Degenerate Configurations, Singularities and the Non-Abelian Nature of Loop Quantum Gravity

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Abstract

Degenerate geometrical configurations in quantum gravity are important to understand if the fate of classical singularities is to be revealed. However, not all degenerate configurations arise on an equal footing, and one must take into account dynamical aspects when interpreting results: While there are many degenerate spatial metrics, not all of them are approached along the dynamical evolution of general relativity or a candidate theory for quantum gravity. For loop quantum gravity, relevant properties and steps in an analysis are summarized and evaluated critically with the currently available information, also elucidating the role of degrees of freedom captured in the sector provided by loop quantum cosmology. This allows an outlook on how singularity removal might be analyzed in a general setting and also in the full theory. The general mechanism of loop quantum cosmology will be shown to be insensitive to recently observed unbounded behavior of inverse volume in the full theory. Moreover, significant features of this unboundedness are not a consequence of inhomogeneities but of non-Abelian effects which can also be included in homogeneous models.

1 Introduction

In loop quantum cosmology [1, 2] and loop inspired approaches [3, 4] to quantum cosmology one often considers operators for inverse powers of the determinant of the metric such as $(\det q)^{-1/2}$ as they are needed for matter Hamiltonians and can be used for a first guess as to the fate of classical singularities in a quantization. Since many models lead to bounded operators for inverse volume [5, 6, 7, 3, 4, 8], following methods of full loop quantum gravity [9], conclusions for the non-singular nature of quantum gravity have often been drawn in recent years. This is the case generically, for instance, in isotropic cosmology [10, 7] where bounded inverse volume expressions play a central role in the non-singular evolution as it

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follows from a difference equation for the wave function. The issue is already different, however, in anisotropic models [11, 12] where the same quantization gives matter densities or curvatures which are not necessarily bounded when considered on full minisuperspace. In those cases, a non-singular evolution exists nonetheless, which demonstrates that, by itself, boundedness of kinematical operators such as inverse volume is not essential and that its role can only be seen when combined with dynamical information such as that given by the Hamiltonian constraint. Also, implicit in using inverse volume for statements about singularities is the classical relation between curvature divergence and inverse metric components. Even if all inverse powers of the volume could be shown to be finite at the quantum level, one would still have to discuss their relation to space-time curvature, so again dynamical properties, which is more complicated. While classically, at least in isotropic cases, extrinsic curvature is proportional to an inverse power of the volume, this follows only after equations of motion are solved. Thus, also at the quantum level one has to use equations of motion to relate properties of inverse volume expressions to the singularity issue.

In order to obtain a more complete picture, it is important to analyze the behavior of inverse volume in different models and also, as far as possible, in the full theory. Recently [13], partial information from the full theory has been derived for gauge-invariant 3-vertices of spin network states which are always annihilated by the volume operator [14, 15]. Here, inverse volume operators have, as functions of the edge spins, unbounded kinematical expectation values in 3-valent vertex states. The new situation, unlike any example used in isotropic, anisotropic or even some inhomogeneous models so far [10, 11, 12, 16, 17], is that unboundedness occurs already on states which have zero volume eigenvalue. (Which still is an improvement compared to the classical behavior which would predict diverging inverse volume. That this is not happening in the loop quantization is a direct consequence of the techniques introduced in [18, 9].) While it would be possible to construct operators with similar properties in models, in loop quantum cosmology they would break parity invariance of the theory. (Parity, i.e. $\mathrm{sgn}\det E$ for the densitized triad E_i^a, has not been studied in the full calculations done so far since only configurations of vanishing determinant are considered.)

When a difference between symmetric models and the full theory is discovered, the first reflex is often to blame it, without further corroboration, on a breakdown of the minisuperspace approximation. So also in this case, e.g. inhomogeneities have quickly been named as a potential culprit. However, there are many technical and conceptual differences between models and the full theory, and so several reasons for discrepancies can be imagined. For a reasoned and informed judgment a deeper investigation is thus warranted. To illustrate this we mention atomic spectra as an example, where characteristic properties of the hydrogen atom obtained from a spherically symmetric potential remain intact but only become more complicated in details if realistic interactions are switched on. In particular, the atom remains stable unlike its classical counterpart. This is, of course, only true if interactions are restricted (in this case from other theoretical investigations or observations) and not allowed to be arbitrary. Under arbitrary interactions, properties can certainly change dramatically, and indeed do so for atoms of high central charge where the electromagnetic

interaction is stronger. Similarly, in quantum gravity it is not sufficient to look at arbitrary degenerate geometrical configurations unless their meaning is known. There must be an analog of switching on the *correct* gravitational interaction, e.g. between inhomogeneities, which means that constraint equations or observables need to be considered.

While in [13, 19] the main aim was to derive and point out potential differences (also emphasizing, as oftentimes before in loop quantum cosmology, that dynamical information has to be included), here we analyze and contrast different geometrical configurations so as to facilitate developing a better idea for the relation between models and the full theory. We will then first discuss, in Sec. 3.1, aspects of the regularization procedures of loop quantum gravity and differences to analogous procedures in an Abelian truncation which play a role for geometrical operators but have not been recognized thus far. This indicates new sources of contributions to effective equations from non-Abelian properties which can play a role for cosmological phenomenology. An analysis of inverse volume operators in different models and truncations to be discussed in Sec. 3.2 reveals a possible technical origin of bounded behavior which, however, is not directly related to physical information such as degrees of freedom. After comparing the behavior in models with the full theory and its Abelian truncation in Sec. 3.3, we discuss in Sec. 4 aspects of dynamics which in any situation ultimately has to be studied for a conclusion about singularity removal. We will find potential dynamical effects of those properties of the full theory studied in [13].

2   Inverse volume spectra: Models and the full theory

We briefly recall now which operators have been studied for quantizations of $(\det q)^{-1/2}$ and their main properties. All these constructions follow the general scheme of classically expressing inverse metric components through Poisson brackets between connection components A_a^i and a positive power of volume based on [18, 9]

$$2\pi\gamma G\epsilon^{ijk}\epsilon_{abc}\frac{E_j^b E_k^c}{\sqrt{|\det E|}}\,\mathrm{sgn}\det E = \{A_a^i, \int \sqrt{|\det E|}\mathrm{d}^3x\}\,, \tag{1}$$

(γ is the Barbero–Immirzi parameter [20, 21] and G the gravitational constant). Instead of the spatial metric q_{ab}, a densitized triad E_i^a is used which is canonically conjugate to the connection [22, 20]. This relation is then quantized using holonomies of A_a^i, the volume operator, and turning the Poisson bracket into a commutator. The resulting operator is densely defined, and in some models it was found to be even bounded. A first explicit expression for a spectrum was computed in the isotropic context [6] where, e.g.,

$$(|\det E|^{-1/2})_\mu = \left(4\gamma^{-1}\ell_\mathrm{P}^{-2}(\sqrt{V_{\mu+1}} - \sqrt{V_{\mu-1}})\right)^6 \tag{2}$$

on states $|\mu\rangle$ with a single label $\mu \in \mathbb{R}$ whose geometrical meaning is given through the densitized triad component p with eigenvalue $p_\mu = \frac{1}{6}\gamma\ell_\mathrm{P}^2\mu$ ($\ell_\mathrm{P} = \sqrt{8\pi\hbar G}$ is the Planck length and $V_\mu = |p_\mu|^{3/2}$ a volume eigenvalue). The absolute value of μ thus determines the

volume, while its sign is spatial orientation encoded geometrically in the handedness of the triad. The fact that the isotropic inverse scale factor is bounded has played a major role in further developments leading to a demonstration of singularity-free evolution in various models.

Already in anisotropic but still homogeneous models the situation is different since volume can become small even if some metric components are large. The single quantum number μ is replaced in diagonal homogeneous models [11] by three labels μ_I, $I = 1, \ldots, 3$, which determine the three diagonal components of a densitized triad by $p^I_{\mu_1,\mu_2,\mu_3} = \frac{1}{2}\gamma\ell_P^2\mu_I$ and the volume by $V_{\mu_1,\mu_2,\mu_3} = (\frac{1}{2}\gamma\ell_P^2)^{3/2}\sqrt{|\mu_1\mu_2\mu_3|}$ and the orientation by $\mathrm{sgn}(\mu_1\mu_2\mu_3)$. Volume can then become small not only when all μ_I go to zero but also when, say, two of them go to zero while the third diverges at a suitable rate. This can, and generically does, lead to unbounded behavior of inverse volume eigenvalues at smaller volume unless cancellations occur.

For an inverse volume as above, or more generally a power of the volume parameterized by $r > 0$, one arrives at an expression of the form

$$(V^{6r-4})_{\mu_1,\mu_2,\mu_3} = (\tfrac{1}{2}\gamma\ell_P^2)^{9r-6}F_r(\mu_1)|\mu_2\mu_3|^r F_r(\mu_2)|\mu_1\mu_3|^r F_r(\mu_3)|\mu_1\mu_2|^r \tag{3}$$

where F_r is a function behaving as $F_r(\mu) \sim \mu^{r-2}$ for $\mu \gg 1$ but cutting off the classical divergence at small μ. This is obtained using, analogously to Eq. (2.5) of [13] and its generalization to $r \neq \frac{1}{2}$, the classical relation

$$V^{6r-4} = |p^1 p^2 p^3|^{3r-2} = (4\pi\gamma Gr)^{-6}\left(\prod_{I=1}^{3}\{c_I, V^r\}\right)^2$$

where c_I are homogeneous connection components conjugate to the triad components p^I, $\{c_I, p^J\} = 8\pi\gamma G\delta_I^J$. One quantizes the right hand side by expressing the connection components through "holonomies,"[1] e.g. $\{c_I, V^r\} = 2\,\mathrm{tr}(\tau_I e^{c(I)\tau_I}\{e^{-c(I)\tau_I}, V^r\})$ with SU(2) generators $\tau_I = -\frac{i}{2}\sigma_I$, using the volume operator and turning the Poisson bracket into a commutator. This leads to functions

$$F_r(\mu) = r^{-2}(|\mu+1|^{r/2} - |\mu-1|^{r/2})^2\,.$$

Since there are still positive powers of μ_I in (3), unboundedness could be possible when, e.g., μ_1 becomes large. While in this case we can use the large-μ behavior of F_r and see that μ_1 does not lead to unboundedness for an inverse power (i.e. $r < \frac{2}{3}$), other quantizations or different curvature components, where not a product as above occurs but instead a sum, lead to unbounded behavior at small volume. With the above quantization, however, inverse volume or curvature operators are always *bounded, and in fact zero, on states of zero volume* since $F_r(0) = 0$.

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[1]On the relevant Hilbert space of functions on the Bohr compactification of the real line, the components c_I are not represented directly but only their exponentials $e^{i\delta c_I}$ in a manner not continuous in δ [7].

The latter property is not realized in the full theory where a non-zero and unbounded expectation value in 3-valent vertex states, equation (4.19) of [13], has recently been derived. The expression is again available for any positive value of r if one only replaces the powers $\frac{1}{4}$ of that formula by $\frac{r}{2}$. It can thus not only be used for inverse volume ($r = \frac{1}{2}$) but also for other powers such as the volume itself ($r = \frac{5}{6}$). In all these cases the function of the three spins j_I is *unbounded on zero-volume eigenstates*, showing clearly a difference to homogeneous models. This behavior on zero-volume states has been used in [19] to argue that the singularity structure of full loop quantum gravity may not be captured reliably by models.

What should make one suspicious, though, is the fact, unnoticed in [13], that not only the inverse volume ($r = \frac{1}{2}$) is unbounded on zero-volume states but also the determinant of the metric ($r = \frac{5}{6}$), which can be taken as the volume of the vertex. Thus, even though the volume eigenvalue is zero on any such state, there is a quantization of the volume which can be arbitrarily large.[2] This should not come as a big surprise since any such state must be considered deeply quantum and quantization ambiguities can have large effects. But it also shows that the geometrical, let alone dynamical, meaning of these configurations must be understood better before the results can be used for definitive conclusions. In particular the geometrical meaning of the spin labels around a vertex should be better known, similarly to the situation in homogeneous models where the parameters can directly be associated with quantized metric components. Unlike in models, a suggestive identification of those states of zero volume, or other states containing those vertices, as "big bang states" is premature.

One particularly important aspect of a geometrical configuration used for establishing non-singular evolution in models is parity. Since loop quantum gravity is based on a triad formulation of gravity, the orientation of space (not through the background manifold but through the basic variables of the theory) is encoded in sgn det E. The classical formulation is not much different from a metric formulation since the two orientations are not connected by physical trajectories but instead separated by singularities. This changes in loop quantum cosmology where the quantum evolution equation connects a wave function uniquely from one parity side to the other. For this, a region of reversed parity is required since otherwise it would not be possible to extend evolution. This is the main scheme for non-singular evolution, and its main ingredient is the parity degree of freedom and the structure of the evolution equation [5, 11, 24]. It is clear from those considerations that it is not enough to consider only zero-volume states in isolation, i.e. static properties. The main part to a conclusion of singularity removal is a consideration of transitions where zero-volume states appear, which can only come from whatever evolution equation one is using. These are thus dynamical properties, which are crucial for any singularity statement. After providing more details about degenerate configurations, we will come back to

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[2]One can change the quantization by introducing a sign factor for the sign in (1), which would give zero expectation values of the new operator in any zero-volume state [23]. However, this property would equally apply to inverse powers which then also become zero and thus bounded. Though maybe valuable, we will not consider this quantization choice further in this paper because it would remove the differences between models and the full theory related to boundedness of inverse volume on zero-volume states.

dynamics in Sec. 4.

3   Approximations in the full theory

Since exact calculations in the full theory are usually quite involved, approximations other than symmetry reduction are necessary if a wider picture is to be developed. In the present context, it can be helpful to employ an Abelianization by replacing the structure group $SU(2)$ by $U(1)^3$ and using suitable definitions of the volume operator and other objects [25]. Many calculations then simplify, which has already been used to approach zero-volume states in a coherent state and to demonstrate that at least in this case expectation values of inverse volume can remain bounded along classical trajectories [19]. As we will show now, however, the boundedness issue for inverse volume operators is quite different in $U(1)^3$ compared to $SU(2)$. Nevertheless, $U(1)^3$ calculations are not only much simpler but also provide additional information which is of interest for comparison with symmetric models.

3.1   Abelianization vs. Homogeneity

In fact, many symmetric models turn out to be related to $U(1)^3$-vertices. The labels μ_I of a diagonal homogeneous model, for instance, can be viewed as corresponding to a 6-valent $U(1)^3$-vertex with edges given by straight lines entering and leaving the vertex and having charges $(\mu_1, 0, 0)$, $(0, \mu_2, 0)$ and $(0, 0, \mu_3)$, respectively. Different holonomies in the model are still non-commuting and thus take $SU(2)$-properties into account, but as far as geometrical expressions such as the volume are concerned the model can be formulated in an effectively Abelianized manner, which is the main reason for simplifications. (Non-Abelian properties, however, enter the dynamics as given by the Hamiltonian constraint.) Looking at $U(1)^3$-configurations thus provides means to distinguish between different aspects of a symmetry reduction and to see which of them are responsible for special behavior in models. However, since introducing $U(1)^3$ also presents an approximation to the full theory, one must as well understand its range of validity and possible deviations to full $SU(2)$-calculations.

There is indeed one prominent place in our context where the non-Abelian nature plays a role. In the rewriting procedure one makes use of relations of the form

$$\epsilon \dot{e}^a \{A_a^i, V\} \approx 2\,\mathrm{tr}(\tau_i h_e \{h_e^{-1}, V\}) + O(\epsilon^2) \tag{4}$$

where h_e is a holonomy along an edge e of parameter length ϵ and V the volume of a region (τ_i are generators of the structure group). The left bracket can thus be expressed by holonomies up to terms of order ϵ^2 which vanish in a continuum limit. Alternatively, at fixed ϵ the right hand side can be seen as an approximation or regularization of the left hand side. Irrespective of the interpretation of ϵ-terms, the quantization does not depend on this parameter. The Poisson bracket gives then rise to a commutator in a vertex contribution

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to an operator, and the holonomy can be elongated or shortened at the end away from the vertex without changing the action on states.

As mentioned before in homogeneous models (see Sec. 2), the application of identities such as (4) is necessary because the Hilbert space obtained through a background independent quantization does not allow the action of individual connection components but only of holonomies. This is a characteristic property of the quantization and is present irrespective of details of a regularization or its interpretation. It is one possible origin of correction terms in, e.g., an effective Hamiltonian compared to the classical expression just as higher curvature terms often arise in effective actions [26, 27, 28]. For the arguments to follow it is important to notice that there are two different kinds of corrections on the right hand side of (4), which usually are both subsumed in the symbol $O(\epsilon^2)$: First, one has to approximate the connection component in a single point by a suitable edge integral and, secondly, that integral is exponentiated.[3] If the edge length is small, both modifications imply only small changes of the indicated order.

What is different in these two types of corrections is their dependence on the phase space variables. The left hand side of the equation, even though it contains connection components in the Poisson bracket, depends only on triad components after the bracket is computed. Similarly, if the connection component is replaced by an integral, the bracket is just evaluated in different points along the edge and still depends only on triad components. When the integral is exponentiated, however, the situation changes since in a non-Abelian theory the cancellation between h_e outside the bracket and h_e^{-1} inside is not complete: we have, schematically, $\delta h_e / \delta A_a^i(x) = h_0(x)\tau_i h_1(x)$ where $h_{0/1}(x)$ denote the holonomies running from the starting point of the edge to x and from x to the end point, respectively. Only the contribution from h_1 cancels in (4), but we are left with $h_0 \tau_i h_0^{-1}$. Again, this gives only small correction terms, but *they do depend on connection components*, not just on the triad.

In an Abelianization, on the other hand, the second type of correction terms does not appear because now the cancellation between h_e and h_e^{-1} is complete. There are still $O(\epsilon^2)$-corrections of the first kind, but they are independent of connection components. Thus, in the Abelian case a commutator as vertex contribution is a more direct quantization of the classical expression on the left hand side of (4) which is not the case in a non-Abelian framework where regularization has stronger effects. A further consequence is that the commutators obtained in a non-Abelian setting do not commute with the volume operator even though they are supposed to quantize objects which classically do not depend on connection components. Again, this can be seen as resulting from correction terms as above, which do depend on the connection. Indeed, in Abelian quantizations such commutators commute with the volume operator, even though holonomies appear in intermediate stages.

The two different correction terms also have different implications for homogeneous models. The first one, originating in replacing connection components by integrals, is clearly a consequence of a field theory. It does not arise in homogeneous models where

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[3]In non-Abelian cases these are not two independent and subsequent steps in a computational procedure; they are rather mixed up in the path ordered exponential entering the holonomy.

we can simply take exponentials such as $e^{c_{(I)}\tau_I}$ as used before. Thus, if the first correction would be crucial for the properties of [13], it would clearly be an indication that homogeneous models do not reliably capture the full situation. The second type of correction term, on the other hand, is a consequence of non-Abelian behavior. It disappears in an Abelianization irrespective of whether it is introduced as a truncation to the full theory or effectively a consequence of a particular symmetry reduction. Moreover, it even occurs in non-diagonal homogeneous models where not all derivatives with respect to connection components of, e.g., holonomies of the form $e^{c_{(I)}\Lambda_I^i\tau_i}$ (as explained in more detail in Sec. 4.3) are proportional to the same internal direction $\Lambda_I^i\tau_i$. These properties, which we will discuss further in what follows, thus allow one to *distinguish between non-Abelian effects and effects of inhomogeneities.*

The above remarks on regularization are by no means saying that the quantization would be wrong or suspect. The classical limit is not affected since the relation is valid in regions of phase space where semiclassical behavior is expected. In this limit, one obtains similar behavior of Abelian and non-Abelian calculations (which in some cases can also be demonstrated explicitly [29]). But when regimes close to potential singularities are studied, one should pay utmost attention to such correction terms. In fact, the rewriting procedure introduces such corrections which are expected to be strong close to singularities and can significantly influence even qualitative behavior. In addition, quantization ambiguities appear since rewriting is possible in different ways.[4]

One example for such quantization ambiguities in action is the already observed discrepancy, based on [13], between unbounded behavior of a full quantization of $\sqrt{\det q}$ in a vertex even though the basic volume operator annihilates that state. Since both operators correspond to the same classical expression, this different behavior is caused by a quantization ambiguity which can be traced back to the relation (4) discussed above. In this case, it is clear that the volume operator should be considered more basic since it can be quantized directly using only fluxes [14, 15], and as such it is indeed essentially unique [32]. Then again, the fact that it annihilates all 3-vertices is very special, resting solely on gauge invariance, and so considering only 3-vertices is potentially misleading. Such properties of different operators quantizing volume show that a geometrical interpretation of vertex configurations can be complicated. Unless other information is available, it is safest to use only regimes where the volume operator is close to other quantizations of $\sqrt{\det q}$, which in the case of 3-valent vertices means small spins. As for inverse volume operators, they are clearly bounded on any bounded range of spins even though the volume eigenvalue vanishes. Moreover, on those states, the quantized $(\det q)^{-1/2}$ is smaller than $\sqrt{\det q}$ even though this corresponds to states of small volume. There is thus a cut-off of classical divergences on degenerate vertices of the full theory.

In light of the two types of correction terms, the difference of the two volume operators on 3-vertices must be a consequence of the second type of corrections since it does

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[4]Quantization ambiguities have been studied in detail in diagonal homogenous models and found not to be essential for qualitative properties [30, 31]. The new ambiguities arising in a non-Abelian context, however, are of a different type and still have to be analyzed.

not occur, for the same valence of a vertex,[5] in an Abelian truncation (see the following subsection). It is thus not a consequence of inhomogeneities. It is, rather, a consequence of additional correction terms which only appear in a non-Abelian setting and can lead to more distinguishing features between otherwise similar operators. The same conclusions then apply to the behavior noticed in [13] where commutators such as inverse volume or the alternative volume are unbounded on states which are annihilated by the basic volume. It is then crucial to understand which operator among different choices is relevant for the identification of singular states.

Fortunately, the presence of quantization ambiguities does not preclude definitive conclusions. In the present case, where different quantizations of volume lead to different degenerate states, one can use dynamical input as additional information. After all, the singularity issue is a dynamical problem, and isolating classical geometries or quantum states corresponding to a singularity has to be done with knowledge of the evolution equations. In a canonical quantization, the evolution equations are given by the Hamiltonian constraint operators. They do indeed contain the volume operator, but only in combination with holonomies as a commutator [18]. Where explicit evolution equations have been obtained [10, 7, 11, 24], their coefficients are determined by matrix elements of these commutators. It is the potential vanishing of these coefficients which signals the possibility of a singularity in the quantum evolution, and knowing these coefficients is a prerequisite for understanding the removal of singularities at the quantum level. The conclusion is that, from a dynamical point of view, one should not look at states annihilated by the basic volume operator, but at states annihilated by commutators between the volume operator and holonomies as they appear in the constraints. These commutators also make use of the relation (4) and are thus much closer to expressions used for inverse volume. In particular, all such commutators, whether they quantize positive or inverse powers of volume or other metric components, have generically[6] the same kernel, smaller than that of the basic volume operator, since they differ only in containing different powers of the volume operator. Issues of unboundedness on degenerate states, as in [13], then do not arise.

3.2   Degenerate configurations

The situation is simpler in models studied so far since they have volume eigenstates which are identical to eigenstates of commutators of volume and holonomies (see also Sec. 4.3). To shed more light on the relation between the full non-Abelian theory and those models,

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[5]On vertices of higher valence, as dicsussed later, it can happen also in the Abelian case that states annihilated by the basic volume operator are not annihilated by commutators. This is an effect of cancellations in the volume operator occurring at higher valence, which can annihilate states even for large edge labels. Also here, inhomogeneities are not relevant which can be seen from the fact that for individual commutators compared to flux operators no crucial ambiguities arise even on higher-valent Abelian vertices.

[6]There could be accidental cancellations in explicit expressions of matrix elements which could imply slightly different kernels in some cases. This would depend very sensitively on properties of the volume spectrum and show that, if a particular operator would have a bigger kernel, the additional states would be very special.

we study different degenerate configurations in $U(1)^3$ and in particular inverse volume operators. Since, e.g., diagonal homogeneous models can be viewed as special cases of 6-valent $U(1)^3$-vertices, one can directly see where potential differences come from.

In $SU(2)$ one typically has objects such as $\operatorname{tr} \tau_I h[h^{-1}, \hat{V}]$ where holonomies in the commutator when acting on an edge of spin j lead to contributions with higher spin $j + \frac{1}{2}$ and lower spin $j - \frac{1}{2}$. In a symmetric setting such as loop quantum cosmology, whose composite operators are modeled on $SU(2)$ expressions of the full theory, one can take the trace explicitly and obtain products of the form $\sin \frac{c}{2} \hat{V} \cos \frac{c}{2} - \cos \frac{c}{2} \hat{V} \sin \frac{c}{2}$ where c is a connection component appearing in holonomies $h(c) = \exp(c\tau) = \cos \frac{c}{2} + 2\tau \sin \frac{c}{2}$ [6, 10, 7]. Since, e.g., $\cos \frac{c}{2} = \frac{1}{2}(U(c) + U(c)^{-1})$ with $U(c) = \exp(ic/2)$, the resulting operators are close to the behavior in the non-Abelian setting because U as a multiplication operator increases the label μ of a state $\langle c|\mu\rangle = \exp(i\mu c/2)$ while U^{-1} decreases it. Nevertheless, since one can simply interpret $U(c) = \exp(ic/2)$ as an Abelian holonomy, one can also view this expression as obtained in an Abelian setting. However, in a directly Abelianized model one would rather use an expression such as $iU[U^{-1}, \hat{V}]$, simply replacing the non-Abelian holonomies by Abelian ones [25]. The action is then quite different because an Abelian holonomy either increases the charge or decreases it, but does not give two such contributions at the same time as happens with $SU(2)$.

The more symmetric treatment which does model some non-Abelian behavior is precisely what leads to an inverse volume vanishing on zero-volume states in isotropic loop quantum cosmology as can easily be seen from the resulting eigenvalues. In an isotropic model, the purely Abelian version [3] results in eigenvalues of the form $V^r_{\mu+1} - V^r_\mu$ with $V_\mu \propto |\mu|^{3/2}$ while the symmetric form of loop quantum cosmology gives $V^r_{\mu+1} - V^r_{\mu-1}$ as in (2). At zero volume, $\mu = 0$, the first result is non-zero unlike the latter one. Thus, while in loop quantum cosmology the inverse volume is automatically zero on zero-volume states, not by hand but by following constructions of the full theory, the loop inspired treatment of [3] and also the Abelianized version of [13] can give non-zero results. (Ref. [13] also considers vertices of higher valence where even the symmetric treatment can give non-zero results on zero volume, as discussed below. However, those higher valent configurations are not contained in a homogeneous setting.) This demonstrates the importance of following full expressions as closely as possible in models, as advocated in loop quantum cosmology. While basic operators can directly be derived from the full theory [33, 34, 35], which is the crucial difference between loop quantum cosmology and an ordinary minisuperspace quantization, this is more complicated (and so far unfinished) for composite operators such that for them guidance from the full theory must be employed in their constructions.

We can now discuss the issue of zero-volume states in the $U(1)^3$ setting, still using our 6-vertex but now with unrestricted charges collected in a 3×3 matrix n^I_i. The volume of such a configuration is given by $V_n \propto \sqrt{|\det n|}$, such that zero-volume configurations span an 8-parameter subset of the 9-dimensional configuration space.

Following loop quantum cosmology, powers of metric components expressed through commutators will take eigenvalues of the form

$$^{(r)}e^{i_0}_{I_0} \propto V^r_{n^I_i + \delta^I_{I_0} \delta^{i_0}_i} - V^r_{n^I_i - \delta^I_{I_0} \delta^{i_0}_i}$$

where only one coefficient $n_{i_0}^{I_0}$ changes in the volume labels. For any matrix n_i^I, not just diagonal ones, this can easily be seen to be zero using multi-linearity of the determinant since the volume eigenvalues depend only on the absolute value of the determinant of n_i^I. Thus, a whole row of components $^{(r)}e_{I_0}^{i_0}$ vanishes in those cases and so do all volume or inverse volume operators $\det {}^{(r)}e_I^i$ constructed from them. The main difference to diagonal models is that cancellations as in (3) do not happen; but this is not problematic since they have not been made use of in loop quantum cosmology, anyway.

The advantage of using a 6-vertex of the above form is that the geometrical meaning of its labels is clear from the identification of n_i^I with eigenvalues of densitized triad components p_i^I through flux operators. As we have demonstrated, boundedness can be a direct consequence of effectively Abelian behavior, but it is not directly related to symmetry in the form of diagonalization or isotropy which only leads to more special matrices n_i^I. The situation for a 6-vertex in SU(2) of a similar type, which could still be interpreted as corresponding to a homogeneous model, is unfortunately more complicated and so far unknown.[7]

One can generalize the Abelian setting by introducing more edges (and thus more degrees of freedom, albeit still finitely many ones). Volume eigenvalues are then given not by a single determinant but by a sum of determinants of different matrices (for each non-planar triple of edges). In many cases one can reduce this to a single determinant using multi-linearity and gauge invariance, but in general this is not possible and there are explicit examples with unboundedness [36]. However, it is not clear what this means since also the geometrical interpretation of labels would be lost: We would have more independent labels than the nine values contained locally in a classical triad. Moreover, such an Abelian situation cannot come from a geometrically motivated symmetry reduction where any vertex which is more than 6-valent (or 6- or less-valent but without straight edges) could not be embedded in a $U(1)^3$-vertex.[8]

A different generalization brings us to inhomogeneities: So far we assumed edge labels to be identical for opposite edges of the 6-vertex, which can be interpreted as diagonal homogeneous models. When this condition is dropped, one leaves homogeneity but the configurations can still be interpreted as vertices of inhomogeneous models. (Indeed, also in the SU(2)-setting the 3-vertices of [13] can be viewed as extreme forms of such inhomogeneous 6-vertices with three vanishing labels.) For instance, if we have an Abelian vertex with only one such inhomogeneous edge with opposite labels $k_\pm$ and of diagonal form, it can be viewed as a general vertex of a polarized cylindrical wave model [34]. Volume eigenvalues are then of the form $V_{k_\pm,\nu,\mu} \propto \sqrt{|k_+ + k_-|\,\mu\nu}$. The behavior of inverse volume can easily be seen to be unchanged since we simply replace a label by a sum of two labels. Also in non-diagonal cases (which include the configurations used in [25] without any symmetry assumptions), the previous conclusions about boundedness do not change.

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[7]The unboundedness on non-Abelian 3-vertices suggests that also regular 6-vertices with equal spins on opposite edges will have unbounded expectation values of inverse volume.

[8]Effectively Abelian behavior occurs when all independent holonomies have orthogonal internal directions (see, e.g., [34]) such as $e^{c_{(I)}\tau_I}$ in diagonal homogeneous models. This is no longer possible if there are new independent connection components from additional edges.

Thus, even inhomogeneity does not lead directly to unbounded behavior.

3.3 Comparison

From the preceding expressions and discussion it is clear that a direct technical reason for vanishing inverse volume operators on zero volume eigenstates lies in volume eigenvalues which depend on the edge labels only through the absolute value of a multi-linear function. Examples are the multi-linear functions μ in the isotropic case or $\det n$ for a $U(1)^3$ 6-vertex. This is, however, not a necessary condition as demonstrated by the isotropic model where alternative volume eigenvalues $((|\mu|-1)|\mu|(|\mu|+1))^{3/2}$, somewhat closer to $SU(2)$ expressions, have been used [37, 10] which also give zero inverse volume eigenvalues. Nevertheless, the appearance of multi-linear functions is quite common, as realized e.g. in isotropic, diagonal homogeneous and some inhomogeneous models. It is, however, difficult to find a geometrical reason for unboundedness since many different possibilities are realized. Linking unbounded behavior to, say, inhomogeneities is impossible because there are (non-diagonal) homogeneous models whose volume eigenvalues do not depend on edge labels through a multi-linear function [37], but also inhomogeneous models which do have such a dependence [17].

There are different ways to break the multi-linearity by considering configurations with additional parameters. The simplest way is by adding more edges and their labels as discussed in the Abelian setting. Sometimes there are also alternative volume operators in models, e.g. [16], which lead to a different dependence on labels and thus can possibly change the qualitative behavior of eigenvalues. On the other hand, as discussed in Sec. 3.1, there are additional connection dependent terms in identities such as (4). It is then uncertain how important the role of additional degrees of freedom compared to those terms is. New parameters are in fact provided by using general non-Abelian holonomies while in symmetric models often only the Killing norm of su(2) elements is relevant. Additional correction terms are then automatically switched on when non-Abelian degrees of freedom are added.

Additional parameters appear only if one takes into account more degrees of freedom than realized in a given symmetric model. However, the physical role of these additional degrees of freedom is not always clear since the geometrically and physically relevant ones have already been picked out by the symmetry reduction. It is important to keep in mind that, in contrast to a minisuperspace quantization, the selection of degrees of freedom in loop quantum cosmology is not done solely by hand. Common to a minisuperspace reduction is the choice of a symmetry to be imposed, which specifies the physical situation of interest and the corresponding reduced classical phase space. Symmetric states, in the connection representation, are then defined as distributions in the full theory whose support contains all relevant invariant connections as a dense subset [33]. Since the support of a distribution is by definition a closed set, it is not automatic that symmetric states are not supported on some connections which do not appear as invariant ones at the classical level. At this point, crucial information from the full configuration space of generalized connections enters the basic construction of symmetric models. As the analysis reveals, the

classical set of invariant connections is indeed extended to a bigger (compactified) space of generalized invariant connections, which can also be introduced at the minisuperspace level [7]. However, one can prove [33] that this space of generalized invariant connections is a closed subset of the full space of generalized connections and indeed the support of symmetric distributions. Thus, reducing at the kinematical quantum level does not add new degrees of freedom, which shows that kinematically the symmetry reduction followed in loop quantum cosmology is consistent within the full theory. This justifies the treatment along the lines of a minisuperspace quantization, an issue which could never have been addressed without a detailed relation between models and the full theory. More complicated, and still open, is the question of how the dynamics of models and the full theory are related.

4   Evolution

In addition to geometrical, static properties of degenerate configurations one also needs to know their dynamical role. This is achieved in models by using difference equations in internal time representing the Hamiltonian constraint in the triad representation, which show how degenerate configurations are approached during physical evolution. The existence of such difference equations which are globally defined, i.e. on the full mini- or midisuperspace of the model, relies on the removal of some non-Abelian effects. General non-Abelian behavior makes fluxes, representing triad components, non-commutative[9] such that not all of them can be sharp at the same time and no triad representation exists [38]. Nevertheless, local versions of the difference equation, as indeed used in inhomogeneous models [24], can still be possible. In models one can then see that unboundedness, even if it occurs such as in anisotropic models, is no obstruction to non-singular evolution. A non-symmetric ordering as used originally (and its associated dynamical initial conditions [39, 40] at least in their most straightforward incarnation) would not work with non-zero matter densities at zero volume, but such an ordering is already ruled out in inhomogeneous models [24]. It is, however, not necessary for non-singular behavior that inverse volume expressions are zero or bounded on zero-volume states if a symmetric ordering for the constraint is used [10, 41].

4.1   General scheme

The boundedness of inverse volume operators on superspace is not relevant for the general mechanism of singularity removal in models of loop quantum gravity as long as there is a well-defined expression at all. The isotropic case, where these developments have started, is however quite special which has led to some confusion and, occasionally, an overemphasis of certain aspects. There is only one parameter characterizing an isotropic spatial geometry

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[9]This is to be distinguished from the behavior discussed in Sec. 3.1 as a consequence of (4). In this case, correction terms lead to non-commuting operators even when they correspond to metric components smeared along the same surface.

and thus only one way to approach the classical singularity on minisuperspace. In this case, a well-defined behavior of inverse powers of volume would imply boundedness which indeed is realized automatically [6]. The most crucial aspect of non-singular behavior, however, is a unique extension of evolution beyond the classical singularity which is *not* guaranteed even for bounded inverse volume (see, e.g., the discussion in [10]). On the other hand, this extension of evolution has been generalized to non-isotropic cases (making use of difference equations representing the Hamiltonian constraint in a triad representation) even when inverse powers of volume or curvature components are not bounded on the respective superspaces.

The non-trivial task in any such demonstration of non-singular behavior is, first, to identify the classically singular boundaries of superspace and, second, to study the quantum evolution as those boundaries are approached. The key reason for using models is that for them the singularity structure is often clear and the first task can be performed. Other simplifications may then arise which facilitate a direct analysis of the evolution but are not necessarily crucial for the result. For the evolution one has to specify initial values and usually also boundary values for the wave function in the *non-singular* part of superspace, such that one obtains a well-posed initial value problem. In non-singular situations, these data uniquely give the solution to the constraint not only on one classically connected part of superspace, but also on other parts separated by classical singularities. In this sense, one can evolve beyond the classical singularity and the singular boundary is removed.

Note that this is not simply an issue of counting the number of solutions through initial values and that the existence of an extension is non-trivial. Singular evolution in this setting arises when values of the wave function at the singular boundary are not determined (due to vanishing leading coefficients in a difference equation) but would be needed for evolving further. One can, of course, simply include those values as data to be specified, but this does not solve the singularity problem. It would even be possible classically, e.g. in isotropic cosmology where one can specify the scale factor and matter fields at, say, $t = 1$ as well as $t = -1$ when the singularity is at coordinate time[10] $t = 0$, and then solve the Friedmann equation with both sets of initial values and simply glue the solutions. In the classical as well as quantum case this would give a (non-unique) evolution beyond the classical singularity but would only remove a breakdown of evolution by putting in missing information by hand.

Crucial ingredients are a quantum equation for the wave function on a suitable configuration space which requires an explicit construction from quantum gravity and involves analytical techniques. This equation is often called "evolution equation" even though it is not necessary to have a global time evolution picture. One then needs to identify classically singular parts of the configuration space and suitable transversal directions along which classical evolution breaks down. These directions would, in an evolution picture, be parameterized by internal time variables. In this step one uses classical gravity and geometry. All ingredients then have to be combined in order to see if a wave function can be

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[10]We use here an argument in coordinate time since it is usually used in the classical situation. One could equally use internal time.

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extended uniquely along the transversal directions, which would correspond to evolution through the classical singularity (see also [24]). One can conclude non-singular behavior only if the extension is possible for any state or, if one has information on the physical inner product, any state relevant for the physical Hilbert space. In this way, the explicit construction of observables and a reconstruction of the corresponding geometry is by-passed, but nonetheless models can be concluded to be *physically* non-singular: if an extension for the wave function to a new region is known, one can extract information from it on both sides of a classical singularity. This is in particular true for loop quantum cosmology where kinematically both sides differ only by their orientation. If extracting an effective geometry at one side is well-defined, as is required for the correct semiclassical limit, one can do so equally at the other side for any parity invariant observable.

What is not strictly necessary but often useful for an explicit analysis are the existence of a triad representation, a global time variable, or bounded inverse volume on superspace (in general, it is enough that inverse volume remains well-defined along any transversal direction relevant for the quantum evolution; see Fig. 1). If a triad representation exists, one obtains a more intuitive picture through evolution by a difference equation. Bounded inverse volume or curvature can be helpful since then any state is in the domain of definition. The action of the matter Hamiltonian can then be computed in each iteration step. If such an operator is not bounded and only densely defined, it could happen that one needs to act on a state arising during the evolution on which the operator is not defined. This could be a reason for a breakdown of evolution related to unbounded operators. With such a property the analysis would be more complicated, but it does not immediately imply singular behavior. The dynamical behavior would have to be analyzed in a detailed manner, and all initial values leading to such a breakdown must be identified. Even if one would obtain a unique extension for generic initial values, it would be impossible to claim singularity freedom unless one can show that the singular states are removed from the physical Hilbert space. If, on the other hand, arbitrary initial values lead to a uniquely extended state, for the singularity issue it does not matter how exactly the physical Hilbert space is obtained from the space of solutions to the constraint.

4.2   Full theory

So far the observed unboundedness of inverse volume on tri-valent vertices of the full theory has no direct implications for the singularity removal mechanism of loop quantum cosmology. Those states can be considered zero-volume states (even though also this depends on quantization ambiguities, see Sec. 3.1) but it is not clear how they occur in a physical transition. For this, one would also have to consider states of non-zero volume which in some sense are close to tri-valent vertices. This could be conceivable to do in a 4-valent setting, or on a 6-vertex using [42], where one would have to determine the volume spectrum on the intertwiner space. Parity reversal will then be controlled not by spins of outer edges but by intertwiner labels. Changing from positive to negative eigenvalues of $\det E$ would not affect outer edge spins and thus not result in an intermediate 3-vertex. This could only arise after using the full Hamiltonian constraint, which does change edge spins.
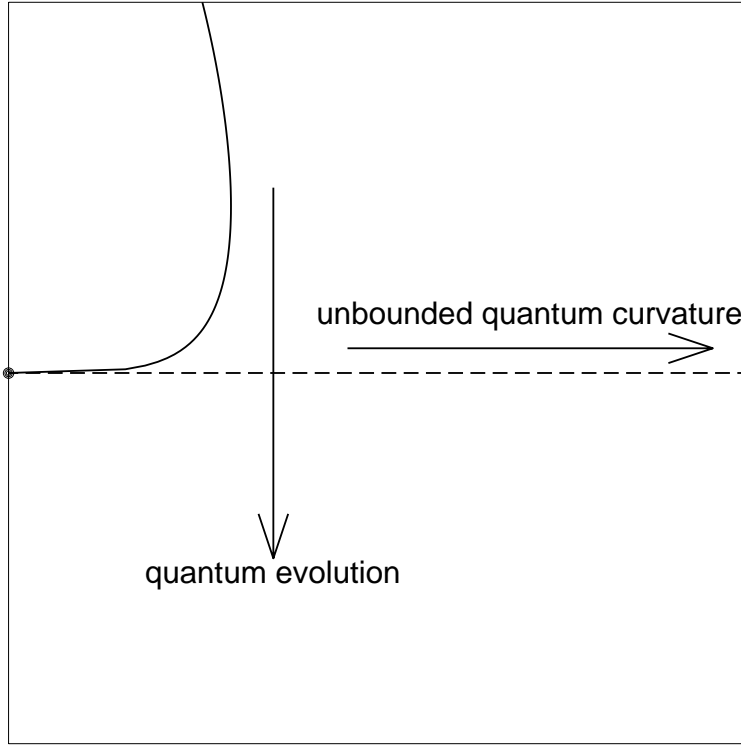
Figure 1: Illustration of the quantum behavior for wave functions on the minisuperspace of an anisotropic model compared to the classical situation (see also [11, 41]). Any classical trajectory (solid curve) runs into the classical singularity (circle). Classical expressions for curvature diverge at the dashed line which contains the classical singularity. Quantum expressions for curvature, however, are well-defined on the dashed line and, if unbounded, grow in a parallel direction (corresponding to increasing spin labels of degenerate states). The quantum evolution proceeds in a direction transversal to the classically singular dashed line and uniquely extends a wave function to the lower part of minisuperspace (obtained through orientation reversal) not reached by the classical evolution. This demonstrates the importance of knowing the singularity structure on superspace, and the irrelevance of boundedness on or close to degenerate states.

After finding suitable such transitions, one has to see if they are required by the dynamical evolution and whether or not evolution breaks down. This could possibly be done locally by a difference equation with an intertwiner spin as clock variable even though due to the non-Abelian nature no global difference equation for the constraint on superspace exists.

A mechanism alternative to using dynamical information from difference equations, which would be much more complicated to obtain in the full theory, was suggested by using coherent states and following their behavior when they are peaked on geometries of ever smaller volume [19]. One can then follow the expectation value of matter energy, which indeed in a $U(1)^3$ calculation has been shown to remain bounded provided that fluxes are bounded along the classical evolution (as realized, e.g., for vacuum Bianchi class A models even when approaching the classical singularity). This coincides with the expectation from anisotropic models.[11]

A similar calculation for $SU(2)$ would be much more involved, but to some degree one can rely on arguments that expectation values in $U(1)^3$ coherent states are close to what one gets with $SU(2)$. Nevertheless, potential differences can occur particularly when unbounded behavior is to be studied. In $U(1)^3$, the expectation value is independent of the connection where the coherent state is peaked since it appears only in a phase factor, but this is not the case in a non-Abelian setting. Moreover, as discussed above, in a non-Abelian calculation there are correction terms to the basic identity which are small for small connection components. In contrast to an Abelian calculation we thus expect an expectation value of inverse volume operators which is also connection dependent, and moreover correction terms which in general are small only for small connections. Accordingly, the Abelian calculation should be reliable when connection components are small, but not necessarily when they are large. While one can give arguments for the boundedness[12] of expectation values of inverse volume even in $SU(2)$ coherent states [23], based on the fact that holonomies are bounded functions of the connection, correction terms are important for more detailed aspects and in particular cosmological phenomenology. Since classically connection components, entering non-Abelian correction terms, become arbitrarily large when a singularity is approached, the $U(1)^3$ truncation of the full theory has to be supported by additional arguments. Similarly, potential correction terms to symmetric models and their dynamical role are still to be studied.

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[11]Indeed, a similar procedure developed in models consists in using effective equations [43, 44, 26, 27, 28]. This has been used, e.g., to study modifications to classical chaos in the Bianchi IX model [45] where the situation is comparable to that in [13, 19]: the effective evolution is governed by an *unbounded* curvature potential which, however, is *bounded* from above at *fixed volume* and in particular *along the evolution* [46, 47].

[12]As in the $U(1)^3$ calculation of [19], this only refers to boundedness of the expectation value as a function of spin labels *provided that all fluxes remain bounded*. These arguments can thus be used to illustrate the expected effective semiclassical behavior, but say nothing about boundedness of operators and the more complicated issue of extending evolution beyond a classical singularity.

4.3 Models

A general extension of wave functions beyond classical singularities has been found to be realized in all models so far where singularities have been investigated [5, 7, 11, 12, 4, 41, 24]. We now give some details in the isotropic case which also show how bringing in additional parameters of su(2) elements could potentially change the picture. Similarly to inverse volume operators, the Hamiltonian constraint [18] is constructed from holonomies and the volume operator which in the flat case gives [48, 10]

$$\hat{H} \propto \sum_{IJK} \epsilon^{IJK} \operatorname{tr}(h_I h_J h_I^{-1} h_J^{-1} h_K [h_K^{-1}, \hat{V}]) \,.$$

Holonomies are of the form $h_I = \exp(c\Lambda_I^i \tau_i)$ with the isotropic connection component c and an SO(3) matrix Λ_I^i specifying internal directions which are pure gauge. Using $h_I = \cos \frac{c}{2} + 2\Lambda_I \sin \frac{c}{2}$ with $\Lambda_I := \Lambda_I^i \tau_i$, the two contributions are

$$\epsilon^{IJK} h_I h_J h_I^{-1} h_J^{-1} = 8\Lambda_K \sin^2 \frac{c}{2} \cos^2 \frac{c}{2} + 4\epsilon^{IJK}(\Lambda_I - \Lambda_J)\sin^3 \frac{c}{2}\cos \frac{c}{2} \tag{5}$$

and

$$\begin{aligned} h_K[h_K^{-1}, \hat{V}] &= \hat{V} - \cos \frac{c}{2}\hat{V}\sin \frac{c}{2} - \sin \frac{c}{2}\hat{V}\cos \frac{c}{2} \\ &\quad -2\Lambda_K(\sin \frac{c}{2}\hat{V}\cos \frac{c}{2} - \cos \frac{c}{2}\hat{V}\sin \frac{c}{2}) \\ &\quad -2\sin \frac{c}{2}\cos \frac{c}{2}[\Lambda_K, \hat{V}] - 4\Lambda_K \sin^2 \frac{c}{2}[\Lambda_K, \hat{V}] \,. \end{aligned} \tag{6}$$

In the isotropic model, Λ_I^i commutes with the volume operator and its columns are orthogonal to each other. The constraint then reduces to

$$\hat{H}_{\text{iso}} \propto \sin^2 \frac{c}{2}\cos^2 \frac{c}{2}(\sin \frac{c}{2}\hat{V}\cos \frac{c}{2} - \cos \frac{c}{2}\hat{V}\sin \frac{c}{2})$$

(or its symmetrization) which has been used for the proof of non-singular behavior [5]. Here, one uses the action

$$\cos \frac{c}{2}|\mu\rangle = \frac{1}{2}(|\mu + 1\rangle + |\mu - 1\rangle) \tag{7}$$

$$\sin \frac{c}{2}|\mu\rangle = -\frac{i}{2}(|\mu + 1\rangle - |\mu - 1\rangle) \tag{8}$$

and derives a difference equation in the triad representation for a wave function $|\psi\rangle = \sum_\mu \psi_\mu |\mu\rangle$.

We can now bring in new parameters by allowing arbitrary Λ_I^i except that their norm must still be one (since otherwise c would be redundant). This allows six rather than three angles in specifying the internal directions, and three of them remain after removing gauge freedom. Analyzing the canonical structure of the new model and the meaning of the additional parameters (which can be related to shape parameters following [49]) is beyond the scope of this paper, but we can already see potential extra terms in the constraint. The

following statements can be interpreted in the context where these additional parameters are switched on perturbatively such that the basic representation of an isotropic model remains unchanged except that states now are also functions of the new angles.

There are two conditions of the isotropic model which are no longer satisfied: (i) $\operatorname{tr}\Lambda_I\Lambda_K = 0$ for $I \neq K$ and (ii) $[\Lambda_K, \hat{V}] = 0$. Dropping the first condition implies that also the second term in (5) contributes to the constraint which changes the difference equation. However, there is no crucial change in structure since the equation remains of the same order (the number of trigonometric functions in each term is the same). Leading coefficients of the difference equation could a priori be vanishing in different places with the new term, but this does not happen since the new contribution is imaginary (the additional sine implies an additional factor of i in a difference operator from (8)). The only change is thus in coefficients becoming complex.

Dropping the second condition is more important since it leads to a higher order of difference equations. (Note that we always have $0 = [\Lambda_K^2, \hat{V}] = \Lambda_K \cdot [\Lambda_K, \hat{V}] + [\Lambda_K, \hat{V}] \cdot \Lambda_K$ using the fact that the Λ_K are normalized. But an individual commutator $[\Lambda_K, \hat{V}]$ can still be non-zero.) Moreover, the last two terms in (6), unlike the first two lines, do not commute with the volume operator. This is different from other homogeneous models where inverse volume operators commute with the volume operator, but it would be analogous to behavior in the full theory as noticed in Sec. 3.1. Indeed, the extra terms in (6) can also contribute to non-zero expectation values of inverse volume operators in zero-volume eigenstates. If we use $\hat{O} := \operatorname{tr}(\Lambda_K h_K[h_K^{-1}, \hat{V}^r])$ for an inverse power, we will have $\langle 0|\hat{O}|0\rangle = 0$ in the isotropic zero volume eigenstate $|0\rangle$ since $\operatorname{tr}[\Lambda_K, \hat{V}] = 0$ and $\langle 0|\sin\frac{c}{2}\cos\frac{c}{2}|0\rangle = 0$. But for $I \neq K$ we can have $\langle 0|\operatorname{tr}(\Lambda_I h_K[h_K^{-1}, \hat{V}^r])|0\rangle \neq 0$ from the last term in (6) which also corresponds to some inverse power of volume. The behavior is thus closer to that in the full theory and provides an explicit example for the role of non-Abelian effects in unboundedness as discussed in general in Sec. 3.1. Still, features of the isotropic model are recognizable. In particular, there is a difference equation which, compared to the isotropic one, is of higher order and has different coefficients. Deriving implications for the singularity issue requires an interpretation of new degrees of freedom in Λ in relation to classical metric components as well as an understanding of the classical singularity structure. This requires a more detailed analysis of the canonical and geometrical structure, but the above considerations already show that kinematical properties such as those discussed in [13] can be studied in models without jumping directly to the full theory.

5 Conclusions

We have demonstrated that, from currently available information, there is no contradiction between models and the full theory of loop quantum gravity. Instead, a consistent picture emerges when all cases, isotropic as well as anisotropic or inhomogeneous models and the full theory, are considered. There is a technical difference between inverse volume in the full theory and similar operators in models, in that the expectation values in the full theory are unbounded on states of vanishing volume eigenvalue, while they are zero on such states

in models considered so far. However, in the full theory the distinction of singular states, in particular their identification with certain degenerate states, is blurred due to non-Abelian effects. There is then no crucial difference between a model where curvature is unbounded *close to singular states* (i.e. for quantum labels close to those of a singular state) and the full theory where inverse volume is unbounded *on degenerate states* which can be argued to be close to, but not necessarily identical with singular states. Any claim of a contradiction between models and the full theory, based on these properties, is thus unsubstantiated.

Nevertheless, there are certainly differences by design since models capture behavior only in a particular, geometrically selected sector of physical interest. What is clear from the considerations is that so far inhomogeneities cannot be made responsible for any explicit discrepancy between models and the full theory. To judge what implications particular properties of inverse volume operators have, their geometrical and dynamical roles must be clear before rushing to conclusions. The latter can be derived, e.g., through difference equations representing the Hamiltonian constraint or observables. But also from the classical side one needs to provide knowledge on the general singularity structure, which becomes exceedingly complicated when symmetry assumptions are dropped (see, e.g., [50]).

The main issue to be checked in further investigations is implications of non-Abelian effects not studied so far in the different sectors of loop quantum gravity (including not only cosmological situations but also black hole horizons and other models or semiclassical states, all of which often exploit possible eliminations of non-Abelian terms in explicit calculations). In models, understanding the meaning of degenerate configurations is achieved by the selection of an appropriate sector of the theory displaying the configurations of interest explicitly. We emphasize again that at this point crucial information from the full theory enters through distributional states. The symmetry is specified for the physical context, and then the relevant quantum degrees of freedom result through derivation. (Similarly, imposing black hole horizons selects the appropriate degrees of freedom, again in an effectively Abelian manner, through physical conditions for an isolated horizon [51] relating horizon degrees of freedom to full flux operators.) This suggests that non-Abelian degrees of freedom are not always crucial physically, which can also be seen from the fact that diagonal homogeneous models, which in contrast to non-diagonal ones can effectively be Abelianized, show the complete behavior of cosmological evolution [49].

Still, additional correction terms in a non-Abelian situation can provide characteristic effects. Better understanding non-Abelian behavior is thus not just important for the relation between models and the full theory, i.e. as a test of approximations, but can also provide new physical insights. The non-commutative behavior of quantum geometry [38] so far has not been made use of in cosmological investigations in loop quantum gravity, while non-commutative geometry itself has given rise to several cosmological applications (see, e.g., [52, 53]). Combining loop cosmological phenomenology with non-commutative behavior thus has the potential of providing further scenarios for the very early universe.

Acknowledgements

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