ON THE CONCEPT OF AN ASYMPTOTIC VELOCITY IN T^3-GOWDY SPACETIMES

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ABSTRACT. This is the first of two papers which together prove strong cosmic censorship in T^3-Gowdy spacetimes. In the end, we prove that there is a set of initial data, open with respect to the $C^2 \times C^1$-topology and dense with respect to the C^∞-topology, such that the corresponding spacetimes have the following properties. Given an inextendible causal geodesic, one direction is complete, the other is incomplete and the Kretschmann scalar, i.e. the Riemann tensor contracted with itself, blows up in the incomplete direction. In fact, it is possible to give a very detailed description of the asymptotic behaviour in the direction of the singularity for the generic solutions. In this paper, we shall however focus on the concept of asymptotic velocity. Under the symmetry assumptions made here, Einstein's equations reduce to a wave map equation with a constraint. The target of the wave map is the hyperbolic plane. There is a natural concept of kinetic and potential energy density, and the perhaps most important result of this paper is that the limit of the potential energy as one lets time tend to the singularity for a fixed spatial point is zero and that the limit exists for the kinetic energy. We define the asymptotic velocity, v_∞, to be the non-negative square root of the limit of the kinetic energy density. The asymptotic velocity has some very important properties. In particular, curvature blow up and the existence of smooth expansions of the solutions close to the singularity can be characterized by the behaviour of v_∞. It also has properties such that if $0 < v_\infty(\theta_0) < 1$, then v_∞ is smooth in a neighbourhood of θ_0. Furthermore, if $v_\infty(\theta_0) > 1$ and v_∞ is continuous in θ_0, then v_∞ is smooth in a neighbourhood of θ_0. Finally, we show that the map from initial data to the asymptotic velocity is continuous under certain circumstances and that what will in the end constitute the generic set of solutions is an open set with respect to the $C^2 \times C^1$-topology on initial data.

1. INTRODUCTION

1.1. Motivation and background. In [5], Yvonne Choquet-Bruhat showed that it is possible to view the Einstein vacuum equations as an initial value problem. Later, Choquet-Bruhat and Geroch [6] proved that, given vacuum initial data, there is a maximal globally hyperbolic development of the data, and that this development is unique up to isometry. There are however examples for which it is possible to extend the maximal globally hyperbolic development in inequivalent ways [7]. Consequently, it is not possible to predict what spacetime one is in simply by looking at initial data. This naturally leads to the strong cosmic censorship conjecture, stating that for generic initial data, the maximal globally hyperbolic development is inextendible. The statement is rather vague, as it does not specify exactly what is meant by generic, and since it does not give a precise definition of inextendibility; a spacetime may be extendible in one differentiability class but inextendible in another. In order to have a precise statement, one has to give a clear definition of these concepts. To prove the conjecture in general is not feasible at this time. For this reason it is tempting to consider the following related problem. Consider a class of initial data satisfying a given set of symmetry conditions. Is it possible to show that the maximal globally hyperbolic development is inextendible for initial data that are generic in this class? Note that, strictly speaking, this problem is unrelated to the original one, since a class of initial data satisfying symmetry conditions is a non-generic class in the full set of initial data. However, this is the problem that will be addressed in this paper and the next.

One way of proving that a spacetime is inextendible is to prove that, given a causal geodesic in the spacetime, there are two possible outcomes in a given time direction; either the geodesic is complete, or it is incomplete but the curvature is unbounded along it, cf. Lemma 30. Note that the natural associated inextendibility concept is that of C^2-inextendibility. Note also that it is of course conceivable that one could get away with proving less and still getting inextendibility. In this paper, we are concerned with the T^3-Gowdy spacetimes, and for these spacetimes it is known that in one time direction, the inextendible causal geodesics are always complete, cf. [21], and in the other, they are always incomplete, cf. Proposition 14. One is thus interested in proving that for generic initial data, the curvature becomes unbounded in the incomplete direction of every causal geodesic. This ties together the strong cosmic censorship conjecture and the problem of trying to understand the structure of singularities in cosmological spacetimes. By the singularity theorems, cosmological spacetimes typically have a singularity in the sense of causal geodesic incompleteness. However, it is of interest to know that one generically also has a singularity in the sense of curvature blow up.

To our knowledge, the only result concerning strong cosmic censorship in an inhomogeneous cosmological setting is contained in [9]. This paper is concerned with polarized Gowdy spacetimes and contains a proof of the statement that there is an open and dense set of initial data for which the maximal globally hyperbolic development is inextendible. Note however that the authors do not restrict themselves to T^3 topology; all topologies compatible with Gowdy symmetry are allowed. In our setting, polarized T^3-Gowdy corresponds to setting $Q = 0$ in (2)-(3), i.e. one gets a linear PDE for one unknown function. To analyze the asymptotic behaviour of this linear equation is of course easier, but the freedom one has when perturbing the initial data is more restricted. In other words, not all aspects of the problem are simplified by considering the polarized sub case.

1.2. **Objects of study.** For the purposes of this paper, we shall take the Gowdy spacetimes to be defined by (1). The question then arises why it should be natural to consider such a class. Since this question has already been addressed elsewhere, cf. [12] and [8] and, for a brief description, [19], we do not wish to do so here as well. Suffice it to say that there are geometric conditions that lead to this form of the metric. Let

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2] \tag{1}$$

Here, $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on T^3. The Einstein vacuum equations become

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau} Q_\theta^2) = 0 \tag{2}$$
$$Q_{\tau\tau} - e^{-2\tau} Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau} P_\theta Q_\theta) = 0, \tag{3}$$

and

$$\lambda_\tau = P_\tau^2 + e^{-2\tau} P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau} Q_\theta^2) \tag{4}$$
$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P} Q_\theta Q_\tau). \tag{5}$$

Obviously, (2)-(3) do not depend on λ, so the idea is to solve these equations and then find λ by integration. There is however one obstruction to this; the integral of the right hand side of (5) has to be zero. This is a restriction to be imposed on the initial data for P and Q, which is then preserved by the equations. In the end, the equations of interest are however the two non-linear coupled wave equations (2)-(3). In the above parameterization, the singularity corresponds to $\tau \to \infty$, and essentially all the work in this paper concerns the asymptotic behaviour of solutions to (2)-(3) in this time direction. Note that there is a special solution of (2)-(5) given by $P = \tau$, $Q = 0$ and $\lambda = \tau$. The corresponding metric has the property that the curvature tensor is identically zero.

The equations (2)-(3) are wave map equations. In fact, let

$$g_0 = -e^{-2\tau} d\tau^2 + d\theta^2 + e^{-2\tau} d\chi^2$$

be a Lorentz metric on $\mathbb{R} \times T^2$ and let

$$g_R = dP^2 + e^{2P} dQ^2 \tag{6}$$

be a Riemannian metric on $\mathbb{R}^2$. Then (2)-(3) are the wave map equations for a map from $(\mathbb{R} \times T^2, g_0)$ to $(\mathbb{R}^2, g_R)$ which is independent of the χ-coordinate. Note that $(\mathbb{R}^2, g_R)$ is isometric to the upper half plane $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with metric

$$g_H = \frac{dx^2 + dy^2}{y^2} \tag{7}$$

under the map

$$\phi_{RH}(Q, P) = (Q, e^{-P}). \tag{8}$$

Thus the target space is hyperbolic space. In order to formulate the results, we need to introduce some terminology. Note that isometries of hyperbolic space map solutions to solutions. One particular isometry which will be of great use is the inversion, defined by

$$\mathrm{Inv}(Q_0, P_0) = \left[\frac{Q_0}{Q_0^2 + e^{-2P_0}}, P_0 + \ln(Q_0^2 + e^{-2P_0})\right]. \tag{9}$$

The reason for the name is that it corresponds to an inversion in the unit circle with center at the origin in the upper half plane model. A third representation of hyperbolic space which will be very useful is the disc model:

$$g_D = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, \tag{10}$$

where the underlying manifold is the open unit disc D. An isometry from the upper half plane to the disc model is given by

$$\phi_{HD} = \frac{z-i}{z+i}.$$

Composing ϕ_{HD} and ϕ_{RH}, we get what we shall refer to as *the canonical map* from the PQ-plane to the disc model:

$$\phi_{RD}(Q,P) = \frac{Q + i(e^{-P} - 1)}{Q + i(e^{-P} + 1)}. \tag{11}$$

1.3. **Asymptotic expansions.** In the analysis of Gowdy spacetimes, the existence of expansions for the solutions close to the singularity in certain situations is the key starting point. The idea of finding such expansions started with the paper [13] by Grubišić and Moncrief. In our setting, the natural expansions are

$$P(\tau,\theta) = v_a(\theta)\tau + \phi(\theta) + u(\tau,\theta) \tag{12}$$

$$Q(\tau,\theta) = q(\theta) + e^{-2v_a(\theta)\tau}[\psi(\theta) + w(\tau,\theta)] \tag{13}$$

where $w, u \to 0$ as $\tau \to \infty$ and $0 < v_a(\theta) < 1$. Note that if we have a solution with such expansions, then $Q(\tau,\theta)$ converges and $P(\tau,\theta)$ tends to infinity as $\tau \to \infty$. Applying ϕ_{RH}, we see that for a fixed θ the solution roughly speaking goes to the boundary along a geodesic in the upper half plane model, see Figure 1. A heuristic argument motivating the condition on the velocity can be found in [2]. In the non-generic case $Q = 0$, one can prove that (12) holds without any condition on v_a. This special case is called polarized Gowdy and has been studied in [14], which also considers the other topologies for Gowdy spacetimes. In [15] and [17], the authors developed methods for proving that given v_a, ϕ, q, ψ with a suitable degree of regularity and $0 < v_a < 1$, there are unique solutions to (2)-(3) with asymptotics of the form (12)-(13). In [15], the regularity requirement was that of real analyticity, a condition which was relaxed to smoothness in [17]. It is of interest to note that if q is constant, the condition on v_a can be relaxed to $v_a > 0$. In [19], we proved a result going in the other direction, i.e. we provided a condition on initial data which lead to asymptotic expansions of the form (12)-(13). The condition is rather technical, requiring bounds of up to three derivatives in L^2. On the other hand, the arguments are not peculiar to $1+1$-systems of equations. In fact, the results apply to a generalization of (2)-(3) where S^1 is replaced by T^d, the d-dimensional torus, ∂_θ^2 is replaced by Δ and the product of spatial derivatives is replaced by the scalar product of gradients. In [20], we proved another condition which only requires bounds in the C^1-norm on initial data. In this case the methods are based on considering the behaviour along characteristics, and are thus not very easy to generalize to higher spatial dimensions, though it is of course conceivable that some aspects of the argument could be useful in higher dimensions. In this paper, we shall prove yet another condition on initial data that lead to smooth expansions, see Section 4.

1.4. **Asymptotic velocity.** According to our experience, the most important part of the expansions (12)-(13) is the function v_a. This object may seem to be arbitrary and devoid of geometric content. That this is not the case can be seen in the

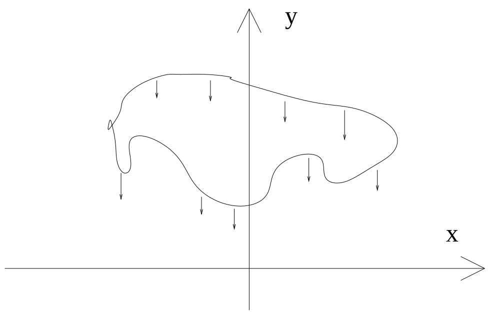

FIGURE 1. The asymptotic behaviour of a solution satisfying asymptotics of the form (12)-(13) as seen in the upper half plane model.

following way. Define the potential and kinetic energy densities by

$$\mathcal{P}(\tau,\theta) = e^{-2\tau}(P_\theta^2 + e^{2P}Q_\theta^2)(\tau,\theta) \tag{14}$$

$$\mathcal{K}(\tau,\theta) = (P_\tau^2 + e^{2P}Q_\tau^2)(\tau,\theta). \tag{15}$$

Naively differentiating the expansions and computing $\mathcal{K}$, one sees that this expression converges to v_a^2. In this sense, v_a^2 has a geometric significance. Due to Corollary 6, the point wise limit of the kinetic energy density always exists. This naturally leads to the following definition.

Definition 1. Let $x = (Q, P)$ be a solution to (2)-(3) and let $\theta_0 \in S^1$. Then we define *the asymptotic velocity at* θ_0 to be

$$v_\infty(\theta_0) = \left[\lim_{\tau\to\infty} \mathcal{K}(\tau,\theta_0)\right]^{1/2}.$$

If we wish to make the dependence on the solution explicit, we shall write $v_\infty[x]$.

There is another perspective on this object that will be of interest. Let d_R be the topological metric induced by the Riemannian metric (6) and let $(Q_0, P_0) \in \mathbb{R}^2$ be some reference point. Given a solution to (2)-(3), we can then define

$$\rho(\tau,\theta) = d_R\{[Q(\tau,\theta), P(\tau,\theta)], [Q_0, P_0]\}.$$

This is the hyperbolic distance from a reference point to the solution at a space time point. We shall be interested in the limit $\rho(\tau,\theta)/\tau$ as $\tau \to \infty$. Note that if this limit exists, it is independent of the base point (Q_0, P_0). Furthermore, it coincides for solutions that are related by an isometry.

Theorem 1. *Consider a solution to (2)-(3) and let* $\theta_0 \in S^1$. *Then*

$$\lim_{\tau\to\infty} \frac{\rho(\tau,\theta_0)}{\tau} = v_\infty(\theta_0).$$

Furthermore, v_∞ *is semi continuous in the sense that given* θ_0*, there is for every* $\epsilon > 0$ *a* $\delta > 0$ *such that for all* $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$

$$v_\infty(\theta) \le v_\infty(\theta_0) + \epsilon.$$

Proof. This follows from Corollary 6 and 7. □

The importance of the asymptotic velocity comes from the fact that if $v_\infty(\theta_0) \neq 1$, then the curvature blows up along any causal curve ending at θ_0. We refer the reader to Section 11 for a precise statement. Note that the solution $P = \tau$, $Q = 0$ has the property that $v_\infty \equiv 1$. Furthermore, the corresponding metric, with $\lambda = \tau$, has a curvature tensor which is identically zero. In other words, if $v_\infty(\theta_0) = 1$, the curvature need not necessarily blow up along a causal curve ending at θ_0. As a consequence of the above theorem, one can prove that for $z = \phi_{RD} \circ x$, the limit

$$\lim_{\tau\to\infty} \left[\frac{z}{|z|}\frac{\rho}{\tau}\right](\tau,\theta)$$

always exists, cf. Lemma 8. Note here that $\rho/|z|$ is a real analytic function from the open unit disc to the real numbers if ρ is the hyperbolic distance from the origin of the unit disc to the solution, cf. (31). Let us call the limit $v(\theta)$. Note that it would be more natural to refer to this function as the asymptotic velocity, since it gives not only the rate at which the solution tends to the boundary of hyperbolic space, but also the point of the boundary to which it converges. In the special case $v_\infty(\theta) = 0$, the solution does not tend to the boundary, in fact, $z(\tau,\theta)$ remains in compact subset of the open unit disc for $\tau \geq 0$, cf. [20].

The type of arguments used to prove the above results can also be used to prove statements concerning the asymptotic behaviour of e.g. P_τ. Let us use the notation $\mathcal{D}_{\theta_0,\tau} = [\theta_0 - e^{-\tau}, \theta_0 + e^{-\tau}]$.

Proposition 1. *Consider a solution to (2)-(3) and let $\theta_0 \in S^1$. Then*

$$\lim_{\tau\to\infty} \| |P_\tau(\tau,\cdot)| - v_\infty(\theta_0)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0, \quad \lim_{\tau\to\infty} \|(e^P Q_\tau)(\tau,\cdot)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0$$

and

$$\lim_{\tau\to\infty} \|\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0.$$

In particular, $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$ or to $-v_\infty(\theta_0)$. If $P_\tau(\tau,\theta_0) \to -v_\infty(\theta_0)$, then $(Q_1, P_1) = \mathrm{Inv}(Q,P)$ has the property that $P_{1\tau}(\tau,\theta_0) \to v_\infty(\theta_0)$. Furthermore, if $v_\infty(\theta_0) > 0$, then $Q_1(\tau,\theta_0)$ converges to 0.

Proof. This follows by combining Corollary 5, Proposition 8, 9 and Lemma 7. □

The fact that $P_\tau(\tau,\theta_0)$ sometimes converges to a negative value is a nuisance, and this is associated with the occurrence of so called false spikes, cf. Definition 2. To get a geometric picture of the situation, let us assume that we have a solution with $v_\infty(\theta_0) > 0$. Recall the isometry from the PQ-plane to the upper half plane defined in (8). Assuming $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$, we conclude that $P(\tau,\theta_0)$ tends to infinity, and since $e^P Q_\tau$ is bounded, cf. Lemma 3, $Q(\tau,\theta_0)$ converges exponentially. The corresponding picture in the upper half plane is that the solution at (τ,θ_0) converges to a point of the boundary in the upper half plane. In particular, it converges to the part of the boundary constituted by the real line. If $P_\tau(\tau,\theta_0)$ converges to $-v_\infty(\theta_0)$, $P(\tau,\theta_0)$ tends to $-\infty$ and the corresponding behaviour in the upper half plane is that the solution at (τ,θ_0) tends to infinity. In other words, the solution again tends to the boundary. Note however that by applying an inversion to the solution, we get convergence to the origin in the upper half plane. In this sense, the fact that $P_\tau(\tau,\theta_0)$ sometimes converges to $v_\infty(\theta_0)$ and sometimes to $-v_\infty(\theta_0)$ is associated with the fact that in the upper half plane there is a distinguished boundary point, namely infinity. In the disc model, there is no distinguished boundary point, and therefore, there is no problem of this type. However, we shall later use the Gowdy to Ernst transformation which takes a very simple form in the PQ-variables. There are in other words different advantages of the different points of view.

It is interesting to note that it is important to restrict one's attention to regions of the form $\mathcal{D}_{\theta_0,\tau}$ in order to get conclusions such as the ones given in Proposition 1. One conclusion of this proposition is that $\mathcal{P}$ converges to zero point wise everywhere. By Corollary 11, $\mathcal{P}$ does however not in general converge to zero uniformly. Only considering $\mathcal{D}_{\theta_0,\tau}$ thus simplifies the behaviour significantly. Note that this is the smallest region one can consider if one wants to say something about what happens at the singularity at θ_0.

1.5. **Asymptotic expansions.**

Let us give examples of how the asymptotic velocity can be used as a criterion for the existence of expansions.

Proposition 2. *Let (Q,P) be a solution to (2)-(3) and assume $v_\infty = 0$ in a compact interval K with non-empty interior. Then there are $q,\phi \in C^\infty(K,\mathbb{R})$, polynomials Ξ_k and a T such that for all $\tau \geq T$*

$$\|P_\tau(\tau,\cdot)\|_{C^k(K,\mathbb{R})} + \|P(\tau,\cdot)-\phi\|_{C^k(K,\mathbb{R})} \leq \Xi_k e^{-2\tau}, \tag{16}$$

$$\|Q_\tau(\tau,\cdot)\|_{C^k(K,\mathbb{R})} + \|Q(\tau,\cdot)-q\|_{C^k(K,\mathbb{R})} \leq \Xi_k e^{-2\tau}. \tag{17}$$

Proof. See the latter parts of Section 9. □

Comparing with (13), one sees that if $v_a = 0$, then q and ψ cannot be distinguished in any natural way. The following proposition was essentially already proved in [20]. The proof is to be found at the end of Section 6.

Proposition 3. *Let (Q,P) be a solution to (2)-(3) and assume $0 < v_\infty(\theta_0) < 1$. If $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$, then there is an open interval I containing θ_0, $v_a,\phi,q,r \in C^\infty(I,\mathbb{R})$, $0 < v_a < 1$, polynomials Ξ_k and a T such that for all $\tau \geq T$*

$$\|P_\tau(\tau,\cdot)-v_a\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{18}$$

$$\|P(\tau,\cdot)-p(\tau,\cdot)\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{19}$$

$$\left\|e^{2p(\tau,\cdot)}Q_\tau(\tau,\cdot)-r\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{20}$$

$$\left\|e^{2p(\tau,\cdot)}[Q(\tau,\cdot)-q]+\frac{r}{2v_a}\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau} \tag{21}$$

where $p(\tau,\cdot) = v_a\cdot\tau+\phi$ and $\alpha > 0$. If $P_\tau(\tau,\theta_0)$ converges to $-v_\infty(\theta_0)$, then $\mathrm{Inv}(Q,P)$ has expansions of the above form in a neighbourhood of θ_0.

Remark. One consequence of the above proposition is that if $0 < v_\infty(\theta_0) < 1$, then v_∞ is smooth in a neighbourhood of θ_0. This is a very important property of the above result; we only make assumptions concerning the spatial point θ_0, but we get conclusions in a neighbourhood.

The relation between (19) and (12) is quite clear, but in order to see the relation between (21) and (13), let us define the object inside the norm on the left hand side of (21) to be $\tilde{w}$. Then

$$Q = q + e^{-2p}\left[-\frac{r}{2v_a}+\tilde{w}\right].$$

From this one can see the relation between ψ and r and between w and $\tilde{w}$. The reason for including the estimates (18) and (20) is that, strictly speaking, one cannot draw any conclusions concerning the first time derivatives of (Q, P) from asymptotic statements of the form (12)-(13). The expansions (18)-(21) together with the equations (2)-(3) are however sufficient for carrying out any computation concerning higher order time derivatives.

Theorem 2. *Let (Q, P) solve (2)-(3) and assume that $k \leq v_\infty(\theta) < k+2$ for all $\theta \in K$, where K is a compact interval with non-empty interior and $k \in \mathbb{N}$. Then either (Q, P) has expansions in K of the form (18)-(21) or $\text{Inv}(Q, P)$ has such expansions. Furthermore, the q appearing in the expansions is a constant and we can take $\alpha = 2$.*

Remark. Note in particular that if $v_\infty(\theta_0) > 1$ and v_∞ is continuous in θ_0, then v_∞ is smooth in a neighbourhood of θ_0.

Proof. See the latter part of Section 9. □

Proposition 4. *Let (Q, P) solve (2)-(3). Then there is a subset $\mathcal{E}$ of S^1 which is open and dense, and for each $\theta_0 \in \mathcal{E}$, there is an open neighbourhood of θ_0 such that either (Q, P) or $\text{Inv}(Q, P)$ has expansions of the form (16)-(17) or (18)-(21). If $v_\infty(\theta_0) \geq 1$, then the q appearing in the expansions is a constant and $\alpha = 2$.*

Remark. This sort of result was already obtained in [4].

Proof. See the latter part of Section 9. □

1.6. **Generic solutions.** In order to be able to define the generic set of solutions, we first need to define what we mean by non-degenerate true and false spikes. To our knowledge, spikes were first discussed in [2], a paper concerned with numerical studies of the Gowdy equations, see also [3]. The basic reference for the perspective taken here is however [18], and we refer the reader to this paper for more details. Note also the recent numerical work on higher velocity spikes in [11].

Definition 2. Consider a solution (Q, P) to (2)-(3). Assume $0 < v_\infty(\theta_0) < 1$ for some $\theta_0 \in S^1$ and that

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta_0) = -v_\infty(\theta_0).$$

Let $(Q_1, P_1) = \text{Inv}(Q, P)$. By Proposition 3, we get the conclusion that (Q_1, P_1) has smooth expansions in a neighbourhood I of θ_0. In particular, Q_1 converges to a smooth function q_1 in I, and the convergence is exponential in any C^k-norm. By Proposition 1, $q_1(\theta_0) = 0$. We call θ_0 a *non-degenerate false spike* if $\partial_\theta q_1(\theta_0) \neq 0$.

The reason for the terminology false spike is that the property of being a false spike is not invariant under isometries. Note that in the above setting, $0 < v_\infty(\theta) < 1$ in a neighbourhood of θ_0 and

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta)$$

in a punctured neighbourhood of θ_0. This follows from the analysis presented in Section 8. In Figure 2, Q is plotted for a solution that is about to develop a false spike. This figure was obtained by putting $u = w = 0$ in (12)-(13), choosing v_a, ϕ, q and ψ with $q(0) = 0$, performing an inversion and then plotting the result. In order to see how P_τ looks for a false spike, one only has to take 1 minus the plot of P_τ for a true spike, see Figure 3.

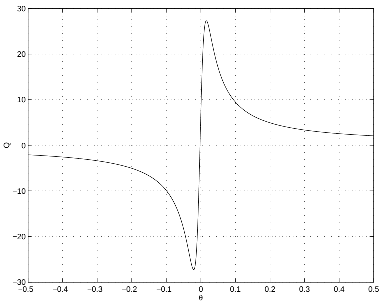

FIGURE 2. Q for a false spike.

In order to define the concept true spike, we need to define the Gowdy to Ernst transformation. Consider a solution (Q, P) to (2)-(3) with $\theta \in \mathbb{R}$ instead of S^1. Then the conditions

$$P_1 = \tau - P, \quad Q_{1\tau} = -e^{2(P-\tau)}Q_\theta, \quad Q_{1\theta} = -e^{2P}Q_\tau \tag{22}$$

define a new solution to the equations up to a constant translation in Q. We shall write $(Q_1, P_1) = \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q, P)$, where the role of the constants q_0, τ_0, θ_0 is to specify that $Q_1(\tau_0, \theta_0) = q_0$. In Section 8, we give a more precise definition of the Gowdy to Ernst transformation, and we also describe the basic properties of it. It is important to note that it does not necessarily preserve periodicity. Sometimes we shall apply the Gowdy to Ernst transformation to solutions with $\theta \in S^1$, meaning that we apply the transformation to the naturally associated 2π-periodic solution with $\theta \in \mathbb{R}$. Let us make some observations in preparation for the definition of non-degenerate true spikes. Assume that (Q, P) is a solution, $1 < v_\infty(\theta_0) < 2$ and that $P_\tau(\tau, \theta_0) \to v_\infty(\theta_0)$. Let $(Q_1, P_1) = \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q, P)$. By (22), we see that $P_{1\tau}(\tau, \theta_0) \to 1 - v_\infty(\theta_0)$. Since the limit is negative, we can apply an inversion to change the sign, cf. Proposition 1. In other words, $(Q_2, P_2) = \mathrm{Inv}(Q_1, P_1)$ has the property that $P_{2\tau}(\tau, \theta_0) \to v_\infty(\theta_0) - 1$ and $Q_2(\tau, \theta_0) \to 0$. By Proposition 3, we get the conclusion that (Q_2, P_2) have smooth expansions in a neighbourhood I of θ_0. In particular, Q_2 converges to a smooth function q_2, and the convergence is exponential in any C^k-norm. By the above, $q_2(\theta_0) = 0$.

Definition 3. Consider a solution (Q, P) to (2)-(3). Assume $1 < v_\infty(\theta_0) < 2$ for some $\theta_0 \in S^1$ and that

$$\lim_{\tau\to\infty} P_\tau(\tau, \theta_0) = v_\infty(\theta_0).$$

Let $(Q_2, P_2) = \mathrm{Inv} \circ \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q, P)$. By the observations made prior to the definition, (Q_2, P_2) has smooth expansions in a neighbourhood I of θ_0. In particular Q_2 converges to a smooth function q_2 in I and the convergence is exponential in any C^k-norm. We call θ_0 a *non-degenerate true spike* if $\partial_\theta q_2(\theta_0) \neq 0$.

Remark. The choice of q_0, τ_0, θ_0 is unimportant, cf. Lemma 13.

Note that in the above setting, $0 < v_\infty(\theta) < 1$ in a punctured neighbourhood of θ_0 and

$$\lim_{\tau\to\infty} P_\tau(\tau, \theta) = v_\infty(\theta)$$

in a neighbourhood of θ_0, cf. Lemma 14. Observe also that $\mathcal{P}$ does not converge to zero uniformly in a neighbourhood of θ_0, cf. Corollary 11, even though $\mathcal{P}$ converges to zero point wise everywhere by Proposition 1. In Figure 3, we have plotted P_τ for a solution which is about to develop a true spike. In this case, Q converges nicely, so we have not bothered to plot it. The figures presented here should be compared with the ones in [2].

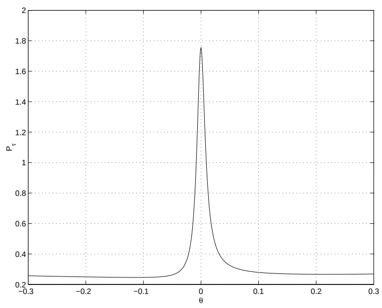

FIGURE 3. P_τ for a true spike.

Definition 4. Let $\mathcal{G}_{l,m}$ be the set of smooth solutions (Q,P) to (2)-(3) on $\mathbb{R}\times S^1$ with l non-degenerate true spikes $\theta_1,...,\theta_l$ and m non-degenerate false spikes $\theta_1',...,\theta_m'$ such that

$$\lim_{\tau\to\infty} P_\tau(\tau,\theta) = v_\infty(\theta),$$

for all $\theta \notin \{\theta_1',...,\theta_m'\}$ and $0 < v_\infty(\theta) < 1$ for all $\theta \notin \{\theta_1,...,\theta_l\}$. Let $\mathcal{G}_{l,m,c}$ be the set of $(Q,P)\in\mathcal{G}_{l,m}$ such that

$$\int_{S^1}(P_\tau P_\theta + e^{2P}Q_\tau Q_\theta)d\theta = 0. \tag{23}$$

Finally

$$\mathcal{G} = \bigcup_{l=0}^{\infty}\bigcup_{m=0}^{\infty}\mathcal{G}_{l,m}, \quad \mathcal{G}_c = \bigcup_{l=0}^{\infty}\bigcup_{m=0}^{\infty}\mathcal{G}_{l,m,c}.$$

Note that, except for the true and false spikes, the solutions have the property that there are smooth expansions of the form (18)-(21) in a neighbourhood of every spatial point.

Proposition 5. *$\mathcal{G}_{l,m}$ is open in the $C^2\times C^1$-topology on initial data and $\mathcal{G}_{l,m,c}$ is open in the $C^2\times C^1$-topology in the subset of initial data satisfying (23).*

Proof. The proof is to be found at the end of Section 10. □

Proposition 6. *Given $z\in\mathcal{G}_{l,m}$, there is an open neighbourhood of the initial data for z in the $C^1\times C^0$ topology such that for each corresponding solution $\hat{z}$, $0<[1-v_\infty[\hat{z}](\theta)]^2<1$ for all $\theta\in S^1$.*

Remark. Note that the solutions in the open neighbourhood have the property that the curvature blows up everywhere on the singularity.

Proof. The proof is to be found at the end of Section 10. □

Finally, let us state the properties of spacetimes corresponding to generic initial data.

Definition 5. Let (M, g) be a connected Lorentz manifold which is at least C^2. Assume there is a connected C^2 Lorentz manifold $(\hat{M}, \hat{g})$ of the same dimension as M and an isometric embedding $i : M \to \hat{M}$ such that $i(M) \neq \hat{M}$. Then we say that M is C^2-*extendible.* If (M, g) is not C^2-extendible, we say that it is C^2-*inextendible.*

Proposition 7. *Consider the set of smooth initial data $\mathcal{S}_{i,p,c}$ to (2)-(3) satisfying (23). There is a subset $\mathcal{G}_{i,c}$ of $\mathcal{S}_{i,p,c}$ with the following properties*

- *$\mathcal{G}_{i,c}$ is open with respect to the $C^1 \times C^0$-topology on $\mathcal{S}_{i,p,c}$,*
- *every spacetime corresponding to initial data in $\mathcal{G}_{i,c}$ has the property that in one time direction, it is causally geodesically complete, and in the opposite time direction, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded along any inextendible causal curve,*
- *for every spacetime corresponding to initial data in $\mathcal{G}_{i,c}$, the maximal globally hyperbolic development is C^2-inextendible.*

Remark. Any T^3-Gowdy spacetime has the property that it is causally geodesically complete to the future and each causal geodesic is incomplete to the past, cf. [21] and Proposition 14.

Proof. Define $\mathcal{G}_{i,c}$ to be the union of the neighbourhoods constructed in Proposition 6 intersected with $\mathcal{S}_{i,p,c}$. The first statement is immediate and the second statement follows from Lemma 29 and [21]. Combining this with Lemma 30, we obtain the third statement. □

1.7. **Outline of the paper.** In Section 2, we introduce the equations in the disc model, the basic monotonic quantities and the terminology necessary in order to obtain estimates for the derivatives. The reason for introducing the equations in the disc model is to avoid the problems associated with false spikes. The importance of the different monotonic quantities cannot be over emphasized; they are the heart of basically every argument. Furthermore, we generalize the solution concept. To consider non-periodic solutions is natural in view of the fact that we wish to apply the Gowdy to Ernst transformation, an operation which does not respect periodicity. It is also natural to weaken the differentiability conditions if we wish to prove that the generic set is open in $C^2 \times C^1$. In Section 3, we discuss suitable topologies on the set of solutions and some properties of the generalized solutions introduced in Section 2. The sort of properties we are interested in are continuity properties with respect to the $C^k \times C^{k-1}$-topology of the map taking initial data at one hypersurface to initial data at another hypersurface. In Section 4, we give a new condition on initial data yielding smooth expansions at the singularity.

After these preliminary observations, the heart of the paper consists of Sections 5 and 6. In these sections, we prove that the point wise limit of $\mathcal{K}$ exists. The essence of the argument is to consider the solution in regions of the form $\mathcal{D}_{\theta_0,\tau}$ for $\tau \geq \tau_0$. There is a quantity, $e^{-\tau}F_{\theta_0}$, defined in Section 2, which dominates the full energy density $\mathcal{P} + \mathcal{K}$, cf. (14) and (15), in $\mathcal{D}_{\theta_0,\tau}$ and which is monotonically decaying. If the potential energy density is non-zero, this object decreases. On an intuitive level, one then expects the potential energy density to converge to zero.

In order to turn this intuition into a rigorous argument, it is however necessary to have some bound on the variation of $\mathcal{P}$ in the regions of interest. The purpose of Section 5 is to provide the necessary bounds. In the beginning of Section 6, we use this to prove that $\mathcal{P}(\tau, \theta_0 \pm e^{-\tau})$ always converges to zero. One important property of the equations is that if one wants to compute $e^{-\tau}F_{\theta_0}(\tau)$ using initial data in $\mathcal{D}_{\theta_0,\tau_0}$, one only needs to have information concerning the initial data close to the characteristics, i.e. close to $(\tau_0, \theta_0 \pm e^{-\tau_0})$, as the difference $\tau - \tau_0$ becomes large. One thus expects the kinetic energy density along the characteristics to dominate the limit of $e^{-\tau}F_{\theta_0}$, since the potential energy density converges to zero along the characteristics. In fact, it turns out to be possible to prove that the lim inf of the kinetic energy density along the characteristics dominates the limit of $e^{-\tau}F_{\theta_0}$. These are then the necessary tools for proving the existence of the asymptotic velocity, that the limit of P_τ exists etc.

The definition of the asymptotic velocity is by a point wise limit. However, it is often of interest to have uniform control. How to go from point wise to uniform limits under special circumstances is the subject of Section 7. This is not a completely trivial question, as can be seen by the fact that the potential energy density converges to zero point wise everywhere, but in the presence of a true spike, it does not converge to zero uniformly, cf. the discussion following Definition 3. Note that if ρ/τ converges uniformly, then v_∞ is continuous. In Section 7, we prove that if v_∞ is continuous in some compact interval, then the convergence has to be uniform. In Section 8, we introduce the Gowdy to Ernst transformation and combine it with previously obtained results in order to conclude something about the existence of asymptotic expansions when the velocity is not an integer. The reason one can do this is that one can use the Gowdy to Ernst transformation together with an inversion repeatedly in order to reduce this situation to the situation where the velocity is strictly between 0 and 1, something we know how to deal with. If one is interested in obtaining expansions in an interval where the velocity passes through an integer value, one has to come up with a different argument. This is the subject of Section 9. In Section 10 we prove that the map from initial data to asymptotic velocity under certain circumstances is continuous with respect to the $C^1 \times C^0$-topology of initial data. We also prove that under certain circumstances, the map to the limit of Q and the limit of Q_θ are continuous with respect to the $C^2 \times C^1$-topology of initial data. Finally, in Section 11, we prove that if the asymptotic velocity at a point θ_0 is not 1, then the curvature blows up along any causal curve ending at θ_0 on the singularity.

2. Notation and monotonic quantities

2.1. **Notation.** Since we shall be interested in the Gowdy to Ernst transformation, which does not respect periodicity, and since we shall be interested in the continuity of the map taking initial data at one point in time to another point in time with respect to the $C^k \times C^{k-1}$-topology, it is natural to extend the solution concept in the following way.

Definition 6. Let $\mathcal{S}_k$, where $k \in \mathbb{N} \cup \{\infty\}$ satisfies $k \geq 2$, denote the set of $x = (Q, P)$ with $x \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ solving (2)-(3). Let $\mathcal{S}_{p,k}$ be the subset of $\mathcal{S}_k$ consisting of x that are periodic in θ with period 2π. If $k = \infty$, we shall speak of

$\mathcal{S}$ and $\mathcal{S}_p$ and not of $\mathcal{S}_\infty$ and $\mathcal{S}_{p,\infty}$. Given $x = (Q, P) \in \mathcal{S}_{p,2}$, let

$$c_0[x] = \int_{S^1} (P_\tau P_\theta + e^{2P} Q_\tau Q_\theta) d\theta. \tag{24}$$

Note that this quantity is independent of τ due to the equations. Finally, we shall denote the set of $x \in \mathcal{S}_{p,k}$ satisfying $c_0[x] = 0$ by $\mathcal{S}_{p,c,k}$. If $k = \infty$, we shall also use the notation $\mathcal{S}_{p,c}$

The main tool in the analysis consists of studying the behaviour of suitable objects along characteristics. Let us specify the necessary terminology. For $(Q, P) \in \mathcal{S}_k$, define, for $0 \leq j \leq k-1$,

$$\mathcal{A}_{j,\pm} = \frac{1}{2} e^\tau [(\partial_\tau \partial_\theta^j P \pm e^{-\tau} \partial_\theta^{j+1} P)^2 + e^{2P} (\partial_\tau \partial_\theta^j Q \pm e^{-\tau} \partial_\theta^{j+1} Q)^2].$$

We also define $\mathcal{A}_\pm = \mathcal{A}_{0,\pm}$. Sometimes we shall need to refer to the particular solution and for $x = (Q, P)$, we shall write $\mathcal{A}_{k,\pm}[x]$. The important point is that

$$(\partial_\tau \mp e^{-\tau} \partial_\theta) \mathcal{A}_\pm = \frac{1}{2} e^\tau (\mathcal{K} - \mathcal{P}) = \frac{1}{2} (\mathcal{A}_+ + \mathcal{A}_-) - e^\tau \mathcal{P}, \tag{25}$$

where the potential and kinetic energy densities are defined by (14) and (15). In order to obtain estimates for the higher derivatives, we shall need the following computation

$$(\partial_\tau \mp e^{-\tau} \partial_\theta) \mathcal{A}_{k,\pm} = I_{1,k,\pm} + I_{2,k,\pm}, \tag{26}$$

where

$$\begin{aligned} I_{1,k,\pm} \;=\; & \frac{1}{2} e^\tau \{(\partial_\tau \partial_\theta^k P)^2 - e^{-2\tau} (\partial_\theta^{k+1} P)^2 + e^{2P} [(\partial_\tau \partial_\theta^k Q)^2 - e^{-2\tau} (\partial_\theta^{k+1} Q)^2]\} \\ & - e^{2P+\tau} (P_\tau \pm e^{-\tau} P_\theta) [(\partial_\tau \partial_\theta^k Q)^2 - e^{-2\tau} (\partial_\theta^{k+1} Q)^2] \\ & + e^{2P+\tau} (Q_\tau \pm e^{-\tau} Q_\theta) [(\partial_\tau \partial_\theta^k P \pm e^{-\tau} \partial_\theta^{k+1} P)(\partial_\tau \partial_\theta^k Q \mp e^{-\tau} \partial_\theta^{k+1} Q) \\ & - (\partial_\tau \partial_\theta^k P \mp e^{-\tau} \partial_\theta^{k+1} P)(\partial_\tau \partial_\theta^k Q \pm e^{-\tau} \partial_\theta^{k+1} Q)] \end{aligned} \tag{27}$$

and

$$\begin{aligned} I_{2,k,\pm} \;=\; & e^\tau \{\partial_\theta^k [e^{2P} (Q_\tau^2 - e^{-2\tau} Q_\theta^2)] - 2e^{2P} (Q_\tau \partial_\theta^k \partial_\tau Q - e^{-2\tau} Q_\theta \partial_\theta^{k+1} Q)\} \\ & \cdot (\partial_\tau \partial_\theta^k P \pm e^{-\tau} \partial_\theta^{k+1} P) + e^{2P+\tau} \sum_{l=1}^{k-1} \binom{k}{l} [-2\partial_\theta^{k-l} \partial_\tau P \partial_\theta^l \partial_\tau Q \\ & + 2e^{-2\tau} \partial_\theta^{k-l+1} P \partial_\theta^{l+1} Q](\partial_\tau \partial_\theta^k Q \pm e^{-\tau} \partial_\theta^{k+1} Q). \end{aligned} \tag{28}$$

If $k \leq 1$, the sum is taken to be zero. If $I = [a, b]$ is a subinterval of $\mathbb{R}$, let

$$\mathcal{D}_I = \{(\tau, \theta) \in \mathbb{R}^2 : \theta \in [a - e^{-\tau}, b + e^{-\tau}]\}.$$

The definition if I is an open interval is similar. If I only consists of the point θ_0, we shall also write $\mathcal{D}_{\theta_0}$, cf. Figure 4. Let

$$\mathcal{D}_{I,\tau} = [a - e^{-\tau}, b + e^{-\tau}].$$

We define $\mathcal{D}_{\theta_0,\tau}$ similarly. Finally,

$$F_{I,k}(\tau) = \sum_\pm \sup_{\theta \in \mathcal{D}_{I,\tau}} \mathcal{A}_{k,\pm}(\tau, \theta).$$

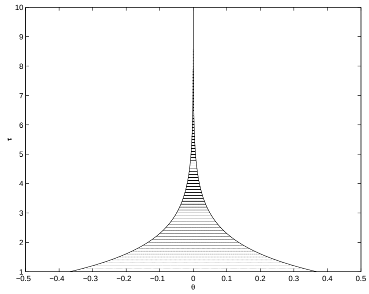

FIGURE 4. Depiction of $\mathcal{D}_{\theta_0}$. The horizontal lines yield $\mathcal{D}_{\theta_0,\tau}$.

If $(Q,P) \in \mathcal{S}_p$ and $I = S^1$ we shall use the notation F_k instead, and we define $F = F_0$, $F_I = F_{I,0}$. If $I = \{\theta_0\}$, we shall write F_{θ_0} instead of F_I. We shall say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ *converges to zero in* $\mathcal{D}_I$ if

$$\lim_{\tau\to\infty} \|f(\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} = 0.$$

2.2. **Equations in the disc model.** When considering the asymptotic behaviour of solutions to (2)-(3), there are some technical complications that arise. These are associated with the occurrence of false spikes and have already been addressed in the introduction. One way around the problems is to consider the equations in the disc model. The equation corresponding to (2)-(3) is

$$\partial_\tau\left(\frac{z_\tau}{(1-|z|^2)^2}\right) - e^{-2\tau}\partial_\theta\left(\frac{z_\theta}{(1-|z|^2)^2}\right) = \frac{2z}{(1-|z|^2)^3}[|z_\tau|^2 - e^{-2\tau}|z_\theta|^2]. \tag{29}$$

The easiest way to see this is to use the action

$$\int\int[P_\tau^2 + e^{2P}Q_\tau^2 - e^{-2\tau}(P_\theta^2 + e^{2P}Q_\theta^2)]d\theta d\tau$$

in order to derive the Gowdy equations. It translates into

$$\int\int\frac{4(|z_\tau|^2 - e^{-2\tau}|z_\theta|^2)}{(1-|z|^2)^2}d\theta d\tau$$

in the disc model. Note that the canonical map given by (11) defines an injective and surjective map of solutions to (2)-(3) to solutions of (29), and we shall use this map to identify solutions to the different equations. Thus, if we have a solution $x = (Q,P)$ to (2)-(3), and suddenly speak of z, we shall take it to be understood that $z = \phi_{RD} \circ x$, and vice versa. Furthermore, when we speak of solutions, we shall take it to be understood that they are smooth unless otherwise specified. One important isometry in the PQ-plane is the inversion defined in (9). One can compute that this corresponds to the isometry $-\bar{z}$ in the disc model, i.e.

$$\phi_{RD} \circ \text{Inv} \circ \phi_{RD}^{-1}(z) = -\bar{z}. \tag{30}$$

We shall also use (ρ,ϕ) as variables. They are defined by

$$z = |z|e^{i\phi}, \quad \rho = \ln\frac{1+|z|}{1-|z|}. \tag{31}$$

Note that ρ is the hyperbolic distance from the origin of the disc to the solution. In the end we are only interested in the absolute value of derivatives of ϕ, and since

$$\phi_\tau^2 = \left|\partial_\tau\left(\frac{z}{|z|}\right)\right|^2,$$

it is clear that these make sense as long as $|z| > 0$. It is useful to keep in mind that

$$\rho_\tau^2 + \sinh^2\rho\phi_\tau^2 = \frac{4|z_\tau|^2}{(1-|z|^2)^2}, \tag{32}$$

and similarly for the θ-derivatives. It will be convenient to have the inverse of the canonical map

$$(Q,P) = [-\frac{2\mathrm{Im}z}{1+|z|^2-2\mathrm{Re}z}, -\ln(1-|z|^2) + 2\ln|1-z|. \tag{33}$$

Using the definition of ρ, this yields

$$P = \rho - 2\ln(1+|z|) + 2\ln|1-z|. \tag{34}$$

Note that a problem arises if z roughly coincides with 1. This is associated with false spikes.

2.3. **Monotonic quantities.** There are several monotonic quantities that are crucial to the argument. Here we wish to define these quantities and specify under what circumstances they are monotonic. Let us prove that the most fundamental quantity, $e^{-\tau}F_I$, is monotonic. Due to (25), we have, for $\tau \geq \tau_0$ and $\theta \in \mathcal{D}_{I,\tau}$,

$$\begin{aligned}\mathcal{A}_\pm(\tau,\theta) &= \mathcal{A}_\pm(\tau_0, \theta \pm e^{-\tau_0} \mp e^{-\tau}) + \int_{\tau_0}^{\tau}[(\partial_u \mp e^{-u}\partial_\theta)\mathcal{A}_\pm](u, \theta \pm e^{-u} \mp e^{-\tau})du \\ &\leq \sup_{\theta\in\mathcal{D}_{I,\tau_0}} \mathcal{A}_\pm(\tau_0,\theta) + \frac{1}{2}\int_{\tau_0}^{\tau} F_I(u)du.\end{aligned}$$

Taking the supremum and adding the estimates, we get

$$F_I(\tau) \leq F_I(\tau_0) + \int_{\tau_0}^{\tau} F_I(u)du. \tag{35}$$

By Grönwall's lemma, we get the conclusion that

$$e^{-\tau}F_I(\tau) \leq e^{-\tau_0}F_I(\tau_0). \tag{36}$$

Note that this inequality gives an apriori bound on $\mathcal{P}$ and $\mathcal{K}$ in $\mathcal{D}_{I,\tau}$ for $\tau \geq T$, where $T \in \mathbb{R}$ and I is a compact interval. In order to introduce another important monotonic quantity, let us define

$$\mathcal{B}_\pm = \frac{1}{2}e^\tau[(P_\tau - 1 \pm e^{-\tau}P_\theta)^2 + e^{2P}(Q_\tau \pm e^{-\tau}Q_\theta)^2]. \tag{37}$$

Then

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{B}_\pm = \frac{1}{2}e^\tau[(P_\tau-1)^2 - e^{2P}Q_\tau^2 - e^{-2\tau}P_\theta^2 + e^{2P-2\tau}Q_\theta^2] \leq \frac{1}{2}(\mathcal{B}_+ + \mathcal{B}_-). \tag{38}$$

If we introduce

$$G_I(\tau) = \sum_\pm \sup_{\theta\in\mathcal{D}_{I,\tau}} \mathcal{B}_\pm(\tau,\theta),$$

and so on, we obtain, similarly to the above,

$$G_I(\tau) \leq G_I(\tau_0) + \int_{\tau_0}^{\tau} G_I(u)du. \tag{39}$$

Consequently $e^{-\tau}G_I$ is monotonically decaying. For $x \in \mathcal{S}_2$, we shall use the notation $G_I[x]$ if we wish to emphasize the particular solution. The analogue of F_I in the disc model is given by

$$F_I[z](\tau) = 2\sum_{\pm}\left\|\frac{z_\tau \pm e^{-\tau}z_\theta}{1-|z|^2}\right\|^2_{C^0(\mathcal{D}_{I,\tau},\mathbb{R}^2)}.$$

Note that the definition of F_I is geometric whereas the definition of G_I is not. There are two more important quantities that are monotonic only under special circumstances. For $x \in \mathcal{S}_2$, let

$$H_I[x](\tau) = \frac{1}{2}\sum_{\pm}\left\|\left(P_\tau - \frac{P}{\tau} \pm e^{-\tau}P_\theta\right)^2 + e^{2P}(Q_\tau \pm e^{-\tau}Q_\theta)^2\right\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})}$$

Then $H_I[x]$ satisfies an estimate

$$H_I[x](\tau) \le \left(\frac{T}{\tau}\right)^2 H_I[x](T) \tag{40}$$

for $\tau \ge T$, assuming $1 \le P(s,\theta) \le s-1$ for all $s \in [T,\tau]$ and $\theta \in \mathcal{D}_{I,s}$. This was proved in Lemma 1 of [20]. For $x \in \mathcal{S}_2$, $z = \phi_{RD}\circ x$, let

$$L_I[z](\tau) = \frac{1}{2}\sum_{\pm}\left\|\frac{2z_\tau}{1-|z|^2} - \frac{\rho}{\tau}\frac{z}{|z|} \pm \frac{2e^{-\tau}z_\theta}{1-|z|^2}\right\|^2_{C^0(\mathcal{D}_{I,\tau},\mathbb{R}^2)}$$

Then

$$L_I(\tau) \le \left(\frac{T}{\tau}\right)^2 L_I(T) \tag{41}$$

assuming $\rho(s,\theta) \le s-2$ for all $s \in [T,\tau]$ and $\theta \in \mathcal{D}_{I,s}$. A proof of this fact was given in Lemma 5 of [20].

3. Properties of generalized solutions

In Section 2, we generalized the solution concept, and in this section, we wish to write down the associated basic facts. The material is quite standard, but we include it for the sake of completeness. As we shall not use it until Section 10, the reader might want to skip this section on a first reading.

In order to be able to define a metric on C^m-functions that are not necessarily bounded, let

$$D_m(f) = \sum_{l=1}^{\infty} 2^{-l}\frac{\|f\|_{C^m([-l,l],\mathbb{R}^2)}}{1+\|f\|_{C^m([-l,l],\mathbb{R}^2)}}$$

for $f \in C^m(\mathbb{R},\mathbb{R}^2)$.

Definition 7. Consider $x_i \in \mathcal{S}_k$, $i = 1,2$, where $k \in \mathbb{N}$ and $k \ge 2$. Define, for $m \in \mathbb{N}$, $m \le k$, and $\tau \in \mathbb{R}$,

$$d_{m,\tau}(x_1,x_2) = D_m[x_1(\tau,\cdot) - x_2(\tau,\cdot)] + D_{m-1}[x_{1\tau}(\tau,\cdot) - x_{2\tau}(\tau,\cdot)].$$

If $x_i \in \mathcal{S}$, $i = 1,2$, we define

$$d_\tau(x_1,x_2) = d_{\infty,\tau}(x_1,x_2) = \sum_{m=1}^{\infty} 2^{-m} d_{m,\tau}(x_1,x_2).$$

Finally, we shall use the notation $d_m = d_{m,0}$ and $d = d_0$.

Lemma 1. *Let $x_n \in \mathcal{S}_l$ be a Cauchy sequence with respect to d_{m,τ_0} for some $m \leq l-1$. Then x_n is a Cauchy sequence with respect to d_{m,τ_1} for any $\tau_1 \in \mathbb{R}$.*

Remark. Note that m and l are allowed to equal ∞.

Proof. There are two cases to consider. Either $\tau_1 \geq \tau_0$, or $\tau_1 \leq \tau_0$. The cases are rather similar, but since we shall study the former case quite extensively in what follows, let us only consider the latter. Let $I = [\theta_1, \theta_2]$ be a compact subinterval of $\mathbb{R}$ and assume for the moment that m is finite. Let

$$\mathcal{D}_{I,\tau_1,\tau} = [\theta_1 + e^{-\tau} - e^{-\tau_1}, \theta_2 - e^{-\tau} + e^{-\tau_1}]$$

for all $\tau \in [\tau_1, \tau_0]$. Define

$$F_{I,\tau_1,k}[x](\tau) = \sum_{\pm} \|\mathcal{A}_{k,\pm}[x](\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau_1,\tau},\mathbb{R})}.$$

For $k=0$, we shall speak of F_{I,τ_1} and not of $F_{I,\tau_1,0}$. Due to (25), we have, for $\tau \in [\tau_1, \tau_0]$ and $\theta \in \mathcal{D}_{I,\tau_1,\tau}$,

$$\begin{aligned} \mathcal{A}_{\pm}(\tau,\theta) &= \mathcal{A}_{\pm}(\tau_0, \theta \pm e^{-\tau_0} \mp e^{-\tau}) \\ &\quad - \int_{\tau}^{\tau_0} (\partial_\tau \mp e^{-s}\partial_\theta)\mathcal{A}_{\pm}(s, \theta \pm e^{-s} \mp e^{-\tau}) ds \\ &\leq \|\mathcal{A}_{\pm}(\tau_0,\cdot)\|_{C^0(\mathcal{D}_{I,\tau_1,\tau_0},\mathbb{R})} + \frac{1}{2}\int_{\tau}^{\tau_0} F_{I,\tau_1}(s) ds. \end{aligned}$$

Taking the supremum over $\mathcal{D}_{I,\tau_1,\tau}$ and adding, we get

$$F_{I,\tau_1}(\tau) \leq F_{I,\tau_1}(\tau_0) + \int_{\tau}^{\tau_0} F_{I,\tau_1}(s) ds.$$

By a Grönwall's lemma type argument, we get

$$e^{\tau} F_{I,\tau_1}[x_n](\tau) \leq e^{\tau_0} F_{I,\tau_1}[x_n](\tau_0)$$

for all $\tau \in [\tau_1, \tau_0]$. Since x_n is a Cauchy sequence with respect to d_{1,τ_0}, the right hand side is uniformly bounded in n. Due to this we have uniform bounds on $P_{n\tau}$, $P_{n\theta}$, $e^{P_n} Q_{n\tau}$ and $e^{P_n} Q_{n\theta}$ in the set

$$\mathcal{D}_{I,[\tau_1,\tau_0]} = \bigcup_{s \in [\tau_1,\tau_0]} \{s\} \times \mathcal{D}_{I,\tau_1,s},$$

cf. Figure 5. Since we have uniform control of P_n in $\{\tau_0\} \times \mathcal{D}_{I,\tau_1,\tau_0}$, this control can be used to first get control of P_n and then Q_n and its first derivatives uniformly in $\mathcal{D}_{I,[\tau_1,\tau_0]}$. Let us make the inductive assumption that we have bounds on up to k derivatives in $\mathcal{D}_{I,[\tau_1,\tau_0]}$ uniformly in n. Due to (26) and the inductive assumption, we can carry out an argument as above in order to get an estimate

$$F_{I,\tau_1,k}[x_n](\tau) \leq F_{I,\tau_1,k}[x_n](\tau_0) + \int_{\tau}^{\tau_0} \{C_k F_{I,\tau_1,k}[x_n](s) + C_k F_{I,\tau_1,k}^{1/2}[x_n](s)\} ds,$$

for all $s \in [\tau_1, \tau_0]$, where C_k is independent of n, but is allowed to depend on τ_1 and τ_0. Using this estimate, a Grönwall's lemma type argument and the fact that we have bounds independent of n for $\tau = \tau_0$, we get bounds on up to $k+1$ derivatives in $\mathcal{D}_{I,[\tau_1,\tau_0]}$ uniformly in n. Since x_n is a Cauchy sequence with respect to d_{m,τ_0}, we thus get bounds on m derivatives of x_n in $\mathcal{D}_{I,[\tau_1,\tau_0]}$ uniformly in n. Strictly

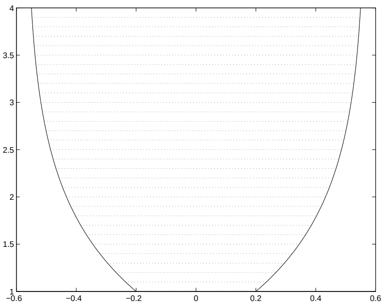

FIGURE 5. Depiction of $\mathcal{D}_{I,[\tau_1,\tau_0]}$. The horizontal lines yield $\mathcal{D}_{I,\tau_1,\tau}$.

speaking, we have only obtained estimates for expressions hit by m derivatives including at most one time derivative. This can however be remedied by using the equations.

The equations (2)-(3) can be written $x_{\tau\tau} - e^{-2\tau}x_{\theta\theta} = J(\tau, x, x_\theta, x_\tau)$ for some J which is smooth in all its variables. Let $\hat{x} = x_{n_2} - x_{n_1}$ for some $n_1, n_2 \in \mathbb{N}$. Note that

$$\begin{aligned}
\hat{x}_{\tau\tau} - e^{-2\tau}\hat{x}_{\theta\theta} &= \int_0^1 \partial_s\{J(\tau, sx_{n_2} + (1-s)x_{n_1}, x_{n_2\theta}, x_{n_2\tau})\}ds \\
&+ \int_0^1 \partial_s\{J(\tau, x_{n_1}, sx_{n_2\theta} + (1-s)x_{n_1\theta}, x_{n_2\tau})\}ds \\
&+ \int_0^1 \partial_s\{J(\tau, x_{n_1}, x_{n_1\theta}, sx_{n_2\tau} + (1-s)x_{n_1\tau})\}ds.
\end{aligned}$$

If we differentiate this equation $k-1$ times, where $k \leq m$, the right hand side consists of terms that can be bounded by

$$C_k \sum_{j=0}^{k} |\partial_\theta^j \hat{x}| + C_k \sum_{j=0}^{k-1} |\partial_\theta^j \partial_\tau \hat{x}|$$

in $\mathcal{D}_{I,[\tau,\tau_0]}$, where the C_k are allowed to depend on τ_0 and τ_1, but not on n. Letting

$$\begin{aligned}
\mathcal{B}_{k,\pm} &= \frac{1}{2}|\partial_\theta^k(\hat{x}_\tau \pm e^{-\tau}\hat{x}_\theta)|^2 + \frac{1}{2}|\hat{x}|^2, \\
\hat{F}_{I,\tau_1,k}(\tau) &= \sum_{l=0}^{k}\sum_{\pm} \|\mathcal{B}_{l,\pm}(\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau_1,\tau},\mathbb{R})}.
\end{aligned}$$

we get for $\tau \in [\tau_1, \tau_0]$, $k \geq 1$ and $\theta \in \mathcal{D}_{I,\tau_1,\tau}$,

$$\begin{aligned}
(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{B}_{k-1,\pm}(\tau,\theta) &\leq C_k \sum_{j=0}^{k} |\partial_\theta^j \hat{x}|^2(\tau,\theta) + C_k \sum_{j=0}^{k-1} |\partial_\theta^j \partial_\tau \hat{x}|^2(\tau,\theta) \\
&\leq C_k \hat{F}_{I,\tau_1,k-1}(\tau).
\end{aligned}$$

By an argument similar to the one proving the uniform bounds on the derivatives, we get

$$\hat{F}_{I,\tau_1,k-1}(\tau) \leq \hat{F}_{I,\tau_1,k-1}(\tau_0) + C_k \int_\tau^{\tau_0} \hat{F}_{I,\tau_1,k-1}(s)ds.$$

By this inequality and a Grönwall's lemma type argument, we get the conclusion that $[x_n(\tau_1,\cdot), x_{n\tau}(\tau_1,\cdot)]$ is a Cauchy sequence in the $C^m \times C^{m-1}$-norm on I. Note that we assumed m to be finite. However, due to the construction of d_{τ_1}, it is enough that we have convergence in any C^m-norm with m finite on any I in order to get the desired conclusion for $m = \infty$. The lemma follows. □

Corollary 1. *Let $(f, g) \in C^k(\mathbb{R}, \mathbb{R}^2) \times C^{k-1}(\mathbb{R}, \mathbb{R}^2)$ for some $k \geq 2$. For any $\tau_0 \in \mathbb{R}$, there is a unique solution $(Q, P) = x \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ to (2)-(3) such that $x(\tau_0, \cdot) = f$ and $x_\tau(\tau_0, \cdot) = g$.*

Proof. Let us first prove that we have global existence for smooth initial data. Let I be some compact subinterval of $\mathbb{R}$ and let $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ equal 1 on I. Consider the initial value problem with (f, g) replaced by $(\phi f, \phi g)$. In this case one can bound any C^k norm of the solution on any compact time interval using estimates similar to the ones in the proof of the previous lemma. Thus one gets global existence for the modified data. Letting the interval tend to $\mathbb{R}$, we get global existence for smooth data due to how the domain of dependence looks. For initial data in $C^k \times C^{k-1}$, we get the desired conclusion by approximating with smooth data and using the proof of the previous lemma; by the proof, we get convergence in the C^k-norm with respect to space and time in regions of the form $\mathcal{D}_{I,[\tau_1,\tau_0]}$ and similarly to the future of τ_0. Uniqueness follows by an argument similar to the end of the proof of the previous lemma. □

Note that if we give the subsets $\mathcal{S}_{p,k} \subset \mathcal{S}_k$ and $\mathcal{S}_p \subset \mathcal{S}$ the induced topology, we get the same topology as if though we had used the $C^k \times C^{k-1}$ topology on initial data on the circle in the former case, and the C^∞ topology on initial data on the circle in the latter case.

4. Existence of expansions

As was mentioned in the introduction, there are several conditions on initial data that lead to the existence of asymptotic expansions of the form (18)-(21). The first condition appeared in [19] and the second in [20]. The first required 3 derivatives in L^2, but the second only involved up to 1 derivative in C^0. However, the second condition involved P/τ. It would be more natural to have a condition that only involves $P_\tau, e^{-\tau}P_\theta, e^P Q_\tau$ and $e^{P-\tau}Q_\theta$. In this section we provide such a condition. Beyond being the most natural of the conditions on initial data, another advantage of it is that the proof is the easiest. The existence of the monotonic quantity $e^{-\tau}F_I$ has been known for a long time, but the existence of the monotonic quantity $e^{-\tau}G_I$ is, as far as we are aware, a contribution of the present paper. When one has this additional monotonic quantity, the task of providing a condition on initial data that leads to the existence of asymptotic expansions becomes much easier. Recall the notation introduced in Section 2.

Lemma 2. *Let $(Q, P) \in \mathcal{S}$ and I be a compact interval with non-empty interior. Assume that for some τ_0 and $0 < \alpha < 1$,*

$$\max\{e^{-\tau_0}G_I(\tau_0), e^{-\tau_0}F_I(\tau_0)\} \leq (1-\alpha)^2. \tag{42}$$

Then

$$\alpha \leq P_\tau(\tau, \theta) \leq 1 - \alpha \tag{43}$$

for all $\tau \geq \tau_0$ *and* $\theta \in \mathcal{D}_{I,\tau}$. *Furthermore, if*

$$\gamma \leq P_\tau(\tau_0,\theta) \leq 1-\gamma, \tag{44}$$

for some $0 < \gamma \leq 1/2$ *and all* $\theta \in \mathcal{D}_{I,\tau_0}$*, and*

$$\left\| \left[3e^{-\tau_0}|P_\theta| + 2e^{2P}(Q_\tau^2 + e^{-2\tau_0}Q_\theta^2)\right](\tau_0,\cdot)\right\|_{C^0(\mathcal{D}_{I,\tau_0},\mathbb{R})} \leq \gamma, \tag{45}$$

then there is an $0 < \alpha < 1$ *satisfying (42). Finally, if (43) is satisfied, there are* $v_a, \phi, q, r \in C^\infty(I,\mathbb{R})$*, with* $0 < \alpha \leq v_a \leq 1-\alpha < 1$ *such that (18)-(21) hold.*

Proof. Once one has (43), the conclusions (18)-(21) follow by Proposition 3. What we need to prove is thus that (42) implies (43) and that (44) and (45) imply (42). Let us start with the first implication. By the arguments presented in Section 2, $e^{-\tau}F_I(\tau)$ and $e^{-\tau}G_I(\tau)$ are monotonically decaying with time. For all $\tau \geq \tau_0$, (42) will thus be satisfied with τ_0 replaced by τ. Thus

$$|P_\tau(\tau,\theta)| \leq e^{-\tau/2}F_I^{1/2}(\tau) \leq 1-\alpha \quad \text{and} \quad |1-P_\tau(\tau,\theta)| \leq e^{-\tau/2}G_I^{1/2}(\tau) \leq 1-\alpha,$$

for all $\tau \geq \tau_0$ and all $\theta \in \mathcal{D}_{I,\tau}$, so that (43) holds. In order to prove that (44) and (45) imply (42), let us first note that

$$e^{-\tau_0}|P_\theta(\tau_0,\theta)|, |P_\tau(\tau_0,\theta)|, |1-P_\tau(\tau_0,\theta)| \leq 1$$

for all $\theta \in \mathcal{D}_{I,\tau_0}$ due to (44) and (45). Estimates of the form $ab \leq (a^2+b^2)/2$ and $a^2 \leq |a|$ for $|a| \leq 1$ then yield the conclusion that

$$\begin{aligned} 2e^{-\tau_0} & \sup_{\theta\in\mathcal{D}_{I,\tau_0}} \mathcal{B}_\pm(\tau_0,\theta) \\ \leq & \sup_{\theta\in\mathcal{D}_{I,\tau_0}} (P_\tau-1)^2(\tau_0,\theta) + \sup_{\theta\in\mathcal{D}_{I,\tau_0}} \left[3e^{-\tau_0}|P_\theta| + 2e^{2P}(Q_\tau^2 + e^{-2\tau_0}Q_\theta^2)\right](\tau_0,\theta) \\ \leq & (1-\gamma)^2 + \gamma. \end{aligned}$$

Adding the two inequalities, we get

$$e^{-\tau_0}G_I(\tau_0) \leq 1-\gamma+\gamma^2.$$

The argument for $e^{-\tau_0}F_I(\tau_0)$ is identical if one replaces $P_\tau - 1$ with P_τ. The lemma follows. □

5. Higher order derivative estimates

The main result of this paper is Theorem 1. It may seem a bit technical, but once one has the ideas needed to prove it, the rest more or less follows automatically. The main point of the argument is to study the behaviour of the solution in regions of the form $\mathcal{D}_{\theta_0}$. One observes that $e^{-\tau}F_{\theta_0}$ decays and that, on an intuitive level, if the potential energy density (14) does not converge to zero, it should in fact tend to $-\infty$. One problem with turning this intuition into a rigorous argument is due to our lack of knowledge concerning the variation of the potential energy density in regions of the form $\mathcal{D}_{\theta_0,\tau}$. Without some apriori control of this variation, it would be impossible to do anything. Fortunately, one can obtain such control by what is reasonably standard arguments, and this is the subject of the present section.

Consider (26). Note that whenever squares of the highest order derivatives appear, the form of the corresponding expression is such that we can integrate partially when integrating along the characteristic $(\tau, \theta_0 \pm e^{-\tau})$ and in doing so obtain terms in which the highest order derivatives only occur with the power one. This observation is essentially all that is needed in order to prove the following lemma.

Lemma 3. *Let $(Q,P) \in \mathcal{S}_{m+1}$ and let I be a compact interval. Then for $\tau \geq 0$ and $k \leq m-1$,*

$$\|(\partial_\tau\partial_\theta^k P)^2 + e^{-2\tau}(\partial_\theta^{k+1}P)^2 + e^{2P}[(\partial_\tau\partial_\theta^k Q)^2 + e^{-2\tau}(\partial_\theta^{k+1}Q)^2]\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} \leq C_k e^{2k\tau}.$$

Furthermore, if $\mathcal{P}$ converges to zero in $\mathcal{D}_I$, then

$$e^{-2k\tau}\{(\partial_\tau\partial_\theta^k P)^2 + e^{-2\tau}(\partial_\theta^{k+1}P)^2 + e^{2P}[(\partial_\tau\partial_\theta^k Q)^2 + e^{-2\tau}(\partial_\theta^{k+1}Q)^2]\}$$

converges to zero in $\mathcal{D}_I$ for $1 \leq k \leq m-1$.

Remark. Note that as a consequence, if for some integer $k \geq 1$, $P(\tau,\theta_0) - k\tau \to \infty$ as $\tau \to \infty$, then $(\partial_\theta^k Q)(\tau,\theta_0)$ tends to zero as τ tends to infinity.

Proof. By (36), the lemma holds for $k=0$. The argument proceeds by induction. In order to deal with the two situations simultaneously, let us introduce the notation

$$\begin{aligned} h_k(\tau) &= \|\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} + \sum_{l=1}^{k-1} e^{-2l\tau}\|(\partial_\tau\partial_\theta^l P)^2 + e^{-2\tau}(\partial_\theta^{l+1}P)^2 \\ &\quad + e^{2P}[(\partial_\tau\partial_\theta^l Q)^2 + e^{-2\tau}(\partial_\theta^{l+1}Q)^2]\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})}. \end{aligned}$$

When $k \leq 1$, we set the sum to be zero. The inductive assumptions are either that h_k is bounded or that it converges to zero, for some $k \geq 1$. We have, for $\theta \in \mathcal{D}_{I,\tau}$,

$$\mathcal{A}_{k,\pm}(\tau,\theta) = \mathcal{A}_{k,\pm}[\gamma_\pm(\tau_0)] + \int_{\tau_0}^{\tau} [(\partial_u \mp e^{-u}\partial_\theta)\mathcal{A}_{k,\pm}][\gamma_\pm(u)]du, \tag{46}$$

where

$$\gamma_\pm(u) = (u, \theta \pm e^{-u} \mp e^{-\tau}).$$

Let us use the notation $f_\pm = f \circ \gamma_\pm$, and note that

$$\partial_u f_\pm = [(\partial_u \mp e^{-u}\partial_\theta)f]_\pm. \tag{47}$$

Consider $I_{1,k,\pm} \circ \gamma_\pm$. All the terms that appear can be written in the form

$$h_\pm[(\partial_u \pm e^{-u}\partial_\theta)f_1]_\pm \partial_u f_{2\pm}$$

for some functions h, f_1, f_2. Compute

$$\begin{aligned} &\int_{\tau_0}^{\tau} \{h_\pm[(\partial_u \pm e^{-u}\partial_\theta)f_1]_\pm \partial_u f_{2\pm}\}(u)du \\ \text{(48)} \quad \leq\ & \{h_\pm[(\partial_u \pm e^{-u}\partial_\theta)f_1]_\pm f_{2\pm}\}_{\tau_0}^{\tau} - \int_{\tau_0}^{\tau}\{\partial_u h_\pm[(\partial_u \pm e^{-u}\partial_\theta)f_1]_\pm f_{2\pm}\}(u)du \\ & - \int_{\tau_0}^{\tau}\{h_\pm[(\partial_u \mp e^{-u}\partial_\theta)(\partial_u \pm e^{-u}\partial_\theta)f_1]_\pm f_{2\pm}\}(u)du. \end{aligned}$$

The possibilities for f_i are $\partial_\theta^k P$ and $\partial_\theta^k Q$. Note that

$$|[e^{u/2}(\partial_u \pm e^{-u}\partial_\theta)\partial_\theta^k P]_\pm| + |[e^{P+u/2}(\partial_u \pm e^{-u}\partial_\theta)\partial_\theta^k Q]_\pm| \leq F_{I,k}^{1/2}(u),$$

and that

$$|[e^{u/2}\partial_\theta^k P]_\pm| + |[e^{P+u/2}\partial_\theta^k Q]_\pm| \leq C h_k^{1/2}(u)\exp\left[\left(k+\frac{1}{2}\right)u\right].$$

The possibilities for h, up to a numerical factor, are

$$e^u,\quad e^{2P+u},\quad e^{2P+u}(P_u \pm e^{-u}P_\theta),\quad e^{2P+u}(Q_u \pm e^{-u}Q_\theta),$$

and the corresponding bounds for $|\partial_u h_\pm|$ are

$$Ce^u,\quad Ce^{2P_\pm+u},\quad Ce^{2P_\pm+u},\quad Ce^{P_\pm+u},$$

respectively, where we have used (47), (2)-(3) and the fact that $\mathcal{P}$ and $\mathcal{K}$ are bounded due to the zeroth step of the induction argument. The first two terms on the right hand side of (48) can thus be estimated by

$$\begin{aligned} C_k \quad &+ \quad C_k \exp\left[\left(k+\frac{1}{2}\right)\tau\right] h_k^{1/2}(\tau)F_{I,k}^{1/2}(\tau) \\ &+ \quad C_k \int_{\tau_0}^{\tau} \exp\left[\left(k+\frac{1}{2}\right)u\right] h_k^{1/2}(u)F_{I,k}^{1/2}(u)du. \end{aligned}$$

Consider the third term on the right hand side of (48). Note that

$$(\partial_u \mp e^{-u}\partial_\theta)(\partial_u \pm e^{-u}\partial_\theta)f_1 = \partial_u^2 f_1 - e^{-2u}\partial_\theta^2 f_1 \mp e^{-u}\partial_\theta f_1. \tag{49}$$

The term arising from $e^{-u}\partial_\theta f_1$ can be estimated similarly to the above. What remains of (49) amounts to applying ∂_θ^k to (2)-(3). If all the k derivatives hit one P_τ, Q_τ, P_θ or Q_θ-factor, we get the same estimate as above. All other terms are bounded by

$$C_k \int_{\tau_0}^{\tau} \exp[(2k+1)u]h_k(u)du,$$

if we assume that h_k is bounded. Thus

$$\begin{aligned} \int_{\tau_0}^{\tau} I_{1,k,\pm}(u,\theta \pm e^{-u})du \le C_k + C_k \exp\left[\left(k+\frac{1}{2}\right)\tau\right] h_k^{1/2}(\tau)F_{I,k}^{1/2}(\tau) \\ +C_k \int_{\tau_0}^{\tau} \left[\exp[(2k+1)u]h_k(u) + \exp\left[\left(k+\frac{1}{2}\right)u\right] h_k^{1/2}(u)F_{I,k}^{1/2}(u)\right] du. \end{aligned}$$

The argument concerning $I_{2,k,\pm}$ is more straightforward. Assuming h_k is bounded, we get

$$\begin{aligned} F_{I,k}(\tau) \quad &\le \quad C_k + C_k \exp\left[\left(k+\frac{1}{2}\right)\tau\right] h_k^{1/2}(\tau)F_{I,k}^{1/2}(\tau) \\ &+ \quad C_k \int_{\tau_0}^{\tau} \left[\exp[(2k+1)u]h_k(u) + \exp\left[\left(k+\frac{1}{2}\right)u\right] h_k^{1/2}(u)F_{I,k}^{1/2}(u)\right] du. \end{aligned} \tag{50}$$

Note that

$$C_k \exp\left[\left(k+\frac{1}{2}\right)\tau\right] h_k^{1/2}(\tau)F_{I,k}^{1/2}(\tau) \le \frac{1}{2}F_{I,k}(\tau) + \frac{1}{2}C_k^2 \exp[(2k+1)\tau]h_k(\tau),$$

so that (50) implies

$$\begin{aligned} F_{I,k}(\tau) \quad &\le \quad C_k + C_k \exp[(2k+1)\tau]h_k(\tau) \\ &+ \quad C_k \int_{\tau_0}^{\tau} \left[\exp[(2k+1)u]h_k(u) + \exp\left[\left(k+\frac{1}{2}\right)u\right] h_k^{1/2}(u)F_{I,k}^{1/2}(u)\right] du. \end{aligned} \tag{51}$$

Using this and the assumption that h_k is bounded, one can conclude that h_{k+1} is bounded using a Grönwall's lemma type argument. Let us assume that h_k converges to zero as $\tau \to \infty$. Let $\epsilon > 0$ and assume that τ_0 is big enough that $C_k h_k, C_k h_k^{1/2} \leq \epsilon/2$ for all $\tau \geq \tau_0$. By the above, we then get

$$F_{I,k}(\tau) \leq C_k + \epsilon \exp[(2k+1)\tau] + \int_{\tau_0}^{\tau} \epsilon \exp\left[\left(k+\frac{1}{2}\right)u\right] F_{I,k}^{1/2}(u)du.$$

Denote the right hand side by g. Note that $\epsilon \exp[(2k+1)\tau] \leq g$ and estimate

$$\begin{aligned} g' &\leq (2k+1)\epsilon \exp[(2k+1)\tau] + \epsilon \exp\left[\left(k+\frac{1}{2}\right)\tau\right] g^{1/2} \\ &\leq \left\{(2k+1)\epsilon^{1/2} \exp\left[\left(k+\frac{1}{2}\right)\tau\right] + \epsilon \exp\left[\left(k+\frac{1}{2}\right)\tau\right]\right\} g^{1/2}. \end{aligned}$$

We conclude that

$$2g^{1/2}(\tau) \leq 2g^{1/2}(\tau_0) + \left[2\epsilon^{1/2} + \frac{2}{2k+1}\epsilon\right] \exp\left[\left(k+\frac{1}{2}\right)\tau\right].$$

Thus

$$\limsup_{\tau\to\infty} \exp\left[-\left(k+\frac{1}{2}\right)\tau\right] g^{1/2}(\tau) \leq \epsilon^{1/2} + \frac{\epsilon}{2k+1}.$$

The lemma follows. □

Corollary 2. *Let $(Q,P) \in \mathcal{S}$ and I be a compact interval. Then, for $\tau \geq 0$,*

$$\|P_{\tau\tau}\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} + \|e^P Q_{\tau\tau}\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} \leq C$$

and

$$\|(\partial_\tau \pm e^{-\tau}\partial_\theta)(e^{-\tau}P_\theta)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} + \|(\partial_\tau \pm e^{-\tau}\partial_\theta)(e^{P-\tau}Q_\theta)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} \leq C.$$

Proof. The statements follow from Lemma 3 and (2)-(3). □

6. Existence of an asymptotic velocity

As has already been observed, $e^{-\tau}F_{\theta_0}$ is decaying, and the essence of the proof of Theorem 1 is to analyze the corresponding estimates in detail. There are two things that cause decay. The first comes from a non-zero $\mathcal{P}$, as has already been observed. The second comes from the fact that if one wants to compute $\mathcal{A}_\pm$ in a region $\mathcal{D}_{\theta_0,\tau_1}$ by integrating along characteristics from $\mathcal{D}_{\theta_0,\tau_0}$, one only has to integrate along a neighbourhood of the characteristics $(\tau, \theta_0 \pm e^{-\tau})$, and this neighbourhood shrinks to the characteristics as the difference $\tau_1 - \tau_0$ tends to infinity. This can be used to translate bounds on the energy density along the characteristics to bounds inside the entire region. Note that the concentration of information to the characteristics is extremely important. The estimates proved in Lemma 3 yield the conclusion that $e^{-\tau}\partial_\theta\mathcal{P}, e^{-\tau}\partial_\theta\mathcal{K}$ are bounded. Thus the variation of $\mathcal{P}$ and $\mathcal{K}$ is bounded in regions of the form $\mathcal{D}_{\theta_0,\tau}$. Since we already know that $\mathcal{P}$ and $\mathcal{K}$ are bounded, this information is not very useful. However, the point is that we only need to analyze the variation in regions which are arbitrarily small multiples of $e^{-\tau}$, and in such a situation the bounds obtained by Lemma 3 are exactly what we need. Let us be more precise.

Lemma 4. *Let $(Q,P) \in \mathcal{S}$, $\theta_0 \in \mathbb{R}$ and $\tau \geq \tau_0$. Then*

$$e^{-\tau}F_{\theta_0}(\tau) \leq e^{-\tau_0}F_{\theta_0}(\tau_0) - D_{\mathcal{P}} - D_{\mathcal{L}}, \tag{52}$$

where

$$\begin{aligned} D_{\mathcal{P}} &= \sum_{\pm} \inf_{\theta\in\mathcal{D}_{\theta_0,\tau}} \int_{\tau_0}^{\tau} e^{u-\tau}\mathcal{P}(u,\theta\pm e^{-u}\mp e^{-\tau})du, \\ D_{\mathcal{L}} &= e^{-\tau}\left[F_{\theta_0}(\tau_0)-\sum_{\pm}\sup_{\theta\in\mathcal{D}_{\theta_0,\tau}}\mathcal{A}_{\pm}(\tau_0,\theta\pm e^{-\tau_0}\mp e^{-\tau})\right]. \end{aligned}$$

Remark. The term $D_{\mathcal{P}}$ represents the decay we get if $\mathcal{P}$ does not go to zero along the characteristics. The term $D_{\mathcal{L}}$ represents the decay resulting if the energy density away from the characteristics is bigger than the energy density along the characteristics. The $\mathcal{L}$ stands for localization.

Proof. Let $\theta\in\mathcal{D}_{\theta_0,\tau}$. By (35),

$$F_{\theta_0}(\tau)\leq F_{\theta_0}(\tau_0)+\int_{\tau_0}^{\tau}F_{\theta_0}(u)du.$$

Denoting the right hand side by $h(\tau)$, we get the conclusion that $h'\leq h$. As a consequence

$$h(\tau)\leq e^{\tau-\tau_0}h(\tau_0)=e^{\tau-\tau_0}F_{\theta_0}(\tau_0).$$

Thus

$$F_{\theta_0}(\tau_0)+\int_{\tau_0}^{\tau}F_{\theta_0}(s)ds\leq F_{\theta_0}(\tau_0)e^{\tau-\tau_0} \tag{53}$$

for all $\tau\geq\tau_0$. However, if $\theta\in\mathcal{D}_{\theta_0,\tau}$, then by (25),

$$\begin{aligned} \mathcal{A}_{\pm}(\tau,\theta) &= \mathcal{A}_{\pm}(\tau_0,\theta\pm e^{-\tau_0}\mp e^{-\tau})+\int_{\tau_0}^{\tau}\frac{1}{2}(\mathcal{A}_{+}+\mathcal{A}_{-})(u,\theta\pm e^{-u}\mp e^{-\tau})du \\ &\quad -\int_{\tau_0}^{\tau}e^{u}\mathcal{P}(u,\theta\pm e^{-u}\mp e^{-\tau})du \\ &\leq \sup_{\theta'\in\mathcal{D}_{\theta_0,\tau_0}}\mathcal{A}_{\pm}(\tau_0,\theta')+\frac{1}{2}\int_{\tau_0}^{\tau}F_{\theta_0}(u)du-\int_{\tau_0}^{\tau}e^{u}\mathcal{P}(u,\theta\pm e^{-u}\mp e^{-\tau})du \\ &\quad -[\sup_{\theta'\in\mathcal{D}_{\theta_0,\tau_0}}\mathcal{A}_{\pm}(\tau_0,\theta')-\mathcal{A}_{\pm}(\tau_0,\theta\pm e^{-\tau_0}\mp e^{-\tau})]. \end{aligned}$$

Taking the supremum over $\theta\in\mathcal{D}_{\theta_0,\tau}$, adding and using (53), we get the conclusion of the lemma. □

Let us demonstrate how this estimate can be used to prove that the potential energy density has to decay to zero along characteristics. In the end we want a lemma which can also be applied to prove uniform convergence. For this reason, the statement is more technical than needed for the immediate applications.

Lemma 5. *Let $(Q,P)\in\mathcal{S}$. Assume there is an $\epsilon>0$ and sequences $\theta_n,u_n\in\mathbb{R}$ with $\{\theta_n\}$ bounded and $u_n\to\infty$ such that*

$$\mathcal{P}(u_n,\theta_n+e^{-u_n})\geq\epsilon \quad or \quad \mathcal{P}(u_n,\theta_n-e^{-u_n})\geq\epsilon. \tag{54}$$

Then there is an $\eta>0$ and a sequence $\tau_n\geq u_n$ such that

$$e^{-\tau_n}F_{\theta_n}(\tau_n)\leq e^{-u_n}F_{\theta_n}(u_n)-\eta. \tag{55}$$

Proof. Let I be a compact interval containing the sequence $\{\theta_n\}$ and let $T \in \mathbb{R}$ be such that $u_n \geq T$ for all n. By Lemma 3 and Corollary 2, there is a constant $C \geq 1$ such that

$$\|(\partial_\tau \pm e^{-\tau}\partial_\theta)\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} + \|e^{-\tau}\partial_\theta\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} \leq C \tag{56}$$

for all $\tau \geq T$. There are two cases to consider, but they are rather similar, so let us assume that the first of the two inequalities in (54) is the one that occurs. Let us use the notation $\mathcal{P}_n(\tau) = \mathcal{P}(\tau, \theta_n + e^{-\tau})$. Note that for $\theta \in \mathcal{D}_{\theta_n,\tau}$,

$$|\theta + e^{-u} - e^{-\tau} - (\theta_n + e^{-u})| \leq 2e^{-\tau}.$$

By (56), there are constants $C_i \geq 1$ such that

$$|\mathcal{P}(u, \theta + e^{-u} - e^{-\tau}) - \mathcal{P}_n(u)| \leq C_1 e^{u-\tau-1}, \quad |\mathcal{P}_n(s_2) - \mathcal{P}_n(s_1)| \leq C_2|s_2 - s_1|,$$

for $\theta \in \mathcal{D}_{\theta_n,\tau}$, $\tau \geq u$ and $u, s_1, s_2 \geq T$. Let $0 < \delta < 1$ and $\tau_n \geq u_n + 1$ be defined by

$$C_2\delta = \frac{\epsilon}{4}, \quad C_1 e^{u_n - \tau_n} = \frac{\epsilon}{4},$$

where we have assumed $\epsilon < 1$ for simplicity. Then $\mathcal{P}_n(u) \geq 3\epsilon/4$ for $u \in [u_n, u_n + \delta]$ and

$$\mathcal{P}(u, \theta + e^{-u} - e^{-\tau_n}) \geq \frac{\epsilon}{2}$$

for all $(u, \theta) \in [u_n, u_n + \delta] \times \mathcal{D}_{\theta_n,\tau_n}$. Combining this with (52), we get

$$e^{-\tau_n} F_{\theta_n}(\tau_n) \leq e^{-u_n} F_{\theta_n}(u_n) - e^{u_n - \tau_n}\frac{\epsilon}{2}\delta = e^{-u_n} F_{\theta_n}(u_n) - \frac{\epsilon^3}{32 C_1 C_2}.$$

We get the conclusion of the lemma with $\eta = \epsilon^3/(32C_1C_2)$. □

Corollary 3. *Let $(Q, P) \in \mathcal{S}$ and $\theta_0 \in \mathbb{R}$. Then*

$$\lim_{\tau\to\infty} \mathcal{P}(\tau, \theta_0 \pm e^{-\tau}) = 0.$$

Proof. Let us assume that $\mathcal{P}(\tau, \theta_0 + e^{-\tau})$ does not converge to zero. Then there is an $\epsilon > 0$ and a sequence $u_n \to \infty$ such that the conditions of Lemma 5 are fulfilled with $\theta_n = \theta_0$. Since $e^{-\tau} F_{\theta_0}(\tau)$ is monotonically decaying and bounded from below, it converges to some $\gamma \geq 0$. By letting n tend to infinity in (55), we thus get $\gamma \leq \gamma - \eta$ for some $\eta > 0$. □

Next, we prove that the kinetic energy density along the characteristics controls the limit of the energy density. Again, we make a technical statement suited for later applications to uniform convergence.

Lemma 6. *Let $(Q, P) \in \mathcal{S}$. Assume there is an $\epsilon > 0$ and sequences $\theta_n, u_n \in \mathbb{R}$ with $\{\theta_n\}$ bounded and $u_n \to \infty$ such that*

$$e^{-u_n} F_{\theta_n}(u_n) - \mathcal{K}(u_n, \theta_n + e^{-u_n}) \geq \epsilon \quad \text{or} \quad e^{-u_n} F_{\theta_n}(u_n) - \mathcal{K}(u_n, \theta_n - e^{-u_n}) \geq \epsilon. \tag{57}$$

Furthermore, let us assume that $\mathcal{P}$ converges to zero uniformly along characteristics ending at θ_n, by which we mean that for every $\xi > 0$ there is a T such that

$$\mathcal{P}(\tau, \theta_n \pm e^{-\tau}) \leq \xi$$

for all $\tau \geq T$ and all n. Then there is an $\eta > 0$, a subsequence u_{n_k} and a sequence $\tau_k \geq u_{n_k}$ such that

$$e^{-\tau_k} F_{\theta_{n_k}}(\tau_k) \leq e^{-u_{n_k}} F_{\theta_{n_k}}(u_{n_k}) - \eta.$$

Proof. There are two cases to consider, but since they are rather similar, let us assume that we have the first of the two inequalities in (57). Let I be a compact interval containing $\{\theta_n\}$ and let T be such that $u_n \geq T$ for all n. By Lemma 3 applied to I, there is a constant C_1 such that

$$|e^{-u_n}\mathcal{A}_\pm(u_n, \theta \pm e^{-u_n} \mp e^{-\tau}) - e^{-u_n}\mathcal{A}_\pm(u_n, \theta_n \pm e^{-u_n})| \leq C_1 e^{u_n - \tau}, \tag{58}$$

assuming $\theta \in \mathcal{D}_{\theta_n,\tau}$ and $\tau \geq u_n$. Since $\mathcal{P}$ converges to zero uniformly along characteristics ending at θ_n, there is, for every $\xi > 0$, a T' such that

$$|e^{-\tau}\mathcal{A}_+(\tau, \theta_n \pm e^{-\tau}) - e^{-\tau}\mathcal{A}_-(\tau, \theta_n \pm e^{-\tau})| \leq \xi \tag{59}$$

for all $\tau \geq T'$ and all n. By choosing a subsequence if necessary, we can assume that either

$$e^{-u_n}\mathcal{A}_-(u_n, \theta_n - e^{-u_n}) \geq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) \tag{60}$$

for all n, or that the opposite inequality holds for all n. The idea behind making this division is the following. If the opposite inequality to (60) holds, we should be in good shape, since $e^{-u_n}\mathcal{A}_+(u_n, \theta_n + e^{-u_n})$ satisfies this sort of bound with a margin, due to the assumptions; note that

$$e^{-u_n}\mathcal{A}_+(u_n, \theta_n + e^{-u_n}) - \frac{1}{2}\mathcal{K}(u_n, \theta_n + e^{-u_n})$$

can be assumed to be arbitrarily small by assuming n to be large enough. If (60) holds, then due to (59), the difference between the supremum of $\mathcal{A}_+(u_n, \cdot)$ over $\mathcal{D}_{\theta_n,u_n}$ and $\mathcal{A}_+(u_n, \theta_n + e^{-\tau_n})$ is some positive number depending on ϵ. This also yields decay.

1. Assume that (60) holds for all n. By (59), we then conclude that for n big enough,

$$e^{-u_n}\mathcal{A}_+(u_n, \theta_n - e^{-u_n}) \geq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon}{16},$$

so that

$$e^{-u_n}\sup_{\theta \in \mathcal{D}_{\theta_n,u_n}} \mathcal{A}_+(u_n, \theta) \geq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon}{16}.$$

Let $\tau_n \geq u_n$ be such that $C_1 e^{u_n - \tau_n} = \epsilon/8$, and assume n to be large enough that

$$e^{-u_n}\mathcal{A}_+(u_n, \theta_n + e^{-u_n}) \leq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon}{4}.$$

This is possible by the assumptions and the fact that $\mathcal{P}$ converges to zero uniformly on characteristics ending at θ_n. Then, due to (58),

$$\sup_{\theta \in \mathcal{D}_{\theta_n,\tau_n}} e^{-u_n}\mathcal{A}_+(u_n, \theta + e^{-u_n} - e^{-\tau_n}) \leq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon}{8}.$$

We get

$$\begin{aligned} & e^{-u_n}F_{\theta_n}(u_n) - \sum_\pm e^{-u_n}\sup_{\theta \in \mathcal{D}_{\theta_n,\tau_n}} \mathcal{A}_\pm(u_n, \theta \pm e^{-u_n} \mp e^{-\tau_n}) \\ \geq \; & e^{-u_n}\sup_{\theta \in \mathcal{D}_{\theta_n,u_n}} \mathcal{A}_+(u_n, \theta) - e^{-u_n}\sup_{\theta \in \mathcal{D}_{\theta_n,\tau_n}} \mathcal{A}_+(u_n, \theta + e^{-u_n} - e^{-\tau_n}) \geq \frac{\epsilon}{16}. \end{aligned}$$

Combining this with (52), we get

$$e^{-\tau_n}F_{\theta_n}(\tau_n) \leq e^{-u_n}F_{\theta_n}(u_n) - e^{u_n - \tau_n}\frac{\epsilon}{16} = e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon^2}{128C_1}.$$

The lemma follows under the present assumptions.

2. Assume that the inequality (60) does not hold for any n. Let $\tau_n \geq u_n$ be such that $C_1 e^{u_n-\tau_n} = \epsilon/16$ and assume n to be big enough that

$$e^{-u_n}\mathcal{A}_+(u_n,\theta_n+e^{-u_n}) \leq \frac{1}{2}e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon}{4}.$$

Then, using this, (58) and the negation of (60), we get, for n large enough,

$$e^{-u_n}F_{\theta_n}(u_n) - \sum_{\pm} e^{-u_n} \sup_{\theta\in\mathcal{D}_{\theta_n,\tau_n}} \mathcal{A}_\pm(u_n,\theta\pm e^{-u_n}\mp e^{-\tau_n}) \geq \frac{\epsilon}{8}.$$

Thus

$$e^{-\tau_n}F_{\theta_n}(\tau_n) \leq e^{-u_n}F_{\theta_n}(u_n) - e^{u_n-\tau_n}\frac{\epsilon}{8} = e^{-u_n}F_{\theta_n}(u_n) - \frac{\epsilon^2}{128C_1}.$$

The lemma follows under the present assumptions. □

Corollary 4. *Let* $(Q,P)\in\mathcal{S}$ *and* $\theta_0\in\mathbb{R}$. *Then*

$$\lim_{\tau\to\infty} e^{-\tau}F_{\theta_0}(\tau) = \min\{\liminf_{\tau\to\infty}\mathcal{K}(\tau,\theta_0+e^{-\tau}), \liminf_{\tau\to\infty}\mathcal{K}(\tau,\theta_0-e^{-\tau})\}.$$

Proof. Let us denote the left hand side by γ and the right hand side by δ. We know that $\gamma\geq\delta$. Assume that $\delta<\gamma$. There are again two similar cases to consider. Let us assume $\delta = \liminf_{\tau\to\infty}\mathcal{K}(\tau,\theta_0+e^{-\tau})$. Then there is an $\epsilon>0$ and a sequence $u_n\to\infty$ such that

$$e^{-u_n}F_{\theta_0}(u_n) \geq \lim_{\tau\to\infty} e^{-\tau}F_{\theta_0}(\tau) \geq \mathcal{K}(u_n,\theta_0+e^{-u_n})+\epsilon.$$

By Corollary 3, Lemma 6 is applicable with $\theta_n=\theta_0$. We obtain an $\eta>0$, a subsequence u_{n_k} and a sequence $\tau_k\geq u_{n_k}$ such that

$$e^{-\tau_k}F_{\theta_0}(\tau_k) \leq e^{-u_{n_k}}F_{\theta_0}(u_{n_k}) - \eta.$$

Letting k tend to infinity in this inequality, we obtain $\gamma\leq\gamma-\eta$ for some $\eta>0$. □

Proposition 8. *Let* $(Q,P)\in\mathcal{S}$. *Then*

$$\lim_{\tau\to\infty} \|\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0.$$

Proof. Assume the contrary. Then there exists an $\epsilon>0$ and (τ_n,θ_n), $n\geq1$, with $\theta_n\in\mathcal{D}_{\theta_0,\tau_n}$ and $\tau_n\to\infty$ such that $\mathcal{P}(\tau_n,\theta_n)\geq\epsilon$. By choosing a subsequence, we can assume that $\theta_n\geq\theta_0$ or the opposite holds for all n. Assume the former. Let

$$s_n = -\ln\left[\frac{1}{2}(e^{-\tau_n}+\theta_n-\theta_0)\right].$$

We are interested in

$$\mathcal{U}_{n,\tau} = [\theta_n - e^{-\tau}+e^{-\tau_n}, \theta_0+e^{-\tau}]$$

for $\tau\in[\tau_n,s_n]$, cf. Figure 6. Note that $\mathcal{U}_{n,s_n}$ consists of a point and that we can assume

$$\theta_n-\theta_0 \leq (1-2\delta)e^{-\tau_n},$$

where $\delta>0$ depends on ϵ. The reason for the latter is that $\mathcal{P}(\tau,\theta_0+e^{-\tau})$ converges to zero and that $e^{-\tau}\partial_\theta\mathcal{P}$ is bounded. We conclude that

$$-\ln(1-\delta) \leq s_n-\tau_n \leq \ln 2. \tag{61}$$

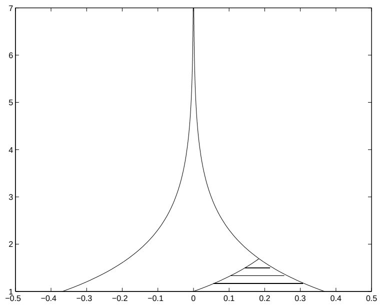

FIGURE 6. Depiction of the union of the $\mathcal{U}_{1,\tau}$ with $\theta_1 = \theta_0 = 0$ and $\tau_1 = 1$.

Let

$$H_n(\tau) = \sum_{\pm} \sup_{\theta \in \mathcal{U}_{n,\tau}} \mathcal{A}_\pm(\tau, \theta).$$

By an argument similar to the derivation of (52), we have

$$(\mathcal{K} + \mathcal{P})(s_n, \theta_0 + e^{-s_n}) \leq e^{-\tau_n} H_n(\tau_n) - \int_{\tau_n}^{s_n} e^{u - s_n} \mathcal{P}(u, \theta_n - e^{-u} + e^{-\tau_n}) du.$$

Due to (61), the fact that $\mathcal{P}(\tau_n, \theta_n) \geq \epsilon$ and the fact that $(\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{P}$ is bounded, we conclude the existence of an $\eta > 0$ depending on ϵ but not on n, such that

$$(\mathcal{P} + \mathcal{K})(s_n, \theta_0 + e^{-s_n}) \leq e^{-\tau_n} H_n(\tau_n) - \eta.$$

Note that $e^{-\tau} F_{\theta_0}(\tau)$ converges and denote the limit by γ. Then

$$\limsup_{n \to \infty} e^{-\tau_n} H_n(\tau_n) \leq \gamma \quad \text{and} \quad \limsup_{n \to \infty} \mathcal{K}(s_n, \theta_0 + e^{-s_n}) \leq \gamma - \eta.$$

Combining this with Corollary 4, we get a contradiction. The case $\theta_n \leq \theta_0$ for all n is similar. □

Corollary 5. *Consider $(Q, P) \in \mathcal{S}$ and let $\theta_0 \in \mathbb{R}$. Then*

$$\lim_{\tau \to \infty} e^{-2\tau} \| P_{\tau\theta}^2 + e^{-2\tau} P_{\theta\theta}^2 + e^{2P}(Q_{\tau\theta}^2 + e^{-2\tau} Q_{\theta\theta}^2) \|_{C^0(\mathcal{D}_{\theta_0,\tau}, \mathbb{R})} = 0.$$

Proof. This is a direct consequence of Lemma 3 and Proposition 8. □

Corollary 6. *Let $(Q, P) \in \mathcal{S}$ and $\theta_0 \in \mathbb{R}$. Then*

$$\lim_{\tau \to \infty} \mathcal{K}(\tau, \theta_0) \tag{62}$$

exists, and we denote the non-negative square root of the limit by $v_\infty(\theta_0)$. Furthermore

$$\lim_{\tau \to \infty} e^{-\tau} F_{\theta_0}(\tau) = v_\infty^2(\theta_0) \quad \text{and} \quad \| \mathcal{K}(\tau, \cdot) - v_\infty^2(\theta_0) \|_{C^0(\mathcal{D}_{\theta_0,\tau}, \mathbb{R})} = 0. \tag{63}$$

Finally, the function v_∞ is semi continuous in the sense that given θ_0, there is for every $\epsilon > 0$ a $\delta > 0$ such that for all $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$,

$$v_\infty(\theta) \leq v_\infty(\theta_0) + \epsilon. \tag{64}$$

Remark. If we wish to make the dependence on the solution explicit, we shall use the notation $v_\infty[z]$ or $v_\infty[x]$ for $x \in \mathcal{S}$.

Proof. First of all, the limit of $e^{-\tau}F_{\theta_0}(\tau)$ exists, since this quantity is monotonic and bounded from below. We define $v_\infty(\theta_0)$ to be the non-negative square root of the limit. The first of (63) follows. Due to Proposition 8 and Corollary 5, the variation of $e^{-\tau}\mathcal{A}_\pm$ inside $\mathcal{D}_{\theta_0,\tau}$ converges to zero. Consequently

$$e^{-\tau}F_{\theta_0}(\tau) - e^{-\tau}(\mathcal{A}_+ + \mathcal{A}_-)(\tau, \theta_0 + e^{-\tau})$$

converges to zero. Since $e^{-\tau}(\mathcal{A}_+ + \mathcal{A}_-) = \mathcal{P} + \mathcal{K}$ and we have Proposition 8, $\mathcal{K}(\tau, \theta_0 + e^{-\tau})$ converges to $v_\infty^2(\theta_0)$. Since the spatial variation of $\mathcal{K}$ inside $\mathcal{D}_{\theta_0,\tau}$ converges to zero due to Proposition 8 and Corollary 5, we conclude that the limit (62) exists and that the second of (63) holds. In order to prove the semi continuity, let $\epsilon > 0$. By (63), there is a T such that

$$e^{-T}F_{\theta_0}(T) \leq \left[v_\infty(\theta_0) + \frac{\epsilon}{2}\right]^2.$$

By continuity, there is a $\delta > 0$ such that

$$e^{-T}F_{I_\delta}(T) \leq [v_\infty(\theta_0) + \epsilon]^2,$$

where $I_\delta = [\theta_0 - \delta, \theta_0 + \delta]$. Since $e^{-\tau}F_{I_\delta}(\tau)$ is monotonic, we get the desired conclusion. □

Proposition 9. *Consider* $(Q, P) \in \mathcal{S}$ *and let* $\theta_0 \in \mathbb{R}$. *Then*

$$\lim_{\tau\to\infty} P_\tau(\tau, \theta_0) \tag{65}$$

exists and equals $\pm v_\infty(\theta_0)$. *Furthermore*

$$\lim_{\tau\to\infty} (e^P Q_\tau)(\tau, \theta_0) = 0. \tag{66}$$

Remark. Due to Corollary 5 and Proposition 8, the variation of P_τ and $e^P Q_\tau$ in sets of the form $\mathcal{D}_{\theta_0,\tau}$ tends to zero.

Proof. Consider

$$f(\tau) = (P_\tau + e^{-\tau}P_\theta)(\tau, \theta_0 + e^{-\tau}).$$

Let

$$\alpha = \liminf_{\tau\to\infty} |f(\tau)|, \quad \beta = \limsup_{\tau\to\infty} |f(\tau)|.$$

Assume $\alpha < \beta$. Then there must be a sequence $\tau_k \to \infty$ such that $|f(\tau_k)| \to \alpha$ and $f'(\tau_k) \leq 0$. Since

$$f' = P_{\tau\tau} - e^{-\tau}P_\theta - e^{-2\tau}P_{\theta\theta} = e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) - e^{-\tau}P_\theta,$$

we conclude that $\mathcal{K}(\tau_k, \theta_0 + e^{-\tau_k})$ converges to α^2 due to Corollary 3. By Corollary 4 we get a contradiction to $\alpha < \beta$. We conclude that the limit (65) exists, due to the fact that the variation of P_τ inside $\mathcal{D}_{\theta_0,\tau}$ tends to zero. If $e^P Q_\tau$ does not converge to zero, we conclude that f' does not converge to zero since $\mathcal{P}(\tau, \theta_0 + e^{-\tau}) \to 0$. Since f'' is bounded we conclude that f cannot converge. Thus (66) holds and the limit (65) has to be $\pm v_\infty(\theta_0)$ due to Corollary 6. □

Lemma 7. *Let* $(Q, P) \in \mathcal{S}$ *and* $\theta_0 \in \mathbb{R}$. *Assume that* $P_\tau(\tau, \theta_0) \to -v_\infty(\theta_0)$ *and that* $v_\infty(\theta_0) > 0$. *Then, if* $(Q_1, P_1) = \text{Inv}(Q, P)$,

$$\lim_{\tau\to\infty} P_{1\tau}(\tau, \theta_0) = v_\infty(\theta_0), \quad \lim_{\tau\to\infty} Q_1(\tau, \theta_0) = 0.$$

Proof. By Proposition 9, we know that $P_{1\tau}(\tau, \theta_0)$ converges to $\pm v_\infty(\theta_0)$. Let us call the limit α. Note that $P_1(\tau, \theta_0)/\tau$ also has to converge to α. Since $P(\tau, \theta_0)$ tends to $-\infty$ and

$$e^{-P_1} = \frac{e^{-P}}{Q^2 + e^{-2P}} \leq e^P,$$

we conclude that the right hand side converges to zero in θ_0. Consequently, α must be positive, and the first conclusion of the lemma follows. Since

$$Q_1 = \frac{Q}{Q^2 + e^{-2P}},$$

the second statement follows from the fact that $P(\tau, \theta_0)$ tends to $-\infty$. □

Corollary 7. *Consider a solution z to (29) and let $\theta_0 \in \mathbb{R}$. Then*

$$\lim_{\tau \to \infty} \frac{\rho(\tau, \theta_0)}{\tau} = v_\infty[x](\theta_0), \tag{67}$$

where ρ is defined in (31) and $(Q, P) = x = \phi_{RD}^{-1} \circ z$.

Proof. By (32) and Corollary 6, we conclude that if $v_\infty[x](\theta_0) = 0$, then $\rho_\tau(\tau, \theta_0)$ converges to zero. Consequently (67) holds in that case. Assume that $v_\infty(\theta_0) > 0$. By applying an inversion, if necessary, we can assume that $P_\tau(\tau, \theta_0)$ converges to $v_\infty(\theta_0)$. Note that applying an inversion does not affect ρ, cf. (31) and (30). We know that $P(\tau, \theta_0)$ converges to infinity linearly and since $e^P Q_\tau$ is bounded due to Lemma 3, we conclude that Q converges to some q_0. Thus $\phi_{RD}(Q, P)$ converges to some z_0, with $|z_0| = 1$ but $z_0 \neq 1$. Consider (34). We know that if we divide the left hand side by τ and compute the limit, we get $v_\infty(\theta_0)$, and on the right hand side, all the terms except ρ/τ converge to zero; for the second term on the right hand side of (34), this is clear since it can be bounded by $2 \ln 2/\tau$, and for the third term, this follows from the fact that $|z|$ converges to 1, but $-2\text{Re}z$ converges to $-2\text{Re}z_0 > -2$. The corollary follows. □

Corollary 8. *Consider a solution z to (29) and let $\theta_0 \in \mathbb{R}$. If $v_\infty(\theta_0) > 0$, then*

$$\varphi_\infty(\theta_0) = \lim_{\tau \to \infty} z(\tau, \theta_0)$$

exists and $|\varphi_\infty(\theta_0)| = 1$.

Proof. We know apriori that

$$\sinh^2 \rho \left| \partial_\tau \left(\frac{z}{|z|} \right) \right|^2$$

is bounded. By the assumptions, $\sinh \rho$ tends to infinity exponentially. Consequently $z/|z|$ converges exponentially. On the other hand, $|z|$ converges exponentially to 1. □

Definition 8. Consider a solution to (29) with $\theta \in \mathbb{R}$. Then the function v from $\mathbb{R}$ to $\mathbb{R}^2$ is defined to be

$$v(\theta) = \varphi_\infty(\theta) v_\infty(\theta),$$

for $v_\infty(\theta) \neq 0$ and to be 0 if $v_\infty(\theta) = 0$. If we wish to make the dependence on the solution z explicit, we shall use the notation $v[z]$ or $v[x]$ for $x \in \mathcal{S}$.

Lemma 8. *Consider a solution z to (29) with $\theta \in \mathbb{R}$. Then, for all $\theta \in \mathbb{R}$,*

$$\lim_{\tau\to\infty}\left[\frac{z}{|z|}\frac{\rho}{\tau}\right](\tau,\theta) = v(\theta).$$

Remark. There is no problem in defining $z\rho/|z|$, since

$$\frac{1}{|z|}\ln\frac{1+|z|}{1-|z|}$$

can be considered to be a real analytic function from the open unit disc to the real numbers by defining its value at the origin to be 2.

Proof. If $v_\infty(\theta) = 0$, then the statement is obvious. If $v_\infty(\theta) > 0$, then $\rho(\tau,\theta)/\tau$ converges to $v_\infty(\theta)$ and $z(\tau,\theta)$ converges to $\varphi_\infty(\theta)$. Since $|z(\tau,\theta)|$ converges to 1, the lemma follows in this case as well. □

Proof of Proposition 3. By Theorem 3 of [20], we get the conclusion of the proposition. In fact, the statement of Theorem 3 does not completely suffice; it only gives the existence of an isometry such that one obtains expansions in a neighbourhood. Here we claim that it suffices to use an inversion. However, going through the proof of Lemma 6 of [20], keeping (30) in mind, one sees that the isometry can be chosen to be an inversion. □

7. Uniform convergence

Let us consider a situation in which v_∞ is continuous in a compact set. We here wish to prove that under this assumption, it is possible to go from point wise to uniform convergence.

Lemma 9. *Consider $(Q,P) \in \mathcal{S}$ and assume that v_∞ is continuous in a compact set K. For every $\eta > 0$, there exists a T and open intervals $I_1, ..., I_m$ such that K is contained in the union of the I_i and if $\theta_i \in I_i \cap K$ and $\tau \geq T$, then*

$$e^{-\tau}F_{I_i}(\tau) \leq v_\infty^2(\theta_i) + \eta.$$

Proof. Let $\eta > 0$. For every $\theta \in K$ we can choose a T_θ and an interval I_θ containing θ in its interior such that

$$e^{-\tau}F_{I_\theta}(\tau) \leq v_\infty^2(\theta) + \eta/2$$

for all $\tau \geq T_\theta$. For $I_\theta = \{\theta\}$, this follows from (63), with an even better constant. At T_θ, one can then increase I_θ to some interval containing θ in its interior by continuity. The rest then follows from the monotonicity of the left hand side. Decreasing the interval, if necessary, we get

$$e^{-\tau}F_{I_\theta}(\tau) \leq v_\infty^2(\theta') + \eta$$

for any $\theta' \in I_\theta \cap K$, due to the continuity of v_∞. The interiors of the I_θ form an open covering of K and this covering has a finite sub covering, say $I_{\theta_1}, ..., I_{\theta_m}$. The desired T is then given by $T = \max\{T_{\theta_1}, ..., T_{\theta_m}\}$ and the I_i are given by the interiors of the I_{θ_i}. □

Proposition 10. *Consider $(Q,P) \in \mathcal{S}$ and assume that v_∞ is continuous in a compact interval $K = [\theta_-, \theta_+]$. Then*

$$\lim_{\tau\to\infty} ||\mathcal{P}(\tau,\cdot)||_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} = 0.$$

Proof. The argument is by contradiction. Assume there is an $\eta > 0$ and a sequence (τ_n, θ_n) with $\tau_n \to \infty$, $\theta_n \in \mathcal{D}_{K,\tau_n}$ and $\mathcal{P}(\tau_n, \theta_n) \geq \eta$ and let $\theta_n' = \theta_n - e^{-\tau_n}$. By choosing a subsequence, if necessary, we can assume that $\theta_n \to \theta_*$. Since $\mathcal{P}$ converges to zero in $\mathcal{D}_{\theta_\pm}$, only a finite number of (τ_n, θ_n) can belong to $\mathcal{D}_{\theta_\pm}$. Consequently, $\theta_n \in [\theta_- + e^{-\tau_n}, \theta_+ - e^{-\tau_n}]$ for n large enough, so that $\theta_n' \in K$ and $v_\infty(\theta_n') \to v_\infty(\theta_*)$ by the assumptions of the proposition. Given $\xi > 0$, there is a T and open intervals $I_1, ..., I_m$ as in Lemma 9. There is an i such that $\theta_* \in I_i$ and for n great enough, we must have $\theta_n' \in I_i$. By Lemma 9, we thus have

$$e^{-\tau_n} F_{\theta_n'}(\tau_n) \leq v_\infty^2(\theta_*) + \xi$$

for n large enough. Since ξ can be chosen to be arbitrarily small, we get

$$\limsup_{n\to\infty} e^{-\tau_n} F_{\theta_n'}(\tau_n) \leq v_\infty^2(\theta_*). \tag{68}$$

Since $\mathcal{P}(\tau_n, \theta_n' + e^{-\tau_n}) \geq \eta$, we can apply Lemma 5 in order to obtain an $\epsilon > 0$ and a sequence $u_n \geq \tau_n$ such that

$$e^{-u_n} F_{\theta_n'}(u_n) \leq e^{-\tau_n} F_{\theta_n'}(\tau_n) - \epsilon.$$

By (68), the right hand side is bounded by $v_\infty^2(\theta_*) - \epsilon$ in the limit. The left hand side, on the other hand, bounds $v_\infty^2(\theta_n')$, which converges to $v_\infty^2(\theta_*)$. We have a contradiction. □

Proposition 11. *Consider $(Q, P) \in \mathcal{S}$ and assume that v_∞ is continuous in a compact interval $K = [\theta_-, \theta_+]$. Then*

$$\lim_{\tau\to\infty} \|\mathcal{K}(\tau, \cdot) - v_\infty^2\|_{C^0(K,\mathbb{R})} = 0.$$

Proof. Due to Lemma 9, the only thing that can go wrong is if there is an $\eta > 0$ and a sequence (τ_n, θ_n) with $\tau_n \to \infty$, $\theta_n \in K$ and

$$\mathcal{K}(\tau_n, \theta_n) \leq v_\infty^2(\theta_n) - \eta.$$

By choosing a subsequence, if necessary, we can assume that $\theta_n \to \theta_*$. Since $\mathcal{K}(\tau, \cdot)$ converges to $v_\infty^2(\theta_\pm)$ in $\mathcal{D}_{\theta_\pm}$, and since v_∞ is continuous in K, we have $\theta_n \in [\theta_- + e^{-\tau_n}, \theta_+ - e^{-\tau_n}]$ for n large enough. Consequently $\theta_n' = \theta_n - e^{-\tau_n}$ belongs to K for n large enough. Similarly to the previous proposition, we have (68). Note that

$$e^{-\tau_n} F_{\theta_n'}(\tau_n) - \mathcal{K}(\tau_n, \theta_n' + e^{-\tau_n}) \geq v_\infty^2(\theta_n') - \mathcal{K}(\tau_n, \theta_n) \geq v_\infty^2(\theta_n') - v_\infty^2(\theta_n) + \eta.$$

Since K is compact, v_∞ is uniformly continuous on K, so that the right hand side converges to η as n tends to infinity. By Proposition 10, we know that $\mathcal{P}$ converges to zero along characteristics ending on θ_n'. By Lemma 6, we conclude that there is an $\epsilon > 0$, a subsequence τ_{n_k} and a sequence $u_k \geq \tau_{n_k}$ such that

$$e^{-u_k} F_{\theta_{n_k}'}(u_k) \leq e^{-\tau_{n_k}} F_{\theta_{n_k}'}(\tau_{n_k}) - \epsilon.$$

We get a contradiction in the same way as in the previous proposition. □

It is of interest to note the following behaviour of the second derivatives.

Corollary 9. *Consider $(Q, P) \in \mathcal{S}$ and assume that v_∞ is continuous in a compact interval K. Then*

$$\lim_{\tau\to\infty} e^{-2\tau} \|P_{\tau\theta}^2 + e^{-2\tau} P_{\theta\theta}^2 + e^{2P}(Q_{\tau\theta}^2 + e^{-2\tau} Q_{\theta\theta}^2)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} = 0. \tag{69}$$

Proof. The corollary follows from Lemma 3 and Proposition 10. □

Proposition 12. *Consider* $(Q,P) \in \mathcal{S}$ *and assume that* v_∞ *is continuous in a compact interval* $K = [\theta_-, \theta_+]$. *Then*

$$\lim_{\tau\to\infty} \left\| \frac{\rho(\tau,\cdot)}{\tau} - v_\infty \right\|_{C^0(K,\mathbb{R})} = 0.$$

Proof. Again, we argue by contradiction. Due to Proposition 11, the only problem that can arise is the existence of an $\eta > 0$ and a sequence (τ_n, θ_n) with $\tau_n \to \infty$, $\theta_n \in K$ and

$$\frac{\rho(\tau_n,\theta_n)}{\tau_n} \le v_\infty(\theta_n) - \eta. \tag{70}$$

We can assume, by choosing a subsequence if necessary, that $\theta_n \to \theta_* \in K$. After performing an inversion if necessary, cf. Lemma 7, we can assume that $P_\tau(\tau,\theta_*)$ converges to $v_\infty(\theta_*)$. For any $\xi > 0$, there is, by previous results and a continuity argument, a T and an I containing θ_* in its interior such that

$$[P_\tau - v_\infty(\theta_*)]^2 + \left[\frac{P}{\tau} - v_\infty(\theta_*)\right]^2 + e^{2P}Q_\tau^2 + e^{-2\tau}(P_\theta^2 + e^{2P}Q_\theta^2) \le \xi^2 \tag{71}$$

for $\tau = T$ and $\theta \in \mathcal{D}_{I,T}$. Since $P \le \rho$, cf. (34), we have (70) with ρ replaced by P. Let us go back from (τ_n, θ_n) along the characteristic

$$\gamma_n(\tau) = (\tau, \theta_n + e^{-\tau} - e^{-\tau_n}).$$

Note that for any $\tau \le \tau_n$, $\gamma_n(\tau) \in \{\tau\} \times \mathcal{D}_{K,\tau}$. For n large enough, $\gamma_n(T) \in \{T\} \times \mathcal{D}_{I,T}$. Thus

$$(P_\tau + e^{-\tau}P_\theta)[\gamma_n(T)] \ge v_\infty(\theta_*) - 2\xi.$$

Assuming $3\xi < \eta$, we get the existence of a $T \le T' \le \tau_n$ such that

$$(P_\tau + e^{-\tau}P_\theta)[\gamma_n(T')] = v_\infty(\theta_*) - \eta + \xi, \quad [(\partial_\tau - e^{-\tau}\partial_\theta)(P_\tau + e^{-\cdot}P_\theta)][\gamma_n(T')] \le 0. \tag{72}$$

The reason for this is the following. By choosing T to be great enough, we can assume $\mathcal{P}$ to be arbitrarily small in $\mathcal{D}_{K,\tau}$ for $\tau \ge T$. Consequently, $\mathcal{P}$ can be assumed to be arbitrarily small at $\gamma_n(s)$ for $T \le s \le \tau_n$. If there is no point satisfying (72), we can thus assume that

$$(P_\tau - e^{-\tau}P_\theta)[\gamma_n(\tau)] \ge v_\infty(\theta_*) - \eta + \xi/2$$

for all $\tau \in [T, \tau_n]$. As a consequence, $\partial_\tau(P \circ \gamma_n/\tau)$ is positive if $P \circ \gamma_n/\tau$ is strictly less than $v_\infty(\theta_*) - \eta + \xi/2$. In other words, $P \circ \gamma_n/\tau$ cannot reach the value it has to reach at $\tau = \tau_n$. Note that in the construction of T', we can choose ξ to be arbitrarily small, T to be arbitrarily large and by Proposition 10, $\mathcal{P}$ converges to zero uniformly. This can be used to prove that there must be a sequence (τ_n', θ_n') with $\tau_n' \to \infty$, $\theta_n' \in \mathcal{D}_{K,\tau_n'}$ and $\theta_n' \to \theta_*$ such that

$$\lim_{n\to\infty} \mathcal{K}(\tau_n', \theta_n') = [v_\infty(\theta_*) - \eta]^2. \tag{73}$$

By Corollary 9 and Proposition 10, we conclude that there is a sequence $\theta_n'' \in K$ at a distance less than $e^{-\tau_n'}$ from θ_n' such that (73) holds with θ_n' replaced by θ_n''. Since $v_\infty(\theta_n'')$ converges to $v_\infty(\theta_*)$, we get

$$\mathcal{K}(\tau_n', \theta_n'') \le \left[v_\infty(\theta_n'') - \frac{\eta}{2}\right]^2$$

for n large enough. This contradicts Proposition 11. □

Lemma 10. *Let $(Q,P)\in\mathcal{S}$. Assume that v_∞ is continuous in a compact interval K and that*

$$\lim_{\tau\to\infty}\frac{P(\tau,\theta)}{\tau}=v_\infty(\theta) \tag{74}$$

for all $\theta\in K$. Then

$$\lim_{\tau\to\infty}\left\|(P_\tau-v_\infty)^2+\left(\frac{P}{\tau}-v_\infty\right)^2+e^{2P}Q_\tau^2\right\|_{C^0(K,\mathbb{R})}=0. \tag{75}$$

Proof. By (74), $P_\tau(\tau,\theta)$ converges to $v_\infty(\theta)$ for all $\theta\in I$, cf. Proposition 9. Let us prove that P_τ converges uniformly to v_∞. Due to Proposition 11, the only thing that can go wrong is the existence of an $\eta>0$ and a sequence (τ_n,θ_n) with $\tau_n\to\infty$ and $\theta_n\in K$ such that

$$P_\tau(\tau_n,\theta_n)\le v_\infty(\theta_n)-\eta. \tag{76}$$

We can assume that $\theta_n\to\theta_*$. The remainder of the proof is identical to the proof of Proposition 12, starting from (71); instead of (70), we have (76). We conclude that P/τ converges to v_∞ uniformly. Combining the fact that P_τ converges to v_∞ uniformly and Proposition 11, we conclude that $e^{2P}Q_\tau^2$ converges to zero uniformly. □

Corollary 10. *Let $(Q,P)\in\mathcal{S}$. Assume that v_∞ is continuous in a compact interval K and that $v_\infty(\theta_0)>0$ for some $\theta_0\in K$. Then there is an $\epsilon>0$ such that, after applying an inversion if necessary,*

$$\lim_{\tau\to\infty}\left\|(P_\tau-v_\infty)^2+\left(\frac{P}{\tau}-v_\infty\right)^2+e^{2P}Q_\tau^2\right\|_{C^0(K\cap I_\epsilon,\mathbb{R})}=0.$$

where $I_\epsilon=[\theta_0-\epsilon,\theta_0+\epsilon]$.

Proof. Let z be the associated solution in the disc model. Due to Proposition 12, we know that there is an $\epsilon>0$ and a T such that $\rho(\tau,\theta)/\tau\ge v_\infty(\theta_0)/2$ for $\theta\in K\cap I_\epsilon$ and $\tau\ge T$. Since $\sinh^2\rho\phi_\tau^2$ is bounded, cf. (32), this implies that ϕ converges exponentially to a continuous function on $K\cap I_\epsilon$. By performing an inversion in the disc model, if necessary, and making ϵ smaller, one can ensure that $\mathrm{Re}z\le 1-\delta$ for $\theta\in I_\epsilon$, some $\delta>0$ and $\tau\ge T$ for some T. This implies that we have (74) in $K\cap I_\epsilon$, cf. (34). We are thus in a position to apply Lemma 10. □

8. The Gowdy to Ernst transformation

The Gowdy to Ernst transformation has been put to good use by several authors, see e.g. [18] and [4]. Combining this transformation with the results already obtained in this paper, in particular Proposition 9, we are in a good position to analyze the behaviour of solutions around spatial points θ_0 such that $v_\infty(\theta_0)$ is not an integer. As we shall see below, it is in this case possible to use combinations of the Gowdy to Ernst transformation and inversions in order to reduce the velocity at θ_0 to a value belonging to $(0,1)$. One then obtains expansions of the form (18)-(21) in a neighbourhood of θ_0. In the end one then only has to transform back in order to find out how the original solution behaves asymptotically around θ_0. Since transforming back can sometimes be quite complicated, we shall however not try to do this in all generality.

In order to define the Gowdy to Ernst transformation, let $(Q_1, P_1) \in \mathcal{S}_k$ for some $k \geq 2$ and define (Q_2, P_2) by

$$P_2(\tau, \theta) = -P_1(\tau, \theta) + \tau \tag{77}$$

$$Q_2(\tau, \theta) = q_2 - \int_{\theta_0}^{\theta} [e^{2P_1} Q_{1\tau}](\tau_0, \phi) d\phi - \int_{\tau_0}^{\tau} [e^{2(P_1 - s)} Q_{1\theta}](s, \theta) ds, \tag{78}$$

where q_2, τ_0, θ_0 are given constants. We shall use the notation

$$(Q_2, P_2) = \mathrm{GE}_{q_2, \tau_0, \theta_0}(Q_1, P_1).$$

Lemma 11. $\mathrm{GE}_{q,\tau,\theta}$ *is a continuous map from* $(\mathcal{S}, d_m)$ *to itself for any* $m \in \mathbb{N} \cup \{\infty\}$. *Let* $(Q_2, P_2) = \mathrm{GE}_{q_2,\tau_0,\theta_0}(Q_1, P_1)$ *for some* $(Q_1, P_1) \in \mathcal{S}$. *Then*

$$Q_{2\tau} = -e^{2(P_1 - \tau)} Q_{1\theta}, \quad Q_{2\theta} = -e^{2P_1} Q_{1\tau}.$$

Finally, if $(Q_1, P_1) \in \mathcal{S}_p$, *then* P_2, $Q_{2\tau}$ *and* $Q_{2\theta}$ *are* 2π*-periodic and*

$$Q_2(\tau, \theta + 2\pi) - Q_2(\tau, \theta) = -B_1$$

for all $(\tau, \theta) \in \mathbb{R}^2$, *where the constant* B_1 *is given by*

$$B_1 = \int_{S^1} [e^{2P_1} Q_{1\tau}](\tau_0, \theta) d\theta = \int_{S^1} [e^{2P_1} Q_{1\tau}](\tau, \theta) d\theta.$$

Remark. Note that for a large class of solutions to the Gowdy equations, it is possible to apply an isometry to the solution in order to achieve $B_1 = 0$, cf. Lemma 8.2 of [21]. However, there is also a large class of solutions for which this is not possible.

Proof. The lemma follows by straightforward computations using the fact that (Q_1, P_1) solves the Gowdy equations. In particular, if $(Q_1, P_1) \in \mathcal{S}_{p,k}$, then

$$\partial_\tau \int_{S^1} e^{2P_1} Q_{1\tau} d\theta = e^{-2\tau} \int_{S^1} \partial_\theta [e^{2P_1} Q_{1\theta}] d\theta = 0,$$

so that B_1 is independent of τ. □

Up to a constant, composing a Gowdy to Ernst transformation with itself is the identity. In fact,

$$\mathrm{GE}_{q_2,\tau_2,\theta_2} \circ \mathrm{GE}_{q_1,\tau_1,\theta_1}(Q_1, P_1) = [Q_1 + q_2 - Q_1(\tau_2, \theta_2), P_1]. \tag{79}$$

We shall also be interested in performing inversions on $\mathcal{S}$, defined by (9).

Lemma 12. *The transformation* Inv *is a continuous map from* $\mathcal{S}$ *to itself with respect to* $d_{k,\tau}$ *for any* $k \in \mathbb{N} \cup \{\infty\}$ *and* $\tau \in \mathbb{R}$. *Furthermore* $\mathrm{Inv} \circ \mathrm{Inv} = \mathrm{Id}$, *where* Id *is the identity map.*

Proof. The fact that Inv takes solutions of (2)-(3) to solutions is a consequence of the geometric setting and the fact that Inv defines an isometry of the hyperbolic plane. Another way of proving this fact is by direct computation. The continuity is rather obvious, as well as the last statement of the lemma. □

Let us prove that the concept of non-degenerate true spike does not depend on the constants chosen when applying the Gowdy to Ernst transformation.

Lemma 13. *Consider a solution (Q,P) to (2)-(3). Assume that $1 < v_\infty(\theta_0) < 2$ for some $\theta_0 \in S^1$ and that*

$$\lim_{\tau\to\infty} P_\tau(\tau,\theta_0) = v_\infty(\theta_0).$$

Let $(Q_{i,2}, P_{i,2}) = \mathrm{Inv}\circ\mathrm{GE}_{q_i,\tau_i,\theta_i}(Q,P)$ for two choices of constants q_i, τ_i, θ_i, $i = 1,2$. Then $(Q_{i,2}, P_{i,2})$ has smooth expansions in a neighbourhood of θ_0. In particular $Q_{i,2}$ converges to a smooth function $q_{i,2}$. Furthermore, $q_{i,2}(\theta_0) = 0$ and if $\partial_\theta q_{i,2}(\theta_0) \neq 0$ for $i = 1$, the same is true for $i = 2$.

Proof. Everything stated in the lemma was proved in connection with Definition 3, except for the statement that if $\partial_\theta q_{i,2}(\theta_0) \neq 0$ for $i = 1$, the same is true for $i = 2$. Let $(Q_{i,1}, P_{i,1}) = \mathrm{Inv}(Q_{i,2}, P_{i,2})$. Note that $P_{1,1} = P_{2,1}$ and that $Q_{1,1}$ and $Q_{2,1}$ differ by a constant. Due to this fact and the fact that Inv is an isometry,

$$(\partial_\theta P_{i,j})^2 + e^{2P_{i,j}}(\partial_\theta Q_{i,j})^2 = (\partial_\theta P_{k,l})^2 + e^{2P_{k,l}}(\partial_\theta Q_{k,l})^2$$

for any $i,j,k,l \in \{1,2\}$. The desired statement follows from the observation that

$$\lim_{\tau\to\infty} \exp\{-[v_\infty(\theta_0) - 1]\tau\}[(\partial_\theta P_{i,2})^2 + e^{2P_{i,2}}(\partial_\theta Q_{i,2})^2](\tau,\theta_0) = c_i[\partial_\theta q_{i,2}(\theta_0)]^2$$

for some constants $c_i > 0$. □

The point of the Gowdy to Ernst transformation is that it allows us to reduce the velocity. We already know how to obtain asymptotic expansions if $0 < v_\infty < 1$. The idea is to reduce more general situations to that case. Let us describe a method for transforming solutions to solutions which we shall later on refer to as *reduction of velocity*. Consider $(Q,P) \in \mathcal{S}$ and assume that $v_\infty(\theta_0) \geq 1$ for some $\theta_0 \in \mathbb{R}$. We can assume $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$ by applying an inversion if necessary, cf. Lemma 7. Let $(Q_1,P_1) = \mathrm{GE}_{q_2,\tau_0,\theta_0}(Q,P)$. The choice of constants is not important. Then $P_{1\tau}(\tau,\theta_0) \to 1 - v_\infty(\theta_0)$. Define $(Q_2,P_2) = \mathrm{Inv}(Q_1,P_1)$. By Lemma 7 and Proposition 9,

$$\lim_{\tau\to\infty} P_{2\tau}(\tau,\theta_0) = v_\infty(\theta_0) - 1 \text{ and } \lim_{\tau\to\infty} Q_2(\tau,\theta_0) = 0 \text{ if } v_\infty(\theta_0) > 1.$$

Assuming $1 \leq k \leq v_\infty(\theta_0) < k+1$, we can now iterate this procedure in order to produce solutions (Q_{2i}, P_{2i}) $i = 1,...,k$ to the equations with the property that

$$\lim_{\tau\to\infty} P_{2i\tau}(\tau,\theta_0) = v_\infty(\theta_0) - i \text{ and } \lim_{\tau\to\infty} Q_{2i}(\tau,\theta_0) = 0 \text{ if } v_\infty(\theta_0) > i. \tag{80}$$

Let us consider the case $k < v_\infty(\theta_0) < k+1$. Due to (80) and Proposition 3, we get the conclusion that there are smooth expansions of (Q_{2k}, P_{2k}) in a neighbourhood of θ_0.

Let us derive some basic consequences of this procedure.

Lemma 14. *Let $x = (Q,P) \in \mathcal{S}$, $\theta_0 \in \mathbb{R}$ and assume that*

$$1 < \lim_{\tau\to\infty} P_\tau(\tau,\theta_0) < 2.$$

Then $x_2 = (Q_2,P_2) = \mathrm{Inv}\circ\mathrm{GE}_{q_2,\tau_0,\theta_0}(Q,P)$ has smooth expansions of the form (18)-(21) in a neighbourhood I of θ_0. Let us call the corresponding functions $v_\infty[x_2]$, ϕ_2, r_2 and q_2. Furthermore, for $\theta \in I$,

$$v_\infty[x](\theta) = \lim_{\tau\to\infty} P_\tau(\tau,\theta) = \begin{cases} 1 - v_\infty[x_2](\theta) & \text{if } q_2(\theta) \neq 0 \\ 1 + v_\infty[x_2](\theta) & \text{if } q_2(\theta) = 0. \end{cases}$$

Finally, there is a $\gamma > 0$ and polynomials Ξ_k such that for all $\tau \geq 0$,

$$
\begin{aligned}
(81)\qquad ||Q_\theta(\tau,\cdot) - q_2^2 r_2 + 2q_2 v_\infty[x_2]||_{C^k(I,\mathbb{R})} &\leq \Xi_k e^{-\gamma\tau}, \\
||[(P_\tau - 1)^2 + e^{2P-2\tau}Q_\theta^2](\tau,\cdot) - v_\infty^2[x_2]||_{C^k(I,\mathbb{R})} &\leq \Xi_k e^{-\gamma\tau}, \\
||[e^{-2\tau}P_\theta^2 + e^{2P}Q_\tau^2](\tau,\cdot)||_{C^k(I,\mathbb{R})} &\leq \Xi_k e^{-\gamma\tau}.
\end{aligned}
$$

Remark. One can of course start with a solution (Q_2, P_2) with expansions of the form (18)-(21) and then go in the other direction. This was done systematically in [18], cf. also (82). By the results of [15] and [17], we are free to specify $v_\infty[x_2]$, ϕ_2, r_2 and q_2, as long as $0 < v_\infty[x_2] < 1$ and the functions are smooth. In particular, we can let θ_0 be an accumulation point of non-degenerate zeros of q_2. We then get a solution with an infinite number of non-degenerate true spikes.

Proof. By the arguments preceding the lemma, we conclude that (Q_2, P_2) has expansions as stated. Note that by (79),

$$(82)\qquad (Q, P) = \mathrm{GE}_{Q(\tau_0,\theta_0),\tau_0,\theta_0} \circ \mathrm{Inv}(Q_2, P_2).$$

Let $(Q_1, P_1) = \mathrm{Inv}(Q_2, P_2)$. If $q_2(\theta) \neq 0$, then $P_{1\tau}(\tau,\theta)$ converges to $v_\infty[x_2](\theta)$ so that $P_{2\tau}(\tau,\theta)$ converges to $1 - v_\infty[x_2](\theta)$. Let us consider $\theta \in I$ such that $q_2(\theta) = 0$. By the existence of the expansions, $e^{P_2}Q_{2\tau}$ converges to zero uniformly, and $e^{P_2}Q_2$ converges to zero at θ. We have

$$P_{1\tau} = -P_{2\tau} + \frac{2P_{2\tau}e^{2P_2}Q_2^2 + 2e^{2P_2}Q_2 Q_{2\tau}}{1 + e^{2P_2}Q_2^2},$$

which converges to $-v_\infty[x_2](\theta)$ at θ. The statement concerning $v_\infty[x]$ follows. By Lemma 11 and the above, we get

$$Q_\theta = -e^{2P_1}Q_{1\tau} = Q_2^2 e^{2P_2}Q_{2\tau} - Q_{2\tau} - 2Q_2 P_{2\tau}.$$

By the existence of the expansions for (Q_2, P_2), we get the second conclusion of the lemma. In order to get the third conclusion, note that $P = -P_1 + \tau$ and $Q_\theta = -e^{2P_1}Q_{1\tau}$. Let us compute

$$(P_\tau - 1)^2 + e^{2P-2\tau}Q_\theta^2 = P_{1\tau}^2 + e^{2P_1}Q_{1\tau}^2 = P_{2\tau}^2 + e^{2P_2}Q_{2\tau}^2,$$

where the third equality is due to the fact that Inv is an isometry. Due to the expansions we have for (Q_2, P_2), we get the third conclusion. The fourth conclusion follows by a similar argument. □

Corollary 11. *Let $(Q, P) \in \mathcal{G}_{l,m}$, with non-degenerate true spikes at $\theta_1, ..., \theta_l$. Then*

$$(83)\qquad \limsup_{\tau\to\infty} ||\mathcal{P}(\tau,\cdot)||_{C^0(S^1,\mathbb{R})} = \max\{[1 - v_\infty(\theta_1)]^2, ..., [1 - v_\infty(\theta_l)]^2\},$$

where the right hand side is taken to be zero if $l = 0$.

Remark. It is quite conceivably possible to replace lim sup by lim.

Proof. Let $\epsilon > 0$. By Lemma 14, there is a $\delta > 0$ and a T such that

$$(84)\qquad |[(P_\tau - 1)^2 + e^{2P-2\tau}Q_\theta^2](\tau,\theta) - [1 - v_\infty(\theta_i)]^2| \leq \epsilon$$

for all $\theta \in [\theta_i - \delta, \theta_i + \delta]$ and $\tau \geq T$ (note that $(1 - v_\infty)^2$ is smooth in a neighbourhood of θ_i by Lemma 14). Observe that $e^{-2\tau}P_\theta^2$ converges to zero uniformly in a neighbourhood of θ_i and $\mathcal{P}$ converges to zero uniformly outside of $[\theta_i - \delta, \theta_i + \delta]$, $i = 1, ..., l$. Thus there is a T' such that

$$\|\mathcal{P}(\tau, \cdot)\|_{C^0(S^1,\mathbb{R})} \le \max\{[1 - v_\infty(\theta_1)]^2, ..., [1 - v_\infty(\theta_l)]^2\} + 2\epsilon$$

for all $\tau \ge T'$. Thus the left hand side of (83) is less than or equal to the right hand side. Let i be such that $[1 - v_\infty(\theta_i)]^2$ equals the right hand side of (83). Let $\epsilon > 0$. Let δ and T be such that (84) is satisfied in a δ neighbourhood of θ_i for $\tau \ge T$. Assume furthermore that there are no other true or false spikes in $[\theta_i - \delta, \theta_i + \delta]$. Consider $f(\tau) = P_\tau(\tau, \theta_i + e^{-\tau})$. Note that f converges to $v_\infty(\theta_i)$. Let $T_1 \ge T$ be large enough that $f(T_1) > 1$ and $e^{-T_1} < \delta$. Let $g(\tau) = P_\tau(\tau, \theta_i + e^{-T_1})$. Then $g(T_1) > 1$ and the limit of of $g(\tau)$ as $\tau \to \infty$ is strictly smaller than 1. By continuity, there is a $T_2 \ge T_1$ such that $g(T_2) = 1$. Inserting this information in (84), we get the conclusion that

$$[e^{2P-2\tau} Q_\theta^2](T_2, \theta_i + e^{-T_1}) \ge [1 - v_\infty(\theta_i)]^2 - \epsilon.$$

This proves that the left hand side in (83) is greater than or equal to the right hand side. □

Corollary 12. *Consider $(Q, P) \in \mathcal{S}$ and let $\theta_0 \in \mathbb{R}$. Then the following holds:*

- *if $v_\infty(\theta_0) < 1$, then v_∞ is continuous in a neighbourhood of θ_0,*
- *if $0 < v_\infty(\theta_0) < 1$, then v_∞ is smooth in a neighbourhood of θ_0, and if $P_\tau(\tau, \theta_0)$ converges to $v_\infty(\theta_0)$, then (Q, P) has smooth expansions of the form (18)-(21) in a neighbourhood of θ_0, and if not,* Inv(Q, P) *has such expansions,*
- *if $v_\infty(\theta_0) = 1$, then v_∞ is continuous at θ_0,*
- *if $1 < v_\infty(\theta_0) < 2$, then $(1 - v_\infty)^2$ is smooth in a neighbourhood of θ_0.*

Remark. In particular, if $v_\infty(\theta_0) < 2$, then $(1 - v_\infty)^2$ is continuous in a neighbourhood of θ_0.

Proof. The first statement was proved in [20] and the second statement is contained in Proposition 3. In order to prove the third statement, let $\epsilon > 0$. By Lemma 7, we can assume that $P_\tau(\tau, \theta_0)$ converges to $v_\infty(\theta_0)$. There is furthermore a T and a $\delta > 0$ such that

$$e^{-T} G_{I_\delta}(T) \le \epsilon^2,$$

where $I_\delta = (\theta_0 - \delta, \theta_0 + \delta)$. If we were to replace I_δ by θ_0, this would be clear, even with ϵ^2 replaced by $\epsilon^2/2$ on the right hand side. The statement as it stands then follows by continuity. By the monotonicity of $e^{-\tau} G_{I_\delta}$, we conclude that

$$|P_\tau(\tau, \theta) - 1| \le \epsilon$$

for all $\theta \in I_\delta$ and $\tau \ge T$. The continuity of v_∞ at θ_0 follows. The last statement follows from Lemma 14. □

Corollary 13. *Let $(Q, P) \in \mathcal{S}$ and assume*

$$0 \le \lim_{\tau\to\infty} P_\tau(\tau, \theta) < 2, \quad \lim_{\tau\to\infty} Q(\tau, \theta) = q_0$$

for all $\theta \in I$, where I is an interval and q_0 is a constant. Then v_∞ is continuous in I.

Proof. If $v_\infty(\theta) \le 1$, then v_∞ is continuous at θ due to Corollary 12. Let us assume $1 < v_\infty(\theta_0) < 2$ for some $\theta_0 \in I$ and that v_∞ is discontinuous at θ_0, considered as a function from I to $\mathbb{R}$. Due to Lemma 14, and using the notation of that lemma, we have that $q_2(\theta_0) = 0$, but that q_2 is not identically zero in any neighbourhood of θ_0 with respect to the topology induced on I. By Lemma 14, we know that $Q(\tau, \cdot)$ converges to some smooth function q in any C^k norm in a neighbourhood of θ_0. Let $I \ni \theta_k \to \theta_0$ be a sequence with the property that $q_2(\theta_k) \neq 0$. Note that $v_\infty[x_2](\theta_k)$ converges to a non-zero number, that $q_2(\theta_k)$ converges to zero and that $r_2(\theta_k)$ converges to some number. Inserting this information in (81), we get the conclusion that $q_\theta(\theta_k) \neq 0$ for k large enough. Consequently, the limit of Q cannot be constant in I. We have a contradiction to the assumptions of the corollary. □

Lemma 15. *Let $(Q, P) \in \mathcal{S}$ and $\theta_0 \in \mathbb{R}$. Assume that $k \le v_\infty < k+2$ in a neighbourhood of θ_0. If $P_\tau(\tau, \theta) \to v_\infty(\theta)$ for $\theta = \theta_0$, then the same is true for all θ in a neighbourhood of θ_0.*

Remark. If we have the assumptions in a half neighbourhood, i.e. for $\theta \in [\theta_0, \theta_0 + \epsilon)$ or $(\theta_0 - \epsilon, \theta_0]$, we get the same conclusions, but in a half neighbourhood.

Proof. By the assumptions there is an interval I containing θ_0 in its interior, an $\epsilon > 0$ and a T such that $e^{-\tau} G_I(\tau) \le (k+1-\epsilon)^2$ for all $\tau \ge T$. Assuming $P_\tau(\tau, \theta) \to -v_\infty(\theta)$ for some θ in I leads to the conclusion that $v_\infty(\theta) + 1 \le k + 1 - \epsilon$, a contradiction to the assumptions. □

Corollary 14. *Consider $(Q, P) \in \mathcal{S}$ and assume that $k \le v_\infty(\theta) < k+2$ for some $k \in \mathbb{N}$, all $\theta \in K$ and some compact interval K. Then v_∞ is continuous in K.*

Proof. Let $\theta_0 \in K$. By carrying out an inversion, if necessary, we can assume that $P_\tau(\tau, \theta)$ converges to $v_\infty(\theta)$ for $\theta = \theta_0$. By Lemma 15, the same is true of all θ in a neighbourhood of θ_0 with respect to K. Reducing the velocity as above, we get

$$\lim_{\tau \to \infty} Q_{2i}(\tau, \theta) = 0, \quad \lim_{\tau \to \infty} P_{2i\tau}(\tau, \theta) = v_\infty(\theta) - i$$

for θ in a neighbourhood of θ_0 and $i = 1, \dots, k$. Thus we can apply Corollary 13 to (Q_{2k}, P_{2k}) in order to achieve the conclusion that $v_\infty - i$ is continuous in a neighbourhood of θ_0. We conclude that v_∞ is continuous in K. □

Corollary 15. *Consider $(Q, P) \in \mathcal{S}$ and assume that $k < v_\infty(\theta) < k+2$ for some $k \in \mathbb{N}$, all $\theta \in K$ and some compact interval K. Then v_∞ is continuous in K. Furthermore, either*

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta), \quad \lim_{\tau \to \infty} Q(\tau, \theta) = q_0 \tag{85}$$

for all $\theta \in K$ and some constant q_0, or the same holds with (Q, P) replaced by $\mathrm{Inv}(Q, P)$.

Proof. The continuity follows from the previous corollary. Let z be the solution to (29) associated to (Q, P). Due to Proposition 12, we conclude that ρ/τ converges to v_∞ uniformly in K. Since $\sinh^2 \rho |\partial_\tau (z/|z|)|^2$ is bounded, we conclude that $z/|z|$ converges exponentially and uniformly to some φ_∞ in K. Since $e^{-2\tau} \sinh^2 \rho |\partial_\theta (z/|z|)|^2$ is bounded and $v_\infty > k \ge 1$, we conclude that φ_∞ is constant in K. The corollary follows. □

Corollary 16. *Let $(Q,P)\in\mathcal{S}$. If $1\le v_\infty(\theta)<2$ for all $\theta\in I$, where I is some interval, then v_∞ is continuous in I.*

Proof. By Lemma 14, $(1-v_\infty)^2$ is continuous in I. The conclusion follows. □

In Section 4, we obtained expansions for $0<v_\infty<1$. Later on, we would like to derive the existence of expansions for $v_\infty>1$, given that we have some additional information. In that context, the following result will be useful.

Lemma 16. *Consider $(Q,P)\in\mathcal{S}$. Assume that there exists a compact subinterval I of $\mathbb{R}$ with non-empty interior and a $\gamma>0$ such that*

$$\|P_\tau(\tau,\cdot)-v_a\|_{C^k(I,\mathbb{R})} \le C_k e^{-\gamma\tau}, \tag{86}$$

$$\|e^{2P-2\tau}Q_\theta\|_{C^k(I,\mathbb{R})} \le C_k e^{-\gamma\tau}, \tag{87}$$

$$\lim_{\tau\to\infty} Q(\tau,\theta) = q_0 \tag{88}$$

for all $\theta\in I$, where $v_a>0$ is in $C^\infty(I,\mathbb{R})$ and q_0 is a constant. Then there exists $\phi,\ r\in C^\infty(I,\mathbb{R})$ and polynomials Π_k for $k\ge 0$ such that

$$|\partial_\theta^k(P_\tau-v_a)(\tau,\theta)| \le \Pi_k(\tau)[e^{-2\tau}+e^{-2v_a(\theta)\tau}], \tag{89}$$

$$|\partial_\theta^k(P-v_a\tau-\phi)(\tau,\theta)| \le \Pi_k(\tau)[e^{-2\tau}+e^{-2v_a(\theta)\tau}], \tag{90}$$

$$|\partial_\theta^k[e^{2p}Q_\tau-r](\tau,\theta)| \le \Pi_k(\tau)[e^{-2\tau}+e^{-2v_a(\theta)\tau}], \tag{91}$$

$$\left|\partial_\theta^k\left[e^{2p}(Q-q_0)+\frac{r}{2v_a}\right](\tau,\theta)\right| \le \Pi_k(\tau)[e^{-2\tau}+e^{-2v_a(\theta)\tau}], \tag{92}$$

for all $\tau\ge 0$, all $\theta\in I$ and all $k\ge 0$, where Π_k is a polynomial in τ and $p=v_a\tau+\phi$.

Proof. Due to (86), we get the conclusion that there is a $\phi\in C^\infty(I,\mathbb{R})$ such that

$$\|P(\tau,\cdot)-v_a\tau-\phi\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau} \tag{93}$$

for all $\tau\ge 0$. Due to (3) and (87), we get the conclusion that

$$\|\partial_\tau[e^{2P}Q_\tau](\tau,\cdot)\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau}.$$

We conclude that there is an $r\in C^\infty(I,\mathbb{R})$ such that

$$\|[e^{2P}Q_\tau](\tau,\cdot)-r\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau}.$$

Let $p=v_a\tau+\phi$. Since

$$\|e^{2p(\tau,\cdot)-2P(\tau,\cdot)}-1\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau}, \tag{94}$$

we get

$$\|[e^{2p}Q_\tau](\tau,\cdot)-r\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau}. \tag{95}$$

Since

$$e^{2p(\tau,\cdot)}[Q(\tau,\cdot)-q_0]=-\int_\tau^\infty \exp[2p(\tau,\cdot)-2p(s,\cdot)]\{[e^{2p}Q_\tau](s,\cdot)-r\}ds-\frac{r}{2v_a}$$

we get, by differentiating under the integral sign and using Hölder's inequality,

$$\left\|e^{2p(\tau,\cdot)}[Q(\tau,\cdot)-q_0]+\frac{r}{2v_a}\right\|_{C^k(I,\mathbb{R})}\le C_k e^{-\gamma\tau}. \tag{96}$$

Note that (95) and (96) imply the estimate

$$|\partial_\theta^k Q_\tau(\tau,\theta)|+|\partial_\theta^k Q_\theta(\tau,\theta)|\le \Pi_k(\tau)\exp[-2v_a(\theta)\tau]$$

for all $\theta \in I$ and all $\tau \geq 0$, where Π_k is a polynomial in τ. Inserting this into (2), using (93), we get the conclusion that

$$|\partial_\theta^k \partial_\tau^2 P(\tau,\theta)| \leq \Pi_k(\tau)(e^{-2\tau} + e^{-2v_a(\theta)\tau}).$$

The inequalities (89) and (90) follow. Due to (93) and (96), we draw the conclusion that

$$\|[e^{2P}Q_\theta](\tau,\cdot)\|_{C^k(I,\mathbb{R})} \leq C_k \tau$$

for all $\tau \geq 1$. Going through the same steps with the above improved estimates, we conclude that (91) and (92) hold. □

Lemma 17. *Consider a solution $(Q_0, P_0) \in \mathcal{S}$ satisfying the conditions of Lemma 16 where I is a compact interval with non-empty interior and $q_0 = 0$. Define*

$$(Q_1, P_1) = \mathrm{Inv}(Q_0, P_0), \quad (Q_2, P_2) = \mathrm{GE}_{q_2,\tau_0,\theta_0}(Q_1, P_1),$$

for some q_2, τ_0, θ_0 with $\theta_0 \in I$. Then (Q_2, P_2) satisfies the conditions of Lemma 16 with v_a replaced by $v_a + 1$.

Proof. We have

$$P_2 = P_0 + \tau - \ln[1 + e^{2P_0}Q_0^2],$$

so that, using the conclusions of Lemma 16,

$$|\partial_\theta^k[P_{2\tau} - (v_a+1)](\tau,\theta)| \leq \Pi_k(e^{-2\tau} + e^{-2v_a(\theta)\tau}).$$

In particular (86) is fulfilled. Consider

$$e^{2P_2 - 2\tau}Q_{2\theta} = -e^{2P_2 + 2P_1 - 2\tau}Q_{1\tau} = -Q_{1\tau}.$$

Let us compute

$$Q_{1\tau} = \frac{(e^{2P_0}Q_{0\tau} - r) + 2(P_{0\tau} - v_a)e^{2P_0}Q_0 + 2v_a(e^{2P_0}Q_0 + \frac{r}{2v_a}) - e^{4P_0}Q_0^2 Q_{0\tau}}{(1 + e^{2P_0}Q_0^2)^2}.$$

By arguments similar to ones already presented and the fact that Q_0 converges to zero uniformly, we get the conclusion that $Q_{1\tau}$ converges to zero uniformly and exponentially in any C^k norm. Consequently, the same holds for $e^{2P_2-2\tau}Q_{2\theta}$, so that we have reproduced (87) for Q_2. As a consequence, $Q_{2\theta}$ converges to zero uniformly and exponentially, since $P_{2\tau}$ converges to $v_a + 1$, where $v_a > 0$. Since $e^{2P_2}Q_{2\tau}$ converges (using (3) and (87)), we also get the conclusion that Q_2 converges uniformly. Combining these two facts, we conclude that Q_2 converges to a constant. □

Lemma 18. *Let $(Q, P) \in \mathcal{S}$ and assume that $P_\tau(\tau,\theta)$ converges to $v_\infty(\theta)$ for $\theta \in I$, where I is a compact interval with non-empty interior, and that $k < v_\infty(\theta) < k+1$ for all $\theta \in I$, where $k \in \mathbb{N}$. Then there are expansions of the form given as a result in Lemma 16.*

Proof. Carrying out the above sort of reduction of velocity, we obtain

$$\lim_{\tau\to\infty} Q_{2i}(\tau,\theta) = 0, \quad \lim_{\tau\to\infty} P_{2i\tau}(\tau,\theta) = v_\infty(\theta) - i \tag{97}$$

for $\theta \in I$ and $i = 1, ..., k$. As a consequence, we get smooth expansions for (Q_{2k}, P_{2k}) due to Proposition 3, and by (97), Q_{2k} converges to 0. Consequently, the conditions of Lemma 16 are fulfilled and we are allowed to use Lemma 17. Note that when we go backward to (Q_{2k-2i}, P_{2k-2i}) we reproduce not only the conditions of Lemma 16, but also the statement that Q_{2k-2i} converges to 0 for $i < k$, due to (97). Applying Lemma 17 k times, we obtain the conclusions of the lemma. □

9. Asymptotic expansions

In the previous section, we obtained some information concerning the existence of expansions under the assumption that the velocity stays inside an interval of the form $(k, k+1)$, see e.g. Lemma 18. We would however like to know something about what happens in a neighbourhood of a point where the velocity is integer valued. Note that by Corollary 12, if $v_\infty(\theta_0) = 1$, then v_∞ is continuous at θ_0. It is however to be expected that v_∞ is typically discontinuous in any neighbourhood of θ_0, though we are not in a position to prove that here. The Corollaries 13 and 16 give conditions under which v_∞ is continuous in a neighbourhood of θ_0 even though $v_\infty(\theta_0)$ may equal 1. In this section, we are concerned with the question if it is possible to obtain expansions in a neighbourhood of a θ_0 with $v_\infty(\theta_0) = 1$ if we make the additional assumptions that are obtained as conclusions in Corollaries 13 and 16.

Lemma 19. *Let $(Q, P) \in \mathcal{S}$. Assume that v_∞ is continuous in a compact interval K with non-empty interior and let z be the associated solution in the disc model. We shall assume that one of the following holds for all $\theta \in K$,*

$$1 \leq v_\infty(\theta) < 2 \tag{98}$$

$$0 < v_\infty(\theta) < 2, \quad \varphi_\infty(\theta) = \varphi_0, \tag{99}$$

where $\varphi_0 \neq 1$ is a constant and φ_∞ was defined in Corollary 8. Then, after applying an inversion in the first case if necessary, there exists a $\beta > 0$ and a function $\phi \in C^0(K, \mathbb{R})$ such that

$$\begin{aligned} \|P(\tau,\cdot) - v_\infty\tau - \phi\|_{C^0(K,\mathbb{R})} + \|P_\tau(\tau,\cdot) - v_\infty\|_{C^0(K,\mathbb{R})} &\leq Ce^{-\beta\tau}, \\ \|e^{-2\tau}P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau}Q_\theta^2)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} &\leq Ce^{-\beta\tau}, \\ \|e^{-2\tau}P_{\tau\theta}^2 + e^{-4\tau}P_{\theta\theta}^2 + e^{2P-2\tau}(Q_{\tau\theta}^2 + e^{-2\tau}Q_{\theta\theta}^2)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} &\leq Ce^{-\beta\tau}. \end{aligned} \tag{100}$$

Furthermore, Q converges uniformly to a constant.

Proof. In the first case, we can, after applying an inversion if necessary, for each $\theta \in K$ find a compact subset K' containing θ in its interior (with respect to the topology induced from K) such that (75) holds. This is a consequence of Corollary 10. In the second case, we can assume that (75) holds for $K' = K$, cf. Lemma 10 and (34). Note that we also have Proposition 10 and Corollary 9. Consequently $e^{2P}Q_\tau^2$ converges to zero in $\mathcal{D}_{K'}$, and since the variation of P_τ in intervals of length less than $e^{-\tau}$ inside $\mathcal{D}_{K',\tau}$ converges to zero, bounds for P_τ in K' are for all practical purposes as good as bounds for P_τ in $\mathcal{D}_{K',\tau}$. Fix some $\eta > 0$ and define

$$\mathcal{A}^c_{1,\pm} = \mathcal{A}_{1,\pm} + \frac{1}{2}\eta^4 e^\tau P_\theta^2.$$

Consider (26). Due to (75) and Proposition 10, we have

$$|I_{2,1,\pm}| \leq o(1)(\mathcal{A}^c_{1,+} + \mathcal{A}^c_{1,-}), \quad |I_{1,1,\pm}| \leq \frac{1}{2}[\alpha + o(1)](\mathcal{A}_{1,+} + \mathcal{A}_{1,-}),$$

where

$$\alpha = \max\{2 \sup_{\theta\in K'} v_\infty(\theta) - 1, 1\}. \tag{101}$$

Compute

$$\begin{aligned}(\partial_\tau \mp e^{-\tau}\partial_\theta)[\frac{1}{2}\eta^4 e^\tau P_\theta^2] &= \frac{1}{2}\eta^4 e^\tau P_\theta^2 + \eta^4 e^\tau P_\theta(P_{\tau\theta} \mp e^{-\tau}P_{\theta\theta}) \\ &\leq \frac{1}{2}\eta^4 e^\tau P_\theta^2 + \frac{1}{2}\eta^6 e^\tau P_\theta^2 + \frac{1}{2}\eta^2 e^\tau (P_{\tau\theta} \mp e^{-\tau}P_{\theta\theta})^2 \\ &\leq \frac{1}{2}\eta^4 e^\tau P_\theta^2 + \eta^2 \mathcal{A}^c_{1,\mp}.\end{aligned}$$

Adding up, we get

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}^c_{1,\pm} \leq \frac{1}{2}(\alpha + 4\eta^2)(\mathcal{A}^c_{1,-} + \mathcal{A}^c_{1,+}),$$

where we have assumed τ to be great enough. By arguments of the type given in Subsection 2.3 and these estimates, we conclude that for every $\eta > 0$, there is a C_η such that

$$\|P_{\tau\theta}^2 + e^{-2\tau}P_{\theta\theta}^2 + e^{2P}(Q_{\tau\theta}^2 + e^{-2\tau}Q_{\theta\theta}^2) + \eta^2 P_\theta^2\|_{C^0(\mathcal{D}_{K',\tau},\mathbb{R})} \leq C_\eta e^{(2-\delta)\tau} \tag{102}$$

for all $\tau \geq 0$, where $\delta = 3 - \alpha - 4\eta^2$. Since $\alpha < 3$, we have $\delta > 0$ for η small enough. In particular, $e^{-\tau}P_\theta$ and $e^{-2\tau}P_{\theta\theta}$ converge to zero exponentially. Let us draw some conclusions from this. Let $g = e^{4P}Q_\tau^2$. By the above one can then for every $\eta > 0$ find a C_η such that

$$\begin{aligned}\partial_\tau g &= 2e^{2P}Q_\tau\partial_\theta(e^{2P-2\tau}Q_\theta) = 2e^{2P}Q_\tau(2P_\theta e^{2P-2\tau}Q_\theta + e^{2P-2\tau}Q_{\theta\theta}) \\ &\leq C_\eta g^{1/2}\exp\left[\frac{1}{2}[2v_\infty(\theta) - \delta + o(1)]\tau\right],\end{aligned}$$

since $P/\tau = v_\infty + o(1)$ uniformly in K'. We conclude that

$$2[g^{1/2}(\tau,\theta) - g^{1/2}(\tau_0,\theta)] \leq C_\eta \exp\left[\frac{1}{2}(2v_\infty(\theta) - \delta + 2\eta^2)\tau\right]$$

for all $\tau \geq \tau_0$, assuming τ_0 is great enough and $\delta < 2\inf_{\theta\in K'} v_\infty(\theta)$. Assuming first η to be small enough that $2\eta^2 < \delta$ and then τ_0 to be great enough, we get the conclusion that $e^P Q_\tau$ converges to zero exponentially and uniformly in K'. Since $e^{-\tau}P_\theta$ and $e^{P-\tau}Q_{\tau\theta}$ converge to zero exponentially and uniformly in $\mathcal{D}_{K'}$, we get the conclusion that the same is true of $e^P Q_\tau$. Due to (102), we also know that $e^{-2\tau}P_{\theta\theta}$ decays exponentially. Considering (2), we conclude that there is a $\beta > 0$ and a C such that

$$P_{\tau\tau}(\tau,\theta) \leq Ce^{-\beta\tau}$$

for all $\tau \in K'$. Integrating this inequality, we obtain

$$P_\tau(\tau_1,\theta) - P_\tau(\tau,\theta) \leq \beta^{-1}Ce^{-\beta\tau}$$

for $\tau_1 \geq \tau$. Letting $\tau_1 \to \infty$ and integrating again, we obtain

$$P(\tau,\theta) \geq P(\tau_0,\theta) + v_\infty(\theta)(\tau - \tau_0) - \beta^{-2}Ce^{-\beta\tau_0}.$$

In other words, $P - \tau$ is uniformly bounded from below by a constant on $\mathcal{D}_{K'}$, assuming (98) holds. Since $e^{P-\tau}Q_\theta$ converges to zero uniformly on $\mathcal{D}_{K'}$, this means that Q_θ converges to zero uniformly on $\mathcal{D}_K$. Thus Q converges to a constant in K'.

Consequently, even in the first case, we conclude that Q converges to a constant in all of K, so that all the conclusions above hold for $K' = K$ in both cases.

Let $\eta > 0$. For every $\theta \in K$, there is a compact $K_\theta \subseteq K$ containing θ in its interior, with respect to the topology induced by K, which is such that

$$\alpha - 1 - 2v_\infty(\theta'') \leq -2\min\{\inf_{\theta' \in K_\theta} v_\infty(\theta'), 1\} + 2\eta^2$$

for all $\theta'' \in K_\theta$, where α is defined as in (101), but with $K' = K_\theta$. Note that we have used the continuity of v_∞ in K in order to get this conclusion. By (102), there is for every $\eta > 0$ a C_η such that

$$|Q_{\tau\theta}(\tau, \theta')| \leq C_\eta \exp\left[\frac{1}{2}(\alpha - 1 - 2v_\infty(\theta') + 6\eta^2)\tau\right]$$

for $\theta' \in K_\theta$ assuming τ is great enough. Assuming η to be small enough and then τ to be great enough, the expression appearing in the exponential is negative, say $-\gamma < 0$. This implies

$$|Q_\theta(\tau_1, \theta') - Q_\theta(\tau, \theta')| \leq \gamma^{-1} C_\eta e^{-\gamma\tau}$$

for $\tau_1 \geq \tau$. This implies that Q_θ converges uniformly in K_θ, but since the limit of Q is constant in K, the limit of Q_θ has to be zero. Letting $\tau_1 \to \infty$, we get the conclusion that

$$|Q_\theta(\tau, \theta)| \leq \gamma^{-1} C_\eta e^{-\gamma\tau}.$$

In fact, choosing η small enough, we get the conclusion that $e^{P-\tau} Q_\theta$ converges to zero exponentially in K_θ. By a compactness argument, we get the same conclusion on all of K, and by (102), we get the conclusion in $\mathcal{D}_{K,\tau}$. We conclude that $P_{\tau\tau}$ converges to zero exponentially on K. Thus P_τ converges to v_∞ on K with an exponentially small error, and

$$P(\tau, \theta) = v_\infty(\theta)\tau + \phi(\theta) + O(e^{-\beta\tau})$$

for some $\beta > 0$, where the convergence is uniform in K. □

In the following results until but not including Lemma 25, we shall assume that we have the following setup. Let $(Q, P) \in \mathcal{S}$, K be a compact interval with non-empty interior, $v_\infty \in C^0(K, \mathbb{R})$ and $1 - \epsilon \leq v_\infty(\theta) \leq 1 + \epsilon$ for all $\theta \in K$ and some $\epsilon \in (0, 1/32]$. Finally, we shall assume that $P_\tau(\tau, \cdot)$ converges to v_∞ in K and that $Q(\tau, \cdot)$ converges to a constant. Consequently, we are allowed to apply Lemma 19. The result we are heading for in the end is Proposition 13, but it is convenient to break down the proof into several smaller steps.

Lemma 20. *If there is a T and a polynomial Π_k such that for all $\tau \geq T$,*

$$\|e^{2P}(\partial_\theta^k \partial_\tau Q)^2 + e^{2P-2\tau}(\partial_\theta^{k+1} Q)^2\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Pi_k e^{2j\epsilon\tau},$$

for some $k \in \mathbb{N}$ and $0 \leq j \leq 2$, then there is a polynomial Ξ_k such that for all $\tau \geq T$

$$\|e^{2P-2\tau}(\partial_\theta^k Q)^2\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Xi_k \exp\{-2[1 - (j+2)\epsilon]\tau\}. \tag{103}$$

Remark. In the proofs below, we shall use Π and Π_k to denote any polynomial. They will be allowed to change from line to line.

Proof. Due to (100) and the assumptions, we have

$$|\partial_\theta^k \partial_\tau Q(\tau, \theta)| \leq \Pi_k \exp\{-[1 - (j+1)\epsilon]\tau\}$$

for $\theta \in K$. We can integrate this inequality to obtain

$$|\partial_\theta^k Q(\tau_2,\theta) - \partial_\theta^k Q(\tau_1,\theta)| \leq \Pi_k \exp\{-[1-(j+1)\epsilon]\tau_1\} \tag{104}$$

for $\tau_2 \geq \tau_1$. We conclude that $\partial_\theta^k Q$ converges uniformly to a continuous function. Since Q converges to a constant by assumption, this function has to be zero, so that

$$|\partial_\theta^k Q(\tau,\theta)| \leq \Pi_k \exp\{-[1-(j+1)\epsilon]\tau\}$$

for all $\theta \in K$. In order to get the desired estimate for $\theta \in \mathcal{D}_{K,\tau} - K$, let $\theta' \in K$ be the point in K closest to θ. Note that as a consequence, $|\theta - \theta'| \leq e^{-\tau}$. Thus

$$|\partial_\theta^k Q(\tau,\theta) - \partial_\theta^k Q(\tau,\theta')| \leq \Pi_k \exp\{-[1-(j+1)\epsilon]\tau\}$$

by the assumptions. The lemma follows due to the fact that the variation of P in intervals of length $e^{-\tau}$ is bounded and the assumptions. □

Note that

$$\|\{(\partial_\theta^k\partial_\tau P)^2 + e^{-2\tau}(\partial_\theta^{k+1}P)^2 + e^{2P}[(\partial_\theta^k\partial_\tau Q)^2 + e^{-2\tau}(\partial_\theta^{k+1}Q)^2]\}(\tau,\cdot)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})}$$

can be bounded from below by $e^{-\tau}F_{K,k}(\tau)/2$ and from above by $e^{-\tau}F_{K,k}(\tau)$. In other words, $e^{-\tau}F_{K,k}(\tau)$ considered as a norm is equivalent to this expression.

Lemma 21. *Assume there are polynomials Π_k and a T such that for $\tau \geq T$,*

$$e^{-\tau}F_{K,k}(\tau) \leq \Pi_k(\tau)\exp[2j_k\epsilon\tau]$$

where $k = 0,...,l$, $l \geq 1$, $2j_{l-1} \leq j_l \leq 2$ and $j_k = 0$ if $k \leq \max\{0, l-2\}$. Then there are polynomials $\Xi_{m,\epsilon}$ such that for $m \leq l-1$ and $\tau \geq T$,

$$\|e^{2P}(\partial_\theta^m\partial_\tau Q)^2\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Xi_{m,\epsilon}(\tau)\exp\{-2[1-(j_{m+1}+2)\epsilon]\tau\}. \tag{105}$$

Remark. The conditions on the j_k are for technical convenience.

Proof. Let us compute, using (3),

$$\begin{aligned}\partial_\tau(e^{2P}\partial_\theta^m\partial_\tau Q) &= e^{2P-2\tau}\partial_\theta^{m+2}Q - 2e^{2P}\sum_{n=1}^{m}\binom{m}{n}\partial_\theta^n\partial_\tau P\partial_\theta^{m-n}\partial_\tau Q \\ &\quad +2e^{2P-2\tau}\sum_{n=0}^{m}\binom{m}{n}\partial_\theta^{n+1}P\partial_\theta^{m-n+1}Q.\end{aligned} \tag{106}$$

We shall prove the statement by induction on m. The inductive assumption is

$$\|e^{2P}\partial_\theta^o\partial_\tau Q\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Pi_{o,\epsilon}\exp(i_o\epsilon\tau) \tag{107}$$

for $o = 0,...,m-1$. The argument will yield expressions for the i_o. Note that by Lemma 20 and the assumptions,

$$\begin{aligned}\|e^{P-\tau}\partial_\theta^{m-n+1}Q\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} &\leq \Pi\exp\{-[1-(j_{m-n+1}+2)\epsilon]\tau\} \\ \|\partial_\theta^{n+1}P\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} &\leq \Pi_\epsilon\exp(j_{n+1}\epsilon\tau),\end{aligned}$$

where we integrated the assumed inequality in order to achieve the last estimate. Note that this is allowed, since $1 \leq n+1$, $m-n+1 \leq l$. Inserting this information, the inductive assumption and the assumptions of the lemma into (106), we get

$$\begin{aligned}|\partial_\tau(e^{2P}\partial_\theta^m\partial_\tau Q)(\tau,\theta)| &\leq \Pi\exp[(j_{m+1}+1)\epsilon\tau]+\Pi\sum_{n=1}^{m}\exp[(j_n+i_{m-n})\epsilon\tau]\\ &\quad+\Pi\sum_{n=0}^{m}\exp\{-[1-(j_{m-n+1}+j_{n+1}+3)\epsilon]\tau\}.\end{aligned}$$

The third term on the right hand side is always exponentially decaying due to the assumptions, so it need not concern us. If $m=0$, the second term does not appear, and we obtain (107) with $m=0$, $i_0=j_1+1$ and $\mathcal{D}_{K,\tau}$ replaced with K. Consider the case $m=1$. Then the first and second terms have exponents of the form $(j_2+1)\epsilon\tau$ and $(j_1+i_0)\epsilon\tau$. Since $i_0=j_1+1$ and $2j_1\leq j_2$ by assumption, we get (107) with $m=1$, $i_1=j_2+1$ and $\mathcal{D}_{K,\tau}$ replaced with K. Assume inductively that we have (107) up to and including $m-1\geq 1$, with $i_o=j_{o+1}+1$ and $\mathcal{D}_{K,\tau}$ replaced with K. Consider m. Note that $l\geq 3$ and consider j_n+i_{m-n}. If $n\leq l-2$, $j_n=0$ and $i_{m-n}=j_{m-n+1}+1\leq j_{m+1}+1$, which is OK. If $m\geq n\geq l-1\geq 2$, then $i_{m-n}=1$ and $j_n\leq j_{m+1}$, which is also OK. We have (107) up to and including $o=l-1$, with $i_o=j_{o+1}+1$ and $\mathcal{D}_{K,\tau}$ replaced with K. In order to take the step from K to $\mathcal{D}_{K,\tau}$, consider

$$\begin{aligned}|e^{-\tau}\partial_\theta(e^{2P}\partial_\theta^o\partial_\tau Q)| &\leq 2|e^{-\tau}P_\theta e^{2P}\partial_\theta^o\partial_\tau Q|+|e^{-\tau}e^{2P}\partial_\theta^{o+1}\partial_\tau Q|\\ &\leq \Pi\exp[(j_o+j_1+1)\epsilon\tau]+\Pi\exp[(j_{o+1}+1)\epsilon\tau]\\ &\leq \Pi\exp[(j_{o+1}+1)\epsilon\tau],\end{aligned}$$

where the last inequality is due to the fact that $j_o+j_1\leq j_{o+1}$. This yields the conclusion of the lemma. □

Define

$$\hat{\mathcal{A}}_{k,\pm}=\mathcal{A}_{k,\pm}+\exp(\tau-\beta\tau)(\partial_\theta^k P)^2,\quad \hat{F}_{K,k}(\tau)=\sum_{\pm}\sup_{\theta\in\mathcal{D}_{K,\tau}}\hat{\mathcal{A}}_{k,\pm}(\tau,\theta),$$

where $\beta>0$ is the constant obtained in Lemma 19. Consider (26), with $I_{1,k,\pm}$ and $I_{2,k,\pm}$ defined in (27) and (28). Note that

$$I_{1,k,\pm}\leq\left[\frac{1}{2}+\epsilon+C\exp\left(-\frac{1}{2}\beta\tau\right)\right](\mathcal{A}_{k,+}+\mathcal{A}_{k,-}). \tag{108}$$

Observe also that if we have estimates of the form

$$\|e^{2P}(\partial_\theta^k\partial_\tau Q)^2+e^{2P-2\tau}(\partial_\theta^{k+1}Q)^2\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})}\leq\Pi_k e^{-\gamma\tau}, \tag{109}$$

for some $\gamma>0$, then the term $\epsilon(\mathcal{A}_{k,+}+\mathcal{A}_{k,-})$ is replaced by a term of the form $\Pi_k e^{\tau-\gamma\tau}$. Finally, note that

$$\begin{aligned}(\partial_\tau\mp e^{-\tau}\partial_\theta)\left[\exp(\tau-\beta\tau)(\partial_\theta^k P)^2\right] &\leq \exp(\tau-\beta\tau)(\partial_\theta^k P)^2\\ &\quad+C\exp\left(-\frac{1}{2}\beta\tau\right)(\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-}).\end{aligned} \tag{110}$$

Using (26), (108) and (110), we conclude that

$$(\partial_\tau\mp e^{-\tau}\partial_\theta)\hat{\mathcal{A}}_{k,\pm}\leq\left[\frac{1}{2}+\epsilon+C\exp\left(-\frac{1}{2}\beta\tau\right)\right](\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-})+I_{2,k,\pm}. \tag{111}$$

If we have (109), we can improve this estimate to

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\hat{\mathcal{A}}_{k,\pm} \leq \left[\frac{1}{2} + C\exp\left(-\frac{1}{2}\beta\tau\right)\right](\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-})+\Pi_k e^{\tau-\gamma\tau}+I_{2,k,\pm}. \tag{112}$$

Lemma 22. *Assume that*

$$I_{2,k,\pm} \leq Ce^{-\gamma\tau}(\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-})$$

in $\mathcal{D}_K$ for $\tau \geq T$, where $\gamma > 0$. Then

$$e^{-\tau}F_{K,k}(\tau) \leq C_k \exp[2\epsilon\tau].$$

Proof. By (111) and the assumptions of the lemma, we have

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\hat{\mathcal{A}}_{k,\pm} \leq \left[\frac{1}{2} + \epsilon + Ce^{-\delta\tau}\right](\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-}),$$

where $\delta = \min\{\beta/2, \gamma\}$. Thus

$$\hat{F}_{K,k}(\tau) \leq \hat{F}_{K,k}(T) + \int_T^\tau \left[1 + 2\epsilon + Ce^{-\delta s}\right]\hat{F}_{K,k}(s)ds.$$

A Grönwall's lemma type argument yields the conclusion. □

Lemma 23. *Assume that*

$$e^{-\tau}F_{K,l}(\tau) \leq C_l \exp[2j_l\epsilon\tau], \tag{113}$$

for $l = 0, ..., k+1$, where $k \geq 0$, $2j_k \leq j_{k+1} \leq 2$ and $j_l = 0$ if $l \leq \max\{0, k-1\}$, and that

$$I_{2,k,\pm} \leq C\exp\left(-\frac{1}{2}\gamma\tau\right)(\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-}) + C\exp\left[\left(\frac{1}{2}-\gamma\right)\tau\right](\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-})^{1/2} \tag{114}$$

in $\mathcal{D}_K$ for $\tau \geq T$, where $\gamma > 0$. Then

$$e^{-\tau}F_{K,k}(\tau) \leq C_k.$$

Proof. Combining (113) with Lemma 20 and 21, we conclude that (109) holds. Consequently, we have (112). Combining this with (114), we get

$$\begin{aligned}\hat{F}_{K,k}(\tau) \quad \leq \quad & \hat{F}_{K,k}(T) + \int_T^\tau \Big\{\left[1 + Ce^{-\delta s}\right]\hat{F}_{K,k}(s) \\ & + C\exp\left[\left(\frac{1}{2}-\gamma\right)s\right]\hat{F}_{K,k}^{1/2}(s) + \Pi_k \exp[(1-\gamma)s]\Big\}\, ds\end{aligned} \tag{115}$$

for some $\delta > 0$. Let F denote the right hand side. Let h be a function such that $h' = 1 + Ce^{-\delta\tau}$. Finally, let $g = e^{-h}F + 1$. Using (115) and the fact that $h = \tau + O(1)$, one can estimate that

$$g' \leq Ce^{-\gamma\tau}g^{1/2} + \Pi e^{-\gamma\tau} \leq \Pi e^{-\zeta\tau}g^{1/2}$$

for some $\zeta > 0$, where we used the fact that $g \geq 1$ in the last step. Thus g is bounded, and the lemma follows. □

Lemma 24. *Assume that*

$$I_{2,k,\pm} \leq C\exp\left(-\frac{1}{2}\gamma\tau\right)(\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-}) + C\exp\left[\left(\frac{1}{2}+2j\epsilon\right)\tau\right](\hat{\mathcal{A}}_{k,+}+\hat{\mathcal{A}}_{k,-})^{1/2}$$

in $\mathcal{D}_K$, for $\tau \geq T$, where $j \leq 2$. Then

$$e^{-\tau} F_{K,k}(\tau) \leq C_k \exp(2 j_k \epsilon \tau),$$

where $j_k = \max\{1, 2j\}$.

Proof. The argument is similar to the proof of the previous two lemmas. □

Proposition 13. *There is a T and for each $k \geq 0$ a constant C_k such that for $\tau \geq T$,*

$$e^{-\tau} F_{K,k}(\tau) \leq C_k. \tag{116}$$

Proof. Consider the case $k = 1$. We have

$$I_{2,1,\pm} = 2e^{\tau} P_\theta e^{2P} (Q_\tau^2 - e^{-2\tau} Q_\theta^2)(P_{\tau\theta} \pm e^{-\tau} P_{\theta\theta}) \leq C \exp\left(-\frac{1}{2}\beta\tau\right)(\hat{\mathcal{A}}_{1,+} + \hat{\mathcal{A}}_{1,-}).$$

By Lemma 22, we get

$$e^{-\tau} F_{K,k}(\tau) \leq C_k \exp(2 j_k \epsilon \tau) \tag{117}$$

for $k = 1$, $j_1 = 1$ and $\tau \geq T$.

Consider the case $k = 2$. As a consequence of the estimates for $k = 1$, we have

$$\|e^{2P}[(\partial_\theta^l \partial_\tau Q)^2 + e^{-2\tau}(\partial_\theta^{l+1} Q)^2]\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Pi_l(\tau) \exp\{-2[1-(j_{l+1}+2)\epsilon]\tau\}, \tag{118}$$

for $l = 0$ due to Lemmas 20 and 21. We also know that P_θ has a bound of the form $\Pi \exp(j_1 \epsilon \tau)$. Consider the two different terms of $I_{2,2,\pm}$. Let us start with

$$\begin{aligned} & e^{\tau}\{\partial_\theta^{m+1}[e^{2P}(Q_\tau^2 - e^{-2\tau} Q_\theta^2)] \\ & -2e^{2P}(Q_\tau \partial_\theta^{m+1} \partial_\tau Q - e^{-2\tau} Q_\theta \partial_\theta^{m+2} Q)\}(\partial_\tau \partial_\theta^{m+1} P \pm e^{-\tau} \partial_\theta^{m+2} P) \\ = & \sum_{i,j,l} a_{ijl} \partial_\theta^i (e^{2P+\tau})[(\partial_\theta^j \partial_\tau Q)(\partial_\theta^l \partial_\tau Q) - e^{-2\tau}(\partial_\theta^{j+1} Q)(\partial_\theta^{l+1} Q)] \\ & \cdot(\partial_\tau \partial_\theta^{m+1} P \pm e^{-\tau} \partial_\theta^{m+2} P), \end{aligned} \tag{119}$$

where $m = 1$, $i+j+l = m+1$ and $j, l \leq m$. Assume first that $i = m+1$. Depending on whether the derivatives hit one or two of the P:s, we have the following estimates

$$C \exp\left(-\frac{1}{2}\beta\tau\right)(\hat{\mathcal{A}}_{2,+} + \hat{\mathcal{A}}_{2,-}), \quad \Pi \exp\left\{-\left[\frac{3}{2} - (4j_1 + 4)\epsilon\right]\tau\right\}(\mathcal{A}_{2,+} + \mathcal{A}_{2,-})^{1/2}. \tag{120}$$

If $i = 1$, we get a bound of the form

$$\Pi \exp\left\{-\left[\frac{1}{2} - (3j_1 + 2)\epsilon\right]\tau\right\}(\mathcal{A}_{2,+} + \mathcal{A}_{2,-})^{1/2}. \tag{121}$$

If $j = l = 1$, we get an estimate of the form

$$C \exp\left[\left(\frac{1}{2} + 2j_1\epsilon\right)\tau\right](\mathcal{A}_{2,+} + \mathcal{A}_{2,-})^{1/2}. \tag{122}$$

The remaining terms in $I_{2,2,\pm}$ can be estimated similarly, and by Lemma 24, we get (117) with $k = 2$ and $j_2 = 2j_1$. Due to Lemma 23, we get (117) with $k = 1$ and $j_1 = 0$. Consequently, in the estimates of the form (122), $\exp[(1/2 + 2j_1\epsilon)\tau]$ can be replaced with $e^{\tau/2}$ so that we obtain (117) with $k = 2$ and $j_2 = 1$ by Lemma 24.

Induction hypothesis. Let us assume that we have (117) for $k \leq m$ with $j_k = 0$ for $k \leq m-1$ and $j_m = 1$, where $m \geq 2$. We wish to prove the same statement with m replaced by $m+1$. Note that as a consequence of the induction hypothesis,

$$\|\partial_\theta^k P(\tau,\cdot)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq \Pi_k, \quad \|\partial_\theta^m P(\tau,\cdot)\|_{C^0(\mathcal{D}_{K,\tau},\mathbb{R})} \leq C\exp(j_m\epsilon\tau),$$

for all $\tau \geq T$ and $k \leq m-1$. Furthermore, we have (118) for $l \leq m-1$ due to Lemma 20 and 21. Consider (119). Assume $i = m+1$. If all the derivatives hit one P, we have an estimate similar to the first estimate in (120). The remaining cases lead to an estimate similar to the second estimate in (120). The case $i = m$ yields an estimate similar to the second of (120). If $i \leq m-1$, all the corresponding derivatives of P have polynomial bounds, so that we need not concern ourselves with them. Since $j+l \leq m+1$, $j,l \leq m$ and $m \geq 2$, we get the conclusion that the remaining terms can be estimated as the second term in (120) or as in (121). Consider the second term in $I_{2,m,\pm}$. We need to estimate terms of the form

$$e^{2P+\tau}[-2\partial_\theta^{m+1-l}\partial_\tau P\partial_\theta^l\partial_\tau Q + 2e^{-2\tau}\partial_\theta^{m-l+2}P\partial_\theta^{l+1}Q](\partial_\tau\partial_\theta^{m+1}Q \pm e^{-\tau}\partial_\theta^{m+2}Q),$$

where $1 \leq l \leq m$. If $l \leq m-1$, we get an estimate similar to (121). If $l = m$, we get an estimate of the form

$$C\exp\left[\left(\frac{1}{2} + 2j_m\epsilon\right)\tau\right](\mathcal{A}_{m+1,+} + \mathcal{A}_{m+1,-})^{1/2}.$$

In fact, we get the estimate with $2j_m$ replaced by j_m, but we choose this estimate since we wish to apply Lemma 24. By Lemma 24, we conclude that (117) holds for $k = m+1$ and $j_{m+1} = 2$. Using this information with (118) for $l \leq m-1$, we can go through the estimates in order to see that Lemma 23 is applicable for $k = m$. This yields (117) for m with $j_m = 0$. Going through the estimates for $m+1$ with this added information, one then gets (117) for $m+1$ with $j_{m+1} = 1$. This completes the induction and proves the proposition. □

Lemma 25. *Let $(Q,P) \in \mathcal{S}$ and assume that*

$$1-\epsilon \leq \lim_{\tau\to\infty} P_\tau(\tau,\theta) \leq 1+\epsilon, \quad \lim_{\tau\to\infty} Q(\tau,\theta) = q_0$$

for all $\theta \in K$, where $\epsilon \in (0,1/32]$, K is a compact interval with non-empty interior and q_0 is a constant. Then we have expansions in K as stated in the conclusions of Lemma 16.

Proof. Due to Corollary 13, we conclude that v_∞ is continuous in K. The setup described prior to Lemma 20 thus applies. By Proposition 13, we conclude that (116) holds. We conclude that $\partial_\theta^k P$ does not grow faster than linearly for any k. By Lemma 20, $e^{P-\tau}\partial_\theta^k Q$ and $e^{2P-2\tau}\partial_\theta^k Q$ are exponentially decaying for all k. By Lemma 21, $e^P\partial_\theta^k\partial_\tau Q$ is exponentially decaying for all k. Combining these facts with (2)-(3), we conclude that $\partial_\theta^k\partial_\tau^2 P$ and $\partial_\theta^k(e^{2P-2\tau}Q_\theta)$ decay exponentially for all k. Since Q converges to a constant, all the conditions of Lemma 16 are fulfilled, so that we obtain the conclusions of that lemma. □

Before proving Theorem 2 it is convenient to prove a somewhat weaker statement.

Theorem 3. *Let (Q,P) solve (2)-(3) and assume that $k < v_\infty(\theta) < k+2$ for all $\theta \in K$, where K is a compact interval with non-empty interior and $k \in \mathbb{N}$. Then either (Q,P) has expansions in K of the form (18)-(21) or* $\mathrm{Inv}(Q,P)$ *has such* *expansions. Furthermore, the q appearing in the expansions is a constant and we can take $\alpha = 2$.*

Proof. Due to Corollary 15, we conclude that v_∞ is continuous and that (85) holds (possibly after having carried out an inversion). Reducing velocity, we obtain

$$\lim_{\tau\to\infty} Q_{2i}(\tau,\theta) = 0, \quad \lim_{\tau\to\infty} P_{2i\tau}(\tau,\theta) = v_\infty(\theta) - i$$

for $i = 1, ..., k$. Let us use the notation $x_i = (Q_{2i}, P_{2i})$. The subset S of K consisting of θ such that $v_\infty[x_k](\theta) = 1$ is compact. For each $\theta \in S$, there is a compact interval I_θ containing θ in its interior with respect to the topology induced on K, such that $v_\infty[x_k](\theta') \in [1 - 1/32, 1 + 1/32]$ for all $\theta' \in I_\theta$. By compactness, there are θ_i, $i = 1, ..., n$ such that the interiors of $I_i = I_{\theta_i}$ cover S. Let

$$S_c = K - \bigcup_{i=1}^{n} \mathrm{int} I_i,$$

where the interiors are computed with respect to the topology induced on K. This set consists of two compact sets S_+ and S_- with $v_\infty > 1$ in S_+ and $v_\infty < 1$ in S_-. The sets $S_\pm$ can similarly be covered by a finite number of intervals $I_{i,\pm}$, $i = 1, ..., n_\pm$, where we assume $v_\infty[x_k] > 1$ in $I_{i,+}$ and $v_\infty[x_k] < 1$ in $I_{i,-}$. In each of the intervals we get smooth expansions of (Q_{2k}, P_{2k}) as in the conclusions of Lemma 16, with $q_0 = 0$, due to Lemma 18, 25 or Proposition 3. Since the different intervals have non-empty intersection, we get smooth expansions in all of K, with Q converging to zero. Applying Lemma 17 k times, we get the conclusions of the theorem. □

Proof of Theorem 2. Let $\theta \in K$. If $k < v_\infty(\theta) < k+2$, there is a neighbourhood of θ with respect to K such that the same condition is fulfilled in this neighbourhood, cf. Corollary 14. Assuming the neighbourhood is connected, we can apply Theorem 3 to it in order to get smooth expansions. Assume $v_\infty(\theta) = k$. Then there is a subinterval I of K containing θ in its interior, with respect to K, such that $v_\infty \leq k+1/64$ in I. There are two cases to consider. If $k \geq 2$, then $k-1 < v_\infty < k+1$ in I and we can apply Theorem 3 in order to obtain expansions in I. If $k = 1$, we can apply Lemma 19 in order to conclude that Q converges to a constant, possibly after having applied an inversion. By Lemma 25 we get expansions in I. In particular, if we view the solution in the disc model, there is for each $\theta \in K$ a neighbourhood of θ such that $z(\tau, \cdot)$ converges to a constant. The limit of $z(\tau, \cdot)$ therefore has to be constant in K. After applying an inversion, if necessary, we can thus assume that $z(\tau, \cdot)$ converges to a constant different from 1 in all of K. By the above, this solution has expansions in a neighbourhood of each of its points. By compactness, it has expansions in all of K. □

Corollary 17. *Let $(Q, P) \in \mathcal{S}$ and $\theta_0 \in \mathbb{R}$. Assume that $v_\infty(\theta_0) > 1$ and that v_∞ is continuous at θ_0. If $P_\tau(\tau, \theta_0) \to v_\infty(\theta_0)$, (Q, P) has expansions as in the conclusions of Lemma 16 in a neighbourhood of θ_0. If $P_\tau(\tau, \theta_0) \to -v_\infty(\theta_0)$, then* $\mathrm{Inv}(Q, P)$ *has such expansions. In particular, v_∞ is smooth in a neighbourhood of θ_0.*

Proof of Proposition 2. Let $(Q_1, P_1) = \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q, P)$. By Lemma 19 and 25, we conclude that there are functions $\phi, r \in C^\infty(K, \mathbb{R})$ and a constant q_0 such that

$$\begin{aligned} \|P_{1\tau}(\tau,\cdot) - 1\|_{C^k(K,\mathbb{R})} + \|P_1(\tau,\cdot) - \tau + \phi\|_{C^k(K,\mathbb{R})} &\leq \Xi_k e^{-2\tau}, \\ \|e^{2p} Q_{1\tau}(\tau,\cdot) - r\|_{C^k(K,\mathbb{R})} + \left\| e^{2p}[Q_1(\tau,\cdot) - q_0] + \frac{r}{2} \right\|_{C^k(K,\mathbb{R})} &\leq \Xi_k e^{-2\tau}, \end{aligned}$$

where $p = \tau + \phi$. Since $P = -P_1 + \tau$, we get the first conclusion of the proposition. We also have

$$Q_\tau = -e^{2P_1 - 2\tau} Q_{1\theta}.$$

This can be used together with the above expansions in order to obtain the second conclusion of the proposition. □

Consider $(Q, P) \in \mathcal{S}$. Let $\mathcal{E}$ be the subset of $\mathbb{R}$ consisting of points θ_0 such that one of the following holds

- $v_\infty = 0$ in a neighbourhood of θ_0,
- $0 < v_\infty(\theta_0) < 1$,
- $v_\infty = 1$ in a neighbourhood of θ_0,
- $v_\infty(\theta_0) > 1$, and v_∞ is continuous in θ_0.

The first three conditions are open and the fourth one as well, due to Corollary 17. Note that if $\theta_0 \in \mathcal{E}$, then there are smooth expansions of one form or another in a neighbourhood of θ_0, possibly after having applied an inversion to the solution.

Lemma 26. *Let $(Q, P) \in \mathcal{S}$ and let $\mathcal{E}$ be as above. Then $\mathcal{E}$ is open and dense.*

Proof. Consider $\bar{\mathcal{E}}$, the closure of $\mathcal{E}$. Assume $v_\infty(\theta_0) = 0$. Either there is a neighbourhood of θ_0 such that $v_\infty = 0$, in which case $\theta_0 \in \mathcal{E}$, or there is a sequence $\theta_k \to \theta_0$ such that $0 < v_\infty(\theta_k) < 1$. In either case $\theta_0 \in \bar{\mathcal{E}}$. If $0 < v_\infty(\theta_0) < 1$, then $\theta_0 \in \mathcal{E}$. Assume $v_\infty(\theta_0) = 1$. If there is a sequence $\theta_k \to \theta_0$ such that $v_\infty(\theta_k) < 1$, we are done, so assume not. Then $v_\infty \geq 1$ in a neighbourhood of θ_0. By Corollary 16 and the semi continuity of v_∞, v_∞ is continuous in a neighbourhood of θ_0. If $v_\infty = 1$ in a neighbourhood of θ_0, $\theta_0 \in \mathcal{E}$. If this is not the case, $\theta_0 \in \bar{\mathcal{E}}$. Assume inductively that all θ such that $v_\infty(\theta) \leq k$ belong to $\bar{\mathcal{E}}$. Let $k < v_\infty(\theta_0) < k + 2$. By the semi continuity of the velocity, $v_\infty < k + 2$ in a neighbourhood of θ_0. If there is a sequence of $\theta_l \to \theta_0$ such that $v_\infty(\theta_l) \leq k$, then $\theta_0 \in \bar{\mathcal{E}}$. If there is no such sequence, v_∞ is continuous in a neighbourhood of θ_0 by Corollary 15 so that $\theta_0 \in \mathcal{E}$. The lemma follows by induction. □

Proof of Proposition 4. Let $\mathcal{E}$ be as in Lemma 26 and let $\theta_0 \in \mathcal{E}$. If $v_\infty = 0$ in a neighbourhood of θ_0, we can use Proposition 2. If $0 < v_\infty(\theta_0) < 1$, we can use Proposition 3. If $v_\infty = 1$ in a neighbourhood of θ_0, then we can apply Lemma 19 and Lemma 25. Finally, if $v_\infty(\theta_0) > 1$ and v_∞ is continuous at θ_0, then we can apply Theorem 2. □

10. Continuous dependence on initial data

In order to be able to prove that the generic set of solutions is open with respect to d_2, it is necessary to prove that the map from initial data to certain quantities on the singularity is continuous under special circumstances. Let us start with the asymptotic velocity.

Lemma 27. *Consider a solution z to (29), where $\theta \in \mathbb{R}$, and let $z_l \to z$ with respect to d_1. Assume $v_\infty[z](\theta) < 1$ for all $\theta \in I = [\theta_1, \theta_2]$. Then $v[z]$ is continuous in I, as well as $v[z_l]$ for l large enough, and*

$$\lim_{l\to\infty} \|v[z] - v[z_l]\|_{C^0(I,\mathbb{R}^2)} = 0.$$

Remark. Below, we shall use Lemma 1 freely. Recall that v was defined in Definition 8.

Proof. Note that since I is compact and $v_\infty[z]$ is continuous in I, there is a $\delta > 0$ such that $v_\infty[z](\theta) \le 1 - 2\delta$ for all $\theta \in I$. Recall the notation of Subsection 2.3. Let us first prove that for every $\theta \in I$, there is an M_θ, a T_θ and a closed interval I_θ, containing θ in its interior, such that

$$e^{-\tau} F_{I_\theta}[z_l](\tau) \le \left(1 - \frac{3}{2}\delta\right)^2$$

for all $\tau \ge T_\theta$ and $l \ge M_\theta$. Here we consider z to be included; $z_\infty = z$. If we replace I_θ with the point θ and z_l with z, this is clear, in fact we can get a somewhat better estimate. At T_θ, one can then extend the estimate to a larger interval I_θ, by continuity, at the expense of increasing the constant on the right hand side slightly. Since the z_l converge to z, and since the left hand side is monotonic, one then gets the desired estimate by demanding that l be large enough. By (32), we conclude that

$$\|\rho_{l,\tau}\|_{C^0(\mathcal{D}_{I_\theta,\tau},\mathbb{R})} \le 1 - \frac{3}{2}\delta$$

for all $\tau \ge T_\theta$ and $l \ge M_\theta$. By increasing T_θ if necessary, we can assume that

$$\frac{\rho_l(\tau,\theta)}{\tau} \le 1 - \delta$$

for all $\tau \ge T_\theta$, $l \ge M_\theta$ and $\theta \in I_\theta$. The interiors of the I_θ form an open covering of the compact interval I. Thus there is a finite number $I_{\theta_1}, ..., I_{\theta_k}$ of intervals such that $I_0 = \cup_{i=1}^k I_{\theta_i}$ contains I in its interior. Let $M_0 = \max\{M_{\theta_1}, ..., M_{\theta_k}\}$ and $T_0 \ge \max\{T_{\theta_1}, ..., T_{\theta_k}\}$ be such that $\mathcal{D}_{I,\tau} \subseteq I_0$ for all $\tau \ge T_0$. Consequently,

$$\frac{\rho_l(\tau,\theta)}{\tau} \le 1 - \delta$$

for all $l \ge M_0$, $\tau \ge T_0$ and $\theta \in \mathcal{D}_{I,\tau}$. In order to get monotonicity of $G_I[z_l]$ defined below, it will be convenient to assume that T_0 is big enough that $(1-\delta)\tau \le \tau - 2$ for all $\tau \ge T_0$. By (41),

$$L_I[z_l](\tau) \le \left(\frac{T}{\tau}\right)^2 L_I[z_l](T)$$

for all $\tau \ge T \ge T_0$ and $l \ge M_0$. By arguments given in the end of Lemma 5 of [20], we have

$$\left\| \frac{\rho_l(\tau_1,\cdot)}{\tau_1} \frac{z_l(\tau_1,\cdot)}{|z_l(\tau_1,\cdot)|} - \frac{\rho_l(\tau_2,\cdot)}{\tau_2} \frac{z_l(\tau_2,\cdot)}{|z_l(\tau_2,\cdot)|} \right\|_{C^0(I,\mathbb{R})} \le 2 L_I^{1/2}[z_l](T_0) \frac{T_0}{\tau_1} \tag{123}$$

for all $\tau_2 \ge \tau_1 \ge T_0$ and $l \ge M_0$. Note that this proves that $v[z]$ is continuous, as well as $v[z_l]$ for l large enough. Let us take $\tau_2 = \infty$ in this estimate. We get

$$\left\| \frac{\rho_l(\tau_1,\cdot)}{\tau_1} \frac{z_l(\tau_1,\cdot)}{|z_l(\tau_1,\cdot)|} - v[z_l] \right\|_{C^0(I,\mathbb{R})} \le 2 L_I^{1/2}[z_l](T_0) \frac{T_0}{\tau_1}.$$

Note that $L_I[z_l](T_0)$ converges to a real number so that the right hand side can be assumed to be arbitrarily small uniformly in l by demanding that τ_1 be large enough. Note also that

$$\lim_{l\to\infty}\left\|\frac{\rho(\tau_1,\cdot)}{\tau_1}\frac{z(\tau_1,\cdot)}{|z(\tau_1,\cdot)|}-\frac{\rho_l(\tau_1,\cdot)}{\tau_1}\frac{z_l(\tau_1,\cdot)}{|z_l(\tau_1,\cdot)|}\right\|_{C^0(I,\mathbb{R})}=0$$

for a fixed τ_1. These facts together yield the desired conclusion. □

In order to prove that the generic set of solutions is open in the presence of spikes, we need to know that the limit of $Q(\tau,\cdot)$ and its first derivative under certain circumstances depend continuously on the initial data.

Lemma 28. *Consider a sequence $x_l=(Q_l,P_l)\in\mathcal{S}$ converging to $x=(Q,P)\in\mathcal{S}$. Let us assume that there is a compact interval I with non-empty interior such that*

$$\lim_{\tau\to\infty}P_\tau(\tau,\theta)=v_\infty[x](\theta)$$

for all $\theta\in I$, and $0<v_\infty<1$ in I. Then the same is true for the limit of $P_{l\tau}$ for l large enough, and we shall denote the limit of Q_l on I by $q[x_l]$. If x_l converges to x with respect to d_1, then

$$\lim_{l\to\infty}\|q[x]-q[x_l]\|_{C^0(I,\mathbb{R})}=0,$$

and if the convergence is with respect to d_2, then

$$\lim_{l\to\infty}\|q_\theta[x]-q_\theta[x_l]\|_{C^0(I,\mathbb{R})}=0.$$

Proof. The argument is similar to the proof of the previous lemma. Due to the continuity of $v_\infty[x]$ on I and the conditions of the lemma, we conclude that there is a $\delta>0$ such that

$$2\delta\leq v_\infty[x](\theta)\leq 1-2\delta$$

for all $\theta\in I$. Recall the notation of Subsection 2.3. Note that $e^{-\tau}F_J[x]$ and $e^{-\tau}G_J[x]$ are monotonically decaying functions and that $H_J[x]$ satisfies an estimate (40) for all $\tau\geq T$, assuming $1\leq P(s,\theta)\leq s-1$ for all $s\in[T,\tau]$ and $\theta\in\mathcal{D}_{J,s}$. Similarly to the proof of the previous lemma, there is for every $\theta\in I$ an M_θ, an I_θ and a T_θ, where I_θ is a compact interval containing θ in its interior, such that

$$e^{-\tau}F_{I_\theta}[x_l](\tau),\ e^{-\tau}G_{I_\theta}[x_l](\tau)\leq\left(1-\frac{3}{2}\delta\right)^2$$

for all $l\geq M_\theta$ (including $l=\infty$) and all $\tau\geq T_\theta$. Since the interiors of the I_θ constitute an open covering of the compact interval I, there is a finite set of points $\theta_1,...,\theta_k$ such that the union I_0 of the interiors of the I_{θ_i} contains I. Let $M=\max\{M_{\theta_1},...,M_{\theta_k}\}$ and let $T\geq\max\{T_{\theta_1},...,T_{\theta_k}\}$ be such that $\mathcal{D}_{I,T}$ is contained in I_0. Thus

$$\frac{3}{2}\delta\leq P_{l\tau}(\tau,\theta)\leq 1-\frac{3}{2}\delta$$

for all $\tau\geq T$, $\theta\in\mathcal{D}_{I,\tau}$ and $l\geq M$. By increasing T if necessary, we can assume that

$$\delta\leq\frac{P_l(\tau,\theta)}{\tau}\leq 1-\delta \tag{124}$$

for all $\tau\geq T$, $\theta\in\mathcal{D}_{I,\tau}$ and $l\geq M$, and that $\delta T\geq 1$. We conclude that $P_{l\tau}$ has to converge to $v_\infty[x_l]$ in I. Note that we can assume $H_I[x_l](T)$ to be bounded by a constant C independent of l, since x_l converges to x with respect to d_1. Due to this fact, (124) and (40), we have

$$H_I[x_l](\tau) \leq C\left(\frac{T}{\tau}\right)^2 \tag{125}$$

for all $l \geq M$ and all $\tau \geq T$. Note in particular that

$$e^{2\delta\tau}Q_{l\tau}^2(\tau,\theta) \leq e^{2P_l}Q_{l\tau}^2(\tau,\theta) \leq H_I[x_l](\tau) \leq C$$

for all $\theta \in I$, $\tau \geq T$ and $l \geq M$. Consequently

$$\begin{aligned}\|q[x]-q[x_l]\|_{C^0(I,\mathbb{R})} &\leq \|Q(\tau,\cdot)-q[x]\|_{C^0(I,\mathbb{R})}+\|Q(\tau,\cdot)-Q_l(\tau,\cdot)\|_{C^0(I,\mathbb{R})}\\ &\quad+\|Q_l(\tau,\cdot)-q[x_l]\|_{C^0(I,\mathbb{R})}\\ &\leq 2C\delta^{-1}e^{-\delta\tau}+\|Q_l(\tau,\cdot)-Q(\tau,\cdot)\|_{C^0(I,\mathbb{R})}.\end{aligned}$$

The first conclusion of the lemma follows. Using the fact that $H_I[x_l]$ converges to zero uniformly in l and (124), we can assume T_1 to be big enough that

$$\frac{\delta}{2} \leq P_{l\tau} \pm e^{-\tau}P_{l\theta} \leq 1-\frac{\delta}{2}$$

for all $\tau \geq T_1$, $\theta \in \mathcal{D}_{I,\tau}$ and $l \geq M$. Due to this estimate and arguments given in Lemma 1 and Lemma 2 of [20], the function

$$K_I[x](\tau) = \frac{1}{2}\sum_{\pm}\|e^{2P}(Q_\tau \pm e^{-\tau}Q_\theta)^2\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})}$$

satisfies

$$K_I[x](\tau) \leq \exp[-\delta(\tau-T_1)]K_I[x](T_1)$$

for all $\tau \geq T_1$, and similarly with x replaced with x_l. Note that $K_I[x_l](T_1)$ is bounded by a constant independent of l. Let

$$\mathcal{A}_{1,\pm}^c[x] = \mathcal{A}_{1,\pm}[x]+\frac{1}{2}e^{(1-\delta)\tau}P_\theta^2, \quad F_{I,1}^c[x](\tau) = \sum_{\pm}\|\mathcal{A}_{1,\pm}^c[x]\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})}.$$

Note that we have (26). Due to the estimates we have, we get

$$\begin{aligned}(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_{1,\pm}[x_l] &\leq \left(\frac{1}{2}+Ce^{-\delta(\tau-T_1)/2}\right)(\mathcal{A}_{1,+}[x_l]+\mathcal{A}_{1,-}[x_l])\\ &\quad+Ce^{-\delta(\tau-2T_1)/2}(\mathcal{A}_{1,+}^c[x_l]+\mathcal{A}_{1,-}^c[x_l]).\end{aligned}$$

The second term comes from $I_{2,1,\pm}$ whereas the first term comes from $I_{1,1,\pm}$. Note that the constants are independent of l. Let us estimate

$$(\partial_\tau \pm e^{-\tau}\partial_\theta)\left[\frac{1}{2}e^{(1-\delta)\tau}P_{l\theta}^2\right] \leq \frac{1}{2}e^{(1-\delta)\tau}P_{l\theta}^2+e^{-\delta\tau/2}\mathcal{A}_{1,\pm}^c[x_l].$$

Adding up and assuming $T_1 \geq 0$, we get the conclusion that

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_{1,\pm}^c[x_l] \leq \left(\frac{1}{2}+Ce^{-\delta(\tau-2T_1)/2}\right)(\mathcal{A}_{1,+}^c[x_l]+\mathcal{A}_{1,-}^c[x_l])$$

so that

$$F_{I,1}^c[x_l](\tau) \leq F_{I,1}^c[x_l](T_2)+\int_{T_2}^{\tau}\left[1+Ce^{-\delta(s-2T_1)/2}\right]F_{I,1}^c[x_l](s)ds.$$

Assuming $\tau \geq T_2 \geq 2T_1$, and using Grönwall's lemma, we conclude that

$$e^{-\tau}F_{I,1}^c[x_l](\tau) \leq C_\delta e^{-T_2}F_{I,1}^c[x_l](T_2)$$

where C_δ is independent of l. Since the right hand side is bounded by a constant independent of l if we assume x_l to converge to x with respect to d_2, we get a bound of the form

$$e^{2P_l}Q_{l\theta\tau}^2(\tau,\theta) \leq C_\delta$$

for all $l \geq M$, $\tau \geq T_2$ and $\theta \in I$. Due to (124), we can argue as before in order to obtain the second conclusion of the lemma. □

Proof of Proposition 5. Let $x \in \mathcal{G}_{l,m}$ and assume that x_k is a sequence of solutions converging to x with respect to d_2. Let $\theta_1, ..., \theta_l$ be the non-degenerate true spikes and $\theta_1', ..., \theta_m'$ be the non-degenerate false spikes. Consider a non-degenerate true spike θ_i. Letting $(\tilde{Q}, \tilde{P}) = T(Q, P)$ where

$$T = \text{Inv} \circ \text{GE}_{q_0,\tau_0,\theta_0}, \tag{126}$$

for some constants q_0, τ_0, θ_0, we get smooth expansions in a neighbourhood of θ_i by Proposition 3. Furthermore $\tilde{Q}(\tau,\cdot)$ converges to $\tilde{q}$ with $\tilde{q}(\theta_i) = 0$ but $\tilde{q}_\theta(\theta_i) \neq 0$. Let I_i be a compact interval containing θ_i in its interior such that $\tilde{P}_\tau(\tau,\cdot)$ converges to $v_\infty[\tilde{z}]$ in I_i and that $\tilde{q}_\theta \neq 0$ in I_i. Due to Lemma 28 and the fact that T is a continuous map from solutions to solutions with respect to d_2, we conclude that for k large enough, x_k has exactly one non-degenerate true spike in the interior of I_i but no false ones, and that except for the non-degenerate true spike, the velocity belongs to the interval $(0, 1)$. The non-degenerate false spikes can be dealt with similarly by using $T = \text{Inv}$. In this way we get compact intervals $I_1, ..., I_l$ and $I_1', ..., I_m'$ containing $\theta_1, ..., \theta_l$ and $\theta_1', ..., \theta_m'$ respectively in their interiors. Furthermore, for k large enough, there is exactly one non-degenerate true spike in each of $\text{int}I_i$ and exactly one non-degenerate false spike in each of $\text{int}I_i'$. Except for these spike points, all other elements of the intervals I_i and I_i' have the property that $P_{k\tau}(\tau,\cdot)$ converges to a number in the interval $(0, 1)$. Let S be the complement of the interiors of I_i and I_i' in S^1. Since S is a compact set and $v_\infty[x]$ is continuous on S, there is a $\delta > 0$ such that $2\delta \leq v_\infty[x](\theta) \leq 1 - 2\delta$ for all $\theta \in S$. For each $\theta \in S$, there is a T_θ and an I_θ, where I_θ is an interval containing θ in its interior, such that

$$e^{-\tau}F_{I_\theta}[x](\tau) \leq (1-\delta)^2, \quad e^{-\tau}G_{I_\theta}[x](\tau) \leq (1-\delta)^2,$$

for all $\tau \geq T_\theta$, using the notation of Subsection 2.3. Since the interiors of the I_θ constitute an open covering, there is a finite sub covering, consisting of the interiors of $J_i = I_{\theta_i''}$, $i = 1, ..., n$, Letting $T = \max\{T_{\theta_1''}, ..., T_{\theta_n''}\}$, we get

$$e^{-\tau}F_{J_i}[x_k](\tau) \leq \left(1 - \frac{\delta}{2}\right)^2, \quad e^{-\tau}G_{J_i}[x_k](\tau) \leq \left(1 - \frac{\delta}{2}\right)^2,$$

for $\tau \geq T$ as long as k is great enough. The reason this is true is the fact that it is true for $\tau = T$ for k great enough and the fact that the left hand sides are monotonically decaying with time. We conclude that for k large enough, $x_k \in \mathcal{G}_{l,m}$. Consequently, the complement of $\mathcal{G}_{l,m}$ is closed with respect to d_2, and the proposition follows. □

Proof of Proposition 6. Let $x \in \mathcal{G}_{l,m}$, θ_i be a non-degenerate true spike and $x_2 = T \circ x$, where T is defined in (126). Then there is a $\delta_i > 0$ such that $v[x_2](I_{\delta_i})$, where $I_{\delta_i} = [\theta_i - \delta_i, \theta_i + \delta_i]$, is contained in the open unit disc and is bounded away from the origin. Let U_i be an open neighbourhood of $v[x_2](I_{\delta_i})$ which does not contain the origin. Let O_i be the set of $\hat{x} \in \mathcal{S}_p$ such that $v[T \circ \hat{x}](I_{\delta_i}) \subseteq U_i$. Assume that $\hat{x}_k$ converges to $\hat{x} \in O_i$ with respect to d_1. Due to the continuity of T and Lemma

27, we conclude that $\hat{x}_k \in O_i$ for k large enough. Consequently, O_i is open with respect to d_1. Note that for $\hat{x} \in O_i$, $0 < (1 - v_\infty[\hat{x}])^2 < 1$ in I_{δ_i}. The complement S of the I_{δ_i} consists of finitely many intervals on which $v_\infty[x] \in (0,1)$. Arguing similarly to the above, there is an open neighbourhood O of x with respect to d_1 such that for $\hat{x} \in O$ $v_\infty[\hat{x}](S) \in (0,1)$. Taking the intersection of O and the O_i, $i = 1, ..., l$, we get the desired neighbourhood. □

11. Geometric properties of T^3-Gowdy metrics

Let us state the results we need concerning the behaviour of the curvature. First we need to introduce some terminology. Let $M = \mathbb{R} \times T^3$. This is the spacetime manifold. Let $t : M \to \mathbb{R}$ be defined by $t(s,x) = s$. Let $\gamma : (a,b) \to M$ be a future oriented inextendible causal curve, i.e. $\langle \gamma', \partial_\tau \rangle > 0$. Note that the regularity we have in mind here is that of piecewise smoothness. Then

$$\lim_{s \to a} t[\gamma(s)] = \infty.$$

The reason is that $t[\gamma(s)]$ is a monotonically decreasing function. Thus the limit above must exist. If it equals $\alpha \in \mathbb{R}$, then the causal character of the curve and the fact that P, Q, λ are smooth functions on

$$[t\{\gamma[(a+b)/2]\}, \alpha] \times T^3$$

yield the conclusion that the curve is extendible. Let us use the notation

$$\tau(s) = t[\gamma(s)], \quad \theta(s) = \theta[\gamma(s)].$$

Here θ is of course only defined locally, but this will not be a problem for our considerations. Due to the causality of the curve, we have

$$\left(\frac{d\theta}{ds}\right)^2 \leq e^{-2\tau} \left(\frac{d\tau}{ds}\right)^2.$$

Thus

$$\int_a^s \left|\frac{d\theta}{ds}\right| ds \leq \int_a^s e^{-\tau}(-\frac{d\tau}{ds}) ds = \int_\infty^{\tau(s)} (-e^{-\tau}) d\tau = e^{-\tau(s)}.$$

In particular, this proves that $\theta(s)$ converges to something as $s \to a$. We shall call this limit $\theta_0[\gamma]$. Furthermore, we see that $\theta[\gamma(s)]$ always belongs to $\mathcal{D}_{\theta_0[\gamma], t[\gamma(s)]}$.

Lemma 29. *Consider a T^3-Gowdy spacetime (M,g), i.e. g is of the form (1) and P, Q, λ satisfy (2)-(5). Let $\gamma : (a,b) \to M$ be an inextendible future directed causal curve. Then, using the above terminology, if $v_\infty(\theta_0[\gamma]) \neq 1$,*

$$\lim_{s \to a} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})[\gamma(s)] = \infty.$$

Proof. As was noted before the statement of the lemma, $\theta[\gamma(s)]$ always belongs to $\mathcal{D}_{\theta_0[\gamma], t[\gamma(s)]}$. Thus we only need to concern ourselves with a region defined by $\tau \geq \tau_0$ and $\theta \in \mathcal{D}_{\theta_0[\gamma], \tau}$. Due to Proposition 9, Corollary 5 and Proposition 8, we have quite good understanding for how the solution behaves in such a region. This is enough to obtain the conclusion of the lemma. □

In the proof of strong cosmic censorship, the following result is useful.

Lemma 30. *Consider a Lorentz manifold (M,g) such that for each timelike geodesic, the Kretschmann scalar $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ is unbounded in every incomplete direction. Then (M,g) is C^2-inextendible.*

Remark. The concept of C^2-inextendibility is specified in Definition 5.

Proof. Assume the spacetime is extendible and recall the notation of Definition 5. Let us first prove that there is a timelike geodesic beginning in $i(M)$ and ending outside of $i(M)$. Since $i(M) \neq \hat{M}$ and $\hat{M}$ is connected, there must be a point $p \in \partial i(M)$, and since $i(M)$ is an open subset of $\hat{M}$, $p \notin i(M)$. Let U be a convex neighbourhood of p and let $q \in U$ be such that there is a timelike geodesic from q to p with non-zero length. If $q \in i(M)$, we already have what we desire. Assume $q \notin i(M)$. Note that there is a neigbourhood V of p such that there is a timelike geodesic between any point in V and q. Since V is a neighbourhood of a boundary point of $i(M)$, we conclude that there is a timelike geodesic with the desired properties. In other words, we can assume that there is a timelike geodesic $\gamma : [0,1] \to \hat{M}$ with the property that $\gamma([0,1)) \subset i(M)$ and $\gamma(1) \notin i(M)$. Then $\gamma|_{[0,1)}$ is an inextendible timelike geodesic when viewed in M. Consequently, it either has to have infinite length or the Kretschmann scalar has to become unbounded along it. Both possibilities lead to a contradiction. □

Lemma 31. *The spacetime defined by the metric (1) on $M = \mathbb{R} \times T^3$ is globally hyperbolic.*

Proof. Let $\gamma : (s_-, s_+) \to M$ be an inextendible causal curve in (M, g) and let $M_\tau = \{\tau\} \times T^3$ for some $\tau \in \mathbb{R}$. Assume γ does not intersect M_τ. There are two possibilities; either $t[\gamma(s)] < \tau$ for all s or the opposite inequality holds for all s, cf. the notation above. Since the cases are similar, let us assume the former. Assuming $\langle \gamma', \partial_\tau \rangle < 0$, we get the conclusion that $\gamma(s)$ is contained in a compact set for all $s \in [s_0, s_+)$ for a fixed $s_0 \in (s_-, s_+)$. Since $t[\gamma(s)]$ is monotonically increasing and bounded from above, we conclude that it converges. Let $\tau_0 = t[\gamma(s_0)]$. Since $[\tau_0, \tau] \times S^1$ is compact, P, Q and λ and their derivatives are bounded on this set. Since $t[\gamma(s)]$ converges and γ is a causal curve, we conclude that the spatial component of γ also has to converge. □

Proposition 14. *Consider a causal geodesic $\gamma : (s_-, s_+) \to M$, where $M = \mathbb{R} \times T^3$ with a metric of the form (1) where P, Q and λ satisfy (2)-(5). Assume $\langle \gamma'(s), \partial_\tau|_{\gamma(s)} \rangle < 0$. Then γ is past incomplete.*

Remark. In the Gowdy spacetimes, $\tau \to -\infty$ corresponds to the expanding direction, so it is natural to consider the vectorfield ∂_τ to be past directed. A causal geodesic with the properties stated in the proposition is thus past directed, i.e. increasing s corresponds to going into the past.

Proof. We proceed as in [19]. Consider the orthonormal basis given by

$$e_0 = e^{\lambda/4+3\tau/4}\partial_\tau, \quad e_1 = e^{\lambda/4-\tau/4}\partial_\theta, \quad e_2 = e^{\tau/2-P/2}\partial_\sigma, \quad e_3 = e^{\tau/2+P/2}(\partial_\delta - Q\partial_\sigma).$$

Define

$$\phi = e^{-\lambda/4-3\tau/4}, \quad f_0 = -\langle \gamma', e_0|_\gamma \rangle, \quad f_k = \langle \gamma', e_k|_\gamma \rangle$$

for $k = 1, 2, 3$. Observe that $\sum f_k^2 \leq f_0^2$ due to causality. Let $\gamma_0 = t \circ \gamma$. By the arguments given above, $\gamma_0(s) \to \infty$ as $s \to s_+-$. Furthermore, the θ-coordinate of γ converges to $\theta_0[\gamma]$. We shall here omit the reference to γ and simply write θ_0. Consider

$$\frac{df_0}{ds} = -\langle \gamma', \nabla_{\gamma'} e_0 \rangle = -\sum_{\mu,\nu} f_\mu f_\nu \langle e_\mu, \nabla_{e_\nu} e_0 \rangle \circ \gamma.$$

Using the knowledge we have concerning the asymptotics, cf. Proposition 8 and 9,

$$\phi\langle e_1, \nabla_{e_1} e_0\rangle = -\frac{1}{4}(\lambda_\tau - 1),\ \phi\langle e_2, \nabla_{e_2} e_0\rangle = -\frac{1}{2}(1 - P_\tau),\ \phi\langle e_3, \nabla_{e_3} e_0\rangle = -\frac{1}{2}(1 + P_\tau)$$

and all other elements of the matrix $\phi\langle e_\mu, \nabla_{e_\nu} e_0\rangle$ converge to zero as $s \to s_+-$. Note that the θ-coordinate of $\gamma(s)$ belongs to $\mathcal{D}_{\theta_0, t[\gamma(s)]}$. Let

$$\theta_k = \phi\circ\gamma\langle e_k, \nabla_{e_k} e_0\rangle \circ \gamma.$$

There are in principle three different cases to consider. If $v_\infty(\theta_0) \leq 1$, then $\theta_1 \geq 0$ and $\theta_2, \theta_3 \leq 0$, if we neglect an error which converges to zero. If $v_\infty(\theta_0) > 1$, then $\theta_1 < 0$ in the limit and one of θ_2, θ_3 converges to $[v_\infty(\theta_0) - 1]/2$ and the other is negative in the limit. We have

$$\frac{df_0}{ds} \geq -\psi\circ\gamma \sum_k f_k^2 \theta_k + \psi\circ\gamma\delta f_0^2$$

where we here and below shall use the notation δ for any function such that $\delta(s) \to 0$ as $s \to s_+-$ and $\psi = 1/\phi$. Depending on the different cases we get one of

$$\frac{df_0}{ds} \geq \psi\circ\gamma f_0^2 \frac{1}{4}[v_\infty^2(\theta_0) - 1] + \psi\circ\gamma\delta f_0^2,\quad \frac{df_0}{ds} \geq -\psi\circ\gamma f_0^2 \frac{1}{2}[v_\infty(\theta_0) - 1] + \psi\circ\gamma\delta f_0^2.$$

Compute

$$\frac{d\psi\circ\gamma}{ds} = \frac{\partial\psi}{\partial\tau}\circ\gamma\frac{d\gamma_0}{ds} + \frac{\partial\psi}{\partial\theta}\circ\gamma\frac{d\gamma_1}{ds}$$

where γ_1 is the θ-coordinate of γ (observe that even though this is not well defined, the derivative is). However,

$$\frac{d\gamma_0}{ds} = \psi\circ\gamma f_0,\quad \frac{d\gamma_1}{ds} = \exp[\lambda\circ\gamma/4 - \gamma_0/4] f_1$$

so that

$$\frac{d\psi\circ\gamma}{ds} \geq \psi^2\circ\gamma f_0[\frac{1}{4}(v_\infty^2(\theta_0) + 3) + \delta].$$

Letting $h = f_0 \cdot \psi\circ\gamma$, we get

$$\frac{dh}{ds} = \frac{df_0}{ds}\psi\circ\gamma + f_0\frac{d\psi\circ\gamma}{ds}.$$

Depending on the different cases we thus get one of the following inequalities

$$\begin{aligned}\frac{dh}{ds} &\geq \frac{1}{2}h^2[v_\infty^2(\theta_0) + 1 + \delta],\\ \frac{dh}{ds} &\geq \frac{1}{4}h^2[2 - 2v_\infty(\theta_0) + v_\infty^2 + 3 + \delta] \geq \frac{1}{4}h^2[4 + \delta].\end{aligned}$$

Thus there is an s_1 such that for $s \geq s_1$

$$\frac{dh}{ds}(s) \geq \frac{1}{3}h^2(s).$$

We get the conclusion that the geodesic is past incomplete. □

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