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Consistency check on volume and triad operator quantization in loop quantum gravity: II

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Abstract
In this paper, we provide the techniques and proofs for the results presented in our companion paper concerning the consistency check on volume and triad operator quantization in loop quantum gravity.

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Contents

1. Introduction

In this paper, we deliver the necessary techniques and proofs for the results discussed in our companion paper [20]. The consistency check on the method of quantizing triads by means of the so-called Poisson bracket identity is performed. This identity allows us to replace triads by the Poisson brackets between the Ashtekar connection and the classical volume and places a prominent role in the dynamics of LQG [3]. The consistency check is made by constructing an alternative flux operator based on the Poisson bracket identity whose action is then compared with the action of the usual flux operator, quantized in a standard way as a differential operator.

In particular, we show that one must consider the electric field of LQG as a pseudo-2-form, since otherwise no consistent alternative flux operator can be obtained. Note that, classically, the electric field can be considered either as a 2-form or as a pseudo 2-form; the symplectic structure is insensitive to that when simultaneously changing the relation between the extrinsic curvature and the connection appropriately. Furthermore, a consistent alternative flux operator can only be achieved if one uses the volume operator introduced by Ashtekar and Lewandowski $\widehat{V}_{\mathrm{AL}}$ [6]. The Rovelli–Smolin volume operator $\widehat{V}_{\mathrm{RS}}$ [5] is inconsistent with the usual flux operator. The ambiguity of $\widehat{V}_{\mathrm{AL}}$ caused by regularization can be uniquely fixed by this consistency check. Moreover, since we apply the formula for matrix elements of the volume operator developed in [16], this formula is tested independently through our analysis here. Additionally, we could demonstrate that when considering higher representation weights than the fundamental one of $SU(2)$ for the holonomies involved in the alternative flux operator the results stay invariant. Hence, we get no ambiguities in the quantization process. Finally, the factor ordering of the alternative flux operator is unique if one insists on the principle of minimality.

These results show that instead of taking holonomies and fluxes as fundamental operators one could instead use holonomies and volumes as fundamental operators. It also confirms that the method to quantize the triad developed in [3] is mathematically consistent.

This paper is organized as follows. In section 2 we review the regularization and definition of the fundamental flux operator for the benefit of the reader and in order to make the comparison with the alternative quantization easier. In section 3 we derive the classical expression for the alternative flux operator. In section 4 we describe in detail the regularization of the alternative flux operator and arrive at its explicit action on spin network functions. In section 5 we draw first conclusions about and determine general properties of the expression obtained in section 4. In section 6 we compute the full matrix elements of the alternative flux operator. In section 7 we show that the chosen factor ordering is unique within the minimalistic class of factor orderings mentioned above. In section 8 we compute the matrix elements of the fundamental flux operator. In section 9 we compare the two flux operators and discover that there is a perfect match for any value of ℓ if and only if $C_{\rm reg} = 1/48$, if and only if the electric field is a pseudo-2-form and if and only if we use the AL volume operator. In section 10 we rule out the RS volume operator explicitly. In particular, we stress that the fact that the RS volume operator is inconsistent could not have been guessed from the outset. The consistency check performed in this paper is non-trivial and should not be taken as criticism of the RS volume but rather as a mechanism to tighten the mathematical structure of LQG. In section 11 we summarize and conclude. Finally, in appendices A–E we supply the detailed calculations and proofs for the claims that we have made in the main text.

2. Review of the usual flux operator in LQG

The classical electric flux $E_k(S)$ through a surface S in LQG is given by the integral of the densitized triad E_k^a over a 2-surface S

$$E_k(S) = \int_S E_k^a n_a^S, \tag{2.1}$$

where n_a^S is the conormal vector with respect to the surface S. In order to define a corresponding flux operator in the quantum theory, we have to consider the Poisson brackets between the classical electric flux and an arbitrary cylindrical function $f_\gamma : G^{|E(\gamma)|} \to \mathbb{C}$, where G is the corresponding gauge group, namely $SU(2)$ in our case:

$$\{E_k(S), f_\gamma(\{h_e(A)\}_{e\in E(\gamma)})\} = \sum_{e\in E(\gamma)} \left\{E_k^a, (h_e)_{AB}\right\} \frac{\partial f_\gamma}{\partial (h_e)_{AB}}. \tag{2.2}$$

We experience that the Poisson brackets between E_k^a and f_γ can be calculated whenever the Poisson brackets between E_k^a and the holonomy $(h_e)_{AB}$ are known. As the latter cannot be calculated on the manifold directly because terms including distributions would appear, we have to regularize our electric flux and also the holonomy. Then we will investigate the regularized Poisson brackets, remove the regulator afterwards and hope that at the end of the day we will obtain a well-defined operator. The regularization can be implemented by smearing the 2-surface S into the third dimension, shown in figure 1, so that we get an array of surfaces S_t. The surface associated with $t = 0$ is our original surface S.

We define our regularized classical flux as

$$E_k^\epsilon(S) := \frac{1}{2\epsilon} \int_{-\epsilon}^{+\epsilon} {\rm d}t\, E_k(S_t). \tag{2.3}$$

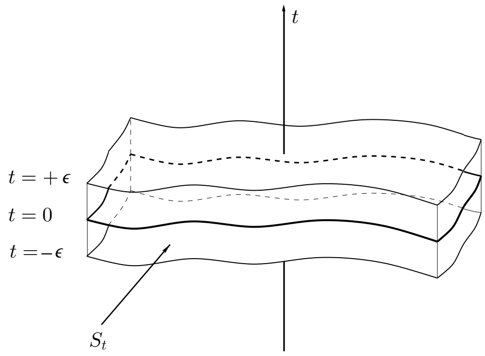

Figure 1. Smearing of the surface S into the third dimension. We obtain an array of surfaces S_t labelled by the parameter t with $t \in \{-\epsilon, +\epsilon\}$. The original surface S is associated with $t = 0$.

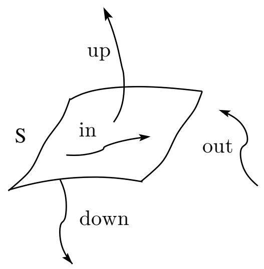

Figure 2. Edges of type up, down, in and out with respect to the surface S.

The corresponding operator $\widehat{E}_k(S)$ in the quantum theory is then defined as

$$\widehat{E}_k(S) f_\gamma := \mathrm{i}\hbar \lim_{\epsilon \to 0} \left\{ E_k^\epsilon(S), f_\gamma \right\}. \tag{2.4}$$

We have to derive the Poisson brackets between $E_k^\epsilon(S)$ and any possible cylindrical function f_γ. For this purpose, we can reduce the problem to investigating the Poisson brackets for any possible edge that is contained in the graph labelling the cylindrical function. The edges appearing can be classified as (i) up, (ii) down, (iii) in and (iv) out. Therefore, if we know the Poisson brackets for any of these types of edges, we will be able to derive the Poisson brackets between E_k^ϵ and any arbitrary f_γ. The calculation of the regularized Poisson brackets can be found, for example, in the second paper of [1]. After having removed the regulator we end up with the following action of the flux operator on an arbitrary cylindrical function f_γ,

$$\widehat{E}_k(S) f_\gamma = \frac{\mathrm{i}}{2} \ell_p^2 \sum_{e \in E(\gamma)} \epsilon(e, S) \left[\frac{\tau_k}{2} \right]_{AB} \frac{\partial f_\gamma (h_{e'})_{e' \in E(\gamma)}}{\partial (h_e)_{AB}}, \tag{2.5}$$

where τ_k is related to the Pauli matrices by $\tau_k := -\mathrm{i}\sigma_k$. The sum is taken over all edges of the graph γ associated with f_γ. The function $\epsilon(e, S)$ can take the values $\{-1, 0, +1\}$ depending on the type of edge that is considered. It is +1 for edges of type up, −1 one for down and 0 for edges of type in or out (see figure 2).

If we introduce right invariant vector fields X_k^e, defined by $\left(X_k^e f\right)(h) := \frac{\mathrm{d}}{\mathrm{d}t} f(e^{t\tau_k} h)\big|_{t=0}$, we can express the action of the flux operator by

$$\widehat{E}_k(S) f_\gamma = \frac{\mathrm{i}}{4} \ell_p^2 \sum_{e \in E(\gamma)} \epsilon(e, S) X_k^e f_\gamma. \tag{2.6}$$

The right invariant vector fields fulfil the following commutator relations:

$$\left[X_e^r, X_e^s\right] = -2\epsilon_{rst} X_e^t. \tag{2.7}$$

By means of introducing the self-adjoint right invariant vector field $Y_e^k := -\frac{\mathrm{i}}{2} X_e^k$, we achieve commutator relations for Y_e^k which are similar to that of the angular momentum operators in quantum mechanics

$$\left[Y_e^r, Y_e^s\right] = \mathrm{i}\epsilon_{rst} Y_e^t. \tag{2.8}$$

Consequently, we can describe the action of $\widehat{E}_k(S)$ by the action of the self-adjoint right invariant vector field Y_e^k on f_γ,

$$\widehat{E}_k(S) f_\gamma = -\frac{1}{2}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e, S) Y_e^k f_\gamma. \tag{2.9}$$

3. Idea and motivation of the alternative quantization of the flux operator

Recall the definition of the regularized classical flux $E_k^\epsilon(S)$ in equation (2.3). We take the Poisson brackets of the Ashtekar connection A_a^j and the densitized triad E_k^b given by

$$\left\{A_a^j(x), E_k^b(y)\right\} = \delta^3(x, y)\delta_b^a\delta_j^k \tag{3.1}$$

as our fundamental starting point. If we use a canonical transformation in order to go from the ADM formalism to the formulation in terms of Ashtekar variables, we have two possibilities in choosing such a canonical transformation that both lead to the Poisson brackets above. These two possibilities are

$$\begin{array}{lll} \mathrm{I} & A_a^j = \Gamma_a^j + \gamma\,\mathrm{sgn}(\det(e)) K_a^j, & E_k^a = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} e_b^s e_c^t \\ \mathrm{II} & A_a^j = \Gamma_a^j + \gamma K_a^j, & E_k^a = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} e_b^s e_c^t\,\mathrm{sgn}(\det(e)). \end{array} \tag{3.2}$$

Here Γ_a^j is the $SU(2)$-spin connection, K_a^j is the extrinsic curvature, and γ is the Immirzi parameter. Due to the two possible canonical transformations, we also have two possibilities in defining an alternative densitized triad

$$E_k^a = \begin{cases} \det(e) e_k^a = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} e_b^s e_c^t \\ \sqrt{\det(q)} e_k^a = \frac{1}{2}\epsilon_{kst}\epsilon^{abc} \underbrace{\mathrm{sgn}(\det(e))}_{=:\mathcal{S}} e_b^s e_c^t \end{cases} =: \begin{cases} E_k^{a,\mathrm{I}} \\ E_k^{a,\mathrm{II}}, \end{cases} \tag{3.3}$$

where e_a^j is the cotriad related to the intrinsic metric as $q_{ab} = e_a^j e_b^j$. From now on, we will use $E_k^{a,\mathrm{I}}$ and $E_k^{a,\mathrm{II}}$, respectively, for the two cases.

Now the idea of defining an alternative regularized flux

$$E_k^{\epsilon,\mathrm{I/II}}(S) := \frac{1}{2\epsilon}\int_{-\epsilon}^{+\epsilon} \mathrm{d}t\, \widetilde{E}_k^{\mathrm{I/II}}(S_t); \qquad \widetilde{E}_k^{\mathrm{I/II}}(S_t) = \int_{S_t} E_k^{a,\mathrm{I/II}} n_a^{S_t} \tag{3.4}$$

is to express the densitized triad E_k^a in terms of the triads as above. So instead of quantizing the densitized triad directly, we could use the above classical identities, quantize them via the Poisson bracket identity, and check whether both quantization procedures are consistent. The main difference between these two definitions is basically a sign factor which we will denote by $\mathcal{S}$. From the mathematical point of view, both definitions in equation (3.4) are equally viable, thus we will keep both possibilities and emphasize the differences that occur when we choose one or the other definition. However, note that case I leads to the anholonomic constraint $\det(E) \geqslant 0$ emphasized also in [13], which already seems unlikely to be reproduced by quantizing E as a vector field on some space of connections. Note that the precise distinction between case I and case II is often forgotten in the LQG literature where one treats $*E$ as a 2-form (I) when convenient and $*E$ as a vector density (II) when convenient. While this is classically immaterial as long as Σ is orientable, we will see that in the quantum theory this becomes crucial.

If we parametrize the surface integral over S_t, we obtain for the alternative flux

$$\widetilde{E}_k^{\mathrm{I/II}}(S_t) = \begin{cases} \int_{S_t} \mathrm{d}^2u\, \epsilon_{kst}\big[e_b^s(X(u))X^b_{,u_3}(u)\big]\big[e_c^t(X(u))X^c_{,u_4}(u)\big], & E_k^{a,\mathrm{I}} = \det(e)e_k^a \\ \int_{S_t} \mathrm{d}^2u\, \epsilon_{kst}\big[e_b^s(X(u))X^b_{,u_3}(u)\big]\mathcal{S}\big[e_c^t(X(u))X^c_{,u_4}(u)\big], & E_k^{a,\mathrm{II}} = \sqrt{\det(q)}e_k^a, \end{cases} \tag{3.5}$$

where we used the expression of the conormal vector $n_a^{S_t} = \epsilon_{aqr}X^q_{,u_3}(u)X^r_{,u_4}(u)$ associated with the surface S_t in terms of an arbitrary embedding $X : \left(-\frac{1}{2}, +\frac{1}{2}\right)^2 \to S;\ (u_3, u_4) \mapsto X(u_3, u_4)$.

Our strategy in quantizing the alternative expression of the electric flux will be as follows. First of all, we express the triads such as e_b^s in equation (3.5) in terms of the Poisson brackets between the components of the connection A_b^s and the volume $V(R)$, given by $\left\{A_b^s, V(R)\right\}$. Here $V(R) = \int_R \mathrm{d}^3x \sqrt{\det q}$ is the volume of the region R. This kind of quantization procedure was first introduced in [3] in order to derive a well-defined expression for the Hamiltonian constraint in the quantum theory and is used in various applications in LQG nowadays. By comparing the action of the alternative flux operator with that for the usual flux operator later on, we are able to verify whether this particular way of quantizing leads to the correct and expected result. Therefore, this can be seen as an independent check of this particular method of quantization. As the second step, we replace the connection by holonomies, for which well-defined operators exist. For this reason, we will have to partition each surface S_t and consider the limit where the partition gets finer and finer. This will be explained in more detail later. Before we apply canonical quantization and replace the Poisson brackets by the corresponding commutators, we want to get various issues out of the way.

3.1. Replacement of the triads by means of the Poisson brackets

As before we derive the relation between the Poisson brackets $\left\{A_b^s, V(R)\right\}$ and the cotriads for both expressions of E_k^a in equation (3.5). The explicit definition of the densitized triad E_k^a in terms of the e_k^a enters the calculation. Thus it is not surprising that the final result is different for the two cases:

$$\left\{A_b^s, V(R)\right\} = \begin{cases} -\frac{\kappa}{2}\mathcal{S}e_b^s, & E_k^{a,\mathrm{I}} = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}e_b^s e_c^t \\ -\frac{\kappa}{2}e_b^s, & E_k^{a,II} = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}\mathcal{S}e_b^s e_c^t. \end{cases} \tag{3.6}$$

By using the above identity and inserting it into equation (3.5) we get

$$\widetilde{E}_k^{\mathrm{I/II}}(S_t) = \begin{cases} \frac{4}{\kappa^2}\int_{S_t} \mathrm{d}^2u\, \epsilon_{kst}\big\{A_b^s(X(u))X^b_{,u_3}(u), V(R)\big\}\big\{A_c^t(X(u))X^c_{,u_4}(u), V(R)\big\}, \\ \quad E_k^{a,\mathrm{I}} = \det(e)e_k^a \\ \frac{4}{\kappa^2}\int_{S_t} \mathrm{d}^2u\, \epsilon_{kst}\big\{A_b^s(X(u))X^b_{,u_3}(u), V(R)\big\}\mathcal{S}\big\{A_c^t(X(u))X^c_{,u_4}(u), V(R)\big\}, \\ \quad E_k^{a,\mathrm{II}} = \sqrt{\det(q)}e_k^a. \end{cases} \tag{3.7}$$

Here $V(R)$ is any region containing $\cup S_t$, $t \in [-\epsilon, \epsilon]$, and we used $\mathcal{S}\mathcal{S} \in \{0, +1\}$ and thus could completely neglect the sign factor in the case of $E_k^a = \det(e)e_k^a$, because classically

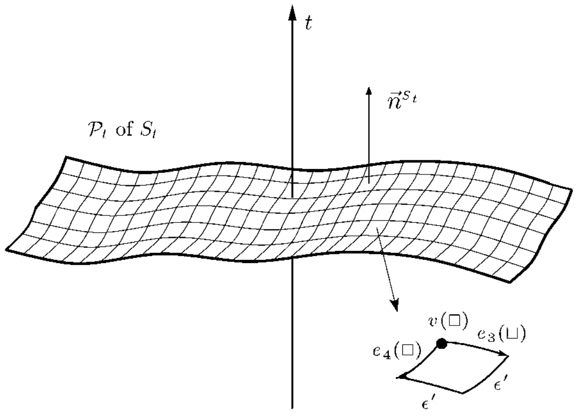

Figure 3. Partition $\mathcal{P}_t$ of the surface S_t into small squares with a parameter edge length ϵ'.

the 3-metric is non-degenerate, hence $\mathcal{S}^2 = 1$. At this stage, we already see that the main difference between the two expressions of E^a_k is whether we will have a sign factor in the final (classical) expression or not. Exactly this feature will be very important in the quantum expression, because the action of the corresponding operator differs remarkably if the operator contains a corresponding sign operator or if it does not.

3.2. Replacement of the connections by holonomies

Our main aim is to express the components of the connections $A^s_b(X_S(u))$ in terms of holonomies for which well-defined operators on the quantum level are known. For this reason, we partition each surface S_t into small squares with an parameter edge length ϵ' as shown in figure 3. We can therefore express the integral over S_t as the sum over the integrals over all small squares in the limit where the partition gets infinitesimally small. Consequently, we can rewrite equation (3.5) as

$$\widetilde{E}^{\mathrm{I/II}}_k(S_t) = \begin{cases} \lim\limits_{\mathcal{P}_t \to S_t} \sum\limits_{\square \in \mathcal{P}_t} \dfrac{4}{\kappa^2} \epsilon_{kst} \left\{A^s_3(\square), V(R_{v(\square)})\right\}\left\{A^t_4(\square), V(R_{v(\square)})\right\}, \\ \quad E^{a,\mathrm{I}}_k = \det(e) e^a_k \\ \lim\limits_{\mathcal{P}_t \to S_t} \sum\limits_{\square \in \mathcal{P}_t} \dfrac{4}{\kappa^2} \epsilon_{kst} \left\{A^s_3(\square), V(R_{v(\square)})\right\}\mathcal{S}\left\{A^t_4(\square), V(R_{v(\square)})\right\}, \\ \quad E^{a,\mathrm{II}}_k = \sqrt{\det(q)} e^a_k, \end{cases} \tag{3.8}$$

where we introduced the notation $A^s_I(\square) = \int_{e_\mathrm{I}(\square)} A^s, I = 3, 4$ for the integral over the connection along the edge $e_\mathrm{I}(\square)$ of $\square$. Here $R_{v_\square}$ is any region containing the point $e_3(\square) \cap e_4(\square)$ and in the limit $\epsilon' \to 0$ also $R_{v_\square} \to v(\square)$.

If we choose ϵ' small enough, we can use the following approximation:

$$\left\{A^s_I(\square), V(R_{v(\square)})\right\}\frac{\tau_s}{2} + o(\epsilon'^2) = +h_{e_\mathrm{I}}\left\{h^{-1}_{e_\mathrm{I}}, V(R_{v(\square)})\right\}. \tag{3.9}$$

The above equation holds for holonomies in the spin-$\frac{1}{2}$ representation. We would like to generalize this relation to the case of holonomies with an arbitrary weight ℓ in order to construct an operator that could contain arbitrary spin representations. This could be useful in the sense that we are then able to analyse whether the result of our alternative flux operator is sensitive to the chosen weight. That is, we investigate the effect of this particular kind of factor ordering ambiguity in the classical limit. The generalization of equation (3.9) is straightforward and leads to

$$\left\{A_I^s(\Box), V(R_{v(\Box)})\right\}\tfrac{1}{2}\pi_\ell(\tau_s) + o(\epsilon'^2) = +\pi_\ell\big(h_{e_I}\big)\left\{\pi_\ell\big(h_{e_I}^{-1}\big), V(R_{v(\Box)}\right), \tag{3.10}$$

where we denote a representation with weight ℓ by π_ℓ. By choosing ϵ' small enough, we are allowed to replace the Poisson brackets $\{A_3^s(\Box), V(R_{v(\Box)})\}$ and $\{A_4^t(\Box), V(R_{v(\Box)})\}$, respectively, by Poisson brackets including holonomies. Thus the basis of the alternative flux operator will be the following classical identity:

$$\begin{aligned}
{}^{(\ell)}\widetilde{E}_k^{\mathrm{I/II}}(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\epsilon_{kst}\frac{4}{\kappa^2}\left\{A_3^s(\Box), V(R_{v(\Box)})\right\}\mathcal{S}\left\{A_4^t(\Box), V(R_{v(\Box)})\right\}\\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\mathrm{Tr}\Big(\pi_\ell\big(h_{e_3(\Box)}\big)\left\{\pi_\ell\big(h_{e_3(\Box)}^{-1}\big), V(R_{v(\Box)})\right\}\\
&\quad\times\pi_\ell(\tau_k)\boxed{\mathcal{S}}\,\pi_\ell\big(h_{e_4(\Box)}\big)\left\{\pi_\ell\big(h_{e_4(\Box)}^{-1}\big), V(R_{v(\Box)})\right\}\Big).
\end{aligned} \tag{3.11}$$

The box around the sign factor $\mathcal{S}$ indicates that it is not contained in the equation if we choose $E^{a,\mathrm{I}}$, but occurs when we use $E_k^{a,\mathrm{II}}$. If one wants to show the correctness of the above identity, one has to use the following identity, $\mathrm{tr}\,(\pi_\ell(\tau_s)\pi_\ell(\tau_k)\pi_\ell(\tau_t)) = -\frac{4}{3}\ell(\ell+1)(2\ell+1)\epsilon_{skt}$ which is derived in appendix B.

Hence, we managed to derive an alternative expression for the flux operator on the classical level which we are able to quantize by means of well-known operators

$$\begin{aligned}
{}^{(\ell)}\widetilde{E}_k^{\mathrm{I/II}}(S_t) = \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}&\mathrm{Tr}\Big(\pi_\ell\big(h_{e_3(\Box)}\big)\left\{\pi_\ell\big(h_{e_3(\Box)}^{-1}\big), V(R_{v(\Box)})\right\}\\
&\times\pi_\ell(\tau_k)\boxed{\mathcal{S}}\,\pi_\ell\big(h_{e_4(\Box)}\big)\left\{\pi_\ell\big(h_{e_4(\Box)}^{-1}\big), V(R_{v(\Box)})\right\}\Big).
\end{aligned} \tag{3.12}$$

From now on, we will neglect the dependence of the edges $e_I(\Box)$ on the particular point $P_I(\Box)$ in order to keep the expressions clearer.

3.3. Notion of convergence and factor ordering

Let us now discuss in which sense the limit $\epsilon\to 0$ is to be understood. First of all, we formally have for any spin network state T_s

$$^{(\ell)}\hat{E}_k^\epsilon T_s := \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\sum_{s'}\langle T_{s'}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)|T_s\rangle T_{s'} \tag{3.13}$$

where we sum over all spin network labels s' (resolution of unity). Note that the sum $\sum_{s'}$ must be taken under the integral as otherwise the result would automatically be zero. Moreover, note that for each t the number of s' contributing is finite. Next we have

$$\langle T_{s'}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)|T_s\rangle = \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\langle T_{s'}|{}^{(\ell)}\widehat{\widetilde{E}}_k(\Box)|T_s\rangle. \tag{3.14}$$

In order to simplify the notation, let us assume that for all t the limit $\epsilon'\to 0$ implies $\mathcal{P}_t\to S_t$ while the parameter area of the squares within the partitions decays to zero as $(\epsilon')^2$. Then we can combine the two formulae and write

$$^{(\ell)}\hat{\widetilde{E}}_k^{\epsilon,\epsilon'} T_s := \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\mathrm{d}t\sum_{s'}\sum_{\Box\in\mathcal{P}_t}\langle T_{s'}|{}^{(\ell)}\widehat{\widetilde{E}}_k(\Box)|T_s\rangle T_{s'}. \tag{3.15}$$

It is easy to see that the Hilbert norm of this object vanishes with respect to the Hilbert space $\mathcal{H}_{\rm Kin} = L_2(\overline{\mathcal{A}}, {\rm d}\mu_{\rm AL})$ of LQG where $\overline{\mathcal{A}}$ is the Ashtekar–Isham space of *generalized* connections and $\mu_{\rm AL}$ is the Ashtekar–Lewandowski measure. Basically, this happens because the norm squared involves a double integral over t, t' while the integrand has support only on the measure zero subset $t = t'$. Hence, we cannot use the strong operator topology as a notion of convergence. The same applies to the weak operator topology. Rather, we will use the same notion of convergence as that which has been used for the fundamental flux operator: given a point $A \in \mathcal{A}$ in the space of *smooth* connections, we may evaluate the above expression at A and obtain a function on $\mathcal{A}$:

$$\left[{}^{(\ell)}\hat{\tilde{E}}_k^{\epsilon,\epsilon'} T_s\right](A) := \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} {\rm d}t \sum_{s'} \sum_{\Box\in\mathcal{P}_t} \langle T_{s'}|{}^{(\ell)}\widehat{\tilde{E}}_k(\Box)|T_s\rangle T_{s'}(A). \tag{3.16}$$

We now take the limit $\epsilon' \to 0$ *before* the limit $\epsilon \to 0$ in the following sense: we say that

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} {}^{(\ell)}\hat{\tilde{E}}_k{}^{\epsilon,\epsilon'}(S) = {}^{(\ell)}\widehat{\tilde{E}}_k(S) \tag{3.17}$$

provided that for any $A \in \mathcal{A}$ and any spin network label s

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} \left|\left[{}^{(\ell)}\hat{\tilde{E}}_k^{\epsilon,\epsilon'}(S) T_s\right](A) - [{}^{(\ell)}\widehat{\tilde{E}}_k(S) T_s](A)\right| = 0. \tag{3.18}$$

Note that the limit is pointwise in A, s and not uniform. Note also that this is a limit from the space of operators on the space of functions of smooth connections to operators on $\mathcal{H}_{\rm Kin}$ and not a convergence of operators on $\mathcal{H}_{\rm Kin}$.

With these preparations out of the way we may now draw some first conclusions about the action of the final operator ${}^{(\ell)}\widehat{\tilde{E}}_k(S)$. We may assume without loss of generality that both graphs $\gamma = \gamma(s), \gamma' = \gamma(s')$ underlying $\langle T_{s'}|{}^{(\ell)}\widehat{\tilde{E}}_k(\Box)|T_s\rangle$ are adapted to S in the sense that each of their edges has well-defined type with respect to S. If an edge e is of type up or down, respectively, then $S_t \cap e \neq \emptyset$ only for $t \geqslant 0$ or $t \leqslant 0$ respectively. If e is of type in or out, respectively, then for sufficiently small ϵ we have $S_t \cap e \neq \emptyset$ only for $t = 0$ or for no t at all respectively. Now consider $\langle T_{s'}|{}^{(\ell)}\widehat{\tilde{E}}_k(\Box)|T_s\rangle$ at finite ϵ'. Since $\widehat{\tilde{E}}_k(\Box)$ involves the volume operator which has non-trivial action only when the state on which it acts has at least one trivalent vertex, no matter whether we use the RS or AL volume operator, it easily follows that $\left[h_{e_I(\Box)}^{-1}, \hat{V}_{v(\Box)}\right]$, $I = 3, 4$ annihilates T_s unless $e_I(\Box), \gamma$ intersect each other. Hence, in order to obtain a non-vanishing contribution, we must refine $\mathcal{P}_t$ in such a way that each edge $e \in E(\gamma)$ intersecting S_t completely does so by intersecting with at least one of the $e_I(\Box)$ for some $I \in \{3, 4\}$ and at least one $\Box \in \mathcal{P}_t$. Making use of the fact that classically the limit $\mathcal{P}_t \to S_t$ is independent of the refinement, we refine $\mathcal{P}_t$ graph by demanding that eventually $e \cap S_t$ coincide with precisely one of the $v(\Box)$ if e is of type up or down and $t \geqslant 0$ or $t \leqslant 0$ respectively. This is motivated by the fact that otherwise no such edge would contribute if we use the AL version. If e is of type in and $t = 0$ then the number of intersections of e with the $e_I(\Box)$ necessarily diverges as $\epsilon' \to 0$. However, if we use the AL volume, all these contributions vanish because, in order that its action be non-trivial, it needs non-coplanar vertices, except if $v(\Box)$ coincides with an endpoint of e where there might be additional edges of adjacent e which are transversal to S. If we use the RS volume then all these intersections contribute and the sum over $\Box$ diverges for suitable s' as $\epsilon' \to 0$. However, since we perform the integral over t before taking $\epsilon' \to 0$ and the support of the integrand for type in edges consists of the measure zero set $t = 0$, the contribution vanishes, again no matter whether we use the RS or AL volume.

We conclude that for both versions of the volume operator only edges of type up or down will contribute, exactly as for the fundamental flux operator. However, for the AL volume the required ordering is more restrictive because there must be terms with both edges $e_3(\square)$, $e_4(\square)$ to the right of $\hat{V}_{v(\square)}$. For the RS volume there are more possibilities available which we will discuss in a later part of the paper.

Now let us derive which s' contribute to $\langle T_{s'}|^{(\ell)}\widehat{\widetilde{E}}_k(\square)|T_s\rangle$ for given s and $\square \in S_t$. We may restrict ourselves to edges of type up or down as just discussed. The factors $\pi_\ell(h_{e_I(\square)}), \pi_\ell(h_{e_I(\square)})^{-1}$ involved could *a priori* change the graph γ by adding the edge $e_I(\square)$ with spin $J = 0, 1, \ldots, 2\ell$. However, the operator $^{(\ell)}\widehat{\widetilde{E}}_k(\square)$ is invariant under gauge transformations at the endpoints of the $e_I(\square)$ by construction, hence we must necessarily have $J = 0$. Thus, even at finite ϵ' the operator $^{(\ell)}\widehat{\widetilde{E}}_k(\square)$ does not change the range of the graph γ. Hence, the only difference between s', s is that $\gamma' = \gamma$ but the edge $e \in E(\gamma')$ appears split into e_1^t, e_2^t with $e = (e_2^t)^{-1} \circ (e_1^t)$ and $e_1^t \cap e_2^t = v(\square) = e \cap S_t$. Note also that with d$t$ measure 1 the point $S_t \cap e$ is an interior point of e. This is important because the contribution of $\langle T_{s'}|^{(\ell)}\widehat{\widetilde{E}}_k(\square)|T_s\rangle$ for $\square \in S$ differs from that for $\square \in S_t, t \neq 0$ because in the former case $v(\square)$ may be a vertex of higher valence than 4.

Finally, note that $^{(\ell)}\widehat{\widetilde{E}}_k(\square)|T_s\rangle$ transforms in the spin-1 representation at $v(\square)$ because T_s is gauge invariant there. Hence $T_{s'}$ must have a spin-1 intertwiner at $v(\square)$.

What happens now when we take the limit as discussed is as follows. For each value of t the sum over $\square$ can be replaced by a finite number of terms, one for each $e \in E(\gamma)$ of type up or down and taking the limit $\epsilon' \to 0$ becomes trivial. Next, for each value of t and each edge $e \in E(\gamma)$ there will be a finite number of states $T_{s'_{e,t}}$ which contribute to the sum over s' and which are mutually orthogonal for different e, t. The numbers $\langle T_{s'_{e,t}}|^{(\ell)}\widehat{\widetilde{E}}_k(\square)|T_s\rangle$ do not depend on t (thanks to the diffeomorphism invariance of the measure); however, the states $T_{s'_{e,t}}$ do. Fortunately, considered as functions of smooth connections, the limit $\epsilon \to 0$ converges and results in states $T_{s'_e}$ where $\gamma(s'_e) = \gamma$ not only have the same range but also the same edge sets. Then s, s'_e differ only by the intertwiner at the point $v = b(e)$.

3.4. Classical identity

Collecting all the arguments of the discussion of the last section, we end up with the following ordering of the classical terms[3],

$$
\begin{aligned}
^{(\ell)}\widetilde{E}_k^{\mathrm{I/II}}(S_t) = &-\lim_{\mathcal{P}_t \to S_t} \sum_{\square \in \mathcal{P}_t} \frac{16}{\kappa^2} \frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \pi_\ell(\tau_k)_{CB} \pi_\ell(h_{e_4})_{CD} \\
&\times \{\pi_\ell(h_{e_3}^{-1}), V(R_{v(\square)})\}_{AB} \boxed{\mathcal{S}} \{V(R_{v(\square)}), \pi_\ell(h_{e_4}^{-1})\}_{DE} \pi_\ell(h_{e_3})_{EA},
\end{aligned}
\tag{3.19}
$$

where the indices $\{A, B, C, D, E\} \in \{-\ell, \ldots, +\ell\}$.

4. Construction of the alternative flux operator

The Poisson brackets in equation (3.19) includes the classical volume function $V(R_{v(\square)})$, therefore the corresponding alternative flux operator will contain the volume operator $\widehat{V}$. As mentioned in the introduction, in LQG there exist two different volume operators, $\widehat{V}_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$. Thus for each case I and II we have two different alternative flux operators depending on the choice of $\widehat{V}_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ respectively. Hence, after canonical quantization, we end up with four different versions of the alternative flux operator. For these four operators, we use

[3] In the case of $\widehat{V}_{\mathrm{RS}}$ there exists more than this symmetric factor ordering. We will discuss this aspect later in the paper.

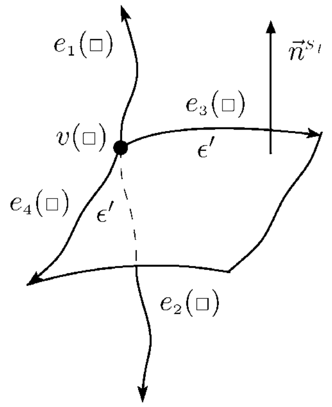

Figure 4. A non-vanishing contribution to $\langle T_{s'}|^{(\ell)}\widehat{E}_k(\square)|T_s\rangle$ can only be achieved if T_s contains edges of type up and/or down, respectively, with respect to the surface S_t. Moreover, the edges $e_3(\square)$, $e_4(\square)$ have to be attached to T_s in this specific way.

the following notation:

$$\widehat{E}^{\mathrm{I}}_k(S_t) \longrightarrow \widehat{E}^{\mathrm{I,AL}}_k(S_t), \qquad \widehat{E}^{\mathrm{I,RS}}_k(S_t) \tag{4.1}$$

$$\widehat{E}^{\mathrm{II}}_k(S_t) \longrightarrow \widehat{E}^{\mathrm{II,AL}}_k(S_t), \qquad \widehat{E}^{\mathrm{II,RS}}_k(S_t). \tag{4.2}$$

Before we apply canonical quantization on the classical identity in equation (3.19) we will discuss the two volume operators $\widehat{V}_{\mathrm{RS}}$, $\widehat{V}_{\mathrm{AL}}$ in more detail.

4.1. The two volume operators of LQG

4.1.1. The volume operator $\widehat{V}_{\mathrm{RS}}$ of Rovelli and Smolin. The idea that the volume operator acts only on the vertices of a given graph was first mentioned in [17]. The first version of a volume operator can be found in [5] and is given by

$$\begin{aligned}
\widehat{V}(R)_\gamma &= \int_R \mathrm{d}^3 p\, \widehat{V}(p)_\gamma \\
\widehat{V}(p)_\gamma &= \ell_p^3 \sum_{v\in V(\gamma)} \delta^{(3)}(p,v)\widehat{V}_{v,\gamma} \\
\widehat{V}^{\mathrm{RS}}_{v,\gamma} &= \sum_{I,J,K}\sqrt{\left|\frac{\mathrm{i}}{8}C_{\mathrm{reg}}\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K}\right|}.
\end{aligned} \tag{4.3}$$

Here we sum over all triples of edges at the vertex $v \in V(\gamma)$ of a given graph γ. $\widehat{V}_{\mathrm{RS}}$ is not sensitive to the orientation of the edges, thus also linearly dependent triples have to be considered in the sum. Moreover, we introduced a constant $C_{\mathrm{reg}} \in \mathbb{R}$ that we will keep arbitrary for the moment and that is basically fixed by the particular regularization scheme one chooses. As for the usual flux operator, we express $\widehat{V}$ in terms of self-adjoint vector fields $Y^k_e := -\frac{\mathrm{i}}{2}X^k_e$. Hence, we have

$$\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K} = -8\mathrm{i}\epsilon_{ijk}Y^i_{e_I}Y^j_{e_J}Y^k_{e_K} \tag{4.4}$$

and thus

$$\widehat{V}^{\mathrm{RS}}_{v,\gamma} = \sum_{I,J,K}\sqrt{\left|\mathrm{i}C_{\mathrm{reg}}\epsilon_{ijk}Y^i_{e_I}Y^j_{e_J}Y^k_{e_K}\right|}. \tag{4.5}$$

In order to select the gauge invariant states properly, we have to express our abstract angular momentum states in terms of the recoupling basis. The following identity [12] holds,

$$\tfrac{1}{8}\epsilon_{ijk}X^i_{e_I}X^j_{e_J}X^k_{e_K} = \tfrac{1}{4}\big[Y^2_{IJ}, Y^2_{JK}\big] =: \tfrac{1}{4}q^Y_{IJK}, \tag{4.6}$$

where $Y_{IJ} := Y_I + Y_J$. Consequently, we get

$$\widehat{V}(R)^{Y,\mathrm{RS}}_{\gamma}|JM;M'\rangle = \ell_p^3 \sum_{v\in V(\gamma)\cap R}\sum_{I<J<K} \underbrace{3!\sqrt{\left|\frac{\mathrm{i}}{4}C_{\mathrm{reg}}\widehat{q}^Y_{IJK}\right|}}_{\widehat{V}^{\mathrm{RS}}_{v,\gamma}}|JM;M'\rangle. \tag{4.7}$$

The additional factor of 3! is due to the fact that we sum only over ordered triples $I<J<K$ now. The way to calculate eigenstates and eigenvalues of $\widehat{V}$ is as follows. Let us introduce the operator $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ as

$$\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK} := \ell_p^6\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}\widehat{q}^Y_{IJK}. \tag{4.8}$$

As the first step, we have to calculate the eigenvalues and corresponding eigenstates for $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$. If, for example, $|\phi\rangle$ is an eigenstate of $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ with corresponding eigenvalue λ, then we obtain $\widehat{V}|\phi\rangle = \sqrt{|\lambda|}|\phi\rangle$. Consequently, we see that while $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ can have positive and negative eigenvalues, $\widehat{V}$ has only positive ones. Furthermore, if we consider the eigenvalues $\pm\lambda$ of $\widehat{Q}^{Y,\mathrm{RS}}_{v,IJK}$ and the corresponding eigenstate $|\phi_{+\lambda}\rangle, |\phi_{-\lambda}\rangle$, we note that these eigenvalues will be degenerate in the case of the operator $\widehat{V}$, as $\sqrt{|+\lambda|} = \sqrt{|-\lambda|}$.

4.1.2. The volume operator $\widehat{V}_{\mathrm{AL}}$ of Ashtekar and Lewandowski. Another version of the volume operator which differs by the chosen regularization scheme was defined in [6]

$$\widehat{V}(R)^{Y,\mathrm{AL}}_{\gamma}|JM;M'\rangle = \ell_p^3 \sum_{v\in V(\gamma)\cap R} \underbrace{\sqrt{\left|\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}\sum_{I<J<K}\epsilon(e_I,e_J,e_K)\widehat{q}^Y_{IJK}\right|}}_{\widehat{V}^{\mathrm{AL}}_{v,\gamma}}|JM;M'\rangle. \tag{4.9}$$

The major difference between $\widehat{V}_{\mathrm{AL}}$ and $\widehat{V}_{\mathrm{RS}}$ is the factor $\epsilon(e_I,e_J,e_K)$ that is sensitive to the orientation of the tangent vectors of the edges $\{e_I,e_J,e_K\}$. $\epsilon(e_I,e_J,e_K)$ is +1 for right-handed, −1 for left-handed and 0 for linearly dependent triples of edges. In the case of $\widehat{V}_{\mathrm{AL}}$ it is convenient to introduce an operator $\widehat{Q}^{Y,\mathrm{AL}}_{v}$ that is defined as the expression that appears inside the absolute value under the square root in $\widehat{V}^{\mathrm{AL}}_{v,\gamma}$,

$$\widehat{Q}^{Y,\mathrm{AL}}_{v} := \ell_p^6\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}\sum_{I<J<K}\epsilon(e_I,e_J,e_K)\widehat{q}^Y_{IJK} \tag{4.10}$$

By comparing equation (4.7) with (4.9) we note that another difference between $\widehat{V}_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ is the fact that for the first one, we have to sum over the triples of edges outside the square root, while for the latter one, we sum inside the absolute value under the square root. In addition to the difference of the sign factor, the difference in the summation will play an important role later on. Note that one arrives at (4.9) also from a usual point splitting regularization [12].

4.2. Canonical quantization

Usually the densitized triads, appearing in the classical flux $E_k(S)$ are quantized as differential operators, while holonomies are quantized as multiplication operators. If we choose the alternative expression $\widetilde{E}_k(S)$ we will instead get the scalar volume $\widehat{V}$ and the so-called sign $\widehat{\mathcal{S}}$ operator into our quantized expression. The properties of this $\widehat{\mathcal{S}}$ will be explained in more detail below. Moreover, we have to replace Poisson brackets by commutators, following the replacement rule $\{\cdot,\cdot\} \to (1/\mathrm{i}\hbar)[\cdot,\cdot]$. In order to simplify the following calculations, we achieve a form of the operator such that on the left-hand side only inverses of the holonomies appear while right beneath the product of operators $\widehat{V}\boxed{\widehat{\mathcal{S}}}\widehat{V}$ only holonomies appear. Thus, we make use of the identities $\widehat{\pi}_\ell\big(h_{e_1}^{-1}\big)_{AB} = \pi_\ell(\epsilon)_{AC}\pi_\ell(\epsilon)_{BD}\widehat{\pi}_\ell\big(h_{e_1}\big)_{DC}$ and $\widehat{\pi}_\ell\big(h_{e_1}\big)_{AB} = \pi_\ell(\epsilon)_{CA}\pi_\ell(\epsilon)_{DB}\widehat{\pi}_\ell\big(h_{e_1}^{-1}\big)_{DC}$, where $\pi_\ell(\epsilon)$ stands for the ϵ_{AB} of $SU(2)$ in a higher representation with weight ℓ. The explicit form can be derived from equation (B.1) in appendix B and is given by $\pi_\ell(\epsilon)_{AB} = (-1)^{\ell-A}\delta_{A+B,0}$. Clearly, we want the total operator to be self-adjoint, so we will calculate the adjoint of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S_t)$ and define the total and final operator to be ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t) = \frac{1}{2}(\widehat{\widetilde{E}}_k(S_t) + \widehat{\widetilde{E}^\dagger}_k(S_t))$ that is self-adjoint by construction. Hence, the final operator for $\widehat{V}_{\mathrm{RS}}$ which we will use through the calculation of this paper is given by

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I/II,RS}}_{k,\mathrm{tot}}(S_t) = \lim_{\mathcal{P}_t\to S_t} \sum_{\square\in\mathcal{P}_t} \frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI}\big\{+\pi_\ell(\epsilon)_{FC}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\big]^\dagger \\
\times \big[\big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{BA}\big]^\dagger, \widehat{V}_{\mathrm{RS}}\big]\boxed{\widehat{\mathcal{S}}}\big[\widehat{V}_{\mathrm{RS}}, \widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\big]\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} - \pi_\ell(\epsilon)_{FB}\big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{IG}\big]^\dagger \\
\times \big[\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{EA}\big]^\dagger, \widehat{V}_{\mathrm{RS}}\big]\boxed{\widehat{\mathcal{S}}}\big[\widehat{V}_{\mathrm{RS}}, \widehat{\pi}_\ell\big(h_{e_3}\big)_{FG}\big]\widehat{\pi}_\ell\big(h_{e_4}\big)_{CA}\big\},
\end{aligned} \tag{4.11}
$$

whereby we used the identity $\pi_\ell\big(h_{e_1}^{-1}\big)_{AB} = \big[\pi_\ell\big(h_{e_1}\big)_{BA}\big]^\dagger$, the definition of the Planck length $\ell_p^{-4} := (\hbar\kappa)^{-2}$, and additionally, $\pi_\ell(\epsilon)_{GD}\pi_\ell(\epsilon)_{DH} = (-1)^{2\ell}\delta_{G,H}$.

Considering the operator $\widehat{V}_{\mathrm{AL}}$, we know that for each commutator only one term will contribute, because otherwise we cannot construct linearly independent triples of edges since $\{e_1, e_2, e_{3/4}\}$ are linearly dependent. Therefore in the case of $\widehat{V}_{\mathrm{AL}}$ we obtain the following final expression:

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I/II,AL}}_{k,\mathrm{tot}}(S_t) = \lim_{\mathcal{P}_t\to S_t} \sum_{\square\in\mathcal{P}_t} \frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI}\big\{+\pi_\ell(\epsilon)_{FC}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\big]^\dagger \\
\times \big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{BA}\big]^\dagger\widehat{V}_{\mathrm{AL}}\boxed{\widehat{\mathcal{S}}}\widehat{V}_{\mathrm{AL}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} - \pi_\ell(\epsilon)_{FB}\big[\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\big]^\dagger \\
\times \big[\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA}\big]^\dagger\widehat{V}_{\mathrm{AL}}\boxed{\widehat{\mathcal{S}}}\widehat{V}_{\mathrm{AL}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{CA}\big\}.
\end{aligned} \tag{4.12}
$$

Here again for case II the sign operator is included, whereas in case I it is not.

By looking at the equation above, we see that the operator contains a lot of sums, so it does not seem to be that trivial to actually compute expectation values. However, we will show in the next section how we can use the given structure of the operator and derive some properties from it that will simplify the summation and therefore the calculation of expectation values.

5. General properties of the operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$

In this section, we will discuss some general properties of the alternative flux operator. Since these properties are valid independent of the choice of $\widehat{V}_{\mathrm{AL}}$ or $\widehat{V}_{\mathrm{RS}}$ we will drop this labelling of the volume operator here. If not explicitly mentioned otherwise these properties also hold independently of the fact whether we are considering case I or case II. Thus, we will only talk about the operator ${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$.

5.1. Correspondence between the AL and the abstract angular momentum system Hilbert space

Going back to the action of the usual flux operator in equation (2.9) we see that the action of the flux operator can be expressed in terms of self-adjoint right invariant vector fields Y_e^k. The same is true for the volume operator appearing in the new alternative flux operator. Since we would like to utilize the technology of Clebsch–Gordan coefficients (CGC), $6j$-symbols and the like in order to calculate matrix elements of these operators with respect to spin network states, we will discuss in detail how the AL-Hilbert space and the abstract angular momentum system Hilbert space are related.

Consider the explicit expression for the matrix elements of the unitary transformation matrix $[\pi_j(g)]_{mn}$ for the components ψ_m of a totally symmetric spinor of rank $2j$ under $SU(2)$ gauge transformations reviewed in appendix E, that is, $\psi'_m = \sum_{n=-j}^{j}[\pi_j(g)]_{mn}\psi_n$. By elementary linear algebra, the unitary representation $g \mapsto U(g)$ of $SU(2)$ on the linear span of the standard angular momentum states $|jm\rangle$ is obtained by transposition, i.e. $U(g)|jm\rangle = \sum_{n=-j}^{j}[\pi_j(g)]_{nm}|jn\rangle$. To see this, it is enough to check that the standard angular momentum operators J^k when written in terms of ladder operators have the same action as the infinitesimal generators of the one-parameter groups $t \mapsto U(\exp(\mathrm{i}t\tau_k/2))$. (Recall that $\mathrm{i}\tau_k = \sigma_k$ are the Pauli matrices.) Explicitly, we find

$$J^k|jm\rangle = +\sum_n \frac{\mathrm{i}}{2}[\pi_j(\tau_k)]_{nm}|jn\rangle$$

where $\pi_j(\tau_k)$ are the matrices derived in appendix E.

Now consider the functions

$$\langle h_e|jm\rangle_{m'} := \sqrt{2j+1}[\pi_j(h_e)]_{mm'}, \tag{5.1}$$

where h_e denotes the holonomy along some edge e. For fixed m' they are orthonormal just as the $|jm\rangle$. Moreover, the operators $Y_e^k := -\mathrm{i}X_e^k/2$, where X_e^k are the right invariant vector fields on $SU(2)$, satisfy the same algebra as the J^k. Let us drop the label e for the purpose of this paragraph. From the explicit representation of the gauge transformation on the $\langle h|jm\rangle_{m'}$ given by $V(g)\langle h|jm\rangle_{m'} = \langle gh|jm\rangle_{m'} = [\pi_j(g)]_{mn}\langle h|jn\rangle_{m'}$ we can explicitly calculate that the Y^k are the infinitesimal generators of the one-parameter groups $t \mapsto V(\exp(\mathrm{i}t\tau_k/2))$, explicitly

$$Y^k|jm\rangle_{m'} = -\sum_n \frac{\mathrm{i}}{2}[\pi_j(\tau_k)]_{mn}|jn\rangle_{m'}. \tag{5.2}$$

It is instructive to verify the angular momentum algebra for J^k, Y^k.

The fluxes are expressed in terms of the Y_e^k and the spin network states are expressed in terms of the $|jm\rangle_{m'}^e$ (the superscript e reminds us of the edge which the state $|jm\rangle_{m'}$ is associated with). In order to write these in terms of J^k and $|jm\rangle$ we must determine the unitary operator

$$W : \mathcal{H}^{jm'} \to \mathcal{H}^j_{m'}; \qquad W|jm;m'\rangle = \sum_n W_{jmn}|jn\rangle_{m'} \tag{5.3}$$

such that $WJ^kW^{-1} = Y^k$. Here $\mathcal{H}^{jm'}$ is the linear span of abstract angular momentum eigenstates $|jm;m'\rangle$ which for fixed $m' \in \{-j, -j+1, \ldots, j\}$ are just the $|jm\rangle$ with additional label m' while $\mathcal{H}^j_{m'}$ is the linear span of the spin network states $|jm\rangle_{m'}$, and W_{jmn} is a unitary matrix.

It is not difficult to see from the above formulae that $W_{jmn} = [\pi_j(\epsilon)]_{mn}$ where $\epsilon = -\tau_2$. Therefore

$$W|jm; m'\rangle = [\pi_j(\epsilon)]_{mn}|jn\rangle_{m'} \quad \Leftrightarrow \quad W^{-1}|jm\rangle_{m'} = [\pi_j(\epsilon^{-1})]_{mn}|jn; m'\rangle \tag{5.4}$$

and we will make frequent use of the identities $\epsilon^{-1} = \epsilon^T = -\epsilon, \epsilon g^T \epsilon^T = g^{-1}$ valid for any $g \in SL(2,\mathbb{C})$ such as $g = \tau_k$ and $\tau_k^{-1} = -\tau_k = \overline{\tau_k}^T$.

Now in order to use these identities, consider some spin network states $T_{\tilde{\gamma},\vec{\tilde{j}},\vec{\tilde{m}},\vec{\tilde{m}}'}$, $T_{\gamma,\vec{j},\vec{m},\vec{m}'}$ and some operator $\hat{O}_Y$ which we think of as a function in the operators Y^k_e. Then by unitarity

$$\begin{aligned}\langle T_{\tilde{\gamma},\vec{\tilde{j}},\vec{\tilde{m}},\vec{\tilde{m}}'}|\hat{O}_Y|T_{\gamma,\vec{j},\vec{m},\vec{m}'}\rangle_{\mathrm{SNF}} &= \sum_{\vec{\tilde{n}},\vec{n}} \prod_{\tilde{e}\in E(\tilde{\gamma})} [\pi_{j_{\tilde{e}}}(\epsilon^{-1})]_{\tilde{m}_{\tilde{e}}\tilde{n}_{\tilde{e}}} \\ &\times \prod_{e\in E(\gamma)} [\pi_{j_e}(\epsilon^{-1})]_{m_e n_e} \langle T'_{\tilde{\gamma},\vec{\tilde{j}},\vec{\tilde{n}},\vec{\tilde{m}}'}|\hat{O}_J|T'_{\gamma,\vec{j},\vec{n},\vec{m}'}\rangle_{\mathrm{ABS}}\end{aligned} \tag{5.5}$$

where SNF stands for the spin network Hilbert space and ABS for the abstract angular momentum system Hilbert space. We use the following notation. Whenever we address SNF we call them T and express them in terms of $|jm\rangle_{m'}$. In contrast, if we refer to states in the abstract angular momentum system Hilbert space, we use the notation T' for the abstract angular momentum system functions which result from T upon substituting $|jm\rangle_{m'}$ by $|jm; m'\rangle$. The operator $\hat{O}_J$ is the same as $\hat{O}_Y$ except that Y^k_e is everywhere replaced by J^k_e.

The discussion above shows that we have to map the holonomies $\pi_\ell(h)_{AB}$ in the alternative flux operator in equation (4.12) into the abstract angular momentum system Hilbert space via the unitary map W in equation (5.4) in order to apply technical tools of usual angular momentum recoupling theory. Thus, if we apply the unitary map W (summation convention is assumed)

$$W\widehat{\pi}_\ell(h)_{AB} = \frac{\pi_\ell(\epsilon^{-1})_{AC}}{\sqrt{2\ell+1}}\langle h|\ell C; B\rangle, \tag{5.6}$$

use the fact that $\pi_\ell(\tau_k)_{BC} = -[\pi_\ell(\epsilon)\pi_\ell(\tau_k)\pi_\ell(\epsilon^{-1})]_{CB}$ and the following properties of $\pi_\ell(\epsilon^{-1})$

$$\begin{aligned}&\pi_\ell(\epsilon^{-1})_{AB} = (-1)^{2\ell}\pi_\ell(\epsilon)_{AB} \qquad \pi_\ell(\epsilon)_{AB}\pi_\ell(\epsilon)_{BC} = (-1)^{2\ell}\delta_{AC} \\ &\pi_\ell(\epsilon)_{AB} = (-1)^{2\ell}\pi_\ell(\epsilon)_{BA}\end{aligned} \tag{5.7}$$

that can easily be derived from the explicit expression of $\pi_\ell(\epsilon)_{AB}$, we end up with

$$\begin{aligned}{}^{(\ell)}\widehat{E}_{k,\mathrm{tot}}(S_t) &= -\lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)^2}\pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{EI} \\ &\times \{+\pi_\ell(\epsilon)_{FC}(\otimes\langle_{e_3}\ell B; A| \otimes \langle_{e_4}\ell F; G|\widehat{O}_1|\ell I; G\rangle_{e_4} \otimes |\ell E; A\rangle_{e_3}\otimes) \\ &- \pi_\ell(\epsilon)_{FB}(\otimes\langle_{e_3}\ell E; A| \otimes \langle_{e_4}\ell I; G|\widehat{O}_2|\ell F; G\rangle_{e_4} \otimes |\ell C; A\rangle_{e_3}\otimes)\}.\end{aligned} \tag{5.8}$$

The definitions of the operators $\widehat{O}_1$ and $\widehat{O}_2$ in the four different cases are shown in equation (5.9). We introduced the notation V_{qIJK} in the RS case meaning that only the contribution of the triple $\{e_I, e_J, e_K\}$ is taken into account. Why $\widehat{O}_{1,2}$ have this particular structure in the case of RS will be explained in more detail in appendix E. Basically, the structure displayed is due to the various contributions from the four terms involved in the product of two commutators in equation (4.11):

$$\begin{aligned}O_1^{\mathrm{I,AL}} &= \widehat{V}^2_{\mathrm{AL}} \\ O_1^{\mathrm{I,RS}} &= \widehat{V}^2_{\mathrm{RS}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{123}} - \widehat{V}_{q_{124}}\widehat{V}_{\mathrm{RS}} - \widehat{V}_{\mathrm{RS}}\widehat{V}_{q_{123}}\end{aligned}$$

$$\begin{aligned}
O_2^{\mathrm{I,AL}} &= \widehat{V}_{\mathrm{AL}}^2 \\
O_2^{\mathrm{I,RS}} &= \widehat{V}_{\mathrm{RS}}^2 + \widehat{V}_{q_{123}}\widehat{V}_{q_{124}} - \widehat{V}_{q_{123}}\widehat{V}_{\mathrm{RS}} - \widehat{V}_{\mathrm{RS}}\widehat{V}_{q_{124}} \\
O_1^{\mathrm{II,AL}} &= \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}} \\
O_1^{\mathrm{II,RS}} &= \widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{RS}} + \widehat{V}_{q_{124}}\widehat{\mathcal{S}}\widehat{V}_{q_{123}} - \widehat{V}_{q_{124}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{RS}} - \widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}\widehat{V}_{q_{123}} \\
O_2^{\mathrm{II,AL}} &= \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}} \\
O_2^{\mathrm{II,RS}} &= \widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{RS}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}}\widehat{V}_{q_{124}} - \widehat{V}_{q_{123}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{RS}} - \widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}\widehat{V}_{q_{124}}.
\end{aligned} \tag{5.9}$$

Recall from the discussion in section 3.3 that the action of both operators $\widehat{E}_k(S)$, ${}^{(\ell)}\widehat{\widehat{E}}_k(S)$ on any SNF was totally determined by its action on single edges of type up, down, in and out, and that the latter two were annihilated by this operator. The surface $\Box$ which intersects an edge e of type up or down necessarily transversally splits e as $e = e_2(\Box)^{-1} \circ e_1(\Box)$ where $e_1(\Box), e_2(\Box)$ is of type up or down with respect to $\Box$ (or S_t) if e is of type up with respect to S and conversely if e is of type down. Note that $e_I(\Box)$, $I = 1, 2$ inherit from e the same spin label j coupling to total spin j_{12} at the point $v(\Box) = e_1(\Box) \cap e_2(\Box)$.

As the operators $\widehat{O}_1$ and $\widehat{O}_2$ in equation (5.8) contain the volume operator $\widehat{V}(R_{v(\Box)})$, at some point we will have to calculate matrix elements of $\widehat{V}$. With this in mind, it is advisable to work in the so-called recoupling basis right from the beginning, because the formula for matrix elements of $\widehat{V}$ derived in [16] applies only to states in that particular basis[4]. The particular SNF we want to work with can be characterized in the recoupling basis by its total angular momentum j_{12} and its magnetic quantum number n_{12} (and two additional labels m_1', m_2') since the first intermediate coupling a_1 is equivalent to the spin label of the first edge which is fixed and j in our case. Therefore, we will call those states $|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'} := |a_1 = j a_2 = j_{12} n_{12}\rangle_{m_1' m_2'}$ where $n_{12} \in \{-j_{12}, \ldots, j_{12}\}$ and m_1', m_2' can be treated as additional indices unimportant for the recoupling procedure. This means that to a fixed choice of j_{12} we have $(2j_{12}+1)(2n_{12}+1)$ orthogonal states $|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'}$ being a basis of the Hilbert space for this particular value of j_{12}. This SNF is also shown in figure 5.

As before we map the SNF $|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'}$ and the operators $\widehat{O}^Y_{1/2}$ into the abstract angular momentum system Hilbert space

$$W|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'} = \sum_{m_{12}} \pi_{j_{12}}(\epsilon^{-1})_{n_{12}m_{12}} |\beta^{j_{12}}, m_{12}; m_1', m_2'\rangle. \tag{5.10}$$

Consequently, the map W has the following effect on the matrix element of ${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$,

$$\begin{aligned}
{}_{\widetilde{m}_1',\widetilde{m}_2'}\langle\beta^{\tilde{j}_{12}}, \widetilde{n}_{12}|{}^{(\ell)}\hat{\hat{E}}^Y{}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'} &= \sum_{m_{12},\widetilde{m}_{12}} \pi_{\tilde{j}_{12}}(\epsilon^{-1})_{\tilde{n}_{12}\widetilde{m}_{12}} \pi_{j_{12}}(\epsilon^{-1})_{n_{12}m_{12}} \\
&\quad \times \langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}; \widetilde{m}_1'\widetilde{m}_2'|{}^{(\ell)}\hat{\hat{E}}^J{}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}; m_1' m_2'\rangle
\end{aligned}$$

where for reasons of clarity we denoted by superscripts Y, J the same algebraic expression in terms of the Y, J operators respectively. In what follows, we will drop this label and

[4] Recall that in the tensor basis a state is characterized by the spin labels j_i and the magnetic quantum numbers m_i and an additional label m_i' that are attached to the edges e_i of a particular vertex of the corresponding graph γ. We express a given SNF in this basis by tensor products between states $|jm_i\rangle^e_{m_i'}$ multiplied by corresponding intertwiners. In contrast, in the recoupling basis states are characterized by the total angular momentum J, the total magnetic quantum number M to which the edges couple at a particular vertex of the graph γ and the value of the intermediate couplings. In order to know what kind of intermediate couplings are possible, we have to fix an order in which we want to couple the edges associated at one particular vertex from the very beginning. Then the intermediate couplings a_i are successively defined by $a_{i+1} := \{|a_i - j_{i+1}|, \ldots, a_i + j_{i+1}\}$ with $a_1 := j_1$. If we choose a different order of coupling, we will end up with a different recoupling scheme, where these two recoupling schemes are related by so-called $3nj$-symbols. (For a brief introduction to recoupling theory see, for example, [12, 16].)

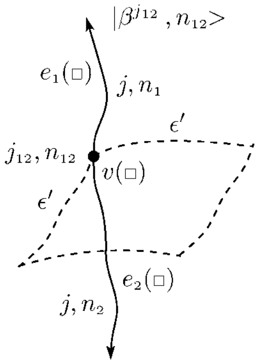

Figure 5. SNF $|\beta^{j_{12}}, n_{12}\rangle$ that consists of two edges, whereby one is of type up and the other of type down with respect to the surface S_t. These two edges carry both a spin label j and couple at the vertex $v(\square)$ to an resulting angular momentum j_{12}.

it will be understood that we will be working in the abstract angular momentum space only. The same transformation applies to the matrix element of the usual flux operator. Since the inverse of the matrices $\pi_{j_{12}}(\epsilon^{-1})$ exists, we can conclude that in order to show that the matrix element of the usual flux operator and that of ${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$ are identical, we only have to show that after taking the limits $\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}$ the matrix element $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12};$ $m_1'm_2'|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}; m_1'm_2'\rangle$ agrees with the matrix element of the usual flux operator $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}; m_1'm_2'|\widehat{E}_k(S)|\beta^{j_{12}}, m_{12}; m_1'm_2'\rangle$ for every possible value of $\widetilde{m}_{12}, m_{12}$. Thus, we do not have to consider the two additional $\pi_{j_{12}}(\epsilon^{-1})$.

Note that also the explicit value of the matrix element of the usual flux operator will be contracted by these $\pi_j(\epsilon^{-1})$. Considering gauge invariant states ($j = 0$) only, $\pi_0(\epsilon^{-1}) = 1$ is only a single number. Thus, if one would work with gauge invariant operators only, all $\pi_j(\epsilon^{-1})$ would drop out in equation (5.11).

For the further calculation of $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}; \widetilde{m}_1'\widetilde{m}_2'|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}; m_1'm_2'\rangle$ we will introduce the following abbreviations:

$$\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}; m_1'm_2'| := \langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}| \qquad |\beta^{j_{12}}, m_{12}; m_1'm_2'\rangle := |\beta^{j_{12}}, m_{12}\rangle. \tag{5.11}$$

5.2. The explicit action of ${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$

If the operator acts on such a state $|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'}$ it will basically add two additional edges e_3 and e_4 to the SNF. These edges lie in the surface S_t as can be seen in figure 6. Consequently, applying the operator to the states $|\beta^{j_{12}}, n_{12}\rangle$ means nothing else than coupling the two additional edges e_3, e_4 to the already existing edges e_1, e_2 and constructing a new SNF with four edges that we will call $|\alpha_i^J, M\rangle$. We label these new states $|\alpha_i^J, M\rangle$ again by their resulting total angular momentum J and their corresponding magnetic quantum number M. The two additional edges both carry a spin label ℓ. These states $|\alpha_i^J, M\rangle$ include three intermediate couplings a_1, a_2, a_3, and a_4 is equal to the total angular momentum J. In contrast to $|\beta^{j_{12}}, m_{12}\rangle$ we need an additional index i here for distinguishing all possible states $|\alpha_i^J, M\rangle$, because it will be the case that for a particular value of J several values of intermediate couplings a_2, a_3 exist. (This becomes clearer when we explicitly describe the set of states that belong to a particular total angular momentum J and that build a basis of the corresponding Hilbert space.)

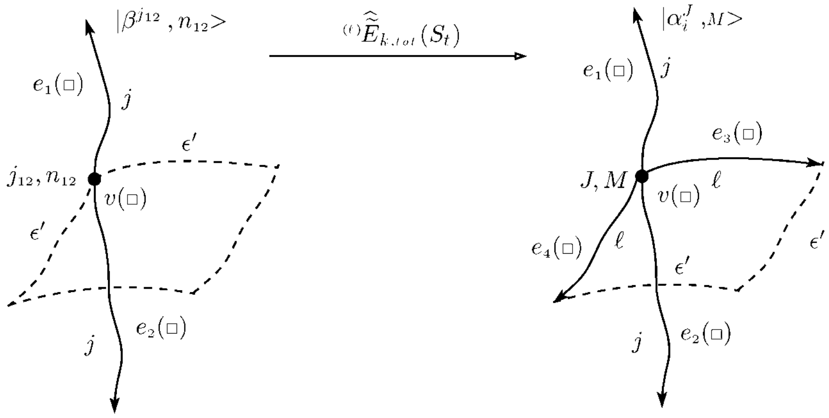

Figure 6. The SNF $|\beta^{j_{12}}, n_{12}\rangle$ is transformed into a new SNF $|\alpha_i^J M\rangle$ by the action of $^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$.

Therefore the action of $^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$ can be expressed in terms of the recoupling basis states $|\alpha_i^J, M\rangle$, where the expansion coefficients are the corresponding CGC.

Therefore the action and consequently the matrix element of $^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$ can be described by the following expression:

$$\begin{aligned}
\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle &= -\lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)^2}\\
&\times\sum_{A,B,C,E,F,G=-\ell}^{+\ell}\Bigg\{\pi_\ell(\tau_k)_{CB}\pi_\ell(\epsilon)_{E-E}\sum_{\widetilde{J}=|\widetilde{a}_3-\ell|}^{\widetilde{a}_3+\ell}\sum_{J=|a_3-\ell|}^{a_3+\ell}\sum_{\widetilde{a}_3=|\tilde{j}_{12}-\ell|}^{\tilde{j}_{12}+\ell}\sum_{a_3=|j_{12}-\ell|}^{j_{12}+\ell}\delta_{\widetilde{J},J}\\
&\times\big[+\pi_\ell(\epsilon)_{FC}\langle\tilde{j}_{12}\widetilde{m}_{12};\,\ell B|\widetilde{a}_3\widetilde{m}_{12}+B\rangle\langle\widetilde{a}_3\widetilde{m}_{12}+B;\,\ell F|\widetilde{J}\widetilde{m}_{12}+B+F\rangle\\
&\times\langle j_{12}m_{12};\,\ell E|a_3m_{12}+E\rangle\langle a_3m_{12}+E;\,\ell-E|Jm_{12}\rangle\delta_{\widetilde{m}_{12}+F+B,m_{12}}\\
&\times\big\langle\alpha_{\tilde{i}}^{\widetilde{J}}, M=\widetilde{m}_{12}+B+F;\,\widetilde{m}_1'\widetilde{m}_2'AG\big|\widehat{O}_1\big|\alpha_i^J, M=m_{12};\,m_1'm_2'AG\big\rangle\\
&-\pi_\ell(\epsilon)_{FB}\langle\tilde{j}_{12}\widetilde{m}_{12};\,\ell E|\widetilde{a}_3\widetilde{m}_{12}+E\rangle\langle\widetilde{a}_3\widetilde{m}_{12}+E;\,\ell-E|\widetilde{J}\widetilde{m}_{12}\rangle\\
&\times\langle j_{12}m_{12};\,\ell C|a_3m_{12}+C\rangle\langle a_3m_{12}+C;\,\ell F|Jm_{12}+C+F\rangle\delta_{m_{12}+C+F,\widetilde{m}_{12}}\\
&\times\big\langle\alpha_{\tilde{i}}^{\widetilde{J}}, M=\widetilde{m}_{12};\,\widetilde{m}_1'\widetilde{m}_2'AG\big|\widehat{O}_2\big|\alpha_i^J, M=m_{12}+C+F;\,m_1'm_2'AG\big\rangle\big]\Bigg\}.
\end{aligned}\tag{5.12}$$

Here $\langle j_1m_1; j_2m_2|JM\rangle$ denotes the CGC that describes the coupling of the angular momentum j_1 and j_2 with magnetic quantum numbers m_1, m_2 to a resulting angular momentum J with magnetic quantum number M.

Since the states $|\alpha_i^J, M\rangle$ for different angular momenta and different magnetic quantum numbers are orthogonal to each other, meaning $\langle\alpha_{\tilde{i}}^{\widetilde{J}}, \widetilde{M}\,|\,\alpha_i^J, M\rangle = \delta_{\widetilde{J},J}\delta_{\widetilde{M},M}\langle\alpha_{\tilde{i}}^J, M|\alpha_i^J, M\rangle$ and the operator $\widehat{O}$ leaves J and M invariant, we replaced $\widetilde{J}$ and $\widetilde{M}$ by J and M and added the necessary δ-function $\delta_{\widetilde{J},J}$. Furthermore, we used the definition $\pi_\ell(\epsilon)_{EI} = (-1)^{\ell-E}\delta_{E+I,0}$ and substituted I by $-E$ in the whole equation. This restriction of I together with the constraint that $|\alpha_i^J, M\rangle$ and $\langle\alpha_{\tilde{i}}^J, M|$ must have the same magnetic quantum number leads to two other δ-functions including m_{12} and $\widetilde{m}_{12}$. Although the δ-functions above will surely simplify the summation, we still have 11 sums in total and some even depend on each other. Especially, the summation over $\widetilde{J}$ and J contains many terms. But fortunately due to the structure of the operator we can reduce these sums.

Theorem 5.1. *The resulting angular momenta J and $\widetilde{J}$ of the states $|\alpha_i^J, M\rangle$ and $\langle\alpha_{\tilde{i}}^{\widetilde{J}}, M|$ that do contribute to the matrix element $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ are only j_{12} and $\tilde{j}_{12}$ of the incoming states, respectively.*

More precisely, the only contribution to ${}^{(\ell)}\widehat{\widehat{E}}_k(S_t)$ is the angular momentum $J = j_{12}$, while the only contribution to ${}^{(\ell)}\widehat{\widehat{E^\dagger}}_k(S_t)$ is $\widetilde{J} = \tilde{j}_{12}$.

(Recall that the first term of the sum in equation (5.12) is caused by ${}^{(\ell)}\widehat{\widehat{E}}_k(S_t)$ and the second and negative part belongs to ${}^{(\ell)}\widehat{\widehat{E^\dagger}}_k(S_t)$.)

Proof 5.1. First of all we will prove the following lemma. Afterwards, we will use it so as to be able to prove the theorem just stated. □

Lemma 5.2.

$$\sum_{E=-\ell}^{+\ell} \pi_\ell(\epsilon)_{E-E}\langle j_{12}m_{12};\ell E|a_3 m_{12}+E\rangle\langle a_3 m_{12}+E;\ell-E|J m_{12}\rangle$$
$$= (-1)^{-j_{12}-\ell-3a_3}\frac{\sqrt{2a_3+1}}{\sqrt{2j_{12}+1}}\delta_{J,j_{12}}\big(\delta_{m_{12},-j_{12}}+\delta_{m_{12},-j_{12}+1}+\cdots+\delta_{m_{12},j_{12}}\big). \tag{5.13}$$

The proof of lemma 5.2 is shown in appendix A.

We use lemma 5.2 for performing the sum over E in equation (5.12). The summation over $\widetilde{J}$ and J contains only one term now, so we can easily carry out these two sums. Moreover, since the operator $\widehat{O}$ does not change the m'-indices, we can trivially sum over the indices A, G. This leads to an additional factor of $(2\ell+1)^2$. Accordingly, the final version of the matrix element of the operator ${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)$ with which we will start in the next section is

$$\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$$
$$= -\lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^{-4}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\sum_{B,C,F=-\ell}^{+\ell}\Bigg\{\pi_\ell(\tau_k)_{CB}\sum_{\tilde{a}_3=|\tilde{j}_{12}-\ell|}^{\tilde{j}_{12}+\ell}\;\sum_{a_3=|j_{12}-\ell|}^{j_{12}+\ell}$$
$$\times\Bigg[+(-1)^{-F}\delta_{F+C,0}(-1)^{-j_{12}-3a_3}\frac{\sqrt{2a_3+1}}{\sqrt{2j_{12}+1}}\delta_{\widetilde{m}_{12}+F+B,m_{12}}$$
$$\times\langle\tilde{j}_{12}\widetilde{m}_{12};\ell B|\tilde{a}_3\widetilde{m}_{12}+B\rangle\langle\tilde{a}_3\widetilde{m}_{12}+B;\ell F|j_{12}\widetilde{m}_{12}+B+F\rangle$$
$$\times\big\langle\alpha_{\tilde{i}}^{j_{12}}, M=\widetilde{m}_{12}+B+F;\widetilde{m}'_1\widetilde{m}'_2\big|\widehat{O}_1\big|\alpha_i^{j_{12}}, M=m_{12};m'_1m'_2\big\rangle$$
$$-(-1)^{-F}\delta_{F+B,0}(-1)^{-\tilde{j}_{12}-3\tilde{a}_3}\frac{\sqrt{2\tilde{a}_3+1}}{\sqrt{2\tilde{j}_{12}+1}}\delta_{m_{12}+C+F,\widetilde{m}_{12}}$$
$$\times\langle j_{12}m_{12};\ell C|a_3m_{12}+C\rangle\langle a_3m_{12}+F;\ell C|\tilde{j}_{12}m_{12}+C+F\rangle$$
$$\times\big\langle\alpha_{\tilde{i}}^{\tilde{j}_{12}}, M=\widetilde{m}_{12};\widetilde{m}'_1\widetilde{m}'_2\big|\widehat{O}_2\big|\alpha_i^{\tilde{j}_{12}}, M=m_{12}+C+F;m'_1m'_2\big\rangle\Bigg]\Bigg\}, \tag{5.14}$$

where we used $\delta_{J,\widetilde{J}}\delta_{J,j_{12}} = \delta_{\widetilde{J},j_{12}}$ and $(-1)^{4\ell} = +1$. We omitted the sum over the δ-function acting on the magnetic quantum number m_{12} and $\widetilde{m}_{12}$ respectively (see lemma 5.2). This is possible as long as we keep in mind that the action of the operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ is identical for each fixed m_{12} and $\widetilde{m}_{12}$ of the states $|\beta^{j_{12}}, m_{12}\rangle$ and $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|$.

However, by simply looking at equation (5.14) we see that only the resulting angular momentum $J = \widetilde{J} = j_{12}, \tilde{j}_{12}$ contributes to the matrix element $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$.

Consequently, we have proven theorem (5.1).

We can read off from equation (5.14) that we have already managed to reduce the number of summations down to 5 just by investigating the physical properties of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$.

5.3. Behaviour of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S)$ under gauge transformations

Now we will take a closer look at the behaviour of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$ under gauge transformations and see that this will constrain the possible values of $\tilde{j}_{12}$. Applying a gauge transformation on equation (3.11) under which h_{e_I} transforms as $h^g_{e_\mathrm{I}} \to g(b(e_\mathrm{I}))g^{-1}(f(e_\mathrm{I}))$ with $b(e_\mathrm{I})$ and $f(e_\mathrm{I})$ being the beginning and the final point of the edge e_I respectively, we obtain

$$\left[{}^{(\ell)}\widetilde{E}_k(S_t)_{k,\mathrm{tot}}\right]^g = \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{16}{\kappa^2}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\,\mathrm{Tr}\Big(\pi_\ell\big(h_{e_3}\big)\big\{\pi_\ell\big(h^{-1}_{e_3}\big), V(R_{v(\Box)})\big\}$$
$$\times\, g^{-1}(b(e))\pi_\ell(\tau_k)g(b(e))\boxed{\mathcal{S}}\,\pi_\ell\big(h_{e_4}\big)\,\big\{\pi_\ell\big(h^{-1}_{e_4}\big), V(R_{v(\Box)})\big\}\Big). \tag{5.15}$$

Thus the classical expression transforms in the spin-1 representation, due to the term $g^{-1}(b(e))\pi_\ell(\tau_k)g(b(e))$. Consequently, we know that if we applied the corresponding operator on an incoming state $|\beta^{j_{12}}, m_{12}\rangle$, the action of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$ would change the intertwiner at the vertex $v(\gamma)$ by $0, \pm 1$. Therefore, if we consider matrix elements of the kind $\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_k(S)|\beta^{j_{12}}, m_{12}\rangle$ the only non-vanishing values for $\tilde{j}_{12}$ are $\tilde{j}_{12} = j_{12}, j_{12}\pm 1$. In the specific case where $j_{12} = 0$, $\tilde{j}_{12}$ can only take the value $\tilde{j}_{12} = j_{12}+1$. Of course, we only want to consider incoming states that are physically relevant. Therefore we have to choose an incoming state $|\beta^{j_{12}}, m_{12}\rangle$ with a total angular momentum $j_{12} = 0$ in order to ensure that this state is gauge invariant. Hence, the transformation property of ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$ leads to the restriction of $\tilde{j}_{12} = 1$. Therefore, by means of theorem (5.1), the only total angular momentum J of the states $\left|\alpha^J_i, M\right\rangle$ that contribute to the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)$ is $J = 0, 1$. Therefore equation (5.14) can be rewritten, according to our particular choices of $j_{12} = 0$ and $\tilde{j}_{12} = 1$, as

$$\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^0, m_{12}\rangle = -\lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{8\ell_p^{-4}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\sum_{B,C,F=-\ell}^{+\ell}$$
$$\times\Bigg\{\pi_\ell(\tau_k)_{CB}\sum_{\tilde{a}_3=|1-\ell|}^{1+\ell}\sum_{a_3=|-\ell|}^{+\ell}\Big[+(-1)^{-F}\delta_{F+C,0}(-1)^{-3a_3}\sqrt{2a_3+1}\,\delta_{\widetilde{m}_{12}+F+B,m_{12}}$$
$$\times\langle 1\widetilde{m}_{12};\ell B|\tilde{a}_3\widetilde{m}_{12}+B\rangle\langle\tilde{a}_3\widetilde{m}_{12}+B;\ell F|0\widetilde{m}_{12}+B+F\rangle$$
$$\times\big\langle\alpha^0_{\tilde{i}}, M=\widetilde{m}_{12}+B+F;\widetilde{m}'_1\widetilde{m}'_2\big|\widehat{O}_1\big|\alpha^0_i, M=m_{12};m'_1m'_2\big\rangle$$
$$-(-1)^{-F}\delta_{F+B,0}(-1)^{-1-3\tilde{a}_3}\frac{\sqrt{2\tilde{a}_3+1}}{\sqrt{3}}\delta_{m_{12}+C+F,\widetilde{m}_{12}}$$
$$\times\langle 0m_{12};\ell C|a_3m_{12}+C\rangle\langle a_3m_{12}+C;\ell F|1m_{12}+C+F\rangle$$
$$\times\big\langle\alpha^1_{\tilde{i}}, M=\widetilde{m}_{12};\widetilde{m}'_1\widetilde{m}'_2\big|\widehat{O}_2\big|\alpha^1_i, M=m_{12}+C+F;m'_1m'_2\big\rangle\Big]\Bigg\}, \tag{5.16}$$

where $\widetilde{m}_{12} = \{-1, 0, 1\}$ and $m_{12} = 0$ is the only possible value of the magnetic quantum number for $|\beta^0, m_{12}\rangle$.

In the next section, we will calculate the matrix elements $\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\text{tot}}(S_t)|\beta^0, m_{12}\rangle$ of all four versions ${}^{(\ell)}\widehat{\widetilde{E}}^{\text{I,AL}}_{k,\text{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}^{\text{I,RS}}_{k,\text{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}^{\text{II,AL}}_{k,\text{tot}}(S_t)$, ${}^{(\ell)}\widehat{\widetilde{E}}^{\text{II,RS}}_{k,\text{tot}}(S_t)$ of the new flux operator.

6. Matrix elements of the new flux operator ${}^{(\ell)}\widehat{\widetilde{E}}_{k,\text{tot}}(S_t)$

Before we explicitly calculate the necessary matrix elements of $\widehat{O}_1$, $\widehat{O}_2$, the question arises: what are the matrix elements that we need, or rather what kind of matrix elements will appear in the recoupling procedure of equation (5.16)? As the action of $\widehat{V}$ and accordingly also the action of $\widehat{q}_{IJK}$ leave the total angular momentum J of a state $|\alpha_i^J, M\rangle$ invariant, the whole matrix that includes the elements of all possible values of J belonging to a particular choice of j_{12} and $\tilde{j}_{12}$ would be divided into orthogonal submatrices for each fixed total angular momentum J. Consequently, we can actually calculate the eigenvalues and eigenstates separately for every possible value of J. Hence, in our case we should take a detailed look at the corresponding Hilbert spaces of $J = 0, 1$. Similarly to $|\beta^{j_{12}}, m_{12}\rangle$ the spin labels of e_1 and e_2 of $|\alpha_i^J, M\rangle$ are identical ($j_1 = j_2 = j$). Therefore, we already know that $a_2 = j \otimes j \in \{0, +1, \ldots, 2j\} = j_{12}$ can only be an integer. Hence, a basis of the Hilbert space belonging to $J = 0$ is given by

$$\begin{aligned}
&|\alpha_1^0, M\rangle := |a_1 = j\; a_2 = 0\; a_3 = \ell\; J = 0\rangle\\
&|\alpha_2^0, M\rangle := |a_1 = j\; a_2 = 1\; a_3 = \ell\; J = 0\rangle\\
&|\alpha_3^0, M\rangle := |a_1 = j\; a_2 = 2\; a_3 = \ell\; J = 0\rangle\\
&\cdots\\
&|\alpha_{2j+1}^0, M\rangle := |a_1 = j\; a_2 = 2j\; a_3 = \ell\; J = 0\rangle.
\end{aligned} \tag{6.1}$$

Here, the only possible value for a_3 is $a_3 = \ell$, because otherwise a_3 and $j_4 = \ell$ could not couple to a resulting angular momentum $J = 0$. Furthermore, we have assumed that the condition $a_2 \leqslant 2\ell$ has to be fulfilled to ensure that a resulting total angular momentum of $J = 0$ can be achieved. If this is not the case, the number of states reduces down to the number of states where the condition $a_2 \leqslant 2\ell$ is still true[5]. Fortunately, we will not have to calculate matrix elements of all possible combinations of states. In our case, we already know that $\tilde{j}_{12} = 1$ and $j_{12} = 0$. This is equivalent to $\widetilde{a}_2 = 1$ and $a_2 = 0$ and we realize that we only have to calculate the matrix element $\langle\alpha_2^0, M|\widehat{q}_{134}|\alpha_1^0, M\rangle$ here.

The transformation properties of the operator ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$, discussed in section 5.3, led us to this restriction $\tilde{j}_{12} = 1$. Even if we have not at all been worried about any transformation properties of the operator before, we see at this point by simply looking at equation (6.6) that all other possible matrix elements $\langle\alpha_i^0, M|\widehat{q}_{134}|\alpha_1^0, M\rangle$ where $i > 2$ will vanish anyway. This is due to the fact that for $i > 2\; \Delta a_2 := |\widetilde{a}_2 - a_2| > 1$. In this case, the $6j$-symbols in equation (6.6) in the last brackets will be zero and this makes the whole matrix element vanish. Summarizing, if we start with a gauge invariant state $|\beta^0, 0\rangle$ there exists only one non-vanishing matrix element for the case $J = 0$ which is $\langle\alpha_2^0, M|\widehat{q}_{134}|\alpha_1^0, M\rangle$ in our notation.

Let us analyse the case of a total angular momentum $J = 1$ now. In this case, we have three different values of the intermediate coupling $a_3 = \{\ell - 1, \ell, \ell + 1\}$ to ensure that a total angular momentum of $J = 1$ can be achieved. Hence, a basis of the corresponding Hilbert

[5] Consequently, only for large enough ℓ the Hilbert space belonging to a zero total angular momentum will be $(2j+1)$ dimensional; for example, for the simplest case $\ell = \frac{1}{2}$ it is only two dimensional.

space is given by

$$
\begin{aligned}
&\left|\alpha_1^1, M\right\rangle := |a_1 = j\ a_2 = 0\ a_3 = \ell\ J = 1\rangle \\
&\left|\alpha_2^1, M\right\rangle := |a_1 = j\ a_2 = 1\ a_3 = \ell - 1 J = 1\rangle \\
&\left|\alpha_3^1, M\right\rangle := |a_1 = j\ a_2 = 1\ a_3 = \ell J = 1\rangle \\
&\left|\alpha_4^1, M\right\rangle := |a_1 = j\ a_2 = 1\ a_3 = \ell + 1 J = 1\rangle \\
&\ldots \\
&\left|\alpha_{6j-1}^1, M\right\rangle := |a_1 = j\ a_2 = 2j\ a_3 = \ell - 1 J = 1\rangle \\
&\left|\alpha_{6j}^1, M\right\rangle := |a_1 = j\ a_2 = 2j\ a_3 = \ell J = 1\rangle \\
&\left|\alpha_{6j+1}^1, M\right\rangle := |a_1 = j\ a_2 = 2j\ a_3 = \ell + 1 J = 1\rangle.
\end{aligned}
\tag{6.2}
$$

Here the condition on a_2 and ℓ is $a_2 \leqslant 2\ell + 1$. Note that in the special and simplest case where $\ell = \frac{1}{2}$ the intermediate coupling $a_3 = \ell - \frac{1}{2}$ is not sensible, therefore this state has to be dropped here and the Hilbert space includes only $5 \times 3 = 15$ states. Again, due to the construction of the operator ${}^{(\ell)}\widehat{\widetilde{E}}_k(S)$, we only have to consider the matrix elements with $\widetilde{a}_2 = 1, a_2 = 0$ and these are precisely $\left\langle\alpha_i^1, M\middle|\widehat{q}_{134}\middle|\alpha_1^1, M\right\rangle$ where $i = 2, 3, 4$. Hence, we see that for $J = 1$ three different matrix elements will contribute to the final result. As in the case $J = 0$ all matrix elements $\left\langle\alpha_i^1, M\middle|\widehat{q}_{134}\middle|\alpha_1^1, M\right\rangle$ for $i > 4$ vanish, because then $\Delta a_2 := |\widetilde{a}_2 - a_2| > 1$.

We will now go back to equation (5.16) and apply our new results. Furthermore, the discussion above showed that in the first term of equation (5.16) the only possible value for $a_3, \widetilde{a}_3$ is ℓ (case $J = 0$). In the second term $a_3 = \ell$ is still valid, but here $\widetilde{a}_3$ can take the values $\widetilde{a}_3 = \{\ell - 1, \ell, \ell + 1\}$. Therefore equation (5.16) simplifies to

$$
\begin{aligned}
\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^0, 0\rangle &= -\lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{8\ell_p^{-4}(-1)^{3\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\sum_{B,C,F=-\ell}^{+\ell}\Bigg\{\pi_\ell(\tau_k)_{CB} \\
&\times\Bigg[+(-1)^{-F}\delta_{F+C,0}\sqrt{2\ell+1}\,\delta_{\widetilde{m}_{12}+B+F,0}\langle 1\widetilde{m}_{12};\ \ell B|\ell\widetilde{m}_{12}+B\rangle\langle\ell\widetilde{m}_{12}+B;\ \ell F|00\rangle \\
&\times\left\langle\alpha_2^0, M = \widetilde{m}_{12}+B+F;\ \widetilde{m}_1'\widetilde{m}_2'\middle|\widehat{O}_1\middle|\alpha_1^0, M = 0;\ m_1'm_2'\right\rangle \\
&-(-1)^{-F}\delta_{F+B,0}\delta_{C+F,\widetilde{m}_{12}}\langle 00;\ \ell C|\ell C\rangle\langle\ell C;\ \ell F|1C+F\rangle \\
&\times\Bigg[+\frac{\sqrt{2\ell-1}}{\sqrt{3}}\left\langle\alpha_2^1, M = \widetilde{m}_{12};\ \widetilde{m}_1'\widetilde{m}_2'\middle|\widehat{O}_2\middle|\alpha_1^1, M = C+F;\ m_1'm_2'\right\rangle \\
&-\frac{\sqrt{2\ell+1}}{\sqrt{3}}\left\langle\alpha_3^1, M = \widetilde{m}_{12};\ \widetilde{m}_1'\widetilde{m}_2'\middle|\widehat{O}_2\middle|\alpha_1^1, M = C+F;\ m_1'm_2'\right\rangle \\
&+\frac{\sqrt{2\ell+3}}{\sqrt{3}}\left\langle\alpha_4^1, M = \widetilde{m}_{12};\ \widetilde{m}_1'\widetilde{m}_2'\middle|\widehat{O}_2\middle|\alpha_1^1, M = C+F;\ m_1'm_2'\right\rangle\Bigg]\Bigg]\Bigg\}.
\end{aligned}
\tag{6.3}
$$

6.1. *Matrix elements of* $\widehat{Q}_v^{\mathrm{AL}}$ *and* $\widehat{Q}_{v,\mathrm{IJK}}^{\mathrm{RS}}$

In order to calculate the matrix elements of $\widehat{O}_1, \widehat{O}_2$ we have to calculate the matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$ and $\widehat{Q}_{v,IJK}^{\mathrm{RS}}$ as an intermediate step. Thus, we will discuss this calculation first before we talk about the four different cases separately.

6.1.1. Matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$*.* First, we have to apply the map W in equation (5.4) to $\widehat{Q}_v^{Y,\mathrm{AL}}$ since we need the corresponding operator in the abstract angular momentum system Hilbert space depending on J,

$$\widehat{Q}_v^{J,\mathrm{AL}} := \ell_p^6 \frac{3!\mathrm{i}}{4} C_{\mathrm{reg}} \sum_{I<J<K} \epsilon(e_I, e_J, e_K) \widehat{q}_{IJK}^J, \tag{6.4}$$

whereby $\widehat{q}_{IJK}^J$ results from $\widehat{q}_{IJK}^Y$ upon replacing Y_e^k everywhere by J_e^k. From now on, we will neglect the explicit label J for $\widehat{Q}_v^{J,\mathrm{AL}}$ and keep in mind that we are working in the abstract angular momentum system Hilbert space. Our SNF under consideration $|\alpha_i^J, M\rangle$ contains two linearly independent triples constructed from the edges $\{e_1, e_3, e_4\}$ and $\{e_2, e_3, e_4\}$ for which the sign factor is non-vanishing. Here we split an edge of type up or down as $e = e_2^{-1} \circ e_1$ and then $\epsilon(e_1, e_3, e_4) = -\epsilon(e_2, e_3, e_4) = \pm 1$ for edges of type up and down respectively. Hence, we have

$$\widehat{Q}_v^{J,\mathrm{AL}} := \ell_p^6 \frac{3!\mathrm{i}}{4} C_{\mathrm{reg}} \left(\widehat{q}_{134}^J - \widehat{q}_{234}^J\right). \tag{6.5}$$

We note that the matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$ apart from the constant pre-factor $\ell_p^6 \frac{3!\mathrm{i}}{4} C_{\mathrm{reg}}$ are basically equal to the matrix elements of $\widehat{q}_{IJK}^J$. In [16] a general formula for the matrix element $\langle \alpha_i^J, M | \widehat{q}_{IJK} | \alpha_{\tilde{i}}^J, M \rangle$ for an arbitrary n-valent vertex was derived (equation (47) in [16]). We can use this result in order to get the desired matrix element of $\widehat{q}_{134}$ and $\widehat{q}_{234}$. We obtain

$$\begin{aligned} \langle \alpha_i^J, M | \widehat{q}_{134} | \alpha_{\tilde{i}}^J, M \rangle &= \frac{1}{4}(-1)^{+2j+\ell+J} \sqrt{2j(2j+1)(2j+2)} [2\ell(2\ell+1)(2\ell+2)]^{\frac{3}{2}} \\ &\quad \times \sqrt{(2a_2+1)(2\tilde{a}_2+1)} \sqrt{(2a_3+1)(2\tilde{a}_3+1)} \begin{Bmatrix} j & j & a_2 \\ 1 & \tilde{a}_2 & j \end{Bmatrix} \begin{Bmatrix} J & \ell & a_3 \\ 1 & \tilde{a}_3 & \ell \end{Bmatrix} \\ &\quad \times \left[(-1)^{\tilde{a}_3+\tilde{a}_2} \begin{Bmatrix} \tilde{a}_2 & \tilde{a}_3 & \ell \\ 1 & \ell & a_3 \end{Bmatrix} \begin{Bmatrix} a_3 & \ell & \tilde{a}_2 \\ 1 & a_2 & \ell \end{Bmatrix} \right. \\ &\quad \left. - (-1)^{a_3+a_2} \begin{Bmatrix} a_2 & a_3 & \ell \\ 1 & \ell & \tilde{a}_3 \end{Bmatrix} \begin{Bmatrix} \tilde{a}_3 & \ell & a_2 \\ 1 & \tilde{a}_2 & \ell \end{Bmatrix} \right] \end{aligned} \tag{6.6}$$

$$\begin{aligned} \langle \alpha_i^J, M | q_{234} | \alpha_{\tilde{i}}^J, M \rangle &= +\frac{1}{4}(-1)^{+2j+\ell+J} \sqrt{2j(2j+1)(2j+2)} [2\ell(2\ell+1)(2\ell+2)]^{\frac{3}{2}} \\ &\quad \times \sqrt{(2\tilde{a}_2+1)(2a_2+1)} \sqrt{2(a_3+1)(2\tilde{a}_3+1)} \\ &\quad \times \begin{Bmatrix} j & j & a_2 \\ 1 & \tilde{a}_2 & j \end{Bmatrix} \begin{Bmatrix} J & \ell & a_3 \\ 1 & \tilde{a}_3 & \ell \end{Bmatrix} \left[(-1)^{a_2+\tilde{a}_3} \begin{Bmatrix} a_3 & \ell & a_2 \\ 1 & \tilde{a}_2 & \ell \end{Bmatrix} \begin{Bmatrix} \tilde{a}_2 & \ell & a_3 \\ 1 & \tilde{a}_3 & \ell \end{Bmatrix} \right. \\ &\quad \left. - (-1)^{\tilde{a}_2+a_3} \begin{Bmatrix} \tilde{a}_3 & \ell & a_2 \\ 1 & \tilde{a}_2 & \ell \end{Bmatrix} \begin{Bmatrix} a_2 & \ell & a_3 \\ 1 & \tilde{a}_3 & \ell \end{Bmatrix} \right] \end{aligned} \tag{6.7}$$

The explicit derivation can be found in appendix C. Here we already used that $j_1 = j_2 = j$, $j_3 = j_4 = \ell$ and $a_4 = J$ and $\left\{\begin{smallmatrix} a & c & e \\ b & d & f \end{smallmatrix}\right\}$ are the $6j$-symbols defined in equation (120) in [16].

6.1.2. Matrix elements of $\widehat{Q}_{v,IJK}^{\mathrm{RS}}$*.* If we consider the operator $\widehat{Q}_{v,IJK}^{\mathrm{RS}}$ we also have to consider linearly dependent triples. Therefore also the triples $\{e_1, e_2, e_3\}$ and $\{e_1, e_2, e_4\}$ will contribute. Since the sum over the triples is positioned outside the square root and the absolute value in the case of RS (see equation (4.7) for details), we moreover have to deal with four separated operators, namely $\widehat{Q}_{v,134}^{\mathrm{RS}}$, $\widehat{Q}_{v,234}^{\mathrm{RS}}$, $\widehat{Q}_{v,123}^{\mathrm{RS}}$, $\widehat{Q}_{v,124}^{\mathrm{RS}}$. From equation (4.8) we can read off that the matrix element of $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ is derived from the matrix element of $\widehat{q}_{IJK}$ multiplied by the constant $\ell_p^6 \frac{3!\mathrm{i}}{4} C_{\mathrm{reg}}$. Thus, here we also need the matrix elements of $\widehat{q}_{123}$ and $\widehat{q}_{124}$ which are presented below,

$$\langle \alpha_i^J, M | \widehat{q}_{123} | \alpha_{\tilde{i}}^J, M\rangle = +\frac{1}{2}(-1)^{+2j+\ell+1}(-1)^{\widetilde{a}_2 - a_2 + a_3} X(j,\ell)^{\frac{1}{2}} A(a_2, \widetilde{a}_2) \times \begin{Bmatrix} j & j & a_2 \\ 1 & \widetilde{a}_2 & j \end{Bmatrix} \begin{Bmatrix} a_3 & \ell & a_2 \\ 1 & \widetilde{a}_2 & \ell \end{Bmatrix} [a_2(a_2-1) - \widetilde{a}_2(\widetilde{a}_2 - 1)]\delta_{a_3,\widetilde{a}_3} \tag{6.8}$$

and

$$\langle \alpha_i^J, M | \widehat{q}_{124} | \alpha_{\tilde{i}}^J, M\rangle = +\frac{1}{2}(-1)^{+2j+J} X(j,\ell)^{\frac{1}{2}} A(a_2, \widetilde{a}_2) A(a_3, \widetilde{a}_3) \times \begin{Bmatrix} j & j & a_2 \\ 1 & \widetilde{a}_2 & j \end{Bmatrix} \begin{Bmatrix} \ell & a_2 & a_3 \\ 1 & \widetilde{a}_2 & \widetilde{a}_3 \end{Bmatrix} \begin{Bmatrix} a_4 & \ell & a_3 \\ 1 & \widetilde{a}_3 & \ell \end{Bmatrix} [a_2(a_2-1) - \widetilde{a}_2(\widetilde{a}_2 + 1)]. \tag{6.9}$$

6.2. Case $^{(\ell)}\widehat{E}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$*, i.e.* $E_k^{a,\mathrm{I}} = \det(e) e_k^a$ *and* $\widehat{V}_{\mathrm{AL}}$

If we consider the case of $^{(\ell)}\widehat{E}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$, the operators $\widehat{O}_1, \widehat{O}_2$ in equation (6.3) are $\widehat{O}_1 = \widehat{O}_2 = \widehat{V}^2_{\mathrm{AL}}$. Going back to equations (4.7) and (6.4), we see that $\widehat{V}^2_{\mathrm{AL}} = |\widehat{Q}^{\mathrm{AL}}_v|$. Consequently, the task of calculating matrix elements of $\widehat{V}^2_{\mathrm{AL}}$ can be treated in the following way. As the first step we compute the eigenvalues λ_j^Q and eigenstates $\{\vec{e}_j\}$ of $\widehat{Q}^{\mathrm{AL}}_v$. Afterwards, we expand the matrix elements of $\widehat{V}^2_{\mathrm{AL}}$ in terms of the eigenvectors of $\widehat{Q}^{\mathrm{AL}}_v$,

$$\langle \alpha_{\tilde{i}}^{\widetilde{J}}, \widetilde{M} | \widehat{V}^2_{\mathrm{AL}} | \alpha_i^J, M\rangle = \sum_j |\lambda_j^Q| \langle \alpha_{\tilde{i}}^{\widetilde{J}}, \widetilde{M} | \vec{e}_j\rangle\langle \vec{e}_j | \alpha_i^J, M\rangle, \tag{6.10}$$

wherein we took $\widehat{V}^2_{\mathrm{AL}}$ and $\widehat{Q}^{\mathrm{AL}}_v$ to have the same eigenvectors, and if λ_j^Q is an eigenvalue of $\widehat{Q}^{\mathrm{AL}}_v$, so is $|\lambda_j^Q|$ an eigenvalue of $\widehat{V}^2_{\mathrm{AL}}$.

The four matrix elements of $\widehat{V}^2_{\mathrm{AL}}$ that occur in equation (6.3) are $\langle \alpha_2^0, 0 | \widehat{V}^2_{\mathrm{AL}} | \alpha_1^0, 0\rangle$ and $\langle \alpha_i^1, M | \widehat{V}^2_{\mathrm{AL}} | \alpha_1^1, M\rangle$ where $i = 2, 3, 4$. As the operator $\widehat{V}^2_{\mathrm{AL}}$ does not change the total angular momentum J and magnetic quantum number M of the states $|\alpha_i^J, M\rangle$ and, moreover, the Hilbert spaces belonging to different J are orthogonal to each other, we can calculate the cases of $J = 0$ and $J = 1$ separately. Since these Hilbert spaces for arbitrary spin ℓ of the edges e_3, e_4 in general are $(2j+1)$ and $(6j+1) \times 3$ dimensional for $J = 0$ and $J = 1$, respectively (see also equation (6.1) and (6.2) for this) there is much work to do. The diagonalization of $\widehat{Q}^{\mathrm{AL}}_v$ for the two easiest cases $\ell = 0.5, 1$, where the dimension of the Hilbert spaces in these cases is so small that we were still able to calculate the eigensystems of $\widehat{Q}^{\mathrm{AL}}_v$ analytically, can be found in appendix D. Applying the eigenvector expansion, we obtain the following matrix elements[6] for $\widehat{V}^2_{\mathrm{AL}}$:

Surprisingly, all matrix elements turned out to be identical to zero. Therefore the operator $^{(\ell)}\widehat{E}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$, at least for the spin labels $\ell = 0.5, 1$, becomes the zero operator! Consequently, it is not consistent with the usual flux operator $\widehat{E}_k(S_t)$, which is definitely not the zero operator.

6.3. Case $^{(\ell)}\widehat{E}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$*, i.e.* $E_k^{a,\mathrm{I}} = \det(e) e_k^a$ *and* $\widehat{V}_{\mathrm{RS}}$

As pointed out before, we have to take into account the linearly dependent triples. The total $\widehat{V}_{\mathrm{RS}}$ is then given by

$$\widehat{V}_{\mathrm{RS}} = \widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}, \tag{6.11}$$

[6] Note that in the case $\ell = 1/2$ the state $|\alpha_1^1 M\rangle$ does not exist (see equation (6.2) for the definition of $|\alpha_1^1 M\rangle$). That is the reason why we do not have to consider this particular matrix element.

$\ell = 0.5$	$\ell = 1$
$\langle\alpha_2^0, 0\| \widehat{V}_{\rm AL}^2 \|\alpha_1^0, 0\rangle = 0$	$\langle\alpha_2^0, 0\| \widehat{V}_{\rm AL}^2 \|\alpha_1^0, 0\rangle = 0$
	$\langle\alpha_2^1, M\| \widehat{V}_{\rm AL}^2 \|\alpha_1^1, M\rangle = 0$
$\langle\alpha_3^1, M\| \widehat{V}_{\rm AL}^2 \|\alpha_1^1, M\rangle = 0$	$\langle\alpha_3^1, M\| \widehat{V}_{\rm AL}^2 \|\alpha_1^1, M\rangle = 0$
$\langle\alpha_4^1, M\| \widehat{V}_{\rm AL}^2 \|\alpha_1^1, M\rangle = 0$	$\langle\alpha_4^1, M\| \widehat{V}_{\rm AL}^2 \|\alpha_1^1, M\rangle = 0$

whereby for each $\widehat{V}_{q_{IJK}}$ the operator identity $\widehat{V}_{q_{IJK}} = \sqrt{\left|Q^{\rm RS}_{v,IJK}\right|}$ holds. If we consider the expression of $\widehat{V}_{\rm RS}$ in equation (6.11) together with the definition of the operators $\widehat{O}_1, \widehat{O}_2$ in equation (5.9), we can rewrite the operators $\widehat{O}_1, \widehat{O}_2$ in the following way:

$$\begin{aligned}\widehat{O}_1^{\rm I,RS} &= \widehat{V}_{q_{134}}^2 + \widehat{V}_{q_{234}}^2 + \widehat{V}_{q_{134}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{134}} \\ &\quad + \widehat{V}_{q_{234}}\widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{124}}\widehat{V}_{q_{123}} \\ \widehat{O}_2^{\rm I,RS} &= \widehat{V}_{q_{134}}^2 + \widehat{V}_{q_{234}}^2 + \widehat{V}_{q_{234}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{123}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{V}_{q_{124}} \\ &\quad + \widehat{V}_{q_{123}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{V}_{q_{124}}.\end{aligned} \tag{6.12}$$

Similar to $\widehat{V}_{\rm AL}$ we are restricted to the spin labels $\ell = 0.5, 1$ of the additional edges e_3, e_4, because for higher spin labels the matrices of $Q^{\rm RS}_{v,IJK}$ cannot be diagonalized analytically anymore. Using the operator identity $\widehat{V}_{q_{IJK}} = \sqrt{|Q^{\rm RS}_{v,IJK}|}$, we can, as before, expand each $\widehat{V}_{q_{IJK}}$ in terms of the eigenvectors of $Q^{\rm RS}_{v,IJK}$ and use that if λ_j^Q is an eigenvalue of $Q^{\rm RS}_{v,IJK}$, then $\sqrt{\left|\lambda_j^Q\right|}$ is also an eigenvalue of $\widehat{V}_{q_{IJK}}$,

$$\left\langle\alpha_i^J, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_i^J, M\right\rangle = \sum_j \sqrt{\left|\lambda_j^Q\right|}\left\langle\alpha_i^J, M\middle|\vec{e}_j\right\rangle\left\langle\vec{e}_j\middle|\alpha_i^J, M\right\rangle. \tag{6.13}$$

The detailed calculations of the matrix elements of $\widehat{O}_1, \widehat{O}_2$ can be found in appendix E. Here we will list only the final results. The matrix elements that are included in ${}^{(\ell)}\widehat{\widetilde{E}}^{\rm I,RS}_{k,\rm tot}(S_t)$ are precisely $\left\langle\alpha_2^0, 0\middle|\widehat{O}_1^{\rm I,RS}\middle|\alpha_1^0, 0\right\rangle$ and $\left\langle\alpha_i^1, M\middle|\widehat{O}_2^{\rm I,RS}\middle|\alpha_1^1, M\right\rangle$ where $i = 3, 4$ for $\ell = 0.5$ and $i = 2, 3, 4$ if $\ell = 1$, respectively. We get

$\ell = 0.5$	$\ell = 1$
$\langle\alpha_2^0, 0\| \widehat{O}_1^{\rm I,RS} \|\alpha_1^0, 0\rangle = 0$	$\langle\alpha_2^0, 0\| \widehat{O}_1^{\rm I,RS} \|\alpha_1^0, 0\rangle = 0$
	$\langle\alpha_2^1, M\| \widehat{O}_2^{\rm I,RS} \|\alpha_1^1, M\rangle = 0$
$\langle\alpha_3^1, M\| \widehat{O}_2^{\rm I,RS} \|\alpha_1^1, M\rangle = 0$	$\langle\alpha_3^1, M\| \widehat{O}_2^{\rm I,RS} \|\alpha_1^1, M\rangle = 0$
$\langle\alpha_4^1, M\| \widehat{O}_2^{\rm I,RS} \|\alpha_1^1, M\rangle = 0$	$\langle\alpha_4^1, M\| \widehat{O}_2^{\rm I,RS} \|\alpha_1^1, M\rangle = 0$

Consequently, similar to our previous calculations with $V_{\rm AL}^2$, we obtain only vanishing matrix elements of $\widehat{O}_1^{\rm I,RS}, \widehat{O}_2^{\rm I,RS}$. Thus the matrix element $\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}^{\rm I,RS}_{k,\rm tot}(S_t)|\beta^0, 0\rangle$ is zero as well. Consequently, analogous to the case of $\widehat{V}_{\rm AL}$, ${}^{(\ell)}\widehat{\widetilde{E}}^{\rm I,RS}_{k,\rm tot}(S_t)$ becomes the zero operator.

It is true that due to the absence of the factor $\epsilon(e_{\rm I}, e_J, e_K)$, other orderings for the RS volume operator are available in which not $V_{\rm RS}^2$ but rather two factors of $V_{\rm RS}$ sandwiched between holonomies appear and such orderings could potentially lead to non-vanishing matrix elements. Unfortunately, all these orderings also lead to identically vanishing matrix elements as we prove explicitly in appendix E.

6.4. Summarizing the results of case I

The analysis of the last two sections showed that either the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ or the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ are consistent with the usual flux operator, because both of them are the zero operators. This is due to the fact that all matrix elements of the operators $\widehat{O}_1, \widehat{O}_2$ that occur in equation (6.3) vanish. Since the action on an arbitrary SNF can be determined from the matrix element $\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL/RS}}_{k,\mathrm{tot}}(S_t)|\beta^0, 0\rangle$, we know that the vanishing of this matrix element is equivalent to the fact that ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,AL/RS}}_{k,\mathrm{tot}}(S_t)$ becomes the zero operator. For this reason we can conclude, at least in the cases where we choose $\ell = 0.5, 1$, that the choice of $E^{a,\mathrm{I}}_k(S_t) = \det(e)e^a_k$ does not lead to an alternative flux operator that is consistent with the usual one. To rule out the choice $E^a_k(S_t) = \det(e)e^a_k$ completely, we need to investigate the matrix element for arbitrary representation weights ℓ. For higher values of ℓ the calculation cannot be done analytically anymore simply due to the fact that the roots of the characteristic polynomial of Hermitian matrices of the form $Q = \mathrm{i}A$, $A^T = -A$ can be found by quadratures in general only up to rank 9. However, the results for $\ell = 0.5, 1$ indicate that there is an abstract reason which leads to the vanishing of the matrix elements for *any* ℓ. We were not able to find such an abstract argument yet. However, even if that was not the case and there was a range of values for ℓ for which not all of the matrix elements would vanish, it would be awkward that the classical theory is independent of ℓ while the quantum theory strongly depends on ℓ even in the correspondence limit of large j.

6.5. Case ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, *i.e.* $E^{a,\mathrm{II}}_k = \mathcal{S}\det(e)e^a_k$ *and* $\widehat{V}_{\mathrm{AL}}$

Considering the case of the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, we can read off from equation (5.9) the expressions $\widehat{O}_1 = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}} = \widehat{O}_2$. Hence, again we have to compute special matrix elements of the operators $\widehat{O}_1, \widehat{O}_2$. Since the sign operator $\widehat{\mathcal{S}}$ that corresponds to the classical expression $\mathcal{S} := \mathrm{sgn}(\det(e))$ does not exist in the literature so far, we will explain in detail how the operator $\widehat{\mathcal{S}}$ has to be understood.

6.5.1. The sign operator $\widehat{\mathcal{S}}$. We are dealing now with case II meaning that the densitized triad is given by $E^{a,\mathrm{II}}_k = \mathcal{S}\det(e)e^a_k$, where $\mathcal{S} := \det(e)$. Applying the determinant onto $E^{a,\mathrm{II}}_k$, we get

$$\det(E) = \mathrm{sgn}(\det(e))\det(q) \qquad \text{with} \qquad \det(q) = [\det(e)]^2 \geqslant 0. \tag{6.14}$$

Therefore, we obtain

$$\mathrm{sgn}(\det(E)) = \mathrm{sgn}(\det(e)) = \mathcal{S}. \tag{6.15}$$

In the following, we show that $\mathcal{S} = \mathrm{sgn}(\det(E))$ can be identified with the sign of the expression inside the absolute value under the square roots in the definition of the volume. For this purpose, let us first discuss this issue on the classical level and afterwards go back into the quantum theory and see how the corresponding operator $\widehat{\mathcal{S}}$ is connected with the operator $\widehat{Q}^{\mathrm{AL}}_v$ in equation (6.4).

In order to do this let us consider equation (3.19). This equation contains the classical volume $V(R_{v(\Box)})$ where $R_{v(\Box)}$ denotes a region centred around the vertex $v(\Box)$.

The volume of such a cube is given by

$$V(R_{v(\Box)}) = \int_{R_{v(\Box)}} \sqrt{\det(q)}\,\mathrm{d}^3x = \int_{R_{v(\Box)}} \sqrt{|\det(E)|}\,\mathrm{d}^3x, \tag{6.16}$$

where we used $\det(q) = |\det(E)|$ from equation (6.14). Introducing a parametrization of the cube now, we end up with

$$V(R_{v(\square)}) = \int_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} \left|\frac{\partial X^I(u)}{\partial u_J}\right| \sqrt{|\det(E)(u)|}\mathrm{d}^3u = \int_{[-\frac{\epsilon'}{2},+\frac{\epsilon'}{2}]^3} |\det(X)|\sqrt{|\det(E)(u)|}\,\mathrm{d}^3u. \tag{6.17}$$

In order to be able to carry out the integral, we choose the cube $R_{v(\square)}$ small enough and thus the volume can be approximated by

$$V(R_{v(\square)}) \approx \epsilon'^3 \left|\det\left(\frac{\partial X}{\partial u}\right)(v)\right| \sqrt{|\det(E)(v)|}. \tag{6.18}$$

Using the definition of $\det(E) = \frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l$, we can rewrite equation (6.16) as

$$V(R_{v(\square)}) = \int_\square \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{jkl}E^a_jE^b_kE^c_l\right|}\,\mathrm{d}^3x. \tag{6.19}$$

If we again choose $R_{v(\square)}$ small enough and define the square surfaces of the cube as S^{I}, we can re-express the volume integral over the densitized triads in terms of their corresponding electric fluxes through the surfaces S^{I},

$$V(R_{v(\square)}) \approx \sqrt{\left|\frac{1}{3!}\epsilon_{IJK}\epsilon^{jkl}E_j(S^{\mathrm{I}})E_k(S^J)E_l(S^K)\right|}. \tag{6.20}$$

The flux through a particular surfaces S^{I} is defined as

$$E_j(S^{\mathrm{I}}) = \int_{S^{\mathrm{I}}} E^a_j n^{S^{\mathrm{I}}}_a \qquad n^{S^{\mathrm{I}}}_a = \frac{1}{2}\epsilon^{IJK}\epsilon_{abc}X^b_{,u_J}X^c_{,u_K}\big|_{n^{\mathrm{I}}=0}. \tag{6.21}$$

Here $n^{S^{\mathrm{I}}}_a$ denotes the conormal vector associated with the surface S^{I}. Regarding equation (6.20) we realize that inside the absolute value in equation (6.20) appears exactly the definition of $\det(E_j(S^{\mathrm{I}}))$. Therefore we get

$$V(R_{v(\square)}) \approx \sqrt{|\det(E_j(S^{\mathrm{I}}))|}. \tag{6.22}$$

On the other hand, by taking advantage of the fact that the surfaces S^{I} are small enough so that the integral can be approximated by the value at the vertex times the size of the surface itself, we obtain for $\det(E_j(S^{\mathrm{I}}))$

$$\begin{aligned}\det(E_j(S^{\mathrm{I}})) &\approx \det\left(E^a_j(v)n^{S^{\mathrm{I}}}_a(v)\epsilon'^2\right)\\ &= \det\left(E^a_j(v)\right)\det\left(n^{S^{\mathrm{I}}}_a(v)\right)\epsilon'^6\\ &= \det\left(E(v)\right)\det\left(n^{S^{\mathrm{I}}}_a(v)\right)\epsilon'^6.\end{aligned} \tag{6.23}$$

If we consider the definition of the normal vector in equation (6.21), we can show the following identity:

$$\begin{aligned}n^{S^{\mathrm{I}}}_a &= \det(X)X^{S^{\mathrm{I}}}_a\\ \det\left(n^{S^{\mathrm{I}}}_a\right) &= \det(X)^3\det(X^{-1}) = \frac{\det(X)^3}{\det(X)} = \det(X)^2.\end{aligned} \tag{6.24}$$

Inserting equation (6.24) back into equation (6.23) we have

$$\det(E_j(S^{\mathrm{I}})) \approx \det(E(v))[\det(X(v))]^2\epsilon'^6 \tag{6.25}$$

and can conclude that equation (6.22) is consistent with the usual definition of the volume in equation (6.18).

Since we want to identify $\mathcal{S} := \mathrm{sgn}(\det(E))$ with the sign that appears inside the absolute value under the square root in the definition of the volume, we can read off from equation (6.22) that we still have to show $\mathrm{sgn}(\det(E)) = \mathrm{sgn}(\det(E_j(S^{\mathrm{I}})))$. However, this can be done by means of equation (6.25),

$$\begin{aligned}\operatorname{sgn}(\det(E_j(S^{\mathrm{I}}))) &\approx \operatorname{sgn}(\det(E(v))[\det(X(v))]^2\epsilon'^6)\\ &= \operatorname{sgn}(\det(E(v)))\operatorname{sgn}([\det(X(v))]^2)\operatorname{sgn}(\epsilon'^6)\\ &= \operatorname{sgn}(\det(E(v))).\end{aligned} \tag{6.26}$$

Consequently, we can identify $\mathcal{S}$ with the sign that appears inside the absolute value under the square root in the definition of the volume V in the classical theory, because it was precisely the expression $\det(E_j(S_I))$ that was used in the construction of the volume operator, defined as the square root of absolute value of $\det(E)$. In the quantum theory, we introduced the operator $\widehat{Q}$ in equation (6.4), which is basically the expression inside the absolute value in the definition of the volume operator. Hence, it can be seen as the squared version of the volume operator that additionally contains information about the sign of the expression inside the absolute values. Consequently, we can identify the operator $\widehat{Q}_v^{\mathrm{AL}}$ with $\widehat{Q}_v^{\mathrm{AL}} = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. Now we will be left with the task to calculate particular matrix elements for $\widehat{Q}_v^{\mathrm{AL}}$ which can be done by means of the formula derived in [16].

In order to apply operator $\widehat{\mathcal{S}}$ onto states expressed in terms of abstract angular momentum states, we have to use the W map defined in equation (5.4). Classically, it is the sign of $\det(E)$ which is quantized by smearing the E_j^a with surfaces upon which we obtain fluxes. Using that $\det((E_j(S^I))) \approx [\det((\partial X^a/\partial u^I))]^2 \det\left(E_j^a\right)$ as the surfaces shrink to a point v as we saw above, the sign of $\det(E)$ is the sign of the determinant of the fluxes which in turn gives the operator $\hat{Q}_v$ which is related to $\hat{V}_v$ by $\hat{V}_v = \sqrt{|\hat{Q}_v|}$. Now $\widehat{Q}_v^{\mathrm{AL}}$ is given by

$$\begin{aligned}\widehat{Q}_v^{Y,\mathrm{AL}} &= C_{\mathrm{reg}}\sum_{I,J,K}\epsilon(e_I,e_J,e_K)\epsilon_{ijk}\left(\mathrm{i}\ell_p^2X_{e_I}^i\right)\left(\mathrm{i}\ell_p^2X_{e_J}^j\right)\left(\mathrm{i}\ell_p^2X_{e_K}^k\right)\\ &= -8C_{\mathrm{reg}}\ell_p^6\sum_{I,J,K}\epsilon(e_I,e_J,e_K)\epsilon_{ijk}Y_{e_I}^iY_{e_J}^jY_{e_K}^k,\end{aligned} \tag{6.27}$$

because $\widehat{E}_j(S) = \mathrm{i}\ell_p^2\sum_e\sigma(e,S)X_e^j$. Applying the map W then simply transforms Y into J. Due to the global minus sign in the above equation, we will obtain a global minus sign in front of the whole operator ${}^{(\ell)}\widehat{\tilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$. Thus, the minus sign in equation (6.3) gets cancelled.

6.5.2. Matrix elements of $\widehat{O}_1$, $\widehat{O}_2$ in the case of ${}^{(\ell)}\widehat{\tilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$. For ${}^{(\ell)}\widehat{\tilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ the operators $\widehat{O}_1, \widehat{O}_2 = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. We showed in the last section, where $\widehat{\mathcal{S}}$ was introduced, the following operator identity: $\widehat{Q}_v^{\mathrm{AL}} = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. Therefore calculating matrix elements of $\widehat{O}_{1/2}$ is equivalent to calculating matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$. Hence, in order to get the matrix element for ${}^{(\ell)}\widehat{\tilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, we need to compute the matrix elements $\langle\alpha_2^0,0|\widehat{Q}_v^{\mathrm{AL}}|\alpha_1^0,0\rangle$ and $\langle\alpha_i^1,M|\widehat{Q}_v^{\mathrm{AL}}|\alpha_1^1,M\rangle$ with $i = 2,3,4$. And now one big advantage of the occurrence of the sign operator $\widehat{\mathcal{S}}$ can be observed. In case I, when we were forced to compute particular matrix elements of $\widehat{V}_{\mathrm{AL}}^2$, we had to calculate the whole eigensystem of $\widehat{Q}_v^{\mathrm{AL}}$ as the first step in order to use an eigenstate expansion for the matrix elements of $\widehat{V}_{\mathrm{AL}}^2$. Here, since ${}^{(\ell)}\widehat{\tilde{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ includes matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$, we can use the formula derived in [16] to get $\langle\alpha_2^0,0|\widehat{Q}_v^{\mathrm{AL}}|\alpha_1^0,0\rangle$ and $\langle\alpha_i^1,M|\widehat{Q}_v^{\mathrm{AL}}|\alpha_1^1,M\rangle$ and no involved diagonalization of $\widehat{Q}_v^{\mathrm{AL}}$ is needed anymore.

Moreover, we are only considering matrix elements with $|\alpha_1^1,M\rangle$ as an incoming state $|\alpha_1^1,M\rangle$. This state has the property that the intermediate coupling a_2 of the edges e_1, e_2 is zero. Thus, we have $J_{e_1} = -J_{e_2}$ and therefore obtain in these cases $\widehat{q}_{134} = -\widehat{q}_{234}$. Hence we only have to deal with one of the triples. So, in our special case we get

$$\widehat{Q}_v^{\mathrm{AL}} = \ell_p^6\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}(\epsilon(e_1,e_3,e_4)\widehat{q}_{134}+\epsilon(e_2,e_3,e_4)\widehat{q}_{234}) = \sigma\ell_p^6\frac{3!\mathrm{i}}{2}C_{\mathrm{reg}}\widehat{q}_{134}, \tag{6.28}$$

where we introduced $\sigma = +1$ for edges of type up and $\sigma = -1$ for edges of type down. Moreover, we have chosen to take $\widehat{q}_{134}$ without loss of generality. In the following calculation, we will consider the case of an up edge, so we choose $\sigma = +1$. The whole calculation is analogous for an edge of type down with the only difference that all subsequent formulae have to be multiplied by a factor of -1. Taking the formulae[7] for the matrix elements of $\widehat{q}_{134}, \widehat{q}_{234}$ in equations (6.6), (6.7) we obtain the following result:

$$\begin{aligned}
\langle\alpha_2^0, M|\widehat{q}_{134}|\alpha_1^0, M\rangle &= \frac{4}{\sqrt{3}}\sqrt{j(j+1)}\sqrt{\ell(\ell+1)} \\
\langle\alpha_2^1, M|\widehat{q}_{134}|\alpha_1^1, M\rangle &= \frac{4}{\sqrt{3}}\sqrt{j(j+1)}\frac{\sqrt{(\ell+1)^3(2\ell-1)}}{\sqrt{\ell(2\ell+1)}} \\
\langle\alpha_3^1, M|\widehat{q}_{134}|\alpha_1^1, M\rangle &= \frac{4}{\sqrt{3}}\sqrt{j(j+1)}\frac{(\ell(\ell+1)-1)}{\sqrt{\ell(\ell+1)}} \\
\langle\alpha_4^1, M|\widehat{q}_{134}|\alpha_1^1, M\rangle &= \frac{4}{\sqrt{3}}\sqrt{j(j+1)}\frac{\sqrt{\ell^3(2\ell+3)}}{\sqrt{(\ell+1)(2\ell+1)}}.
\end{aligned} \tag{6.29}$$

The matrix elements do not depend on the magnetic quantum number M and are therefore identical for any chosen value of M. From equation (6.28) we can read off that the matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$ are given by equation (6.29) multiplied by a factor of $\left(\mathrm{i}\ell_p^6\frac{3!\mathrm{i}}{2}C_{\mathrm{reg}}\right)$. Quite promising at this stage is the fact that the j and ℓ dependence of the matrix elements factorizes, because it might be a slight indication that the whole ℓ dependence will cancel exactly in the end. With the result of the matrix elements we can go ahead in computing the matrix element of ${}^{(\ell)}\widehat{\widehat{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ by inserting the matrix elements above into equation (6.3).

6.5.3. Explicit calculation of the matrix elements of ${}^{(\ell)}\widehat{\widehat{E}}{}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$. Multiplying the matrix elements in equation (6.29) by the necessary factor of $\left(\mathrm{i}\ell_p^6\frac{3!\mathrm{i}}{2}C_{\mathrm{reg}}\right)$, inserting them into equation (6.3) and taking into account the global factor of -1 due to the W map of $\widehat{\mathcal{S}}$, we obtain

$$\begin{aligned}
\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^0, 0\rangle &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^2C_{\mathrm{reg}}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{(-1)^{3\ell}3!2\mathrm{i}}{\sqrt{3}}\sqrt{j(j+1)} \\
&\times\left[+\sum_{B=-\ell}^{+\ell}\{\pi_\ell(\tau_k)_{B(\widetilde{m}_{12}+B)}(-1)^{B+\widetilde{m}_{12}}\sqrt{2\ell+1}\sqrt{\ell(\ell+1)}\right. \\
&\times\langle 1\widetilde{m}_{12};\,\ell B\mid\ell\widetilde{m}_{12}{+}B\rangle\langle\ell\widetilde{m}_{12}{+}B;\,\ell{-}(\widetilde{m}_{12}{+}B)|00\rangle\} \\
&-\sum_{C=-\ell}^{+\ell}\left\{\pi_\ell(\tau_k)_{(C-\widetilde{m}_{12})C}(-1)^{C-\widetilde{m}_{12}}\langle 00;\,\ell\widetilde{m}_{12}C\mid\ell\widetilde{m}_{12}C\rangle\langle\ell\widetilde{m}_{12}C;\,\ell\widetilde{m}_{12}{-}C|1\widetilde{m}_{12}\rangle\right. \\
&\times\left(\frac{\sqrt{2\ell-1}}{\sqrt{3}}\frac{\sqrt{(\ell+1)^3(2\ell-1)}}{\sqrt{\ell(2\ell+1)}}-\frac{\sqrt{2\ell+1}}{\sqrt{3}}\frac{(\ell(\ell+1)-1)}{\sqrt{\ell(\ell+1)}}\right. \\
&\left.\left.\left.+\frac{\sqrt{2\ell+3}}{\sqrt{3}}\frac{\sqrt{\ell^3(2\ell+3)}}{\sqrt{(\ell+1)(2\ell+1)}}\right)\right\}\right],
\end{aligned} \tag{6.30}$$

where we put a global factor of $\left(\ell_p^6\frac{3!\mathrm{i}}{2}C_{\mathrm{reg}}\right)\frac{(-1)^{3\ell}4}{\sqrt{3}}\sqrt{j(j+1)}$ in front of the summation. In order to get rid of the δ-functions, we performed the sum over the indices C, F in the first term

[7] This formula was originally derived for gauge invariant SNF only, but can easily be extended to gauge variant states with a total angular momentum different from zero [19].

and the sum over B, F, in the last term. Hence, only one summation is left. Compared to our starting point equation (6.30), equation (5.12) has become effectively simplified. Nevertheless, for carrying out the last sum, we have to insert the explicit expressions for the remaining CGC. They are given by

$$\langle 1\widetilde{m}_{12};\ell B \mid \ell\widetilde{m}_{12}+B\rangle = \frac{1}{\sqrt{\ell(\ell+1)}}\left\{\begin{array}{l} -\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B-1)}\delta_{\widetilde{m}_{12},-1} \\ \quad -B\delta_{\widetilde{m}_{12},0} \\ +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B+1)}\delta_{\widetilde{m}_{12},+1}\end{array}\right\}$$

$$\langle \ell\widetilde{m}_{12}+B;\ell-(\widetilde{m}_{12}+B)|00\rangle = \frac{(-1)^{\ell-B+\widetilde{m}_{12}}}{\sqrt{2\ell+1}}$$

$$\langle 00;\ell C|\ell C\rangle = 1$$

$$\langle \ell C;\ell\widetilde{m}_{12}-C \mid 1\widetilde{m}_{12}\rangle = \frac{(-1)^{\ell-C}\sqrt{3}}{\sqrt{\ell(\ell+1)(2\ell+1)}}\left\{\begin{array}{l} -\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C+1)}\delta_{\widetilde{m}_{12},-1} \\ +C\delta_{\widetilde{m}_{12},0} \\ +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C-1)}\delta_{\widetilde{m}_{12},+1}\end{array}\right\}. \tag{6.31}$$

If we insert these CGC into equation (6.30), we will get an additional factor of $(-1)^{\ell}$ which, combined with the already existing factor of $(-1)^{3\ell}$, leads to a total of $(-1)^{4\ell}=+1$ and can therefore be neglected. Furthermore, the factors $(-1)^B$ and $(-1)^C$ are cancelled by the corresponding inverse factors included in the CGC in equation (6.31). Hence, we get

$$\begin{aligned}\langle\beta^1,\widetilde{m}_{12}|^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^0,0 &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{8\ell_p^2C_{\mathrm{reg}}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{3!2\mathrm{i}}{\sqrt{3}}\sqrt{j(j+1)} \\ &\times\left[+\sum_{B=-\ell}^{+\ell}\left\{\pi_\ell(\tau_k)_{B(\widetilde{m}_{12}+b)}\left\{\begin{array}{l} -\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B-1)}\delta_{\widetilde{m}_{12},-1} \\ -B\delta_{\widetilde{m}_{12},0} \\ +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B+1)}\delta_{\widetilde{m}_{12},+1}\end{array}\right\}\right\}\right. \\ &-\sum_{C=-\ell}^{+\ell}\{\pi_\ell(\tau_k)_{(c-\widetilde{m}_{12})c}\left\{\begin{array}{l} +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C+1)}\delta_{\widetilde{m}_{12},-1} \\ +C\delta_{\widetilde{m}_{12},0} \\ -\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C-1)}\delta_{\widetilde{m}_{12},+1}\end{array}\right\} \\ &\left.\times\left[\frac{(2\ell-1)}{(2\ell+1)}\frac{(\ell+1)}{\ell}-\left(1-\frac{1}{\ell(\ell+1)}\right)+\frac{(2\ell+3)}{(2\ell+1)}\frac{\ell}{(\ell+1)}\right]\Bigg\}\right].\end{aligned} \tag{6.32}$$

Here we have used $(-1)^{2m_{12}}=+1$ in the first term, absorbed the factor of $(-1)^{m_{12}}$ in a change of sign in the CGC for $\widetilde{m}_{12}=\pm 1$ and combined and cancelled square roots where appropriate. Fortunately, the expression in the square brackets in the second sum is identical to one, so equation (6.32) simplifies to

$$\langle\beta^1,\widetilde{m}_{12}|^{(\ell)}\widehat{\widetilde{E}}_{k,\mathrm{tot}}(S_t)|\beta^0,0\rangle = \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{3!16i\ell_p^2C_{\mathrm{reg}}\sqrt{j(j+1)}}{\sqrt{3}\frac{4}{3}\ell(\ell+1)(2\ell+1)}$$

$$\times\left[\sum_{B=-\ell}^{+\ell}\pi_\ell(\tau_k)_{B(\widetilde{m}_{12}+B)}\begin{Bmatrix}-\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B-1)}\delta_{\widetilde{m}_{12},-1}\\ -B\delta_{\widetilde{m}_{12},0}\\ +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-B(B+1)}\delta_{\widetilde{m}_{12},+1}\end{Bmatrix}\right.$$
$$\left.+\sum_{C=-\ell}^{+\ell}\pi_\ell(\tau_k)_{(C-\widetilde{m}_{12})C}\begin{Bmatrix}-\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C+1)}\delta_{\widetilde{m}_{12},-1}\\ -C\delta_{\widetilde{m}_{12},0}\\ +\frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)-C(C-1)}\delta_{\widetilde{m}_{12},+1}\end{Bmatrix}\right], \tag{6.33}$$

where we absorbed the minus sign in front of the second sum into the GCG. The τ-matrices for an arbitrary $SU(2)$ representation with weight ℓ are derived in appendix B

$$\begin{aligned}\pi_\ell(\tau_1)_{mn} &= -\mathrm{i}\sqrt{\ell(\ell+1)-m(m-1)}\delta_{m-n,1}-\mathrm{i}\sqrt{\ell(\ell+1)-m(m+1)}\delta_{m-n,-1}\\ \pi_\ell(\tau_2)_{mn} &= \sqrt{\ell(\ell+1)-m(m+1)}\delta_{m-n,-1}-\sqrt{\ell(\ell+1)-m(m-1)}\delta_{m-n,1}\\ \pi_\ell(\tau_3)_{mn} &= -2\mathrm{i}m\delta_{m-n,0}.\end{aligned} \tag{6.34}$$

Taking a closer look at the structure of these τ-matrices, we realize that a different choice of $\widetilde{m}_{12}$ in equation (6.33) projects onto different τ-matrices; for example, only $\pi_\ell(\tau_3)$ will contribute to the case $\widetilde{m}_{12}=0$, while in the case $\widetilde{m}_{12}=\pm1$ only $\pi_\ell(\tau_1)$ and $\pi_\ell(\tau_2)$ have to be considered. Formulating this fact in terms of δ-functions and using the explicit expressions for the τ-matrices in equation (6.34), we obtain

$$\langle\beta^1,\widetilde{m}_{12}|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S_t)|\beta^0,0\rangle = \lim_{\mathcal{P}_t\to S_t}\sum_{\Box\in\mathcal{P}_t}\frac{3!16\mathrm{i}\ell_p^2C_{\mathrm{reg}}}{\sqrt{3}}\sqrt{j(j+1)}\times\begin{Bmatrix}-\frac{1}{\sqrt{2}}\delta_{\widetilde{m}_{12},-1}\{-\mathrm{i}\delta_{k,1}+\delta_{k,2}\}\\ +\mathrm{i}\delta_{\widetilde{m}_{12},0}\delta_{k,3}\\ +\frac{1}{\sqrt{2}}\delta_{\widetilde{m}_{12},1}\{-\mathrm{i}\delta_{k,1}-\delta_{k,2}\}\end{Bmatrix}, \tag{6.35}$$

whereby we used $\sum_{B=-\ell}^{\ell}B^2=\frac{1}{3}\ell(\ell+1)(2\ell+1)$.

Now we take the limit $\lim_{\epsilon\to0}\lim_{\epsilon'\to0}$. The discussion in section 3.3 showed that taking the $\lim_{\epsilon'\to0}$ (that is equivalent to $\lim_{\mathcal{P}_t\to S_t}$) is trivial and taking the $\lim_{\epsilon\to0}$ leads to an additional overall factor of $1/2$. So, when calculating the action of the alternative flux operator on the state $|\beta^0,0\rangle$, we use the expansion

$${}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle = \sum_{\widetilde{m}_{12}=-1}^{+1}\langle\beta^1,\widetilde{m}_{12}|{}^{(\ell)}\widehat{\widehat{E}}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle|\beta^1,\widetilde{m}_{12}\rangle \tag{6.36}$$

and end up with the final result

$$\begin{aligned}{}^{(\ell)}\widehat{\widehat{E}}{}^{\mathrm{II,AL}}_{1,\mathrm{tot}}(S)|\beta^0,0\rangle &= -\frac{3!8\ell_p^2C_{\mathrm{reg}}}{\sqrt{6}}\sqrt{j(j+1)}\{|\beta^1,-1\rangle-|\beta^1,+1\rangle\}\\ {}^{(\ell)}\widehat{\widehat{E}}{}^{\mathrm{II,AL}}_{2,\mathrm{tot}}(S)|\beta^0,0\rangle &= -\frac{3!8\mathrm{i}\ell_p^2C_{\mathrm{reg}}}{\sqrt{6}}\sqrt{j(j+1)}\{|\beta^1,-1\rangle+|\beta^1,+1\rangle\}\\ {}^{(\ell)}\widehat{\widehat{E}}{}^{\mathrm{II,AL}}_{3,\mathrm{tot}}(S)|\beta^0,0\rangle &= -\frac{3!8\ell_p^2C_{\mathrm{reg}}}{\sqrt{3}}\sqrt{j(j+1)}|\beta^1,0\rangle.\end{aligned} \tag{6.37}$$

Remarkably, in the final result the ℓ dependence drops out completely.

6.6. *Case* ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$, *i.e.* $E^{a,\mathrm{II}}_k = \mathcal{S}\det(e)e^a_k$ *and* $\widehat{V}_{\mathrm{RS}}$

In this case, the operators $\widehat{O}_1$, $\widehat{O}_2$ have the following form:

$$\begin{aligned}\widehat{O}_1 = &+\widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{123}}\\ &+\widehat{V}_{q_{124}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{124}}\widehat{\mathcal{S}}\widehat{V}_{q_{123}}\\ \widehat{O}_2 = &+\widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}} + \widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}}\widehat{V}_{q_{134}}\\ &+\widehat{V}_{q_{134}}\widehat{\mathcal{S}}\widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}}\widehat{V}_{q_{234}} + \widehat{V}_{q_{234}}\widehat{\mathcal{S}}\widehat{V}_{q_{124}} + \widehat{V}_{q_{123}}\widehat{\mathcal{S}}\widehat{V}_{q_{124}}.\end{aligned} \tag{6.38}$$

But, before continuing we want to discuss some difficulties that occur if one uses the volume operator $\widehat{V}_{\mathrm{RS}}$ in this case.

6.6.1. Problems with the sign operator $\widehat{\mathcal{S}}$ in the case of RS. When we introduced the quantization of $\mathcal{S} \to \widehat{\mathcal{S}}$ in section 6.5.1, we realized that $\widehat{\mathcal{S}}$ has a precise relation to the operator $\widehat{Q}^{\mathrm{AL}}_v$, i.e. $\widehat{Q}^{\mathrm{AL}}_v = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. However, this was possible because $\widehat{V}_{\mathrm{AL}}$ sums over the triples inside the absolute value under the square root (see equation (4.9)). In contrast, $\widehat{V}_{\mathrm{RS}}$, defined in equation (4.7) consists of a sum of single square roots. Consequently, we are not able to repeat the calculations done in section 6.5.1 if we choose $\widehat{V}_{\mathrm{RS}}$, because there is no possible origin for a sign. This means that there exists no sign operator $\widehat{\mathcal{S}}$ that is quantized in the same way as $\widehat{V}_{\mathrm{RS}}$ is quantized. Accordingly, in a strict sense the operator ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ does not exist, because $\widehat{\mathcal{S}}$ cannot be implemented in the quantum theory just using the regularization that leads to $\widehat{V}_{\mathrm{RS}}$. The conclusion is that ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ is inconsistent with the usual flux operator. In retrospect there is a simple argument why the only possibility ${}^{(\ell)}\widehat{E}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ (since ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ does not exist) is ruled out $\widehat{V}_{\mathrm{RS}}$ without further calculation: namely, the lack of a factor of orientation in $\widehat{V}_{\mathrm{RS}}$, like $\epsilon(e_{\mathrm{I}}, e_J, e_K)$ in $\widehat{V}_{\mathrm{AL}}$, leads to the following basic disagreement with the usual flux operator. Suppose we had chosen the orientation of the surface S in the opposite way. Then the type of the edge e switches between up and down and similarly for e_1, e_2. Then, the result of the usual flux operator would differ by a minus sign. In the case of $\widehat{V}_{\mathrm{AL}}$ we would get this minus sign as well due to $\epsilon(e_{\mathrm{I}}, e_J, e_K)$, whereas a change of the orientation of e_1, e_2 would not modify the result of the alternative flux operator if we used $\widehat{V}_{\mathrm{RS}}$ instead, because it is not sensitive to the orientation of the edges.

A way out would be to use the somehow 'artificial' construction $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$, where $\widehat{\mathcal{S}}$ denotes the sign operator $\widehat{\mathcal{S}}$ introduced in section 6.5.1. We attached the label AL to it in order to emphasize that its quantization is in agreement with $\widehat{V}_{\mathrm{AL}}$. This is artificial for the following reason. Suppose we have a classical quantity $A := \det(E)$ and two different functions $f_1 := \sqrt{|A|}$ and $f_2 := \mathrm{sgn}(A)$. If we want to quantize the functions f_1 and f_2, we do this with the help of the corresponding operator $\widehat{A}$ and obtain due to the spectral theorem $\widehat{f}_1 = \sqrt{|\widehat{A}|}$ and $\widehat{f}_2 = \mathrm{sgn}(\widehat{A})$. The product of operators $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ rather corresponds to $\widehat{g}_1 = \widehat{A}'$ and $\widehat{g}_2 = \mathrm{sgn}(\widehat{A})$, because $\widehat{V}_{\mathrm{RS}}$ is quantized with a different regularization scheme than $\widehat{\mathcal{S}}$ is. This would only be justified if $\sqrt{|\widehat{A}|}$ and $\widehat{A}'$ would agree semi-classically. However they do not: if we compare the expressions for V_{AL} and V_{RS} then, schematically, they are related in the following way when restricted to a vertex, $\widehat{V}_{v,\mathrm{AL}} = \left|\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}\sum_{I<J<K}\epsilon(e_{\mathrm{I}}, e_J, e_K)\widehat{q}_{IJK}\right|^{1/2}$ while $\widehat{V}_{v,\mathrm{RS}} = \sum_{I<J<K}\left|\frac{3!\mathrm{i}}{4}C_{\mathrm{reg}}\widehat{q}_{IJK}\right|^{1/2}$. It is clear that apart from the sign $\epsilon(e_{\mathrm{I}}, e_J, e_K)$ the two operators can agree at most on states where only one of the $\widehat{q}_{IJK}$ is non-vanishing (3- or 4-valent graphs) simply because $\sqrt{|a+b|} \neq \sqrt{|a|} + \sqrt{|b|}$ for generic real numbers a, b.

6.6.2. Matrix elements of $\widehat{O}_1$, $\widehat{O}_2$ in the case of ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$. Nevertheless, we can analyse whether ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ including $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ is consistent with the usual flux operator $\widehat{E}_k(S)$.

In the case of ${}^{(\ell)}\widehat{E}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, no diagonalization of the $\widehat{Q}^{\mathrm{AL}}_v$ matrices was necessary because of the operator identification $\widehat{Q}^{\mathrm{AL}}_v = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. Since this is not possible for ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ we have to diagonalize the $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ in order to get the eigenvalues and eigenvectors. Then we can compute the matrix elements, for instance, $\langle\alpha^0_2, M|\widehat{O}_1|\alpha^0_1, M\rangle$ by an eigenvector expansion for each operator contained in $\widehat{O}_1$,

$$\begin{aligned}\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}\widehat{\mathcal{S}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha^0_1, M\rangle &= \sum_{|\alpha'\rangle,|\alpha''\rangle}\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\alpha'\rangle\langle\alpha'|\widehat{\mathcal{S}}|\alpha''\rangle\langle\alpha''|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha^0_1, M\rangle\\ &= \sum_{|\alpha'\rangle,|\alpha''\rangle}\sum_{k,k',k''}\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha'\rangle\langle\alpha'|\widehat{\mathcal{S}}|\vec{e}_{k'}\rangle\langle\vec{e}_{k'}|\alpha''\rangle\\ &\quad\times\langle\alpha''|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\vec{e}_{k''}\rangle\langle\vec{e}_{k''}|\alpha^0_1, M\rangle,\end{aligned}\tag{6.39}$$

whereby $|\vec{e}_k\rangle$ are the eigenvectors of the corresponding operators and $|\alpha'\rangle$ are all states belonging to the Hilbert space $\mathcal{H}^J$. We calculated the matrix elements $\langle\alpha^0_2, M|\widehat{O}_1|\alpha^0_1, M\rangle, \langle\alpha^1_3, M|\widehat{O}_2|\alpha^1_1, M\rangle, \langle\alpha^1_4, M|\widehat{O}_2|\alpha^1_1, M\rangle$ that occur in equation (6.3) for $\ell = 0.5$. The details can be found in appendix E. The results are shown below:

$$\begin{aligned}\langle\alpha^0_2, M|\widehat{O}^{\mathrm{RS}}_1|\alpha^0_1, M\rangle &= C_1(\ell)\langle\alpha^0_2, M|\widehat{O}^{\mathrm{AL}}_1|\alpha^0_1, M\rangle\\ \langle\alpha^1_3, M|\widehat{O}^{\mathrm{RS}}_2|\alpha^1_1, M\rangle &= C_3(j,\ell)\langle\alpha^1_3, M|\widehat{O}^{\mathrm{AL}}_2|\alpha^1_1, M\rangle\\ \langle\alpha^1_4, M|\widehat{O}^{\mathrm{RS}}_2|\alpha^1_1, M\rangle &= C_4(j,\ell)\langle\alpha^1_4, M|\widehat{O}^{\mathrm{AL}}_2|\alpha^1_1, M\rangle.\end{aligned}\tag{6.40}$$

Here $C_1(\ell), C_i(j,\ell) \in \mathbb{R}$ and the explicit expression can be found in equations (E.82) and (E.105). Furthermore, we expressed the matrix elements in terms of the associated AL-matrix elements, because the whole calculation has already been done for ${}^{(\ell)}\widehat{E}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ and therefore in this way of writing we can easily note where differences occur. $C_i(j,\ell)$ are real constants whose values depend on the explicit value of the spin labels j and ℓ. In the case of ${}^{(\ell)}\widehat{E}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ we could see that the whole dependence on the spin label ℓ drops out in the final result. Hence, if in the case of ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ we obtain not exactly the same matrix elements as for ${}^{(\ell)}\widehat{E}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, we already know that the ℓ dependence will not be cancelled in the final result here. The j dependence is basically caused by terms proportional to $(\sqrt{j(j+1)+c})(\sqrt{j(j+1)})^{-1}$, whereby $c \in \mathbb{N}$. Thus semi-classically, i.e. in the limit of large j, the numerator and the denominator become equal and accordingly the j dependence vanishes, $C_i(j,\ell) \to C_i(\ell)$. By reinserting the matrix elements from equation (6.40) into equation (6.3) and repeating all the steps of the former ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ calculation for $\ell = 0.5$, we end up with

$$\begin{aligned}{}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{1,\mathrm{tot}}(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{6}}C(j,\ell)C_{\mathrm{reg}}\sqrt{j(j+1)}\{|\beta^1, -1\rangle - |\beta^1, +1\rangle\}\\ {}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{2,\mathrm{tot}}(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{6}}C(j,\ell)C_{\mathrm{reg}}\sqrt{j(j+1)}\{|\beta^1, -1\rangle + |\beta^1, +1\rangle\}\\ {}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{3,\mathrm{tot}}(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{3}}C(j,\ell)C_{\mathrm{reg}}\sqrt{j(j+1)}|\beta^1, 0\rangle.\end{aligned}\tag{6.41}$$

Here, $C(j,\ell) \in \mathbb{R}$ with $C(j,\ell) \to C(\ell)$ semi-classically and

$$C(\ell) = \left[C_1\left(\frac{1}{2}\right) - C_3\left(\frac{1}{2}\right)\left(1 - \frac{1}{\ell(\ell+1)}\right) + C_4\left(\frac{1}{2}\right)\left(\frac{(2\ell+3)}{(2\ell+1)}\frac{\ell}{(\ell+1)}\right)\right]_{\ell=0.5}.\tag{6.42}$$

The functions $C_1(\ell), C_2(\ell), C_3(\ell), C_4(\ell)$ can be computed analytically only for $\ell = 0.5, 1$. Note that $C_2(\ell)$ is zero for $\ell = 1/2$ since the state $\left|\alpha_2^1, M\right\rangle$ does not exist for $\ell = 0.5$ (see equation (D.4)). For this reason it does not occur in equation (6.42). However, if we know these constants the precise ℓ dependence of $C(\ell)$ would be

$$C(\ell) = C_1(\ell) + C_2(\ell)\left(\frac{(2\ell-1)}{(2\ell+1)}\frac{(\ell+1)}{\ell}\right) - C_3(\ell)\left(1 - \frac{1}{\ell(\ell+1)}\right) + C_4(\ell)\left(\frac{(2\ell+3)}{(2\ell+1)}\frac{\ell}{(\ell+1)}\right). \tag{6.43}$$

Since the ℓ dependence of $C_i(\ell)$ should result from the ℓ dependence of $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ which is non-trivial in general, it is very unlikely that the whole ℓ dependence is cancelled for arbitrary ℓ as in the case of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, where $C_i(\ell) = 1$ for $i = 1, 2, 3, 4$.

Thus, we conclude that the volume operator introduced by Rovelli and Smolin is not appropriate to reproduce the result of the usual flux operator $\widehat{E}_k(S_t)$ and therefore cannot be used to construct the alternative flux operator. In other words, the RS operator is inconsistent with the fundamental flux operator on which it is based.

6.7. Summarizing the results of case II

Considering the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, the operators $\widehat{O}_1, \widehat{O}_2$ whose matrix elements are included in equation (6.3) are given by $\widehat{O}_1 = \widehat{O}_2 = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. Thus we have to implement the sign operator $\mathcal{S} = \mathrm{sgn}(\det(e)) \to \widehat{\mathcal{S}}$ on the quantum level. In section 6.5.1 we showed in detail that $\widehat{\mathcal{S}}$ has a well-defined relation with $\widehat{Q}^{\mathrm{AL}}_v$, in particular $\widehat{\mathcal{S}} = \mathrm{sgn}\big(\widehat{Q}^{\mathrm{AL}}_v\big)$. This relation is equivalent to the operator identity $\widehat{Q}^{\mathrm{AL}}_v = \widehat{V}_{\mathrm{AL}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$. Consequently, it remarkably turns out that the operators $\widehat{O}_1, \widehat{O}_2$ are identical to the operator $\widehat{Q}^{\mathrm{AL}}_v$ in the case of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$. Along with this comes the nice side effect that a diagonalization of the operator $\widehat{Q}^{\mathrm{AL}}_v$ is no longer necessary since now the matrix elements of $\widehat{Q}^{\mathrm{AL}}_v$ instead of matrix elements of $\widehat{V}_{\mathrm{AL}}$ contribute to the calculation. Therefore, we can apply the general formula for matrix elements of $\widehat{Q}^{\mathrm{AL}}_v$ derived in [16], even for arbitrary spin labels ℓ, and we are done. The expression for the matrix elements of $\widehat{Q}^{\mathrm{AL}}_v$ is given in equation (6.29). By reinserting these matrix elements into equation (6.3) and following the intermediate steps discussed in section 6.5.3, we end up with the final result in equation (6.37). For ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ the whole dependence on the spin label ℓ that is associated with the two additional edges e_3, e_4 drops out in the final result. Hence, the result is independent of the chosen representation of the holonomies in the alternative flux operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$.

In the case of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ the operators $\widehat{O}_1, \widehat{O}_2$ have quite lengthly expressions that can be found in equation (6.38). $\widehat{O}_1, \widehat{O}_2$ are both given by a sum of operators that have the form $\widehat{V}_{q_{IJK}}\widehat{\mathcal{S}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}$, whereby $\widehat{V}_{q_{IJK}}$ denotes the operator $\widehat{V}_{\mathrm{RS}}$ when only the contribution of the triple $\{e_I, e_J, e_K\}$ is considered. In contrast to $\widehat{Q}^{\mathrm{AL}}_v$, for $\sum_{IJK}\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ no relation with the sign operator $\widehat{\mathcal{S}}$ can be derived. This fact is dealt with in section 6.6.1. Consequently, it is impossible to quantize $\widehat{\mathcal{S}}$ in an analogous way as $\widehat{V}_{\mathrm{RS}}$ is quantized. This is a big difference to $\widehat{V}_{\mathrm{AL}}$ where $\widehat{\mathcal{S}}$ and $\widehat{V}_{\mathrm{AL}}$ could be quantized in the same manner. Therefore, the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ cannot be defined rigorously since $\widehat{\mathcal{S}}$ does not exist for $\widehat{V}_{\mathrm{RS}}$. Thus, the operator ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ is inconsistent with the usual flux operator $\widehat{E}_k(S_t)$.

Nevertheless, we analysed the artificial construction $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}\widehat{V}_{\mathrm{AL}}$ for ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$. It is artificial because the operator $\widehat{V}_{\mathrm{RS}}$ and the operator $\widehat{\mathcal{S}}$ are quantized with respect to different regularization schemes and are not semi-classically consistent with each other. The results are shown in equation (6.41).

In the next section, we will calculate the matrix element of the usual flux operator $\widehat{E}_k(S)$ in order to compare it with the results in equations (6.37) and (6.41) afterwards.

7. Matrix elements of the usual flux operator $\widehat{E}_k(S)$

In this section, we will calculate the action of the usual flux operator on our SNF $|\beta^0, 0\rangle$ that was used through all the calculations of the alternative flux operator before[8]. If we want to use the technical tools of angular momentum recoupling theory (e.g. CGC) we have to apply the W map in equation (5.4) to all states in the SNF Hilbert space in order to justify to work in the angular momentum system Hilbert space. Therefore a matrix element of the usual flux operator is given by

$$
{}_{m_1',m_2'}\langle\beta^{\tilde{j}_{12}}, \widetilde{n}_{12}|\widehat{E}_k^Y(S)|\beta^{j_{12}}, n_{12}\rangle_{m_1',m_2'} = \sum_{m_{12},\widetilde{m}_{12}} \pi_{\tilde{j}_{12}}(\epsilon^{-1})_{\tilde{n}_{12}\widetilde{m}_{12}}\pi_{j_{12}}(\epsilon^{-1})_{n_{12}m_{12}}
\times\langle\beta^{\tilde{j}_{12}}, \widetilde{m}_{12}; m_1'm_2'|\widehat{E}_k^J(S)|\beta^{j_{12}}, m_{12}; m_1'm_2'\rangle.
$$

As has been pointed out before, this mapping is similar for the alternative and the usual flux operator. Therefore, we will only consider the matrix elements of $\widehat{E}_k$ in the abstract angular momentum system Hilbert space here. Since the inverse of $\pi_\ell(\epsilon^{-1})$ exists, a possible difference between the usual and the alternative flux operator can only occur in the matrix element in the abstract angular momentum Hilbert space. Throughout this section, we will neglect the additional indices m_1', m_2' of the states $|\beta^{j_{12}}, m_{12}\rangle$ as we did in the calculation of the alternative flux operator. Working in the abstract angular momentum system Hilbert space now, we can re-express the action of $\widehat{E}_k(S)$ in terms of angular momentum operators the actions of which, on the other hand, are well known for states expressed in the tensor basis. Thus, it is suggestive to transform the recoupling states $|\beta^0, 0\rangle$ back into the tensor basis and apply the operator $\widehat{E}_k(S)$ onto it afterwards. Thereafter, we have to reformulate the result again in terms of the recoupling basis in order to be able to compare this result of the usual flux operator with the calculations of $\widehat{\widehat{E}}_k(S)$ in the last sections.

The state $|\beta^0, 0\rangle$ transforms into the tensor basis according to the following linear combination,

$$
|\beta^0, 0\rangle = \sum_{m=-j}^{+j} \langle jm, j-m|00\rangle |jm; m'_{e_1}\rangle_{e_1} \otimes |jm; m'_{e2}\rangle_{e_2} \tag{7.1}
$$

$$
= \sum_{m=-j}^{+j} \frac{(-1)^{j-m}}{\sqrt{2j+1}} |jm; m'_{e_1}\rangle_{e_1} \otimes |j-m; m'_{e_2}\rangle_{e_2}, \tag{7.2}
$$

where we have used the explicit expression for the CGC $\langle jm, j-m|00\rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}}$. Furthermore, the two edges e_1 and e_2 of our graph γ couple to a resulting angular momentum $j_{12} = 0$. Therefore, we have $\widehat{J}^k_{e_1} = -\widehat{J}^k_{e_2}$. Additionally, the tangent vectors $\dot{e}_1(t)$ and

[8] Note that it so happens that for $\widehat{O} = \widehat{V}\widehat{S}\widehat{V} = \widehat{Q}$ an explicit diagonalization of $\widehat{Q}$ is not necessary so we may refrain from using the recoupling basis and can work directly in the tensor basis. The associated calculations are of a similar length but sidestep the use of CGCs and hence may be used as an independent check of our result. We did this and the result completely agrees with the recoupling basis calculation. However, for $\widehat{O} = \widehat{V}^2 \neq \widehat{Q}$ it is necessary to diagonalize $\widehat{Q}$ and the use of the recoupling basis becomes calculationally mandatory, which is why we have done all calculations in this paper in the recoupling basis.

$\dot{e}_2(t)$ have opposite orientations with respect to the surface S, from which it follows that $\epsilon(e_1, S) = -\epsilon(e_2, S) = \sigma$, where $\sigma = +1$ for edges of type up and $\sigma = -1$ for type down edges. Hence, we obtain

$$\begin{aligned}\widehat{E}_k(S)|\beta^0, 0\rangle &= -\frac{1}{2}\ell_p^2\big[\epsilon(e_1, S)\widehat{J}^k_{e_1} + \epsilon(e_2, S)\widehat{J}^k_{e_2}\big]|\beta^0, 0\rangle \\ &= -\frac{1}{2}\ell_p^2\sum_{m=-j}^{+j}\big[\epsilon(e_1, S)\widehat{J}^k_{e_1} + \epsilon(e_2, S)\widehat{J}^k_{e_2}\big]\frac{(-1)^{j-m}}{\sqrt{2j+1}}|jm; m'_{e_1}\rangle_{e_1}\otimes|j-m; m'_{e_2}\rangle_{e_2} \\ &= -\frac{\ell_p^2}{\sqrt{2j+1}}\sum_{m=-j}^{+j}(-1)^{j-m}\big(\widehat{J}^k_{e_1}|jm; m'_{e_1}\rangle_{e_1}\big)\otimes|j-m; m'_{e_2}\rangle_{e_2}.\end{aligned} \tag{7.3}$$

By applying equation (7.3), we calculate the action of $\widehat{E}_k(S)$ on our SNF for each $k = 1, 2, 3$ separately with the case $k = 1$ being the first one. From elementary quantum mechanics we know that we can introduce ladder angular momentum operators $\widehat{J}^+$ and $\widehat{J}^-$ defined by $\widehat{J}^+ := \widehat{J}^1 + \mathrm{i}\widehat{J}^2$ and $\widehat{J}^- := \widehat{J}^1 - \mathrm{i}\widehat{J}^2$, respectively. Hence, we can express $\widehat{J}^1$ as $\widehat{J}^1 = \frac{1}{2}(\widehat{J}^+ + \widehat{J}^-)$. The action of the ladder operators on a state in the abstract spin system $|jm; m'\rangle$ with spin j and magnetic quantum number m is given by

$$\begin{aligned}\widehat{J}^+|jm; m'\rangle &= \sqrt{j(j+1) - m(m+1)}|jm+1; m'\rangle \\ \widehat{J}^-|jm; m'\rangle &= \sqrt{j(j+1) - m(m-1)}|jm-1; m'\rangle.\end{aligned} \tag{7.4}$$

Therefore, by means of equation (7.3) we obtain for the $k = 1$ component of the flux operator $\widehat{E}_1(S)$ acting on the SNF $|\beta^0, 0\rangle$ the following result:

$$\begin{aligned}\widehat{E}_1(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{2j+1}}\sum_{m=-j}^{+j}(-1)^{j-m}\big(\widehat{J}^1_{e_1}|jm; m'_{e_1}\rangle_{e_1}\big)\otimes|j-m; m'_{e_2}\rangle_{e_2} \\ &= -\frac{\ell_p^2}{2\sqrt{2j+1}}\sum_{m=-j}^{+j}(-1)^{j-m}\big\{+\sqrt{j(j+1) - m(m+1)}|jm+1; m'_{e_1}\rangle_{e_1} \\ &\otimes|j-m; m'_{e_2}\rangle_{e_2} + \sqrt{j(j+1) - m(m-1)}|jm-1; m'_{e_1}\rangle_{e_1}\otimes|j-m; m'_{e_2}\rangle_{e_2}\big\}.\end{aligned} \tag{7.5}$$

We wish to express the final result in terms of recoupling states. Consequently, we have to transform the tensor product $|jm\pm 1; m'_{e_1}\rangle_{e_1}\otimes|j-m; m'_{e_2}\rangle_{e_2}$ back into the recoupling basis,

$$\begin{aligned}|jm+1; m'\rangle_{e_1}\otimes|j-m; -m'\rangle_{e_2} &= -(-1)^{j-m}\sqrt{\frac{3}{2}}\sqrt{\frac{j(j+1) - m(m+1)}{j(j+1)(2j+1)}}|\beta^1, 1\rangle \\ &\quad + \sum_{\widetilde{j}_{12}=2}^{2j}\langle\widetilde{j}_{12}\widetilde{m}_2 = 1|jm+1; j-m\rangle|\beta^{\widetilde{j}_{12}}, 1\rangle \\ |jm-1; m'\rangle_{e_1}\otimes|j-m; -m'\rangle_{e_2} &= (-1)^{j-m}\sqrt{\frac{3}{2}}\sqrt{\frac{j(j+1) - m(m-1)}{j(j+1)(2j+1)}}|\beta^1, -1\rangle \\ &\quad + \sum_{\widetilde{j}_{12}=2}^{2j}\langle\widetilde{j}_{12}\widetilde{m}_2 = 1|jm+1; j-m\rangle|\beta^{\widetilde{j}_{12}}, -1\rangle,\end{aligned} \tag{7.6}$$

where we used the definition $|\beta^{j_{12}}, m_{12}\rangle := |a_1 = ja_2 = j_{12}m_{12}; m'_{e_1}m'_{e_2}\rangle$ as we did during the whole calculation of the new flux operator. We want to expand the action of $\widehat{E}_1(S)$ on $|\beta^0, 0\rangle$ in terms of the states $|\beta^1, m_{12}\rangle$,

$$\widehat{E}_1(S)|\beta^0, 0\rangle = \sum_{\widetilde{m}_{12}=-1}^{+1} \langle\beta^1, \widetilde{m}_{12}|\widehat{E}_1(S)|\beta^0, 0\rangle|\beta^1, \widetilde{m}_{12}\rangle. \tag{7.7}$$

As the next step, we insert equation (7.6) into equation (7.7). As j_{12} denotes the total angular momentum of the state $|\beta^{j_{12}}, m_{12}\rangle$, we know that two states with different values of j_{12} and m_{12} are orthogonal to each other, meaning $\langle\beta^{\widetilde{j}_{12}}, \widetilde{m}_{12}|\beta^{j_{12}}, m_{12}\rangle = \delta_{\widetilde{j}_{12}, j_{12}}\delta_{\widetilde{m}_{12}, m_{12}}$. Taking this into account, we obtain

$$\begin{aligned}\widehat{E}_1(S)|\beta^0, 0\rangle = \frac{-\ell_p^2}{2\sqrt{2j+1}} \sum_{m=-j}^{+j} \Bigg\{ &-(-1)^{2(j-m)}\sqrt{\frac{3}{2}}\sqrt{\frac{(j(j+1)-m(m+1))^2}{j(j+1)(2j+1)}}|\beta^1, 1\rangle \\ &+ (-1)^{2(j-m)}\sqrt{\frac{3}{2}}\sqrt{\frac{(j(j+1)-m(m-1))^2}{j(j+1)(2j+1)}}|\beta^1, -1\rangle \\ = -\frac{\ell_p^2}{\sqrt{6}}\sqrt{j(j+1)}\{|\beta^1, -1\rangle - |\beta^1, 1\rangle\Bigg\}.\end{aligned} \tag{7.8}$$

Here we used $(-1)^{2(j-m)} = +1$, as $(j-m) \in \mathbb{Z}$ and $\sum_{m=-j}^{j} m^2 = (1/3)j(j+1)(2j+1)$.

Analogous to $\widehat{J}^1$, we can formulate $\widehat{J}^2$ in terms of ladder operators $\widehat{J}^2 = \frac{1}{2\mathrm{i}}(\widehat{J}^+ - J^-)$. Hence, the action of the $k = 2$ component of the flux operator $\widehat{E}_2(S)$ acting on $|\beta^0, 0\rangle$ is given by

$$\begin{aligned}\widehat{E}_2(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m} \left(\widehat{J}^2_{e_1}|jm; m'_1\rangle_{e_1}\right) \otimes |j-m; m'_2\rangle_{e_2} \\ &= -\frac{\ell_p^2}{2\mathrm{i}\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m}\{+\sqrt{j(j+1)-m(m+1)}|jm+1; m'_1\rangle_{e_1} \\ &\quad \otimes |j-m; m'_2\rangle_{e_2} - \sqrt{j(j+1)-m(m-1)}|jm-1; m'_1\rangle_{e_1} \otimes |j-m; m'_2\rangle_{e_2}\}.\end{aligned} \tag{7.9}$$

Again, we want to transform the tensor product $|jm \pm 1; m'_1\rangle_{e_1} \otimes |j-m; m'_2\rangle_{e_2}$ into the recoupling basis by means of the necessary CGC that can be found in equation (7.6). Inserting equation (7.6) into the equation above and taking advantage of the orthogonality relation concerning different $m's$ and $j'_{12}s$, we get

$$\begin{aligned}\widehat{E}_2(S)|\beta^0, 0\rangle = -\frac{\ell_p^2}{2\mathrm{i}\sqrt{2j+1}} \sum_{m=-j}^{+j} \Bigg\{ &-(-1)^{2(j-m)}\sqrt{\frac{3}{2}}\sqrt{\frac{(j(j+1)-m(m+1))^2}{j(j+1)(2j+1)}}|\beta^1, 1\rangle \\ &- (-1)^{2(j-m)}\sqrt{\frac{3}{2}}\sqrt{\frac{(j(j+1)-m(m-1))^2}{j(j+1)(2j+1)}}|\beta^1, -1\rangle\Bigg\} \\ = -\frac{\mathrm{i}\ell_p^2}{\sqrt{6}}&\sqrt{j(j+1)}\{|\beta^1, -1\rangle + |\beta^1, 1\rangle\},\end{aligned} \tag{7.10}$$

where we again used $(-1)^{2(j-m)} = +1$, as $(j-m) \in \mathbb{Z}$ and $\sum_{m=-j}^{j} m^2 = (1/3)j(j+1)(2j+1)$.

It remains to calculate the $k = 3$ component of $\widehat{E}_k(S)$. This case is easier than the other two components as $\widehat{J}^3_{e_1}$ does not change the magnetic quantum number m. Rather $|jm; m'\rangle$ is already an eigenstate of $\widehat{J}^3_{e_1}$:

$$\widehat{J}^3|jm\rangle = m|jm\rangle. \tag{7.11}$$

Using the eigenvalue above, we can evaluate the action of $\widehat{E}_3(S)$ on the SNF $|\beta^0, 0\rangle$:

$$\begin{aligned}\widehat{E}_3(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m} \left(\widehat{J}^3_{e_1}|jm; m'_1\rangle_{e_1}\right) \otimes |j-m; m'_2\rangle_{e_2} \\ &= -\frac{\ell_p^2}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m} m|jm; m'_1\rangle_{e_1} \otimes |j-m; m'_2\rangle_{e_2}.\end{aligned} \tag{7.12}$$

As we have a different tensor product $|jm; m'_1\rangle_{e_1} \otimes |j-m; m'_2\rangle_{e_2}$ than in the $k = 1, 2$ component case, we will consequently have a different expansion in terms of the recoupling basis states, in particular different in terms of the CGG appearing:

$$\begin{aligned}|jm; m'_1\rangle_{e_1} \otimes |j-m; m'_2\rangle_{e_2} &= +\frac{(-1)^{j-m}}{\sqrt{2j+1}}|\beta^0, 0\rangle + \frac{(-1)^{j-m} m\sqrt{3}}{\sqrt{j(j+1)(2j+1)}}|\beta^1, 0\rangle \\ &\quad + \sum_{\widetilde{j}_{12}=2}^{2j} \langle \widetilde{j}_{12}\widetilde{m}_{12} = 0|jm; j-m\rangle|\beta^{\widetilde{j}_{12}}, 0\rangle.\end{aligned} \tag{7.13}$$

Here, we can neglect the first two summands in equation (7.13). As for the $k = 1, 2$ component, we will expand the final result in terms of the states $|\beta^1, m_{12}\rangle$ (see also equation (7.7) for this). Because $|\beta^0, 0\rangle$ and $|\beta^1, m_{12}\rangle$ are orthogonal to each other, the scalar product $\langle\beta^1, m_{12}|\beta^0, 0\rangle$ vanishes. Additionally, the first summand in equation (7.13) leads to an expression proportional to $\sum_{m=-j}^{+j} m = 0$ when inserting it into equation (7.12). Therefore, we will just consider the second summand of equation (7.13) as all the other terms of the remaining sum vanish as well, because of the orthogonality relation concerning $\widetilde{j}_{12}$. Hence, we get

$$\begin{aligned}\widehat{E}_3(S)|\beta^0, 0\rangle &= -(-1)^{j-m}\sqrt{\frac{3}{2}}\sqrt{\frac{(j(j+1) - m(m-1))^2}{j(j+1)(2j+1)}}|\beta^1, 0\rangle \\ &= -\frac{\ell_p^2}{\sqrt{3}}\sqrt{j(j+1)}|\beta^1, 0\rangle,\end{aligned} \tag{7.14}$$

where we have taken advantage of the fact that $(-1)^{2(j-m)} = +1$, as $(j-m) \in \mathbb{Z}$ and used $\sum_{m=-j}^{j} m^2 = (1/3)j(j+1)(2j+1)$. Summarizing, the results of this section we can extract from equations (7.8), (7.10) and (7.14) the following results for the three components of the flux operator $\widehat{E}_k(S)$:

$$\begin{aligned}\widehat{E}_1(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{6}}\sqrt{j(j+1)}\{|\beta^1, -1\rangle - |\beta^1, +1\rangle\} \\ \widehat{E}_2(S)|\beta^0, 0\rangle &= -\frac{\mathrm{i}\ell_p^2}{\sqrt{6}}\sqrt{j(j+1)}\{|\beta^1, -1\rangle + |\beta^1, +1\rangle\} \\ \widehat{E}_3(S)|\beta^0, 0\rangle &= -\frac{\ell_p^2}{\sqrt{3}}\sqrt{j(j+1)}|\beta^1, 0\rangle.\end{aligned} \tag{7.15}$$

8. Comparison of the two flux operators

By comparing equation (7.15) with the results of ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ in equation (6.37) and the results of ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ shown in equation (6.41), we can judge whether our newly constructed flux operators ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$, ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ are consistent with the action of the usual one $\widehat{E}_k(S)$.[9]

Let us first discuss the operator ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$. It transpires that

$$ {}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle = 3!8C_{\mathrm{reg}}\widehat{E}_k(S)|\beta^0,0\rangle. \tag{8.1} $$

Therefore the two operators differ only by a positive integer constant. As there is still the regularization constant C_{reg} in the above equation, we can now fix it by requiring that both operators do exactly agree with each other. In fact, there is no other choice than exact agreement because the difference would be a global constant which does not decrease as we take the corresponding limit of large quantum numbers j. Thus, we can remove the regularization ambiguity of the volume operator in this way and choose C_{reg} to be $C_{\mathrm{reg}} := \frac{1}{3!8} = \frac{1}{48}$.

This is exactly the value of C_{reg} that was obtained in [6] by a completely different argument. Thus the geometrical interpretation of the value we have to choose for C_{reg} is perfectly provided[10].

Note that the consistency check holds in the full theory and not only in the semi-classical sector. Consequently, the operator ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ is consistent with the usual flux operator.

Now, considering the operator ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ things look differently. Here, a quantization that is consistent with $\widehat{V}_{\mathrm{RS}}$ of the sign operator $\widehat{\mathcal{S}}$ cannot be found. Accordingly, we should stop here and draw the conclusion that ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ is not consistent with $\widehat{E}_k(S)$. A way out of this problem is to use artificially $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ for ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$. In doing so, we obtain

$$ {}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S)|\beta^0,0\rangle = C(j,\ell)C_{\mathrm{reg}}\widehat{E}_k(S)|\beta^0,0\rangle, \tag{8.2} $$

whereby $C(j,\ell)\in\mathbb{R}$ is a constant depending on the spin labels j,ℓ in general. Precisely, the dependence on the spin label j causes a discrepancy of ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ with respect to $\widehat{E}_k(S)$. But since $C(j,\ell)\to C(\ell)$ semi-classically, i.e. in the limit of large j, which is shown in appendix E and discussed in section 6.6.2 of [20], ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ including that the artificial operator $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ is consistent with $\widehat{E}_k(S)$ within the semi-classical regime of the theory if we choose $C_{\mathrm{reg}} = 1/C(\ell)$. Unfortunately, $C(\ell)$ has a non-trivial ℓ dependence which is not acceptable because it is absent in the classical theory. Moreover, we do not see any geometrical interpretation available for the chosen value of C_{reg} in this case. One could possibly get rid of the ℓ dependence by simply cancelling the linearly dependent triples by hand from the definition of $\widehat{V}_{\mathrm{RS}}$. But then the so-modified $\widehat{V}'_{\mathrm{RS}}$ and $\widehat{V}_{\mathrm{AL}}$ would practically become identical on 3- and 4-valent vertices, and moreover $\widehat{V}'_{\mathrm{RS}}$ would now depend on the differentiable structure of Σ.

9. Uniqueness of the chosen factor ordering

Since the analysis here holds for $\widehat{V}_{\mathrm{AL}}$ as well as for $\widehat{V}_{\mathrm{RS}}$ we neglect the explicit labelling in this section. Now, we discuss to which extent the factor ordering chosen by us in section 3.3 is unique. For this purpose let us go back to equation (3.11). Instead of using the classical identity shown in that equation we could have used the following identity,

[9] The operators ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\tilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ have been ruled out before since they are the zero operator and not consistent with the usual flux operator $\widehat{E}_k(S)$.

[10] The factor $8 = 2^3$ comes from the fact that during the regularization one integrates a product of 3 δ-distributions on $\mathbb{R}$ over $\mathbb{R}^+$ only. The factor $6 = 3!$ is due to the fact that one should sum over ordered triples of edges only.

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}'_k(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{kst}\frac{4}{\kappa^2}\left\{A_3^s, V(R_{v(\square)})\right\}\mathcal{S}\left\{A_4^t, V(R_{v(\square)})\right\} && (9.1)\\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{16}{\kappa^2}\epsilon_{skt}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)}\mathrm{Tr}\big(\pi_\ell(\tau_s)\pi_\ell\big(h_{e_3}\big)\big\{\pi_\ell\big(h_{e_3}^{-1}\big),\\
&\quad V(R_{v(\square)})\big\}\big)\mathrm{Tr}\big(\boxed{\mathcal{S}}\,\mathbb{1}_{(2\ell+1)}\big)\,\mathrm{tr}\big(\pi_\ell(\tau_t)\pi_\ell\big(h_{e_4}\big)\big\{\pi_\ell\big(h_{e_4}^{-1}\big), V(R_{v(\square)})\big\}\big), && (9.2)
\end{aligned}
$$

where we used $\mathrm{Tr}(\pi_\ell(\tau_s)\pi_\ell(\tau'_s)) = -\frac{4}{3}2\ell(\ell+1)\ell(\ell+1)\delta_{s,s'}$. Surely, the operator corresponding to equation (3.11) would lead to a flux operator with a trivial action so far, for the reason that only one edge is added to $|\beta^0, 0\rangle$ before $\widehat{V}$ acts. Nevertheless, as the holonomies commute classically, and additionally the trace is invariant under cyclic permutations, we are allowed to insert a well-chosen unitary matrix in every trace:

$$
\begin{aligned}
&\mathrm{Tr}\big(\pi_\ell(\tau_t)\pi_\ell\big(h_{e_4}\big)\big\{\pi_\ell l\big(h_{e_4}^{-1}\big), V(R_{v(\square)})\big\}\big)\\
&\qquad = \mathrm{Tr}\big(\widehat{\pi}_\ell(\tau_t)\pi_\ell\big(h_{e_4}\big)\big\{\pi_\ell\big(h_{e_4}^{-1}\big), V(R_{v(\square)})\big\}\pi_\ell\big(h_{e_3}\big)\pi_\ell\big(h_{e_3}^{-1}\big)\big)\\
&\qquad = \mathrm{Tr}\big(\pi_\ell\big(h_{e_3}^{-1}\big)\pi_\ell(\tau_t)\pi_\ell\big(h_{e_4}\big)\big\{\pi_\ell\big(h_{e_4}^{-1}\big), V(R_{v(\square)})\big\}\pi_\ell\big(h_{e_3}\big)\big). && (9.3)
\end{aligned}
$$

Considering the trace that includes the sign factor $\widehat{S}$, we note that we have to insert two unitary matrices here, in order to avoid a trivial action of the corresponding operator. Accordingly, we end up with

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}'_k(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{kst}\frac{4}{\kappa^2}\left\{A_3^s, V(R_{v(\square)})\right\}\mathcal{S}\left\{A_4^t, V(R_{v(\square)})\right\}\\
&= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\frac{16}{\kappa^2}\epsilon_{skt}\frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)}\\
&\quad\times\mathrm{Tr}\big(\pi_\ell\big(h_{e_4}^{-1}\big)\pi_\ell(\tau_s)\pi_\ell\big(h_{e_3}\big)\big\{\pi_\ell\big(h_{e_3}^{-1}\big), V(R_{v(\square)})\big\}\pi_\ell\big(h_{e_4}\big)\big)\\
&\quad\times\mathrm{Tr}\big(\widehat{\pi}_\ell\big(h_{e_4}\big)\pi_\ell\big(h_{e_3}^{-1}\big)\boxed{\mathcal{S}}\,\mathbb{1}_{(2\ell+1)}\pi_\ell\big(h_{e_3}\big)\pi_\ell\big(h_{e_4}^{-1}\big)\big)\\
&\quad\times\mathrm{Tr}\big(\pi_\ell\big(h_{e_3}^{-1}\big)\pi_\ell(\tau_t)\pi_\ell\big(h_{e_4}\big)\big\{\pi_\ell\big(h_{e_4}^{-1}\big), V(R_{v(\square)})\big\}\pi_\ell\big(h_{e_3}\big)\big). && (9.4)
\end{aligned}
$$

When we apply the formalism of canonical quantization now, we get an operator with a different factor ordering than the one we used before,

$$
\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}'_k(S_t) &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{skt}\frac{-4\ell_p^{-4}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)}\\
&\quad\times\mathrm{Tr}\big(\widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\pi_\ell(\tau_s)\widehat{\pi}_\ell\big(h_{e_3}\big)\big[\widehat{V}(R_{v(\square)}), \widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\big]\widehat{\pi}_\ell\big(h_{e_4}\big)\big)\\
&\quad\times\mathrm{Tr}\big(\widehat{\pi}_\ell\big(h_{e_4}\big)\widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\boxed{\widehat{\mathcal{S}}}\,\mathbb{1}_{(2\ell+1)}\widehat{\pi}_\ell\big(h_{e_3}\big)\widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\big)\\
&\quad\times\mathrm{Tr}\big(\widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\widehat{\pi}_\ell(\tau_t)\widehat{\pi}_\ell\big(h_{e_4}\big)\big[\widehat{V}\big(R_{v(\square)}, \widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\big)\big]\widehat{\pi}_\ell(h_{e_3})\big). && (9.5)
\end{aligned}
$$

Hence, the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}'_k(S_t)$ can be calculated in the following way:

$$
\begin{aligned}
\langle\beta^1, \widetilde{m}_{12}|{}^{(\ell)}\widehat{\widetilde{E}}'_k(S_t)|\beta^0, 0\rangle &= \lim_{\mathcal{P}_t\to S_t}\sum_{\square\in\mathcal{P}_t}\epsilon_{skt}\frac{-16\ell_p^{-4}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)}\frac{1}{(2\ell+1)}\\
&\quad\times\sum_{\widetilde{m}'_{12}=-1}^{+1}\langle\beta^1, \widetilde{m}_{12}|\,\mathrm{tr}\big(\widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\pi_\ell(\tau_s)\widehat{\pi}_\ell\big(h_{e_3}\big)\big[\widehat{V}(R_{v(\square)}), \widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\big]\\
&\quad\times\widehat{\pi}_\ell\big(h_{e_4}\big)\big)|\beta^0, 0\rangle\langle\beta^0, 0|\,\mathrm{tr}\big(\widehat{\pi}_\ell\big(h_{e_4}\big)\widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\boxed{\widehat{\mathcal{S}}}\,\mathbb{1}_{(2\ell+1)}\widehat{\pi}_\ell\big(h_{e_3}\big)\widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\big)
\end{aligned}
$$

$$\times |\beta^1, \widetilde{m}'_{12}\rangle\langle\beta^1, \widetilde{m}'_{12}|\,\mathrm{tr}\big(\widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\widehat{\pi}_\ell(\tau_t)\widehat{\pi}_\ell\big(h_{e_4}\big)$$
$$\times \big[\widehat{V}(R_{v(\square)}), \widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\big]\widehat{\pi}_\ell\big(h_{e_3}\big)\big)|\beta^0, 0\rangle. \tag{9.6}$$

In order to show why this factor ordering is not appropriate to construct an alternative flux operator, we take a closer look at the trace terms, for instance the one on the rightmost side. Carrying out this trace leads to

$$\langle\beta^1, \widetilde{m}'_{12}|\,\mathrm{tr}\big(\widehat{\pi}_\ell\big(h_{e_3}^{-1}\big)\widehat{\pi}_\ell(\tau_t)\widehat{\pi}_\ell\big(h_{e_4}\big)\big[\widehat{V}(R_{v(\square)}), \widehat{\pi}_\ell\big(h_{e_4}^{-1}\big)\big]\widehat{\pi}_\ell\big(h_{e_3}\big)\big)|\beta^0, 0\rangle$$
$$= \lim_{\mathcal{P}_t \to S_t} \sum_{\square \in \mathcal{P}_t} \frac{16\ell_p^{-4}(-1)^{2\ell}}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \pi_\ell(\tau_k)_{CB}$$
$$\times \langle\beta^1, \widetilde{m}'_{12}|\pi_\ell(\epsilon)_{EI}\pi_\ell(\epsilon)_{FC}\widehat{\pi}_\ell\big(h_{e_4}^\dagger\big)_{FG}\widehat{\pi}_\ell\big(h_{e_3}^\dagger\big)_{BA}\widehat{V}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA}|\beta^0, 0\rangle. \tag{9.7}$$

However, this is exactly the expression of the former operator in equation (4.12) with the small but important difference that in this case the operator $\widehat{O} = \{\widehat{V}\widehat{S}\widehat{V}, \widehat{V}^2\}$ is replaced by the volume operator $\widehat{V}$ itself. As $\widehat{V}^2$ and the operator $\widehat{V}$ have the same eigenvectors, we can conclude from the discussion about the case where $\widehat{O} = \widehat{V}^2$ in sections 6.2 and appendix D that the matrix element is zero. Consequently, the whole flux operator ${}^{(\ell)}\widehat{\widetilde{E}}{}'_k(S_t)$ has a trivial action. Therefore this factor ordering cannot be used. Moreover, one can show that the other trace terms vanish as well, so that the trivial action of ${}^{(\ell)}\widehat{\widetilde{E}}{}'_k(S_t)$ is not only due to the disappearing of the matrix element which we took as an example.

Another idea could be to put an additional trace including additional holonomies around the already existing traces. We did this for a trace including one more holonomy and calculated the case where all three edges that are added to $|\beta^0, 0\rangle$ carry a spin label of $\ell = \frac{1}{2}$, and it turned out that the result is zero, too.

10. Conclusion

In contrast to our companion paper [20], we focused in this paper on the technical and mathematical aspects of the consistency check. By following the technical details step by step we hope to have provided a possibility to present, among other things, the robustness of this consistency check. For instance, the fact that case I where the densitized triad is given by $E_k^a = \det(e)e_k^a$ leads to an alternative flux operator for $\widehat{V}_{\mathrm{AL}}$ as well as $\widehat{V}_{\mathrm{RS}}$, that is, the zero operator could not have been guessed from the outset. This seems to be caused by an abstract symmetry of the volume operator that we were not aware of until now. We would like to be able to understand this issue from a more abstract perspective. Nevertheless, since the quantization of the momentum operator $i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ on $L_2(\mathbb{R}^+, \mathrm{d}x)$ is also not possible, the result that E_k^a cannot be considered as a 2-form fits perfectly well.

Quite unexpectedly, the quantization of the sign operator becomes necessary in order to perform the consistency check. Furthermore, the explicit relation to the operator $\widehat{Q}_v^{\mathrm{AL}}$, namely, $\widehat{\mathcal{S}} = \mathrm{sgn}\big(\widehat{Q}_v^{\mathrm{AL}}\big)$ which is equivalent to the operator identity $\widehat{Q}_v^{\mathrm{AL}} = \widehat{V}_{\mathrm{AL}}\mathcal{S}\widehat{V}_{\mathrm{AL}}$, provides us with (i) the possibility of performing the check for arbitrary spin labels ℓ, thanks to the techniques developed in [16], and (ii) conclusion that $\widehat{V}_{\mathrm{RS}}$ is not consistent with the usual flux operator, because there is no way to quantize a sign operator by using the regularization that was employed when $\widehat{V}_{\mathrm{RS}}$ was defined. Even the artificial construction where one uses $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ leads to an alternative flux operator that also differs from the usual one semi-classically since it contains a regularization constant still dependent on the spin label ℓ.

By comparing the detailed calculation of cases I and II one realizes that the sign operator $\widehat{\mathcal{S}}$, roughly speaking, acts like a 'switch' which either leads to cancellation or survival of terms in the eigenvector expansions.

The regularization of the alternative and the usual flux operator is based on the same method and it turns out that the classification of edges in types up, down, in and out that is sensible for the usual flux operator is also meaningful for the alternative one. Moreover, the meaning of the limit as we remove the regulator and define the alternative flux operator has to be understood in the same way as for the usual flux operator, otherwise the alternative flux operator is identical to zero. Moreover, without the additional smearing we would be missing a crucial factor of $1/2$ and our C_{reg} would be off the value found in [6].

The correspondence between the Ashtekar–Lewandowski ($\mathcal{H}_{\text{AL}}$) and the abstract angular momentum system Hilbert space has to be taken into account and has a large impact on the final result. If we had not introduced the unitary map W that allows us to transform between $\mathcal{H}_{\text{AL}}$ and the abstract angular momentum Hilbert space the result of the alternative flux operator would differ from the result for the usual one.

Finally, all the ℓ dependence cancels at the end. Since many ℓ-dependent terms are involved in the calculation as, for instance, Clebsch–Gordan coefficients, τ-matrices and the matrix elements of $\widehat{Q}_v^{\text{AL}}$, this is rather astonishing and demonstrates that all the ingredients of this consistency check fit together harmonically.

This paper along with our companion paper [20] is one of the first papers that tightens the mathematical structure of full LQG by using the kind of consistency argument used here. Many more such checks should be performed in the future to remove ambiguities of LQG and to make the theory more rigid, in particular those connected with quantum dynamics.

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Appendix A. Proof of lemma 5.2 in section 5.2

In order to keep the proof comprehensible, we express the CGC in equation (5.13) in terms of Wigner $3j$-symbols, because the symmetry properties of the Wigner 3j-symbols are easier to handle than those of the CGC itself[11]. The relation between the CGC and the corresponding $3j$-symbol is given by

$$\langle j_{12}m_{12};\ell E|a_3 m_{12}+E\rangle = (-1)^{m_{12}+E+j_{12}-\ell}\sqrt{2a_3+1}\begin{pmatrix} j_{12} & \ell & a_3 \\ m_{12} & E & -(m_{12}+E)\end{pmatrix}. \tag{A.1}$$

Replacing the first CGC in equation (5.2) by the corresponding $3j$-symbol and using the definition of $\pi_\ell(\epsilon)_{E-E}$, we get

$$\sum_{E=-\ell}^{+\ell}\pi_\ell(\epsilon)_{E-E}\langle j_{12}m_{12};\ell E|a_3m_{12}+E\rangle\langle a_3m_{12}+E;\ell-E|Jm_{12}\rangle$$
$$= \sum_{E=-\ell}^{+\ell}(-1)^{m_{12}+j_{12}}\sqrt{2a_3+1}\begin{pmatrix} j_{12} & \ell & a_3 \\ m_{12} & E & -(m_{12}+E)\end{pmatrix}\langle a_3m_{12}+E;\ell-E|Jm_{12}\rangle. \tag{A.2}$$

[11] Note that we have already used the replacement $I=-E$ in the lemma. We could have left both indices E,I independent, but due to $\pi_\ell(\epsilon)_{EI}=(-1)^{\ell-E}\delta_{E+I,0}$ all terms in which $I\neq -E$ will vanish anyway.

With the help of the symmetry properties of the $3j$-symbol, we are able to show that $\begin{pmatrix} j_{12} & \ell & a_3 \\ m_{12} & E & -(m_{12}+E) \end{pmatrix}$ is proportional to the CGC $\langle a_3 m_{12} + E; \ell - E | j_{12} m_{12} \rangle$

$$\langle a_3 m_{12} + E; \ell - E | j_{12} m_{12} \rangle = (-1)^{m_{12}+3a_3+\ell+2j_{12}} \sqrt{2j_{12}+1} \begin{pmatrix} j_{12} & \ell & a_3 \\ m_{12} & E & -(m_{12}+E) \end{pmatrix}. \tag{A.3}$$

Hence, rearranging the equation above leads to the desired proportionality

$$\begin{pmatrix} j_{12} & \ell & a_3 \\ m_{12} & E & -(m_{12}+E) \end{pmatrix} = \frac{(-1)^{-m_{12}-3a_3-\ell-2j_{12}}}{\sqrt{2j_{12}+1}} \langle a_3 m_{12} + E; \ell - E | j_{12} m_{12} \rangle. \tag{A.4}$$

The next step will be to insert equation (A.4) into equation (A.2) in order to use the orthogonality relation of the CGC for the remaining two CGC of the rewritten version of equation (A.2),

$$\begin{aligned} &\sum_{E=-\ell}^{+\ell} \pi_\ell(\epsilon)_{E-E} \langle j_{12} m_{12}; \ell E | a_3 m_{12} + E \rangle \langle a_3 m_{12} + E; \ell - E | J m_{12} \rangle \\ &\quad = (-1)^{-\ell-3a_3-j_{12}} \frac{\sqrt{2a_3+1}}{\sqrt{2j_{12}+1}} \sum_{E=-\ell}^{\ell} \langle j_{12} m_{12} | a_3 m_{12} + E; \ell - E \rangle \\ &\qquad \times \langle a_3 m_{12} + E; \ell - E | J m_{12} \rangle, \end{aligned} \tag{A.5}$$

where we utilized that the CGC are real by convention in the last step. Now, we can take advantage of the orthogonality relation of the CGC which is given by

$$\begin{aligned} &\sum_{E=-\ell}^{\ell} \langle j_{12} m_{12} | a_3 m_{12} + E; \ell - E \rangle \langle a_3 m_{12} + E; \ell - E | J m_{12} \rangle \\ &\quad = \delta_{J, j_{12}} \left(\delta_{m_{12}, -j_{12}} + \delta_{m_{12}, -j_{12}+1} + \cdots + \delta_{m_{12}, j_{12}} \right). \end{aligned} \tag{A.6}$$

Replacing the sum in equation (A.5) by the means of equation (A.6), we are able to show that lemma (5.2) is true

$$\begin{aligned} &\sum_{E=-\ell}^{+\ell} \pi_\ell(\epsilon)_{E-E} \langle j_{12} m_{12}; \ell E | a_3 m_{12} + E \rangle \langle a_3 m_{12} + E; \ell - E | J m_{12} \rangle \\ &\quad = (-1)^{-j_{12}-\ell-3a_3} \frac{\sqrt{2a_3+1}}{\sqrt{2j_{12}+1}} \delta_{J, j_{12}} \left(\delta_{m_{12}, -j_{12}} + \delta_{m_{12}, -j_{12}+1} + \cdots + \delta_{m_{12}, j_{12}} \right). \end{aligned} \tag{A.7}$$

Appendix B. τ-matrices in arbitrary representation with weight ℓ

In order to be able to define the alternative flux ${}^{(\ell)}\widetilde{E}_k(S)$ on the classical level, we need to derive the matrix elements $\pi_\ell(\tau_k)_{mn}$ for the three τ-matrices in an arbitrary representation with weight ℓ. For this purpose, we will use a formula for the matrix elements suitable for general $SL(2, \mathbb{C})$ matrices $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{C}$ and $\det(h) = ad - bc = 1$, given in [18].

Let $\pi_\ell(h)$ be the $(2\ell + 1)$-dimensional matrix for h in a particular representation with weight ℓ. It is the transformation matrix between totally symmetric spinors of rank 2ℓ. The $\pi_\ell(h)_{mn}$, where $m, n = \{-\ell, \ldots, \ell\}$, are given by

$$\pi_\ell(h)_{mn} = \sum_s \frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}{(\ell-m-s)!(\ell+n-s)!(m-n+s)!s!} a^{\ell+n-s} b^{m-n+s} c^s \, \mathrm{d}^{\ell-m-s}. \tag{B.1}$$

Here the sum has to be taken over all integers s that do not cause negative factorials. Using the definition of the matrix element of the τ-matrices in a particular representation with weight ℓ

$$\pi_\ell(\tau_k)_{mn} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi_\ell(\mathrm{e}^{t\tau_k})_{mn}, \tag{B.2}$$

where $\tau_k := -\mathrm{i}\sigma_k$, and we can write down the three matrices $\mathrm{e}^{t\tau_k}$ for $k = 1, 2, 3$ that are shown in equation (B.3),

$$\begin{aligned} \mathrm{e}^{t\tau_1} &= \cos(t)1_1 + \sin(t)\tau_1 \\ \mathrm{e}^{t\tau_2} &= \cos(t)1_2 + \sin(t)\tau_2 \\ \mathrm{e}^{t\tau_3} &= \cos(t)1_2 + \sin(t)\tau_3. \end{aligned} \tag{B.3}$$

Inserting the above matrices into the formula in equation (B.1) and taking the derivative at the point $t = 0$, we achieve a general expression for the matrix elements of the three τ-matrices $\pi_\ell(\tau_k)$ in a particular representation with weight ℓ,

$$\begin{aligned} \pi_\ell(\tau_1)_{mn} &= -\mathrm{i}\sqrt{\ell(\ell+1) - m(m-1)}\delta_{m-n,1} - \mathrm{i}\sqrt{\ell(\ell+1) - m(m+1)}\delta_{m-n,-1} \\ \pi_\ell(\tau_2)_{mn} &= \sqrt{\ell(\ell+1) - m(m+1)}\delta_{m-n,-1} - \sqrt{\ell(\ell+1) - m(m-1)}\delta_{m-n,1} \\ \pi_\ell(\tau_3)_{mn} &= -2\mathrm{i}m\delta_{m-n,0}. \end{aligned} \tag{B.4}$$

During the derivation of the alternative flux ${}^{(\ell)}\widetilde{E}_k(S)$, we will need the following property of the τ-matrices $\pi_\ell(\tau_k)$.

Lemma B.1. *Let $\pi_\ell(\tau_k)$ be the $(2\ell+1)$-dimensional matrix for $\tau_k := -\mathrm{i}\sigma_k$ in a particular representation with weight ℓ, then the following identity holds:*

$$\mathrm{tr}(\pi_\ell(\tau_k)\pi_\ell(\tau_r)\pi_\ell(\tau_s)) = -\tfrac{4}{3}\ell(\ell+1)(2\ell+1)\epsilon_{krs}. \tag{B.5}$$

We desist from writing the proof of lemma B.1 here, since the lemma can be easily proven by using basic algebraic tools and explicitly calculating the identity for the various cases.

Appendix C. Derivation of the formulae for the matrix elements of $\widehat{q}_{IJK}$

In this section, we will derive the explicit formulae for the matrix elements of $\widehat{q}_{IJK}$, namely equations (6.6)–(6.9), because it turned out [19] that these are two special cases in which the general formula in [16] is not applicable. Therefore we have to start from the very beginning and use the definition of q_{IJK} in equation (4.6). In the following, we will adopt the notation introduced in [16] and denote different recoupling schemes by $\vec{g}(IJ)$ where I, J label the momenta that are coupled together at first. Therefore, often $\vec{g}(12)$ is called the standard recoupling scheme. The intermediate couplings of particular scheme $\vec{g}(IJ)$ will be called g_i, while the intermediate couplings of our states $|\alpha_i^J, M\rangle$ and $|\alpha_{\tilde{i}}^J, M\rangle$ are still a_i and $\widetilde{a}_i$, respectively. Using equation (4.6) for the case of $I = 1$, $J = 3$, $K = 4$, we obtain

$$\begin{aligned} \langle\alpha_i^J, M|q_{134}|\alpha_{\tilde{i}}^J, M\rangle &= \langle\alpha_i^J, M|[(J_{13})^2, (J_{34})^2]|\alpha_{\tilde{i}}^J, M\rangle \\ &= \langle\alpha_i^J, M|(J_{13})^2(J_{34})^2|\alpha_{\tilde{i}}^J, M\rangle - \langle\alpha_i^J, M|(J_{34})^2(J_{13})^2]|\alpha_{\tilde{i}}^J, M\rangle \\ &= \sum_{\vec{g}''(12)}\Bigg\{\sum_{\vec{g}(13),\vec{g}(34)} g_2(13)(g_2(13)+1)g_2(34)(g_2(34)+1)\langle\vec{g}(13)|\vec{g}''(12)\rangle \\ &\quad\times\langle\vec{g}''(12)|\vec{g}(34)\rangle\big[\langle\vec{g}(13)|\alpha_i^J, M\rangle\langle\vec{g}(34)|\alpha_{\tilde{i}}^J, M\rangle \\ &\quad - \langle\vec{g}(34)|\alpha_i^J, M\rangle\langle\vec{g}(13)|\alpha_{\tilde{i}}^J, M\rangle\big] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\vec{g}''(12)} \Bigg\{ \sum_{\vec{g}(13)} g_2(13)(g_2(13)+1)\langle \vec{g}(13)|\vec{g}''(12)\rangle \big\langle \vec{g}(13)\big|\alpha_i^J, M\big\rangle \\
&\quad \times \sum_{\vec{g}(34)} g_2(34)(g_2(34)+1)\langle \vec{g}(34)|\vec{g}''(12)\rangle \big\langle \vec{g}(34)\big|\alpha_{\tilde{i}}^J, M\big\rangle \Bigg\} \\
&\quad - \Big[\big|\alpha_i^J, M\big\rangle \longleftrightarrow \big|\alpha_{\tilde{i}}^J, M\big\rangle\Big] \Bigg\}, \qquad \text{(C.1)}
\end{aligned}
$$

where the last term has to be understood as the analogue of the first term when the states $\big|\alpha_i^J, M\big\rangle$ and $\big|\alpha_{\tilde{i}}^J, M\big\rangle$ are interchanged. It was demonstrated in [16] that by means of the Elliot–Biedenharn identity, one can actually carry out the sum over $\vec{g}(13)$ and $\vec{g}(34)$ in the above equation. Hence, we will take the result from [16, 19],

$$
\begin{aligned}
&\sum_{\vec{g}(13)} g_2(13)(g_2(13)+1)\langle \vec{g}(13)|\vec{g}''(12)\rangle \big\langle \vec{g}(13)\big|\alpha_i^J, M\big\rangle \\
&\quad = \left[\frac{1}{2}(-1)^{-j_1-j_2+j_3+1} X(j_1,j_3)^{\frac{1}{2}} A(g_2'',a_2) \begin{Bmatrix} j_2 & j_1 & g_2'' \\ 1 & a_2 & j_1 \end{Bmatrix} (-1)^{a_3} \right. \\
&\quad \left. \times \begin{Bmatrix} a_3 & j_3 & g_2'' \\ 1 & a_2 & j_3 \end{Bmatrix} + C(j_1,j_3)\delta_{g_2'',a_2} \right] \delta_{g_3'',a_3}\delta_{g_4'',a_4}. \qquad \text{(C.2)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\vec{g}(34)} g_2(34)(g_2(34)+1)\langle \vec{g}(34)|\vec{g}''(12)\rangle \big\langle \vec{g}(34)\big|\alpha_i^J, M\big\rangle \\
&\quad = \left[\frac{1}{2}(-1)^{-2(j_1+j_2)+j_4-j_3}(-1)^{a_2+1}(-1)^{a_3-g_3''} X(j_3,j_4)^{\frac{1}{2}} A(g_3'',a_3) \right. \\
&\quad \left. \times \begin{Bmatrix} a_2 & j_3 & g_3'' \\ 1 & a_3 & j_3 \end{Bmatrix} (-1)^{a_4} \begin{Bmatrix} a_4 & j_4 & g_3'' \\ 1 & a_3 & j_4 \end{Bmatrix} + C(j_3,j_4)\prod_{k=2}^{3}\delta_{g_k'',a_k} \right] \delta_{g_2'',a_2}\delta_{g_4'',a_4}. \qquad \text{(C.3)}
\end{aligned}
$$

In order to keep the equation comprehensible, we introduced the following abbreviations:

$$
\begin{aligned}
C(a,b) &:= a(a+1) + b(b+1) \\
X(a,b) &:= 2a(2a+1)(2a+2)2b(2b+1)(2b+2) \qquad \text{(C.4)} \\
A(a,b) &:= \sqrt{(2a+1)(2b+1)}.
\end{aligned}
$$

The next step is to insert equations (C.2) and (C.3) back into equation (C.1). By doing so, we recognize that the term containing $C(a,b)$ is symmetric under the interchange of $a_i \leftrightarrow \tilde{a}_i$ and accordingly will be cancelled, because we subtract the terms where $\big|\alpha_i^J, M\big\rangle$ and $\big|\alpha_{\tilde{i}}^J, M\big\rangle$ are interchanged from each other. Consequently, only the first term of (C.2) and (C.3) survives and we end up with

$$
\begin{aligned}
\big\langle \alpha_i^J, M\big| q_{134}\big|\alpha_{\tilde{i}}^J, M\big\rangle &= \sum_{\vec{g}''(12)} \Bigg\{ +\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}(-1)^{\tilde{a}_2+1}(-1)^{\tilde{a}_3-g_3''}(-1)^{a_3+\tilde{a}_4} X(j_1,j_3)^{\frac{1}{2}} \\
&\quad \times X(j_3,j_4)^{\frac{1}{2}} A(g_2'',a_2)A(g_3'',\tilde{a}_3) \begin{Bmatrix} j_2 & j_1 & g_2'' \\ 1 & a_2 & j_1 \end{Bmatrix} \begin{Bmatrix} a_3 & j_3 & g_2'' \\ 1 & a_2 & j_3 \end{Bmatrix} \\
&\quad \times \begin{Bmatrix} \tilde{a}_2 & j_3 & g_3'' \\ 1 & \tilde{a}_3 & j_3 \end{Bmatrix} \begin{Bmatrix} \tilde{a}_4 & j_4 & g_3'' \\ 1 & \tilde{a}_3 & j_4 \end{Bmatrix} \delta_{g_2'',\tilde{a}_2}\delta_{g_4'',\tilde{a}_4}\delta_{g_3'',a_3}\delta_{g_4'',a_4}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}(-1)^{a_2+1}(-1)^{a_3-g''_3}(-1)^{\widetilde{a}_3+a_4}X(j_1,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}\\
&\times A(g''_2,\widetilde{a}_2)A(g''_3,a_3)\begin{Bmatrix} j_2 & j_1 & g''_2\\ 1 & \widetilde{a}_2 & j_1\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_3 & j_3 & g''_2\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}\begin{Bmatrix} a_2 & j_3 & g''_3\\ 1 & a_3 & j_3\end{Bmatrix}\\
&\times\begin{Bmatrix} a_4 & j_4 & g''_3\\ 1 & a_3 & j_4\end{Bmatrix}\delta_{g''_2,a_2}\delta_{g''_4,a_4}\delta_{g''_3,\widetilde{a}_3}\delta_{g''_4,\widetilde{a}_4}\Bigg\}\\
=&+\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}(-1)^{\widetilde{a}_2+1}(-1)^{\widetilde{a}_3-a_3}(-1)^{a_3+a_4}X(j_1,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}\\
&\times A(\widetilde{a}_2,a_2)A(a_3,\widetilde{a}_3)\begin{Bmatrix} j_2 & j_1 & \widetilde{a}_2\\ 1 & a_2 & j_1\end{Bmatrix}\begin{Bmatrix} a_3 & j_3 & \widetilde{a}_2\\ 1 & a_2 & j_3\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_2 & j_3 & a_3\\ 1 & \widetilde{a}_3 & j_3\end{Bmatrix}\\
&\times\begin{Bmatrix} a_4 & j_4 & a_3\\ 1 & \widetilde{a}_3 & j_4\end{Bmatrix}-\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}(-1)^{a_2+1}(-1)^{a_3-\widetilde{a}_3}(-1)^{\widetilde{a}_3+a_4}\\
&\times X(j_1,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}A(a_2,\widetilde{a}_2)A(\widetilde{a}_3,a_3)\begin{Bmatrix} j_2 & j_1 & a_2\\ 1 & \widetilde{a}_2 & j_1\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_3 & j_3 & a_2\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}\\
&\times\begin{Bmatrix} a_2 & j_3 & \widetilde{a}_3\\ 1 & a_3 & j_3\end{Bmatrix}\begin{Bmatrix} a_4 & j_4 & \widetilde{a}_3\\ 1 & a_3 & j_4\end{Bmatrix}\\
=&+\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+a_4}X(j_2,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}\\
&\times A(\widetilde{a}_2,a_2)A(a_3,\widetilde{a}_3)\begin{Bmatrix} j_1 & j_2 & a_2\\ 1 & \widetilde{a}_2 & j_2\end{Bmatrix}\begin{Bmatrix} a_4 & j_4 & a_3\\ 1 & \widetilde{a}_3 & j_4\end{Bmatrix}\Bigg[(-1)^{\widetilde{a}_2+\widetilde{a}_3}\begin{Bmatrix} a_3 & j_3 & a_2\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}\\
&\times\begin{Bmatrix} \widetilde{a}_2 & j_3 & a_3\\ 1 & \widetilde{a}_3 & j_3\end{Bmatrix}-(-1)^{a_2+a_3}\begin{Bmatrix} \widetilde{a}_3 & j_3 & a_2\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}\begin{Bmatrix} a_2 & j_3 & a_3\\ 1 & \widetilde{a}_3 & j_3\end{Bmatrix}\Bigg]. \qquad \text{(C.5)}
\end{aligned}
$$

In the last line we used the symmetry properties of the $6j$-symbols, in particular the fact that

$$
\begin{Bmatrix} a & b & c\\ d & e & b\end{Bmatrix}=\begin{Bmatrix} a & b & e\\ d & c & b\end{Bmatrix}.
$$

Recalling that for our SNF $\left|\alpha_i^J, M\right\rangle$ we have $j_1=j_2=j$, $j_3=j_4=\ell$ and $a_4=J$, the matrix element of q_{134} can be expressed as

$$
\begin{aligned}
\left\langle\alpha_i^J, M\right|q_{134}\left|\alpha_{\tilde{i}}^J, M\right\rangle =&+\frac{1}{4}(-1)^{+2j+\ell+J}\sqrt{2j(2j+1)(2j+2)}[2\ell(2\ell+1)(2\ell+2)]^{\frac{3}{2}}\\
&\times\sqrt{(2\widetilde{a}_2+1)(2a_2+1)}\sqrt{2(a_3+1)(2\widetilde{a}_3+1)}\begin{Bmatrix} j & j & a_2\\ 1 & \widetilde{a}_2 & j\end{Bmatrix}\begin{Bmatrix} J & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}\\
&\times\Bigg[(-1)^{\widetilde{a}_2+\widetilde{a}_3}\begin{Bmatrix} a_3 & \ell & a_2\\ 1 & \widetilde{a}_2 & \ell\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_2 & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}-(-1)^{a_2+a_3}\\
&\times\begin{Bmatrix} \widetilde{a}_3 & \ell & a_2\\ 1 & \widetilde{a}_2 & \ell\end{Bmatrix}\begin{Bmatrix} a_2 & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}\Bigg], \qquad \text{(C.6)}
\end{aligned}
$$

where we used additionally that $2j\in\mathbb{Z}$ and therefore $(-1)^{-6j}=(-1)^{-2j}=(-1)^{+2j}$. Now we have to repeat the whole calculation for the case $I=2$, $J=3$, $K=4$ in order to derive the formula for q_{234}. Since the intermediate steps in the calculation are analogous to q_{134} not all details will be given. Here equation (4.6) leads to

$$
\begin{aligned}
\left\langle\alpha_i^J, M\right|q_{234}\left|\alpha_{\tilde{i}}^J, M\right\rangle &= \left\langle\alpha_i^J, M\right|[(J_{23})^2,(J_{34})^2]\left|\alpha_{\tilde{i}}^J, M\right\rangle\\
&=\sum_{\vec{g}''(12)}\Bigg\{\sum_{\vec{g}(23)}g_2(23)(g_2(23)+1)\langle\vec{g}(23)|\vec{g}''(12)\rangle\left\langle\vec{g}(23)\right|\alpha_i^J, M\right\rangle
\end{aligned}
$$

$$\times \sum_{\vec{g}(34)} g_2(34)(g_2(34)+1)\langle\vec{g}(34)|\vec{g}''(12)\rangle\langle\vec{g}(34)\big|\alpha_{\tilde{i}}^{J}, M\rangle\Bigg\}$$
$$-\left[\big|\alpha_i^J, M\rangle \longleftrightarrow \big|\alpha_{\tilde{i}}^J, M\rangle\right]. \tag{C.7}$$

Thus, we need to know the result of the summation over $\vec{g}(23)$ in this case. It is given by [16, 19]

$$\sum_{\vec{g}(23)} g_2(23)(g_2(23)+1)\langle\vec{g}(23)|\vec{g}''(12)\rangle\langle\vec{g}(23)\big|\alpha_i^J, M\rangle$$
$$= \Bigg[\frac{1}{2}(-1)^{-j_1-j_2+j_3+1}(-1)^{a_2-g_2''}X(j_2,j_3)^{\frac{1}{2}}A(g_2'',a_2)$$
$$\times \begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & a_2 & j_2 \end{Bmatrix}(-1)^{a_3}\begin{Bmatrix} a_3 & j_3 & g_2'' \\ 1 & a_2 & j_3 \end{Bmatrix} + C(j_2,j_3)\delta_{g_2'',a_2}\Bigg]\delta_{g_3'',a_3}\delta_{g_4'',a_4}. \tag{C.8}$$

Reinserting the equation above and the result of the summation over $\vec{g}(34)$ from equation (C.3) into equation (C.7), we get

$$\langle\alpha_i^J, M\big|q_{234}\big|\alpha_{\tilde{i}}^J, M\rangle = \sum_{\vec{g}''(12)}\Bigg\{+\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}(-1)^{a_2-g_2''+\tilde{a}_2+1}(-1)^{\tilde{a}_3-g_3''}(-1)^{a_3+\tilde{a}_4}$$
$$\times X(j_2,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}A(g_2'',a_2)A(g_3'',\tilde{a}_3)\begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & a_2 & j_2 \end{Bmatrix}\begin{Bmatrix} a_3 & j_3 & g_2'' \\ 1 & a_2 & j_3 \end{Bmatrix}$$
$$\times \begin{Bmatrix} \tilde{a}_2 & j_3 & g_3'' \\ 1 & \tilde{a}_3 & j_3 \end{Bmatrix}\begin{Bmatrix} \tilde{a}_4 & j_4 & g_3'' \\ 1 & \tilde{a}_3 & j_4 \end{Bmatrix}\delta_{g_2'',\tilde{a}_2}\delta_{g_4'',\tilde{a}_4}\delta_{g_3'',a_3}\delta_{g_4'',a_4} - \frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+1}$$
$$\times(-1)^{\tilde{a}_2-g_2''+a_2+1}(-1)^{a_3-g_3''}(-1)^{\tilde{a}_3+a_4}X(j_2,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}A(g_2'',\tilde{a}_2)A(g_3'',a_3)$$
$$\times \begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & \tilde{a}_2 & j_2 \end{Bmatrix}\begin{Bmatrix} \tilde{a}_3 & j_3 & g_2'' \\ 1 & \tilde{a}_2 & j_3 \end{Bmatrix}\begin{Bmatrix} a_2 & j_3 & g_3'' \\ 1 & a_3 & j_3 \end{Bmatrix}\begin{Bmatrix} a_4 & j_4 & g_3'' \\ 1 & a_3 & j_4 \end{Bmatrix}$$
$$\times \delta_{g_2'',a_2}\delta_{g_4'',a_4}\delta_{g_3'',\tilde{a}_3}\delta_{g_4'',\tilde{a}_4}\Bigg\}$$
$$= +\frac{1}{4}(-1)^{-3(j_1+j_2)+j_4+a_4}X(j_2,j_3)^{\frac{1}{2}}X(j_3,j_4)^{\frac{1}{2}}$$
$$\times A(\tilde{a}_2,a_2)A(a_3,\tilde{a}_3)\begin{Bmatrix} j_1 & j_2 & a_2 \\ 1 & \tilde{a}_2 & j_2 \end{Bmatrix}\begin{Bmatrix} a_4 & j_4 & a_3 \\ 1 & \tilde{a}_3 & j_4 \end{Bmatrix}$$
$$\times\Bigg[(-1)^{a_2+\tilde{a}_3}\begin{Bmatrix} a_3 & j_3 & a_2 \\ 1 & \tilde{a}_2 & j_3 \end{Bmatrix}\begin{Bmatrix} \tilde{a}_2 & j_3 & a_3 \\ 1 & \tilde{a}_3 & j_3 \end{Bmatrix} - (-1)^{\tilde{a}_2+a_3}$$
$$\times\begin{Bmatrix} \tilde{a}_3 & j_3 & a_2 \\ 1 & \tilde{a}_2 & j_3 \end{Bmatrix}\begin{Bmatrix} a_2 & j_3 & a_3 \\ 1 & \tilde{a}_3 & j_3 \end{Bmatrix}\Bigg]. \tag{C.9}$$

In the first step, we used again the symmetry of the term containing $C(a,b)$ in equations (C.3) and (C.8) under the interchange of $a_i \leftrightarrow \tilde{a}_i$. Furthermore, as before, we took advantage of the symmetry properties of the $6j$-symbols in order to be able to write the equation more compactly in the last line. Let us as a last step implement the spin labels of our states $\big|\alpha_i^J, M\rangle$, namely $j_1 = j_2 = j$, $j_3 = j_4 = \ell$ and $a_4 = J$. Moreover, as before, we rewrite $(-1)^{-6j}$ as $(-1)^{+2j}$.

Considering all this we have our final result in equation (C.10),

$$\begin{aligned}\left\langle\alpha_i^J, M\middle|q_{234}\middle|\alpha_{\tilde{i}}^J, M\right\rangle &= +\frac{1}{4}(-1)^{+2j+\ell+J}\sqrt{2j(2j+1)(2j+2)}[2\ell(2\ell+1)(2\ell+2)]^{\frac{3}{2}}\\ &\times\sqrt{(2\widetilde{a}_2+1)(2a_2+1)}\sqrt{2(a_3+1)(2\widetilde{a}_3+1)}\begin{Bmatrix} j & j & a_2\\ 1 & \widetilde{a}_2 & j\end{Bmatrix}\begin{Bmatrix} J & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}\\ &\times\left[(-1)^{a_2+\widetilde{a}_3}\begin{Bmatrix} a_3 & \ell & a_2\\ 1 & \widetilde{a}_2 & \ell\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_2 & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}\right.\\ &\left.-(-1)^{\widetilde{a}_2+a_3}\begin{Bmatrix} \widetilde{a}_3 & \ell & a_2\\ 1 & \widetilde{a}_2 & \ell\end{Bmatrix}\begin{Bmatrix} a_2 & \ell & a_3\\ 1 & \widetilde{a}_3 & \ell\end{Bmatrix}\right].\end{aligned}\tag{C.10}$$

In the case of $\widehat{q}_{123}$ the matrix element can be expressed as

$$\begin{aligned}\left\langle\alpha_i^J, M\middle|q_{123}\middle|\alpha_{\tilde{i}}^J, M\right\rangle &= \left\langle\alpha_i^J, M\middle|[(J_{12})^2, (J_{23})^2]\middle|\alpha_{\tilde{i}}^J, M\right\rangle\\ &= \sum_{\vec{g}''(12)}\left\{\sum_{\vec{g}(12)} g_2(12)(g_2(12)+1)\langle\vec{g}(12)|\vec{g}''(12)\rangle\left\langle\vec{g}(12)\middle|\alpha_i^J, M\right\rangle\right.\\ &\left.\times\sum_{\vec{g}(23)} g_2(23)(g_2(23)+1)\langle\vec{g}(23)|\vec{g}''(12)\rangle\left\langle\vec{g}(23)\middle|\alpha_{\tilde{i}}^J, M\right\rangle\right\}\\ &-\left[\left|\alpha_i^J, M\right\rangle\longleftrightarrow\left|\alpha_{\tilde{i}}^J, M\right\rangle\right].\end{aligned}\tag{C.11}$$

In order to perform the sums appearing in the equation above, we use equation (C.8) and take advantage of

$$\sum_{\vec{g}(12)} g_2(12)(g_2(12)+1)\langle\vec{g}(12)|\vec{g}''(12)\rangle\left\langle\vec{g}(12)\middle|\alpha_i^J, M\right\rangle = a_2(a_2+1)\prod_{k=2}^{4}\delta_{g_k'', a_k}.\tag{C.12}$$

Hence, we obtain

$$\begin{aligned}\left\langle\alpha_i^J, M\middle|q_{123}\middle|\alpha_{\tilde{i}}^J, M\right\rangle &= \sum_{\vec{g}''(12)}\left\{+\frac{1}{2}(-1)^{-j_1-j_2+j_3+1}(-1)^{\widetilde{a}_2-g_2''+\widetilde{a}_3}X(j_2,j_3)^{\frac{1}{2}}A(g_2'',\widetilde{a}_2)\right.\\ &\times\begin{Bmatrix} j_1 & j_2 & g_2''\\ 1 & \widetilde{a}_2 & j_2\end{Bmatrix}\begin{Bmatrix} \widetilde{a}_3 & j_3 & g_2''\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}[a_2(a_2-1)]\delta_{g_3'',\widetilde{a}_3}\delta_{g_4'',\widetilde{a}_4}\delta_{g_2'',a_2}\delta_{g_3'',a_3}\delta_{g_4'',a_4}\\ &-\frac{1}{2}(-1)^{-j_1-j_2+j_3+1}(-1)^{a_2-g_2''+a_3}X(j_2,j_3)^{\frac{1}{2}}A(g_2'',a_2)\\ &\left.\times\begin{Bmatrix} j_1 & j_2 & g_2''\\ 1 & a_2 & j_2\end{Bmatrix}\begin{Bmatrix} a_3 & j_3 & g_2''\\ 1 & a_2 & j_3\end{Bmatrix}[\widetilde{a}_2(\widetilde{a}_2-1)]\delta_{g_3'',a_3}\delta_{g_4'',a_4}\delta_{g_2'',\widetilde{a}_2}\delta_{g_3'',\widetilde{a}_3}\delta_{g_4'',\widetilde{a}_4}\right.\\ &= +\frac{1}{2}(-1)^{-j_1-j_2+j_3+1}(-1)^{\widetilde{a}_2-a_2+a_3}X(j_2,j_3)^{\frac{1}{2}}A(a_2,\widetilde{a}_2)\begin{Bmatrix} j_1 & j_2 & a_2\\ 1 & \widetilde{a}_2 & j_2\end{Bmatrix}\\ &\times\begin{Bmatrix} a_3 & j_3 & a_2\\ 1 & \widetilde{a}_2 & j_3\end{Bmatrix}[a_2(a_2-1)-\widetilde{a}_2(\widetilde{a}_2-1)]\delta_{a_3,\widetilde{a}_3},\end{aligned}\tag{C.13}$$

where we used again that the term proportional to $C(a,b)$ in equation (C.8) is cancelled in the first step. As we did before, we omitted the δ-function $\delta_{a_4,\widetilde{a}_4}$, because a_4 is equal to the total angular momentum J of our states $\left|\alpha_i^J, M\right\rangle$ and therefore we consider only cases where $a_4 = \widetilde{a}_4$ anyway. However, this is different for the intermediate coupling $a_3, \widetilde{a}_3$. For this reason, we have to consider $\delta_{a_3,\widetilde{a}_3}$ in the above equation. Applying the above equation to our particular case where $j_1 = j_2 = j$ and $j_3 = \ell$ yields

$$\langle \alpha_i^J, M | q_{123} | \alpha_{\tilde{i}}^J, M\rangle = +\frac{1}{2}(-1)^{+2j+\ell+1}(-1)^{\widetilde{a}_2-a_2+a_3} X(j,\ell)^{\frac{1}{2}} A(a_2,\widetilde{a}_2) \begin{Bmatrix} j & j & a_2 \\ 1 & \widetilde{a}_2 & j \end{Bmatrix} \times \begin{Bmatrix} a_3 & \ell & a_2 \\ 1 & \widetilde{a}_2 & \ell \end{Bmatrix} [a_2(a_2-1) - \widetilde{a}_2(\widetilde{a}_2-1)]\delta_{a_3,\widetilde{a}_3}. \tag{C.14}$$

Again, we rewrote $(-1)^{-2j}$ as $(-1)^{+2j}$ that is allowed due to $2j \in \mathbb{Z}$ and used the symmetry properties of the $6j$-symbol where appropriate. Now, we consider the last triple q_{124}. The corresponding matrix element is given by

$$\begin{aligned} \langle \alpha_i^J, M | q_{124} | \alpha_{\tilde{i}}^J, M\rangle &= \langle \alpha_i^J, M | [(J_{12})^2, (J_{24})^2] | \alpha_{\tilde{i}}^J, M\rangle \\ &= \sum_{\vec{g}''(12)} \Bigg\{ \sum_{\vec{g}(12)} g_2(12)(g_2(12)+1)\langle \vec{g}(12)|\vec{g}''(12)\rangle \langle \vec{g}(12)|\alpha_i^J, M\rangle \\ &\quad \times \sum_{\vec{g}(24)} g_2(24)(g_2(24)+1)\langle \vec{g}(24)|\vec{g}''(12)\rangle\langle \vec{g}(24)|\alpha_{\tilde{i}}^J, M\rangle \Bigg\} \\ &\quad - [|\alpha_i^J, M\rangle \longleftrightarrow |\alpha_{\tilde{i}}^J, M\rangle]. \end{aligned} \tag{C.15}$$

The summation including $\vec{g}(24)$ can be performed and leads to [19]

$$\begin{aligned} &\sum_{\vec{g}(24)} g_2(24)(g_2(24)+1)\langle \vec{g}(24)|\vec{g}''(12)\rangle\langle \vec{g}(24)|\alpha_{\tilde{i}}^J, M\rangle \\ &= \Bigg[\frac{1}{2}(-1)^{-j_1-j_2+a_4}(-1)^{a_2-g_2''}(-1)^{+g_2''+a_2} X(j_2,j_4)^{\frac{1}{2}} A(g_2'',a_2)A(g_3'',a_3) \\ &\quad \times \begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & a_2 & j_2 \end{Bmatrix} \begin{Bmatrix} j_3 & g_2'' & g_3'' \\ 1 & a_2 & a_3 \end{Bmatrix} \begin{Bmatrix} a_4 & j_4 & g_3'' \\ 1 & a_3 & j_4 \end{Bmatrix} \\ &\quad + C(j_2,j_4)\delta_{g_2'',a_2}\delta_{g_3'',a_3}\Bigg]\delta_{g_4'',a_4}. \end{aligned} \tag{C.16}$$

Inserting equations (C.12) and (C.8) into equation (C.15) and using again that the term proportional to $C(a,b)$ is antisymmetric under the interchange of $|\alpha_i^J, M\rangle \leftrightarrow |\alpha_{\tilde{i}}^J, M\rangle$, we get

$$\begin{aligned} \langle \alpha_i^J, M | q_{124} | \alpha_{\tilde{i}}^J, M\rangle &= \sum_{\vec{g}''(12)} \Bigg\{ +\frac{1}{2}(-1)^{-j_1-j_2+\widetilde{a}_4}(-1)^{\widetilde{a}_2-g_2''}(-1)^{+g_2''+\widetilde{a}_2} X(j_2,j_4)^{\frac{1}{2}} \\ &\quad \times A(g_2'',\widetilde{a}_2)A(g_3'',\widetilde{a}_3) \begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & \widetilde{a}_2 & j_2 \end{Bmatrix} \begin{Bmatrix} j_3 & g_2'' & g_3'' \\ 1 & \widetilde{a}_2 & \widetilde{a}_3 \end{Bmatrix} \begin{Bmatrix} \widetilde{a}_4 & j_4 & g_3'' \\ 1 & \widetilde{a}_3 & j_4 \end{Bmatrix} \\ &\quad \times [a_2(a_2-1)]\delta_{g_2'',a_2}\delta_{g_3'',a_3}\delta_{g_4'',a_4}\delta_{g_4'',\widetilde{a}_4} - \frac{1}{2}(-1)^{-j_1-j_2+a_4}(-1)^{a_2-g_2''}(-1)^{+g_2''+a_2} \\ &\quad \times X(j_2,j_4)^{\frac{1}{2}} A(g_2'',a_2)A(g_3'',a_3) \begin{Bmatrix} j_1 & j_2 & g_2'' \\ 1 & a_2 & j_2 \end{Bmatrix} \begin{Bmatrix} j_3 & g_2'' & g_3'' \\ 1 & a_2 & a_3 \end{Bmatrix} \begin{Bmatrix} a_4 & j_4 & g_3'' \\ 1 & a_3 & j_4 \end{Bmatrix} \\ &\quad \times [\widetilde{a}_2(\widetilde{a}_2-1)]\delta_{g_2'',\widetilde{a}_2}\delta_{g_3'',\widetilde{a}_3}\delta_{g_4'',\widetilde{a}_4}\delta_{g_4'',a_4} \\ &= +\frac{1}{2}(-1)^{-j_1-j_2+a_4} X(j_2,j_4)^{\frac{1}{2}} \\ &\quad \times A(a_2,\widetilde{a}_2)A(a_3,\widetilde{a}_3) \begin{Bmatrix} j_1 & j_2 & a_2 \\ 1 & \widetilde{a}_2 & j_2 \end{Bmatrix} \begin{Bmatrix} j_3 & a_2 & a_3 \\ 1 & \widetilde{a}_2 & \widetilde{a}_3 \end{Bmatrix} \begin{Bmatrix} a_4 & j_4 & a_3 \\ 1 & \widetilde{a}_3 & j_4 \end{Bmatrix} \\ &\quad \times [(-1)^{+2\widetilde{a}_2} a_2(a_2-1) - (-1)^{+2a_2}\widetilde{a}_2(\widetilde{a}_2+1)]. \end{aligned} \tag{C.17}$$

In the last line, we took advantage of the symmetry properties of the $6j$-symbol and, moreover, used that $(-1)^{\widetilde{a}_2-a_2} = (-1)^{a_2-\widetilde{a}_2}$ as $\widetilde{a}_2 - a_2 \in \mathbb{Z}$. In our special situation where $j_1 = j_2 = j$ the value of the intermediate coupling a_2 and $\widetilde{a}_2$ can only be an integer and thus $(-1)^{+2\widetilde{a}_2} = (-1)^{+2a_2} = +1$. Accordingly, we can completely neglect these factors and obtain

$$\langle \alpha_i^J, M | q_{124} | \alpha_{\widetilde{i}}^J, M\rangle = +\frac{1}{2}(-1)^{+2j+J} X(j,\ell)^{\frac{1}{2}} A(a_2,\widetilde{a}_2) A(a_3,\widetilde{a}_3) \begin{Bmatrix} j & j & a_2 \\ 1 & \widetilde{a}_2 & j \end{Bmatrix} \begin{Bmatrix} \ell & a_2 & a_3 \\ 1 & \widetilde{a}_2 & \widetilde{a}_3 \end{Bmatrix}$$
$$\times \begin{Bmatrix} a_4 & \ell & a_3 \\ 1 & \widetilde{a}_3 & \ell \end{Bmatrix} [a_2(a_2-1) - \widetilde{a}_2(\widetilde{a}_2+1)], \qquad \text{(C.18)}$$

where we additionally inserted $j_3 = j_4 = \ell$ and $a_4 = J$ in the equation above.

Appendix D. Case ${}^{(\ell)}\widehat{\widetilde{E}}{}^{\mathbf{I,AL}}_{k,\mathbf{tot}}(S_t)$: detailed calculation of the matrix elements of $\widehat{O}_1 = \widehat{O}_2 = V^2_{\mathbf{AL}}$

The aim of this section is to calculate the four matrix elements $\langle \alpha_2^0, 0 | \widehat{V}^2_{\mathrm{AL}} | \alpha_1^0, 0\rangle$ and $\langle \alpha_i^1, M | \widehat{V}^2_{\mathrm{AL}} | \alpha_1^1, M\rangle$ with $i = 2, 3, 4$. As has been mentioned before, in order to calculate the matrix elements of $\widehat{V}^2_{\mathrm{AL}}$ we first have to calculate the matrix elements of $\widehat{Q}^{\mathrm{AL}}_v$ and derive the eigenvalues and eigenvectors for $\widehat{Q}^{\mathrm{AL}}_v$. If λ_Q is an eigenvalue of $\widehat{Q}^{\mathrm{AL}}_v$ with a corresponding eigenvector $|\phi\rangle$, then $|\lambda_Q|$ is an eigenvalue of V^2 with the same eigenvector. Consequently, we have to calculate all possible matrix elements of $\widehat{Q}^{\mathrm{AL}}_v$ for each fixed total angular momentum J. For this reason, we are not able to evaluate matrix elements for the case of arbitrary spin representation, as we could do in the case of $\widehat{O}_1 = \widehat{O}_2 = \widehat{V}_{\mathrm{AL}}\widehat{S}\widehat{V}_{\mathrm{AL}} = \widehat{Q}^{\mathrm{AL}}_v$. In that case, we needed only particular matrix elements of $\widehat{Q}^{\mathrm{AL}}$ but not the knowledge of the spectrum of $\widehat{Q}^{\mathrm{AL}}_v$ itself. However, we will calculate the matrix elements of $\widehat{V}^2_{\mathrm{AL}}$ for the case of $\ell = \frac{1}{2}, 1$ here. Fortunately, they already show the major difference between the case of $\widehat{O}_1 = \widehat{O}_2 = \widehat{Q}^{\mathrm{AL}}_v$ and $\widehat{O}_1 = \widehat{O}_2 = \widehat{V}^2_{\mathrm{AL}}$.

When we calculated the five necessary matrix elements $\langle \alpha_2^0, 0 | \widehat{Q}^{\mathrm{AL}}_v | \alpha_1^0, 0\rangle$, $\langle \alpha_2^1, M | \widehat{Q}^{\mathrm{AL}}_v | \alpha_1^1, M\rangle$ and $\langle \alpha_i^1, M | \widehat{Q}^{\mathrm{AL}}_v | \alpha_1^1, M\rangle$ in the case of $\widehat{O}_1 = \widehat{O}_2 = \widehat{Q}^{\mathrm{AL}}_v$ with i being 2, 3, 4 in section 6.5, we took advantage of the fact that the intermediate coupling a_2 of $|\alpha_1^1, M\rangle$ is identical to zero and therefore we have $J_{e_1} = -J_{e_2}$. Furthermore, the orientations of the two triples $\{e_1, e_3, e_4\}$, $\{e_2, e_3, e_4\}$ were exactly opposite to each other. Accordingly, we only needed to consider one triple, e.g. $\widehat{q}_{134}$, and multiply the result by a factor of 2, because the second triple had exactly the same contribution as the first one. Now, in contrast, we will also have to consider matrix elements where the incoming state has intermediate couplings a_2 different from zero. Consequently, in these cases we will have to consider the contribution of the second triple exactly, as it might not just be a trivial factor of 2.

By comparing the formulae for general matrix elements of $\widehat{q}_{134}$ in equation (6.6) and of $\widehat{q}_{234}$ in equation (6.7), respectively, we note that the only difference between these two formulae is due to different pre-factors in the square brackets in front of the $6j$-symbols. Before we can actually calculate the matrix elements, we have to know how the corresponding Hilbert space looks like.

D.1. Matrix elements for the case of a spin-$\frac{1}{2}$ representation

Let us begin with the case $\ell = \frac{1}{2}$ and a total angular momentum $J = 0$. From equation (6.1) we can easily extract the basis states of this Hilbert space

$$\begin{aligned} |\alpha_1^0, 0\rangle &:= |a_1 = j a_2 = 0\; a_3 = \tfrac{1}{2} J = 0\rangle \\ |\alpha_2^0, 0\rangle &:= |a_1 = j a_2 = 1\; a_3 = \tfrac{1}{2} J = 0\rangle. \end{aligned} \qquad \text{(D.1)}$$

With the Hilbert space being only two dimensional and the fact that $\widehat{Q}_v^{\rm AL}$ and consequently $\widehat{q}_{134}, \widehat{q}_{234}$ are antisymmetric, we know that $\langle\alpha_2^0, 0|\widehat{Q}_v^{\rm AL}|\alpha_1^0, 0\rangle = -\langle\alpha_1^0, 0|\widehat{Q}_v^{\rm AL}|\alpha_2^0, 0\rangle$ are the only non-vanishing matrix elements. Moreover, we have $\epsilon(e_1, e_3, e_4)\langle\alpha_2^0, 0|\widehat{q}_{134}|\alpha_1^0, 0\rangle = \epsilon(e_2, e_3, e_4)\langle\alpha_2^0, 0|\widehat{q}_{234}|\alpha_1^0, 0\rangle$, because the intermediate coupling a_2 of $|\alpha_1^0, 0\rangle$ is zero. Therefore, we obtain the following matrix structure for $\widehat{Q}_v^{\rm AL}$,

$$\widehat{Q}_{\rm AL}^{J=0} = \begin{pmatrix} 0 & -{\rm i}a \\ {\rm i}a & 0 \end{pmatrix}, \tag{D.2}$$

where $a := \left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\frac{1}{2}\sqrt{j(j+1)}$. We labelled the rows of $\widehat{Q}_{\rm AL}^{J=0}$ by $|\alpha_i^J, M\rangle$ and columns by $|\alpha_{\tilde{i}}^J, M\rangle$. The eigenvalues of $\widehat{Q}_v^{\rm AL}$ are given by $\lambda_1 = -a, \lambda_2 = +a$ with corresponding eigenvectors $\vec{v}_1 = (i, 1), \vec{v}_2 = (-i, 1)$. Hence, for $\widehat{V}_{\rm AL}^2$ we have one degenerated eigenvalue $\lambda = |a|$ and the two corresponding eigenvector components $\vec{v}_1, \vec{v}_2$ in the basis $\{|\alpha_1^0, 0\rangle, |\alpha_2^0, 0\rangle\}$. The matrix element $\langle\alpha_2^0, 0|\widehat{V}_{\rm AL}^2|\alpha_1^0, 0\rangle$ is thus given by

$$\langle\alpha_2^0, 0|\widehat{V}_{\rm AL}^2|\alpha_1^0, 0\rangle = \sum_{k=1}^{2}\langle\alpha_2^0, 0|\widehat{V}_{\rm AL}^2|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^0, 0\rangle = 0. \tag{D.3}$$

Here the vectors $\vec{e}_k$ denote the corresponding normed eigenvectors of $\widehat{V}_{\rm AL}^2$. The surprising issue is that in contrast to the matrix element of $\widehat{Q}_v^{\rm AL}$, the analogous matrix element of $\widehat{V}_{\rm AL}^2$ vanishes. In this special situation where we chose $\ell = \frac{1}{2}$ and the total angular momentum J to be zero, we realize that $\widehat{Q}_v^{\rm AL}$ purely consists of off-diagonal entries, while $\widehat{V}_{\rm AL}^2$ is a diagonal matrix. In order to calculate the remaining matrix elements, we have to consider the Hilbert space for a total angular momentum of $J = 1$ in the case of $\ell = \frac{1}{2}$. Because we consider the special case of $\ell = \frac{1}{2}$ the intermediate coupling $a_3 = \ell - 1$ is not sensible here. Therefore the matrix element $\langle\alpha_2^1, M|\widehat{V}_{\rm AL}^2|\alpha_1^1, M\rangle$ does not exist. Thus, we will only have two remaining matrix elements. Using equation (6.2) for this purpose, we end up with a 4×3-dimensional Hilbert space,

$$\begin{aligned} |\alpha_1^1, M\rangle &:= |a_1 = j\ a_2 = 0\ a_3 = \tfrac{1}{2} J = 1\rangle \\ |\alpha_3^1, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = \tfrac{1}{2} J = 1\rangle \\ |\alpha_4^1, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = \tfrac{3}{2} J = 1\rangle \\ |\alpha_5^1, M\rangle &:= |a_1 = j\ a_2 = 2\ a_3 = \tfrac{3}{2} J = 1\rangle, \end{aligned} \tag{D.4}$$

where we skipped number 2 in labelling the states in order to keep our notation consistent with the former calculations. With the states $|\alpha_1^1, M\rangle$ being orthogonal for different values of the magnetic quantum number M and the knowledge that $\widehat{Q}_v^{\rm AL}$ does not change the magnetic quantum number, we can treat the calculation separately for each value of $M = \{-1, 0, 1\}$. Furthermore, we know that the result is equal for each value of M.

Thus, we have a 4×4 matrix, but as $\widehat{Q}_v^{\rm AL}$ is antisymmetric, its diagonal entries are zero and $\left(\widehat{Q}_v^{\rm AL}\right)_{AB} = -\left(\widehat{Q}_v^{\rm AL}\right)_{BA}$. Hence, we only have to calculate six different matrix elements. Two out of these six matrix elements have already been calculated before and can be extracted from equation (6.29) if we set $\ell = \frac{1}{2}$. For both matrix elements, we have $\epsilon(e_1, e_3, e_4)\widehat{q}_{134} = \epsilon(e_2, e_3, e_4)\widehat{q}_{234}$, so that the contribution coming from the second triple is again only a factor of 2. So, we are left with four matrix elements that have to be evaluated, namely $\langle\alpha_5^1, M|\widehat{Q}_v^{\rm AL}|\alpha_1^1, M\rangle, \langle\alpha_4^1, M|\widehat{Q}_v^{\rm AL}|\alpha_3^1, M\rangle, \langle\alpha_5^1, M|\widehat{Q}_v^{\rm AL}|\alpha_3^1, M\rangle$ and $\langle\alpha_5^1, M|\widehat{Q}_v^{\rm AL}|\alpha_4^1, M\rangle$. By simply looking at equations (6.7) and (6.6) we see that $\langle\alpha_5^1, M|\widehat{Q}_v^{\rm AL}|\alpha_1^1, M\rangle$ vanishes, because $\widetilde{a}_2 - a_2 = 2$ here and therefore the $6j$-symbols in the corresponding square brackets will be zero. Consequently, the matrix elements of $\widehat{q}_{134}$ and $\widehat{q}_{234}$ are zero and thus the corresponding matrix elements of $\widehat{Q}_v^{\mathrm{AL}}$ disappear. In the case of $\langle\alpha_4^1, M|\widehat{Q}_v^{\mathrm{AL}}|\alpha_3^1, M\rangle$ we have $\widetilde{a}_2 = a_2$. Implementing this condition into equations (6.7) and (6.6), we realize that both equations become identical. Accordingly, we get $\epsilon(e_1, e_3, e_4)\widehat{q}_{134} = -\epsilon(e_2, e_3, e_4)\widehat{q}_{234}$ which leads to a vanishing matrix element for $\widehat{Q}_v^{\mathrm{AL}}$. With all this in mind, we end up with the following expression for $\widehat{Q}_v^{\mathrm{AL}}$,

$$\widehat{Q}_{\mathrm{AL}}^{J=1} = \begin{pmatrix} 0 & +\mathrm{i}a & -\mathrm{i}\sqrt{2}a & 0 \\ -\mathrm{i}a & 0 & 0 & +\frac{ib}{\sqrt{2}} \\ +\mathrm{i}\sqrt{2}a & 0 & 0 & -\mathrm{i}b \\ 0 & -\frac{ib}{\sqrt{2}} & +\mathrm{i}b & 0 \end{pmatrix}, \tag{D.5}$$

where we defined $a := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right)^{\frac{2}{3}} \sqrt{j(j+1)}$ and $b := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right)^{\frac{2}{3}} \sqrt{4j(j+1)-3}$. In this case, the eigenvalues of $\widehat{Q}_v^{\mathrm{AL}}$ are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -\lambda_4 = -\sqrt{\frac{3}{2}}\sqrt{2a^2+b^2} =: -\lambda$. The corresponding eigenvectors are given by

$$\begin{aligned} \vec{v}_1 &= \left(\frac{b}{\sqrt{2}a}, 0, 0, 1\right) & \vec{v}_3 &= (0, \sqrt{2}, 0, 1) \\ \vec{v}_4 &= \frac{1}{b}\left(-\sqrt{2}a, -\mathrm{i}\frac{\sqrt{2}}{3}\lambda, +\mathrm{i}\lambda, b\right) & \vec{v}_5 &= \frac{1}{b}\left(-\sqrt{2}a, +\mathrm{i}\frac{\sqrt{2}}{3}\lambda, -\mathrm{i}\lambda, b\right). \end{aligned} \tag{D.6}$$

With the first two eigenvalues being identical to zero, we do not have to consider them when we calculate the matrix element of $\widehat{V}^2$. Hence, we obtain

$$\langle\alpha_3^1, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^1, M\rangle = \sum_{k=3}^{4} \langle\alpha_3^1, M|\widehat{V}_{\mathrm{AL}}^2|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0 \tag{D.7}$$

$$\langle\alpha_4^1, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^1, M\rangle = \sum_{k=3}^{4} \langle\alpha_4^1, M|\widehat{V}_{\mathrm{AL}}^2|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0. \tag{D.8}$$

As in the case of $J = 0$ all matrix elements that occur in the action of the operator $\frac{1}{2}\widehat{\widetilde{E}}_{k,\mathrm{tot}}^{\mathrm{I,AL}}(S_t)$ vanish. Consequently, the whole matrix element $\langle\beta^{\widetilde{j}_{12}}, \widetilde{m}_{12}|\frac{1}{2}\widehat{\widetilde{E}}_{k,\mathrm{tot}}^{\mathrm{I,AL}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ vanishes.

In order to see whether the vanishing of $\langle\beta^{\widetilde{j}_{12}}, \widetilde{m}_{12}|\frac{1}{2}\widehat{\widetilde{E}}_{k,\mathrm{tot}}^{\mathrm{I,AL}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ is somehow connected with the fact that we chose the most easiest case where $\ell = \frac{1}{2}$, we will investigate the matrix elements for the case of $\ell = 1$ as well.

D.2. Matrix elements for the case of a spin-1 representation

In the case when both additional edges carry a spin-1 representation ($\ell = 1$), the Hilbert space belonging to a total angular momentum $J = 0$ is three dimensional,

$$\begin{aligned} |\alpha_1^0, 0\rangle &:= |a_1 = j\ a_2 = 0\ a_3 = 1 J = 0\rangle \\ |\alpha_2^0, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = 1 J = 0\rangle \\ |\alpha_3^0, M\rangle &:= |a_1 = j\ a_2 = 2\ a_3 = 1 J = 0\rangle. \end{aligned} \tag{D.9}$$

Again, the matrix element $\langle\alpha_3^0, 0|\widehat{Q}|\alpha_1^0, 0\rangle$ vanishes, because $\Delta a_2 := |\widetilde{a}_2 - a_2| > 1$. Consequently, the $6j$-symbols including $\widetilde{a}_2$ and a_2 become zero. Thus the whole matrix element is zero. Considering the matrix element $\langle\alpha_3^0, 0|\widehat{Q}_v^{\mathrm{AL}}|\alpha_2^0, 0\rangle$ we see that we have $\widetilde{a}_3 = a_3$ and $\widetilde{a}_2 = a_2 + 1$ here. Inserting this into equations (6.7) and (6.6), we get

$\langle\alpha_3^0, 0|\widehat{q}_{134}|\alpha_1^0, 0\rangle = -\langle\alpha_3^0, 0|\widehat{q}_{234}|\alpha_1^0, 0\rangle$, so that we only have to calculate one of the triples and multiply it by 2. Hence, the operator $\widehat{Q}_v^{\mathrm{AL}}$ can be described by the following matrix,

$$\widehat{Q}^{J=0} = \begin{pmatrix} 0 & -\mathrm{i}a & 0 \\ +\mathrm{i}a & 0 & -\mathrm{i}b \\ 0 & -\mathrm{i}b & 0 \end{pmatrix}, \tag{D.10}$$

where $a := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right) \frac{4}{\sqrt{3}} \sqrt{2}\sqrt{j(j+1)}$ and $b := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right) \frac{4}{\sqrt{3}} \sqrt{4j(j+1)-3}$. The eigenvalues are given by $\lambda_1 = 0$, $\lambda_2 = -\sqrt{a^2+b^2}$, $\lambda_3 = +\sqrt{a^2+b^2} =: \lambda$. The corresponding eigenvectors can be found in the equation below:

$$\vec{v}_1 = \left(\frac{b}{\sqrt{a}}, 0, 1\right) \qquad \vec{v}_2 = \left(-\frac{a}{b}, \frac{\mathrm{i}}{b}\lambda, 1\right) \qquad \vec{v}_3 = \left(-\frac{a}{b}, -\frac{\mathrm{i}}{b}\lambda, 1\right). \tag{D.11}$$

With the eigenvectors again having only purely real and purely imaginary entries, we can again guess that the matrix elements of $\widehat{V}_{\mathrm{AL}}^2$ will vanish. This is indeed the case, as can be seen in the following lines:

$$\langle\alpha_2^0, 0|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^0, 0\rangle = \sum_{k=2}^{3} \langle\alpha_2^0, 0|\widehat{V}_{\mathrm{AL}}^2|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^0, 0\rangle = 0. \tag{D.12}$$

Thus, as long as we have eigenvectors that do have only purely imaginary and purely real components and we are furthermore forced to consider matrix element of $\langle\alpha_i^J, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^J, M\rangle$ such that one of the states has an imaginary expansion coefficient in terms of the eigenvectors, while the other has a real expansion coefficient, we will obtain a vanishing matrix element for $\widehat{V}_{\mathrm{AL}}^2$. Note that this is not the case for the operator $\widehat{Q}_v^{\mathrm{AL}}$, because there the eigenvectors have different eigenvalues $+\lambda$ and $-\lambda$. Accordingly, the corresponding terms would not be cancelled by each other, but would just be added up.

Let us consider a total angular momentum of $J = 1$ now and investigate whether we will get the same behaviour of the eigenvectors as well. For $J = 1$ the associated Hilbert space is already (7×3) dimensional:

$$\begin{aligned}
|\alpha_1^1, M\rangle &:= |a_1 = j\ a_2 = 0\ a_3 = 1\ J = 1\rangle \\
|\alpha_2^1, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = 0\ J = 1\rangle \\
|\alpha_3^1, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = 1\ J = 1\rangle \\
|\alpha_4^1, M\rangle &:= |a_1 = j\ a_2 = 1\ a_3 = 2\ J = 1\rangle \\
|\alpha_5^1, M\rangle &:= |a_1 = j\ a_2 = 2\ a_3 = 1\ J = 1\rangle \\
|\alpha_6^1, M\rangle &:= |a_1 = j\ a_2 = 2\ a_3 = 2\ J = 1\rangle \\
|\alpha_7^1, M\rangle &:= |a_1 = j\ a_2 = 3\ a_3 = 2\ J = 1\rangle.
\end{aligned} \tag{D.13}$$

In order to minimize the amount of computation, we will discuss some particular matrix elements in advance, especially those for which we can read off the result easily from equations (6.7) and (6.6). With a_2 being zero for $|\alpha_1^1, M\rangle$, we know that for all matrix elements

$$\epsilon(e_1, e_3, e_4)\langle\alpha_i^1, M|\widehat{q}_{134}|\alpha_1^1, M\rangle = \epsilon(e_2, e_2, e_3)\langle\alpha_i^1, M|\widehat{q}_{234}|\alpha_1^1, M\rangle, \quad i = 2, \ldots, 7. \tag{D.14}$$

Furthermore, we have $\langle\alpha_i^1, M|\widehat{q}_{134}|\alpha_1^1, M\rangle = 0$ for $i > 4$, because then $\Delta a_2 = |\widetilde{a}_2 - a_2| > 1$. For the same reason the matrix elements $\langle\alpha_7^1, M|\widehat{Q}_v^{\mathrm{AL}}|\alpha_i^1, M\rangle = 0$ with $i = 1, \ldots, 4$ vanish. As $\Delta a_3 = |\widetilde{a}_3 - a_3| > 1$ for $\langle\alpha_i^1, M|\widehat{Q}_v^{\mathrm{AL}}|\alpha_2^1, M\rangle = 0$ for $i = 4, 6, 7$ these matrix elements are zero as well. Then we can find several matrix elements where $\widetilde{a}_2 = a_2$ and $\widetilde{a}_3 = a_3 + 1$. In this case we get $\epsilon(e_1, e_3, e_4)\langle\alpha_i^1, M|\widehat{q}_{134}|\alpha_{i+1}^1, M\rangle = -\epsilon(e_2, e_2, e_3)\langle\alpha_{i+1}^1, M|\widehat{q}_{234}|\alpha_i^1, M\rangle$ with $i = 2, 3, 5$ and consequently the matrix element of $\widehat{Q}_v^{\mathrm{AL}}$ is zero for this particular combination of states $|\alpha_i^1, M\rangle$. Considering these arguments, we obtain the following kind of matrix for $\widehat{Q}_v^{\rm AL}$,

$$\widehat{Q}_{\rm AL}^{J=1} = \begin{pmatrix} 0 & -{\rm i}\frac{8}{3}\sqrt{2}a & -{\rm i}2\sqrt{\frac{2}{3}}a & -{\rm i}\frac{2}{3}\sqrt{10}a & 0 & 0 & 0 \\ +{\rm i}\frac{8}{3}\sqrt{2}a & 0 & 0 & 0 & +{\rm i}\frac{4}{3}b & 0 & 0 \\ +{\rm i}2\sqrt{\frac{2}{3}}a & 0 & 0 & 0 & -{\rm i}\frac{2}{\sqrt{3}}b & 0 & 0 \\ +{\rm i}\frac{2}{3}\sqrt{10}a & 0 & 0 & 0 & -{\rm i}\frac{4}{3\sqrt{5}}b & -{\rm i}2\sqrt{\frac{3}{5}}b & 0 \\ 0 & -{\rm i}\frac{4}{3}b & +\frac{2}{\sqrt{3}}b & +{\rm i}\frac{4}{3\sqrt{5}}b & 0 & 0 & +{\rm i}2\sqrt{\frac{6}{5}}c \\ 0 & 0 & 0 & +{\rm i}2\sqrt{\frac{3}{5}}b & 0 & 0 & -{\rm i}6\sqrt{\frac{2}{5}}c \\ 0 & 0 & 0 & 0 & -{\rm i}2\sqrt{\frac{6}{5}}c & +{\rm i}6\sqrt{\frac{2}{5}}c & 0 \end{pmatrix}, \tag{D.15}$$

where we introduced

$$a := \left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{j(j+1)} \qquad b := \left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{4j(j+1)-3}$$
$$c := \left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{j(j+1)-2}. \tag{D.16}$$

The seven eigenvalues of $\widehat{Q}_{\rm AL}^{J=1}$ are

$$\lambda_1 = 0$$
$$\lambda_2 = -2\left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{4j(j+1)-3} = -\lambda_3$$
$$\lambda_4 = -\left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{24j(j+1) - 2\left(11 + \sqrt{121 + 8j(j+1)(2j(j+1)-5)}\right)} = -\lambda_5$$
$$\lambda_6 = -\left(\ell_p^6 \frac{3!}{2} C_{\rm reg}\right)\sqrt{2}\sqrt{12j(j+1) - 11 + \sqrt{121 + 8j(j+1)(2j(j+1)-5)}} = -\lambda_7. \tag{D.17}$$

The corresponding eigenvectors can be given in the following form:

$$\begin{aligned} \vec{v}_1 &= (0, 0, \gamma_1, \delta_1, 0, 0, 1) \\ \vec{v}_2 &= (0, -{\rm i}\beta_2, +{\rm i}, +{\rm i}\delta_2, \epsilon_2, 1, 0) \\ \vec{v}_3 &= (0, +{\rm i}\beta_2, -{\rm i}, -{\rm i}\delta_2, \epsilon_2, 1, 0) \\ \vec{v}_4 &= (+{\rm i}\alpha_3, \beta_3, \gamma_3, \delta_3, -{\rm i}\epsilon_3, +{\rm i}\phi, 1) \\ \vec{v}_5 &= (-{\rm i}\alpha_3, \beta_3, \gamma_3, \delta_3, +{\rm i}\epsilon_3, -{\rm i}\phi, 1) \\ \vec{v}_6 &= (+{\rm i}\alpha_4, \beta_4, \gamma_4, \delta_4, -{\rm i}\epsilon_4, +{\rm i}\phi, 1) \\ \vec{v}_7 &= (-{\rm i}\alpha_4, \beta_4, \gamma_4, \delta_4, +{\rm i}\epsilon_4, -{\rm i}\phi, 1). \end{aligned} \tag{D.18}$$

All Greek letters appearing in the equation above represent real numbers. Our aim is to calculate the matrix elements $\langle\alpha_i^1, M|\widehat{V}_{\rm AL}^2|\alpha_1^1, M\rangle$ where $i = 2, 3, 4$. Hence, similar to the calculations before we have to expand the states $|\alpha_i^1, M\rangle$ in terms of the eigenvectors $\vec{e}_k$ that are the normed versions of the vectors $\vec{v}_k$. But only by looking at the structure of the eigenvectors $\vec{e}_k$ we can already read off that the three matrix elements will vanish for the following reasons. First of all, the first three eigenvectors do not contribute to the matrix elements at all, because their expansion coefficient for $|\alpha_1^1, M\rangle$ is zero. Additionally, we have $\vec{e}_4^* = \vec{e}_5$ and $\vec{e}_6^* = \vec{e}_7$. As the expansion coefficient for $|\alpha_1^1, M\rangle$ is purely imaginary, while the one for $|\alpha_i^1, M\rangle$ with $i = 2, 3, 4$ is real and, moreover, the two eigenvectors have the same eigenvalue, namely $|\lambda_3|$ and $|\lambda_5|$ respectively, the contribution of $\vec{e}_4$ cancels the contribution of $\vec{e}_5$. The same is true for $\vec{e}_6$ and $\vec{e}_7$. Accordingly, we get

$$\langle\alpha_2^1, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^1, M\rangle = \langle\alpha_3^1, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^1, M\rangle = \langle\alpha_4^1, M|\widehat{V}_{\mathrm{AL}}^2|\alpha_1^1, M\rangle = 0. \tag{D.19}$$

Unfortunately, we are not able to repeat the analysis for arbitrary spin representation ℓ, because the matrices representing $\widehat{Q}$ cannot be solved analytically anymore. Nevertheless, as the structure of the basis states in the Hilbert spaces stays the same, only the amount of states is changed, we would guess that the eigenvalues and eigenvectors look analogous also in the general case. Hence, we would expect a vanishing of the matrix elements of $\widehat{V}_{\mathrm{AL}}^2$ that are contained in $\langle\beta^{\widetilde{j}_{12}}, \widetilde{m}_{12}|^1\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ and therefore expect that the result of $\langle\beta^{\widetilde{j}_{12}}, \widetilde{m}_{12}|^1\widehat{\widetilde{E}}^{\mathrm{I,AL}}_{k,\mathrm{tot}}(S_t)|\beta^{j_{12}}, m_{12}\rangle$ is zero. In any case since the choice of ℓ should not be important in the semi-classical limit of large j, we can rule out the choice $\widehat{V}_{\mathrm{AL}}^2$ already based on the result of the present section.

Appendix E. Detailed calculation in the case of the volume operator $\widehat{V}_{\mathrm{RS}}$ introduced by Rovelli and Smolin

As the first step, we will derive the explicit expressions of the operators $\widehat{O}_1$ and $\widehat{O}_2$ in the case of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$, shown in equation (6.12) and in equation (6.38), respectively. Let us begin with ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$. Apart from some pre-factors including numbers, which are not important for our argument, the precise expression of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ is given by a sum consisting of eight terms,

$$\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t) \propto\, & \pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\big(+\widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{V}_{\mathrm{RS}}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{\pi}_\ell \\
&\times\big(h_{e_3}\big)_{EA} + \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} \\
&- \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA} \\
&- \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA}\big) \\
&- \pi_\ell(\epsilon)_{FB}\big(+\widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{IG}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{EA}\widehat{V}_{\mathrm{RS}}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{CA} \\
&+ \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{IG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{EA}\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_3}\big)_{CA} \\
&- \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{IG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{EA}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h_{e_3}\big)_{CA} \\
&- \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{IG}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{EA}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{FG}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_3}\big)_{CA}\big)\big].
\end{aligned} \tag{E.1}$$

As mentioned before, the volume operator $\widehat{V}_{\mathrm{RS}}$ is the sum of the contributing triples

$$\widehat{V}_{\mathrm{RS}} = \widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{123}} + \widehat{V}_{q_{124}}. \tag{E.2}$$

Recall that the SNF $|\beta^{j_{12}}, m_1\rangle$ consists of two edges e_1, e_2 only. Therefore if $\widehat{V}_{\mathrm{RS}}$ acts before, for instance, $\widehat{\pi}_\ell\big(h_{e_4}\big)$ acts, the only non-vanishing contribution in $\widehat{V}_{\mathrm{RS}}$ is due to $\widehat{V}_{q_{123}}$, because for $\widehat{V}_{q_{134}}$, $\widehat{V}_{q_{234}}$ and $\widehat{V}_{q_{124}}$ the edge e_4 is missing. Analogously, only $\widehat{V}_{q_{124}}$ contributes to $\widehat{V}_{\mathrm{RS}}$ when the latter is applied to $|\beta^{j_{12}}, m_1\rangle$ before $\widehat{\pi}_\ell\big(h_{e_3}\big)$ has acted. Consequently, equation (E.1) reduces to

$$\begin{aligned}
{}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t) \propto\, & \pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\big(+\widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{V}_{\mathrm{RS}}\widehat{V}_{\mathrm{RS}}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{\pi}_\ell \\
&\times\big(h_{e_3}\big)_{EA} + \widehat{\pi}_\ell\big(h^\dagger_{e_4}\big)_{FG}\widehat{V}_{q_{124}}\widehat{\pi}_\ell\big(h^\dagger_{e_3}\big)_{BA}\widehat{\pi}_\ell\big(h_{e_4}\big)_{IG}\widehat{V}_{q123}\widehat{\pi}_\ell\big(h_{e_3}\big)_{EA}
\end{aligned}$$

$$
\begin{aligned}
&-\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{V}_{q124}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\\
&-\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{V}_{q123}\widehat{\pi}_\ell(h_{e_3})_{EA})\\
&-\pi_\ell(\epsilon)_{FB}\big(+\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{\rm RS}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&+\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{V}_{q123}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{V}_{q124}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&-\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{V}_{q123}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&-\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{V}_{q124}\widehat{\pi}_\ell(h_{e_4})_{CA}\big)\big]. \qquad \text{(E.3)}
\end{aligned}
$$

Furthermore, $\widehat{V}_{q123}$ commutes with $\widehat{\pi}_\ell(h_{e_4})$ as well as $\widehat{V}_{q124}$ commutes with $\widehat{\pi}_\ell(h_{e_3})$. Using this, we get

$$
\begin{aligned}
{}^{(\ell)}\widehat{E}^{\rm I,RS}_{k,\rm tot}(S_t) \propto\ & \pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\big(+\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{\rm RS}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell\\
&\times(h_{e_3})_{EA}+\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{q124}\widehat{V}_{q123}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\\
&-\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{q124}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\\
&-\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{V}_{\rm RS}\widehat{V}_{q123}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\big)\\
&-\pi_\ell(\epsilon)_{FB}\big(+\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{\rm RS}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&+\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{q123}\widehat{V}_{q124}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&-\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{q123}\widehat{V}_{\rm RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\\
&-\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{V}_{\rm RS}\widehat{V}_{q124}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\big)\big]\\
=\ &\pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}(\widehat{V}_{\rm RS}\widehat{V}_{\rm RS}+\widehat{V}_{q124}\widehat{V}_{q123}\\
&-\widehat{V}_{q124}\widehat{V}_{\rm RS}-\widehat{V}_{\rm RS}\widehat{V}_{q123})\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}-\pi_\ell(\epsilon)_{FB}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\\
&\times(\widehat{V}_{\rm RS}\widehat{V}_{\rm RS}+\widehat{V}_{q123}\widehat{V}_{q124}-\widehat{V}_{q123}\widehat{V}_{\rm RS}-\widehat{V}_{\rm RS}\widehat{V}_{q124})\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\big]\\
=\ &\pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{O}_1^{\rm I,RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\\
&-\pi_\ell(\epsilon)_{FB}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{O}_2^{\rm I,RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\big], \qquad \text{(E.4)}
\end{aligned}
$$

whereby we used the definition of $\widehat{O}_1^{\rm I,RS}$ and $\widehat{O}_2^{\rm I,RS}$ from equation (6.12) as well as the definition of $\widehat{V}_{\rm RS}$ in equation (E.2) in the last step. The calculation for ${}^{(\ell)}\widehat{E}^{\rm II,RS}_{k,\rm tot}(S_t)$ is similar to the small difference that the sign operator $\widehat{\mathcal{S}}$ is sandwiched between the two volume operators $\widehat{V}_{\rm RS}$. Hence, we end up with

$$
\begin{aligned}
{}^{(\ell)}\widehat{E}^{\rm II,RS}_{k,\rm tot}(S_t) \propto\ & \pi_\ell(\tau_k)_{BC}\pi_\ell(\epsilon)_{EI}\big[+\pi_\ell(\epsilon)_{FC}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{FG}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{BA}\widehat{O}_1^{\rm II,RS}\widehat{\pi}_\ell(h_{e_4})_{IG}\widehat{\pi}_\ell(h_{e_3})_{EA}\\
&-\pi_\ell(\epsilon)_{FB}\widehat{\pi}_\ell(h_{e_3}^\dagger)_{IG}\widehat{\pi}_\ell(h_{e_4}^\dagger)_{EA}\widehat{O}_2^{\rm II,RS}\widehat{\pi}_\ell(h_{e_3})_{FG}\widehat{\pi}_\ell(h_{e_4})_{CA}\big]. \qquad \text{(E.5)}
\end{aligned}
$$

Here we used the expressions for $\widehat{O}_1^{\rm II,RS}$ and $\widehat{O}_2^{\rm II,RS}$ in equation (6.38).

E.1. Case ${}^{(\ell)}\widehat{E}^{\rm I,RS}_{k,\rm tot}(S_t)$: detailed calculation of the matrix elements of $\widehat{O}_1^{\rm I,RS}$ and $\widehat{O}_2^{\rm I,RS}$

As in section appendix D we will investigate the case of a spin representation $\ell = \frac{1}{2}, 1$ for the reason that these are the two easiest cases where the matrices of $\widehat{Q}^{\rm RS}_{v,IJK}$ can still be solved analytically. Here, we keep the discussion succinct and mainly present our results, for the reason that section appendix D already explains in a quite detailed way how matrix elements of the volume operator are actually calculated.

E.1.1. Matrix elements for the case of a spin-$\frac{1}{2}$ representation. With $\ell = \frac{1}{2}$, the matrix elements $\langle\alpha^0_M, 0|\widehat{O}^{\mathrm{I,RS}}_1|\alpha^0_1, 0\rangle$ and $\langle\alpha^1_i, M|\widehat{O}^{\mathrm{I,RS}}_2|\alpha^1_1, M\rangle$, where $i = 3, 4$, contribute to the matrix element of ${}^{(\ell)}\widehat{\widetilde{E}}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$. The matrix elements are given by

$$\begin{aligned}\langle\alpha^0_2, M|\widehat{O}^{I,RS}_1|\alpha^0_1, M\rangle &= +\langle\alpha^0_2, M|\widehat{V}^2_{q_{134}}|\alpha^0_1, M\rangle + \langle\alpha^0_2, M|\widehat{V}^2_{q_{234}}|\alpha^0_1, M\rangle \\ &\quad + \langle\alpha^0_2, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{234}}|\alpha^0_1, M\rangle + \langle\alpha^0_2, M|\widehat{V}_{q_{234}}\widehat{V}_{q_{134}}|\alpha^0_1, M\rangle \\ &\quad + \langle\alpha^0_2, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{123}}|\alpha^0_1, M\rangle + \langle\alpha^0_2, M|\widehat{V}_{q_{124}}\widehat{V}_{q_{134}}|\alpha^0_1, M\rangle \\ &\quad + \langle\alpha^0_2, M|\widehat{V}_{q_{234}}\widehat{V}_{q_{123}}|\alpha^0_1, M\rangle + \langle\alpha^0_2, M|\widehat{V}_{q_{124}}\widehat{V}_{q_{234}}|\alpha^0_1, M\rangle \\ &\quad + \langle\alpha^0_2, M|\widehat{V}_{q_{124}}\widehat{V}_{q_{123}}|\alpha^0_1, M\rangle\end{aligned} \tag{E.6}$$

and with $i = 3, 4$,

$$\begin{aligned}\langle\alpha^1_i, M|\widehat{O}^{I,RS}_2|\alpha^1_1, M\rangle &= +\langle\alpha^1_i, M|\widehat{V}^2_{q_{134}}|\alpha^1_1, M\rangle + \langle\alpha^1_i, M|\widehat{V}^2_{q_{234}}|\alpha^1_1, M\rangle \\ &\quad + \langle\alpha^1_i, M|\widehat{V}_{q_{234}}\widehat{V}_{q_{134}}|\alpha^1_1, M\rangle + \langle\alpha^1_i, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{234}}|\alpha^1_1, M\rangle \\ &\quad + \langle\alpha^1_i, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{134}}|\alpha^1_1, M\rangle + \langle\alpha^1_i, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle \\ &\quad + \langle\alpha^1_i, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{234}}|\alpha^1_1, M\rangle + \langle\alpha^1_i, M|\widehat{V}_{q_{234}}\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle \\ &\quad + \langle\alpha^1_i, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle.\end{aligned} \tag{E.7}$$

Here we used the definitions of the operators $\widehat{O}^{I,RS}_1$, $\widehat{O}^{I,RS}_2$ in equation (6.12). These matrix elements for $\widehat{O}^{I,RS}_1$ consist of the sum of the matrix elements with the following structure,

$$\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha^0_1, M\rangle = \sum_{|\alpha'\rangle}\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\alpha'\rangle\langle\alpha'|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha^0_1, M\rangle, \tag{E.8}$$

where we expanded the matrix element in terms of basis vectors $|\alpha'\rangle$ of the Hilbert space $\mathcal{H}^{J=0}$.

Now each $\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\alpha'\rangle$ can be calculated through an eigenvector expansion

$$\begin{aligned}\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\alpha'\rangle &= \sum_k\langle\alpha^0_2, M|\widehat{V}_{q_{IJK}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha'\rangle \\ \langle\alpha'|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha^0_1, M\rangle &= \sum_k\langle\alpha'|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^0_1, M\rangle\end{aligned} \tag{E.9}$$

where $\vec{e}_k$ denotes the normed eigenvectors of $\widehat{V}_{q_{IJK}}$. As in section appendix D, the eigenvectors of $\widehat{V}_{q_{IJK}}$ are equal to the eigenvectors of $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$, and $\sqrt{|\lambda|}$ is an eigenvalue of $\widehat{V}_{q_{IJK}}$ assuming that the corresponding eigenvalue of $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ is λ. (See also the discussion in section 4.1.)

The states $|\alpha^0_i, M\rangle$ that have to be taken into account in order to derive the matrix of $\widehat{Q}^{\mathrm{RS}}_{v,IJK}$ can be found in equation (D.1). Using equations (6.6)–(6.9) we get the following matrices, eigenvalues and eigenvectors for a total angular momentum $J = 0$,

$$\begin{aligned}&\widehat{Q}^{J=0}_{\mathrm{RS},134} = \begin{pmatrix} 0 & -2\mathrm{i}a \\ +2\mathrm{i}a & 0\end{pmatrix}, \quad \lambda_1 = -2a = -\lambda_2, \quad \vec{v}_1 = (= +\mathrm{i}, 1), \quad \vec{v}_2 = (-\mathrm{i}, 1) \\ &\widehat{Q}^{J=0}_{\mathrm{RS},234} = \widehat{Q}^{J=0}_{\mathrm{RS},123} = \widehat{Q}^{J=0}_{\mathrm{RS},124} = \begin{pmatrix} 0 & +2\mathrm{i}a \\ -2\mathrm{i}a & 0\end{pmatrix}, \quad \lambda_1 = -2a = -\lambda_2, \quad \vec{v}_1 = (= -\mathrm{i}, 1), \\ &\qquad \vec{v}_2 = (+\mathrm{i}, 1),\end{aligned} \tag{E.10}$$

where we defined $a := \left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right)\sqrt{j(j+1)}$ and labelled the rows by $\left|\alpha_i^J, M\right\rangle$, whereas columns are labelled by $\left|\alpha_{\tilde{i}}^J, M\right\rangle$. Inserting these eigenvectors above into equation (E.9), yields to vanishing off-diagonal matrix elements of $\widehat{V}_{q_{IJK}}$

$$\begin{aligned}
&\left\langle\alpha_2^0, 0\right|\widehat{V}_{q_{134}}\left|\alpha_1^0, 0\right\rangle = \left\langle\alpha_2^0, 0\right|\widehat{V}_{q_{234}}\left|\alpha_1^0, 0\right\rangle = \left\langle\alpha_2^0, 0\right|\widehat{V}_{q_{123}}\left|\alpha_1^0, 0\right\rangle = \left\langle\alpha_2^0, 0\right|\widehat{V}_{q_{124}}\left|\alpha_1^0, 0\right\rangle = 0 \\
&\left\langle\alpha_1^0, 0\right|\widehat{V}_{q_{134}}\left|\alpha_2^0, 0\right\rangle = \left\langle\alpha_1^0, 0\right|\widehat{V}_{q_{234}}\left|\alpha_2^0, 0\right\rangle = \left\langle\alpha_1^0, 0\right|\widehat{V}_{q_{123}}\left|\alpha_2^0, 0\right\rangle = \left\langle\alpha_1^0, 0\right|\widehat{V}_{q_{124}}\left|\alpha_2^0, 0\right\rangle = 0
\end{aligned} \tag{E.11}$$

for the reason that the expansion coefficient of $\left|\alpha_1^0, 0\right\rangle$ is purely imaginary, whereas the one of $\left|\alpha_2^0, M\right\rangle$ is real and therefore the terms appearing in the sum of the expansion will cancel each other (see also the explicit calculations done in section appendix D for this).

Accordingly, if we sum over $|\alpha'\rangle$ in equation (E.8) either the first or the second matrix element of $\widehat{V}_{q_{IJK}}$ in the product is zero. Thus each $\left\langle\alpha_2^0, M\right|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}\left|\alpha_1^0, M\right\rangle = 0$ and therefore the whole sum in equation (E.6) is equivalent to zero and we have

$$\left\langle\alpha_2^0, M\right|\widehat{O}_1^{\text{I,RS}}\left|\alpha_1^0, M\right\rangle = 0. \tag{E.12}$$

Let us discuss the matrix element $\left\langle\alpha_i^1, M\right|\widehat{O}_2^{I,RS}\left|\alpha_1^1, M\right\rangle$ from equation (E.6) now. The four states $|\alpha^1 i M\rangle$ that are included in the Hilbert space belonging to a total angular momentum of $J = 1$ are written down in equation (D.4). Inserting them into equation (6.6) leads to

$$\widehat{Q}_{\text{RS},134}^{J=1} = \begin{pmatrix} 0 & +\mathrm{i}\frac{2}{3}a & -\mathrm{i}\frac{2}{3}\sqrt{2}a & 0 \\ -\mathrm{i}\frac{2}{3}a & 0 & -\mathrm{i}\sqrt{2} & +\mathrm{i}\frac{1}{3}\sqrt{2}b \\ +\mathrm{i}\frac{2}{3}\sqrt{2}a & +\mathrm{i}\sqrt{2} & 0 & -\mathrm{i}\frac{2}{3}b \\ 0 & -\mathrm{i}\frac{1}{3}\sqrt{2}b & +\mathrm{i}\frac{2}{3}b & 0 \end{pmatrix}, \qquad \lambda_1 = 0 = \lambda_2,$$
$$\lambda_3 = -\sqrt{\frac{2}{3}}\sqrt{3 + 2a^2 + b^2} = -\lambda_4, \tag{E.13}$$

with $a := \left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right)\sqrt{j(j+1)}$, $b := \left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right)\sqrt{4j(j+1) - 3}$, while the corresponding eigenvectors are given by

$$\begin{aligned}
\vec{v}_1 &= \left(+\frac{b}{\sqrt{2}a}, 0, 0, 1\right) \\
\vec{v}_3 &= \left(-\frac{3}{\sqrt{2}a}, \sqrt{2}, 1, 0\right) \\
\vec{v}_4 &= \left(-\frac{\sqrt{2}a}{b}, \frac{-3\sqrt{2} - \mathrm{i}\sqrt{3}\sqrt{3 + 2a^2 + b^2}}{3b}, \frac{-3 + \mathrm{i}\sqrt{6}\sqrt{3 + 2a^2 + b^2}}{3b}, 1\right) \\
\vec{v}_5 &= \left(-\frac{\sqrt{2}a}{b}, \frac{-3\sqrt{2} + \mathrm{i}\sqrt{3}\sqrt{3 + 2a^2 + b^2}}{3b}, \frac{-3 - \mathrm{i}\sqrt{6}\sqrt{3 + 2a^2 + b^2}}{3b}, 1\right).
\end{aligned} \tag{E.14}$$

In contrast to the case of a total angular momentum $J = 0$ the expansion coefficients of $\left|\alpha_i^1, M\right\rangle$, where $i = 3, 4$, have a real and an imaginary part, while the one of $\left|\alpha_1^1, M\right\rangle$ is real. Consequently, we get a result different from zero here. Additionally, we show the result of $\left\langle\alpha_i^1, M\right|\widehat{V}_{q_{134}}\left|\alpha_5^1, M\right\rangle$ with $i = 3, 4$, because we need these matrix elements later when we expand $\langle\alpha_i^1, M|\widehat{O}_2^{I,RS}|\alpha_1^1, M\rangle$ in terms of $|\alpha_i^1, M\rangle$,

$$
\begin{aligned}
\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +\sqrt{|\lambda_3|}\frac{12a}{|\lambda_3|^2}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +\sqrt{|\lambda_3|}\frac{6\sqrt{2}a}{|\lambda_3|^2}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle &= \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle = -\sqrt{|\lambda_3|}\frac{6\sqrt{2}b}{|\lambda_3|^2}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle &= \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle = -\sqrt{|\lambda_3|}\frac{6b}{|\lambda_3|^2},
\end{aligned}
\tag{E.15}
$$

whereby $\vec{e}_k$ are the normed eigenvectors of $\widehat{Q}^{J=1}_{\mathrm{RS},134}$. The same is true for the triple $\widehat{q}_{234}$ in which case we have the following matrix and eigenvalues,

$$
\begin{aligned}
\widehat{Q}^{J=1}_{\mathrm{RS},234} &= \begin{pmatrix} 0 & -\mathrm{i}\frac{2}{3}a & +\mathrm{i}\frac{2}{3}\sqrt{2}a & 0\\ +\mathrm{i}\frac{2}{3}a & 0 & -\mathrm{i}\sqrt{2} & -\mathrm{i}\frac{1}{3}\sqrt{2}b\\ -\mathrm{i}\frac{2}{3}\sqrt{2}a & +\mathrm{i}\sqrt{2} & 0 & +\mathrm{i}\frac{2}{3}b\\ 0 & +\mathrm{i}\frac{1}{3}\sqrt{2}b & -\mathrm{i}\frac{2}{3}b & 0 \end{pmatrix}, \qquad \lambda_1 = 0 = \lambda_2,\\
\lambda_3 &= -\sqrt{\frac{2}{3}}\sqrt{3+2a^2+b^2} = -\lambda_4
\end{aligned}
\tag{E.16}
$$

and the corresponding eigenvectors

$$
\begin{aligned}
\vec{v}_1 &= \left(+\frac{b}{\sqrt{2}a}, 0, 0, 1\right)\\
\vec{v}_3 &= \left(+\frac{3}{\sqrt{2}a}, \sqrt{2}, 1, 0\right)\\
\vec{v}_4 &= \left(-\frac{\sqrt{2}a}{b}, \frac{3\sqrt{2}+\mathrm{i}\sqrt{3}\sqrt{3+2a^2+b^2}}{3b}, \frac{3-\mathrm{i}\sqrt{6}\sqrt{3+2a^2+b^2}}{3b}, 1\right)\\
\vec{v}_5 &= \left(-\frac{\sqrt{2}a}{b}, \frac{3\sqrt{2}-\mathrm{i}\sqrt{3}\sqrt{3+2a^2+b^2}}{3b}, \frac{3+\mathrm{i}\sqrt{6}\sqrt{3+2a^2+b^2}}{3b}, 1\right).
\end{aligned}
\tag{E.17}
$$

Accordingly, these eigenvectors yield a non-vanishing matrix element for the states $|\alpha_i^1, M\rangle$ where $i = 3, 4$ as well,

$$
\begin{aligned}
\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -\sqrt{|\lambda_3|}\frac{12a}{|\lambda_3|^2}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -\sqrt{|\lambda_3|}\frac{6\sqrt{2}a}{|\lambda_3|^2}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle &= \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle = +\sqrt{|\lambda_3|}\frac{6\sqrt{2}b}{|\lambda_3|^2}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle &= \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle = +\sqrt{|\lambda_3|}\frac{6b}{|\lambda_3|^2}.
\end{aligned}
\tag{E.18}
$$

The situation is different if we consider the triple $\widehat{q}_{123}$. In this case the matrix obtained from equation (6.8) includes more entries that are zero due to the $\delta_{\widetilde{a}_3,a_3}$ in equation (6.8),

$$\widehat{Q}^{J=1}_{\mathrm{RS},123} = \begin{pmatrix} 0 & -\mathrm{i}2a & 0 & 0 \\ +\mathrm{i}2a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}2b \\ 0 & 0 & -\mathrm{i}2b & 0 \end{pmatrix}, \quad \lambda_1 = -2a = \lambda_2, \quad \lambda_3 = -2b = -\lambda_4. \tag{E.19}$$

Therefore, the eigenvectors look much simpler

$$\vec{v}_1 = (-\mathrm{i}, 1, 0, 0), \quad \vec{v}_3 = (+\mathrm{i}, 1, 0, 0), \quad \vec{v}_4 = (0, 0, -\mathrm{i}, 1), \quad \vec{v}_5 = (0, 0, +\mathrm{i}, 1) \tag{E.20}$$

and we can easily extract from them

$$\begin{aligned}
\langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \alpha_1^1, M \rangle &= \sum_k \langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_1^1, M \rangle = 0 \\
\langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \alpha_5^1, M \rangle &= \sum_k \langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_5^1, M \rangle = 0 \\
\langle \alpha_3^1, M | \widehat{V}_{q_{123}} | \alpha_3^1, M \rangle &= \sum_k \langle \alpha_3^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_3^1, M \rangle = +\sqrt{2a} \\
\langle \alpha_3^1, M | \widehat{V}_{q_{123}} | \alpha_4^1, M \rangle &= \sum_k \langle \alpha_3^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_4^1, M \rangle = 0 \\
\langle \alpha_4^1, M | \widehat{V}_{q_{123}} | \alpha_3^1, M \rangle &= \sum_k \langle \alpha_4^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_3^1, M \rangle = 0 \\
\langle \alpha_4^1, M | \widehat{V}_{q_{123}} | \alpha_4^1, M \rangle &= \sum_k \langle \alpha_4^1, M | \widehat{V}_{q_{123}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_4^1, M \rangle = +\sqrt{2b}.
\end{aligned} \tag{E.21}$$

Here $i = 3, 4$. The matrix of the last triple can be evaluated by using equation (6.9). The matrix itself and its eigenvalues can be found in the equation below:

$$\begin{aligned}
\widehat{Q}^{J=1}_{\mathrm{RS},124} &= \begin{pmatrix} 0 & -\mathrm{i}\frac{2}{3}a & -\mathrm{i}\frac{4}{3}\sqrt{2}a & 0 \\ +\mathrm{i}\frac{2}{3}a & 0 & 0 & -\mathrm{i}\frac{4}{3}\sqrt{2}b \\ +\mathrm{i}\frac{4}{3}\sqrt{2}a & 0 & 0 & +\mathrm{i}\frac{2}{3}b \\ 0 & +\mathrm{i}\frac{4}{3}\sqrt{2}b & -\mathrm{i}\frac{2}{3}b & 0 \end{pmatrix}, \qquad \lambda_1 = 0 = \lambda_2, \\
\lambda_3 &= -\sqrt{\frac{2}{3}}\sqrt{3 + 2a^2 + b^2} = -\lambda_4.
\end{aligned} \tag{E.22}$$

The corresponding eigenvectors are given by

$$\begin{aligned}
\vec{v}_1 &= \left(+\mathrm{i}\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 1, 0\right), & \vec{v}_3 &= \left(-\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 1, 0\right), \\
\vec{v}_4 &= \left(0, +\mathrm{i}\frac{2\sqrt{2}}{3}, -\frac{\mathrm{i}}{3}, 1\right), & \vec{v}_5 &= \left(0, -\mathrm{i}\frac{2\sqrt{2}}{3}, +\frac{\mathrm{i}}{3}, 1\right).
\end{aligned} \tag{E.23}$$

In this case, either the eigenvalues or the expansion coefficients of $|\alpha_1^1, M\rangle$ and $|\alpha_5^1, M\rangle$ are zero, so that we also end up with only trivial matrix elements. Only the diagonal matrix elements of $\left|\alpha_3^1, M\right\rangle$, $\left|\alpha_4^1, M\right\rangle$ are non-vanishing:

$$
\begin{aligned}
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle &= \sum_k \left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_1^1, M\right\rangle = 0 \\
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_5^1, M\right\rangle &= \sum_k \left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_5^1, M\right\rangle = 0 \\
\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_3^1, M\right\rangle &= \sum_k \left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_3^1, M\right\rangle = +\sqrt{2}a \\
\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_4^1, M\right\rangle &= \sum_k \left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_4^1, M\right\rangle = 0 \\
\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_3^1, M\right\rangle &= \sum_k \left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_3^1, M\right\rangle = 0 \\
\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_4^1, M\right\rangle &= \sum_k \left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_4^1, M\right\rangle = +\sqrt{2}b \\
\left\langle\alpha_5^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle &= \sum_k \left\langle\alpha_5^1, M\middle|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k\middle|\alpha_1^1, M\right\rangle = 0.
\end{aligned}
\tag{E.24}
$$

After having calculated all the necessary matrix elements of $\widehat{V}_{q_{IJK}}$, we will expand each matrix element $\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}\middle|\alpha_1^1, M\right\rangle$ included in $\left\langle\alpha_i^1, M\middle|\widehat{O}_2^{I,RS}\middle|\alpha_1^1, M\right\rangle$ in terms of the basis states $\left|\alpha_j^1, M\right\rangle$,

$$\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}\middle|\alpha_1^1, M\right\rangle = \sum_{|\alpha'\rangle}\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}|\alpha'\rangle\langle\alpha'|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}\middle|\alpha_1^1, M\right\rangle. \tag{E.25}$$

Considering the operator $\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}$, where $IJK \in \{134, 234\}$. The expansion is given by

$$
\begin{aligned}
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle = &+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_1^1, M\right\rangle\left\langle\alpha_1^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_3^1, M\right\rangle\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_4^1, M\right\rangle\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_5^1, M\right\rangle\left\langle\alpha_5^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle.
\end{aligned}
\tag{E.26}
$$

We can read off from equation (E.21)

$$
\begin{aligned}
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_1^1, M\right\rangle &= \left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_5^1, M\right\rangle = \left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_4^1, M\right\rangle \\
&= \left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_3^1, M\right\rangle = 0.
\end{aligned}
\tag{E.27}
$$

Therefore the expansion reduces to

$$
\begin{aligned}
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle = &+\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_3^1, M\right\rangle\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_3^1, M\right\rangle \\
&+\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{123}}\middle|\alpha_4^1, M\right\rangle\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle.
\end{aligned}
\tag{E.28}
$$

If we choose $\tilde{I}\tilde{J}\tilde{K} = 124$ we see due to $\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle = 0$ (see equation (E.24)) that the matrix element of $\widehat{V}_{q_{123}}\widehat{V}_{q_{124}}$ vanishes. In the case of $\tilde{I}\tilde{J}\tilde{K} = 134, 234$ by comparing equation (E.15) with equation (E.18) we realize $\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{134}}\middle|\alpha_1^1, M\right\rangle = -\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{234}}\middle|\alpha_1^1, M\right\rangle = 0$. Thus, the two contributions cancel each other. Consequently,

$$\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{123}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{124}}\right)\middle|\alpha_1^1, M\right\rangle = 0. \tag{E.29}$$

In the case of operator $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}$, where $IJK \in \{134, 234\}$, the expansion in terms of the basis states $\left|\alpha_i^1, M\right\rangle$ can be written as

$$
\begin{aligned}
\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle = &+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_1^1, M\right\rangle\left\langle\alpha_1^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_3^1, M\right\rangle\left\langle\alpha_3^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_4^1, M\right\rangle\left\langle\alpha_4^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle \\
&+\left\langle\alpha_i^1, M\middle|\widehat{V}_{q_{IJK}}\middle|\alpha_5^1, M\right\rangle\left\langle\alpha_5^1, M\middle|\widehat{V}_{q_{124}}\middle|\alpha_1^1, M\right\rangle.
\end{aligned}
\tag{E.30}
$$

The matrix elements $\langle \alpha_j^1, M | \widehat{V}_{q_{124}} | \alpha_1^1, M\rangle$ with $j = 3, 4, 5$ are identical to zero, as can be seen from equation (E.24). Hence, only the first term in the expansion survives. Moreover, if we choose for $IJK = \{134, 234\}$, we have $\langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle = -\langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M\rangle$, as can be seen in equations (E.15) and (E.18), so that the non-vanishing contributions get cancelled by each other. Accordingly, we obtain

$$\langle \alpha_i^1, M | \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) \widehat{V}_{q_{124}} | \alpha_1^1, M\rangle = 0. \tag{E.31}$$

Analysing the operator $\widehat{V}_{q_{134}} \widehat{V}_{q_{IJK}}$ with $IJK \in \{134, 234\}$, we get

$$\begin{aligned}\langle \alpha_i^1, M | \widehat{V}_{q_{134}} \widehat{V}_{q_{IJK}} | \alpha_1^1, M\rangle = &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle \langle \alpha_1^1, M | \widehat{V}_{q_{IJK}} | \alpha_1^1, M\rangle \\ &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_3^1, M\rangle \langle \alpha_3^1, M | \widehat{V}_{q_{IJK}} | \alpha_1^1, M\rangle \\ &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_4^1, M\rangle \langle \alpha_4^1, M | \widehat{V}_{q_{IJK}} | \alpha_1^1, M\rangle \\ &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_5^1, M\rangle \langle \alpha_5^1, M | \widehat{V}_{q_{IJK}} | \alpha_1^1, M\rangle. \end{aligned} \tag{E.32}$$

From equations (E.15) and (E.18) we can extract for $i = 3, 4$ $\langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle = -\langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M\rangle$. Hence, the expansion above yields

$$\begin{aligned}\langle \alpha_i^1, M | \widehat{V}_{q_{134}} \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) | \alpha_1^1, M\rangle = &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle \langle \alpha_1^1, M | (\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}) | \alpha_1^1, M\rangle \\ &+ \langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_5^1, M\rangle \langle \alpha_5^1, M | (\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}) | \alpha_1^1, M\rangle. \end{aligned} \tag{E.33}$$

The same argument applies to the operator $\widehat{V}_{q_{234}} \widehat{V}_{q_{IJK}}$ with $IJK \in \{134, 234\}$. Therefore we obtain here

$$\begin{aligned}\langle \alpha_i^1, M | \widehat{V}_{q_{234}} \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) | \alpha_1^1, M\rangle = &+ \langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M\rangle \langle \alpha_1^1, M | \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) | \alpha_1^1, M\rangle \\ &+ \langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_5^1, M\rangle \langle \alpha_5^1, M | \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) | \alpha_1^1, M\rangle. \end{aligned} \tag{E.34}$$

Using the fact that $\langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle = -\langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M\rangle$ and $\langle \alpha_i^1, M | \widehat{V}_{q_{134}} | \alpha_5^1, M\rangle = -\langle \alpha_i^1, M | \widehat{V}_{q_{234}} | \alpha_5^1, M\rangle$, which can be seen by comparing equation (E.15) with equation (E.18), we get

$$\langle \alpha_i^1, M | \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) \big(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\big) | \alpha_1^1, M\rangle = 0. \tag{E.35}$$

Now, we add equations (E.29), (E.31) and (E.35) and note that the sum is precisely the operator $\widehat{O}_2^{\mathrm{I,RS}}$. Accordingly,

$$\langle \alpha_i^1, M | \widehat{O}_2^{\mathrm{I,RS}} | \alpha_1^1, M\rangle = 0 \qquad i = 3, 4. \tag{E.36}$$

Since the matrix element of $\widehat{O}_1^{\mathrm{I,RS}}$ as well as the matrix element of $\widehat{O}_2^{\mathrm{I,RS}}$ vanishes and exactly these matrix elements are the only ones that contribute to ${}^{\frac{1}{2}}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$, the operator ${}^{\frac{1}{2}}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ becomes the zero operator.

E.1.2. Matrix elements for the case of a spin-1 representation. In this subsection we will repeat the calculation of the last subsection for the case of a spin representation $\ell = 1$. The matrix elements that are included in the calculations of the alternative flux operator ${}^{1}\widehat{\widetilde{E}}{}^{\mathrm{I,RS}}_{k,\mathrm{tot}}(S_t)$ are $\langle \alpha_2^0, M | \widehat{O}_1^{\mathrm{I,RS}} | \alpha_1^0, M\rangle$ and $\langle \alpha_i^1, M | \widehat{O}_2^{\mathrm{I,RS}} | \alpha_1^1, M\rangle$ where $i = 2, 3, 4$. The definition of $\widehat{O}_1^{\mathrm{I,RS}}$ and $\widehat{O}_2^{\mathrm{I,RS}}$ can be found in equation (6.12), whereas we have to, as before, derive the value of the matrix element of each single triple.

A basis of the Hilbert space associated with a total angular momentum $J = 0$ can be found in equation (D.9) and consists of four states. Hence, for every single triple $\widehat{Q}_{v,IJK}$ we obtain a 4×4 matrix. Starting with $\widehat{Q}^{J=0}_{\mathrm{RS},134}$ we have

$$\widehat{Q}^{J=0}_{\mathrm{RS},134} = \begin{pmatrix} 0 & -\mathrm{i}4\sqrt{\frac{2}{3}}a & 0 \\ +\mathrm{i}4\sqrt{\frac{2}{3}}a & 0 & -\mathrm{i}4\sqrt{\frac{1}{3}}b \\ 0 & +\mathrm{i}4\sqrt{\frac{1}{3}}b & 0 \end{pmatrix}, \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{4}{\sqrt{3}}\sqrt{2a^2+b^2} = -\lambda_3, \tag{E.37}$$

where we used equation (6.6) in order to obtain the matrix. The corresponding four eigenvectors are given by

$$\begin{aligned} \vec{v}_1 &= \left(+\mathrm{i}\frac{b}{\sqrt{2}a}, 0, 1\right), \qquad \vec{v}_2 = \left(-\frac{\sqrt{2}a}{b}, +\mathrm{i}\frac{\sqrt{2a^2+b^2}}{b}, 1\right) \\ \vec{v}_3 &= \left(-\frac{\sqrt{2}a}{b}, -\mathrm{i}\frac{\sqrt{2a^2+b^2}}{b}, 1\right). \end{aligned} \tag{E.38}$$

As in the situation of $\ell = \frac{1}{2}$ for $J = 0$ the remaining three triples are identical and, moreover, just the negative of the matrix of $\widehat{Q}^{J=0}_{\mathrm{RS},234}$,

$$\begin{aligned} \widehat{Q}^{J=0}_{\mathrm{RS},234} = \widehat{Q}^{J=0}_{\mathrm{RS},123} = \widehat{Q}^{J=0}_{\mathrm{RS},124} &= \begin{pmatrix} 0 & +\mathrm{i}4\sqrt{\frac{2}{3}}a & 0 \\ -\mathrm{i}4\sqrt{\frac{2}{3}}a & 0 & +\mathrm{i}4\sqrt{\frac{1}{3}}b \\ 0 & -\mathrm{i}4\sqrt{\frac{1}{3}}b & 0 \end{pmatrix}, \qquad \lambda_1 = 0, \\ \lambda_2 &= -\frac{4}{\sqrt{3}}\sqrt{2a^2+b^2} = -\lambda_3, \end{aligned} \tag{E.39}$$

while the corresponding eigenvectors are

$$\begin{aligned} \vec{v}_1 &= \left(-\mathrm{i}\frac{b}{\sqrt{2}a}, 0, 1\right), \qquad \vec{v}_2 = \left(-\frac{\sqrt{2}a}{b}, -\mathrm{i}\frac{\sqrt{2a^2+b^2}}{b}, 1\right), \\ \vec{v}_3 &= \left(-\frac{\sqrt{2}a}{b}, +\mathrm{i}\frac{\sqrt{2a^2+b^2}}{b}, 1\right). \end{aligned} \tag{E.40}$$

Now, we expand each operator $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}$ included in $\langle\alpha_2^0, M|\widehat{O}_1^{\mathrm{I,RS}}|\alpha_1^0, M\rangle$ with the help of the states $|\alpha_j^0, M\rangle$

$$\begin{aligned} \langle\alpha_2^0, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^0, M\rangle = &+\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^1, M\rangle \\ &+\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_2^1, M\rangle\langle\alpha_2^1, M|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^1, M\rangle \\ &+\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^1, M\rangle. \end{aligned} \tag{E.41}$$

From the eigenvector expansion, we get

$$\begin{aligned} \langle\alpha_2^0, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_2^0, M|\widehat{V}_{q_{IJK}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0 \\ \langle\alpha_2^0, M|\widehat{V}_{q_{IJK}}|\alpha_3^1, M\rangle &= \sum_k \langle\alpha_2^0, M|\widehat{V}_{q_{IJK}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_3^1, M\rangle = 0, \end{aligned} \tag{E.42}$$

where by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ result from $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by just dividing each vector by its norm. Consequently, each term in the expansion in equation (E.41) vanishes separately and we obtain, as in the case of $\ell = 0.5$,

$$\langle\alpha_2^0, M|\widehat{O}_1^{\mathrm{I,RS}}|\alpha_1^0, M\rangle = 0. \tag{E.43}$$

If we consider a total angular momentum of $J = 1$ equation (D.13) tells us that we already have to deal with seven states and consequently get a 7×7 matrix for each triple $\widehat{q}_{IJK}$. Again we have to discuss the matrix elements $\langle \alpha_i^1, M | \widehat{O}_2^{\mathrm{I,RS}} | \alpha_1^1, M\rangle$ with $i = 2, 3, 4$. Starting with $\widehat{Q}^{\mathrm{RS}}_{v,134}$ and using equation (6.6) yields

$$\widehat{Q}^{J=1}_{\mathrm{RS},134} = \begin{pmatrix} 0 & -\mathrm{i}\frac{8}{3}\sqrt{2}\tilde{a} & -\mathrm{i}2\sqrt{\frac{2}{3}}\tilde{a} & -\mathrm{i}\frac{2}{3}\sqrt{10}\tilde{a} & 0 & 0 & 0 \\ +\mathrm{i}\frac{8}{3}\sqrt{2}\tilde{a} & 0 & -\mathrm{i}\frac{4}{\sqrt{3}} & 0 & +\mathrm{i}\frac{4}{3}\tilde{b} & 0 & 0 \\ +\mathrm{i}2\sqrt{\frac{2}{3}}\tilde{a} & +\mathrm{i}\frac{4}{\sqrt{3}} & 0 & -\mathrm{i}2\sqrt{\frac{5}{3}} & -\mathrm{i}\frac{2}{\sqrt{3}}\tilde{b} & 0 & 0 \\ +\mathrm{i}\frac{2}{3}\sqrt{10}\tilde{a} & 0 & +\mathrm{i}2\sqrt{\frac{5}{3}} & 0 & -\mathrm{i}\frac{4}{3\sqrt{5}}\tilde{b} & -\mathrm{i}2\sqrt{\frac{3}{5}}\tilde{b} & 0 \\ 0 & -\mathrm{i}\frac{4}{3}\tilde{b} & +\mathrm{i}\frac{2}{\sqrt{3}}\tilde{b} & +\mathrm{i}\frac{4}{3\sqrt{5}}\tilde{b} & 0 & -\mathrm{i}2\sqrt{3} & +\mathrm{i}2\sqrt{\frac{6}{5}}\tilde{c} \\ 0 & 0 & 0 & +\mathrm{i}2\sqrt{\frac{3}{5}}\tilde{b} & +\mathrm{i}2\sqrt{3} & 0 & -\mathrm{i}6\sqrt{\frac{2}{5}}\tilde{c} \\ 0 & 0 & 0 & 0 & -\mathrm{i}2\sqrt{\frac{6}{5}}\tilde{c} & +\mathrm{i}6\sqrt{\frac{2}{5}}\tilde{c} & 0 \end{pmatrix}, \tag{E.44}$$

where we introduced

$$\tilde{a} := \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j+1)} \qquad \tilde{b} := \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{4j(j+1)-3}$$
$$\tilde{c} := \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j+1)-2}. \tag{E.45}$$

The seven eigenvalues of $\widehat{Q}^{\mathrm{RS}}_{v,134}$ are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j-1)} = -\lambda_3 \\ \lambda_4 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j+2)} = -\lambda_5 \\ \lambda_6 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{2j(j+1)-1} = -\lambda_7. \end{aligned} \tag{E.46}$$

The corresponding eigenvectors can be written in the following form:

$$\begin{aligned} \vec{v}_1 &= (0, 0, -\sqrt{15}a''', +3a''', -\sqrt{3}b''', -b''', 1) \\ \vec{v}_2 &= (+\mathrm{i}c, a - \mathrm{i}b, d + \mathrm{i}e, f + \mathrm{i}g, h - \mathrm{i}\ell, m + \mathrm{i}n, 1) \\ \vec{v}_3 &= (-\mathrm{i}c, a + \mathrm{i}b, d - \mathrm{i}e, f - \mathrm{i}g, h + \mathrm{i}\ell, m - \mathrm{i}n, 1) \\ \vec{v}_4 &= (+\mathrm{i}c', a' + \mathrm{i}b', d' - \mathrm{i}e', f' - \mathrm{i}g', -h' - \mathrm{i}\ell', -m' + \mathrm{i}n', 1) \\ \vec{v}_5 &= (-\mathrm{i}c', a' - \mathrm{i}b', d' + \mathrm{i}e', f' + \mathrm{i}g', -h' + \mathrm{i}\ell', -m' - \mathrm{i}n', 1) \\ \vec{v}_6 &= (-\mathrm{i}c'', -a'' + \mathrm{i}b'', d'' - \mathrm{i}e'', -f'' - \mathrm{i}g'', -h'' - \mathrm{i}\ell'', -m'' + \mathrm{i}n'', 1) \\ \vec{v}_7 &= (+\mathrm{i}c'', -a'' - \mathrm{i}b'', d'' + \mathrm{i}e'', -f'' + \mathrm{i}g'', -h'' + \mathrm{i}\ell'', -m'' - \mathrm{i}n'', 1). \end{aligned} \tag{E.47}$$

Here all letters $\{a, \ldots, n''\}$ denote real numbers which depend on the chosen value for the spin label j that is attached to the edges e_1, e_2. Using the expansion in terms of eigenvectors in equation (E.9), we get

$$\langle \alpha_2^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle = \sum_{k=1}^{4} \langle \alpha_2^1, M | \widehat{V}_{q_{134}} | \vec{e}_k\rangle \langle \vec{e}_k | \alpha_1^1, M\rangle = -2c_2 bc + 2c_4 b'c' - 2c_6 b''c''$$

$$\langle \alpha_3^1, M | \widehat{V}_{q_{134}} | \alpha_1^1, M\rangle = \sum_{k=1}^{4} \langle \alpha_3^1, M | \widehat{V}_{q_{134}} | \vec{e}_k\rangle \langle \vec{e}_k | \alpha_1^1, M\rangle = +2c_2 ce - 2c_4 c'e' + 2c_6 c''e''$$

$$\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +2c_2cg - 2c_4c'g' - 2c_6c''g''$$

$$\langle\alpha_5^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \sum_{k=1}^{4}\langle\alpha_5^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -2c_2c\ell - 2c_4c'\ell' + 2c_6c''\ell''$$

$$\langle\alpha_6^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \sum_{k=1}^{4}\langle\alpha_6^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +2c_2cn + 2c_4c'n' - 2c_6c''n''$$

$$\langle\alpha_7^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \sum_{k=1}^{4}\langle\alpha_7^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0$$

$$\begin{aligned}\langle\alpha_2^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_2^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\ &= +2c_2(ah + b\ell) - 2c_4(a'h' + b'\ell') + 2c_6(a''h'' - b''\ell'') \\ \langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\ &= +2c_2(dh - e\ell) - 2c_4(d'h' - e'\ell') - 2c_6(d''h'' - e''\ell'') \\ \langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\ &= +2c_2(fh - g\ell) - 2c_4(f'h' - g'\ell') + 2c_6(f''h'' + g''\ell'') \\ \langle\alpha_2^1, M|\widehat{V}_{q_{134}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_2^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\ &= +2c_2(am - bn) - 2c_4(a'm' - b'n') + 2c_6(a''m'' + b''n'') \\ \langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\ &= +2c_2(dm - en) - 2c_4(d'm' + e'n') - 2c_6(d''m'' + e''n'') \\ \langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\ &= +2c_2(fm - gn) - 2c_4(f'm' - g'n') + 2c_6(f''m'' + g''n''). \end{aligned} \tag{E.48}$$

Here we introduced the constants c_1, c_2, c_3 that are defined by

$$\begin{aligned} c_2 &:= \frac{\sqrt{|\lambda_2|}}{1 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + \ell^2 + m^2 + n^2} \\ c_4 &:= \frac{\sqrt{|\lambda_4|}}{1 + a'^2 + b'^2 + c'^2 + d'^2 + e'^2 + f'^2 + g'^2 + h'^2 + \ell'^2 + m'^2 + n'^2} \\ c_6 &:= \frac{\sqrt{|\lambda_6|}}{1 + a''^2 + b''^2 + c''^2 + d''^2 + e''^2 + f''^2 + g''^2 + h''^2 + \ell''^2 + m''^2 + n''^2}. \end{aligned} \tag{E.49}$$

The non-vanishing of these matrix elements is as in the case of $\ell = \frac{1}{2}$ caused by the fact that the expansion coefficients of $|\alpha_i^1, M\rangle$, where $i = 2, 3, 4$, have real as well as imaginary parts.

When we apply equation (6.7) to the states in equation (D.13) we obtain the following matrix,

$$\widehat{Q}^{J=1}_{\mathrm{RS},234} = \begin{pmatrix} 0 & +\mathrm{i}\frac{8}{3}\sqrt{2}\tilde{a} & +\mathrm{i}2\sqrt{\frac{2}{3}}\tilde{a} & +\mathrm{i}\frac{2}{3}\sqrt{10}\tilde{a} & 0 & 0 & 0 \\ -\mathrm{i}\frac{8}{3}\sqrt{2}\tilde{a} & 0 & -\mathrm{i}\frac{4}{\sqrt{3}} & 0 & -\mathrm{i}\frac{4}{3}\tilde{b} & 0 & 0 \\ -\mathrm{i}2\sqrt{\frac{2}{3}}\tilde{a} & +\mathrm{i}\frac{4}{\sqrt{3}} & 0 & -\mathrm{i}2\sqrt{\frac{5}{3}} & +\mathrm{i}\frac{2}{\sqrt{3}}\tilde{b} & 0 & 0 \\ -\mathrm{i}\frac{2}{3}\sqrt{10}\tilde{a} & 0 & +\mathrm{i}2\sqrt{\frac{5}{3}} & 0 & +\mathrm{i}\frac{4}{3\sqrt{5}}\tilde{b} & +\mathrm{i}2\sqrt{\frac{3}{5}}\tilde{b} & 0 \\ 0 & +\mathrm{i}\frac{4}{3}\tilde{b} & -\mathrm{i}\frac{2}{\sqrt{3}}\tilde{b} & -\mathrm{i}\frac{4}{3\sqrt{5}}\tilde{b} & 0 & -\mathrm{i}2\sqrt{3} & -\mathrm{i}2\sqrt{\frac{6}{5}}\tilde{c} \\ 0 & 0 & 0 & -\mathrm{i}2\sqrt{\frac{3}{5}}\tilde{b} & +\mathrm{i}2\sqrt{3} & 0 & +\mathrm{i}6\sqrt{\frac{2}{5}}\tilde{c} \\ 0 & 0 & 0 & 0 & +\mathrm{i}2\sqrt{\frac{6}{5}}\tilde{c} & -\mathrm{i}6\sqrt{\frac{2}{5}}\tilde{c} & 0 \end{pmatrix}, \tag{E.50}$$

where we used the abbreviations

$$\begin{aligned} \tilde{a} &:= \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right) \sqrt{j(j+1)} \\ \tilde{b} &:= \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right) \sqrt{4j(j+1)-3} \\ \tilde{c} &:= \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right) \sqrt{j(j+1)-2}. \end{aligned} \tag{E.51}$$

The eigenvalues are similar to those of $\widehat{q}_{134}$

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right) \sqrt{j(j-1)} = -\lambda_3 \\ \lambda_4 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right) \sqrt{j(j+2)} = -\lambda_5 \\ \lambda_6 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right) \sqrt{2j(j+1)-1} = -\lambda_7 \end{aligned} \tag{E.52}$$

and can be used to derive the corresponding eigenvectors

$$\begin{aligned} \vec{v}_1 &= (0, 0, -\sqrt{15}a''', +3a''', -\sqrt{3}b''', -b''', 1) \\ \vec{v}_2 &= (-\mathrm{i}c, a - \mathrm{i}b, d + \mathrm{i}e, f + \mathrm{i}g, -h + \mathrm{i}\ell, -m - \mathrm{i}n, 1) \\ \vec{v}_3 &= (+\mathrm{i}c, a + \mathrm{i}b, d - \mathrm{i}e, f - \mathrm{i}g, -h - \mathrm{i}\ell, -m + \mathrm{i}n, 1) \\ \vec{v}_4 &= (-\mathrm{i}c', a' + \mathrm{i}b', d' - \mathrm{i}e', f' - \mathrm{i}g', h' + \mathrm{i}\ell', m' - \mathrm{i}n', 1) \\ \vec{v}_5 &= (+\mathrm{i}c', a' - \mathrm{i}b', d' + \mathrm{i}e', f' + \mathrm{i}g', h' - \mathrm{i}\ell', m' + \mathrm{i}n', 1) \\ \vec{v}_6 &= (+\mathrm{i}c'', -a'' + \mathrm{i}b'', d'' - \mathrm{i}e'', -f'' - \mathrm{i}g'', h'' + \mathrm{i}\ell'', m'' - \mathrm{i}n'', 1) \\ \vec{v}_7 &= (-\mathrm{i}c'', -a'' - \mathrm{i}b'', d'' + \mathrm{i}e'', -f'' + \mathrm{i}g'', h'' - \mathrm{i}\ell'', m'' + \mathrm{i}n'', 1) \end{aligned} \tag{E.53}$$

where we again suppose that $\{a, \ldots, n''\}$ denote real numbers. Thus the desired matrix elements are

$$\begin{aligned} \langle \alpha_2^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M \rangle &= \sum_{k=1}^{4} \langle \alpha_2^1, M | \widehat{V}_{q_{234}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_1^1, M \rangle = +2c_2 bc - 2c_4 b'c' + 2c_6 b''c'' \\ \langle \alpha_3^1, M | \widehat{V}_{q_{234}} | \alpha_1^1, M \rangle &= \sum_{k=1}^{4} \langle \alpha_3^1, M | \widehat{V}_{q_{234}} | \vec{e}_k \rangle \langle \vec{e}_k | \alpha_1^1, M \rangle = -2c_2 ce + 2c_4 c'e' - 2c_6 c''e'' \end{aligned}$$

$$
\begin{aligned}
\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -2c_2cg + 2c_4c'g' + 2c_6c''g'' \\
\langle\alpha_5^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_5^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -2c_2c\ell - 2c_4c'\ell' + 2c_6c''\ell'' \\
\langle\alpha_6^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_6^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +2c_2cn + 2c_4c'n' - 2c_6c''n'' \\
\langle\alpha_7^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_7^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0 \\
\langle\alpha_2^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_2^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\
&= -2c_2(ah + b\ell) + 2c_4(a'h' + b'\ell') - 2c_6(a''h'' - b''\ell'') \\
\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\
&= -2c_2(dh - e\ell) + 2c_4(d'h' - e'\ell') + 2c_6(d''h'' - e''\ell'') \\
\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_5^1, M\rangle \\
&= -2c_2(fh - g\ell) + 2c_4(f'h' - g'\ell') - 2c_6(f''h'' + g''\ell'') \\
\langle\alpha_2^1, M|\widehat{V}_{q_{234}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_2^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\
&= -2c_2(am - bn) + 2c_4(a'm' - b'n') - 2c_6(a''m'' + b''n'') \\
\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\
&= -2c_2(dm - en) + 2c_4(d'm' + e'n') + 2c_6(d''m'' + e''n'') \\
\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_6^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_6^1, M\rangle \\
&= -2c_2(fm - gn) + 2c_4(f'm' - g'n') - 2c_6(f''m'' + g''n'').
\end{aligned}
\tag{E.54}
$$

In the case of $\widehat{Q}^{\mathrm{RS}}_{v,123}$ equation (6.8) leads to a matrix that looks less complicated,

$$
\widehat{Q}^{J=1}_{\mathrm{RS},123} = \begin{pmatrix}
0 & 0 & = \mathrm{i}\sqrt{\frac{2}{3}}a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mathrm{i}\sqrt{\frac{2}{3}}a & 0 & 0 & 0 & +\mathrm{i}\frac{4}{\sqrt{3}}b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & +\mathrm{i}4\sqrt{\frac{3}{5}}b & 0 \\
0 & 0 & -\mathrm{i}\frac{4}{\sqrt{3}}b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}4\sqrt{\frac{3}{5}}b & 0 & 0 & +\mathrm{i}12\sqrt{\frac{2}{5}}c \\
0 & 0 & 0 & 0 & 0 & -\mathrm{i}12\sqrt{\frac{2}{5}}c & 0
\end{pmatrix}, \tag{E.55}
$$

with

$$a := \left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right) \sqrt{j(j+1)}$$
$$b := \left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right) \sqrt{4j(j+1)-3} \tag{E.56}$$
$$c := \left(\ell_p^6 \frac{3!}{2} C_{\text{reg}}\right) \sqrt{j(j+1)-2},$$

and the corresponding eigenvalues

$$\lambda_1 = 0 = \lambda_2 = \lambda_3$$
$$\lambda_4 = -4\left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right) \sqrt{3}\sqrt{2j(j+1)-3} = -\lambda_5 \tag{E.57}$$
$$\lambda_6 = -4\left(\ell_p^6 \frac{3!}{4} C_{\text{reg}}\right) \sqrt{2j(j+1)-1} = -\lambda_7.$$

Unsurprisingly, the eigenvectors are simpler as well and are shown below:

$$\vec{v}_1 = \left(0, 0, 0, \frac{\sqrt{6}}{\alpha}, 0, 0, 1\right), \qquad \vec{v}_2 + \left(\frac{1}{\sqrt{2}\beta}, 0, 0, 0, 1, 0, 0\right), \qquad \vec{v}_3 = (0, 1, 0, 0, 0, 0, 0)$$
$$\vec{v}_4 = \left(0, 0, 0, -\frac{1}{\sqrt{6}}\alpha, 0, -\mathrm{i}\sqrt{\frac{5}{6}}\gamma, 1\right), \qquad \vec{v}_5 = \left(0, 0, 0, -\frac{1}{\sqrt{6}}\alpha, 0, +\mathrm{i}\sqrt{\frac{5}{6}}\gamma, 1\right)$$
$$\vec{v}_6 = (-\sqrt{2}\beta, 0, -\mathrm{i}\delta, 0, 1, 0, 0), \qquad \vec{v}_7 = (-\sqrt{2}\beta, 0, +\mathrm{i}\delta, 0, 1, 0, 0). \tag{E.58}$$

Here, the dependence on j of the components of $\vec{v}_k$ is less tricky and therefore we mention them explicitly for those interested,

$$\alpha := \frac{b}{c}, \quad \beta := \frac{a}{b}, \qquad \gamma := \frac{\sqrt{2a^2-3}}{c}, \qquad \delta := \frac{\sqrt{7a^2-3}}{b}. \tag{E.59}$$

These eigenvectors demonstrate that all matrix elements $\langle \alpha_i^1, M | \widehat{V}_{q_{123}}^2 | \alpha_1^1, M\rangle$, where $i = 2, 3, 4$, are zero,

$$\langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \alpha_1^1, M\rangle = \sum_{k=1}^{4} \langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \vec{e}_k\rangle\langle \vec{e}_k | \alpha_1^1, M\rangle = 0$$
$$\langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \alpha_5^1, M\rangle = \sum_{k=1}^{4} \langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \vec{e}_k\rangle\langle \vec{e}_k | \alpha_5^1, M\rangle = 0 \tag{E.60}$$
$$\langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \alpha_6^1, M\rangle = \sum_{k=1}^{4} \langle \alpha_i^1, M | \widehat{V}_{q_{123}} | \vec{e}_k\rangle\langle \vec{e}_k | \alpha_1^1, M\rangle = 0,$$

as we have either an expansion coefficient equal to zero or the combination of a real and a purely imaginary expansion coefficient. The last triple that has to be discussed is $\widehat{Q}_{v,124}^{\text{RS}}$.

Considering equation (6.9) we get

$$\widehat{Q}^{J=1}_{\mathrm{RS},124} = \begin{pmatrix} 0 & +\mathrm{i}\frac{4}{3}\sqrt{2}a & +\mathrm{i}2\sqrt{\frac{2}{3}}a & -\mathrm{i}\frac{2}{3}\sqrt{10}a & 0 & 0 & 0 \\ -\mathrm{i}\frac{4}{3}\sqrt{2}a & 0 & 0 & 0 & -\mathrm{i}\frac{8}{3}b & 0 & 0 \\ -\mathrm{i}2\sqrt{\frac{2}{3}}a & 0 & 0 & 0 & +\mathrm{i}\frac{2}{\sqrt{3}}b & -2\mathrm{i}b & 0 \\ +\mathrm{i}\frac{2}{3}\sqrt{10}a & 0 & 0 & 0 & +\mathrm{i}\frac{2}{3\sqrt{5}}b & +\mathrm{i}2\sqrt{\frac{3}{5}}b & 0 \\ 0 & +\mathrm{i}\frac{8}{3}b & -\mathrm{i}\frac{2}{\sqrt{3}}b & -\mathrm{i}\frac{4}{3\sqrt{5}}b & 0 & 0 & -\mathrm{i}6\sqrt{\frac{6}{5}}c \\ 0 & 0 & +2\mathrm{i}b & -\mathrm{i}2\sqrt{\frac{3}{5}}b & 0 & 0 & +\mathrm{i}6\sqrt{\frac{2}{5}}c \\ 0 & 0 & 0 & 0 & +\mathrm{i}6\sqrt{\frac{6}{5}}c & -\mathrm{i}6\sqrt{\frac{2}{5}}c & 0 \end{pmatrix}, \tag{E.61}$$

where we introduced

$$\begin{aligned} a &:= \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j+1)} \\ b &:= \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{4j(j+1)-3} \\ c &:= \left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{j(j+1)-2}. \end{aligned} \tag{E.62}$$

The seven eigenvalues of $\widehat{q}_{124}$ are

$$\begin{aligned} \lambda_1 &= 0 = \lambda_2 = \lambda_3 \\ \lambda_4 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{3}\sqrt{2j(j+1)-3} = -\lambda_5 \\ \lambda_6 &= -4\left(\ell_p^6 \frac{3!}{4} C_{\mathrm{reg}}\right)\sqrt{2j(j+1)-1} = -\lambda_7 \end{aligned} \tag{E.63}$$

and the corresponding eigenvectors can be expressed as

$$\begin{aligned} \vec{v}_1 &= \left(0, 3\sqrt{\frac{3}{10}}\frac{1}{\alpha}, -3\sqrt{\frac{2}{5}}\frac{1}{\alpha}, 0, 0, 0, 1\right), \qquad \vec{v}_2 = \left(-\sqrt{\frac{2}{3}}\frac{1}{\beta}, 0, 0, 0, \frac{1}{\sqrt{3}}, 1, 0\right), \\ \vec{v}_3 &= \left(0, \frac{1}{\sqrt{5}}, \sqrt{\frac{3}{5}}, 1, 0, 0, 0\right) \\ \vec{v}_4 &= \left(0, -\frac{1}{3}\sqrt{\frac{5}{6}}\alpha, \sqrt{\frac{5}{2}}\frac{1}{6}\alpha, -\frac{1}{6\sqrt{6}}\alpha, +\mathrm{i}\sqrt{\frac{5}{2}}\gamma, -\mathrm{i}\sqrt{\frac{5}{6}}\gamma, 1\right) \\ \vec{v}_5 &= \left(0, -\frac{1}{3}\sqrt{\frac{5}{6}}\alpha, \sqrt{\frac{5}{2}}\frac{1}{6}\alpha, -\frac{1}{2\sqrt{6}}\alpha, -\mathrm{i}\sqrt{\frac{5}{2}}\gamma, +\mathrm{i}\sqrt{\frac{5}{6}}\gamma, 1\right) \\ \vec{v}_6 &= \left(2\sqrt{\frac{2}{3}}\beta, +\mathrm{i}\frac{2}{\sqrt{3}}\delta, +\mathrm{i}\delta, -\mathrm{i}\sqrt{\frac{5}{3}}\delta, \frac{1}{\sqrt{3}}, 1, 0\right) \\ \vec{v}_7 &= \left(2\sqrt{\frac{2}{3}}\beta, -\mathrm{i}\frac{2}{\sqrt{3}}\delta, -\mathrm{i}\delta, +\mathrm{i}\sqrt{\frac{5}{3}}\delta, \frac{1}{\sqrt{3}}, 1, 0\right), \end{aligned} \tag{E.64}$$

with the following abbreviations:

$$\alpha := \frac{b}{c}, \quad \beta := \frac{a}{b}, \qquad \gamma := \frac{\sqrt{2a^2-3}}{2c}, \qquad \delta := \frac{\sqrt{2a^2-1}}{b}. \tag{E.65}$$

For this particular triple the matrix elements disappear as well, because the first three eigenvalues are zero, the eigenvectors $\vec{v}_4, \vec{v}_5$ have an expansion coefficient for $|\alpha_1^1, M\rangle$ which is zero, and the vectors $\vec{v}_6, \vec{v}_7$ have a real expansion coefficient for $|\alpha_1^1, M\rangle$, while the one for the states $|\alpha_i^1, M\rangle$ with i being 2, 3, 4 is purely imaginary. Consequently, we have

$$\begin{aligned}\langle\alpha_i^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_i^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0\\ \langle\alpha_7^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= \sum_{k=1}^{4}\langle\alpha_7^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = 0.\end{aligned} \tag{E.66}$$

The expansion of each operator $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}$ that occurs in the operator $\widehat{O}_2^{\mathrm{I,RS}}$ is given by

$$\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^1, M\rangle = \sum_{k=1}^{7}\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_k^1, M\rangle\langle\alpha_k^1, M|\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}|\alpha_1^1, M\rangle. \tag{E.67}$$

Considering the operator $\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}$ where $IJK \in \{134, 234, 124\}$, the expansion above leads to

$$\begin{aligned}\langle\alpha_i^1, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_2^1, M\rangle\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_7^1, M\rangle\langle\alpha_7^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle.\end{aligned} \tag{E.68}$$

We can read off from equation (E.60) $\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_1^1, M\rangle = 0$ with $i = 2, 3, 4$ and $j = 1, 5, 6$. Consequently, the expansion reduces to

$$\begin{aligned}\langle\alpha_i^1, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_2^1, M\rangle\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_7^1, M\rangle\langle\alpha_7^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle.\end{aligned} \tag{E.69}$$

Since $\langle\alpha_7^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = 0$ for $IJK \in \{134, 234, 124\}$ as can be seen in equations (E.48), (E.54) and (E.66), the last term in the sum drops out. Furthermore, $\langle\alpha_i^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle = 0$, whereas $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle$. Accordingly, the non-vanishing contributions of the triples $\{e_1, e_3, e_4\}$ and $\{e_2, e_3, e_4\}$ cancel each other. Hence, we get

$$\langle\alpha_i^1, M|\widehat{V}_{q_{123}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle = 0. \tag{E.70}$$

In the case of the operator $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}$ with $IJK \in \{134, 234\}$, we can expand the matrix elements as

$$\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle = +\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle$$

$$
\begin{aligned}
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_2^1, M\rangle\langle\alpha_2^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_7^1, M\rangle\langle\alpha_7^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.71}
$$

In equation (E.66) is shown $\langle\alpha_j^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle = 0$ with $j = 2, 3, 4, 7$. Therefore, we can neglect four terms in the sum above and get

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= +\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.72}
$$

By comparing the results in equation (E.48) with that in equation (E.48), we note that $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_j^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_j^1, M\rangle$ whereby $i = 2, 3, 4$ and $j = 1, 5, 6$. Accordingly, this yields

$$
\langle\alpha_i^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle = 0. \tag{E.73}
$$

The expansion in terms of $|\alpha_k^1, M\rangle$ of the operator $\widehat{V}_{q_{134}}\widehat{V}_{q_{IJK}}$ whereby $IJK \in \{134, 234\}$ can be found below:

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle &= +\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_2^1, M\rangle\langle\alpha_2^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_7^1, M\rangle\langle\alpha_7^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.74}
$$

If we compare equation (E.48) with equation (E.54), we realize that $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_j^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle$ with $i = 2, 3, 4$ and $|\alpha_7^1, M\rangle\langle\alpha_7^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = 0$ Therefore, we have

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{134}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle &= +\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.75}
$$

The same argument applies to the operator $\widehat{V}_{q_{234}}\widehat{V}_{q_{IJK}}$ whereby $IJK \in \{134, 234\}$, so that its expansion is given by

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{234}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle &= +\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_6^1, M\rangle\langle\alpha_6^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.76}
$$

By using $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_j^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_j^1, M\rangle$ where $i = 2, 3, 4$ and $j = 1, 5, 6$ which can be easily extracted from equations (E.48) and (E.54), we obtain

$$
\langle\alpha_i^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{124}}\right)|\alpha_1^1, M\rangle = 0. \tag{E.77}
$$

If we add up equations (E.70), (E.73) and (E.77) the operators $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}$ add up to the operator $\widehat{O}_2^{\mathrm{I,RS}}$. Hence, we can conclude

$$\big\langle \alpha_i^1, M\big|\widehat{O}_2^{\mathrm{I,RS}}\big|\alpha_1^1, M\big\rangle = 0 \qquad i = 2, 3, 4. \tag{E.78}$$

Since, the matrix elements of $\widehat{O}_1^{\mathrm{I,RS}}$ and $\widehat{O}_2^{\mathrm{I,RS}}$ also vanish in the case of a spin label $\ell = 1$, the operator ${}^{1}\widehat{E}_{k,\mathrm{tot}}^{\mathrm{I,RS}}(S_t)$ is the zero operator as well.

E.2. Case ${}^{(\ell)}\widehat{E}_{k,\mathrm{tot}}^{\mathrm{II,RS}}(S_t)$: detailed calculation of the matrix elements of $\widehat{O}_1^{\mathrm{II,RS}}$ and $\widehat{O}_2^{\mathrm{II,RS}}$

In this section, we discuss the matrix elements $\big\langle \alpha_2^0, M\big|\widehat{O}_1^{\mathrm{II,RS}}\big|\alpha_1^0, M\big\rangle$ and $\big\langle \alpha_2^0, M\big|\widehat{O}_2^{\mathrm{II,RS}}\big|\alpha_1^0, M\big\rangle$ that contribute to the matrix element of the alternative flux operator ${}^{(\ell)}\widehat{E}_{k,\mathrm{tot}}^{\mathrm{II,RS}}(S_t)$. As discussed in section 6.6.1, from our point of view the operator ${}^{(\ell)}\widehat{E}_{k,\mathrm{tot}}^{\mathrm{II,RS}}(S_t)$ including the combination $\widehat{V}_{\mathrm{RS}}\widehat{\mathcal{S}}_{\mathrm{AL}}\widehat{V}_{\mathrm{RS}}$ is highly artificial. Nevertheless, we investigate this operator in detail for a spin label $\ell = 0.5$ here.

E.2.1. Matrix elements for the case of a spin-$\frac{1}{2}$ representation. In order to calculate the matrix element of ${}^{(\ell)}\widehat{E}_{k,\mathrm{tot}}^{\mathrm{II,RS}}(S_t)$, we have to know the matrix elements $\big\langle \alpha_2^0, M\big|\widehat{O}_1^{\mathrm{II,RS}}\big|\alpha_1^0, M\big\rangle$ and $\big\langle \alpha_i^1, M\big|\widehat{O}_2^{\mathrm{II,RS}}\big|\alpha_1^0, M\big\rangle$. This can be seen in equation (6.3). The explicit definition of the operators $\widehat{O}_1^{\mathrm{II,RS}}$, $\widehat{O}_2^{\mathrm{II,RS}}$ are shown in equation (6.38). Here, the calculation for $\widehat{O}_1^{\mathrm{II,RS}}$, $\widehat{O}_2^{\mathrm{II,RS}}$ differs from the discussion of $\widehat{O}_1^{\mathrm{I,RS}}$, $\widehat{O}_2^{\mathrm{I,RS}}$ in the last section, because now additionally the sign operator $\widehat{\mathcal{S}}$ occurs sandwiched between the two volume operators $\widehat{V}_{\mathrm{RS}}$. Since the matrices of the operators $\widehat{Q}_{v,IJK}^{\mathrm{RS}}$ and their corresponding eigenvectors and eigenvalues are already given in the last section, we will not show them here again, but only refer to the results of the last section.

The expansion of each operator $\widehat{V}_{q_{IJK}}\widehat{\mathcal{S}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}$ that contributes to $\widehat{O}_2^{\mathrm{II,RS}}$ is given by

$$\begin{aligned}\big\langle \alpha_2^0, M\big|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}\big|\alpha_1^0, M\big\rangle &= \sum_{i,j=1}^{2}\big\langle \alpha_2^0, M\big|\widehat{V}_{q_{IJK}}\big|\alpha_i^0, M\big\rangle\big\langle \alpha_i^0, M\big|\widehat{S}\big|\alpha_j^0, M\big\rangle\big\langle \alpha_j^0, M\big|\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}\big|\alpha_1^0, M\big\rangle \\ &= \big\langle \alpha_2^0, M\big|\widehat{V}_{q_{IJK}}\big|\alpha_2^0, M\big\rangle\big\langle \alpha_2^0, M\big|\widehat{S}\big|\alpha_1^0, M\big\rangle\big\langle \alpha_1^0, M\big|\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}\big|\alpha_1^0, M\big\rangle,\end{aligned} \tag{E.79}$$

because $\widehat{V}_{q_{\mathrm{RS}}}$ is diagonal in this case.

The matrix elements of the sign operator $\widehat{\mathcal{S}}$ can be calculated by

$$\big\langle \alpha_i^0, M\big|\widehat{S}\big|\alpha_j^0, M\big\rangle = \sum_k \big\langle \alpha_i^0, M\big|\widehat{S}\big|\vec{e}_k\big\rangle\big\langle \vec{e}_k\big|\alpha_j^0, M\big\rangle = \sum_k \mathrm{sgn}\big(\lambda_k^Q\big)\big\langle \alpha_i^0, M\big|\vec{e}_k\big\rangle\big\langle \vec{e}_k\big|\alpha_j^0, M\big\rangle, \tag{E.80}$$

whereby λ_k^Q denotes the eigenvalue of the operator $\widehat{Q}_{v,AL}^{J=0}$ associated with the eigenvector $\vec{e}_k$. Using the results of $\widehat{Q}_{\mathrm{RS},IJK}^{J=0}$ in equation (E.10), we end up with

$$\begin{aligned}&\big\langle \alpha_2^0, M\big|\widehat{V}_{q_{IJK}}\big|\alpha_2^0, M\big\rangle = \big\langle \alpha_1^0, M\big|\widehat{V}_{q_{IJK}}\big|\alpha_1^0, M\big\rangle = \sqrt{2a} \\ &IJK \in \{134, 234, 123, 124\} \qquad \big\langle \alpha_2^0, M\big|\widehat{S}\big|\alpha_1^0, M\big\rangle = +\mathrm{i}.\end{aligned} \tag{E.81}$$

Considering the definition of $\widehat{O}_1^{\mathrm{II,RS}}$ in equation (6.38) and the results above, we obtain

$$\begin{aligned}\big\langle \alpha_2^0, M\big|\widehat{O}_1^{\mathrm{II,RS}}\big|\alpha_1^0, M\big\rangle &= +\mathrm{i}18a = +\mathrm{i}9a_{\mathrm{AL}} = 9\big\langle \alpha_2^0, M\big|\widehat{O}_1^{\mathrm{II,AL}}\big|\alpha_1^0, M\big\rangle \\ &=: C_1(\ell)\big\langle \alpha_2^0, M\big|\widehat{O}_1^{\mathrm{II,AL}}\big|\alpha_1^0, M\big\rangle.\end{aligned} \tag{E.82}$$

Here we used $a = \frac{1}{2}a_{\mathrm{AL}}$ that can be found by comparing the matrix entries of $opQ^{J=0}_{v,AL}$ with the one of $\widehat{Q}^{J=0}_{\mathrm{RS},IJK}$. We want to express everything in terms of the AL parameters here in order to compare the results of ${}^{(\ell)}\widehat{E}^{\mathrm{II,RS}}_{k,\mathrm{tot}}(S_t)$ and ${}^{(\ell)}\widehat{E}^{\mathrm{II,AL}}_{k,\mathrm{tot}}(S_t)$ directly.

For the operator $\widehat{O}^{\mathrm{II,RS}}_{2}$, we have to consider the case of a total angular momentum $J = 1$. The expansion of $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}$ in terms of the basis states $|\alpha^1_k, M\rangle$ of $\mathcal{H}^{J=1}$ is shown below:

$$\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}|\alpha^1_1, M\rangle = \sum_{j,k}\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}|\alpha^1_j, M\rangle\langle\alpha^1_j, M|\widehat{S}|\alpha^1_k, M\rangle\langle\alpha^1_k, M|\widehat{V}_{q_{\bar{I}\bar{J}\bar{K}}}|\alpha^1_1, M\rangle. \tag{E.83}$$

In this case the matrix elements of the sign operator $\widehat{S}$ are given by

$$\langle\alpha^1_j, M|\widehat{S}|\alpha^1_k, M\rangle = \sum_{k'}\langle\alpha^1_j, M|\widehat{S}|\vec{e}_{k'}\rangle\langle\vec{e}_{k'}|\alpha^1_k, M\rangle = \sum_{k'}\mathrm{sgn}(\lambda^Q_{k'})\langle\alpha^1_j, M|\vec{e}_{k'}\rangle\langle\vec{e}_{k'}|\alpha^1_k, M\rangle. \tag{E.84}$$

By using the results shown in section 6.5, we obtain

$$\begin{aligned}
\langle\alpha^1_j, M|\widehat{S}|\alpha^1_j, M\rangle &= \sum_k\langle\alpha^1_j, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_j, M\rangle = 0\\
\langle\alpha^1_j, M|\widehat{S}|\alpha^1_i, M\rangle &= \sum_k\langle\alpha^1_j, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_i, M\rangle = -\langle\alpha^1_i, M|\widehat{S}|\alpha^1_j, M\rangle\\
\langle\alpha^1_3, M|\widehat{S}|\alpha^1_1, M\rangle &= \sum_k\langle\alpha^1_3, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_1, M\rangle = -\mathrm{i}\frac{a_{\mathrm{AL}}}{\lambda_{\mathrm{AL}}}\\
\langle\alpha^1_4, M|\widehat{S}|\alpha^1_1, M\rangle &= \sum_k\langle\alpha^1_4, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_1, M\rangle = -\mathrm{i}\frac{a_{\mathrm{AL}}}{\lambda_{\mathrm{AL}}}\\
\langle\alpha^1_5, M|\widehat{S}|\alpha^1_1, M\rangle &= \sum_k\langle\alpha^1_5, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_1, M\rangle = 0\\
\langle\alpha^1_3, M|\widehat{S}|\alpha^1_5, M\rangle &= \sum_k\langle\alpha^1_3, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_5, M\rangle = -\mathrm{i}\frac{b_{\mathrm{AL}}}{\sqrt{2}\lambda_{\mathrm{AL}}}\\
\langle\alpha^1_4, M|\widehat{S}|\alpha^1_3, M\rangle &= \sum_k\langle\alpha^1_4, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_3, M\rangle = 0\\
\langle\alpha^1_4, M|\widehat{S}|\alpha^1_5, M\rangle &= \sum_k\langle\alpha^1_4, M|\widehat{S}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha^1_5, M\rangle = -\mathrm{i}\frac{b_{\mathrm{AL}}}{\lambda_{\mathrm{AL}}}.
\end{aligned} \tag{E.85}$$

Here we explicitly labelled the constants $a_{\mathrm{AL}}, b_{\mathrm{AL}}$ by AL, because they differ from the constants a, b used in the case of $\widehat{V}_{\mathrm{RS}}$. The relation between these two constants is for $J = 1$ only a factor of $2/3$, namely $a_{\mathrm{AL}} = \left(\frac{2}{3}\right)a$ and $b_{\mathrm{AL}} = \left(\frac{2}{3}\right)b$. This can be easily seen by comparing the matrix entries of $\widehat{Q}^{J=1}_{v,\mathrm{AL}}$ with the one of $\widehat{Q}^{J=1}_{RS;IJK}$. Additionally, we labelled the eigenvalue λ_{AL} by AL, because $\widehat{Q}^{J=1}_{v,\mathrm{AL}}$ and $\widehat{Q}^{J=1}_{RS;IJK}$ have different eigenvalues.

Starting with the operator $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}$ with $IJK \in \{134, 234, 123\}$ and taking into account the vanishing of certain matrix elements of $\widehat{S}$ shown in equation (E.85), we get the following expansion:

$$\begin{aligned}
\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle = &+\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}|\alpha^1_3, M\rangle\langle\alpha^1_3, M|\widehat{S}|\alpha^1_1, M\rangle\langle\alpha^1_1, M|\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle\\
&+\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}|\alpha^1_4, M\rangle\langle\alpha^1_4, M|\widehat{S}|\alpha^1_1, M\rangle\langle\alpha^1_1, M|\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle\\
&+\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}|\alpha^1_1, M\rangle\langle\alpha^1_1, M|\widehat{S}|\alpha^1_3, M\rangle\langle\alpha^1_3, M|\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle\\
&+\langle\alpha^1_i, M|\widehat{V}_{q_{IJK}}|\alpha^1_5, M\rangle\langle\alpha^1_5, M|\widehat{S}|\alpha^1_3, M\rangle\langle\alpha^1_3, M|\widehat{V}_{q_{124}}|\alpha^1_1, M\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle. \qquad \text{(E.86)}
\end{aligned}
$$

From equation (E.24) we can read off $\langle\alpha_i^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle = 0$ with $i = 3, 4, 5$. Hence, only the first two terms of the sum are not zero,

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= +\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle\\
&= +\langle\alpha_i^1, M|\widehat{V}_{q_{IJK}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle. \qquad \text{(E.87)}
\end{aligned}
$$

The matrix elements that are necessary to know in order to calculate the matrix element of $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{124}}$ explicitly are given below:

$$
\begin{aligned}
\langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= \sum_k \langle\alpha_1^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = \sqrt{2a}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle &= \langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_3^1, M\rangle\sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_3^1, M\rangle\\
&= \left(3 + \frac{12}{\lambda^2}\right)\sqrt{\lambda}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle &= \langle\alpha_3^1, M|\widehat{V}_{q_{234}}|\alpha_4^1, M\rangle\sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_4^1, M\rangle\\
&= \left(\frac{6\sqrt{2}}{\lambda^2} - 3\sqrt{2}\right)\sqrt{\lambda}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle &= \langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_3^1, M\rangle\sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_3^1, M\rangle\\
&= \left(\frac{6\sqrt{2}}{\lambda^2} - 3\sqrt{2}\right)\sqrt{\lambda}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle &= \langle\alpha_4^1, M|\widehat{V}_{q_{234}}|\alpha_4^1, M\rangle\sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_4^1, M\rangle\\
&= \left(6 + \frac{6}{\lambda^2}\right)\sqrt{\lambda}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle &= \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_3^1, M\rangle = \sqrt{2a}\\
\langle\alpha_4^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle &= \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_4^1, M\rangle = \sqrt{2b}\\
\langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle &= \langle\alpha_4^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle = \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{124}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_4^1, M\rangle = 0.
\end{aligned}
\qquad \text{(E.88)}
$$

Thus, we obtain

$$
\begin{aligned}
\langle\alpha_3^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{123}}\right)\widehat{V}_{q_{124}}|\alpha_1^1, M\rangle &= -\mathrm{i}12a_{\mathrm{AL}}\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}} - \mathrm{i}2a_{\mathrm{AL}}\frac{a}{\lambda_{\mathrm{AL}}}\\
\langle\alpha_4^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}} + \widehat{V}_{q_{123}}\right)\widehat{V}_{q_{123}}|\alpha_1^1, M\rangle &= +\mathrm{i}\sqrt{2}18a_{\mathrm{AL}}\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}} + \mathrm{i}\sqrt{2}a_{\mathrm{AL}}\frac{2\sqrt{ab}}{\lambda_{\mathrm{AL}}}.
\end{aligned}
\qquad \text{(E.89)}
$$

The operator $\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}$ where $IJK \in \{134, 234\}$ can be expanded as

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{123}}\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = &+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{S}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{S}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle.
\end{aligned}
\tag{E.90}
$$

The results in equation (E.21) show $\langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle = \langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle = 0$. Moreover, by comparing equation (E.15) with equation (E.18), we note $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle$ where $i = 3, 4$. Consequently, we have

$$
\begin{aligned}
&\langle\alpha_i^1, M|\widehat{V}_{q_{123}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&\quad = +\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle\\
&\qquad +\langle\alpha_i^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle.
\end{aligned}
\tag{E.91}
$$

The particular matrix elements that contribute to the expansion above are

$$
\begin{aligned}
&\langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\alpha_3^1, M\rangle = \sum_k \langle\alpha_3^1, M|\widehat{V}_{q_{123}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_3^1, M\rangle = \sqrt{2a}\\
&\langle\alpha_4^1, M|\widehat{V}_{q_{123}}|\alpha_4^1, M\rangle = \sum_k \langle\alpha_4^1, M|\widehat{V}_{q_{123}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_4^1, M\rangle = \sqrt{2b}\\
&\langle\alpha_1^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \langle\alpha_1^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle = \sum_k \langle\alpha_1^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = +\frac{12a^2}{\lambda^2}\sqrt{\lambda}\\
&\langle\alpha_5^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = \langle\alpha_5^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle = \sum_k \langle\alpha_5^1, M|\widehat{V}_{q_{134}}|\vec{e}_k\rangle\langle\vec{e}_k|\alpha_1^1, M\rangle = -\frac{6\sqrt{2}ab}{\lambda^2}\sqrt{\lambda}.
\end{aligned}
\tag{E.92}
$$

Using the results above, we obtain

$$
\begin{aligned}
&\langle\alpha_3^1, M|\widehat{V}_{q_{123}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle = -\mathrm{i}12a_{\mathrm{AL}}\frac{(2a^2+b^2)\sqrt{2a}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2}\\
&\langle\alpha_4^1, M|\widehat{V}_{q_{123}}\left(\widehat{V}_{q_{134}} + \widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle = +\mathrm{i}\sqrt{2}12a_{\mathrm{AL}}\frac{(2a^2+b^2)\sqrt{2b}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2},
\end{aligned}
\tag{E.93}
$$

whereby we used $a = (3/2)a_{\mathrm{AL}}$ and $b_{\mathrm{AL}} = (2/3)b$. Expanding the operator $\widehat{V}_{q_{134}}\widehat{V}_{q_{IJK}}$ where $IJK \in \{134, 234\}$ yields

$$
\begin{aligned}
\langle\alpha_i^1, M|\widehat{V}_{q_{134}}\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle = &+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{S}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle\\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{S}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle \\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{S}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle \\
&+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle \\
=\ &+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\widehat{V}_{q_{IJK}}|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.94}
$$

As before by using $\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_1^1, M\rangle = -\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_1^1, M\rangle$ where $i = 3, 4$, the expansion reduces to

$$
\begin{aligned}
&\langle\alpha_i^1, M|\widehat{V}_{q_{134}}\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle \\
&\quad= +\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}})|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}})|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}})|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{134}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}})|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.95}
$$

The same is true for $\widehat{V}_{q_{234}}\widehat{V}_{q_{IJK}}$, thus

$$
\begin{aligned}
&\langle\alpha_i^1, M|\widehat{V}_{q_{234}}\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle \\
&\quad= +\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_1^1, M\rangle\langle\alpha_1^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_3^1, M\rangle\langle\alpha_3^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle \\
&\qquad+\langle\alpha_i^1, M|\widehat{V}_{q_{234}}|\alpha_4^1, M\rangle\langle\alpha_4^1, M|\widehat{S}|\alpha_5^1, M\rangle\langle\alpha_5^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle.
\end{aligned} \tag{E.96}
$$

Inserting the explicit results of the matrix elements of $\widehat{V}_{q_{134}}$ and $\widehat{V}_{q_{234}}$, we get

$$
\begin{aligned}
&\langle\alpha_3^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle = -\mathrm{i}12a_{\mathrm{AL}}18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right) \\
&\langle\alpha_4^1, M|\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)\left(\widehat{V}_{q_{134}}+\widehat{V}_{q_{234}}\right)|\alpha_1^1, M\rangle = +\mathrm{i}12\sqrt{2}a_{\mathrm{AL}}18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right),
\end{aligned} \tag{E.97}
$$

where for the latter matrix element we used $a = (3/2)a_{\mathrm{AL}}$ and $b_{\mathrm{AL}} = (2/3)b$. By summing the results in equations (E.89), (E.93) and (E.97), we obtain the result of the operator $\widehat{O}_2^{\mathrm{II,RS}}$, because the separated operators $\widehat{V}_{q_{IJK}}\widehat{V}_{q_{\tilde{I}\tilde{J}\tilde{K}}}$ exactly add up to $\widehat{O}_2^{\mathrm{II,RS}}$:

$$
\begin{aligned}
\langle\alpha_3^1, M|\widehat{O}_2^{\mathrm{II,RS}}|\alpha_1^1, M\rangle &= -\mathrm{i}a_{\mathrm{AL}}\left(12\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}}+12\frac{a}{\lambda_{\mathrm{AL}}}\right. \\
&\quad\left.+12\frac{(2a^2+b^2)\sqrt{2a}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2}+12\cdot 18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right)\right) \\
\langle\alpha_4^1, M|\widehat{O}_2^{\mathrm{II,RS}}|\alpha_1^1, M\rangle &= +\mathrm{i}\sqrt{2}a_{\mathrm{AL}}\left(18\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}}+\frac{2\sqrt{ab}}{\lambda_{\mathrm{AL}}}\right. \\
&\quad\left.+12\frac{(2a^2+b^2)\sqrt{2b}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2}+12\cdot 18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right)\right).
\end{aligned} \tag{E.98}
$$

Since the eigenvalues

$$
\lambda_{\mathrm{AL}} = \sqrt{\frac{3}{2}}\sqrt{2a_{\mathrm{AL}}^2+b_{\mathrm{AL}}^2} \qquad \text{and} \qquad \lambda = \sqrt{\frac{2}{3}}\sqrt{2a^2+b^2+3}, \tag{E.99}
$$

the matrix elements of $\widehat{O}_2^{\mathrm{II,RS}}$ will depend on the spin label j in general. The relation between the matrix elements of $\widehat{O}_2^{\mathrm{II,RS}}$ and $\widehat{O}_2^{\mathrm{II,AL}}$ is given by

$$\begin{aligned}\langle\alpha_3^1, M|\widehat{O}_2^{\mathrm{II,RS}}|\alpha_1^1, M\rangle &= C_3\left(j, \tfrac{1}{2}\right)\langle\alpha_3^1, M|\widehat{O}_2^{\mathrm{II,AL}}|\alpha_1^1, M\rangle \\ \langle\alpha_4^1, M|\widehat{O}_2^{\mathrm{II,RS}}|\alpha_1^1, M\rangle &= C_4\left(j, \tfrac{1}{2}\right)\langle\alpha_4^1, M|\widehat{O}_2^{\mathrm{II,AL}}|\alpha_1^1, M\rangle,\end{aligned} \tag{E.100}$$

whereby

$$\begin{aligned}C_3\left(j, \frac{1}{2}\right) &= \left(12\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}} + 12\frac{a}{\lambda_{\mathrm{AL}}} + 12\frac{(2a^2+b^2)\sqrt{2a}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2} + 12\cdot 18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right)\right) \\ C_4\left(j, \frac{1}{2}\right) &= \left(18\frac{\sqrt{\lambda}\sqrt{2a}}{\lambda_{\mathrm{AL}}} + \frac{2\sqrt{ab}}{\lambda_{\mathrm{AL}}} + 12\frac{(2a^2+b^2)\sqrt{2b}\sqrt{\lambda}}{\lambda_{\mathrm{AL}}\lambda^2} + 12\cdot 18\left(\frac{2a^2+b^2}{\lambda_{\mathrm{AL}}\lambda}\right)\right).\end{aligned} \tag{E.101}$$

In order to see whether this dependence vanishes in the semi-classical regime of the theory, i.e. in the limit of large j, we will analyse this limit now.

E.2.2. Semi-classical limit of the matrix elements of $\widehat{O}_2^{\mathrm{II,RS}}$. First, let us investigate the semi-classical behaviour of the eigenvalues λ_{AL} and λ,

$$\begin{aligned}\lambda_{\mathrm{AL}} &= \sqrt{\frac{3}{2}}\sqrt{2a_{\mathrm{AL}}^2 + b_{\mathrm{AL}}^2} \\ \lambda &= \sqrt{\frac{2}{3}}\sqrt{2a^2+b^2+3} = \sqrt{\frac{3}{2}}\sqrt{2a_{\mathrm{AL}}^2 + b_{\mathrm{AL}}^2 + \frac{4}{3}},\end{aligned} \tag{E.102}$$

whereby we used $a_{\mathrm{AL}} = (2/3)a$ and $b_{\mathrm{AL}} = (2/3)b$. Hence, semi-classically, we get $\lambda \to \lambda_{\mathrm{AL}}$. The constants $a_{\mathrm{AL}}, b_{\mathrm{AL}}$ are given by

$$a_{\mathrm{AL}} := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right)\frac{2}{3}\sqrt{j(j+1)} \qquad b_{\mathrm{AL}} := \left(\ell_p^6 \frac{3!}{2} C_{\mathrm{reg}}\right)\frac{2}{3}\sqrt{4j(j+1)-3}. \tag{E.103}$$

Accordingly, in the semi-classical limit $b_{\mathrm{AL}} \to 2a_{\mathrm{AL}}$.

Summarizing, in the semi-classical sector of the theory, we have

$$\lambda \to \lambda_{\mathrm{AL}}, \qquad b_{\mathrm{AL}} \to 2_{\mathrm{AL}} \quad\Rightarrow\quad \lambda_{\mathrm{AL}} \to 3a_{\mathrm{AL}}. \tag{E.104}$$

If we express all a, b occurring in $C_3\left(j, \frac{1}{2}\right), C_4\left(j, \frac{1}{2}\right)$ in terms of a_{AL} and b_{AL}, and afterwards take the semi-classical limit, we end up with

$$C_3\left(j, \tfrac{1}{2}\right) \to C_3\left(\tfrac{1}{2}\right) = 9\cdot 42 \qquad C_4\left(j, \tfrac{1}{2}\right) \to C_4\left(\tfrac{1}{2}\right) = (18+1)(18+\sqrt{2}). \tag{E.105}$$

It is precisely due to the linearly dependent triples that the awkward $\sqrt{2}$ term appears which certainly lacks any combinatorial or geometrical interpretation.

References

[1] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)
Thiemann T 2005 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press) (*Preprint* gr-qc/0110034)

[2] Rovelli C 1998 Loop quantum gravity *Living Rev. Rel.* **1** 1 (*Preprint* gr-qc/9710008)
Thiemann T 2003 Lectures on loop quantum gravity *Lecture Notes in Physics* vol 631 (Berlin: Springer) pp 41–135 (*Preprint* gr-qc/0210094)
Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* **21** R53 (*Preprint* gr-qc/0404018)
Smolin L 2004 An invitation to loop quantum gravity *Preprint* hep-th/0408048

[3] Thiemann T 1996 Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity *Phys. Lett.* B **380** 257–64 (*Preprint* gr-qc/9606088)

Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839–73 (*Preprint* gr-qc/9606089)

Thiemann T 1998 Quantum spin dynamics (QSD): II. The kernel of the Wheeler–Dewitt constraint operator *Class. Quantum Grav.* **15** 875–905 (*Preprint* gr-qc/9606090)

Thiemann T 1998 Quantum spin dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity *Class. Quantum Grav.* **15** 1207–47 (*Preprint* gr-qc/9705017)

Thiemann T 1998 Quantum spin dynamics (QSD): IV. 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity *Class. Quantum Grav.* **15** 1249–80 (*Preprint* gr-qc/9705018)

Thiemann T 1998 Quantum spin dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories *Class. Quantum Grav.* **15** 1281–314 (*Preprint* gr-qc/9705019)

[4] Thiemann T 2003 The Phoenix project: master constraint programme for loop quantum gravity *Preprint* gr-qc/0305080

Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: I. General framework *Preprint* gr-qc/0411138

Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: II. Finite dimensional systems *Preprint* gr-qc/0411139

Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: III. SL(2,R) models *Preprint* gr-qc/0411140

Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: IV. Free field theories *Preprint* gr-qc/0411141

Dittrich B and Thiemann T 2004 Testing the master constraint programme for loop quantum gravity: V. Interacting field theories *Preprint* gr-qc/0411142

[5] Rovelli C and Smolin L 1995 Discreteness of volume and area in quantum gravity *Nucl. Phys.* B **442** 593

Rovelli C and Smolin L 1995 Discreteness of volume and area in quantum gravity *Nucl. Phys.* B **456** 734 (erratum)

[6] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry II: volume operators *Adv. Theor. Math. Phys.* **1** 388–429

[7] Thiemann T 1998 A length operator for canonical quantum gravity *J. Math. Phys.* **39** 3372–92 (*Preprint* gr-qc/9606092)

[8] Thiemann T 2001 Quantum spin dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories *Class. Quantum Grav.* **18** 3293–338 (*Preprint* hep-th/0005232)

Thiemann T 2001 Gauge field theory coherent states (GCS): I. General properties *Class. Quantum Grav.* **18** 2025–64 (*Preprint* hep-th/0005233)

Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): II. Peakedness properties *Class. Quantum Grav.* **18** 2561–636 (*Preprint* hep-th/0005237)

Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): III. Ehrenfest theorems *Class. Quantum Grav.* **18** 4629–81 (*Preprint* hep-th/0005234)

Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamic limit *Class. Quantum Grav.* **18** 4997–5033 (*Preprint* hep-th/0005235)

Sahlmann H, Thiemann T and Winkler O 2001 Coherent states for canonical quantum general relativity and the infinite tensor product extension *Nucl. Phys.* B **606** 401–40 (*Preprint* gr-qc/0102038)

Thiemann T 2002 Complexifier coherent states for canonical quantum general relativity *Preprint* gr-qc/0206037

[9] Sahlmann H and Thiemann T 2002 Towards the QFT on curved spacetime limit of QGR: 1. A general scheme *Preprint* gr-qc/0207030

Sahlmann H and Thiemann T 2002 Towards the QFT on curved spacetime limit of QGR: 2. A concrete implementation *Preprint* gr-qc/0207031

[10] Sahlmann H 2002 When do measures on the space of connections support the triad operators of loop quantum gravity? *Preprint* gr-qc/0207112

Sahlmann H 2002 Some comments on the representation theory of the algebra underlying loop quantum gravity *Preprint* gr-qc/0207111

Sahlmann H and Thiemann T 2003 On the superselection theory of the Weyl algebra for diffeomorphism invariant quantum gauge theories *Preprint* gr-qc/0302090

Sahlmann H and Thiemann T 2003 Irreducibility of the Ashtekar–Isham–Lewandowski representation *Preprint* gr-qc/0303074

Okolow A and Lewandowski J 2003 Diffeomorphism covariant representations of the holonomy flux algebra *Preprint* gr-qc/0302059

Fleischhack C 2004 Representations of the Weyl algebra in quantum geometry *Preprint* math-ph/0407006

Lewandowski J, Okolow A, Sahlmann H and Thiemann T Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, *Comm. Math. Phys.* at press (*Preprint* gr-gc/0504147)

[11] Ashtekar A and Isham C J 1992 Representations of the holonomy algebras of gravity and non-Abelian gauge theories *Class. Quantum Grav.* **9** 1433 (*Preprint* hep-th/9202053)

Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy $C^{\star}$ algebras *Knots and Quantum Gravity* ed J Baez (Oxford: Oxford University Press)

[12] Thiemann T 1998 Closed formula for the matrix elements of the volume operator in canonical quantum gravity *J. Math. Phys.* **39** 3347–71 (*Preprint* gr-qc/9606091)

[13] Loll R 1997 Imposing det E $>$ 0 in discrete quantum gravity *Phys. Lett.* B **399** 227 (*Preprint* gr-qc/9703033)

[14] Immirzi G 1997 Quantum gravity and Regge calculus *Nucl. Phys. Proc. Suppl.* **57** 65 (*Preprint* gr-qc/9701052)

Rovelli C and Thiemann T 1998 The Immirzi parameter in quantum general relativity *Phys. Rev.* D **57** 1009–14 (*Preprint* gr-qc/9705059)

[15] Ashtekar A, Baez J C and Krasnov K 2001 Quantum geometry of isolated horizons and black hole entropy *Adv. Theor. Math. Phys.* **4** 1–94 (*Preprint* gr-qc/0005126)

Meissner K 2004 Black hole entropy in loop quantum gravity *Class. Quantum Grav.* **21** 5245–52 (*Preprint* gr-qc/0407052)

Domagala M and Lewandowski J 2004 Black hole entropy from quantum geometry *Class. Quantum Grav.* **21** 5233–44 (*Preprint* gr-qc/0407051)

[16] Brunnemann J and Thiemann T 2004 Simplification of the spectral analysis of the volume operator in loop quantum gravity *Preprint* gr-qc/0405060

[17] Smolin L Recent developments in non-perturbative quantum gravity *Preprint* hep-th/9202022

[18] Sexl R and Urbantke H 1984 *Relativität, Gruppen und Teilchen* (Berlin: Springer)

[19] Brunnemann J Private communication

[20] Giesel K and Thiemann T 2005 Consistency check on volume and triad operator quantisation in loop quantum gravity: I *Class. Quantum Grav.* **23** 5667–91 (*Preprint* gr-qc/0507036)

[21] Fairbairn W and Rovelli C 2004 Separable Hilbert space in loop quantum gravity *J. Math. Phys.* **45** 2802 (*Preprint* gr-qc/0403047)

[22] Grot N and Rovelli C 1996 Moduli-space structure of knots with intersections *J. Math. Phys.* **37** 3014 (*Preprint* gr-qc/9604010)