $\mathfrak{g}$. As was shown in Section 2, studying the twisted loop Lie algebra $\mathcal{L}_a(\mathfrak{g})$, one can assume without any loss of generality that $a^K = \mathrm{id}_\mathfrak{g}$ for some positive integer K and consider instead of $\mathcal{L}_a(\mathfrak{g})$ the corresponding subalgebra $\mathcal{L}_{a,K}(\mathfrak{g})$ of $\mathcal{L}(\mathfrak{g})$.

A linear homeomorphism A from a Fréchet Lie algebra $\mathfrak{G}$ to itself is said to be an *automorphism* of $\mathfrak{G}$ if

$$A[\xi, \eta] = [A\xi, A\eta]$$

for any $\xi, \eta \in \mathfrak{G}$. Treating $\mathfrak{G}$ as a smooth manifold, we see that since A is linear and continuous, it is smooth.

There are two main classes of automorphisms of the Fréchet Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$. The automorphisms of the first class are generated by diffeomorphisms of S^1.

Let us recall that the group $\mathrm{Diff}(S^1)$ of smooth diffeomorphisms of the circle S^1 can be supplied with the structure of a smooth infinite dimensional manifold in such a way that it becomes a Lie group. Necessary information on groups of diffeomorphisms of compact manifolds and some relevant references are given in Appendix A. The Lie algebra of the Lie group $\operatorname{Diff}(S^1)$ is the Lie algebra $\operatorname{Der} C^\infty(S^1)$ of smooth vector fields on S^1. Here the one-parameter subgroup associated with a vector field X is actually the flow generated by X.

Let f be a diffeomorphism of S^1. Consider a linear continuous mapping $A_f : \mathcal{L}_{a,K}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$ defined by the equality

$$A_f \xi = \xi \circ f^{-1}.$$

It is easy to see that if $\eta = A_f \xi$, then

$$\eta(f(\varepsilon_K f^{-1}(p))) = a(\eta(p)).$$

Hence, if

$$f(\varepsilon_K p) = \varepsilon_K f(p)$$

for any $p \in S^1$, then A_f can be considered as a mapping from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathcal{L}_{a,K}(\mathfrak{g})$. In this case A_f is an automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$. Conversely, if the mapping A_f is a mapping from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathcal{L}_{a,K}(\mathfrak{g})$, then f satisfies the above condition and A_f is an automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$.

One can show that the diffeomorphisms satisfying the condition $f(\varepsilon_K p) = \varepsilon_K f(p)$ form a Lie subgroup of the Lie group $\operatorname{Diff}(S^1)$. We denote it by $\operatorname{Diff}_K(S^1)$. The Lie algebra of $\operatorname{Diff}_K(S^1)$ is the subalgebra of $\operatorname{Der} C^\infty(S^1)$ formed by the vector fields X such that

$$(X(\varphi))(\varepsilon_K p) = (X(\varphi))(p)$$

for any function $\varphi \in C^\infty(S^1)$ satisfying the condition

$$\varphi(\varepsilon_K p) = \varphi(p).$$

Denote this subalgebra by $\operatorname{Der}_K C^\infty(S^1)$. It is clear that we have a left action of $\operatorname{Diff}_K(S^1)$ on $\mathcal{L}_{a,K}(\mathfrak{g})$ realised by automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$. Since this action is effective, we can say that the group of automorphisms $\operatorname{Aut} \mathcal{L}_{a,K}(\mathfrak{g})$ has a subgroup which can be identified with the Lie group $\operatorname{Diff}_K(S^1)$.

The Lie group $\operatorname{Diff}_K(S^1)$ can be identified with a subgroup of the group $\operatorname{Diff}(\mathbb{R})$. To construct this identification we start with consideration of general smooth mappings from S^1 to S^1.

For any $f \in C^\infty(S^1, S^1)$ one can find a smooth mapping $\tilde{f} \in C^\infty(\mathbb{R}, \mathbb{R})$, connected with f by the equality

$$f(\mathrm{e}^{\mathrm{i}\sigma}) = \mathrm{e}^{\mathrm{i}\tilde{f}(\sigma)}.$$

The function $\tilde{f}$ satisfies the relation

$$\tilde{f}(\sigma + 2\pi) - \tilde{f}(\sigma) = 2\pi k,$$

where k is an integer, called the *degree* of f. From the other hand, any smooth mapping $\tilde{f} \in C^\infty(\mathbb{R}, \mathbb{R})$ which satisfies the above relation induces a smooth mapping from S^1 to S^1. It is evident that two functions differing by a multiple of 2π induce the same mapping.

If f is a diffeomorphism, then its degree is 1 for an orientation preserving mapping, and it is -1 for an orientation reversing mapping. Note that in this case the corresponding function $\tilde{f}$ is strictly monotonic, and that any smooth strictly monotonic function satisfying the relation

$$\tilde{f}(\sigma + 2\pi) - \tilde{f}(\sigma) = \pm 2\pi,$$

induces a diffeomorphism of S^1.

If $f \in \mathrm{Diff}_K(S^1)$ one obtains that

$$\tilde{f}(\sigma + 2\pi/K) = \tilde{f}(\sigma) + 2\pi/K,$$

for $K \geq 2$ and that

$$\tilde{f}(\sigma + \pi) = \tilde{f}(\sigma) \pm \pi$$

for $K = 2$. Note that if ξ is an element of $\mathcal{L}_{a,K}(\mathfrak{g})$ and $f \in \mathrm{Diff}_K(S^1)$ then

$$\widetilde{A_f \xi} = \tilde{\xi} \circ \tilde{f}^{-1},$$

where A_f is the automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$ induced by f.

The second interesting class of automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$ is formed by automorphisms generated by automorphisms of $\mathfrak{g}$ acting on the elements of $\mathcal{L}_{a,K}(\mathfrak{g})$ pointwise.

Let α be an element of the Lie group $\mathcal{L}(\mathrm{Aut}\,\mathfrak{g})$. Consider a linear mapping from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathcal{L}(\mathfrak{g})$ defined by the equality

$$A_\alpha \xi = \alpha \xi,$$

where

$$(\alpha\xi)(p) = \alpha(p)(\xi(p)).$$

It is clear that A_α is a homomorphism from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathcal{L}(\mathfrak{g})$. Moreover, if α satisfies the relation

$$\alpha(\varepsilon_K p) = a\,\alpha(p) a^{-1},$$

then the mapping A_α is an automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$. In other words, any element of the Lie group $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ induces an automorphism of the Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$, and we have a left action of $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ on $\mathcal{L}_{a,K}(\mathfrak{g})$ realised by automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$. This action is again effective and, therefore, $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$ has a subgroup which can be identified with the Lie group $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$.

Actually, if for $f \in \mathrm{Diff}_K(S^1)$ and $\alpha \in \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ we define the automorphism $A_{(f,\alpha)}$ of $\mathcal{L}_{a,K}(\mathfrak{g})$ by

$$A_{(f,\alpha)}\xi = \alpha(\xi \circ f^{-1}),$$

we obtain a left effective action of the semidirect product $\mathrm{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ on $\mathcal{L}_{a,K}(\mathfrak{g})$ realised by automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$. Here the group operations in $\mathrm{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ are given by

$$(f_1, \alpha_1)(f_2, \alpha_2) = (f, \alpha),$$

where

$$f = f_1 \circ f_2, \qquad \alpha = \alpha_1(\alpha_2 \circ f_1^{-1}),$$

and

$$(f, \alpha)^{-1} = (f^{-1}, \alpha^{-1} \circ f^{-1}).$$

Thus, we see that $\mathrm{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$ can be identified with a subgroup of the group $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$. In fact, this subgroup exhausts the whole group $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$.

Theorem 3.1 *The group of automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$ can be naturally identified with the semidirect product* $\mathrm{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$.

Proof. The main idea of the proof is borrowed from [9]. Let A be an automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$. Fix a point $p \in S^1$ and consider the mapping A_p from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathfrak{g}$ defined by the equality

$$A_p(\xi) = (A\xi)(p).$$

This mapping is linear and continuous. Some necessary information on such mappings are given in Appendix B. Certainly, A_p is a homomorphism from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathfrak{g}$.

Let m be a nonnegative integer. Denote by χ_p^m a smooth function on S^1 such that

$$\chi_p^{m(m)}(p) = 1, \qquad \chi_p^{m(k)}(p) = 0, \quad k \neq m,$$

and $\operatorname{supp} \chi_p^m \cap \varepsilon_K \operatorname{supp} \chi_p^m = \varnothing$. Let x be an arbitrary element of $\mathfrak{g}$. It is not difficult to get convinced that for any nonnegative integer m the mapping

$$\eta_{p,x}^m = \sum_{l=0}^{K-1} \chi_{\varepsilon_K^l p}^m a^{-l}(x)$$

is an element of $\mathcal{L}_{a,K}(\mathfrak{g})$ satisfying the conditions

$$\eta_{p,x}^{m(m)}(p) = x, \qquad \eta_{p,x}^{m(k)}(p) = 0, \quad k \neq m.$$

The linear mapping A is invertible by definition. Therefore, there is an element $\xi_{p,x}^0 \in \mathcal{L}_{a,K}(\mathfrak{g})$ such that $A(\xi_{p,x}^0) = \eta_{p,x}^0$. This implies that $A_p(\xi_{p,x}^0) = x$. Thus the mapping A_p is surjective.

For any open set $U \subset S^1$ the set

$$\mathcal{L}_{a,K}^U(\mathfrak{g}) = \{\xi \in \mathcal{L}_{a,K}(\mathfrak{g}) \mid \operatorname{supp} \xi \subset U\}$$

is an ideal of $\mathcal{L}_{a,K}(\mathfrak{g})$. If an element $x \in \mathfrak{g}$ belongs to the image of the restriction of A_p to $\mathcal{L}_{a,K}^U(\mathfrak{g})$, then there is an element $\xi \in \mathcal{L}_{a,K}^U(\mathfrak{g})$ such that $A_p(\xi) = x$. Since A_p is surjective, it follows that for any element $y \in \mathfrak{g}$ one can find an element $\eta \in \mathcal{L}_{a,K}(\mathfrak{g})$ such that $A_p(\eta) = y$. Since $[\xi, \eta]$ belongs to $\mathcal{L}_{a,K}^U(\mathfrak{g})$, it follows that $[x, y] = A_p([\xi, \eta])$ belongs to the image of $A_p|_{\mathcal{L}_{a,K}^U(\mathfrak{g})}$. This means that the image of $A_p|_{\mathcal{L}_{a,K}^U(\mathfrak{g})}$ is an ideal of $\mathfrak{g}$. As the Lie algebra $\mathfrak{g}$ is simple, the mapping $A_p|_{\mathcal{L}_{a,K}^U(\mathfrak{g})}$ is either trivial or surjective.

The support of the mapping A_p is the union of sets of the form $\overline{\{q\}}$, where $q \in S^1$ (see Appendix B). Suppose that $\operatorname{supp} A_p = \overline{\{q\}} \cup \overline{\{q'\}}$ and $\overline{\{q\}} \cap \overline{\{q'\}} = \varnothing$. Let U and U' be disjoint neigbourhoods of $\overline{\{q\}}$ and $\overline{\{q'\}}$ respectively. Since S^1 is a normal topological space such neighbourhoods do exist. It is clear that $\mathcal{L}_{a,K}^U(\mathfrak{g})$ and $\mathcal{L}_{a,K}^{U'}(\mathfrak{g})$ are commuting ideals of $\mathcal{L}_{a,K}(\mathfrak{g})$. Therefore, the images $A_p|_{\mathcal{L}_{a,K}^U(\mathfrak{g})}$ and $A_p|_{\mathcal{L}_{a,K}^{U'}(\mathfrak{g})}$ are commuting ideals of $\mathfrak{g}$. Hence, one of the mappings $A_p|_{\mathcal{L}_{a,K}^U(\mathfrak{g})}$ and $A_p|_{\mathcal{L}_{a,K}^{U'}(\mathfrak{g})}$ is surjective, the other one is trivial. Thus, the support of A_p has the form $\overline{\{f'(p)\}}$ for some mapping $f' : S^1 \to S^1$, and we can write

$$A_p(\xi) = \sum_{m=0}^{M} c_p^m(\xi^{(m)}(f'(p))),$$

for some nonnegative integer M and endomorphisms c_p^m (see Appendix B). We assume that the endomorphisms c_p^m are defined for all nonnegative m, but $c_p^m = 0$ for $m > M$.

It is clear that

$$A_p(\eta_{f'(p),x}^m) = c_p^m(x).$$

Using the relations

$$[\eta^m_{p,x},\eta^n_{p,y}]^{(m+n)}(p)=\binom{m+n}{m}[x,y],\qquad [\eta^m_{p,x},\eta^n_{p,y}]^{(k)}(p)=0,\quad k\neq m+n,$$

we obtain

$$A_p([\eta^m_{f'(p),x},\eta^n_{f'(p),y}])=\binom{m+n}{m}c_p^{m+n}([x,y]).$$

Since A_p is a homomorphism, we have

$$A_p([\eta^m_{f'(p),x},\eta^n_{f'(p),y}])=[A_p(\eta^m_{f'(p),x}),A_p(\eta^n_{f'(p),y})],$$

therefore,

$$A_p([\eta^m_{f'(p),x},\eta^n_{f'(p),y}])=[c_p^m(x),c_p^n(y)].$$

Thus, one has the equalities

$$\binom{m+n}{m}c_p^{m+n}([x,y])=[c_p^m(x),c_p^n(y)]. \qquad (*)$$

In particular, for $m=n=0$ the equality

$$c_p^0([x,y])=[c_p^0(x),c_p^0(y)] \qquad (**)$$

is valid. Since $\mathfrak{g}$ is simple, the mapping c_p^0 is either trivial or surjective. Suppose that it is trivial. Putting in the equality (*) $n=0$, we obtain

$$c_p^m([x,y])=[c_p^m(x),c_p^0(y)].$$

Since the Lie algebra $\mathfrak{g}$ is simple, then $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. Therefore, for any m the mapping c_p^m is trivial. Hence, the mapping A_p is also trivial. This contradicts surjectivity of A_p. Thus, c_p^0 is surrjective, and the equality $(**)$ says that it is an automorphism of the Lie algebra $\mathfrak{g}$.

Putting in $(*)$ $m=0$ and $n=1$, we obtain

$$c_p^1([x,y])=[c_p^0(x),c_p^1(y)]$$

for any $x,y\in\mathfrak{g}$. Rewrite this equality as

$$(c_p^0)^{-1}(c_p^1([x,y]))=[x,(c_p^0)^{-1}(c_p^1(y))].$$

Therefore,

$$((c_p^0)^{-1}c_p^1)\,\mathrm{ad}(x)=\mathrm{ad}(x)((c_p^0)^{-1}c_p^1)$$

for any $x\in\mathfrak{g}$. Since the Lie algebra $\mathfrak{g}$ is simple, the linear operator $(c_p^0)^{-1}c_p^1$ is multiplication by some scalar, denote it by ρ. Thus, we have $c_p^1=\rho c_p^0$. Relation $(*)$ for $m=1$ and $n=1$ takes the form

$$2\,c_p^2([x,y])=[c_p^1(x),c_p^1(y)]=\rho^2c_p^0([x,y]).$$

Therefore, $c_p^2=(\rho^2/2)c_p^0$. In general case we have $c_p^m=(\rho^m/m!)c_p^0$ for any positive m. From the other hand, $c_p^m=0$ for $m>M$. It is possible only if $\rho=0$. Hence, $c_p^m=0$ for all $m>0$.

Define a mapping $\alpha: S^1\to\mathrm{Aut}\,\mathfrak{g}$ by

$$\alpha(p)=c_p^0,$$

then one can write

$$A\xi = \alpha(\xi \circ f').$$

Since for any $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ the mapping $A\xi$ belongs to $\mathcal{L}_{a,K}(\mathfrak{g})$, the mappings f' and α must be smooth. The mapping f' is actually an element of $\operatorname{Diff}_K(S^1)$, and α belongs to $\mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$. Hence, defining $f = f'^{-1}$, we see that

$$A\xi = \alpha(\xi \circ f^{-1}).$$

Thus, an arbitrary automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$ has the above form for some $f \in \operatorname{Diff}_K(S^1)$ and some $\alpha \in \mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$. □

Note that in the case where $\mathfrak{g}$ is a complex Lie algebra, the Lie group $\operatorname{Aut}\mathfrak{g}$ is a complex Lie group. In this case $\mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$ is also a complex Lie group. From the other hand, any diffeomorphism from the identity component of the Lie group $\operatorname{Diff}_K(S^1)$ to a complex Lie group is trivial (see, for example, [9]). This implies that $\operatorname{Diff}_K(S^1)$ cannot be endowed with the structure of a complex Lie group. Therefore, even in the case where $\mathfrak{g}$ is a complex Lie algebra we consider $\mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$ as a real Lie group. Thus, the identification described in Theorem 3.1 supplies the group $\operatorname{Aut}\mathcal{L}_{a,K}(\mathfrak{g})$ with the structure of a real Lie group. Here the action of the group $\operatorname{Aut}\mathcal{L}_{a,K}(\mathfrak{g})$ on $\mathcal{L}_{a,K}(\mathfrak{g})$, where $\mathcal{L}_{a,K}(\mathfrak{g})$ is treated as a real manifold, is smooth.

The Lie algebra of the Lie group $\operatorname{Aut}\mathfrak{g}$ is the Lie algebra $\operatorname{Der}\mathfrak{g}$ of derivations of $\mathfrak{g}$. The situation is almost the same for the case of the Lie group $\operatorname{Aut}\mathcal{L}_{a,K}(\mathfrak{g})$. Actually, any element of the Lie algebra of the Lie group $\operatorname{Aut}\mathcal{L}_{a,K}(\mathfrak{g})$ induces a derivation of $\mathcal{L}_{a,K}(\mathfrak{g})$, but in the case where $\mathfrak{g}$ is a complex Lie algebra there are derivations of $\mathcal{L}_{a,K}(\mathfrak{g})$ which cannot be obtained in such a way. To show this, let us consider first the Lie algebra of $\operatorname{Aut}\mathcal{L}_{a,K}(\mathfrak{g})$. Using the identification described in Theorem 3.1, we see that this Lie algebra can be identified with the semidirect product of the Lie algebra of the Lie group $\operatorname{Diff}_K(S^1)$ and the Lie algebra of the Lie group $\mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$. As we already noted the Lie algebra of $\operatorname{Diff}_K(S^1)$ is the subalgebra $\operatorname{Der}_K C^\infty(S^1)$ of the Lie algebra $\operatorname{Der} C^\infty(S^1)$ of smooth vector fields on S^1. The Lie algebra of the Lie group $\mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$ is $\mathcal{L}_{\mathrm{Ad}(a),K}(\operatorname{Der}\mathfrak{g})$. Thus, the Lie algebra of the group of automorphisms of $\mathcal{L}_{a,K}(\mathfrak{g})$ can be naturally identified with the Lie algebra $\operatorname{Der}_K C^\infty(S^1) \ltimes \mathcal{L}_{\mathrm{Ad}(a),K}(\operatorname{Der}\mathfrak{g})$.

By a *derivation* of a Fréchet Lie algebra $\mathfrak{G}$ we mean a continuous linear mapping D from $\mathfrak{G}$ to $\mathfrak{G}$ which satisfies the relation

$$D[\xi, \eta] = [D\xi, \eta] + [\xi, D\eta].$$

Note again that continuity and linearity imply smoothness.

The derivation of $\mathcal{L}_{a,K}(\mathfrak{g})$ corresponding to an element of $\operatorname{Der}_K C^\infty(S^1) \ltimes \mathcal{L}_{\mathrm{Ad}(a),K}(\operatorname{Der}\mathfrak{g})$ is constructed as follows. Define the action of a vector field $X \in \operatorname{Der}_K(S^1)$ on an element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ in the usual way. Let (e_i) be a basis of $\mathfrak{g}$, then for any element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ one can write

$$\xi = \sum_i e_i \xi^i,$$

where ξ^i are smooth functions on S^1. Then one assumes that

$$X(\xi) = \sum_i e_i X(\xi^i).$$

One can get convinced that this definition does not depend on the choice of a basis (e_i). Let (X, δ) be an element of $\operatorname{Der}_K C^\infty(S^1) \ltimes \mathcal{L}_{\mathrm{Ad}(a),K}(\operatorname{Der}\mathfrak{g})$. Consider the corresponding one-parameter subgroup of the Lie group $\operatorname{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\operatorname{Aut}\mathfrak{g})$. It is determined by two mappings $\lambda : \mathbb{R} \to \mathrm{Diff}_K(S^1)$ and $\theta : \mathbb{R} \to \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$. For any fixed element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ one has a curve $\tau \in \mathbb{R} \mapsto \theta(\tau)(\xi \circ (\lambda(\tau))^{-1})$ in $\mathcal{L}_{a,K}(\mathfrak{g})$. The tangent vector to this curve at zero can be treated as the action of a linear operator D on the element ξ. It is clear that

$$D\xi = -X(\xi) + \delta(\xi),$$

where

$$(\delta(\xi))(p) = \delta(p)(\xi(p)).$$

One can verify that D is a derivation of the Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$. In can be shown also that in the case where $\mathfrak{g}$ is a real Lie algebra the derivations of the above form exhaust all possible derivations of the Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$. In the case where $\mathfrak{g}$ is a complex Lie algebra to exhaust all derivations one should assume that the vector field X may be complex.

4 $\mathbb{Z}$-gradations of twisted loop Lie algebras

In general, dealing with $\mathbb{Z}$-gradations of infinite dimensional Lie algebras we confront with necessity to work with infinite series of their elements, or, in other words, with series in Fréchet spaces. The relevant information on such series is given in Appendix C.

Let $\mathfrak{G}$ be a Fréchet Lie algebra. Suppose that for any $k \in \mathbb{Z}$ there is given a closed subspace $\mathfrak{G}_k$ of $\mathfrak{G}$ such that

(a) for any $k, l \in \mathbb{Z}$ one has $[\mathfrak{G}_k, \mathfrak{G}_l] \subset \mathfrak{G}_{k+l}$,

(b) any element ξ of $\mathfrak{G}$ can be uniquely represented as an absolutely convergent series

$$\xi = \sum_{k\in\mathbb{Z}} \xi_k,$$

where $\xi_k \in \mathfrak{G}_k$. In this case we say that the Fréchet Lie algebra $\mathfrak{G}$ is supplied with a $\mathbb{Z}$-*gradation*, and call the subspaces $\mathfrak{G}_k$ the *grading subspaces* of $\mathfrak{G}$ and the elements ξ_k the *grading components* of ξ. If F is an isomorphism from the Fréchet Lie algebra $\mathfrak{G}$ to a Fréchet Lie algebra $\mathfrak{H}$, then taking the subspaces $\mathfrak{H}_k = F(\mathfrak{G}_k)$ of $\mathfrak{H}$ as grading subspaces we endow $\mathfrak{H}$ with a $\mathbb{Z}$-gradation. In this case we say that the $\mathbb{Z}$-gradations of $\mathfrak{G}$ and $\mathfrak{H}$ under consideration are *conjugated* by the isomorphism F. It is clear that if the grading components of an element $\xi \in \mathfrak{G}$ are ξ_k, then the grading components $F(\xi)_k$ of the element $F(\xi) \in \mathfrak{H}$ are $F(\xi_k)$.

As the simplest example, let us consider the so-called *standard gradation* of $\mathcal{L}(\mathfrak{g})$. Denote by λ the standard coordinate function on $\mathbb{C}$ and its restriction to S^1. The grading subspaces for the standard gradation are defined as

$$\mathcal{L}(\mathfrak{g})_k = \{\lambda^k x \mid x \in \mathfrak{g}\},$$

and the expansion of a general element ξ of $\mathcal{L}(\mathfrak{g})$ over grading subspaces is the representation of ξ as a Fourier series:

$$\xi = \sum_{k\in\mathbb{Z}} \lambda^k x_k,$$

that in terms of the mapping $\tilde{\xi}$ has the usual form

$$\tilde{\xi} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{iks} x_k,$$

with

$$x_k = \frac{1}{2\pi} \int_{[0,2\pi]} \mathrm{e}^{-\mathrm{i}ks} \tilde{\xi} \mathrm{d}s.$$

From the theory of Fourier series it follows that the Fourier series of any element $\xi \in \mathcal{L}(\mathfrak{g})$ converges absolutely to ξ as a series in the Fréchet space $\mathcal{L}(\mathfrak{g})$. Hence, we really have a $\mathbb{Z}$-gradation of $\mathcal{L}(\mathfrak{g})$.

The necessity to include the requirement of absolute convergence in the definition of $\mathbb{Z}$-gradation is justified by the following proposition.

Proposition 4.1 *Let a Fréchet Lie algebra $\mathfrak{G}$ be supplied with a $\mathbb{Z}$-gradation. For any two elements of $\mathfrak{G}$,*

$$\xi = \sum_{k \in \mathbb{Z}} \xi_k, \qquad \eta = \sum_{k \in \mathbb{Z}} \eta_k,$$

the grading components of $[\xi, \eta]$ are given by

$$[\xi, \eta]_k = \sum_{l \in \mathbb{Z}} [\xi_{k-l}, \eta_l].$$

Here the series at the right hand side converges absolutely.

Proof. First prove that the series $\sum_{(k,l) \in \mathbb{Z} \times \mathbb{Z}} [\xi_k, \eta_l]$ converges absolutely. Let α be an element of $\mathcal{D}(\mathbb{Z} \times \mathbb{Z})$, fix a positive integer m, and define

$$r_{\alpha,m} = \sum_{(k,l) \in \alpha} \|[\xi_k, \eta_l]\|_m.$$

There are elements $\beta, \gamma \in \mathcal{D}(\mathbb{Z})$ such that $\alpha \subset \beta \times \gamma$. Using Lemma 2.2, we obtain

$$\begin{aligned} r_{\alpha,m} \leqslant \sum_{(k,l) \in \beta \times \gamma} \|[\xi_k, \eta_l]\|_m \leqslant C_m \sum_{(k,l) \in \beta \times \gamma} \|\xi_k\|_m \|\eta_l\|_m \\ = C_m \left(\sum_{k \in \beta} \|\xi_k\|_m \right) \left(\sum_{l \in \gamma} \|\eta_l\|_m \right) \leqslant C_m \left(\sum_{k \in \mathbb{Z}} \|\xi_k\|_m \right) \left(\sum_{l \in \mathbb{Z}} \|\eta_l\|_m \right). \end{aligned}$$

It is clear that for any positive integer m the net $(r_{\alpha,m})_{\alpha \in \mathcal{D}(\mathbb{Z})}$ is monotonically increasing, that means that $r_{\alpha,m} \geq r_{\beta,m}$ if $\alpha \succcurlyeq \beta$. The above inequalities show that it is also bounded above. Similarly as it is for the case of sequences, such a net is convergent. Therefore, the series $\sum_{(k,l) \in \mathbb{Z} \times \mathbb{Z}} [\xi_k, \eta_l]$ converges absolutely.

As follows from Proposition C.2 one can write

$$\sum_{(k,l) \in \mathbb{Z} \times \mathbb{Z}} [\xi_k, \eta_l] = \sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} [\xi_k, \eta_l] \right).$$

For a fixed k the net $\left(\sum_{l \in \alpha} [\xi_k, \eta_l]\right)_{\alpha \in \mathcal{D}(\mathbb{Z})}$ converges absolutely. It is clear that this net coincides with the net $\left([\xi_k, \sum_{l \in \alpha} \eta_l]\right)_{\alpha \in \mathcal{D}(\mathbb{Z})}$. Since the net $\left(\sum_{l \in \alpha} \eta_l\right)_{\alpha \in \mathcal{D}(\mathbb{Z})}$ converges to η and the Lie algebra operation in $\mathfrak{G}$ is continuous, one has

$$\sum_{l \in \mathbb{Z}} [\xi_k, \eta_l] = [\xi_k, \eta].$$

Similarly, one obtains

$$\sum_{k \in \mathbb{Z}} [\xi_k, \eta] = [\xi, \eta].$$

Using again Proposition C.2, we come to the equality

$$[\xi, \eta] = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} [\xi_{k-l}, \eta_l],$$

where for any k the series $\sum_{l \in \mathbb{Z}} [\xi_{k-l}, \eta_l]$ converges absolutely. □

Suppose that a Fréchet Lie algebra $\mathfrak{G}$ is supplied with a $\mathbb{Z}$-gradation such that for any element $\xi = \sum_{k \in \mathbb{Z}} \xi_k$ of $\mathfrak{G}$ the series $\sum_{k \in \mathbb{Z}} k \xi_k$ converges unconditionally. In this case one can define a linear operator Q in $\mathfrak{G}$, acting on an element $\xi = \sum_{k \in \mathbb{Z}} \xi_k$ as

$$Q\xi = \sum_{k \in \mathbb{Z}} k \xi_k.$$

Actually the elements $k \xi_k$ are the grading components of the element $Q\xi$, therefore, the series $\sum_{k \in \mathbb{Z}} k \xi_k$ converges absolutely by the definition of a $\mathbb{Z}$-gradation. It is clear that

$$\mathfrak{G}_k = \{\xi \in \mathfrak{G} \mid Q\xi = k\xi\}.$$

We call the linear operator Q the *grading operator* and say that the $\mathbb{Z}$-gradation under consideration is *generated by grading operator*. If a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ and a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{H}$ are conjugated by an isomorphism F, and the $\mathbb{Z}$-gradation of $\mathfrak{G}$ is generated by a grading operator Q, then the $\mathbb{Z}$-gradation of $\mathfrak{H}$ is generated by the grading operator FQF^{-1}.

The standard gradation of $\mathcal{L}(\mathfrak{g})$ is generated by a grading operator Q such that

$$\widetilde{Q\xi} = -\mathrm{i}\mathrm{d}\tilde{\xi}/\mathrm{d}s.$$

Here the operator Q is a derivation of $\mathcal{L}(\mathfrak{g})$. In general we have the following statement.

Proposition 4.2 *Let a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ be generated by a grading operator Q. The equality*

$$Q[\xi, \eta] = [Q\xi, \eta] + [\xi, Q\eta].$$

is valid for any $\xi, \eta \in \mathfrak{G}$.

Proof. Using Proposition 4.1, one obtains

$$Q[\xi, \eta] = \sum_{k \in \mathbb{Z}} (Q[\xi, \eta])_k = \sum_{k \in \mathbb{Z}} k[\xi, \eta]_k = \sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} k[\xi_{k-l}, \eta_l] \right).$$

In a similar way one comes to the equalities

$$[Q\xi, \eta] = \sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} (k-l)[\xi_{k-l}, \eta_l] \right), \qquad [\xi, Q\eta] = \sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} l[\xi_{k-l}, \eta_l] \right).$$

The three above equalities imply the validity of the statement of the proposition. □

It follows from this lemma that if the grading operator Q generating a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ is continuos, it is a derivation of $\mathfrak{G}$.

We call a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ *integrable* if the mapping Φ from $\mathbb{R} \times \mathfrak{G}$ to $\mathfrak{G}$ defined by the relation

$$\Phi(\tau, \xi) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau} \xi_k$$

is smooth. Here as usually we denote by ξ_k the grading components of the element ξ with respect to the $\mathbb{Z}$-gradation under consideration.

For each fixed $\xi \in \mathfrak{G}$ the mapping Φ induces a smooth curve $\Phi_\xi : \mathbb{R} \to \mathfrak{G}$ given by the equality

$$\Phi_\xi(\tau) = \Phi(\tau, \xi).$$

Proposition 4.3 *Any integrable $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ is generated by grading operator. The corresponding grading operator Q acts on an element $\xi \in \mathfrak{G}$ as*

$$Q\xi = \mathrm{i} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_0 \Phi_\xi,$$

where we denote by t the standard coordinate function on $\mathbb{R}$.

Proof. Since the mapping Φ is smooth and linear in ξ, then Q is a continuous linear operator on $\mathfrak{G}$. Therefore, for any net $(\xi_\alpha)_{\alpha \in \mathcal{D}(\mathbb{Z})}$ in $\mathfrak{G}$ which converges to an element $\xi \in \mathfrak{G}$, the net $(Q\xi_\alpha)_{\alpha \in \mathcal{D}(\mathbb{Z})}$ converges to $Q\xi$. The net $(\xi_\alpha)_{\alpha \in \mathcal{D}(\mathbb{Z})}$, where

$$\xi_\alpha = \sum_{k \in \alpha} \xi_k,$$

where ξ_k are the grading components of ξ, converges to ξ. Since for any $\alpha \in \mathcal{D}(\mathbb{Z})$ the element ξ_α is the sum of a finite number of grading components, one has

$$Q\xi_\alpha = \mathrm{i} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_0 \Phi_{\xi_\alpha} = \sum_{k \in \alpha} k \xi_k.$$

This means that $Q\xi = \sum_{k \in \mathbb{Z}} k \xi_k$. Thus, the linear operator Q generates the $\mathbb{Z}$-gradation under consideration. □

Proposition 4.4 *Let a Fréchet Lie algebra $\mathfrak{G}$ be supplied with an integrable $\mathbb{Z}$-gradation. Then for any fixed $\tau \in \mathbb{R}$ the mapping $\xi \in \mathfrak{G} \mapsto \Phi(\tau, \xi) \in \mathfrak{G}$ is an automorphism of $\mathfrak{G}$. The mapping Φ satisfies the relation*

$$\Phi(\tau_1, \Phi(\tau_2, \xi)) = \Phi(\tau_1 + \tau_2, \xi).$$

Proof. From Proposition 4.1 it follows that one can write

$$[\Phi(\tau, \xi), \Phi(\tau, \eta)] = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} [(\Phi(\tau, \xi))_{k-l}, (\Phi(\tau, \eta))_l].$$

It is clear that

$$(\Phi(\tau, \xi))_{k-l} = \mathrm{e}^{-\mathrm{i}(k-l)\tau} \xi_{k-l}, \qquad (\Phi(\tau, \eta))_l = \mathrm{e}^{-\mathrm{i}l\tau} \eta_l.$$

Therefore, one has

$$[\Phi(\tau, \xi), \Phi(\tau, \eta)] = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau} \sum_{l \in \mathbb{Z}} [\xi_{k-l}, \eta_l] = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau} [\xi, \eta]_k = \Phi(\tau, [\xi, \eta]).$$

That proves the first statement of the proposition. The second statement of the proposition is evident. □

Let us return to consideration of twisted loop Lie algebras. Suppose that $\mathfrak{g}$ is a complex simple Lie algebra, and a is an automorphism of $\mathfrak{g}$ satisfying the relation $a^K = \mathrm{id}_\mathfrak{g}$ for some positive integer K. Assume that the twisted Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$ is endowed with an integrable $\mathbb{Z}$-gradation. Define a mapping φ from $\mathbb{R}$ to the Lie group $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$ by the equality

$$(\varphi(\tau))(\xi) = \Phi(\tau, \xi).$$

It is a curve in the Lie group $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$. Using the identification of $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$ with the Lie group $\mathrm{Diff}_K(S^1) \ltimes \mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$, for any $\tau \in \mathbb{R}$ one can write

$$\varphi(\tau) = (\lambda(\tau), \theta(\tau)),$$

where λ is a mapping from $\mathbb{R}$ to the Lie group $\mathrm{Diff}_K(S^1)$ and θ is a mapping from $\mathbb{R}$ to the Lie group $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$. The mapping λ induces a mapping Λ from $\mathbb{R} \times S^1$ to S^1 given by

$$\Lambda(\tau, p) = (\lambda(\tau))(p),$$

and the mapping θ induces a mapping Θ from $\mathbb{R} \times S^1$ to $\mathrm{Aut}\,\mathfrak{g}$ given by

$$\Theta(\tau, p) = (\theta(\tau))(p).$$

Using the mappings Λ and Θ one can write

$$(\Phi(\tau, \xi))(p) = \Theta(\tau, p)(\xi(\Lambda^{-1}(\tau, p))),$$

where the mapping $\Lambda^{-1} : \mathbb{R} \times S^1 \to S^1$ is defined by the equality

$$\Lambda^{-1}(\tau, \Lambda(\tau, p)) = p.$$

Since the mapping Φ is smooth, also the mappings Λ and Θ are smooth. Therefore, by the exponential law (see Appendix A), the mappings λ and θ are also smooth. Thus, the curve φ is a smooth curve in the Lie group $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$. Actually, as follows from Proposition 4.4, it is a one-parameter subgroup of $\mathrm{Aut}\,\mathcal{L}_{a,K}(\mathfrak{g})$. The tangent vector to the curve φ at zero is a derivation of $\mathcal{L}_{a,K}(\mathfrak{g})$ which coincides with the linear operator $-\mathrm{i}Q$. Therefore, one has the equality

$$Q\xi = -\mathrm{i}X(\xi) + \mathrm{i}\delta(\xi).$$

Here $X \in \mathrm{Der}_K\, C^\infty(S^1)$ is the vector field being the tangent vector at zero to the curve λ in $\mathrm{Diff}_K(S^1)$, and δ is the tangent vector at zero to the curve θ in $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Aut}\,\mathfrak{g})$.

Note that the mapping Λ corresponding to the mapping λ is a flow on S^1, and X is the vector field which generates this flow.

Proposition 4.5 *Either the vector field X is zero vector field, or it has no zeros.*

Proof. It is clear that $\Phi(\tau + 2\pi, \xi) = \Phi(\tau, \xi)$ for any $\xi \in \mathcal{L}(\mathfrak{g})$. It implies that $\Lambda(\tau + 2\pi, p) = \Lambda(\tau, p)$ for any $p \in S^1$. According to the mechanical interpretation of the flow, $\Lambda(\tau, p)$ is the position of a particle at time τ, if its position at zero time is p. Here the velocity of the particle at time τ is $X(\Lambda(\tau, p))$. If $p \in S^1$ is a zero of X, then a particle placed at the point p at some instant of time will forever remain at that point. If $X(p) \neq 0$, a particle placed at the point p will instantly move in the same direction, and it cannot pass any zero of the vector field X. If X has zeros, this contradicts the periodicity of Λ in the first argument. There is no contradiction only if X is zero vector field. □

Recall that any derivation of a simple Lie algebra is an inner derivation. Therefore, if δ is an element of $\mathcal{L}_{\mathrm{Int}(a),K}(\mathrm{Der}\,\mathfrak{g})$, then there exists a unique element η of $\mathcal{L}_{a,K}(\mathfrak{g})$ such that

$$\delta(\xi) = [\eta, \xi].$$

Thus, we come to the following proposition.

Proposition 4.6 *The grading operator Q generating an integrable $\mathbb{Z}$-gradation of a twisted loop Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$ acts on an element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ as*

$$Q\xi = -\mathrm{i}X(\xi) + \mathrm{i}[\eta, \xi].$$

where $X \in \mathrm{Der}_K\, C^\infty(S^1)$, and η is an element of $\mathcal{L}_{a,K}(\mathfrak{g})$.

We will not consider $\mathbb{Z}$-gradations with infinite dimensional grading subspaces. Therefore, the vector field X cannot be zero vector field. Indeed, suppose that $X = 0$ and $Q\xi = \mathrm{i}[\eta, \xi] = k\xi$, then $\|[\eta, \xi]\|_1 = |k|\|\xi\|_1$. From Lemma 2.2 one obtains

$$|k| \leq C\|\eta\|_1.$$

Hence, we have only a finite number of grading subspace, thus, at least some of them must be infinite dimensional.

From now on we identify any element ξ of $\mathcal{L}_{a,K}(\mathfrak{g})$ with the corresponding mapping $\tilde{\xi}$ from $\mathbb{R}$ to $\mathfrak{g}$ omitting the tilde. Similarly, we identify each element of $\mathrm{Diff}_K(S^1)$ with the corresponding mapping $\tilde{f}$ again omitting the tilde. An element X of $\mathrm{Der}_K\, C^\infty(S^1)$ is identified with the vector field on $\mathbb{R}$, which we denote again by X. One has

$$X = v\,\mathrm{d}/\mathrm{d}s,$$

where the function v satisfies the relation

$$v(\sigma + 2\pi/K) = v(\sigma).$$

Proposition 4.7 *Let the twisted loop Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$ be endowed with an integrable $\mathbb{Z}$-gradation with finite dimensional grading subspaces, and Q be the corresponding grading operator, which has the form described in Proposition 4.6. For any diffeomorphism $f \in \mathrm{Diff}_K\, S^1$ one has*

$$A_f Q A_f^{-1}\xi = -\mathrm{i}f_*X(\xi) + \mathrm{i}[\eta, \xi],$$

where A_f is the automorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$ induced by f. Here the diffeomorphism f can be chosen so that

$$f_*X = \kappa\,\mathrm{d}/\mathrm{d}s$$

for some nonzero real constant κ.

Proof. The first statement of the proposition follows from the well known equality

$$f_*X(\varphi) = f^{-1*}X(f^*\varphi)$$

valid for any $\varphi \in C^\infty(S^1)$.

Writing the vector field X as $v\,\mathrm{d}/\mathrm{d}s$, in accordance with Proposition 4.5 we conclude that the function v has no zeros. Thus we can consider a diffeomorphism f of $\mathrm{Diff}_K(S^1)$ with

$$f(\sigma) = \kappa \int_0^\sigma \frac{\mathrm{d}\sigma'}{v(\sigma')}.$$

Here the constant κ is fixed by $f(2\pi/K) = 2\pi/K$. It is easy to verify that $f_*X = \kappa\,\mathrm{d}/\mathrm{d}s$. Thus, the second statement of the proposition is true. □

Without any loss of generality one can assume that the constant κ of the above proposition is positive. Indeed, if it is not the case one can do so performing the mapping $\xi(\sigma) \to \xi(-\sigma)$ which maps $\mathcal{L}_{a,K}(\mathfrak{g})$ isomorphically onto $\mathcal{L}_{a^{-1},K}(\mathfrak{g})$.

Let G be a simply connected Lie group whose Lie algebra coincides with $\mathfrak{g}$. Denote the automorphism of G corresponding to the automorphism a of $\mathfrak{g}$ by the same letter a. Since G is a complex simple Lie group, we will consider it as a linear group. The following proposition is evident.

Proposition 4.8 *Let the twisted loop Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$ be endowed with an integrable $\mathbb{Z}$-gradation, and Q be the corresponding grading operator, which has the form described in Proposition 4.6. Let γ be a smooth mapping from $\mathbb{R}$ to G satisfying the relation*

$$\gamma(\sigma + 2\pi/K) = a(g\gamma(\sigma))$$

for some $g \in G$. Consider a linear mapping A_γ acting on any element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ as

$$A_\gamma \xi = \gamma \xi \gamma^{-1}.$$

The mapping A_γ is an isomorphism from $\mathcal{L}_{a,K}(\mathfrak{g})$ to the Lie algebra of smooth mappings ξ from $\mathbb{R}$ to $\mathfrak{g}$ satisfying the equality

$$\xi(\sigma + 2\pi/K) = a(g\xi(\sigma)g^{-1}).$$

This isomorphism conjugates the $\mathbb{Z}$-gradation of $\mathcal{L}_{a,K}(\mathfrak{g})$ and the $\mathbb{Z}$-gradation generated by the grading operator $A_\gamma Q A_\gamma^{-1}$ which acts as

$$A_\gamma Q A_\gamma^{-1} \xi = -\mathrm{i} X(\xi) + \mathrm{i}[\gamma \eta \gamma^{-1} + X(\gamma)\gamma^{-1}, \xi].$$

Now we are able to prove our main theorem.

Theorem 4.1 *An integrable $\mathbb{Z}$-gradation of a twisted loop Lie algbera $\mathcal{L}_{a,K}(\mathfrak{g})$ with finite dimensional grading subspaces is conjugated by an isomorphism to a $\mathbb{Z}$-gradation of an appropriate twisted loop Lie algebra $\mathcal{L}_{a',K'}(\mathfrak{g})$ generated by grading operator*

$$Q'\xi = -\mathrm{i}\mathrm{d}\xi/\mathrm{d}s.$$

Here the automorphisms a and a' differ by an inner automorphism of $\mathfrak{g}$.

Proof. In accordance with Proposition 4.6 the grading operator of an integrable $\mathbb{Z}$-gradation of $\mathcal{L}_{a,K}(\mathfrak{g})$ with finite dimensional grading subspaces is specified by the choice of a vector field $X \in \mathrm{Der}_K\, C^\infty(S^1)$ and by an element $\eta \in \mathcal{L}_{a,K}(\mathfrak{g})$. Having in mind Proposition 4.7 and the discussion given just below it, we assume without loss of generality that

$$X = \kappa \mathrm{d}/\mathrm{d}s$$

for some positive real constant κ.

Let a mapping $\gamma : \mathbb{R} \to G$ be a solution of the equation

$$\kappa \gamma^{-1} \mathrm{d}\gamma/\mathrm{d}s = -\eta.$$

It is well known that this equation always has solutions, all its solutions are smooth, and if γ and γ' are two solutions then

$$\gamma' = g\gamma$$

for some $g \in G$. Using the equality

$$\eta(\sigma + 2\pi/K) = a(\eta(\sigma)),$$

one concludes that, if γ is a solution, then the mapping γ' defined by the equality

$$\gamma'(\sigma) = a^{-1}(\gamma(\sigma + 2\pi/K))$$

is also a solution. Hence, for some $g \in G$ one has

$$\gamma(\sigma + 2\pi/K) = a(g\gamma(\sigma)).$$

The mapping A_γ, described in Proposition 4.8, accompanied by the transformation $\sigma \to \sigma/K$ maps $\mathcal{L}_{a,K}(\mathfrak{g})$ isomorphically onto the Fréchet Lie algebra $\mathfrak{G}$ formed by smooth mappings ξ from $\mathbb{R}$ to $\mathfrak{g}$ satisfying the condition

$$\xi(\sigma + 2\pi) = a'(\xi(\sigma)),$$

where $a' = a \circ \mathrm{Ad}(g)$. Denote the grading operator generating the corresponding conjugated $\mathbb{Z}$-gradation again by Q. In accordance with Proposition 4.8 the operator Q acts on an element ξ as

$$Q\xi = -\mathrm{i}K'\mathrm{d}\xi/\mathrm{d}s,$$

where $K' = \kappa K$. Suppose that for some integer k the grading subspace $\mathfrak{G}_k$ is nontrivial and $\xi \in \mathfrak{G}_k$ is not equal to zero, then

$$\xi = \exp(\mathrm{i}ks/K')\xi(0)$$

with $\xi(0) \neq 0$. Since ξ is an element of $\mathfrak{G}$, one should have

$$a'(\xi(0)) = \exp(2\pi\mathrm{i}k/K')\xi(0).$$

For any integer l the mapping ξ' defined by

$$\xi' = \exp(\mathrm{i}ls)\xi$$

is a nonzero element of $\mathfrak{G}$. The action of the grading operator Q on ξ' gives $(K'l + k)\xi'$. The number $K'l + k$ should be an integer. Since l is an arbitrary integer, it is possible only if K' is an integer. Actually, due to the remark given after the proof of Proposition 4.7, one can assume without any loss of generality that it is a positive integer.

For any integer k denote by $[k]_{K'}$ the element of the ring $\mathbb{Z}_{K'}$ corresponding to k. Let x be an arbitrary element of $\mathfrak{g}$ and ξ be an element of $\mathfrak{G}$ such that $\xi(0) = x$. Expanding ξ over the grading subspaces,

$$\xi = \sum_{k \in \mathbb{Z}} \xi_k,$$

one obtains

$$x = \sum_{m \in \mathbb{Z}_{K'}} x_m,$$

where

$$x_m = \sum_{\substack{k \in \mathbb{Z} \\ [k]_{K'} = m}} \xi_k(0).$$

Here for any $m \in \mathbb{Z}_{K'}$ we have

$$a'(x_m) = \exp(2\pi\mathrm{i}k/K')x_m,$$

where k is an arbitrary integer such that $[k]_{K'} = m$. Hence, the automorphism a' is semisimple and $a'^{K'} = \mathrm{id}_{\mathfrak{g}}$.

The change $\sigma \to K'\sigma$ induces an isomorphism from $\mathfrak{G}$ to $\mathcal{L}_{a',K'}(\mathfrak{g})$ which conjugates the $\mathbb{Z}$-gradation of $\mathfrak{G}$ under consideration with the $\mathbb{Z}$-gradation of $\mathcal{L}_{a',K'}(\mathfrak{g})$ generated by grading operator $Q' = -\mathrm{id}/\mathrm{d}s$. That was to be proved. □

It follows from the above theorem that to classify all $\mathbb{Z}$-gradations of the twisted loop Lie algebra $\mathcal{L}_{a,K}(\mathfrak{g})$ it suffices to classify the automorphisms of $\mathfrak{g}$ of finite order. The solution of the latter problem can be found, for example in [11, 5], or in [8]. Note here that classification of the automorphisms of $\mathfrak{g}$ of finite order is equivalent to classification of $\mathbb{Z}_K$-gradations of $\mathfrak{g}$. Let us have two $\mathbb{Z}$-gradations of $\mathcal{L}_{a,K}(\mathfrak{g})$ which are conjugated to standard $\mathbb{Z}$-gradations of Lie algebras $\mathcal{L}_{a',K'}(\mathfrak{g})$ and $\mathcal{L}_{a'',K''}(\mathfrak{g})$. It is clear that the initial $\mathbb{Z}$-gradations are congugated by an isomorphism of $\mathcal{L}_{a,K}(\mathfrak{g})$ if and only if $K' = K''$ and the automorphisms a' and a'' are conjugated.

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A Diffeomorphism groups

Let M and N be two finite dimensional manifolds, and M be compact. The space $C^\infty(M, N)$ of all smooth mappings from M to N can be supplied with the structure of a smooth manifold modelled on Fréchet spaces (see, for example, [12, 13, 14]).

Let K, M, N be three finite dimensional manifolds, and let M be compact. Consider a smooth mapping φ from K to $C^\infty(M, N)$. This mapping induces a mapping Φ from $K \times M$ to N defined by the equality

$$\Phi(p, q) = (\varphi(p))(q).$$

One can prove that the mapping Φ is smooth. Conversely, if one has a smooth mapping from $K \times M$ to N, reversing the above equality one can define a mapping from K to $C^\infty(M, N)$, and this mapping is also smooth. Thus, we have the following canonical identification

$$C^\infty(K, C^\infty(M, N)) = C^\infty(K \times M, N).$$

This fact is called the *exponential law* or the *Cartesian clousedness* (see, for example, [13, 14]).

Let M be a compact finite dimensional manifold. The group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M is an open submanifold of the manifold $C^\infty(M, M)$. Here $\mathrm{Diff}(M)$ is a Lie group. The Lie algebra of $\mathrm{Diff}(M)$ is the vector space $\mathrm{Der}\, C^\infty(M)$ of all smooth vector fields on M equipped with the negative of the usual Lie bracket (see, for example, [15, 13, 14]).

Let $\lambda : \mathbb{R} \to \mathrm{Diff}(M)$ be a smooth curve through the point id_M. For each $p \in M$ the curve λ induces a curve $\tau \in \mathbb{R} \mapsto (\lambda(\tau))(p)$ in M through p. The tangent vector to this curve at the point p is an element of $T_p(M)$. In this way we obtain a vector field on M which is the tangent vector to the curve λ at the point id_M.

Let now $\lambda : \mathbb{R} \to \mathrm{Diff}(M)$ be a one-parameter subgroup of $\mathrm{Diff}(M)$. This means that λ is a smooth curve in $\mathrm{Diff}(M)$ which satisfies the equality

$$\lambda(0) = \mathrm{id}_M,$$

and the relation

$$\lambda(\tau_1) \circ \lambda(\tau_2) = \lambda(\tau_1 + \tau_2).$$

The mapping λ is an element of $C^\infty(\mathbb{R}, C^\infty(M, M))$. Denote the corresponding element of $C^\infty(\mathbb{R} \times M, M)$ by $\mathit{\Lambda}$. The mapping Λ satisfies the equality

$$\mathit{\Lambda}(0, p) = p$$

and the relation

$$\mathit{\Lambda}(\tau_1, \mathit{\Lambda}(\tau_2, p)) = \mathit{\Lambda}(\tau_1 + \tau_2, p).$$

Hence, the mapping $\mathit{\Lambda}$ is a flow on M. Here the tangent vector to the curve λ at id_M is the vector field generating the flow $\mathit{\Lambda}$.

Since M is a compact manifold, then for each vector field X there is a flow $\mathit{\Lambda}^X$ generated by X. This flow induces the one-parameter subgroup λ^X of $\mathrm{Diff}(M)$. It is clear that

$$\lambda^X(\tau) = \exp(\tau X).$$

Therefore, in such a way we realize the exponential mapping for $\mathrm{Diff}(M)$.

B Distributions on S^1 and generalisations

A continuous linear functional on the Fréchet space $C^\infty(S^1) = C^\infty(S^1, \mathbb{C})$ is said to be a *distribution* on S^1. For a general presenation of the theory of distributions we refer to the book by Rudin [10].

The *support* of a function $\varphi \in C^\infty(S^1)$ is defined as the closure of the set where φ does not vanish and denoted as $\operatorname{supp} \varphi$. We say that a distribution T vanishes on an open set U if $T(\varphi) = 0$ whenever $\operatorname{supp} \varphi \subset U$. Then the support of T is defined as the complement of the union of all open sets where T vanishes. It is clear that the support of a distribution on S^1 is a closed set.

If the support of a distribution T coincides with a one-point set $\{p\}$, then

$$T(\varphi) = \sum_{m=0}^{n} c_m \varphi^{(m)}(p).$$

for some nonnegative integer n and constants c_m.

Let now T be a continuous linear mapping from $\mathcal{L}(\mathfrak{g})$ to $\mathfrak{g}$. Given a basis (e_i) of $\mathfrak{g}$, denote by (μ^i) the dual basis of $\mathfrak{g}^*$. For any element x of $\mathfrak{g}$ one has

$$x = \sum_i e_i \mu^i(x).$$

Using this equality, one can write

$$T(\xi) = \sum_i e_i \mu^i(T(\xi)).$$

Representing a general element ξ of $\mathcal{L}(\mathfrak{g})$ as $\sum_j e_j \xi^j$, one obtains

$$\mu^i(T(\xi)) = \sum_j \mu^i(T(e_j \xi^j)).$$

Introduce a matrix of distributions $(T^i{}_j)$ on S^1 defined by the relation

$$T^i{}_j(\varphi) = \mu^i(T(e_j \varphi)).$$

Here φ is a smooth function on S^1. Now one can write

$$T(\xi) = \sum_{i,j} e_i T^i{}_j(\xi^j).$$

Thus, the matrix $(T^i{}_j)$ completely determines the mapping T.

The support $\operatorname{supp} \xi$ of the element ξ of $\mathcal{L}(\mathfrak{g})$ is defined as the closure of the set where ξ does not take zero value. Representing ξ as $\sum_i e_i \xi^i$ one concludes that $\operatorname{supp} \xi = \bigcup_i \operatorname{supp} \xi^i$. We say that a continuous smooth mapping T from $\mathcal{L}(\mathfrak{g})$ to $\mathfrak{g}$ vanishes on an open set U if $T(\xi) = 0$ whenever $\operatorname{supp} \xi \subset U$. The support of T is defined as the complement of the union of all open sets where T vanishes. It is clear that $\operatorname{supp} T = \bigcup_{i,j} \operatorname{supp} T^i{}_j$, where $(T^i{}_j)$ is the matrix of distributions on S^1 which determines the mapping T for given dual bases (e_i) and (μ^i) of $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. If the support of T is a one-point set $\{p\}$, then one can easily demonstrate that

$$T(\xi) = \sum_{m=0}^{n} c_m(\xi^{(m)}(p)).$$

for some nonnegative integer n and endomorphisms c_m of $\mathfrak{g}$.

Consider now continuous linear mappings from $\mathcal{L}_a(\mathfrak{g})$ to $\mathfrak{g}$ for the case when $\mathfrak{g}$ is a semisimple Lie algebra and a is an automorphism of $\mathfrak{g}$ satisfying the relation $a^K = \mathrm{id}_{\mathfrak{g}}$ for some positive integer K. In this case $\mathcal{L}_a(\mathfrak{g})$ can be considered as a subalgebra of $\mathcal{L}(\mathfrak{g})$ formed by the elements ξ satisfying the condition

$$\xi(\varepsilon_K p) = a(\xi(p)).$$

We denote this subalgebra as $\mathcal{L}_{a,K}(\mathfrak{g})$. Define a linear operator A in $\mathcal{L}(\mathfrak{g})$ acting on an element ξ in accordance with the relation

$$A\xi(p) = a(\xi(\varepsilon_K^{-1} p)).$$

An element $\xi \in \mathcal{L}(\mathfrak{g})$ belongs to $\mathcal{L}_{a,K}(\mathfrak{g})$ if $A\xi = \xi$.

For an arbitrary element $\xi \in \mathcal{L}(\mathfrak{g})$ the element $\bar{\xi}$ defined as

$$\bar{\xi} = \frac{1}{K} \sum_{m=0}^{K-1} A^{-m} \xi$$

belongs to $\mathcal{L}_{a,K}(\mathfrak{g})$, and one can extend a continuous linear mapping T from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathfrak{g}$ to a continuous linear mapping $\overline{T}$ from $\mathcal{L}(\mathfrak{g})$ to $\mathfrak{g}$ assuming that

$$\overline{T}(\xi) = T(\bar{\xi}).$$

One can easily show that

$$\overline{T} \circ A = \overline{T},$$

and that

$$\operatorname{supp} \overline{T} = \operatorname{supp} T.$$

It is clear that if the support of an element $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$ contains a point $p \in S^1$, then it contains also the point $\varepsilon_K p$. Therefore, the support of an element of $\mathcal{L}_{a,K}(\mathfrak{g})$ is the union of sets of the form

$$\overline{\{p\}} = \{p, \varepsilon_K p, \ldots, \varepsilon_K^{K-1} p\}, \qquad p \in S^1.$$

The same is true for the support of an arbitrary continuous linear mapping from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathfrak{g}$.

Let T be a continuous linear mapping from $\mathcal{L}_{a,K}(\mathfrak{g})$ to $\mathfrak{g}$ whose support is $\overline{\{p\}}$. The corresponding mapping $\overline{T}$ has the same support and is invariant with respect to the action of the automorphism A. Using these facts one can obtain that

$$\overline{T}(\xi) = \frac{1}{K} \sum_{l=0}^{K-1} \sum_{m=0}^{n} c_m(a^{-l}(\xi^{(m)}(\varepsilon_K^l p)))$$

for some nonnegative integer n and endomorphisms c_m of $\mathfrak{g}$. Restricting the mapping $\overline{T}$ again to $\mathcal{L}_{a,K}(\mathfrak{g})$ one has

$$T(\xi) = \sum_{m=0}^{n} c_m(\xi^{(m)}(p)),$$

for any $\xi \in \mathcal{L}_{a,K}(\mathfrak{g})$.

C Convergence and series in Fréchet spaces

A set $\mathcal{D}$ is said to be *directed* if it is supplied with a binary relation $\succcurlyeq$ satisfying the following properties:

(a) for any element $\alpha \in \mathcal{D}$ one has $\alpha \succcurlyeq \alpha$;

(b) if $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \gamma$, then $\alpha \succcurlyeq \gamma$;

(c) for any two elements $\alpha, \beta \in \mathcal{D}$, there exists an element $\gamma \in \mathcal{D}$ such that $\gamma \succcurlyeq \alpha$ and $\gamma \succcurlyeq \beta$.

The relation $\succcurlyeq$ is called a *direction* in $\mathcal{D}$. Below we use the notation $\mathcal{D}$ for a general directed set. Given a countable set S, we denote by $\mathcal{D}(S)$ the set of all finite subspaces of S, considered as a directed set, where $\alpha \succcurlyeq \beta$ if and only if $\alpha \supset \beta$.

A mapping from a directed set $\mathcal{D}$ to a topological space X is called a *net* in X. A net $(x_\alpha)_{\alpha \in \mathcal{D}}$ in a topological space X is said to *converge* to an element $x \in X$, or has *limit* x, if for any neighbourhood U of x there is an element $A \in \mathcal{D}$ such that $x_\alpha \in U$ for all $\alpha \succcurlyeq A$. Here one also says that the net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ is *convergent*. If $\mathcal{D} = \mathbb{N}$ and $\succcurlyeq$ is the ordinary order relation $\geqslant$, nets are sequences with the usual definition of convergence.

Let X and Y be topological spaces, and f be a mapping from X to Y. The mapping f is continuous if and only if for any net $(x_\alpha)_{\alpha \in \mathcal{D}}$ which converges to $x \in X$ the net $(f(x_\alpha))_{\alpha \in \mathcal{D}}$ converges to $f(x) \in Y$.

Let X be a topological vector space, I be some countable set, and $(x_i)_{i \in I}$ be a collection of elements of X indexed by I. The symbol $\sum_{i \in I} x_i$ is called a *series* in X. Consider a net $(s_\alpha)_{\alpha \in \mathcal{D}(I)}$, where

$$s_\alpha = \sum_{i \in \alpha} x_i.$$

If the net (s_α) converges to an element $s \in X$ we say that the series $\sum_{i \in I} x_i$ converges *unconditionally* to s and write

$$s = \sum_{i \in I} x_i.$$

Here the element s is called the *sum* of the series $\sum_{i \in I} x_i$.

The next proposition is a direct generalisation of the corresponding proposition for series in normed spaces (see, for example, [16]).

Proposition C.1 *Let X be a Fréchet space whose topology is induced by a countable collection of seminorms $(\| \cdot \|_m)$, and $\sum_{i \in I} x_i$ be a series in X. If for each m the series $\sum_{i \in I} \|x_i\|_m$ converges unconditionally, then the series $\sum_{i \in I} x_i$ also converges unconditionally.*

Let X be a topological vector space whose topology is induced by a countable family of seminorms $(\| \cdot \|)_m$. If a series $\sum_{i \in I} x_i$ in X converges unconditionally and for each m the series $\sum_{i \in I} \|x_i\|_m$ also converges unconditionally, one says that the series $\sum_{i \in I} x_i$ converges *absolutely*. The above proposition says that for complete X unconditional convergence of the series $\sum_{i \in I} \|x_i\|_m$ leads to unconditional convergence of the series $\sum_{i \in I} x_i$.

For a series whose terms are positive real numbers, unconditional convergence is equivalent to absolute convergence. Therefore, in this case it is customary to say simply about convergence. As for the case of a general series, one sees that absolute convergence, by definition, implies unconditional convergence, but in accordance with the Dvoretzky–Rogers theorem [17], for infinite dimensional topological vector spaces there are series which converge unconditionally, but do not converge absolutely.

The following proposition can be proved along the lines of the proof of the corresponding proposition for series in normed spaces (see, for example, [16]).

Proposition C.2 *Let a series $\sum_{i \in I} x_i$ in a Fréchet space converge absolutely. Assume that the set I is represented as the union of a countable number of nonempty nonintersecting sets I_j, $j \in J$. For any $j \in J$ the series $\sum_{i \in I_j} x_i$ converges absolutely and the series $\sum_{j \in J} y_j$, where*

$$y_j = \sum_{i \in I_j} x_i,$$

converges absolutely. Moreover, one has

$$\sum_{i \in I} x_i = \sum_{j \in J} \left(\sum_{i \in I_j} x_i \right).$$

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