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Consistency check on volume and triad operator quantization in loop quantum gravity: I

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Abstract

The volume operator plays a pivotal role for the quantum dynamics of loop quantum gravity (LQG). It is essential to construct triad operators that enter the Hamiltonian constraint and which become densely defined operators on the full Hilbert space, even though in the classical theory the triad becomes singular when classical GR breaks down. The expression for the volume and triad operators derives from the quantization of the fundamental electric flux operator of LQG by a complicated regularization procedure. In fact, there are two inequivalent volume operators available in the literature and, moreover, both operators are unique only up to a finite, multiplicative constant which should be viewed as a regularization ambiguity. Now on the one hand, classical volumes and triads can be expressed directly in terms of fluxes and this fact was used to construct the corresponding volume and triad operators. On the other hand, fluxes can be expressed in terms of triads and triads can be replaced by Poisson brackets between the holonomy and the volume operators. Therefore one can also view the holonomy operators and the volume operator as fundamental and consider the flux operator as a derived operator. In this paper we mathematically implement this second point of view and thus can examine whether the volume, triad and flux quantizations are consistent with each other. The results of this consistency analysis are rather surprising. Among other findings we show the following. (1) The regularization constant can be uniquely fixed. (2) One of the volume operators can be ruled out as inconsistent. (3) Factor ordering ambiguities in the definition of triad operators are immaterial for the classical limit of the derived flux operator. The results of this paper show that within full LQG triad operators are consistently quantized.

In this paper we merely present ideas and the results of the consistency check. In a companion paper we supply detailed proofs.

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1. Introduction

The major unresolved problem in loop quantum gravity (LQG) (see [1] for books and [2] for reviews) is the satisfactory implementation of quantum dynamics, which in turn is governed by the Wheeler–DeWitt quantum constraint [3, 4], also called the Hamiltonian constraint. The volume operator [5, 6] plays a pivotal role for the very definition of the Hamiltonian constraint, because with its help one can quantize triad functions which enter the classical expression for the Hamiltonian constraint. In particular, one takes advantage of a Poisson bracket identity between the triads and the Poisson brackets between the Ashtekar connection and the classical scalar volume function. Namely, one uses the methods of canonical quantization, that is, the axioms of quantum mechanics, according to which Poisson brackets between classical functions are turned into commutators between the corresponding operators divided by $i\hbar$, at least to lowest order in $\hbar$. Quite surprisingly, the Hamiltonian constraint operator and similarly also length operators [7] are then densely defined on the full kinematical Hilbert space of LQG, although the classical triad becomes singular in physically relevant situations such as black holes or the Big Bang.

While playing such a distinguished role for the most important open problem of LQG, the volume operator and thus the derived triad operators have never been critically examined concerning their physical correctness and mathematical consistency. By the first we mean that it has never been shown within full LQG that the volume operator has the correct classical limit with respect to suitably chosen kinematical semiclassical states, for example those constructed in [8][3].

By the second we mean the following: the fundamental kinematical algebra $\mathfrak{A}$ on which LQG is based is the holonomy–flux algebra [10], and its representation theory together with background independence leads to a unique kinematical Hilbert space [11]. Now classically the volume and triad can be written as limits of functions of the flux. To implement them at the quantum level, one has to go through a complicated regularization procedure and to take the limit. It is surprising that the resulting operators are densely defined at all, and in fact have a discrete spectrum because they are highly non-polynomial expressions as functions of fluxes. This is the pay-off for background independence, since in background-dependent formulations, such as the standard Fock representations, these operators are too singular. On the other hand, since there is little experience with non-Fock representations, it is not at all clear whether the corresponding operators have anything to do with their classical counterpart. In fact, there are at least two ambiguities already at the level of the volume operator. First of all, there are in fact two unitarily inequivalent volume operators [5, 6] which come from two, *a priori* equally justified background independent regularization techniques. We will denote them as Rovelli–Smolin (RS) and Ashtekar–Lewandowski (AL) volumes respectively for the rest of this paper. Secondly, both volume operators are anyway only determined

[3] This has been achieved, so far, only within a certain approximation [9] which essentially consists of replacing $SU(2)$ by $U(1)^3$. The necessary calculations in the non-Abelian case are much more complicated for two reasons: (1) the spectrum of the volume operator is not available in analytical form and (2) the semiclassical analysis is calculationally more difficult. However, work is now in progress in order to fill this gap.

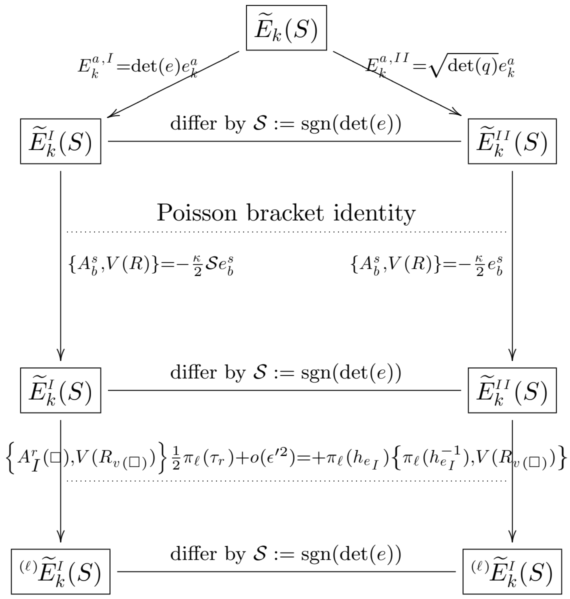

Figure 1. The classical alternative flux $\widetilde{E}_k(S)$ can be defined in two different ways due to the two possible definitions of the densitized triad E^a_k.

up to a multiplicative regularization constant C_{reg} [12] which remains undetermined when taking the limit, quite similar to finite regularization constants that appear in counter terms of standard renormalization of ordinary QFT. The ambiguity is further enhanced by factor ordering ambiguities once we consider triad operators. These ambiguities are parametrized by a spin quantum number $\ell = 1/2, 1, 3/2, \ldots$.

1.1. Logic of the consistency check

In this paper we will be able to remove all those ambiguities by the following consistency check. First we start by reviewing the regularization and definition of the fundamental flux operator $\widehat{E}_k(S)$ in section 2 for the benefit of the reader and in order to make the comparison with the alternative quantization easier. As we mentioned above, the volume and triad can be considered as functions of the fluxes. But the converse is also true: the fluxes can be written in terms of triads, as we explain in detail in section 3. We denote the classical flux expressed in terms of triads by $\widetilde{E}_k(S)$. Moreover, there exist two possible canonical transformations for defining the fluxes in terms of triads, both leading to the same symplectic structure. Hence, we obtain $\widetilde{E}^{\mathrm{I}}_k(S)$ and $\widetilde{E}^{\mathrm{II}}_k(S)$ which differ classically by a sign factor $\mathcal{S} := \mathrm{sgn}(\det(e))$ (see also figure 1). In order to promote $\widetilde{E}^{\mathrm{I}}_k(S)$ and $\widetilde{E}^{\mathrm{II}}_k(S)$ to well-defined operators later on, we have to use the Poisson bracket identity and replace triads by their corresponding Poisson bracket between the Ashtekar connection and the classical scalar volume function. Since the connection cannot be quantized directly, we additionally have to replace the connections by their corresponding holonomies. As already mentioned above, when working with holonomies we obtain an ambiguity labelled by a spin quantum number $\ell = k/2$, where $k \in \mathbb{N}$; thus we obtain ${}^{(\ell)}\widetilde{E}^{\mathrm{I}}_k(S)$ and ${}^{(\ell)}\widetilde{E}^{\mathrm{II}}_k(S)$, respectively. These steps can be found in section 3.1 and are

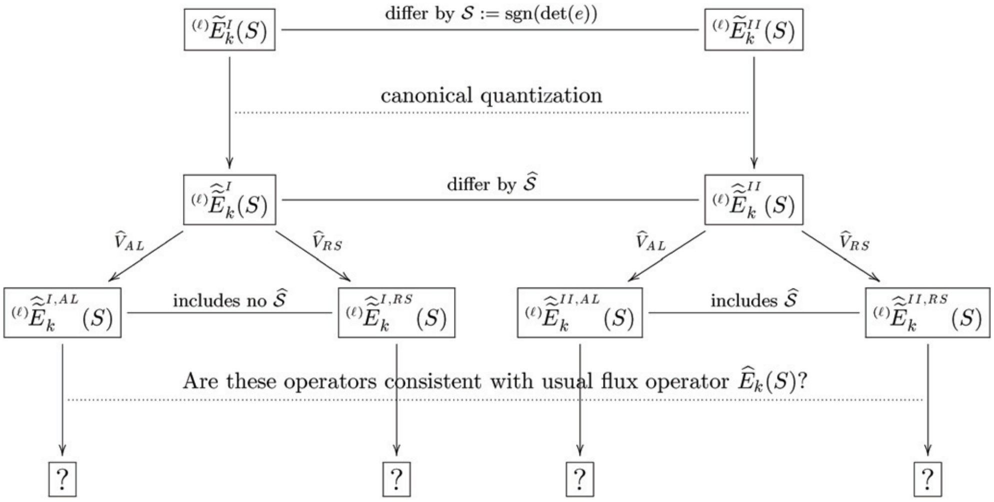

Figure 2. When quantizing the two possible alternative classical fluxes ${}^{(\ell)}\widetilde{E}^{\rm I}_k(S)$ and ${}^{(\ell)}\widetilde{E}^{\rm II}_k(S)$ respectively, we end up with four different operators since in each of the two cases we can either use $\widehat{V}_{\rm AL}$ or $\widehat{V}_{\rm RS}$.

also illustrated graphically in figure 1. As can be seen in section 4, by applying canonical quantization to ${}^{(\ell)}\widetilde{E}^{\rm I}_k(S)$ and ${}^{(\ell)}\widetilde{E}^{\rm II}_k(S)$ we obtain the operators ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm I}_k(S)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm II}_k(S)$. Since their classical counterparts differ by a sign factor $\mathcal{S}$ these operators will differ by the corresponding sign operator $\widehat{\mathcal{S}}$. The quantization of this operator, so far not introduced in the literature, is described in section 6.2. We will show that precisely this operator plays a key role when questions about consistency are asked.

Since in LQG two different volume operators $\widehat{V}_{\rm AL}$ and $\widehat{V}_{\rm RS}$ exist, we have another freedom in defining the alternative flux operator. We could choose either $\widehat{V}_{\rm AL}$ or $\widehat{V}_{\rm RS}$ to promote the classical volume function to a volume operator. Hence, quantum mechanically, we obtain four different operators: ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm I,AL}_k(S)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm I,RS}_k(S)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm II,AL}_k(S)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\rm II,RS}_k(S)$; also see figure 2 for a graphical demonstration. The main question that is addressed in this paper is whether these four operators are consistent with the usual flux operator. The alternative operators can be considered as functions of holonomy operators and the volume operator. This forces us to discuss the topic of factor ordering ambiguities, which is done in section 4.2. (For details concerning the factor ordering ambiguities, see also [20].) Moreover, we know that there exists a regularization constant $C_{\rm reg}$ for the volume operator and we have a freedom in choosing a particular representation weight ℓ of the holonomy operators. We will show in this paper that it is true that there exists a unique regularization constant $C_{\rm reg}$ and a factor ordering such that the corresponding alternative flux operator agrees with the usual flux operator and such that it is independent of the chosen representation weight ℓ. The latter should better be possible at least in the classical limit of large volume, as otherwise the inescapable conclusion would be that the volume operator is inconsistently quantized[4], because the usual flux operator has no ℓ-dependence at all.

[4] In contrast, the triad operator follows from the volume operator by the axioms of quantum mechanics, namely that Poisson brackets be replaced by commutators divided by $i\hbar$, and therefore it is not possible that the source of a possible problem is in the quantization of the triad operator.

The discussion of our results for the four alternative flux operators can be found in sections 5 and 6. Finally, in section 7, we summarize and conclude. For the reader just interested in the results we summarize them here first.

(1) The RS volume operator is inconsistent with the flux operator; the AL volume operator is consistent.
(2) $C_{\mathrm{reg}} = 1/48$ can be uniquely fixed; there is no other choice which is semiclassically acceptable. Remarkably, this is precisely the value that was obtained in [6] by a completely different argument.
(3) The choice of ℓ plays no role semiclassically; in fact it drops out of the final expression for the alternative operator altogether. Therefore the aforementioned factor ordering ambiguity is absent as far as the flux operator is concerned.
(4) There is yet one more ambiguity in LQG which is already present classically: classically, it is possible to consider the electric field either as a two-form or as a pseudo two-form. The corresponding sign of the determinant of the triad is then either encoded in the electric field or in the conjugate connection. One can take either point of view without affecting the symplectic structure of the theory. We will be able to show that one *must* consider the electric field as a pseudo two-form, otherwise the alternative flux operator becomes the zero operator!
(5) As expected, the alternative and fundamental flux operators agree for all values of the Immirzi parameter [14], hence it cannot be fixed by our analysis, which is good because it has been fixed already by arguments coming from quantum black hole physics [15].
(6) The calculations in this paper make extensive use of certain advances in technology concerning the matrix elements of the volume operator [17]. Thus, our calculations provide an independent check of [17] as well.
(7) The factor ordering of the alternative flux operator is unique if one insists on the principle of minimality[5].

These results show that, instead of taking holonomies and fluxes as fundamental operators, one could use holonomies and volumes as fundamental operators, in the sense that any arbitrary operator usually expressed in terms of holonomy and flux operators could now be expressed in terms of holonomy and volume operators by using the alternative definition of the flux operator introduced here. It also confirms that the method to quantize the triad developed in [3] is mathematically consistent.

All the technical details and tools that are needed to perform this consistency check are provided in our companion paper [20].

2. Review of the usual flux operator

In LQG the classical electric flux $E_k(S)$ through a surface S is the integral of the densitized triad E^a_k over a 2-surface S

$$E_k(S) = \int_S E^a_k n^S_a, \tag{2.1}$$

[5] By this we mean that given a function f with self-adjoint quantization $\hat{f}$ and a multiplication operator g for which g^{-1} is defined everywhere on the Hilbert space, we may always consider instead $\hat{f}' = (g\hat{f}g^{-1} + \overline{g^{-1}\hat{f}g})/2$ as the operator corresponding to f if it has self-adjoint extensions as well. By minimalistic we mean the choice $g = 1$. This issue is always present even in ordinary quantum mechanics; however, it is usually not mentioned because one usually deals with polynomials and $g \neq 1$ would destroy polynomiality. In GR the expressions are generically non-polynomial from the outset and thus polynomiality is not available as a simplistic criterion. However, we may still insist on a minimal number of such factor ordering ambiguities.

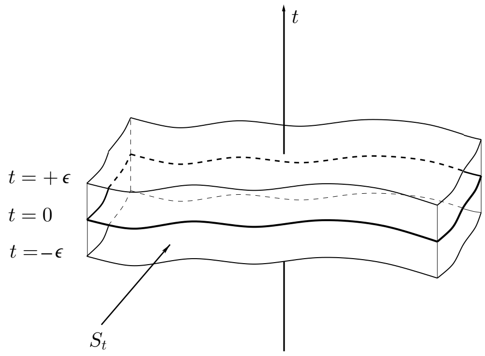

Figure 3. Smearing of the surface S into the third dimension. We obtain an array of surfaces S_t labelled by the parameter t with $t \in \{-\epsilon, +\epsilon\}$. The original surface S is associated with $t = 0$.

where n_a^S is the conormal vector with respect to the surface S. In order to define a corresponding flux operator in the quantum theory, we have to regularize the classical flux and then define the action of the operator on an arbitrary spin network function (SNF) $T_{\gamma,\vec{j},\vec{m},\vec{n}} : G^{|E(\gamma)|} \to \mathbb{C}$, where G is the corresponding gauge group, namely $SU(2)$ in our case, as the action of the regularized expression, denoted by $E_k^\epsilon(S)$ in the limit where the regularization parameter ϵ is removed:

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} := \mathrm{i}\hbar \lim_{\epsilon\to 0} \left\{E_k^\epsilon(S), T_{\gamma,\vec{j},\vec{m},\vec{n}}\right\}. \tag{2.2}$$

Here the limit is to be understood in the following way. The Poisson bracket on the right-hand side of equation (2.2) is calculated by viewing $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ as functions of smooth connections. After taking $\epsilon \to 0$, one extends the result to functions of distributional connections and thus ends up with an operator defined on the Hilbert space of LQG.

Classically, we have

$$\left\{E_k^\epsilon(S), T_{\gamma,\vec{j},\vec{m},\vec{n}}(\{h_e(A)\}_{e\in E(\gamma)})\right\} = \sum_{e\in E(\gamma)} \left\{E_k^{a,\epsilon}, (h_e)_{AB}\right\} \frac{\partial T_{\gamma,\vec{j},\vec{m},\vec{n}}}{\partial (h_e)_{AB}}. \tag{2.3}$$

Here $(h_e)_{AB}$ denotes the $SU(2)$-holonomy. The regularization can be implemented by smearing the 2-surface S into the third dimension, shown in figure 3, so that we get an array of surfaces S_t. The surface associated with $t = 0$ is our original surface S,

$$E_k^\epsilon(S) := \frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon} \mathrm{d}t\, E_k(S_t). \tag{2.4}$$

In order to derive the action of the flux operator on an arbitrary SNF, we would have to analyse the Poisson brackets between the flux and every possible SNF. Fortunately, it turns out that each edge belonging to the associated graph γ of $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ can be classified as (i) up, (ii) down, (iii) in and (iv) out with respect to the surface S. (See figure 4 for a graphical illustration.) An arbitrary $T_{\gamma,\vec{j},\vec{m},\vec{n}}$ contains then a particular amount of edges of each type. Accordingly, if we have the knowledge of the Poisson brackets between the flux and any of these types of edges, we will be able to derive the Poisson brackets between E_k^ϵ and any arbitrary $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. The calculation of the regularized Poisson bracket can be found, e.g., in the second part of [1]. After having removed the regulator we end up with the following action of the flux operator on a SNF $T_{\gamma,\vec{j},\vec{m},\vec{n}}$:

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} = \frac{\mathrm{i}}{2}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S)\left[\frac{\tau_k}{2}\right]_{AB} \frac{\partial T_{\gamma,\vec{j},\vec{m},\vec{n}}(h_{e'})_{e'\in E(\gamma)}}{\partial (h_e)_{AB}}, \tag{2.5}$$

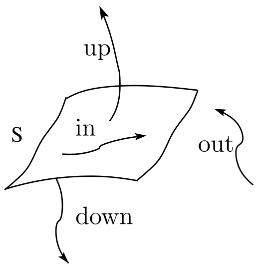

Figure 4. Edges of type up, down, in and out with respect to the surface S.

where τ_k is related to the Pauli matrices by $\tau_k := -\mathrm{i}\sigma_k$. The sum is taken over all edges of the graph γ associated with $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. The function $\epsilon(e, S)$ can take the values $\{-1, 0, +1\}$ depending on the type of edge that is considered. It is +1 for edges of type up, -1 one for down and 0 for edges of type in or out.

By introducing right invariant vector fields X^e_k, defined by $\left(X^e_k f\right)(h) := \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{t\tau_k} h)|_{t=0}$, we can rewrite the action of the flux operator as

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} = \frac{\mathrm{i}}{4}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e, S)X^e_k T_{\gamma,\vec{j},\vec{m},\vec{n}}. \tag{2.6}$$

Note that the right invariant vector fields fulfil the following commutator relations, $\left[X^r_e, X^s_e\right] = -2\epsilon_{rst}X^t_e$. By means of introducing the self-adjoint right invariant vector field $Y^k_e := -\frac{\mathrm{i}}{2}X^k_e$, we achieve commutator relations for Y^k_e that are similar to that of the angular momentum operators in quantum mechanics $\left[Y^r_e, Y^s_e\right] = \mathrm{i}\epsilon_{rst}Y^t_e$. Therefore, we also can describe the action of $\widehat{E}_k(S)$ by the action of the self-adjoint right invariant vector field Y^k_e on $T_{\gamma,\vec{j},\vec{m},\vec{n}}$:

$$\widehat{E}_k(S)T_{\gamma,\vec{j},\vec{m},\vec{n}} = -\frac{1}{2}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e, S)Y^k_e T_{\gamma,\vec{j},\vec{m},\vec{n}}. \tag{2.7}$$

3. Idea of the alternative flux operator

Our starting point will be the Poisson bracket of the Ashtekar connection A^j_a and the densitized triad E^b_k given by

$$\left\{A^j_a(x), E^b_k(y)\right\} = \delta^3(x, y)\delta^a_b\delta^k_j \tag{3.1}$$

which we take as fundamental. In order to go from the ADM-formalism to the formulation in terms of Ashtekar variables, one uses a canonical transformation. There exist two possibilities in choosing such a canonical transformation that both lead to the Poisson bracket above. These two possibilities are

$$\begin{array}{lll} \mathrm{I} & A^j_a = \Gamma^j_a + \gamma\,\mathrm{sgn}(\det(e))K^j_a, & E^a_k = \frac{1}{2}\epsilon_{krs}\epsilon^{abc}e^r_b e^s_c \\ \mathrm{II} & A^j_a = \Gamma^j_a + \gamma K^j_a, & E^a_k = \frac{1}{2}\epsilon_{krs}\epsilon^{abc}e^r_b e^s_c\,\mathrm{sgn}(\det(e)). \end{array} \tag{3.2}$$

Here, Γ^j_a is the $SU(2)$-spin connection, $K_{ab} = K^j_a e^j_b$ the extrinsic curvature (when the Gauss constraint holds) and γ the Immirzi parameter. Recall again the definition of the regularized classical flux $E^\epsilon_k(S)$ in equations (2.4) and (2.1), respectively. Now the idea of defining an alternative flux is motivated by the fact that the densitized triad E^a_k occurring in equation (2.1) can be expressed in terms of the triads. Due to the two possible canonical transformations, we have also two possibilities in defining an alternative densitized triad:

$$E^a_k = \begin{cases} E^{a,\mathrm{I}}_k := \det(e) e^a_k = \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e^r_b e^s_c \\ E^{a,\mathrm{II}}_k := \sqrt{\det(q)} e^a_k = \frac{1}{2}\epsilon_{krs}\epsilon^{abc} \underbrace{\mathrm{sgn}(\det(e))}_{=:\mathcal{S}} e^r_b e^s_c \end{cases}, \tag{3.3}$$

where e^j_a is the cotriad related to the intrinsic metric as $q_{ab} = e^j_a e^j_b$. From now on, we will use $E^{a,\mathrm{I}}_k$ and $E^{a,\mathrm{II}}_k$ respectively for the two cases.

The main difference between these two definitions is basically a sign factor which we will denote by $\mathcal{S}$. From the mathematical point of view, both definitions in equation (3.4) are equally viable; thus we will keep both possibilities and emphasize the differences that occur when we choose one or the other definition. Note, however, that $\det(E^{\mathrm{I}}) = \det(e)^2 \geqslant 0$ gives an anholonomic constraint which appears to be inconsistent with the definition of $\widehat{E}_k(S)$ as a derivative operator as in equation (2.5), as for instance pointed out in [13]. We will see that this is indeed the case.

Hence, we obtain two possible ways of introducing an alternative expression for the classical flux:

$$\widetilde{E}^{\epsilon,\mathrm{I/II}}_k(S) := \frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon} \mathrm{d}t\, \widetilde{E}^{\mathrm{I/II}}_k(S_t); \qquad \widetilde{E}^{\mathrm{I/II}}_k(S_t) = \int_{S_t} E^{a,\mathrm{I/II}}_k n^{S_t}_a. \tag{3.4}$$

Inserting the alternative expression for E^a_k into equation (2.4), we obtain

$$\widetilde{E}^{\mathrm{I,II}}_k(S_t) = \begin{cases} \int_{S_t} \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e^r_b e^s_c n^{S_t}_a, & E^{a,\mathrm{I}}_k = \det(e) e^a_k \\ \int_{S_t} \frac{1}{2}\epsilon_{krs}\epsilon^{abc} e^r_b \mathcal{S} e^s_c n^{S_t}_a, & E^{a,\mathrm{II}}_k = \sqrt{\det(q)} e^a_k \end{cases}. \tag{3.5}$$

So, instead of quantizing $E^{\epsilon,\mathrm{I/II}}_k(S)$ as a derivative operator as is usually done within LQG, we could alternatively use the above classical identities and quantize them via the Poisson bracket identity in equation (3.6). Then we are in the following situation. On the one hand we have the usual flux operator $\widehat{E}_k(S)$ being a derivative operator; on the other hand we have (two) alternative flux operators $\widehat{\widetilde{E}}^{\mathrm{I/II}}_k(S)$ based on the classical equation (3.3). An important question that arises now is whether $\widehat{E}_k(S)$ and $\widehat{\widetilde{E}}^{\mathrm{I/II}}k(S)$ have the same spectrum, because in this case we know that quantizing via Poisson bracket identity does not lead to contradictions and is therefore consistent. Thus, via these alternative fluxes, we are able to apply a theoretical check of the major ingredient of the dynamics of LQG.

3.1. Poisson bracket identity

As a first step towards quantizing the alternative fluxes $\widetilde{E}^{\mathrm{I}}_k(S)$ and $\widetilde{E}^{\mathrm{I/}}_k(S)$ we will apply the Poisson bracket identity in order to replace the triads e^r_b by its associated Poisson brackets between the connection A^r_b and the scalar volume function $V(R) = \int_R \mathrm{d}^3x\sqrt{\det(q)} = \int_R \mathrm{d}^3x\sqrt{|\det(E^{\mathrm{I}})|} = \int_R \mathrm{d}^3x\sqrt{|\det(E^{\mathrm{II}})|}$. The Poisson bracket identity for the two cases is shown below:

$$\{A^s_b, V(R)\} = \begin{cases} -\frac{\kappa}{2}\mathcal{S} e^s_b, & E^{a,\mathrm{I}}_k = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} e^s_b e^t_c \\ -\frac{\kappa}{2} e^s_b, & E^{a,\mathrm{II}}_k = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}\mathcal{S} e^s_b e^t_c \end{cases}. \tag{3.6}$$

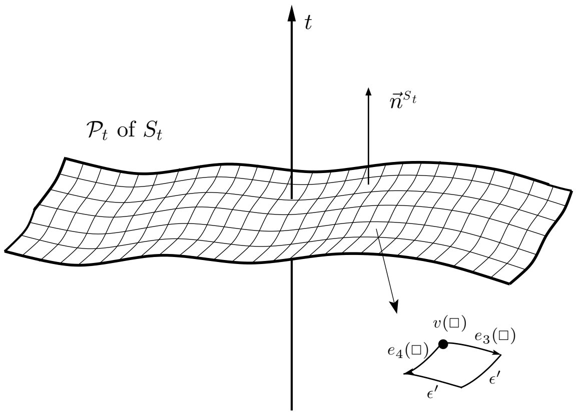

Figure 5. Partition $\mathcal{P}_t$ of the surface S_t into small squares with a parameter edge length ϵ'.

This relation is different for the two cases, because in deriving this relation we have to use the definition of the densitized triad E^a_k in terms of the triads e^s_b, which is different for case I and case II. The difference between $E^{a,\mathrm{I}}_k$ and $E^{a,\mathrm{II}}_k$ is again a sign factor $\mathcal{S}$. Going back to equation (3.5) we note that there is no $\mathcal{S}$ in $E^{a,\mathrm{I}}_k$. Since we have to replace two triads e^r_b, e^s_c by Poisson brackets, we get two factors of $\mathcal{S}$ in the case of $E^{a,\mathrm{I}}_k$. As $\mathcal{S}^2$ is always one classically (since q_{ab} is non-degenerate), it drops out in this case. In contrast, for $E^{a,\mathrm{II}}_k$ we have a sign factor $\mathcal{S}$ occurring in the alternative flux in equation (3.5), but no $\mathcal{S}$ in the Poisson bracket identity. Accordingly, we get only one $\mathcal{S}$ here, which does not disappear classically, because it can take the (constant) values ± 1. Thus

$$\begin{aligned}\widetilde{E}^{\mathrm{I}}_k(S_t) &= \frac{2}{\kappa^2}\int_{S_t}\epsilon_{krs}\epsilon^{abc}\{A^r_b, V(R)\}\{A^s_c, V(R)\}n^{S_t}_a \\ \widetilde{E}^{\mathrm{II}}_k(S_t) &= \frac{2}{\kappa^2}\int_{S_t}\epsilon_{krs}\epsilon^{abc}\{A^r_b, V(R)\}\mathcal{S}\{A^s_c, V(R)\}n^{S_t}_a.\end{aligned} \tag{3.7}$$

When, later on, we replace the classical expressions by their corresponding operators, the main difference between $\widetilde{E}^{\mathrm{I}}_k(S_t)$ and $\widetilde{E}^{\mathrm{II}}_k(S_t)$ will be a so-called sign operator $\widehat{\mathcal{S}}$. Before, we have to replace the connections A^r_b by holonomies since A^r_b cannot be promoted to well-defined operators in the Ashtekar–Lewandowski Hilbert space $\mathcal{H}_{\mathrm{AL}}$. Hence, we choose a partition $\mathcal{P}_t$ of each surface S_t into small squares of area ϵ'^2. In the limit where ϵ' is small enough we are allowed to replace the connection A^r_b along the edge e_I by its associated holonomy $h(e_I)$. The partition is shown in figure 5. Usually this is done for holonomies in the fundamental representation of 1/2. But, as we want to keep our construction of the alternative flux operator as general as possible and to study the effect of factor ordering ambiguities, we will consider holonomies with arbitrary representation weights ℓ. The corresponding relation between the connection integrated along the edge e_I, denoted by $A^r_I(\Box) := \int_{e_I(\Box)} A^s$, where $I = 3, 4$ from now on, and the associated holonomy is given by

$$\{A^r_I(\Box), V(R_{v(\Box)})\}\tfrac{1}{2}\pi_\ell(\tau_r) + o(\epsilon'^2) = +\pi_\ell(h_{e_I})\{\pi_\ell(h^{-1}_{e_I}), V(R_{v(\Box)})\}, \tag{3.8}$$

whereby $R_{v(\Box)}$ is any region containing the point $e_3(\Box)\cap e_4(\Box)$ and in the limit $\epsilon'\to 0$ also $R_{v(\Box)}\to v(\Box)$ and we indicate a representation with weight ℓ by π_ℓ.

Considering equation (3.8), we end up with the following classical identity for ${}^{(\ell)}\widetilde{E}^{\mathrm{I}}_k(S_t)$

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}^{\mathrm{I}}_k(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{krs}\frac{4}{\kappa^2}\left\{A_3^r(\square), V(R_{v(\square)})\right\}\left\{A_4^s(\square), V(R_{v(\square)})\right\} \\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\operatorname{Tr}\Big(\pi_\ell\big(h_{e_3(\square)}\big)\left\{\pi_\ell\big(h^{-1}_{e_3(\square)}\big), V(R_{v(\square)})\right\} \\
&\quad\times \pi_\ell(\tau_k)\pi_\ell\big(h_{e_4(\square)}\big)\left\{\pi_\ell\big(h^{-1}_{e_4(\square)}\big), V(R_{v(\square)})\right\}\Big)
\end{aligned}
\tag{3.9}
$$

and in the case of ${}^{(\ell)}\widetilde{E}^{\mathrm{II}}_k(S_t)$ with

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}^{\mathrm{II}}_k(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{krs}\frac{4}{\kappa^2}\left\{A_3^r(\square), V(R_{v(\square)})\right\}\mathcal{S}\left\{A_4^s(\square), V(R_{v(\square)})\right\} \\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\operatorname{Tr}\Big(\pi_\ell\big(h_{e_3(\square)}\big)\left\{\pi_\ell\big(h^{-1}_{e_3(\square)}\big), V(R_{v(\square)})\right\} \\
&\quad\times \pi_\ell(\tau_k)\mathcal{S}\pi_\ell\big(h_{e_4(\square)}\big)\left\{\pi_\ell\big(h^{-1}_{e_4(\square)}\big), V(R_{v(\square)})\right\}\Big).
\end{aligned}
\tag{3.10}
$$

Here we used $\operatorname{Tr}(\pi_\ell(\tau_{\mathrm{r}})\pi_\ell(\tau_{\mathrm{k}})\pi_\ell(\tau_{\mathrm{s}})) = -\frac{4}{3}\ell(\ell+1)(2\ell+1)\epsilon_{\mathrm{rks}}$ which is derived in appendix B of [20].

Now, we are at the bottom of figure 1 shown in the introduction. We obtained two alternative classical fluxes ${}^{(\ell)}\widetilde{E}^{\mathrm{I}}_k(S_t)$ and ${}^{(\ell)}\widetilde{E}^{\mathrm{II}}_k(S_t)$ which we want to promote to well-defined operators in the quantum theory.

4. Construction of the alternative flux operator

In LQG there exist two volume operators, one introduced by Rovelli and Smolin in 1994 ($\widehat{V}_{\mathrm{RS}}$) [5] and the other one published in 1995 by Ashtekar and Lewandowski ($\widehat{V}_{\mathrm{AL}}$) [6]. Hence, we have actually in each case two different possibilities for the Poisson bracket, because we could either use $\widehat{V}_{\mathrm{RS}}$ or $\widehat{V}_{\mathrm{AL}}$. Thus, case I as well as case II splits into two different versions of alternative flux operators at the quantum level (see also figure 2):

$$
\begin{aligned}
&{}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I}}_k(S_t) \to {}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,AL}}_k(S_t),\ {}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_k(S_t) \\
&{}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II}}_k(S_t) \to {}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_k(S_t),\ {}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_k(S_t).
\end{aligned}
\tag{4.1}
$$

From now on, we will use the notation above for the four different cases. Before we apply canonical quantization, we discuss the two volume operators and their differences in a bit more detail.

4.1. The two volume operators in LQG

4.1.1. The volume operator $\widehat{V}_{\mathrm{RS}}$ of Rovelli and Smolin. The idea that the volume operator acts only on vertices of a given graph was first mentioned in [18]. The first version of a volume operator can be found in [5] and is given by

$$
\begin{aligned}
\widehat{V}(R)_\gamma &= \int_R \mathrm{d}^3p\,\widehat{V}(p)_\gamma \\
\widehat{V}(p)_\gamma &= \ell_p^3\sum_{v\in V(\gamma)}\delta^{(3)}(p,v)\widehat{V}_{v,\gamma} \\
\widehat{V}^{\mathrm{RS}}_{v,\gamma} &= \sum_{I,J,K}\sqrt{\left|\frac{\mathrm{i}}{8}C_{\mathrm{reg}}\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K}\right|}.
\end{aligned}
\tag{4.2}
$$

Here we sum over all triples of edges at the vertex $v \in V(\gamma)$ of a given graph γ. $\widehat{V}_{\mathrm{RS}}$ is not sensitive to the orientation of the edges; thus also linearly dependent triples have to be considered in the sum. Moreover, we introduced a constant $C_{\mathrm{reg}} \in \mathbb{R}^+$ that we will keep arbitrary for the moment and that is basically fixed by the particular regularization scheme one chooses. When working with the volume operator we want to select physically relevant gauge invariant states properly. Hence, it is convenient to express our abstract angular momentum states in terms of the recoupling basis. The following identity [12] holds:

$$\tfrac{1}{8}\epsilon_{ijk} X^i_{e_I} X^j_{e_J} X^k_{e_K} = \tfrac{1}{4}\left[Y^2_{IJ}, Y^2_{JK}\right] =: \tfrac{1}{4} q^Y_{IJK}, \tag{4.3}$$

where $Y_{IJ} := Y^k_{e_I} + Y^k_{e_J}$ and $Y^k_{e_I}$ denotes the self-adjoint vector field $Y^k_{e_I} := -\tfrac{\mathrm{i}}{2} X^k_{e_I}$.

Consequently, we get

$$\widehat{V}(R)^{Y,\mathrm{RS}}_\gamma |JM;M'\rangle = \ell_p^3 \sum_{v\in V(\gamma)\cap R} \underbrace{\sum_{I<J<K} 3!\sqrt{\left|\frac{\mathrm{i}}{4} C_{\mathrm{reg}} \widehat{q}^Y_{IJK}\right|}}_{\widehat{V}^{\mathrm{RS}}_{v,\gamma}} |JM;M'\rangle. \tag{4.4}$$

The additional factor of 3! is due to the fact that we sum only over ordered triples $I < J < K$ now. The way to calculate eigenstates and eigenvalues of $\widehat{V}$ is as follows. Let us introduce the operator $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ as

$$\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK} := \ell_p^6 \frac{3!i}{4} C_{\mathrm{reg}} \widehat{q}^Y_{IJK}. \tag{4.5}$$

As a first step we have to calculate the eigenvalues and corresponding eigenstates for $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$. If, for example, $|\phi\rangle$ is an eigenstate of $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ with corresponding eigenvalue λ, then we obtain $\widehat{V}|\phi\rangle = \sqrt{|\lambda|}|\phi\rangle$. Consequently, we see that while $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ can have positive and negative eigenvalues, $\widehat{V}$ has only positive ones. Furthermore, if we consider the eigenvalues $\pm\lambda$ of $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ and the corresponding eigenstate $|\phi_{+\lambda}\rangle, |\phi_{-\lambda}\rangle$, we notice that these eigenvalues will be degenerate in the case of the operator $\widehat{V}$, as $\sqrt{|+\lambda|} = \sqrt{|-\lambda|}$.

4.1.2. The volume operator $\widehat{V}_{\mathrm{AL}}$ of Ashtekar and Lewandowski. Another version of the volume operator which differs by the chosen regularization scheme was defined in [6]:

$$\widehat{V}(R)^{Y,\mathrm{AL}}_\gamma |JM;M'\rangle = \ell_p^3 \sum_{v\in V(\gamma)\cap R} \underbrace{\sqrt{\left|\frac{3!\mathrm{i}}{4} C_{\mathrm{reg}} \sum_{I<J<K} \epsilon(e_I,e_J,e_K)\widehat{q}^Y_{IJK}\right|}}_{\widehat{V}^{\mathrm{AL}}_{v,\gamma}} |JM;M'\rangle. \tag{4.6}$$

The major difference between $\widehat{V}_{\mathrm{AL}}$ and $\widehat{V}_{\mathrm{RS}}$ is the factor $\epsilon(e_I,e_J,e_K)$ that is sensitive to the orientation of the tangent vectors of the edges $\{e_I,e_J,e_K\}$. $\epsilon(e_I,e_J,e_K)$ is +1 for right-handed, −1 for left-handed and 0 for linearly dependent triples of edges. In the case of $\widehat{V}_{\mathrm{AL}}$ it is convenient to introduce an operator $\widehat{Q}^{Y,\mathrm{AL}}_v$ that is defined as the expression that appears inside the absolute value under the square root in $\widehat{V}^{\mathrm{AL}}_{v,\gamma}$,

$$\widehat{Q}^{Y,\mathrm{AL}}_v := \ell_p^6 \frac{3!\mathrm{i}}{4} C_{\mathrm{reg}} \sum_{I<J<K} \epsilon(e_I,e_J,e_K)\widehat{q}^Y_{IJK}. \tag{4.7}$$

By comparing equation (4.4) with (4.6) we notice that another difference between $\widehat{V}_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ is the fact that for the first one we have to sum over the triples of edges outside the square root, while for the latter one we sum inside the absolute value under the square root. Apart from the difference of the sign factor, the difference in the summation will play an important role later on.

We used the label Y for $\widehat{Q}$ and $\widehat{V}$ here in order to emphasize that we are working with self-adjoint vector fields. Since all the statements and calculations done in this paper are true independently of the considered vector field in $\widehat{V}$ we drop the label Y from now on.

4.1.3. Final remarks on the volume operators and C_{reg}. Finally, we want to remark that both operators are cylindrical consistent operators. In fact, it is the cylindrical consistency that forces the operator to be of the form $\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K}$. Then there is still the freedom left of an arbitrary, but global, constant which we called C_{reg} above. When actually regularizing the classical volume, the regularization constant depends on the type of regularization. If one uses cubes or triangles, one will in general get different results. For $\widehat{V}_{\mathrm{AL}}$, a kind of averaging procedure over all possible kinds of regularization was used this lead to the factor of $\frac{1}{48}$. But, still, any volume operator differing by a global constant would be as possible as $\widehat{V}_{\mathrm{AL}}$. Thus their derivation of the volume operator does not fix the ambiguity. Our point of view is the following: we know that the assumption of cylindrical consistency determines the form of the volume operator only up to a global constant. When comparing the alternative flux operators action with the usual one, we still have this arbitrary constant occurring in our results. Since the action of the usual flux operator is completely known, we can fix C_{reg} in such a way that the action of both operators agree.

The difference between $\widehat{V}_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ occurs, because the single contributions of each small cube, or whatever is used to regularize the volume, is summed up differently at the end in order to get the total volume.

4.2. Factor ordering ambiguities

The discussion in this section is valid for all four different versions of the alternative flux; hence we will neglect the labels I, II, RS, AL here.

The meaning of the limit when the regularization parameter tends to zero, i.e. $\epsilon \to 0$ in combination with the limit of the partition $\mathcal{P}_t$ (see also figure 5) when the edge length parameter $\epsilon' \to 0$ is explained in detail in [20] in section 4.3. It is the same limit as taken for the usual flux operator sketched between equations (2.2) and (2.3). We have, e.g., for those S_t which intersects γ only in an edge of type up schematically

$$\begin{aligned} {}^{(\ell)}\widehat{\widetilde{E}}_k(S)T &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\mathrm{d}t\sum_I\langle T_t^{\mathrm{I}}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)|T\rangle T_t^{\mathrm{I}} \\ &= \frac{1}{2}\lim_{\epsilon\to 0}\sum_I\langle T_\epsilon^{\mathrm{I}}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S_\epsilon)|T\rangle T_\epsilon^{\mathrm{I}}, \end{aligned} \tag{4.8}$$

where T_t^{I} are the SNW states that contribute to ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)T$. $\langle T_\epsilon^{\mathrm{I}}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S_\epsilon)|T\rangle$ is actually independent of ϵ and $T_\epsilon^{\mathrm{I}} \to T_0^{\mathrm{I}}$ as a function of smooth connections which can then be extended to distributional ones. It turns out that the alternative flux operator can be regularized in such a way that each plaquette of the partition $\mathcal{P}_t$ of a surface S_t has an intersection with only one single edge e of a graph γ associated with an arbitrary given SNF $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. If the considered edge lies completely inside the plaquette of S_t or completely outside (up to sets of $\mathrm{d}t$ measure zero if inside) it will lead to a trivial action of the alternative flux operator. This is in full agreement with the usual flux operator. The alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ attaches two additional edges, namely e_3, e_4, to the edge e. By attaching these additional edges e_3, e_4, ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ creates a new vertex at a certain point of e and thus divides e into an

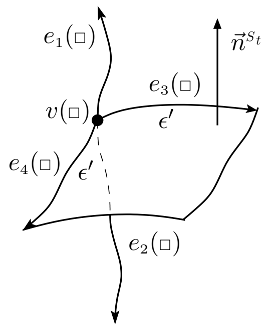

Figure 6. A non-vanishing contribution to the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}_k$ on an arbitrary SNF T_s, i.e. $\langle T_{s'}|{}^{(\ell)}\widehat{\widetilde{E}}_k(\square)|T_s\rangle$ can only be achieved if T_s contains edges of type up and/or down, respectively with respect to the surface S_t. Moreover, the edges $e_3(\square)$, $e_4(\square)$ have to be attached to T_s in this specific way.

edge e_1 of type up and an edge e_2 of type down with respect to S_t.[6] Hence, we can rewrite e as $e = e_1 \circ e_2^{-1}$. The discussion in [20] showed that the additional edges e_3 and e_4 have to lie inside the surface S_t since otherwise the ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ would have only again a trivial action (see also figure 6).

Accordingly, as for the usual flux operator, the action of the alternative flux operator is totally determined by the action on edges of type up and down with respect to the surface S_t. For instance, taking an arbitrary SNF $T_{\gamma,\vec{j},\vec{m},\vec{n}}$, we have to classify the edges of the associated graph γ into edges of types up, down, in and out with respect to the surface S_t. This is no restriction since it can be done for any $T_{\gamma,\vec{j},\vec{m},\vec{n}}$. The action on edges of type in and out is trivial; thus in order to obtain the action of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ on $T_{\gamma,\vec{j},\vec{m},\vec{n}}$, we only have to sum up the contributions of the edges of type up and down, as for the usual flux operator. For this reason, if we can show that $\widehat{E}_k(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ agree for a single edge of type up and down respectively, then they will agree for any arbitrary SNF $T_{\gamma,\vec{j},\vec{m},\vec{n}}$.

Since we are familiar now with the action of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$, we are able to consider the impact of different factor orderings. Going back to the classical identity in equations (3.9) and (3.10) respectively, and considering the results of the discussion above, we know that ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ adds two additional edges e_3, e_4 to a given edge e and divides this edge into an up edge e_1 and an edge of type down e_2. This is illustrated in figure 7. Let us call these SNF that involves the edges e_1, e_2 only $|\beta^{j_{12}}, n_{12}\rangle$. Here we use the so-called recoupling basis to express the SNF and j_{12} denotes the total angular momentum to which the two edges e_1 and e_2 couple at their single vertex v, whereas n_{12} is the associated magnetic quantum number. The SNF that results from $|\beta^{j_{12}}, n_{12}\rangle$ by the action of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ and that contains four edges $\{e_1, e_2, e_3, e_4\}$ will be denoted by $\left|\alpha_i^J, M\right\rangle$. In this case J is the total angular momentum, M the associated magnetic quantum number and i as an index is needed, because with four edges more than one state exists with the same total angular momentum, but different intermediate couplings. For more details, see sections 5.2 and 6 in [20]. Now, still dealing with the classical expression in equations (3.9) and (3.10) respectively, we can apply the trace and rearrange the terms in a certain manner, because classically holonomies commute in order to obtain a sensible operator

[6] The opposite case is also possible of course, but we will restrict our discussion to the first case and emphasize in the following discussion where exactly the choice of e_1 as a down edge and e_2 of type up will make a difference.

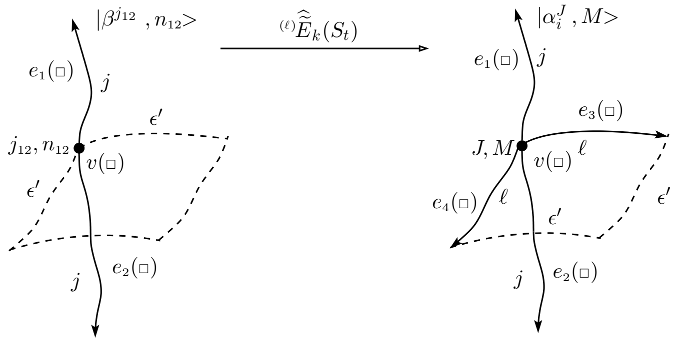

Figure 7. The SNF $|\beta^{j_{12}}, n_{12}\rangle$ is transformed into a new SNF $|\alpha_i^J, M\rangle$ by the action of $^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$.

ordering in the quantum theory. If we consider $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm I,AL}(S_t)$ and $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm II,AL}(S_t)$ that contain $\widehat{V}_{\rm AL}$ we know that due to the sign factor $\epsilon(e_I, e_J, e_K)$ in $\widehat{V}_{\rm AL}$ the action of $\widehat{V}_{\rm AL}$ on linearly dependent triples vanishes. Therefore, it has to be ensured that the two holonomies $\widehat{\pi}_\ell(h_{e_3})$ and $\widehat{\pi}_\ell(h_{e_4})$ act before $\widehat{V}_{\rm AL}$ does. This restriction reduces the number of possible factor orderings down to a single one.

The situation is different for $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm I,RS}(S_t)$ and $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm II,RS}(S_t)$, because here $\widehat{V}_{\rm RS}$ is involved, which has a non-trivial action on linearly dependent triples. Consequently, more than one possible factor ordering exists. Let us discuss the number and differences of these orderings later on and restrict ourselves for $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm I,RS}(S_t)$ as well as $^{(\ell)}\widehat{\widetilde{E}}_k^{\rm II,RS}(S_t)$ to the single ordering that is possible for $\widehat{V}_{\rm AL}$, first. It turns out that for all operators with dt measure 1, the dependence of $\widehat{\widetilde{E}}_k(S_t)$ on t drops out and taking the average $\frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon}{\rm d}t$ becomes trivial resulting in equation (4.8). We will thus keep the label S_t in what follows, but keep in mind that the t-dependence is trivial.

4.3. Canonical quantization

Usually the densitized triads, appearing in the classical flux $E_k(S)$, are quantized as differential operators, while holonomies are quantized as multiplication operators. If we choose the alternative expression $\widetilde{E}_k(S)$ we will instead get the scalar volume $\widehat{V}$ and the so-called sign $\widehat{\mathcal{S}}$ operator into our quantized expression. The properties of this $\widehat{\mathcal{S}}$ will be explained in more detail below. Moreover, we have to replace Poisson brackets by commutators, following the replacement rule $\{.\,,.\} \to (1/i\hbar)[.\,,.]$. The detailed derivation of the final operator can be found in section 4 of [20].

Clearly, we want the total operator to be self-adjoint, so we will calculate the adjoint of $^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ and define the total and final operator to be $^{(\ell)}\widehat{\widetilde{E}}_{k,\rm tot}(S_t) = \frac{1}{2}(\widehat{\widetilde{E}}_k(S_t) + \widehat{\widetilde{E}}^\dagger{}_k(S_t))$ that is self-adjoint by construction. Hence, the final operator for $\widehat{V}_{\rm RS}$ which we will use through the calculation of this paper is given by

$$\begin{aligned}
^{(\ell)}\widehat{\widetilde{E}}_{k,\rm tot}^{\rm I/II,RS}(S_t) = \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI}\big\{+\pi_\ell(\epsilon)_{FC}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\big]^\dagger \\
\times\big[\big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{BA}\big]^\dagger, \widehat{V}_{\rm RS}\big]\boxed{\widehat{\mathcal{S}}}\big[\widehat{V}_{\rm RS}, \widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\big]\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} - \pi_\ell(\epsilon)_{FB} \\
\times\big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{IG}\big]^\dagger\big[\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{EA}\big]^\dagger, \widehat{V}_{\rm RS}\big]\boxed{\widehat{\mathcal{S}}}\big[\widehat{V}_{\rm RS}, \widehat{\pi}_\ell\big(h_{e_3}\big)_{FG}\big]\widehat{\pi}_\ell\big(h_{e_4}\big)_{CA}\big\},
\end{aligned} \tag{4.9}$$

whereby we used the identity $\widehat{\pi}_\ell\big(h_{e_I}^{-1}\big)_{AB} = \big[\widehat{\pi}_\ell\big(h_{e_I}\big)_{BA}\big]^\dagger$, $\pi_\ell(\epsilon)_{AB} = (-1)^{\ell-A}\delta_{A+B,0}$ and the definition of the Planck length $\ell_p^{-4} := (\hbar\kappa)^{-2}$. The $SU(2)$ matrix indices $A, \ldots, I$ run from $-\ell, \ldots, +\ell$. The box surrounding the sign operator $\widehat{\mathcal{S}}$ should indicate that it is contained in the operator in case II, while it is not in case I.

Considering the operator $\widehat{V}_{\mathrm{AL}}$, we know that for each commutator only one term will contribute, because otherwise we cannot construct linearly independent triples of edges since $\{e_1, e_2, e_{3/4}\}$ are linearly dependent. Therefore in the case of $\widehat{V}_{\mathrm{AL}}$ we obtain the following final expression:

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I/II,AL}}_{k,\mathrm{tot}}(S_t) = \lim_{\mathcal{P}_t\to S_t} \sum_{\square\in\mathcal{P}_t} \frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI}\Big\{+\pi_\ell(\epsilon)_{FC}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\big]^\dagger \\
\times \big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{BA}\big]^\dagger \widehat{V}_{\mathrm{AL}} \boxed{\widehat{\mathcal{S}}} \widehat{V}_{\mathrm{AL}} \widehat{\pi}_\ell\big(h_{e_4}\big)_{IG} \widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} \\
- \pi_\ell(\epsilon)_{FB}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\big]^\dagger \big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA}\big]^\dagger \widehat{V}_{\mathrm{AL}} \boxed{\widehat{\mathcal{S}}} \widehat{V}_{\mathrm{AL}} \widehat{\pi}_\ell\big(h_{e_4}\big)_{FG} \widehat{\pi}_\ell\big(h_{e_3}\big)_{CA}\Big\}.
\end{aligned}
\tag{4.10}
$$

Here again for case II the sign operator is included, whereas in case I it is not.

Next, we want to calculate the matrix elements $\langle\beta^{j_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ of all four versions ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ of the new flux operator.

The action of the holonomy operators $\widehat{\pi}_\ell\big(h_{e_I}\big)_{AB}$ on $|\beta^{j_{12}}, m_{12}\rangle$ can be described in the framework of angular momentum recoupling theory with the powerful tool of Clebsch–Gordan coefficients (CGC). The correspondence between the Ashtekar–Lewandowski Hilbert space $\mathcal{H}_{\mathrm{AL}}$ and the abstract angular momentum Hilbert space is discussed in section 5.1 in [20]. Hence, the matrix element will, roughly speaking, have the following structure:

$$
\begin{aligned}
\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle \propto \sum_{\tilde{J},\widetilde{M},J,M} \sum_{\tilde{a}_3,\widetilde{m}_{\tilde{a}_3},a_3,m_{a_3}} C(j_{12}, \ell; a_3 m_{a_3}) \\
\times C^*(\tilde{j}_{12}, \ell; \tilde{a}_3\widetilde{m}_{\tilde{a}_3}) C^*(\tilde{a}_3, \ell; \tilde{J}\widetilde{M}) C(a_3, \ell; JM) \big\langle\alpha_i^{\tilde{J}}, \widetilde{M}\big|\widehat{O}\big|\alpha_j^J, M\big\rangle.
\end{aligned}
\tag{4.11}
$$

Here $C(j_1, j_2; JM)$ denotes the CGC that we get if we couple the angular momenta j_1 and j_2 to a resulting angular momentum J with corresponding magnetic quantum number M. Here $\widehat{O}$ is an operator involving the operators $\widehat{V}_{\mathrm{RS}}$, $\widehat{V}_{\mathrm{AL}}$ and $\widehat{\mathcal{S}}$ according to the case under consideration. The details can be found in section 5.1 of [20]. Due to the symmetry properties of the alternative flux operator that are analysed in section 5.2 of [20], the only total angular momenta that will contribute to the final matrix element are $J = 0, 1$. (See section 5.2 of [20] for more explanations concerning this point.) Since the physically relevant states are gauge-invariant we choose $j_{12} = 0$. The behaviour of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ under gauge transformations, which is discussed in section 5.3 of [20], leads to the restriction $\tilde{j}_{12} = 1$. Consequently, we get the following form of the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$:

$$
\begin{aligned}
\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^0, 0\rangle = -\lim_{\mathcal{P}_t\to S_t} \sum_{\square\in\mathcal{P}_t} \frac{8\ell_p^{-4}(-1)^{3\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \\
\times \sum_{B,C,F=-\ell}^{+\ell} \Big\{ \pi_\ell(\tau_k)_{CB}\big[+(-1)^{-F}\delta_{F+C,0}\sqrt{2\ell+1}\,\delta_{\widetilde{m}_{12}+B+F,0} \\
\times\langle 1\widetilde{m}_{12}; \ell B|\ell\widetilde{m}_{12}{+}B\rangle\langle\ell\widetilde{m}_{12}{+}B; \ell F|00\rangle \\
\times\big\langle\alpha_2^0, M = \widetilde{m}_{12}+B+F; \widetilde{m}_1'\widetilde{m}_2'|\widehat{O}_1|\alpha_1^0, M = 0; m_1'm_2'\big\rangle\big] \\
-(-1)^{-F}\delta_{F+B,0}\delta_{C+F,\widetilde{m}_{12}}\langle 00; \ell C|\ell C\rangle\langle\ell C; \ell F|1 C+F\rangle
\end{aligned}
$$

$$\left[+\frac{\sqrt{2\ell-1}}{\sqrt{3}}\langle\alpha_2^1, M=\widetilde{m}_{12}; \widetilde{m}'_1\widetilde{m}'_2|\widehat{O}_2|\alpha_1^1, M=C+F; m'_1m'_2\rangle\right.$$
$$-\frac{\sqrt{2\ell+1}}{\sqrt{3}}\langle\alpha_3^1, M=\widetilde{m}_{12}; \widetilde{m}'_1\widetilde{m}'_2|\widehat{O}_2|\alpha_1^1, M=C+F; m'_1m'_2\rangle$$
$$\left.\left.+\frac{\sqrt{2\ell+3}}{\sqrt{3}}\langle\alpha_4^1, M=\widetilde{m}_{12}; \widetilde{m}'_1\widetilde{m}'_2|\widehat{O}_2|\alpha_1^1, M=C+F; m'_1m'_2\rangle\right]\right\}, \qquad (4.12)$$

whereby we used the following notation for our basis states

$$\begin{aligned}
\left|\alpha_1^0, M\right\rangle &:= |a_1=j\ a_2=0\ a_3=1\ J=0\rangle\\
\left|\alpha_2^0, M\right\rangle &:= |a_1=j\ a_2=1\ a_3=1\ J=0\rangle\\
\left|\alpha_1^1, M\right\rangle &:= |a_1=j\ a_2=0\ a_3=1\ J=1\rangle\\
\left|\alpha_2^1, M\right\rangle &:= |a_1=j\ a_2=1\ a_3=0\ J=1\rangle\\
\left|\alpha_3^1, M\right\rangle &:= |a_1=j\ a_2=1\ a_3=1\ J=1\rangle\\
\left|\alpha_4^1, M\right\rangle &:= |a_1=j\ a_2=1\ a_3=2\ J=1\rangle
\end{aligned} \qquad (4.13)$$

with a_i being the intermediate coupling steps. For more details concerning the basis states and why only these particular states contribute, we refer the reader to [20].

The four different cases for the operators are encoded in the operators $\widehat{O}_1$ and $\widehat{O}_2$. Explicitly, we have

$$\begin{aligned}
\widehat{O}_1^{\mathrm{I,AL}} &= \widehat{V}_{\mathrm{AL}}^2\\
\widehat{O}_1^{\mathrm{I,RS}} &= \widehat{V}_{q_{134}}^2 + \widehat{V}_{q_{234}}^2 + \widehat{V}_{q_{134}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{134}}\\
&\quad + V_{q_{234}}\widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{123}}\\
O_2^{\mathrm{I,AL}} &= \widehat{V}_{\mathrm{AL}}^2\\
O_2^{\mathrm{I,RS}} &= \widehat{V}_{q_{134}}^2 ms + \widehat{V}_{q_{234}}^2 + \widehat{V}_{q_{234}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{123}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{V}_{q_{234}}\\
&\quad + V_{q_{234}}\widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{V}_{q_{124}}\\
O_1^{\mathrm{II,AL}} &= \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}\widehat{V}}_{\mathrm{AL}}\\
O_1^{\mathrm{II,RS}} &= +\widehat{V}_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{123}}\\
&\quad + V_{q_{124}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{124}}\widehat{\mathcal{S}\widehat{V}}_{q_{123}}\\
O_2^{\mathrm{II,AL}} &= \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}\widehat{V}}_{\mathrm{AL}}\\
O_1^{\mathrm{II,RS}} &= +\widehat{V}_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}\widehat{V}}_{q_{134}}\\
&\quad + V_{q_{134}}\widehat{\mathcal{S}\widehat{V}}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}\widehat{V}}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}\widehat{V}}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}\widehat{V}}_{q_{124}},
\end{aligned} \qquad (4.14)$$

whereby we used the notation $\widehat{V}_{q_{IJK}}$ for $\widehat{V}_{\mathrm{RS}} = \widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}$ when only the triple $\{e_I, e_J, e_K\}$ contributes to $\widehat{V}_{\mathrm{RS}}$. The derivation of the various versions of $\widehat{O}_1$, $\widehat{O}_2$ can be found in [20]. These eight versions result from taking into account (a) the two volume operators, (b) the sign operator $\mathcal{S}$ or not and (c) the adjoint or not in equation (4.12).

Before we discuss our results in the next sections, we explain in a bit more detail what we mean by, say, the consistency check being affirmative or not. We managed to implement an alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ in four different versions, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$. The usual flux operator is quantized as a differential operator in the standard way. The alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ is quantized via the Poisson bracket identity in equation (3.6), analogous to the quantization of the Hamiltonian constraint.

If these two methods of quantization are mathematically consistent with each other, the action of $\widehat{E}_k(S)$ and that of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ should only differ by a constant, namely

$$\widehat{E}_k(S)|\beta^0,0\rangle = C(j,\ell)C_{\mathrm{reg}}\,{}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle = C(j,\ell)C_{\mathrm{reg}} \times \sum_{\widetilde{m}_{12}}\langle\beta^1,\widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle|\beta^1,\widetilde{m}_{12}\rangle, \tag{4.15}$$

whereby ${}^{(\ell)}\widehat{\widetilde{E}}^{0}_{k,\mathrm{tot}}(S)$ denotes the alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S)$, where C_{reg} has been replaced by 1. The constant $C(j,\ell)$ might depend on the spin labels j,ℓ of the edges, but at least semiclassically the dependence on the spin labels j and ℓ must disappear since otherwise the behaviour of $\widehat{E}_k(S)$ and ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S)$ in the correspondence limit of large j would disagree. Note that also a dependence of $\lim_{j\to\infty}C(j,\ell) =: C(\ell)$ on ℓ is unacceptable, because classically the flux is independent of the factor ordering ambiguity ℓ. Moreover, this constant will fix the ambiguity C_{reg} in the volume operator which is due to regularization, as we will see later. Hence, $\lim_{j\to\infty}C(j,\ell) = 1/C_{\mathrm{reg}} = \mathrm{const}$, as will be discussed in more detail in section 6.1.

5. Case I: results for ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$

5.1. Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$

For technical reasons, we consider only a spin label $\ell = 0.5, 1$, because higher spin labels cannot be computed analytically anymore. Fortunately, the main properties of this case already occur when considering small ℓ. The detailed calculation in section 6.2 in [20] shows

$$\begin{aligned}&\langle\alpha_2^0, M=\widetilde{m}_{12};\widetilde{m}_1'\widetilde{m}_2'|\widehat{O}_1^{\mathrm{I,AL}}|\alpha_1^0, M=C+F;m_1'm_2'\rangle = 0\\ &\langle\alpha_i^1 M=\widetilde{m}_{12};\widetilde{m}_1'\widetilde{m}_2'|\widehat{O}_2^{\mathrm{I,AL}}|\alpha_1^1, M=C+F;m_1'm_2'\rangle = 0,\end{aligned} \tag{5.1}$$

where $i = 2,3,4$. Going back to equation (4.12), we note that the vanishing of the matrix elements above has the consequence that the whole matrix element $\langle\beta^{\tilde{j}_{12}},\widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}},m_{12}\rangle$ is zero. Since the action of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ on an arbitrary SNF can be derived from exactly this matrix element, we can conclude that ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ is the zero operator. Accordingly, ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ is not consistent with the usual flux operator.

5.2. Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$

Also for the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ the analogous calculations discussed in section 6.3 of [20] yield only trivial matrix elements

$$\begin{aligned}&\langle\alpha_2^0, M=\widetilde{m}_{12};\widetilde{m}_1'\widetilde{m}_2'|\widehat{O}_1^{\mathrm{I,RS}}|\alpha_1^0, M=C+F;m_1'm_2'\rangle = 0\\ &\langle\alpha_i^1, M=\widetilde{m}_{12};\widetilde{m}_1'\widetilde{m}_2'|\widehat{O}_2^{\mathrm{I,RS}}|\alpha_1^1, M=C+F;m_1'm_2'\rangle = 0.\end{aligned} \tag{5.2}$$

Consequently, we can draw the same conclusion as for ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ and state that ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ is the zero operator and therefore inconsistent with the usual flux operator. Furthermore, as neither ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ nor ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ survives the consistency check, we can rule out, at least for the cases of $\ell = 0.5, 1$, the choice of $E^a_k = \det(e)e^a_k$ on which these operators are based on. To rule out the choice $E^a_k(S_t) = \det(e)e^a_k$ completely, we need to investigate the matrix element for arbitrary representation weights ℓ. For higher values of ℓ, the calculation cannot be done analytically any more. However, the results for $\ell = 0.5, 1$ indicate that there is an abstract reason which leads to the vanishing of the matrix elements for *any* ℓ. We were not able to find such an abstract argument yet. However, even if that was not the case and there would be a range of values for ℓ for which not all of the matrix elements would vanish, it is unacceptable that the classical theory is independent of ℓ while the quantum theory strongly depends on ℓ in the correspondence limit of large j.

6. Case II: results for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$

6.1. Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$

Considering the case of the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, we can read off from equation (4.14) the expressions $\widehat{O}_1 = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}} = \widehat{O}_2$. Since the sign operator $\widehat{\mathcal{S}}$ that corresponds to the classical expression $\mathcal{S} := \mathrm{sgn}(\det(e))$ does not exist in the literature so far, we will explain in detail how the operator $\widehat{\mathcal{S}}$ has to be understood.

6.2. The sign operator $\widehat{\mathcal{S}}$

We are dealing now with case II, meaning that the densitized triad is given by $E^{a,\mathrm{II}}_k = \mathcal{S}\det(e)e^a_k$, where $\mathcal{S} := \mathrm{sgn}(\det(e))$. Applying the determinant onto $E^{a,\mathrm{II}}_k$, we get

$$\det(E) = \mathrm{sgn}(\det(e))\det(q) \qquad \text{with} \qquad \det(q) = [\det(e)]^2 \geqslant 0. \tag{6.1}$$

Therefore, we obtain

$$\mathrm{sgn}(\det(E)) = \mathrm{sgn}(\det(e)) = \mathcal{S}. \tag{6.2}$$

In the following we want to show that $\mathcal{S} = \mathrm{sgn}(\det(E))$ can be identified with the sign of the expression inside the absolute value under the square roots in the definition of the AL-volume. For this purpose, let us first discuss this issue on the classical level and afterwards go back into the quantum theory and see how the corresponding operator $\widehat{\mathcal{S}}$ is connected with the operator $\widehat{Q}^{\mathrm{AL}}_v$ in equation (4.7).

In order to do this let us consider equation (3.10). This equation contains the classical volume $V(R_{v(\square)})$, where $R_{v(\square)}$ denotes a region centred around the vertex $v(\square)$.

The volume of such a cube is given by

$$V(R_{v(\square)}) = \int_{R_{v(\square)}} \sqrt{\det(q)}\,\mathrm{d}^3x = \int_{R_{v(\square)}} \sqrt{|\det(E)|}\,\mathrm{d}^3x, \tag{6.3}$$

where we used $\det(q) = |\det(E)|$ from equation (6.1). Introducing a parametrization of the cube now, we end up with

$$V(R_{v(\square)}) = \int_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} \left|\frac{\partial X^{\mathrm{I}}(u)}{\partial u_J}\right| \sqrt{|\det(E)(u)|}\,\mathrm{d}^3u = \int_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} |\det(X)|\sqrt{|\det(E)(u)|}\,\mathrm{d}^3u. \tag{6.4}$$

In order to be able to carry out the integral we choose the cube $R_{v(\square)}$ small enough and, thus, the volume can be approximated by

$$V(R_{v(\square)}) \approx \epsilon'^3 \left|\det\left(\frac{\partial X}{\partial u}\right)(v)\right| \sqrt{|\det(E)(v)|}. \tag{6.5}$$

Using the definition of $\det(E) = \frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l$, we can rewrite equation (6.3) as

$$V(R_{v(\square)}) = \int_{\square} \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l\right|}\,\mathrm{d}^3x. \tag{6.6}$$

If we again choose $R_{v(\Box)}$ small enough and define the square surfaces of the cube as S^{I}, we can re-express the volume integral over the densitized triads in terms of their corresponding electric fluxes through the surfaces S^{I}

$$V(R_{v(\Box)}) \approx \sqrt{\left|\frac{1}{3!}\epsilon_{IJK}\epsilon^{jkl}E_j(S^{\mathrm{I}})E_k(S^J)E_l(S^K)\right|}. \tag{6.7}$$

The flux through a particular surfaces S^{I} is defined as

$$E_j(S^{\mathrm{I}}) = \int_{S^{\mathrm{I}}} E^a_j n^{S^{\mathrm{I}}}_a \qquad n^{S^{\mathrm{I}}}_a = \frac{1}{2}\epsilon^{IJK}\epsilon_{abc}X^b_{,u_J}X^c_{,u_K}\Big|_{n^{\mathrm{I}}=0}. \tag{6.8}$$

Here $n^{S^{\mathrm{I}}}_a$ denotes the conormal vector associated with the surface S^{I}. Regarding equation (6.7), we realize that inside the absolute value in equation (6.7) appears exactly the definition of $\det(E_j(S^{\mathrm{I}}))$. Therefore we get

$$V(R_{v(\Box)}) \approx \sqrt{\left|\det(E_j(S^{\mathrm{I}}))\right|}. \tag{6.9}$$

On the other hand, by taking advantage of the fact that the surfaces S^{I} are small enough so that the integral can be approximated by the value at the vertex times the size of the surface itself, we obtain for $\det(E_j(S^{\mathrm{I}}))$

$$\begin{aligned}\det(E_j(S^{\mathrm{I}})) &\approx \det\left(E^a_j(v)n^{S^{\mathrm{I}}}_a(v)\epsilon'^2\right)\\ &= \det\left(E^a_j(v)\right)\det\left(n^{S^{\mathrm{I}}}_a(v)\right)\epsilon'^6\\ &= \det(E(v))\det\left(n^{S^{\mathrm{I}}}_a(v)\right)\epsilon'^6.\end{aligned} \tag{6.10}$$

If we consider the definition of the normal vector in equation (6.8),we can show the following identity:

$$\det\left(n^{S^{\mathrm{I}}}_a\right) = \det\left(\det(X)X^{S^{\mathrm{I}}}_a\right) = \det(X)^3\det(X^{-1}) = \frac{\det(X)^3}{\det(X)} = \det(X)^2, \tag{6.11}$$

where $\det(X) := \det(\partial X/\partial u)$. Inserting equation (6.11) back into equation (6.10) we have

$$\det(E_j(S^{\mathrm{I}})) \approx \det(E(v))[\det(X(v))]^2\epsilon'^6 \tag{6.12}$$

and can conclude that equation (6.9) is consistent with the usual definition of the volume in equation (6.5).

Since we want to identify $\mathcal{S} := \mathrm{sgn}(\det(E))$ with the sign that appears inside the absolute value under the square root in the definition of the volume in equation (6.9), namely $\mathrm{sgn}(\det(E_j(S^{\mathrm{I}})))$, we still have to show that $\mathrm{sgn}(\det(E)) = \mathrm{sgn}(\det(E_j(S^{\mathrm{I}})))$. However, this can be done by means of equation (6.12),

$$\begin{aligned}\mathrm{sgn}(\det(E_j(S^{\mathrm{I}}))) &\approx \mathrm{sgn}(\det(E(v))[\det(X(v))]^2\epsilon'^6)\\ &= \mathrm{sgn}(\det(E(v)))\mathrm{sgn}([\det(X(v))]^2)\mathrm{sgn}(\epsilon'^6)\\ &= \mathrm{sgn}(\det(E(v))).\end{aligned} \tag{6.13}$$

Thus, we have shown that $\mathcal{S} = \mathrm{sgn}(\det(E_j(S^{\mathrm{I}})))$. The classical counterpart for the operator $\widehat{Q}^{\mathrm{AL}}_v$ in equation (4.7) is $\mathrm{sgn}(\det(E_j(S^{\mathrm{I}})))\det(E_j(S^{\mathrm{I}})) \propto \mathcal{S}V^2$. Hence $\widehat{Q}^{\mathrm{AL}}_v$ contains, apart from the squared version of the volume, also the information about the sign of $\det(E_j(S^{\mathrm{I}}))$. Consequently, at the quantum level we have the following operator identity $\widehat{Q}^{\mathrm{AL}}_v = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$.

Now we will be left with the task of calculating particular matrix elements for $\widehat{Q}^{\mathrm{AL}}_v$, which can be done by means of the formula derived in [17]. Earlier work referring to the calculation of the matrix elements in the case of $\widehat{Q}^{\mathrm{RS}}_v$ can be found in [16].

6.3. Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$

One big advantage that comes along with the operator identity $\widehat{Q}_v^{\mathrm{AL}} = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$ is that diagonalization of the operator $\widehat{Q}_v^{\mathrm{AL}}$ is no longer necessary as it was in case I for $\widehat{V}_{\mathrm{AL}}^2$, because the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ contains only particular matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$ that can be exactly calculated, even for arbitrary ℓ, by means of the tools developed in [17]. The details of this calculation can be found in [20] as well as the corresponding matrix elements of the usual flux operator $\widehat{E}_k(S)$. If we compare the results of the usual flux operator with those of ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, we can judge whether ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ leads to a result consistent with the usual flux operator. It transpires

$$ {}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle = 3!8C_{\mathrm{reg}}\widehat{E}_k(S)|\beta^0,0\rangle. \tag{6.14} $$

Therefore the two operators differ only by a positive integer constant. As there is still the regularization constant C_{reg} in the above equation we can now fix it by requiring that both operators do exactly agree with each other. In fact there is no other choice than exact agreement because the difference would be a global constant which does not decrease as we take the corresponding limit of large quantum numbers j. Thus, we can remove the regularization ambiguity of the volume operator in this way and choose C_{reg} to be $C_{\mathrm{reg}} := \frac{1}{3!8} = \frac{1}{48}$.

This is exactly the value of C_{reg} that was obtained in [6] by a completely different argument. Thus the geometrical interpretation of the value we have to choose for C_{reg} is perfectly provided[7].

Note that the consistency check holds in the full theory and not only in the semiclassical sector. Consequently, the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ is consistent with the usual flux operator.

6.4. Calculations for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$

Now, considering the operator ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ things look different. Here, a quantization of the sign operator $\widehat{\mathcal{S}}$ that is consistent with $\widehat{V}_{\mathrm{RS}}$ cannot be found for the simple reason that $\widehat{V}_{\mathrm{RS}}$, in contrast to $\widehat{V}_{\mathrm{AL}}$, is a sum of single square roots. Hence, there is no origin for a global sign, where there was in the case of $\widehat{V}_{\mathrm{AL}}$. (See also section 6.6.1 in [20].) In retrospect there is a simple argument why the only possibility ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ (since ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ does not exist) is ruled out without further calculation. Namely, the lack of a factor of orientation in $\widehat{V}_{\mathrm{RS}}$, such as $\epsilon(e_I, e_J, e_K)$ in $\widehat{V}_{\mathrm{AL}}$, leads to the following basic disagreement with the usual flux operator. Suppose we had chosen the orientation of the surface S in the opposite way. Then the type of edge e switches between up and down, as it does for e_1, e_2. Then, the result of the usual flux operator would differ by a minus sign. In the case of $\widehat{V}_{\mathrm{AL}}$ we would get this minus sign as well due to $\epsilon(e_I, e_J, e_K)$ contained in $Q_v^{\mathrm{AL}} = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$, whereas a change of the orientation of e_1, e_2 would not modify the result of the alternative flux operator if we used $\widehat{V}_{\mathrm{RS}}$ instead, because it is not sensitive to the orientation of the edges. Accordingly, we should stop here and draw the conclusion that ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ is not consistent with $\widehat{E}_k(S)$.

One might propose to artificially use $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ for ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$. Note that we attached the label AL to $\widehat{\mathcal{S}}$ to emphasize that its quantization is based on the same regularization method that was used for $\widehat{V}_{\mathrm{AL}}$. This is artificial for the following reason. Suppose we have a classical quantity $A := \det(E)$ and two different functions $f_1 := \sqrt{|A|}$ and $f_2 := \mathrm{sgn}(A)$. If we want to quantize the functions f_1 and f_2, we do this with the help of the corresponding self-adjoint

[7] The factor $8 = 2^3$ comes from the fact that during the regularization one integrates a product of 3 δ-distributions on $\mathbb{R}$ over $\mathbb{R}^+$ only. The factor $6 = 3!$ is due to the fact that one should sum over ordered triples of edges only.

operator $\widehat{A}$ and obtain due to the spectral theorem $\widehat{f_1} = \sqrt{|\widehat{A}|}$ and $\widehat{f_2} = \mathrm{sgn}(\widehat{A})$. The product of operators $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ rather corresponds to $\widehat{g}_1 = \widehat{A}'$ and $\widehat{g}_2 = \mathrm{sgn}(\widehat{A})$, where $\widehat{A}'$ denotes an operator that has the same classical counterpart as $\widehat{A}$ has, but was quantized with a different regularization scheme. Thus, $\widehat{V}_{\mathrm{RS}}$ is quantized with a different regularization scheme than $\widehat{S}$ is. This would only be justified if $\sqrt{|\widehat{A}|}$ and $\widehat{A}'$ agreed semiclassically. However, they do not. If we compare the expressions for $\widehat{V}_{\mathrm{AL}}$ and $\widehat{V}_{\mathrm{RS}}$ then, schematically, they are related in the following way when restricted to a vertex, $\widehat{V}_{v,\mathrm{AL}} = |\frac{3!\mathrm{i}}{4} C_{\mathrm{reg}} \sum_{I<J<K} \epsilon(e_I, e_J, e_K)\widehat{q}_{IJK}|^{1/2}$ while $\widehat{V}_{v,\mathrm{RS}} = \sum_{I<J<K} 3!|\frac{\mathrm{i}}{4} C_{\mathrm{reg}}\widehat{q}_{IJK}|^{1/2}$. It is clear that apart from the sign $\epsilon(e_I, e_J, e_K)$ the two operators can agree at most on states where only one of the $\widehat{q}_{IJK}$ is non-vanishing (three or four valent graphs) simply because $\sqrt{|a+b|} \neq \sqrt{|a|} + \sqrt{|b|}$ for generic real numbers a, b.

Nevertheless, by analysing ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ when the artificial operator $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ is involved, we obtain

$$ {}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S)|\beta^0, 0\rangle = C(j,\ell) C_{\mathrm{reg}} \widehat{E}_k(S)|\beta^0, 0\rangle, \tag{6.15} $$

whereby $C(j,\ell) \in \mathbb{R}$ is a constant depending non-trivially on the spin labels j, ℓ in general. One can show that $C(j,\ell) \to C(\ell)$ semiclassically, i.e. in the limit of large j, which is shown in appendix E and discussed in section 6.6.2 of [20]. Hence ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$, including the artificial operator $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$, would be consistent with $\widehat{E}_k(S)$ within the semiclassical regime of the theory if we chose $C_{\mathrm{reg}} = 1/C(\ell)$ and if $C(\ell)$ were be a universal constant. Unfortunately, $C(\ell)$ has a non-trivial ℓ-dependence which is unacceptable because it is absent in the classical theory. Moreover, we do not see any geometrical interpretation available for the chosen value of C_{reg} for any value of ℓ in this case. One could possibly get rid of the ℓ-dependence by simply cancelling the linearly dependent triples by hand from the definition of $\widehat{V}_{\mathrm{RS}}$. But then the so modified $\widehat{V}'_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ would practically become identical on 3- and 4-valent vertices and moreover $\widehat{V}'_{\mathrm{RS}}$ now depends on the differentiable structure of Σ. See more about this in the conclusion.

7. Conclusion

We hope to have demonstrated in this paper that at least certain aspects of LQG are remarkably tightly defined : certain factor ordering ambiguities turn out to be immaterial, some regularization schemes can be ruled out as unphysical once and for all. The fact that we can exclude the RS volume from now on as far as the quantum dynamics is concerned should not be viewed as a criticism of [5] at all: the regularization performed in [5] is manifestly background independent, natural and intuitively very reasonable. It was the first pioneering paper on quantization of kinematical geometrical operators in LQG and had a deep impact on all papers that followed it. Most of the beautiful ideas spelled out in [5] also entered the regularization performed in [6] and continue to be valid. One could never have guessed that the regularization performed in [5] leads to an inconsistent result. It took a decade to develop the necessary technology in order to perform the consistency check provided in this paper. Hence, the fact that we can define the theory more uniquely now should be taken as a strength of the theory and not as a weakness of [5]. Interestingly, the very first paper on the volume operator [18] that we are aware of does look more like $\widehat{V}_{\mathrm{AL}}$ rather than $\widehat{V}_{\mathrm{RS}}$. We speculate that if one had completed the ideas of [18] one would have ended up with $\widehat{V}_{\mathrm{AL}}$ rather than $\widehat{V}_{\mathrm{RS}}$. Of course, one could take the viewpoint that the consistency check performed here is unnecessary, that one can just take some definition of the volume operator and not worry about triads. However, as triads prominently enter the dynamics of LQG such a point of view would render the quantum dynamics obsolete. In other words, the dynamics and all other

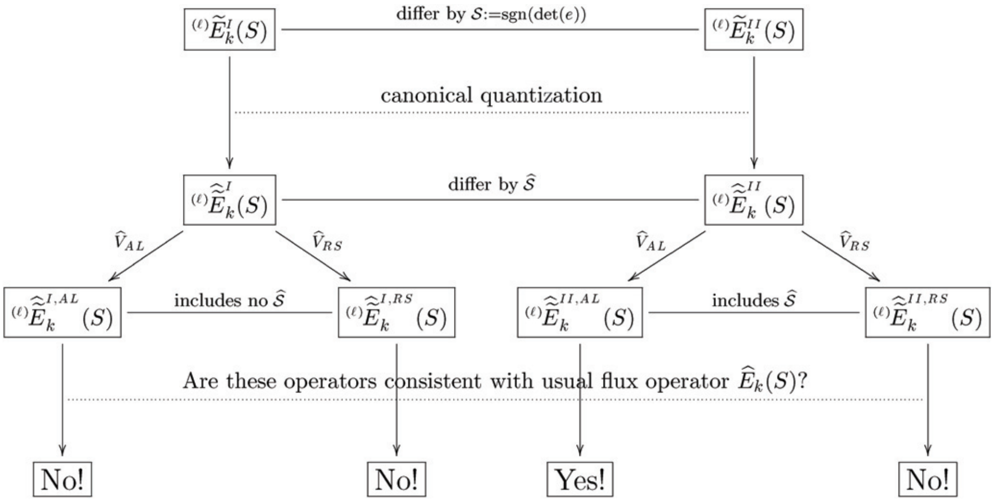

Figure 8. The consistency check done in this paper showed that only the operator ${}^{(\ell)}\widehat{\widetilde{E}}_k^{\rm II,AL}(S)$ is consistent with the usual flux operator $\widehat{E}_k(S)$.

operators which depend on triads such as the length operator [7] or spatially diffeomorphism invariant operators forces us to use $\widehat{V}_{\rm AL}$ rather than $\widehat{V}_{\rm RS}$. Finally, note that one of the motivations for choosing $\widehat{V}_{\rm RS}$ rather than $\widehat{V}_{\rm AL}$ is that $\widehat{V}_{\rm RS}$ does not depend on the differentiable structure of Σ. Hence one could use homeomorphism rather than diffeomorphism in order to define 'diffeomorphism invariant' states, as advertised in [21, 22]. This makes the Hilbert space of such states separable. However, note that homeomorphisms are not a symmetry of the classical theory. Furthermore, there are other possibilities of arriving at a separable Hilbert space: one can decompose the Hilbert space $\mathcal{H}_{\rm Diff}$ corresponding to the strict diffeomorphisms into an uncountably infinite direct sum of separable Hilbert spaces. In the current proposals for the quantum dynamics all of these mutually isomorphic Hilbert spaces are left invariant. If the theory is left invariant by the Dirac observables, then they would be superselected and any one of them would capture the full physics of LQG.

The technical details of the check, mainly displayed in our companion paper [20], required a substantial computational effort. We have also summarized the results of the consistency check in figure 8.

There are an order of ten crucial stages in the calculation where things could have gone badly wrong. Following the details of our analysis, one sees that all the subtle issues mentioned must be properly taken care of in order to get an even qualitatively correct result. These subtleties involve, among other things, the following.

(1) The meaning of the limit as we remove the regulator and to define the alternative flux operator has to be understood in the same way as for the fundamental flux operator; otherwise the alternative flux operator is identical to zero. This issue is discussed in full detail in section 4.3 of [20].
(2) Switching from the spin network basis to the abstract angular momentum basis, discussed elaborately in section 5.1 of [20], has to take care of the precise unitary map between these two representations of the angular momentum algebra; otherwise the two operators differ drastically from each other. This unitary map is not mentioned in the literature because for gauge invariant operators it drops out of the equations. However, for the non-gauge invariant flux it has a large impact.

(3) Very unexpectedly, the sign of the determinant of the triad enters the calculation in a crucial way. Classically one would expect it to be negligible, especially in an orientable manifold. However, had we dropped it from the quantum computation then the alternative flux operator would again have vanished identically. It is very pleasing to see that quantization based on a pseudo-vector density is ruled out. This is for the same reason that one cannot implement the momentum operator $i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ on $L_2(\mathbb{R}^+, \mathrm{d}x)$.
(4) A different ordering than the one we chose would have resulted again in the zero operator being even in case II.
(5) As for the fundamental flux operator, one has to first smear the alternative flux operator into the direction transversal to the surface under consideration. For the fundamental flux this implies that edges of type 'in' are not acted on. Without this additional smearing the fundamental flux would be ill-defined. For the alternative flux this implies furthermore that the classification of edges into the types up, down, in and out is meaningful at all. Indeed, one actually can define the alternative flux without the additional smearing. The result is well defined. However, it would differ drastically from the fundamental flux as soon as there is a vertex of valence higher than two of the graph of the spin network state in question within the surface. The additional smearing has the effect that with $\mathrm{d}t$ measure one all the vertices within the surfaces S_t are bivalent.
(6) Furthermore, the analyses in section 4.3 of [20] show that without the additional smearing we would be missing the crucial factor $1/2$ in equation (4.8) and our C_{reg} would be off the value found in [6].
(7) Just following the tedious calculations step by step as explicitly shown in [20] and evaluating all the CGCs involved, one sees that all the subtle signs have to be there, everything fits only when doing the calculation with 100% accuracy. The calculation is therefore a highly sensitive consistency check.
(8) All the ℓ-dependence disappears even at small values of j. This is especially surprising because the classical approximation of a connection by a holonomy along a given path becomes worse as we let ℓ grow. Of course, in the limit we take paths of infinitesimal length; however, this is done *after* quantization and it could have happened that the quantization is affected by a non-trivial ℓ-dependence which however should disappear in the limit of large j.

It transpires that the reason for getting a zero operator without the sign operator unveils a so far not appreciated symmetry of the volume operator. It would be desirable to understand the symmetry from a more abstract perspective.

This paper along with our companion paper [20] is one of the first papers that tightens the mathematical structure of full LQG by using the kind of consistency argument that we used here. Many more such checks should be performed in the future to remove ambiguities of LQG and to make the theory more rigid, in particular those connected with the quantum dynamics.

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References

[1] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)
Thiemann T 2005 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press) (*Preprint* gr-qc/0110034)

[2] Rovelli C 1998 Loop quantum gravity *Living Rev. Rel.* **1** 1 (*Preprint* gr-qc/9710008)
Thiemann T 2003 Lectures on loop quantum gravity *Lecture Notes Phys.* **631** 41–135 (*Preprint* gr-qc/0210094)
Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* **21** R53 (*Preprint* gr-qc/0404018)
Smolin L 2004 An invitation to loop quantum gravity *Preprint* hep-th/0408048

[3] Thiemann T 1996 Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity *Phys. Lett.* B **380** 257–64 (*Preprint* gr-qc/9606088)
Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839–73 (*Preprint* gr-qc/9606089)
Thiemann T 1998 Quantum spin dynamics (QSD): II. The kernel of the Wheeler–DeWitt constraint operator *Class. Quantum Grav.* **15** 875–905 (*Preprint* gr-qc/9606090)
Thiemann T 1998 Quantum spin dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity *Class. Quantum Grav.* **15** 1207–47 (*Preprint* gr-qc/9705017)
Thiemann T 1998 Quantum spin dynamics (QSD): IV. 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity *Class. Quantum Grav.* **15** 1249–80 (*Preprint* gr-qc/9705018)
Thiemann T 1998 Quantum spin dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories *Class. Quantum Grav.* **15** 1281–314 (*Preprint* gr-qc/9705019)

[4] Thiemann T 2003 The phoenix project: master constraint programme for loop quantum gravity *Preprint* gr-qc/0305080
Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity I. General framework *Preprint* gr-qc/0411138
Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: II. Finite dimensional systems *Preprint* gr-qc/0411139
Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity III. SL(2,R) models *Preprint* gr-qc/0411140
Dittrich B and Thiemann T Testing the master constraint programme for loop quantum gravity: IV. Free field theories *Preprint* gr-qc/0411141
Dittrich B and Thiemann T Testing the master constraint programme for loop quantum gravity: V. Interacting field theories *Preprint* gr-qc/0411142

[5] Rovelli C and Smolin L 1995 Discreteness of volume and area in quantum gravity *Nucl. Phys.* B **442** 593
Rovelli C and Smolin L 1995 Discreteness of volume and area in quantum gravity *Nucl. Phys.* B **456** 734 (erratum)

[6] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry: II. Volume operators *Adv. Theor. Math. Phys.* **1** 388–429

[7] Thiemann T 1998 A length operator for canonical quantum gravity *J. Math. Phys.* **39** 3372–92 (*Preprint* gr-qc/9606092)

[8] Thiemann T 2001 Quantum spin dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of Gauge field theories *Class. Quantum Grav.* **18** 3293–338 (*Preprint* hep-th/0005232)
Thiemann T 2001 Gauge field theory coherent states (GCS): I. General properties *Class. Quantum Grav.* **18** 2025–64 (*Preprint* hep-th/0005233)
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): II. Peakedness properties *Class. Quantum Grav.* **18** 2561–636 (*Preprint* hep-th/0005237)
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): III. Ehrenfest theorems *Class. Quantum Grav.* **18** 4629–81 (*Preprint* hep-th/0005234)
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamic limit *Class. Quantum Grav.* **18** 4997–5033 (*Preprint* hep-th/0005235)
Sahlmann H, Thiemann T and Winkler O 2001 Coherent states for canonical quantum general relativity and the infinite tensor product extension *Nucl. Phys.* B **606** 401–40 (*Preprint* gr-qc/0102038)
Thiemann T 2002 Complexifier coherent states for canonical quantum general relativity *Preprint* gr-qc/0206037

[9] Sahlmann H and Thiemann T 2002 Towards the QFT on curved spacetime limit of QGR: 1. A general scheme *Preprint* gr-qc/0207030
Sahlmann H and Thiemann T 2002 Towards the QFT on curved spacetime limit of QGR: 2. A concrete implementation *Preprint* gr-qc/0207031

[10] Sahlmann H 2002 When do measures on the space of connections support the triad operators of loop quantum gravity? *Preprint* gr-qc/0207112

Sahlmann H 2002 Some comments on the representation theory of the algebra underlying loop quantum gravity *Preprint* gr-qc/0207111
Sahlmann H and Thiemann T 2003 On the superselection theory of the Weyl algebra for diffeomorphism invariant quantum gauge theories *Preprint* gr-qc/0302090
Sahlmann H and Thiemann T 2003 Irreducibility of the Ashtekar–Isham–Lewandowski representation *Preprint* gr-qc/0303074
Okolow A and Lewandowski J 2003 Diffeomorphism covariant representations of the holonomy flux algebra *Preprint* gr-qc/0302059
Fleischhack C 2004 Representations of the Weyl Algebra in quantum geometry *Preprint* math-ph/0407006
Lewandowski J, Okolow A, Sahlmann H and Thiemann T Uniqueness of diffeomorphism invariant states on holonomy–flux algebras *Comm. Math. Phys.* at press *Preprint* gr-qc/0504147
[11] Ashtekar A and Isham C J 1992 Representations of the holonomy algebras of gravity and non-Abelian gauge theories *Class. Quantum Grav.* **9** 1433 (*Preprint* hep-th/9202053)
Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy $C^\star$ algebras *Knots and Quantum Gravity* ed J Baez (Oxford: Oxford University Press)
[12] Thiemann T 1998 Closed formula for the matrix elements of the volume operator in canonical quantum gravity *J. Math. Phys.* **39** 3347–71 (*Preprint* gr-qc/9606091)
[13] Loll R 1997 Imposing $\det e > 0$ in discrete quantum gravity *Phys. Lett.* B **399** 227 (*Preprint* gr-qc/9703033)
[14] Immirzi G 1997 Quantum gravity and Regge calculus *Nucl. Phys.* **57** 65 (*Preprint* gr-qc/9701052)
Rovelli C and Thiemann T 1998 The Immirzi parameter in quantum general relativity *Phys. Rev.* D **57** 1009-14 (*Preprint* gr-qc/9705059)
[15] Ashtekar A, Baez J C and Krasnov K 2001 Quantum geometry of isolated horizons and black hole entropy *Adv. Theor. Math. Phys.* **4** 1–94 (*Preprint* gr-qc/0005126)
Meissner K 2004 Black hole entropy in loop quantum gravity *Class. Quantum Grav.* **21** 5245–52 (*Preprint* gr-qc/0407052)
Domagala M and Lewandowski J 2004 Black hole entropy from quantum geometry *Class. Quantum Grav.* **21** 5233–44 (*Preprint* gr-qc/0407051)
[16] De Pietri R and Rovelli C 1996 Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity *Phys. Rev.* D **54** 2664–90 (*Preprint* gr-qc/9602023)
De Pietri R 1997 On the relation between the connection and the loop representation of quantum gravity *Class. Quantum Gravity* **14** 14 (*Preprint* gr-qc/9605064)
De Pietri R 1996 Spin networks and recoupling in loop quantum gravity *Proc. 2nd Conf. on Constrained Dynamics and Quantum Gravity (Santa Margherita, Italy, 17–21 Sept.)*
De Pietri R 1997 *Nucl. Phys.* B **57** 251 (*Preprint* gr-qc/9701041)
[17] Brunnemann J and Thiemann T 2004 Simplification of the spectral analysis of the volume operator in loop quantum gravity *Preprint* gr-qc/0405060
[18] Smolin L 1992 Recent developments in non-perturbative quantum gravity *Preprint* hep-th/9202022
[19] Brunnemann J 2005 Private communication
[20] Giesel K and Thiemann T 2006 Consistency check on volume and triad operator quantisation in loop quantum gravity: II *Class. Quantum Grav.* **23** 5693–771 (*Preprint* gr-qc/0507037)
[21] Fairbairn W and Rovelli C 2004 *J. Math. Phys.* **45** 2802 (*Preprint* gr-qc/0403047)
[22] Grot N and Rovelli C 1996 *J. Math. Phys.* **37** 3014 (*Preprint* gr-qc/9604010)