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Uniqueness of Diffeomorphism Invariant States on Holonomy–Flux Algebras

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Abstract: Loop quantum gravity is an approach to quantum gravity that starts from the Hamiltonian formulation in terms of a connection and its canonical conjugate. Quantization proceeds in the spirit of Dirac: First one defines an algebra of basic kinematical observables and represents it through operators on a suitable Hilbert space. In a second step, one implements the constraints. The main result of the paper concerns the representation theory of the kinematical algebra: We show that there is only one cyclic representation invariant under spatial diffeomorphisms.

While this result is particularly important for loop quantum gravity, we are rather general: The precise definition of the abstract ∗-algebra of the basic kinematical observables we give could be used for any theory in which the configuration variable is a connection with a compact structure group. The variables are constructed from the holonomy map and from the fluxes of the momentum conjugate to the connection. The uniqueness result is relevant for any such theory invariant under spatial diffeomorphisms or being a part of a diffeomorphism invariant theory.

1. Introduction

In the Hamiltonian analysis of theories of gauge potentials, the configuration space usually is the space $\mathcal{A}$ of connections defined on a principal fiber bundle $\Pi : P \rightarrow \Sigma$ of a compact structure group G. The cotangent bundle $T^*\mathcal{A}$ (appropriately defined) with the natural symplectic structure becomes the phase space.

In addition to the Hamiltonian equations of motion, the theory will exhibit constraint equations. The constraints play a double role in a Hamiltonian theory. On the one hand they generate a group of symmetries of the phase space referred to as the gauge transformations, on the other hand the set of solutions of the constraints defines the constraint surface of the phase space.

The simplest example for such a kind of theory is certainly Maxwell theory, where the structure group is $U(1)$. A more general example is Yang–Mills theory, where the

structure group may be an arbitrary compact Lie group G. In this case the group of the gauge transformations is the group of the fiber preserving automorphisms of the given bundle, homotopic to the identity. The group is often referred to as the "Yang Mills gauge transformations".

Another example, in fact the one which has triggered the present investigations, is gravity, formulated in terms of real Ashtekar variables [1, 4–7]. In the $3 + 1$ case, the structure group is $SU(2)$, the bundle is trivial and defined over a 3-manifold. The group of the gauge transformations generated by the constraints contains *all* the bundle automorphisms homotopic to the identity map. In terms of a local section, the group becomes the semi-direct product of the Yang–Mills gauge transformations and the diffeomorphisms of Σ homotopic to the identity map. This Hamiltonian formulation is the starting point of the loop quantum gravity (LQG, for brevity) program.

To quantize such a theory à la Dirac, one first seeks appropriate basic variables. These are preferred functions separating the points of the phase space which are then quantized. This part of the procedure is called *kinematical* hereafter. The constraints are then imposed as operator equations on the kinematical Hilbert space or in an appropriately selected dual.

In the present paper we are concerned with two issues arising in the kinematical quantization framework of LQG and of every theory of connections whose phase space is $T^*\mathcal{A}$. The first issue concerns the choice of basic classical variables and a definition of a corresponding (abstract) quantum $*$-algebra. We slightly generalize and improve details of the ideas developed in LQG and define a quantum $*$-algebra $\mathfrak{A}$ of basic quantum variables. Our definitions are valid for arbitrary dimension $D \geq 2$ of the base manifold Σ, arbitrary compact structure group G, and arbitrary bundle P.

The second issue arises when we look for representations: If $\mathfrak{A}$ admits more than one representation, which one are we going to choose to carry out the Dirac quantization program? Our result here will hold in a more specific setting than our definition of the algebra $\mathfrak{A}$: We will show that in the case of *diffeomorphism invariant theories*,[1] upon restricting to diffeomorphism invariant representations, this issue will not arise: we find a unique cyclic representation

In the following, let us explain the two results of the paper a bit more in detail and relate them to what has already been achieved elsewhere.

The classical algebra. For the sake of informal presentation, let us choose a (local) trivialization of the bundle P and use the notation of field theory. (The main part of the paper will be kept in the geometric and algebraic style.) Then the phase space consists of pairs (A, E) of fields defined on Σ, where: (i) A is a differential 1-form taking values in the Lie algebra $\mathfrak{g}$ of the gauge group G and (ii) E is a vector density of weight 1 taking values in $\mathfrak{g}^*$, the dual vector space to $\mathfrak{g}$. The non-vanishing Poisson relations between the fields evaluated at points can be written as

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta(x, y), \qquad (1)$$

where in a local coordinate system $(x^1, \ldots, x^D)$ in Σ and in a basis $\{\tau_1, \ldots, \tau_d\}$ of $\mathfrak{g}$ the fields are decomposed into $A = A_a^i dx^a \otimes \tau_i$ and $E = E_i^a \partial_a \otimes \tau^i$ ($\tau^1, \ldots, \tau^d \in \mathfrak{g}^*$ denote the dual basis). The first question to ask is which functionals of A and E should be quantized. A very natural answer is obtained by considering the geometric nature of the fields A and E: A is a 1-form on Σ and therefore integrals of A along 1-dimensional

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[1] In the non-trivial bundle case, we mean invariance with respect to a group of automorphisms of P which induces, by the bundle projection Π, all the diffeomorphisms of Σ homotopic with the identity map.

submanifolds are well defined. E on the other hand, as a vector density of weight one can be turned into a pseudo $(D-1)$-form $\tilde{E}$ (still $\mathfrak{g}^*$ valued) using the totally antisymmetric symbol, namely $\tilde{E} = \frac{1}{(D-1)!} E_i^a \epsilon_{aa_1 \dots a_{D-1}} dx^{a_1} \wedge \dots \wedge dx^{a_{D-1}} \otimes \tau^i$. Hence it can be integrated over $(D-1)$-dimensional hyper-surfaces of Σ. Asking in addition for simple transformation behavior of the functionals of A upon a change of trivialization, that is with respect to

$$A \mapsto g^{-1}Ag + g^{-1}dg, \quad E \mapsto g^{-1}Eg,$$

where g is an arbitrary (locally defined) G valued function in Σ, one is led to consider functionals depending on A via the Wilson loop functionals

$$h_\alpha[A] = \mathcal{P}\left(\exp - \int_\alpha A\right),$$

where α is a path in Σ. A similar requirement applied to the canonical conjugate field E leads to the flux-like variables

$$E_{S,f} = \int_S \tilde{E}_i f^i, \tag{2}$$

where S is a $(D-1)$-dimensional surface and $f : S \to \mathfrak{g}$ is a function of compact support on S. Starting from the bracket (1) these variables can be endowed with a Lie algebra structure with a remarkable geometric flavor which was systematically explored in [9, 10]: The functions $\Psi : \mathcal{A} \to \mathbb{C}$ depending on A via the Wilson loop functionals only form the algebra of cylindrical functions and every flux variable $E_{S,f}$ acts as a derivation $X_{S,f}$ on this algebra, defined by the Poisson bracket[2]

$$X_{S,f}\Psi := \{\Psi, E_{S,f}\}. \tag{3}$$

This is also the approach we will use in the present paper. The product (3) is well defined provided that the intersection between the path α with the surface S contains finitely many isolated points. A simple condition that ensures this property uses a real analytic structure on Σ, analytic paths and analytic surfaces. Correspondingly, analytic diffeomorphisms of Σ are among the natural symmetries inherited from Σ that define automorphisms of the algebra of the basic variables. The analyticity requirement, however, breaks the local character of the non-analytic diffeomorphisms group. Therefore, the most important difference from the treatment of the previous papers on the subject [13–16, 20] is that we will employ here a considerably larger group of symmetries. We will not require that the diffeomorphisms we consider be analytic everywhere but, roughly speaking, analytic only up to submanifolds of lower dimension. Some care has to be taken in the precise definition of this notion, mainly to insure that they form a group and that application of these diffeomorphisms produce surfaces and edges that still have finitely many isolated intersection points. The important point is that this larger symmetry group now contains local diffeomorphisms, and this will be instrumental for proving the uniqueness result.[3] A more technical difference as compared to the LQG literature

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[2] The Poisson bracket in (3) preserves the space of cylindrical functions and the Dirac delta is absorbed completely by the integrations involved in the definitions of the holonomy and flux. This fact was pointed out for the first time in [1, 8]. The specific flux derivation used in this paper was defined in [9].

[3] A more radical enlargement of the symmetry group of the algebra has been advocated for a long time by Zapata (see ex. [17]). Recently, a similar enlargement has been implemented in [11, 29]. See also [18] for a discussion of these questions.

is that, following [20], we will be working with arbitrary space-time dimensions and not assume a trivialization of the G-bundle.

The quantum algebra. The next step in the quantization program is to define the quantum algebra $\mathfrak{A}$. Stated in a heuristic way, we want to define an abstract $*$-algebra of quantum objects $\hat{h}_\alpha$, $\hat{X}_{S,I}$ whose relations reflect (i) the multiplicative structure of the functions of the Wilson loop functionals and the derivations, and (ii) the complex conjugation structure of the functions of the Wilson loop functionals and the flux functionals. Such an algebra has been defined in [13–16] on various levels of rigor and abstraction. Here, we will reach an equivalent, precise definition by using intuition from geometric quantization.

Representations, uniqueness. After one has defined the quantum algebra $\mathfrak{A}$, according to the Dirac quantization program $\mathfrak{A}$ has to be represented on a Hilbert space, the constraints have to be implemented as operators, and solutions to the constraints have to be found. Generically, $\mathfrak{A}$ will admit an infinite number of inequivalent representations, so it is an important question which one of them is the right one to use. Ultimately, this question can only be answered by exhibiting one or more representations in which the program can be followed through to the end, leading to a bona fide quantization of the theory.

However, there are clearly more and less natural choices of representations to try first: Most importantly, if the classical theory has symmetries that act on $\mathfrak{A}$ by a group of automorphisms then it is natural to try to find a representation in which these automorphisms are unitarily implemented. A second natural idea is to first look at irreducible or at least cyclic representations as the simple building blocks, out of which more complicated representations could eventually be built. Finally, if $\mathfrak{A}$ is not a Banach-algebra, one has to worry about domain questions and it is somewhat natural to consider representations first that have simple properties in this respect.

A simple formulation of these properties can be given by asking for a state (i.e. a positive, normalized, linear functional) on $\mathfrak{A}$ that it is invariant under the classical symmetry automorphisms of $\mathfrak{A}$. Given a state on $\mathfrak{A}$ one can define a representation via the GNS construction. This representation will be cyclic by construction. Furthermore, by construction it has a common invariant dense domain for all the operators representing elements of $\mathfrak{A}$. Finally, if the state is invariant under some automorphism of $\mathfrak{A}$, its action is automatically unitarily implemented in the representation.

In this article, we will investigate the class of representations of $\mathfrak{A}$ delineated above in a special case, namely if the theory under consideration is invariant under diffeomorphisms of the manifold Σ in the trivial bundle case, and automorphisms of the bundle in the general case. Most prominently, this is the case for gravity, written in terms of connection variables, as used in loop quantum gravity. It follows from [9, 19], that for the case of interest for loop quantum gravity, $D = 3$ and $G = \mathrm{SU}(2)$, a state with these properties exists. The corresponding representation has subsequently served as a cornerstone in the LQG program. Moreover, this representation can be immediately generalized to arbitrary dimension and arbitrary compact gauge group. Therefore the requirements above do not reduce considerations to the empty set. However, it is an important question for the LQG program, and at least an interesting mathematical question in general, whether there exist other representations with these properties. Our analysis will show that this is not the case: the only state that is invariant under the group of diffeomorphisms described above is the one used in LQG. This is a more satisfying result than the ones obtained in [13–16, 20]. However it relies heavily on the enlargement of the symmetry group of a state not used in earlier publications.

While work on this manuscript was in progress, similar results as the ones that we will present here have been obtained in [29]. The technical setup of [29] differs somewhat from the one used here, and we refer to Sect. 5 for a comparison.
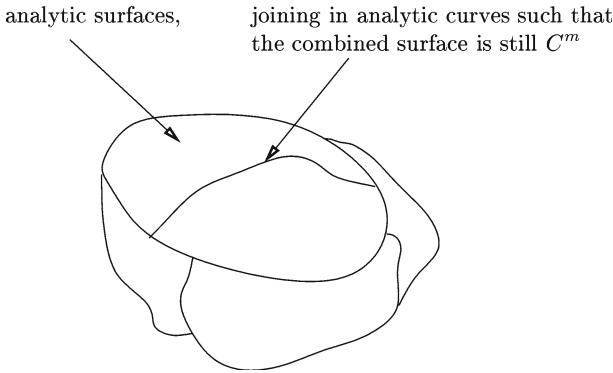
2. The Holonomy–Flux ∗-Algebra

The goal of this section is a definition of the ∗-algebra $\mathfrak{A}$ of basic, quantum observables. We have already mentioned the algebra in the introduction and explained its meaning; in this section we will give a complete definition. As indicated, in our approach we will base all definitions on a new category of manifolds that is larger than the analytic category but smaller than the m-times differentiable one. The technical definitions and proofs in this respect are relegated to the appendix. Let us start here by giving a more intuitive description, justification for this enlargement and outline of the properties relevant in our paper.

2.1. Semianalytic structures. In this work we consider a D-dimensional differential manifold Σ. The differentiability class C^m is fixed, $m \geq 1$. Our elementary variables – already mentioned in the introduction and carefully defined in the following sections – are constructed by using curves (later called edges) and co-dimension one submanifolds (faces) of Σ. A necessary condition for the Poisson bracket between the variables to be finite is that every edge intersects every face in an at most finite number of isolated intersection points plus a finite number of connected segments (i.e. edges in themselves). To ensure this condition we need to carefully define a class of curves and submanifolds we consider. It will be also important that the class be preserved by a sufficiently large subgroup of the diffeomorphisms of Σ. 'Large' means that the subgroup contains sufficiently many diffeomorphisms that act non-trivially only within compact regions. This is not the case, for example, for the analytic diffeomorphism group that has usually been considered in this context. We solve this technical issue by defining an appropriate category of manifolds we will call semianalytic. Next, we assume that the manifold Σ is equipped with a semianalytic structure. Henceforth, all the local maps, diffeomorphisms, submanifolds and functions thereon, are assumed to be C^m and semianalytic. Throughout the paper, submanifolds are assumed to be embedded submanifolds. A semianalytic structure is weaker than an analytic one, therefore it can be determined on Σ for example by choosing an arbitrary analytic structure.

Briefly, 'semianalytic' means 'piecewise analytic'. For example, a semianalytic submanifold would be analytic except for on some lower dimensional sub-manifolds, which in turn have to be piecewise analytic. To convey the idea, Fig. 1 depicts a semianalytic surface in $\mathbb{R}^3$. However, whereas in the case of $\Sigma = \mathbb{R}$ 'piecewise analytic' has a well established interpretation, in a higher dimensional case those words admit a huge ambiguity. Therefore in the appendix we introduce exact definitions and prove relevant properties. We heavily rely on the theory of the semianalytic sets developed by Łojasiewicz [2, 3].

The special property of the semianalytic category so relevant for us in this paper, is that the intersection between every two connected submanifolds, locally, is a finite union of connected submanifolds. Of course, this is also true in the analytic case. But the difference between the analytic and the semianalytic structures is in the local character of the later ones. Technically, the locality is expressed by the fact that every open covering of Σ admits a compatible semianalytic partition of unity.

analytic surfaces,   joining in analytic curves such that
                                the combined surface is still C^m

Fig. 1. A semianalytic surface

2.2. The cylindrical functions. Let us recall from the appendix that by *semianalytic edge* we mean a connected, 1-dimensional semianalytic submanifold of Σ with 2-point boundary.

Definition 2.1. *An edge is an oriented embedded 1-dimensional C^0 submanifold of Σ with a 2-point boundary, given by a finite union of semianalytic edges.*

Over the manifold Σ we fix a principal fiber bundle

$$\Pi : P \to \Sigma. \tag{4}$$

The structure group of P is denoted by G, and it is assumed to be compact and connected. The right action of G on P will be denoted in the usual way as $G \times P \ni (g, p) \mapsto R_g p \in P$. We are assuming the bundle is semianalytic. On P we consider the space of the connections $\mathcal{A}$.

Given an edge e, a connection $A \in \mathcal{A}$ defines a bundle isomorphism

$$A(e) : \Pi^{-1}(x) \to \Pi^{-1}(y), \tag{5}$$

where x and y are the beginning and end points of e, and the fibers of P are considered as pullbacks of the bundle P. The space $\mathcal{A}_e$ of all the bundle isomorphisms $\Pi^{-1}(x) \to \Pi^{-1}(y)$ (in fact, $\mathcal{A}_e$ depends on the points x and y only) can be mapped in a 1-1 way into G,

$$\sigma : \mathcal{A}_e \to G, \tag{6}$$

and the map, called a gauge map, is defined by a choice of two points, $p_x \in \Pi^{-1}(x)$ and $p_y \in \Pi^{-1}(y)$, and by

$$A_e(p_x) = R_{\sigma(A_e)} p_y. \tag{7}$$

Therefore, it is determined up to the left and right multiplication in G by arbitrary elements $g, h \in G$, corresponding to changing the points p_x and p_y. In this way $\mathcal{A}_e$ inherits every structure of G which is left and right invariant (including the topology, the differential manifold structure, the Haar measure).

Definition 2.2. *A function $\Psi : \mathcal{A} \to \mathbb{C}$ is called cylindrical if there exists a finite set $\gamma = \{e_1, \ldots, e_n\}$ of edges and a function $\psi \in C^\infty(\mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n})$ such that for every $A \in \mathcal{A}$,*

$$\Psi(A) = \psi(A(e_1), \ldots, A(e_n)); \tag{8}$$

in this case, we say that Ψ is compatible with γ and ψ.

Every cylindrical function is compatible with many sets of edges. Without lack of generality, we may assume that γ is an embedded graph, that is, if two edges $e_I \neq e_J$ intersect, then the intersection is contained in the boundary of each of them [22].[4] The boundary points of edges constituting a graph γ are called the vertices of γ. It is easy to see that all the cylindrical functions set up a subalgebra of the algebra of all the complex valued functions defined on $\mathcal{A}$; we denote it by Cyl. In a natural way it admits definition of an involution and a norm

$$\Psi^\star := \bar{\Psi}, \quad \|\Psi\| := \sup_{A \in \mathcal{A}} |\Psi(A)|. \tag{9}$$

2.3. The Ashtekar–Isham quantum configuration space $\overline{\mathcal{A}}$. The space of connections is considered here a configuration space. However, promoting the cylindrical functions to the basic position variables on $\mathcal{A}$ (an over-complete set of variables) is equivalent to embedding $\mathcal{A}$ into the Gel'fand spectrum of the unital C*-algebra $\overline{\text{Cyl}}$ defined as the completion of (Cyl, $\| \cdot \|$, $\ast$). Elements of the Gel'fand spectrum of $\overline{\text{Cyl}}$ have a geometric interpretation of generalized (or distributional) connections on $\mathcal{A}$. We recall now the definition of the generalized connections (see [18] for a recent review, and [19, 21–25] for the origins).

Consider the space $\mathcal{E}$ of all the edges in Σ including the trivial one. Certain pairs $(e, e') \in \mathcal{E} \times \mathcal{E}$ can be composed, yielding a new edge. More precisely, let the beginning point of e be the end point of e', then we define

$$e \circ e' := \overline{e \cup e' \setminus (e \cap e')}, \tag{10}$$

provided the result is again an edge, where the line stands for the completion, and the beginning (end) point of $e \circ e'$ is defined to be the beginning (end) point of e (e'). If e' differs from e in orientation only, then $e \circ e'$ is trivial, hence we will also use the notation e^{-1} for the edge e with orientation reversed.

On the other hand, from the principal fiber bundle P, for pairs of points $x, y \in \Sigma$, one has the bundle isomorphisms $\Pi^{-1}(x) \to \Pi^{-1}(y)$, and those from $\Pi^{-1}(x)$ to $\Pi^{-1}(y)$ can be composed with those from $\Pi^{-1}(y)$ to $\Pi^{-1}(z)$, yielding isomorphisms again.

Definition 2.3. *A generalized connection $\bar{A}$ on P assigns to every edge e a bundle isomorphism*

$$\bar{A}(e) : \Pi^{-1}(e_s) \to \Pi^{-1}(e_t),$$

where e_s is the beginning (source) of the edge e, and e_t is its end (target), such that

$$\bar{A}(e \circ e') = \bar{A}(e) \circ \bar{A}(e'), \quad \text{and} \quad \bar{A}(e^{-1}) = \left(\bar{A}(e)\right)^{-1} \tag{11}$$

whenever $e \circ e'$ is defined. We denote the space of generalized connections by $\overline{\mathcal{A}}$.

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[4] In [22] the analyticity was assumed. However, owing to Proposition A.14 the semianalyticity assumption used in the definition of the edges is sufficient.

Every cylindrical function Ψ is naturally extendable to $\overline{\mathcal{A}}$, by using a compatible graph γ and function ψ (see (8)), namely

$$\Psi(\bar{A}) := \psi(\bar{A}(e_1), \ldots, \bar{A}(e_n)), \tag{12}$$

(the result defines a unique function on $\overline{\mathcal{A}}$, independent of choice of the compatible γ and ψ). Given any $\bar{A} \in \overline{\mathcal{A}}$, the map

$$\overline{\mathrm{Cyl}} \ni \Psi \mapsto \Psi(\bar{A}) \in \mathbb{C}, \tag{13}$$

is continuous in $(\overline{\mathrm{Cyl}}, \|\cdot\|)$ and defines a C*-algebra homomorphism, that is an element of the Gel'fand spectrum. Moreover, every element of the Gel'fand spectrum can be represented by a generalized connection in that way [19, 22]. In this way we identify the spectrum with $\overline{\mathcal{A}}$.

Definition 2.4. *The Ashtekar–Isham quantum configuration space for the loop quantization of the theory of connections defined on P is the space $\overline{\mathcal{A}}$ of the generalized connections.*

2.4. Generalized vector fields tangent to $\overline{\mathcal{A}}$. Given a finite dimensional manifold as a configuration space, and the cotangent bundle as the phase space, the momenta correspond to the tangent vector fields[5] and there is available an elegant geometric quantization scheme. This idea is easily generalized to an infinite dimensional $\mathcal{A}$, but for the quantization, one would need a measure on $\mathcal{A}$ in our case required to be invariant with respect to the automorphisms of the bundle P. Instead, Ashtekar and Isham defined $\overline{\mathcal{A}}$ and proposed to embed $\mathcal{A}$ in $\overline{\mathcal{A}}$ because the latter has naturally defined compact topology and is therefor easier to treat. However, $\overline{\mathcal{A}}$ does not have a manifold structure. Nonetheless, the fluxes of the electric field and the corresponding derivations (3) defined in Cyl do lead to a quite precise definition of a generalized vector field tangent to $\overline{\mathcal{A}}$. We introduce it in this subsection in a geometric, manifestly trivialization invariant way.

We define now on $\overline{\mathcal{A}}$ generalized vector fields which correspond to the derivations (3), that is to the smeared fluxes of the frame field E. The generalized vector fields are labeled by *faces*, and appropriate *smearing functions*. A face S is introduced in the appendix (see Definition A.16) as a co-dimension 1 submanifold of Σ, oriented in the sense that the normal bundle of S is equipped with an orientation. Now, we will carefully define the smearing functions. Our emphasis is on the geometric, gauge invariant characteristics, and on careful specification of the class of fields we are going to use. Let S be a face. Consider the bundle

$$P_S := \Pi^{-1}(S) \subset P \tag{14}$$

equipped with the principal fiber bundle structure induced by the bundle P.

Definition 2.5. *Given a face S, a smearing vector field is a compactly supported semi-analytic vector field defined on the bundle P_S, tangent to the fibers of the bundle and invariant under the action of the structure group G in P.*

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[5] Every vector field on a manifold defines naturally a function on the cotangent bundle. If the manifold is a configuration space, and the cotangent bundle with the natural Poisson bracket is the phase space, then the function is linear in 'momenta'.

Let f be a smearing vector field on P_S. Denote by $\exp(\cdot\, f)$ the corresponding flow. The map

$$\exp(tf) : P_S \to P_S \tag{15}$$

assigned by the flow to each $t \in \mathbb{R}$ preserves the fibers of P_S (i.e. $\Pi \circ \exp(tf) = \Pi$) and commutes with the right action of the structure group (i.e. $\exp(tf)R_g = R_g\exp(tf)$, for every $g \in G$). It is easy to show that the flow is semianalytic: in a local trivialization of P_S, the vector field f corresponds to an element of the Lie algebra of G and the flow can be expressed by the usual exponential map. Restricted to each fiber $\Pi^{-1}(x) \subset P_S$, the flow becomes a fiber automorphism $\exp(tf)_x$,

$$\exp(tf)_x := \exp(tf)|_{\Pi^{-1}(x)}. \tag{16}$$

We use the latter one to define below a 1-dimensional group formed by maps $\theta^{(t)}$: $\overline{\mathcal{A}} \to \overline{\mathcal{A}}$ which, briefly speaking, give every generalized connection $\bar{A}$ a 'translation' $\exp(\pm tf)$ supported on those edges which intersect the face S transversally (in the topological sense) where the sign depends on the orientation of the edge with respect to the orientation of S. To define it, note that S admits an open neighborhood $\mathcal{U} \subset \Sigma$ such that

$$\mathcal{U} \setminus S = \mathcal{U}^- \cup \mathcal{U}^+,$$

where $\mathcal{U}^-$ and $\mathcal{U}^+$ are disjoint, each of them is open in Σ, connected and non-empty. The labels '+' and '−' correspond to the orientation of S. An action of the generalized flow $\theta^{(t)}$ on a generalized connection $\bar{A} \in \overline{\mathcal{A}}$, can be defined by using only a subclass of edges taken into account in what follows:

$$\theta^{(t)}(\bar{A})(e) := \begin{cases} \bar{A}(e)\exp(\frac{1}{2}tf)_x & \text{if } e \cap S = \{x\} \text{ and } e \setminus x \subset \mathcal{U}^+ \\ \bar{A}(e)\exp(-\frac{1}{2}tf)_x & \text{if } e \cap S = \{x\} \text{ and } e \setminus x \subset \mathcal{U}^-, \\ \bar{A}(e) & \text{if } e \cap S = \emptyset \text{ or } e \cap \bar{S} = e \end{cases} \tag{17}$$

where x stands for the beginning point of e. Every edge e' can be written as a composition of edges of the type given on the right-hand side of Eq. (17) and their inverses, therefore $\theta^{(t)}(\bar{A})(e')$ is determined by (17) and the requirement that $\theta^{(t)}$ maps generalized connections to generalized connections. Given an orientation of S, the resulting flow is independent of the choice of the neighborhoods $\mathcal{U}, \mathcal{U}^-, \mathcal{U}^+$.

Importantly, the pullback $\theta^{(t)*}$ preserves Cyl and for every cylindrical function Ψ, the derivative

$$X_{S,f}\Psi := \frac{d}{dt}\Psi(\theta^{(t)}(\bar{A}))|_{t=0} \tag{18}$$

is a well defined element of Cyl. This definition is equivalent to (3) and it is its manifestly trivialization independent version. An important observation is that the action of the $X_{S,f}$ on the cylindrical functions is linear in the vector field f, i.e. $X_{S,f_1+f_2} = X_{S,f_1} + X_{S,f_2}$. The explicit formula for the action of the operator can be found for example in [6].

Definition 2.6. *The operator* $X_{S,f}$: Cyl $\to$ Cyl *defined in (18), where S is a face and f is a smearing vector field (see Definition 2.5) will be called the flux vector field corresponding to (S, f).*

The space of the linear combinations of the operators Cyl $\to$ Cyl of the form

$$\Psi \cdot X_{S_1, f_1}, \quad \Psi \cdot [X_{S_1, f_1}, X_{S_2, f_2}], \quad \Psi \cdot [\dots [X_{S_1, f_1}, X_{S_2, f_2}], \dots, X_{S_k, f_k}], \quad (19)$$

where $\Psi \in$ Cyl and $X_{S_1, f_1}, X_{S_2, f_2}, \dots$ are the flux vector fields will be called the space of generalized vector fields tangent to $\overline{\mathcal{A}}$, and denoted by $\Gamma(T\overline{\mathcal{A}})$. It will be also convenient to use the complexification $\Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ of $\Gamma(T\overline{\mathcal{A}})$. Note that every $Y \in \Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ is a derivation in Cyl, that is

$$Y(\Psi\Psi') = Y(\Psi)\Psi' + \Psi Y(\Psi'). \quad (20)$$

Continuing the analogy with the geometric quantization in the finite dimensional case, consider the vector space

$$\mathfrak{A}_{\text{class}} := \text{Cyl} \times \Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}, \quad (21)$$

equipped with:

- the Lie bracket $\{\cdot, \cdot\}$,

$$\{(\Psi, Y), (\Psi', Y')\} := -(Y(\Psi') - Y'(\Psi), [Y, Y']), \quad (22)$$

- the complex conjugation $^-$ defined by the complex conjugations in Cyl and in $\Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ extended to a map

$$\mathfrak{A}_{\text{class}} \ni (\Psi, Y) = a \mapsto \bar{a} := (\bar{\Psi}, \bar{Y}) \in \mathfrak{A}_{\text{class}} \quad (23)$$

(in $\Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ the c.c. is defined naturally as $\bar{Y}(\Psi) := \overline{Y(\bar{\Psi})}$).

Definition 2.7. *The classical Ashtekar–Corichi–Zapata holonomy-flux algebra is the Lie algebra* $(\mathfrak{A}_{\text{class}}, \{\cdot, \cdot\})$ *equipped with* $^-$ *as involution.*

The ACZ algebra $\mathfrak{A}_{\text{class}}$ admits also an action of the the algebra Cyl,

$$\begin{aligned} \text{Cyl} \times \mathfrak{A}_{\text{class}} &\to \mathfrak{A}_{\text{class}}, \\ (\Psi', (\Psi, Y)) &\mapsto \Psi' \cdot (\Psi, Y) := (\Psi'\Psi, \Psi'Y). \end{aligned}$$

2.5. The quantum $*$-algebra. The ACZ classical holonomy-flux Lie algebra $\mathfrak{A}_{\text{class}}$, is used now as a set of labels to define an abstract $*$-algebra. Consider the $*$-algebra of the finite formal linear combinations of all the finite sequences of elements of $\mathfrak{A}_{\text{class}}$ with the obvious vector space structure, the associative product $\cdot$, and involutive anti-linear algebra anti-isomorphism $*$, defined, respectively, as follows

$$\begin{aligned} (a_1, \dots, a_n) \cdot (b_1, \dots, b_m) &= (a_1, \dots, a_n, b_1, \dots, b_m), \quad (24) \\ (a_1, \dots, a_n)^* &= (\overline{a_n}, \dots, \overline{a_1}). \quad (25) \end{aligned}$$

Divide the algebra by a two-sided ideal defined by the following elements (consisting of 1-element and 2-element sequences):

$$\begin{aligned} (\alpha a) - \alpha(a) \ , \ (a+b) - (a) - (b), \quad (26) \\ (a, b) - (b, a) \ -i(\{a, b\}), \quad (27) \\ (\Psi, a) \ -(\Psi a), \quad (28) \end{aligned}$$

given by all $\alpha \in \mathbb{C}, a, b \in \mathfrak{A}_{\text{class}}$ and $\Psi \in \text{Cyl}$. The first class (26) of elements of the ideal relates the linear structure of $\mathfrak{A}_{\text{class}}$ with the linear structure of the resulting quotient. The second class (27) of elements encodes the familiar quantum relation between the bracket $\{\cdot, \cdot\}$ in $\mathfrak{A}_{\text{class}}$ and the commutators in the quantum algebra $\mathfrak{A}$. The third class[6] (28) encodes the module structure of the ACZ Lie algebra $\mathfrak{A}_{\text{class}}$ over the algebra Cyl. Shorter, the algebra $\mathfrak{A}$ may be also viewed as the algebra $\exp(\otimes \mathfrak{A}_{\text{class}})$ divided by the identities (27) and (28). Note that each of the classes (26, 27, 28) is preserved by $*$. Denote the quotient $*$-algebra by $\mathfrak{A}$.

Definition 2.8. *The quantum holonomy-flux $*$-algebra is the (unital) $*$-algebra $(\mathfrak{A}, *)$.*

The classical ACZ algebra $\mathfrak{A}_{\text{class}}$ is naturally mapped in $\mathfrak{A}$,

$$\mathfrak{A}_{\text{class}} \;\to\; \mathfrak{A} \tag{29}$$

in the sense that $\mathfrak{A}$ is isomorphic to the enveloping algebra of $\mathfrak{A}_{\text{class}}$ (see (26), (27)), divided by additional identities (28) to preserve the structure of a Cyl-module. The images in $\mathfrak{A}$ of 1-element sequences $((\Psi, 0))$ or $((0, Y))$, where $\Psi \in \text{Cyl}$ and $Y \in \Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ will be denoted by $\hat{\Psi}$ and $\hat{Y}$ respectively. They generate the algebra $\mathfrak{A}$. In particular, for every cylindrical function Ψ and every flux vector field $X_{S,f}$,

$$\hat{\Psi}^* = \hat{\bar{\Psi}}, \quad \hat{X}_{S,f}^* = \hat{X}_{S,f}. \tag{30}$$

It is easy to see that the map (29) is an embedding. Here is a simple argument.[7] Consider a representation $\pi_0' : \exp(\otimes \mathfrak{A}_{\text{class}}) \to L(\text{Cyl})$, where $L(\text{Cyl})$ is the algebra of the linear maps $\text{Cyl} \to \text{Cyl}$, defined on the generators as follows:

$$\pi_0'((\Psi, X))\Phi = \Psi\Phi - i\{\Phi, X\}. \tag{31}$$

It is easy to check that each of the elements (27) and (28) is in the kernel of π_0'. Therefore, π_0' passes naturally to a representation $\pi_0 : \mathfrak{A} \to L(\text{Cyl})$,

$$\pi_0 \; : \; \mathfrak{A} \;\to\; L(\text{Cyl}). \tag{32}$$

The point is that the composition of the maps (29) and π_0,

$$\mathfrak{A}_{\text{class}} \;\to\; \mathfrak{A} \;\to\; L(\text{Cyl}), \tag{33}$$

is obviously injective. Hence the first map is also injective.

2.6. The elements of $\mathfrak{A}$. Owing to the identities in $\mathfrak{A}$ defined by the third class (28) of elements defining the ideal above, the image $\widehat{\text{Cyl}} \subset \mathfrak{A}$ of Cyl upon the map (29)

$$\text{Cyl} \ni \Psi \;\mapsto\; \hat{\Psi} \in \mathfrak{A} \tag{34}$$

is a $*$-subalgebra. Due to (26), the map is linear, it is multiplicative due to (28), and bijective as the restriction of (29) and hence a $*$-isomorphism between Cyl and $\widehat{\text{Cyl}}$. Every element of the algebra $\mathfrak{A}$ is a finite linear combination of elements of the form

$$\hat{\Psi}, \; \hat{\Psi}_1 \hat{X}_{S_{11}, f_{11}}, \; \hat{\Psi}_2 \hat{X}_{S_{21}, f_{21}} \hat{X}_{S_{22}, f_{22}}, \dots, \hat{\Psi}_k \hat{X}_{S_{k1}, f_{k1}} \dots \hat{X}_{S_{kk}, f_{kk}}, \dots, \tag{35}$$

---

[6] The theorem we formulate and prove in this paper uses only the fact that the two-sided ideal contains elements of the third class for $a \in \text{Cyl}$.

[7] We thank Wojtek Kamiński for help.

where Ψ, $\Psi_i \in$ Cyl and $X_{S_{ij}, f_{ij}}$ are the flux vector fields for all the i, $j = 1, \ldots, k$. For example,

$$a = \hat{X}_{S,f} \hat{\Psi} \hat{X}_{S',f'} = -i\widehat{X_{S,f}(\Psi)}\hat{X}_{S',f'} + \hat{\Psi}\hat{X}_{S,f}\hat{X}_{S',f'}. \tag{36}$$

2.7. Symmetries of $\mathfrak{A}$. The group of the semianalytic automorphisms of the principal fiber bundle P acts naturally in the space $\mathcal{A}$ of connections. The action preserves the algebra Cyl of the cylindrical functions, the norm $\| \cdot \|$ and the $*$ involution. Therefore it induces an action of the bundle automorphism group in the space $\overline{\mathcal{A}}$ of generalized connections. The action of the bundle automorphism group on the flux vector fields can be viewed either as the action on operators $X_{S,f} :$ Cyl $\to$ Cyl (3), or as the action on the field E and its flux functional (2). Both definitions are equivalent and lead to an appropriate action of the bundle automorphisms on the labels, i.e. the faces and the smearing vector fields. In this way, the bundle automorphism group induces an isomorphism of the ACZ classical Lie algebra $\mathfrak{A}_{\mathrm{class}}$, and finally a $*$-isomorphism of the quantum $*$-algebra $\mathfrak{A}$. In this subsection we discuss the action of the automorphisms/diffeomorphisms in detail. But before doing that let us make a remark on the relation between the bundle P automorphisms and the manifold Σ diffeomorphisms. For every bundle automorphism,

$$\tilde{\varphi} : P \to P, \tag{37}$$

there is a unique diffeomorphism

$$\varphi : \Sigma \to \Sigma \tag{38}$$

such that

$$\Pi \circ \tilde{\varphi} = \varphi \circ \Pi. \tag{39}$$

In our case both of them are assumed to be semianalytic. If the diffeomorphism φ is the identity map, then the corresponding automorphism is fiber preserving, and we can call it a Yang–Mills gauge transformation. On the other hand, all the Σ diffeomorphisms homotopic to the identity map are related to the bundle automorphisms via (39). In this sense, the bundle automorphisms represent also the diffeomorphisms of Σ.

Now we turn to the technical details of the action of the bundle automorphism group in the quantum $*$-algebra $\mathfrak{A}$.

For every edge (only the end points of e matter here), the map $\tilde{\varphi}$ defines the following map [20]

$$\mathrm{Ad}_{\tilde{\varphi}} : \mathcal{A}_e \to \mathcal{A}_{\varphi(e)}, \; A(e) \mapsto \tilde{\varphi} \circ A(e) \circ \tilde{\varphi}^{-1}. \tag{40}$$

The map extends naturally to the product space,

$$\begin{aligned}\mathrm{Ad}_{\tilde{\varphi}} : \mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_N} &\to \mathcal{A}_{\varphi(e_1)} \times \cdots \times \mathcal{A}_{\varphi(e_N)}, \\ (A(e_1), \ldots, A(e_N)) &\mapsto (\mathrm{Ad}_{\tilde{\varphi}}(A(e_1)), \ldots, \mathrm{Ad}_{\tilde{\varphi}}(A(e_N))).\end{aligned} \tag{41}$$

It will be relevant later that the map is smooth which can easily be seen by fixing any trivialization of the fibers of P in question. Via Ad, the automorphism $\tilde{\varphi}$ acts in the space of the generalized connections,

$$\overline{\mathrm{Ad}}_{\tilde{\varphi}} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}, \tag{42}$$

$$\left(\overline{\mathrm{Ad}}_{\tilde{\varphi}} \bar{A}\right)(e) := \mathrm{Ad}_{\tilde{\varphi}}(\bar{A}(\varphi^{-1}(e))). \tag{43}$$

The pullback $\overline{\mathrm{Ad}}^*_{\tilde\varphi}$ preserves the space of the cylindrical functions. Indeed, for every cylindrical function Ψ given by (8),

$$(\overline{\mathrm{Ad}}^*_{\tilde\varphi}\Psi)(\bar{A}) = (\mathrm{Ad}^*_{\tilde\varphi}\psi)(\bar{A}(\varphi^{-1}(e_1)), \ldots, \bar{A}(\varphi^{-1}(e_n))), \tag{44}$$

meaning that $\overline{\mathrm{Ad}}^*_{\tilde\varphi}\Psi$ is compatible with the graph $\varphi^{-1}(\gamma) := \{\varphi^{-1}(e_1), \ldots, \varphi^{-1}(e_n)\}$ and the function $\mathrm{Ad}^*_{\tilde\varphi}\psi \in C^\infty(\mathcal{A}_{\varphi^{-1}(e_1)} \times \cdots \times \mathcal{A}_{\varphi^{-1}(e_n)})$.

Given a flux vector field $X_{S,f}$, the map $\overline{\mathrm{Ad}}_{\tilde\varphi}$ defined above maps the generalized flow (17) into a new flow $\overline{\mathrm{Ad}}_{\tilde\varphi}\theta^{(\cdot)}$. It is easy to check that the new flow is the flow of the flux vector field $X_{\varphi(S),\tilde\varphi_* f}$ [20], hence

$$\overline{\mathrm{Ad}}_{\tilde\varphi_*} X_{S,f} = X_{\varphi(S),\tilde\varphi_* f}. \tag{45}$$

Finally, the natural action of $\tilde\varphi$ on $\mathfrak{A}$,

$$\alpha_{\tilde\varphi} : \mathfrak{A} \to \mathfrak{A}, \tag{46}$$

is a $*$-algebra automorphism determined by the following action on the generators $\hat\Psi$ and $\hat{X}_{S,f}$, where $\Psi \in \mathrm{Cyl}$ and $X_{S,f}$ are arbitrary:

$$\alpha_{\tilde\varphi}\hat\Psi := \widehat{\overline{\mathrm{Ad}}^*_{\tilde\varphi^{-1}}\Psi}, \quad \alpha_{\tilde\varphi}\hat{X}_{S,f} := \hat{X}_{\varphi(S),\tilde\varphi_* f}. \tag{47}$$

The action satisfies

$$\alpha_{\tilde\varphi_1 \circ \tilde\varphi_2} = \alpha_{\tilde\varphi_1} \circ \alpha_{\tilde\varphi_2}. \tag{48}$$

It should be pointed out that the quantum holonomy-flux $*$-algebra $\mathfrak{A}$ can potentially admit more symmetries. The relevance of the bundle automorphisms lies in the fact that they are the symmetries of a diffeomorphism invariant classical theory.

3. States, GNS

In all of the following, we will be concerned with states ω on $\mathfrak{A}$ and their GNS representations. Recall that

Definition 3.1. *A state on a $*$-algebra $\mathfrak{A}$ is a functional $\omega : \mathfrak{A} \to \mathbb{C}$, such that for every $\alpha \in \mathbb{C}$, and every $a, b \in \mathfrak{A}$,*

$$\omega(\alpha a + b) = \alpha\omega(a) + \omega(b), \quad \omega(a^*) = \overline{\omega(a)}, \tag{49}$$

$$\omega(a^* a) \geq 0, \quad \omega(I) = 1, \tag{50}$$

where I stands for the unity element of $\mathfrak{A}$.

Given a state on $\mathfrak{A}$, we can construct the corresponding GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega,)$ where $\mathcal{H}_\omega$ is a Hilbert space, π_ω a representation of $\mathfrak{A}$ on $\mathcal{H}_\omega$ and Ω_ω a vector in $\mathcal{H}_\omega$ which, when viewed as a state on $\mathfrak{A}$, coincides with ω. A detailed exposition of the GNS construction for algebras of unbounded operators can be found for example in [12]. Here we will only need the following elements and properties that are easy to prove: (i) the linear space of the equivalence classes

$$[\mathfrak{A}] := \mathfrak{A}/\mathfrak{J}, \tag{51}$$

where $\mathfrak{J}$ is the left ideal formed by all $a \in \mathfrak{A}$ such that $\omega(a^*a) = 0$, is equipped by the state ω with the following product:

$$\langle [a], [b] \rangle := \omega(a^*b),$$

where for every $a \in \mathfrak{A}$, $[a] \in \mathfrak{A}/\mathfrak{J}$ stands for the equivalence class defined by a; (ii) the product provides a norm $\|a\|_\omega = \sqrt{\langle [a], [a] \rangle}$ in $[\mathfrak{A}]$, and the completion

$$\mathcal{H}_\omega := \overline{[\mathfrak{A}]} \tag{52}$$

together with the product $\langle \cdot, \cdot \rangle$ is a Hilbert space; (iii) to every element a of $\mathfrak{A}$ we assign a linear but in general unbounded operator $\pi_\omega(a)$ acting in $[\mathfrak{A}]$,

$$\pi_\omega(a)[b] := [ab], \quad \text{for every} \quad b \in \mathfrak{A}; \tag{53}$$

(iv) the action π_ω preserves the subspace $[\mathfrak{A}]$, hence $[\mathfrak{A}]$ serves as a common, dense domain for all the operators $\pi_\omega(a)$, $a \in \mathfrak{A}$; (v) The representation π_ω satisfies

$$\langle \pi_\omega(a)[b], [c] \rangle = \langle [b], \pi_\omega(a^*)[c] \rangle, \tag{54}$$

for every $a, b, c \in \mathfrak{A}$.

As we explained in the previous section, $\mathfrak{A}$ contains the subalgebra $\widehat{\mathrm{Cyl}} \subset \mathfrak{A}$ isomorphic as a $*$-algebra with the algebra Cyl. Therefore, every state ω defined in $\mathfrak{A}$, restricted to $\widehat{\mathrm{Cyl}}$ defines a state on Cyl. On the other hand, there is known a powerful characterization of states defined on the completion $\overline{\mathrm{Cyl}}$. Fortunately, that characterization applies also to all the states on the $*$-algebra Cyl (and hence $\widehat{\mathrm{Cyl}}$), due to the following fact:

Lemma 3.2. [8] *Suppose that $\omega : \mathrm{Cyl} \to \mathbb{C}$ satisfies for every $\Psi, \Psi' \in \mathrm{Cyl}$ and $\alpha \in \mathbb{C}$, the following equalities and inequality:*

$$\omega(\alpha\Psi + \Psi') = \alpha\omega(\Psi) + \omega(\Psi'), \quad \omega(\bar{\Psi}) = \overline{\omega(\Psi)},$$
$$\omega(\bar{\Psi}\Psi) \geq 0, \quad \omega(I) = 1.$$

Then,

$$|\omega(\Psi)| \leq 2\|\Psi\|. \tag{55}$$

Therefore, ω is continuous with respect to the norm $\| \cdot \|$ and determines a unique extension to a state defined on the C^*-algebra $\overline{\mathrm{Cyl}}$.

Proof. For every $\Psi \in \mathrm{Cyl}$ we have

$$|\omega(\Psi)| = |\omega(\Psi_R) + i\omega(\Psi_I)| \leq |\omega(\Psi_R)| + |\omega(\Psi_I)|, \tag{56}$$

where Ψ_R and Ψ_I are the real and imaginary parts of Ψ, respectively. Let $\Psi' \neq 0$ be Ψ_R or Ψ_I (if both Ψ_R and Ψ_I are zero then (55) is satisfied trivially). For every real number q the following equality holds:

$$\omega(\Psi') = q\|\Psi'\| - \omega(q\|\Psi'\|I - \Psi'). \tag{57}$$

---

[8] This lemma is a modification of similar well known results see for example [30], p. 106,107. We include it for completeness. The factor 2 in the inequality (55) can be probably lowered to 1, but this is not relevant in our paper.

Let $q > 1$. Then, the function $q\|\Psi'\|I - \Psi'$ is strictly positive, and

$$\mathcal{A} \ni A \mapsto \Psi''(A) := \sqrt{q\|\Psi'\|I - \Psi'(A)} \tag{58}$$

is a well defined function on the space of connections. Importantly, this is also a cylindrical function. Indeed, if we represent Ψ' by a compatible graph γ and function ψ' (see (8)), then the function $q\|\psi'\| - \psi'$ is everywhere strictly positive, because the natural map $\mathcal{A} \to \mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n}$ is onto. Therefore the function

$$\psi'' := sqrtq\|\psi'\|I - \psi' \tag{59}$$

is C^∞ and the corresponding cylindrical function is exactly Ψ''. But this means that the second term on the right hand side of the equality (57) (including the sign) is negative. Indeed,

$$-\omega(q\|\Psi'\| - \Psi') = -\omega(\bar{\Psi}''\Psi'') \leq 0. \tag{60}$$

This observation completes the proof of Lemma 3.2. $\square$

4. The Uniqueness Theorem

As explained in the introduction, of particular importance are states invariant with respect to the automorphisms of the principal fiber bundle P. A state ω defined on the algebra $\mathfrak{A}$ is invariant with respect to a bundle automorphism $\tilde{\varphi} : P \to P$, if for every $a \in \mathfrak{A}$,

$$\omega(a) = \omega(\alpha_{\tilde{\varphi}} a). \tag{61}$$

If ω is invariant with respect to all the fiber preserving automorphisms, we call it Yang–Mills gauge invariant.

Definition 4.1. *If a state defined on the quantum $*$-algebra is invariant with respect to all the bundle automorphisms of P that induce, via the bundle projection Π, diffeomorphisms homotopic to the identity, we call it Yang–Mills gauge and diffeomorphism invariant or, if there is no danger of confusion, just invariant.*

Given a $*$-algebra and a symmetry group, assuming the existence of a diffeomorphism invariant state is a strong condition. However, in our case one invariant state is already known; we recall it below. It will be, therefore, natural to ask if there are other states with that property.

Example. A Yang–Mills gauge invariant and diffeomorphism invariant state on $\mathfrak{A}$. Define the action of ω_0 on the elements of $\mathfrak{A}$ of the form $a \cdot \hat{Y}$, where $a \in \mathfrak{A}$ and $Y \in \Gamma(T\overline{\mathcal{A}})^{(\mathbb{C})}$ as simply

$$\omega_0(a \cdot \hat{Y}) := 0 \tag{62}$$

for every a and every vector field Y. To define the action of ω_0 on an element $\hat{\Psi}$ corresponding to $\Psi \in \mathrm{Cyl}$, recall a general form (8) of a cylindrical function. Recall also that each factor $\mathcal{A}_e$ in the domain $\mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n}$ has all the left and right invariant structures of the bundle structure group G. One of them is the probability Haar measure μ_e. We use it to set

$$\omega_0(\hat{\Psi}) := \int_{\mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n}} \psi \, d\mu_{e_1} \otimes \cdots \otimes d\mu_{e_n}. \tag{63}$$

Importantly, this integral is independent of choice of the graph γ compatible with a given Ψ. Due to the general form of $a \in \mathfrak{A}$ given by (35) the equalities (62, 63) determine a state ω_0. The positivity of ω_0 amounts to the positivity of the Haar measure $\mu_{e_1} \otimes \cdots \otimes \mu_{e_n}$ on $C^\infty(\mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n})$ which is obviously true.

The state ω_0 is Yang–Mills gauge and diffeomorphism invariant. To see this, note that every automorphism $\tilde{\varphi}$ of the bundle P maps every flux vector field into another flux vector field, therefore the condition (62) is manifestly invariant. To see the invariance of the part (63) of the definition, consider a graph γ and function $\psi \in C^\infty(\mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n})$ compatible with Ψ (see (8)), and a gauge map $\sigma^{-1}{}_\gamma : G^n \to \mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n}$ defined by the gauge maps (6) and any choice of points $p_x \in \Pi^{-1}(x)$ for every vertex x of γ. Then the definition (63) reads

$$\omega_0(\Psi) = \int_{G^n} \sigma^{-1}{}_\gamma^* \psi \, d\mu_H, \tag{64}$$

where μ_H is the probability Haar measure on G^n. On the other hand, the transformed function $\overline{\mathrm{Ad}}_{\tilde{\varphi}} \Psi$ is compatible with a graph $\varphi^{-1}(\gamma)$, and the function $\psi' = \mathrm{Ad}_{\tilde{\varphi}} \psi$. If we use for the graph $\varphi^{-1}(\gamma)$ the gauge map $\sigma_{\varphi^{-1}(\gamma)}$ given by the points $\tilde{\varphi}^{-1}(p_x)$, then simply

$$\sigma^{-1}{}_{\varphi^{-1}(\gamma)}^* \psi' = \sigma^{-1}{}_\gamma^* \psi. \tag{65}$$

The state ω_0 is well known, and is extensively used in the loop quantization [6, 22].

Given the example of an invariant state above, let us now state and prove our uniqueness result:

Theorem 4.2. *There exists exactly one Yang–Mills gauge invariant and diffeomorphism invariant state on the quantum holonomy-flux $*$-algebra $\mathfrak{A}$.*

Proof. The existence is known, see the example; therefore it suffices to prove the uniqueness.

We will assume from now on that ω is a diffeomorphism invariant state on $\mathfrak{A}$, label the corresponding representation obtained from the GNS construction by $(\mathcal{H}_\omega, \pi_\omega)$ and use the notation introduced in Sect. 3. To simplify the reading, we will break down the proof into two parts. The first of these is rather technical. It will establish a proof of the following fundamental lemma:

Lemma 4.3. *Let ω be an invariant state on $\mathfrak{A}$. Then for every flux vector field $X_{S,f}$, where S is a face, and f a smearing vector field, in the corresponding GNS-representation,*

$$[\hat{X}_{S,f}] = 0. \tag{66}$$

Once the lemma is established, the rest of the proof of the uniqueness is fairly straightforward.

Proof of Lemma 4.3. Let S be a face in Σ and f be a smearing vector field. We will decompose f into a certain finite sum,

$$f = \sum_I \sum_i f_{Ii}, \tag{67}$$

such that each term f_{Ii} is a smearing vector field itself which satisfies

$$[\hat{X}_{S,f_{Ii}}] = 0. \tag{68}$$

Then, (66) follows automatically from the linearity

$$X_{S,f} = \sum_I \sum_i X_{S,f_{Ii}}. \tag{69}$$

To each point $x \in \Pi(\operatorname{supp} f) \subset \Sigma$ choose an open neighborhood $\mathcal{U}_x$ in Σ in such a way that there exists a trivialization $\mathcal{T}_x$ of $\Pi^{-1}(\mathcal{U}_x)$,

$$\mathcal{T}_x : \mathcal{U}_x \times G \; \rightarrow \; \Pi^{-1}(\mathcal{U}_x), \tag{70}$$

and such that there is a chart χ_x containing $\mathcal{U}_x$ in its domain with

$$\chi_x(S \cap \mathcal{U}_x) = \{(x^1, \ldots, x^D) \,|\, x^D = 0, 0 < x^1 < 1, \ldots, 0 < x^{D-1} < 1\}. \tag{71}$$

Since the support of f is compact in P, we can choose from that covering a finite sub-covering $\{\mathcal{U}_I\}_{I=1}^N$ of $\Pi(\operatorname{supp} f)$. We denote the corresponding trivialization by $\mathcal{T}_I$ and the corresponding chart by χ_I. Let $\phi_I : \Sigma \to \mathbb{R}$, where $I = 1, \ldots, N$, be a family of functions such that $\operatorname{supp} \phi_I \subset \mathcal{U}_I$ for every I, and for every $x \in \operatorname{supp} f$,

$$\sum_{I=1}^N \phi_I(x) = 1. \tag{72}$$

We use that partition of unity, to decompose the smearing vector field f,

$$f = \sum_{I=1}^N f_I, \quad f_I := \phi_I \, f. \tag{73}$$

Each f_I $(I = 1, \ldots, N)$ is still a smearing vector field in the sense of Definition (2.5) and additionally has the appropriate support property: $\Pi(\operatorname{supp} f_I) \subset \mathcal{U}_I$.

Now we fix I and decompose the smearing vector field f_I further. Suppose R_1 be a vector field defined on G and right invariant. It defines naturally a vector field on $\mathcal{U}_I \times G$. That vector field is mapped by the trivialization $\mathcal{T}_I$ into a vector field defined in $\Pi^{-1}(\mathcal{U}_I)$, tangent to the fibers and invariant with respect to the group action. Let $R_i, i = 1, \ldots, \dim G$, be a basis in the vector space of the right invariant vector fields defined on G. Then, every smearing vector field f_I defined on $P_S = \Pi^{-1}(S)$ is a sum of the vector fields proportional to the vector fields $\mathcal{T}_{I*} R_i, i = 1, \ldots, \dim G$,

$$f_I = \sum_i f_{Ii}, \quad f_{Ii} = (\Pi^* h_i) \, \mathcal{T}_{I*} R_i, \tag{74}$$

where each coefficient $\Pi^* h_i$ is a function $h_i : S \to \mathbb{R}$ lifted to the bundle $P_S = \Pi^{-1}(S)$. Obviously $\operatorname{supp} h_i \subset \mathcal{U}_I$. Now we can finish the proof of Lemma 4.3, by showing that necessarily (68) is true. To this end, fix indices I, i, and for every compactly supported function $h : S \cap \mathcal{U}_I \to \mathbb{R}$ consider the following smearing vector field defined on S (no summation with respect to i):

$$w(h) := (\Pi^* h) \, \mathcal{T}_{I*} R_i. \tag{75}$$

Consider the following product $(\cdot \,|\, \cdot)$ which, given a pair of compactly supported functions $h, g : S \cap \mathcal{U}_I \to \mathbb{R}$, assigns the following number $(h|g)$:

$$(h|g) := \left\langle [\hat{X}_{S,w(h)}], \, [\hat{X}_{S,w(g)}] \right\rangle.$$

The product $(\cdot \,|\, \cdot)$ has the following properties:

(i) it is bilinear and symmetric,
(ii) it is invariant under diffeomorphisms of Σ which are supported in $\mathcal{U}_I$ and preserve $\mathcal{U}_I$ as well as S,
(iii) for $h = g = h_i$ of (74), it is exactly the norm squared of the Hilbert space $\mathcal{H}_\omega$ element $[\hat{X}_{S,f_{Ii}}]$ under consideration,

$$\|[\hat{X}_{S,f_{Ii}}]\|^2 = \left\langle [\hat{X}_{S,f_{Ii}}], [\hat{X}_{S,f_{Ii}}] \right\rangle = (h_i|h_i). \tag{76}$$

The property ii) above, follows from the fact, that via the trivialization $\mathcal{T}_I$, every diffeomorphism $\varphi : \Sigma \to \Sigma$ preserving $\mathcal{U}_I$ defines an automorphism $\tilde{\varphi}$ of the bundle P which preserves each of the vector fields $\mathcal{T}_{I*}R_i$. Therefore, if φ additionally preserves S, then the action of $\tilde{\varphi}$ (47) on $\hat{X}_{S,w(h)}$ amounts to

$$\alpha_{\tilde{\varphi}}\hat{X}_{S,w(h)} = \hat{X}_{S,w(h\circ\varphi^{-1})}. \tag{77}$$

We will now show that properties i) and ii) already imply that that $(h|h)$ is zero for every function $h : S \to \mathbb{R}$ with compact support in $\mathcal{U}_I$. Then iii) shows that we have reached our goal.

Let us use the chart χ_I to push forward the action arena into $\mathbb{R}^D$:

$$\mathcal{U}_I' := \chi_I(\mathcal{U}_I), \qquad S' = \chi_I(S \cap \mathcal{U}_I), \tag{78}$$
$$h' := h \circ \chi^{-1} : S' \to \mathbb{R}, \tag{79}$$

where h' has compact support and S' is defined by (71).

We want to extend h' to a function defined in $\mathcal{U}_I'$ and of a compact support. Therefore, we choose an arbitrary semianalytic function $\kappa' : \mathbb{R} \to \mathbb{R}$ such that $\kappa'(0) = 1$ and the function

$$(x^1, \ldots, x^D) \mapsto h'(x^1, \ldots, x^{D-1})\kappa'(x^D) \tag{80}$$

has compact support contained in $\mathcal{U}'_I$. Using these ingredients, we can define a map[9] $\varphi'_\lambda : \mathbb{R}^D \to \mathbb{R}^D$, where λ is a real parameter, by

$$\varphi'_\lambda(x^1, \ldots, x^D) := (x^1 + \lambda h'(x^1, \ldots, x^{D-1})\kappa'(x^D), x^2, \ldots, x^D). \tag{81}$$

Lemma 4.4. *There is $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$, φ'_λ is a semianalytic diffeomorphism of $\mathbb{R}^D$ equal to the identity outside of $\mathcal{U}'_I$ and preserving $\mathcal{U}'_I$.*

Proof. The Jacobian of φ'_λ is a triangular matrix and the determinant turns out to be simply $1 + \lambda\kappa'\partial_1 h'$. Since $\lambda\kappa'\partial_1 h'$ has compact support and is semianalytic, it is in particular bounded, and thus there is a $\lambda_0 > 0$ such that $1 + \lambda\kappa'\partial_1 h' > 0$ for every $0 < \lambda < \lambda_0$. Hence φ'_λ is locally a diffeomorphism, provided $0 < \lambda < \lambda_0$. It is also a global diffeomorphism, because outside of the support of $\kappa' h'$ it acts as the identity and thus $\lim_{|x|\to\infty} |\varphi'_\lambda(x)| = \infty$. Then a well known theorem by Hadamard proves the assertion. Because all the functions used in the construction of φ'_λ are assumed to be semianalytic, and all the operations used preserve the semianalyticty (see the Appendix), φ'_λ is also semianalytic. Finally, note that every bijection which is an identity on a certain subset, necessarily preserves the complement. □

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[9] We will use here a modification of the trick mentioned in the Appendix of [27].

Now let us choose a semianalytic function H' with support in $\mathcal{U}'_I$ such that

$$H'(x^1, \ldots, x^D) = x^1 \quad \text{whenever} \quad (x^1, \ldots, x^D) \in \operatorname{supp} \kappa' h'. \tag{82}$$

Such a function can be easily constructed by using an appropriate partition of the unity. Let us see how each of the diffeomorphisms φ'_λ acts on H': Because of the properties of H' in relation to the support of $tail\kappa' h'$ we find

$$\varphi'^*_\lambda H'(x^1, \ldots, x^D) = \begin{cases} x^1 + \lambda h'(x^1, \ldots, x^{D-1}) \kappa'(x^D) & \text{on } \operatorname{supp} \kappa' h' \\ H'(x^1, \ldots, x^D) & \text{otherwise} \end{cases}$$

$$= H'(x^1, \ldots, x^D) + \lambda h'(x^1, \ldots, x^{D-1}) \kappa'(x^D).$$

Now let us pull this relation back to $\mathcal{U}_I$ and the manifold Σ again by using the chart χ_I. Denote the pullbacks of the functions H', h', κ' and φ'_λ, respectively, by H, h, κ and φ_λ. The functions H and $h\kappa$ have support contained in $\mathcal{U}_I$, therefore we can extend them as identically zero to the rest of Σ. Similarly, φ_λ, for every $0 < \lambda < \lambda_0$, is a diffeomorphism defined locally in $\mathcal{U}_I$ that can be extended as the identity to the rest of Σ, and the result is a diffeomorphism of Σ. The above relation then reads

$$\varphi^*_\lambda H = H + \lambda \kappa h. \tag{83}$$

Now we compute

$$(H|H) \overset{ii)}{=} (\varphi^*_\lambda H | \varphi^*_\lambda H) = (H|H) + \lambda(h|H) + \lambda(H|h) + \lambda^2(h|h), \tag{84}$$

where we have used the invariance of the product $(\cdot | \cdot)$ under diffeomorphisms homotopic to the identity (the φ_λ obviously are) and the fact that $\kappa|_S = 1$. Since the equality (84) holds for every value of λ provided $0 < \lambda < \lambda_0$, we conclude that

$$(h|h) = 0. \tag{85}$$

Then, as announced above for $h = h_i$, we get the desired result, and in turn conclude that $[\hat{X}_{S,f}] = 0$ as a vector in the GNS-Hilbert space $\mathcal{H}_\omega$.   □

Now that we have established the fundamental Lemma 4.3 asserting that $[\hat{X}_{S,f}] = 0$ for any face S and any smearing vector field f in any GNS-representation coming from the invariant state ω, we can show that the structure of the GNS-Hilbert space $\mathcal{H}_\omega$ is actually very simple.

Let us start by reminding the reader of the form (35) of elements of $\mathfrak{A}$ whose linear span is $\mathfrak{A}$. It follows immediately that a dense set of vectors in $\mathcal{H}_\omega$ is given by the linear span of all the vectors of the form

$$[\hat{\Psi}], \; \pi_\omega(\hat{\Psi}_1)[\hat{X}_{S_{11}, f_{11}}], \; \pi_\omega(\hat{\Psi}_2 \hat{X}_{S_{21}, f_{21}})[\hat{X}_{S_{22}, f_{22}}], \ldots$$
$$\ldots, \pi_\omega(\hat{\Psi}_k \hat{X}_{S_{k1}, f_{k1}} \ldots)[\hat{X}_{S_{kk}, f_{kk}}], \ldots. \tag{86}$$

But because of Lemma 4.3, of these vectors, only the ones of the form $[\hat{\Psi}]$ are non-zero.

Therefore, all the information on the state ω is contained in the corresponding state defined on the algebra $\widehat{\mathrm{Cyl}}$,

$$\mathrm{Cyl} \ni \Psi \mapsto \omega(\hat{\Psi}) = \left\langle [\hat{I}], [\hat{\Psi}] \right\rangle_{\mathcal{H}_\omega}, \tag{87}$$

where $\hat{I}$ is the unit element of $\mathfrak{A}$. Now we make use of Lemma 3.2 from Sect. 3: the state (87) is actually continuous with respect to the C*-norm on Cyl. Thus, using the representation theorem by Riesz and Markow, there exists a measure μ on $\overline{\mathcal{A}}$ such that

$$\omega(\hat{\Psi}) = \int_{\overline{\mathcal{A}}} \Psi \, d\mu.$$

Notice now that Lemma 4.3 implies what follows:

$$\begin{aligned}
\int_{\overline{\mathcal{A}}} \overline{\Psi} X_{S,f}(\Psi') \, d\mu &= \left\langle [\hat{\Psi}], [\widehat{X_{S,f}(\Psi')}] \right\rangle_{\mathcal{H}_\omega} \\
&= i \left\langle [\hat{\Psi}], [\hat{X}_{S,f}\hat{\Psi}'] - [\hat{\Psi}'\hat{X}_{S,f}] \right\rangle_{\mathcal{H}_\omega} \\
&= i \left\langle [\hat{\Psi}], [\hat{X}_{S,f}\hat{\Psi}'] \right\rangle \\
&= i \left\langle [\hat{X}_{S,f}\hat{\Psi}], [\hat{\Psi}'] \right\rangle \\
&= -\int_{\overline{\mathcal{A}}} \overline{X_{S,f}(\Psi)} \Psi' \, d\mu.
\end{aligned}$$

Setting $\Psi = I$ (i.e. the constant function on $\overline{\mathcal{A}}$ of the value 1) we see that for any face S and any smearing vector field f and for any function $\Psi' \in$ Cyl,

$$\int_{\overline{\mathcal{A}}} X_{S,f}(\Psi') \, d\mu = 0.$$

As it was shown in [15] the only measure satisfying the above condition coincides with the measure defined on $\overline{\mathcal{A}}$ by the state ω_0 described in the example.

   In conclusion,

$$\omega = \omega_0. \tag{88}$$

□

5. Closing Remarks

As we have emphasized the uniqueness result proved in the last section is reassuring for the LQG program, and it shows that diffeomorphism invariance can sometimes be a powerful remedy against complications that one expects based on what one knows about background dependent field theories. Our result is based on certain, albeit reasonable, assumptions, therefore an immediate question is whether it can be generalized.

   Certainly the result holds for any enlargement of the symmetry group that contains the diffeomorphisms considered above. Whether it also holds for smaller extensions, or even for only the analytic diffeomorphisms is an open question. We feel however that as soon as the subgroup of diffeomorphisms is big enough to contain 'local ones', application of the techniques used above should be straightforward.

   Also, even if uniqueness were to break down if only invariance under analytic automorphisms is required, it is not clear how relevant the result would be physically, as it would heavily involve details of the structure of analytic diffeomorphisms on a given manifold Σ.

Another way to generalize the result would be to consider, instead of the flux operators, the unitary groups that they generate, and ask for diffeomorphism invariant representations in which these groups are strongly continuous. In [16] this setting was considered, however the results were not satisfactory due to the more complicated domain questions arising. A more satisfying result was recently obtained by Fleischhack in [29]. It does not make the assumption of a common dense domain for all flux operators and cylindrical functions that is implicit in our treatment. However, it needs an additional assumption on the action of the bundle automorphisms in the representation.

Finally, as for background dependent theories, at least the definition of the kinematical algebra $\mathfrak{A}$ applies in principle. Whether one expects the type of variables used to be well defined in the quantum theory is certainly a difficult question. Still it seems worthwhile to look for non-diffeomorphism invariant representations of $\mathfrak{A}$ and see if they can be put to use in physics.

Another interesting starting point for future work is the observation that the uniqueness theorem fails if a rather innocent looking assumption – that of compact support on S for the smearing functions f used in the flux variables $E_{S,f}$ – is removed: Consider the example $\Sigma = \mathbb{R}^2$, $G =$ U(1), and drop the assumption of compact support for the smearing functions. The hyper-surfaces S are one dimensional in this case. Let us also choose an orientation for Σ. From that orientation, together with the orientation on the normal bundle of a given S we can equip S with an intrinsic orientation, and thus integrate one-forms on S. Then we can define

$$\omega(\hat{\Psi}) := \omega_0(\hat{\Psi}),$$
$$\omega(\hat{\Psi}\hat{X}_{S_1,f_1} \ldots \hat{X}_{S_n,f_n}) := \int_{S_1} df_1 \ldots \int_{S_n} df_n \, \omega_0(\hat{\Psi}).$$

It is easy to check that ω defines a state on $\mathfrak{A}$ and is manifestly invariant under the action of orientation-preserving diffeomorphisms. Obviously it is different from ω_0 and thus would constitute a counterexample to our uniqueness result, were it not for the fact that for smearing functions f with compact support in S, $\int_S df = 0$. Hence under the assumptions made in this paper, $\omega = \omega_0$, and there is no contradiction.

Since obvious generalizations of this state to a higher dimensional situation seem to fail, the existence of ω for smearing functions without compact support might just be a peculiarity of $D = 2$. However, just as for ω the endpoints of the lines S can be used to "anchor" diffeomorphism invariant information in the state, it is not inconceivable that similarly points of the boundary of hyper-surfaces S in which the boundary has a lower differentiability than C^m might be used to that end in higher dimensions.[10] In any case, the restriction to compact support does not seem to be unphysical.[11] It can be viewed as analogous to the smoothness and decay properties assumed for smearing functions in standard quantum field theory. A more detailed investigation into these issues will be carried out elsewhere.

A useful for LQG outcome of our work is introducing the semianalytic category. The corresponding diffeomorphisms form a subgroup of the C^m diffeomorphism group, the

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[10] In a similar way, one might intuitively understand the need to use semi-analytic smearing functions, not just, say, continuous ones: Singular points (from the semianalytic perspective) of the smearing functions could not be removed by semianalytic diffeomorphisms and evaluation of the function at such points would thus constitute diffeomorphism invariant data that could give rise to other diffeomorphism-invariant states.

[11] And, as for applications to Loop Quantum Gravity, all results that use smearing functions with non-compact support (such as the definition of the volume operator) can be recovered by taking appropriate limits once the state is fixed to be ω_0.

group of symmetries induced by the action of the diffeomorphisms of Σ in the classical phase-space and preserving our classical Ashtekar–Corichi–Zapata algebra. The relevance of this symmetry group consists in its local character (as opposed to the analytic diffeomorphisms). For example, the symmetry group provides a new, elegant version of a map Cyl $\rightarrow$ Cyl* (the algebraic dual) which averages with respect to the (allowed) diffeomorphisms of Σ. This application has been recently implemented in [6][12] (see also [5, 26, 27]). There are several similar, non-equivalent extensions of the analytic diffeomorphisms recently introduced in literature. One of them is due to Fleischhack [29] who also applies the theory of the stratifications. Another one was considered by Rovelli and Fairbairn [11]. They even advocate the relevance of non-differentiable homeomorphisms in the classical Einstein's Gravity. The Rovelli–Fairbairn generalized diffeomorphisms, however, are defined to be smooth everywhere except a finite set of points, therefore they would not be useful in our case.

A. Semianalytic Category, Details

A.1. Semianalytic functions in $\mathbb{R}^n$. In this section we introduce semianalytic functions, semianalytic manifolds and semianalytic geometry. We will take advantage of the results of the theory of semianalytic sets [2, 3].[13] Throughout this section, by 'neighborhood' we always mean 'open neighborhood', even if 'open' is dropped.

Briefly speaking, a real valued function f defined on an open subset of $\mathcal{U} \subset \mathbb{R}^n$ will be called semianalytic if it is analytic on an open and dense subset of $\mathcal{U}$, and if the non-analyticity surfaces have also an appropriate analytic structure, and if the restrictions of f to the non-analyticity surfaces are again analytic in an appropriate sense. To introduce our definition, we need a notion of a semianalytic partition of $\mathcal{U}$. Consider in $\mathcal{U}$ a *finite* sequence of equalities and/or inequalities, namely

$$
\begin{aligned}
h_1(x) \quad &\sigma_1 \ 0, \\
&\ldots \\
h_N(x) \quad &\sigma_N \ 0,
\end{aligned}
\tag{89}
$$

where each σ_i is either of the three relations $>$, $<$, $=$, and $\{h_1, \ldots, h_N\}$ is a set of analytic functions defined on a domain containing $\mathcal{U}$. More formally, there is defined a map

$$
\sigma : h = \{h_1, \ldots, h_N\} \ \rightarrow \ \{>, =, <\}
\tag{90}
$$

and in (89) we denoted

$$
\sigma_I := \sigma(h_I),
\tag{91}
$$

---

[12] Except that our definition of the extension of the analytic diffeomorphism group has changed since then.

[13] We thank Christian Fleischhack [28] for drawing our attention to the theory of semianalytic sets.

where the integer I runs from 1 to N. The set of the conditions (89) determines the following subset of $\mathcal{U}$:

$$\mathcal{U}_{h,\sigma} = \{x \in \mathcal{U} : (89)\}. \tag{92}$$

Definition A.1. *Given a finite set h of real valued analytic functions defined on a neighborhood of an open subset $\mathcal{U}$ of $\mathbb{R}^n$, the corresponding semianalytic partition of $\mathcal{U}$ is the set of all the subsets $\mathcal{U}_{h,\sigma} \subset \mathcal{U}$ defined by (89, 92) such that σ is an arbitrary map (90). Given $\mathcal{U}$ and h as above, the partition will be denoted by $\mathcal{P}(\mathcal{U}, h)$.*

Obviously, every semianalytic partition covers $\mathcal{U}$,

$$\mathcal{U} = \bigcup_\sigma \mathcal{U}_{h,\sigma}, \tag{93}$$

where σ runs through all the maps (90). Also,

$$\sigma \neq \sigma' \quad \Rightarrow \quad \mathcal{U}_{h,\sigma} \cap \mathcal{U}_{h,\sigma'} = \emptyset, \tag{94}$$

and a set $\mathcal{U}_{h,\sigma}$ may be empty itself. Another obvious property is that given a semianalytic covering $\mathcal{P}(\mathcal{U}, h)$ and an open subset $\mathcal{V} \subset \mathcal{U}$, the family h of functions defines a semianalytic covering $\mathcal{P}(\mathcal{V}, h)$.

Now, we are in a position to define a semianalytic function:

Definition A.2. *A function $f : \mathcal{U} \to \mathbb{R}^m$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, is called semianalytic if every $x \in \mathcal{U}$ has an open neighborhood $\tilde{\mathcal{U}}$ equipped with a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$, such that for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\tilde{\mathcal{U}}, h)$ there is an analytic function $f_\sigma : \tilde{\mathcal{U}} \to \mathbb{R}^m$, such that*

$$f|_{\tilde{\mathcal{U}}_{h,\sigma}} = f_\sigma|_{\tilde{\mathcal{U}}_{h,\sigma}}, \tag{95}$$

that is, such that f_σ coincides with f on $\tilde{\mathcal{U}}_{h,\sigma}$.

Given a semianalytic function f and a point x in its domain, a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$ which has the properties described in Definition A.2 will be called *compatible* with f at the point x. There are infinitely many semianalytic partitions compatible with a given f at x. Clearly, if $f : \mathcal{U} \to \mathbb{R}^n$ is semianalytic, and $\mathcal{V} \subset \mathcal{U}$ is open, then the restriction function $f_{|\mathcal{V}}$ is semianalytic. Given a semianalytic covering $\mathcal{P}(\tilde{\mathcal{U}}, h)$ compatible with f, and an open subset $\tilde{\mathcal{V}} \subset \tilde{\mathcal{U}} \cap \mathcal{V}$, the semianalytic covering $\mathcal{P}(\tilde{\mathcal{V}}, h)$ is compatible with $f_{|\mathcal{V}}$.

Example. Consider a function $f : \mathbb{R} \to \mathbb{R}$ analytic on every closed interval $[n, n + 1]$. f is semianalytic. A semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$ compatible with f at x_0 is defined for the open interval

$$\tilde{\mathcal{U}} := \,] [x_0] - 1, \, [x_0] + 1 \,[\tag{96}$$

(we denote by $]a, b[$ the open interval bounded by $a, b \in \mathbb{R}$ and by $[a]$ the integer part of a) by the set $\{h_{-1}, h_0, h_1\}$ of functions

$$h_{-1}(x) = x - [x_0] + 1, \quad h_0(x) = x - [x_0], \quad h_1(x) = x - [x_0] - 1. \tag{97}$$

Proposition A.3. *Let $f_1 : \mathcal{U} \to \mathbb{R}$, and $f_2 : \mathcal{U} \to \mathbb{R}^m$ be two semianalytic functions, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Then the functions*

$$\mathcal{U} \ni x \mapsto f_1(x) f_2(x) \in \mathbb{R}^m, \quad \mathcal{U} \ni x \mapsto (f_1(x), f_2(x)) \in \mathbb{R}^{m+1}, \qquad (98)$$

are also semianalytic.

Proof. Let $x \in \mathcal{U}$. Let $\mathcal{P}(\tilde{\mathcal{U}}^{(1)}, h^{(1)})$ be a semianalytic partition compatible with f_1 at x, and $\mathcal{P}(\tilde{\mathcal{U}}^{(2)}, h^{(2)})$ be a semianalytic partition compatible with f_2 at x. The proof becomes obvious if we construct a single semianalytic partition compatible at x with both functions. The natural choice is just the semianalytic partition of the intersection

$$\tilde{\mathcal{U}} := \tilde{\mathcal{U}}^{(1)} \cap \tilde{\mathcal{U}}^{(2)} \qquad (99)$$

defined by the set of functions

$$h := h^{(1)} \cup h^{(2)}. \qquad (100)$$

Indeed, it is enough to notice, that for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\mathcal{U}, h)$ there are some $\tilde{\mathcal{U}}^{(1)}_{h^{(1)},\sigma^{(1)}} \in \mathcal{P}(\tilde{\mathcal{U}}^{(1)}, h^{(1)})$ and $\tilde{\mathcal{U}}^{(2)}_{h^{(2)},\sigma^{(2)}} \in \mathcal{P}(\tilde{\mathcal{U}}^{(2)}, h^{(2)})$ such that

$$\tilde{\mathcal{U}}_{h,\sigma} \subset \tilde{\mathcal{U}}^{(1)}_{h^{(1)},\sigma^{(1)}} \quad \text{and} \quad \tilde{\mathcal{U}}_{h,\sigma} \subset \tilde{\mathcal{U}}^{(2)}_{h^{(2)},\sigma^{(2)}}. \qquad (101)$$

$\square$

It is obvious that if $f : \mathcal{U} \to \mathbb{R}$ is a semianalytic function and it does not vanish on an open set $\mathcal{U}$, then $\frac{1}{f}$ is also semianalytic. This fact will be important in construction of semianalytic partitions of unity. They are useful owing to Proposition A.3.

We turn now to the issue of the morphisms of the semianalytic functions. It is obvious that every analytic map $\phi : \mathcal{U} \to \mathcal{U}'$ between two open subsets $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{U}' \subset \mathbb{R}^{n'}$ pullbacks all the semianalytic functions defined on $\mathcal{U}'$ into semianalytic functions defined on $\mathcal{U}$. The following proposition shows that the same is true for a semianalytic map.

Proposition A.4. *Let $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{U}' \subset \mathbb{R}^{n'}$ be open subsets. Suppose the functions $f' : \mathcal{U}' \to \mathbb{R}^m$ and $\phi : \mathcal{U} \to \mathcal{U}'$ are semianalytic. Then, the composition function $f' \circ \phi : \mathcal{U} \to \mathbb{R}^m$ is semianalytic.*

Proof. The idea of the proof is simple: we construct a suitable partition of $\mathcal{U}$ using the inverse image of a given partition of $\mathcal{U}'$ compatible with f'. The inverse image is not, in general, semianalytic, but we show it can be sub-divided into a semianalytic partition. $\square$

Lemma A.5. *Let $\mathcal{P}(\tilde{\mathcal{U}}', h')$ be a semianalytic partition. Let $\phi : \mathcal{U} \to \tilde{\mathcal{U}}'$ be a semianalytic function, where the subset $\mathcal{U} \subset \mathbb{R}^n$ is open. For every $x_0 \in \mathcal{U}$ there exists an open neighborhood $\tilde{\mathcal{U}}$ and a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ such that for every element of $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$, say $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}}$, there is an element of $\mathcal{P}(\tilde{\mathcal{U}}', h')$, say $\tilde{\mathcal{U}}'_{h',\sigma'}$, such that*

$$\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}} \subset \phi^{-1}\left(\tilde{\mathcal{U}}'_{h',\sigma'}\right). \qquad (102)$$

Proof. Let $x_0 \in \mathcal{U}$. We will construct a partition $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ which satisfies the conclusion. If ϕ is analytic, then the set of the pullbacks of all the functions $h'_I \in h'$ defines a suitable partition of the whole $\mathcal{U}$. In general, ϕ is not analytic. However, it gives rise to a family of analytic functions ϕ_σ defined in some neighborhood $\tilde{\mathcal{U}}$ of x_0 via Definition A.2 (with f being replaced by ϕ). We choose $\tilde{\mathcal{U}}$ small enough, such that all the images $\phi_\sigma(\tilde{\mathcal{U}})$ are contained in the given $\tilde{\mathcal{U}}'$. Hence, consider a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$ compatible with ϕ at x_0, and the corresponding family of analytic functions $\phi_\sigma : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}'$. We define $\tilde{h}$ to be the set of functions formed by (i) all the functions $h'_I \circ \phi_\sigma$ defined by all the functions ϕ_σ and all $h'_I \in h'$, and (ii) all the functions $h_I \in h$. Let us demonstrate that the corresponding semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ satisfies the conclusion. Let $\tilde{\sigma} : \tilde{h} \to \{>, =, <\}$ be an arbitrary map. Denote

$$\sigma := \tilde{\sigma}|_h, \tag{103}$$
$$\sigma' := \tilde{\sigma}|_{\phi_\sigma^*(h')}, \tag{104}$$

where σ in the second line is the one introduced in the first line, and $\phi_\sigma^*(h')$ is the set of pullbacks of the elements of h' by using ϕ_σ^*. Now, it follows directly from (103) (see (89) with h_I and σ_I being themselves as well as being replaced by $\tilde{h}_I$ and $\tilde{\sigma}_I$) that

$$\tilde{\mathcal{U}}_{\tilde{h}, \tilde{\sigma}} \subset \tilde{\mathcal{U}}_{h, \sigma}. \tag{105}$$

On the other hand, the second line (104) means that

$$\phi_\sigma(\tilde{\mathcal{U}}_{\tilde{h}, \tilde{\sigma}}) \subset \tilde{\mathcal{U}}'_{h', \sigma'}. \tag{106}$$

The combination of the last two facts with

$$\phi_\sigma|_{\tilde{\mathcal{U}}_{h, \sigma}} = \phi|_{\tilde{\mathcal{U}}_{h, \sigma}} \tag{107}$$

concludes the proof of the lemma.   □

We go back to the proof of the proposition. Given $x_0 \in \mathcal{U}$, consider the point $\phi(x_0)$ and a partition $\mathcal{P}(\tilde{\mathcal{U}}', h')$ compatible with the function f' at $\phi(x_0)$. Let $\mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ be a partition provided by the lemma. For every $\tilde{\mathcal{U}}_{\tilde{h}, \tilde{\sigma}} \in \mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$ use the pair σ, σ' defined by (103,104). The function $f'_{\sigma'} \circ \phi_\sigma$ is the wanted analytic extension of $f' \circ \phi|_{\tilde{\mathcal{U}}_{\tilde{h}, \tilde{\sigma}}}$.   □

In general, the inverse of an invertible semianalytic function is not necessarily semianalytic. However, a carefully formulated set of assumptions ensures the semianalyticity of the inverse.

Proposition A.6. *Let* $\phi : \mathcal{U} \to \mathcal{U}'$ *be a semianalytic and bijective function, where* $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ *are open. Suppose that for every* $x_0 \in \mathcal{U}$ *there exists a semianalytic partition* $\mathcal{P}(\tilde{\mathcal{U}}, h)$ *compatible with* ϕ *at* x_0, *and such that for every* $\tilde{\mathcal{U}}_{h, \sigma} \in \mathcal{P}(\tilde{\mathcal{U}}, h)$ *the restriction* $\phi|_{\tilde{\mathcal{U}}_{h, \sigma}}$ *is extendable to an analytic, injective function* $\phi_\sigma : \tilde{\mathcal{U}} \to \mathcal{U}'$, *such that:* (i) $\phi_\sigma(\tilde{\mathcal{U}})$ *is an open subset of* $\mathbb{R}^n$, *and* (ii) *the inverse* $\phi_\sigma^{-1} : \phi_\sigma(\tilde{\mathcal{U}}) \to \tilde{\mathcal{U}}$ *is analytic. Then,* ϕ^{-1} *is semianalytic.*

Proof. Given a point $x_0' \in \mathcal{U}'$, let $\mathcal{P}(\tilde{\mathcal{U}}, h)$ be a partition compatible with ϕ at $x_0 = \phi^{-1}(x_0') \in \mathcal{U}$. Suppose $\mathcal{P}(\tilde{\mathcal{U}}, h)$ and ϕ satisfy the assumptions. We have to construct a semianalytic partition of a neighborhood $\tilde{\mathcal{U}}'$ of x_0' compatible with ϕ^{-1}. We choose $\tilde{\mathcal{U}}'$ such that all the inverse functions ϕ_σ^{-1} are well defined, namely

$$\tilde{\mathcal{U}}' = \bigcap_\sigma \phi_\sigma(\tilde{\mathcal{U}}). \tag{108}$$

Mapping with ϕ the partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$ we get a partition of $\tilde{\mathcal{U}}'$ which consists of the sets

$$\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}', \tag{109}$$

given by all the elements $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\tilde{\mathcal{U}}, h)$. For every set $\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'$ we have

$$\phi^{-1}|_{\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'} = \phi_\sigma^{-1}|_{\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'}, \tag{110}$$

where ϕ_σ^{-1} is the analytic function provided by the assumptions. That would be sufficient for the semianalyticity of ϕ^{-1} if the constructed partition were semianalytic. We do not know if it is the case, though. However, we will subdivide the partition in such a way that the result is a semianalytic partition without any doubt. Establishing that refined partition will be enough to complete the proof by referring to (110). The needed semianalytic partition is defined in the following way. First, we fix a subset $\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'$ and use the corresponding analytic function ϕ_σ^{-1} to pullback all the functions $h_I \in h$ from $\tilde{\mathcal{U}}$ onto $\tilde{\mathcal{U}}'$. Denote the resulting set of analytic, real valued functions defined on $\tilde{\mathcal{U}}'$ by $\phi_\sigma^{-1*}h$, and consider the corresponding semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}', \phi_\sigma^{-1*}h)$. It is easy to see that

$$\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}' \in \mathcal{P}(\tilde{\mathcal{U}}', \phi_\sigma^{-1*}h). \tag{111}$$

Next, enlarge the set $\phi_\sigma^{-1*}h$ corresponding to a given σ by taking the union with respect to all the σs (90),

$$h' = \bigcup_\sigma \phi_\sigma^{-1*}h. \tag{112}$$

Consider the semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}', h')$ defined by h'. This partition just divides every $\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'$ into smaller subsets of $\tilde{\mathcal{U}}'$, that is it consists of subsets of the sets $\phi\left(\tilde{\mathcal{U}}_{h,\sigma}\right) \cap \tilde{\mathcal{U}}'$. This concludes the proof. $\square$

Corollary A.7. *Suppose $\phi : \mathcal{U} \to \mathcal{U}'$ is a diffeomorphism of the differentiability class C^m, where $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open and $m > 0$. If ϕ is semianalytic, then so is $\phi^{-1} : \mathcal{U}' \to \mathcal{U}$.*

Proof. Let us assume that ϕ satisfies the assumptions made in Corollary A.7 and consider an arbitrary point x_0 in the domain $\mathcal{U}$. Since ϕ is semianalytic, we can find: (a) a neighborhood $\tilde{\mathcal{U}}$ of x_0, (b) a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, h)$, and (c) for every $\tilde{\mathcal{U}}_{h,\sigma} \in \mathcal{P}(\tilde{\mathcal{U}}, h)$ an analytic function ϕ_σ defined on $\tilde{\mathcal{U}}$, which coincides with ϕ on $\tilde{\mathcal{U}}(h, \sigma)$.

It would be sufficient to show that the data (a)–(c) can be chosen in such a way that every function ϕ_σ of (c) has a non-degenerate derivative $D\phi_\sigma$ at every point of $\tilde{\mathcal{U}}$. Then the hypothesis of Proposition A.6 would be satisfied. Certainly the derivative of ϕ is nowhere degenerate in $\mathcal{U}$. Therefore, for every function ϕ_σ of (c) there is an open subset of $\tilde{\mathcal{U}}$ such that the derivative of ϕ_σ is non-degenerate. The problem is that the subsets of points on which the derivatives are non-degenerate may be too small. They may be smaller than $\tilde{\mathcal{U}}$, and some of them may even not contain the point x_0 at all. Therefore the data (a)–(c) is not yet sufficient to apply Prop. A.6.

We therefore define new data (a')–(c') given by shrinking the neighborhood $\tilde{\mathcal{U}}$ appropriately. For every $\tilde{U}_{h,\sigma} \in \mathcal{P}(\tilde{U}, h)$ consider the subset $S_\sigma \subset \tilde{U}$ of points such that the function ϕ_σ has a nondegenerate derivative. Note that S_σ contains the completion $\overline{\tilde{U}_{h,\sigma}}$. Indeed, it follows from the continuity of $D\phi$ and $D\phi_\sigma$. As a new $\tilde{\mathcal{U}}'$ we take,

$$\tilde{\mathcal{U}}' := \Big(\bigcap_{\sigma:x_0\in\overline{\tilde{U}_{h,\sigma}}} S_\sigma \Big) \setminus \Big(\bigcup_{\sigma:x_0\notin\overline{\tilde{U}_{h,\sigma}}} \overline{\tilde{U}_{h,\sigma}} \Big).$$

The set $\tilde{\mathcal{U}}'$ constitutes new data (a'). A new partition (b') and functions (c') are given just by restricting the previous (b),(c) to $\tilde{\mathcal{U}}'$. (a')–(c') then fulfill the assumptions of Prop. A.6.   □

Once we have generalized the notion of analytic structure into the notion of the semi-analytic structure, it is natural to introduce new partitions by relaxing in Definition A.1 the assumption that the functions constituting the set h are analytic, and replace it by a condition that they be semianalytic. Let us do it, apply the same notation as in Definition A.1 to a finite set of *semi*analytic functions h and call the result a semi-semianalytic partition.

Given any partition of a set into subsets, another partition is called finer if every element of the first partition is a finite union of elements of the second partition.

Lemma A.8. *Suppose $\mathcal{P}(\mathcal{U}, h)$ is a semi-semianalytic partition of an open $\mathcal{U} \subset \mathbb{R}^n$. Then, every $x \in \mathcal{U}$ has a neighborhood $\tilde{\mathcal{U}}$ which admits a semianalytic partition finer than $\mathcal{P}(\tilde{\mathcal{U}}, h)$.*

Proof. Let $x_0 \in \mathcal{U}$. There is a neighborhood $\tilde{\mathcal{U}}$ of x_0 which admits a semianalytic partition $\mathcal{P}(\tilde{\mathcal{U}}, f)$ compatible with all the (semianalytic functions) elements of h. As before, we start with collecting all the analytic functions available. Firstly, all the elements $f_I \in f$ are analytic functions defined on $\tilde{\mathcal{U}}$. Secondly, for every assignment $\sigma : f \to \{>, =, <\}$, every element $h_I \in h$ defines an analytic function $h_{I\sigma}$. Given σ denote the set of the functions $h_{I\sigma}$ such that $h_I \in h$ is arbitrary, by h_σ. The resulting set of the analytic functions is

$$\tilde{h} := f \cup \bigcup_\sigma h_\sigma. \tag{113}$$

Our candidate for a semianalytic partition of $\tilde{\mathcal{U}}$ compatible with $\mathcal{P}(\mathcal{U}, h)$ is the semianalytic partition defined by the set of functions $\tilde{h}$. Consider an arbitrary

$$\tilde{\sigma} : \tilde{h} \to \{>, =, <\}, \tag{114}$$

and the corresponding set $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}} \in \mathcal{P}(\tilde{\mathcal{U}}, \tilde{h})$. We have to point out an element $\mathcal{U}_{h,\sigma'}$ which contains $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}}$. It is defined as follows. Consider

$$\sigma := \tilde{\sigma}|_f. \tag{115}$$

Using this σ select another subset of $\tilde{h}$, namely h_σ. The restriction

$$\tilde{\sigma}|_{h_\sigma}, \tag{116}$$

defines naturally an assignment $\sigma' : h \rightarrow \{>, =, <\}$, namely

$$\sigma'(h_I) := \tilde{\sigma}(h_{I\sigma}). \tag{117}$$

It is easy to check that $\tilde{\mathcal{U}}_{\tilde{h},\tilde{\sigma}} \subset \mathcal{U}_{h,\sigma'}$. $\square$

Finally, our interest in the semi-analytic sets is a consequence of a certain strong result of that theory ([2], see Prop. 2.10 in [3]) which we translate now into the terms of the semianalytic partitions.

We call a semianalytic partition analytic partition if every element of the partition is a connected, analytic submanifold.

The result we are referring to reads:

Proposition A.9. *For every semianalytic partition $\mathcal{P}(\mathcal{U}, h)$ of an open $\mathcal{U} \subset \mathbb{R}^n$, every point $x \in \mathcal{U}$ has a neighborhood $\tilde{\mathcal{U}}$ which admits an analytic partition finer than $\mathcal{P}(\tilde{\mathcal{U}}, h)$.*

A.2. Semianalytic manifolds and submanifolds. In this subsection, Σ is an n dimensional differential manifold. Henceforth we will be assuming that Σ and all the considered functions are of a differentiability class C^m, where $m > 0$.

By analogy with the definitions of an analytic structure, analytic function, and analytic submanifold, we introduce now natural semianalytic generalizations. The generalization is possible due to Propositions A.3, A.4, A.6 of the previous subsection.

We denote below an atlas of Σ by $\{(\mathcal{U}_I, \chi_I)\}_{I \in \mathcal{I}}$, where $\mathcal{I}$ is some labeling set, $\{\mathcal{U}_I\}_{I \in \mathcal{I}}$ is an open covering of Σ, and $\{\chi_I\}_{I \in \mathcal{I}}$ is a family of diffeomorphisms $\chi_I : \mathcal{U}_I \rightarrow \mathcal{U}'_I \subset \mathbb{R}^n$.

Definition A.10. *An atlas $\{(\mathcal{U}_I, \chi_I)\}_{I \in \mathcal{I}}$ of Σ is called semianalytic if for every pair $I, J \in \mathcal{I}$ the map*

$$\chi_J \circ \chi_I^{-1} : \chi_I(\mathcal{U}_I \cap \mathcal{U}_J) \rightarrow \chi_J(\mathcal{U}_I \cap \mathcal{U}_J) \tag{118}$$

is semianalytic. The diffeomorphisms χ_I are called semianalytic charts. A semianalytic structure on Σ is a maximal semianalytic atlas. A semianalytic manifold is a differential manifold endowed with a semianalytic structure.

Definition A.11. *Given two semianalytic manifolds Σ and Σ', a map $f : \Sigma \rightarrow \Sigma'$ is called semianalytic if for every semianalytic chart χ_I of Σ, and every semianalytic chart $\chi'_{I'}$ of Σ' the function $\chi'_{I'} \circ f \circ \chi_I^{-1}$ (whenever the composition can be applied) is semianalytic.*

In particular, if

$$\Sigma' = \mathbb{R}^{n'} \tag{119}$$

and the semianalytic structure is the natural one defined by the atlas $\{(\mathbb{R}^{n'}, \text{id})\}$, then the map f is a semianalytic function defined on Σ.

Definition A.12. *A semianalytic submanifold of a semianalytic manifold Σ is a subset $S \subset \Sigma$ such that for every $x \in S$, there is a semianalytic chart χ_I defined in a neighborhood $\mathcal{U}_I$ of x, such that*

$$\chi_I(S \cap \mathcal{U}_I) = \{(x^1, \dots, x^n) \in \mathbb{R}^n \ : \ x^1 = \dots = x^{n-n'} = 0,$$
$$0 < x^{n-n'+1} < 1, \dots, 0 < x^n < 1\}, \tag{120}$$

where n' is a non-negative integer, $n' \leq n$, and n' is called the dimension of S.

Definition A.13. *An n' dimensional semianalytic submanifold with boundary of Σ is a subset $S \subset \Sigma$ such that for every $x \in S$, there is a semianalytic chart χ_I defined in a neighborhood $\mathcal{U}_I$ of x, such that either* (120) *or*

$$\chi_I(S \cap \mathcal{U}_I) = \{(x^1, \dots, x^n) \in \mathbb{R}^n \ : \ x^1 = \dots = x^{n-n'} = 0,$$
$$0 \leq x^{n-n'+1} < 1, 0 < x^{n-n'+2} < 1, \dots, 0 < x^n < 1\}. \tag{121}$$

It is also assumed that the set of points x such that (121) *is not empty.*

The key property of the semianalytic submanifolds crucial in our work is:

Proposition A.14. *Let S_1 and S_2 be two semianalytic submanifolds of a semianalytic manifold Σ. Suppose $x \in S_1 \cap S_2$. Then, there is an open neighborhood $\mathcal{W}$ of x in Σ, such that $\mathcal{W} \cap S_1 \cap S_2$ is a finite, disjoint union of connected semianalytic submanifolds.*

Remark. What is crucial for us in the conclusion of Proposition A.14 is the *finiteness* of the partition and the *connectedness* of its elements. After all, an infinite set of disjoint, embedded intervals may also form a single submanifold, disconnected though. Those two properties simultaneously hold due to the (semi) analyticity.

Proof. For every point $x \in S_1 \cap S_2$, there is a neighborhood $\mathcal{W}$ which can be mapped by a semianalytic chart into an open subset $\mathcal{U} \subset \mathbb{R}^n$. The intersection $\mathcal{W} \cap S_1 \cap S_2$ is mapped into a subset of $\mathcal{U}$ described by a finite family of equalities of the form (89) defined by some fixed family of semianalytic functions h_I and relations $\sigma_I = $ '=' (the definition of a semianalytic submanifold involves also inequalities, however $\mathcal{W}$ can be chosen such that the latter ones are satisfied at every point in $\mathcal{W}$; we are assuming this is the case). Hence, the intersection is an element of the semi-semianalytic partition defined by the family of the semianalytic functions h_I. Due to Lemma A.8, if we choose the neighborhood $\mathcal{W}$ of the point x appropriately, then the intersection $\mathcal{W} \cap S_1 \cap S_2$ is a finite union of elements of certain semianalytic partition. Finally, via the result quoted in the previous subsection, the neighborhood $\mathcal{W}$ can be chosen such that every element of a semianalytic partition of the image $\mathcal{U}$ is a finite, disjoint union of connected analytic submanifolds. Their inverse image by the chart defines the decomposition of the intersection $\mathcal{W} \cap S_1 \cap S_2$ into semianalytic submanifolds. □

In the paper we are using extensively two particular classes of submanifolds: edges and faces.

Definition A.15. *A semianalytic edge is a connected, 1-dimensional semianalytic submanifold of Σ with 2-point boundary.*

Definition A.16. *A face is a connected, codimension 1 semianalytic submanifold of Σ whose normal bundle is equipped with an orientation.*

The property of the semianalytic structures which distinguishes them so much from the analytic ones is local character of the spaces of the semianalytic functions and semianalytic diffeomorphisms. That feature is guaranteed by the existence of a partition of unity compatible with an arbitrary open covering. We formulate this fact precisely now, in the form we refer to in the proof of our main theorem:

Proposition A.17. *Suppose $\mathcal{W} \subset \Sigma$ is a compact subset. Let $\mathcal{U}_I \subset \Sigma$, $I = 1, \ldots, N$, be a family of open sets which covers $\mathcal{W}$. There exists a family of C^m semianalytic functions $\phi_I : \Sigma \to \mathbb{R}$, $I = 1, \ldots, N$ such that for every I,*

$$\text{supp}\, \phi_I \subset \mathcal{U}_I \tag{122}$$

and

$$\sum_I \phi_I|_{\mathcal{W}} = 1. \tag{123}$$

Proof. The proof is standard owing to the following two properties of the semianalytic functions:

(i) For every open ball in $\mathbb{R}^D$, there is a C^m semianalytic function greater than zero at every point inside the ball and identically zero everywhere else.
(ii) If f is a nowhere vanishing C^m semianalytic function then so is $1/f$. □

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