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Uniform light-cone gauge for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$: solving $\mathfrak{su}(1|1)$ sector

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Abstract: We introduce a uniform light-cone gauge for strings propagating in $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. We use the gauge to analyze strings from the $\mathfrak{su}(1|1)$ sector, and show that the reduced model is described by a quadratic action for two complex fermions. Thus, the uniform light-cone gauge allows us to solve the model exactly. We analyze the near BMN spectrum of states from the $\mathfrak{su}(1|1)$ sector and show that it correctly reproduces the $1/J$ corrections. We also compute the spectrum in the strong coupling limit, and derive the famous $\lambda^{1/4}$ asymptotics. We then show that the same string spectrum can be also derived by solving Bethe ansatz type equations, and discuss their relation to the quantum string Bethe ansatz for the $\mathfrak{su}(1|1)$ sector.

Keywords: Bethe Ansatz, Penrose limit and pp-wave background, AdS-CFT Correspondence.

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Contents

1. Introduction

In recent years we have received an impressive evidence that integrability might provide a key concept to get more insight into the complicated dynamics of the large N gauge and string theories, and, in particular, into the AdS/CFT duality conjecture [1]. Indeed, the $\mathcal{N}=4$ SYM in the large N limit is conjectured to be an integrable model [2, 3], at least in a certain asymptotic approximation. Integrability allows one to formulate the corresponding Bethe ansatz whose solutions encode the spectrum of the model. A nice and distinguished feature of this approach is that at many instances the Bethe equations can be solved exactly or used indirectly to make a comparison with the dual string theory. There has been a lot of discussion in the recent literature concerning construction and applications of the Bethe ansatz to the $\mathcal{N}=4$ SYM theory, we refer the reader to the comprehensive reviews [4].

The sigma-model describing Type IIB superstrings propagating in the $\mathrm{AdS}_5\times\mathrm{S}^5$ space-time [5] is also classically integrable [6]. However, due to the large number of dynamical variables and their involved interactions the quantization problem looks highly non-trivial. An interesting insight into the quantum theory can be gained by studying an expansion around the so-called plane-wave limit [7] where the string theory simplifies dramatically but still allows for a non-trivial comparison to the dual gauge theory [8].

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Perhaps one of the main outcomes of string integrability is that the string sigma-model admits a rich variety of explicit soliton solutions [9, 10]. In fact the whole classical (finite-gap) spectrum is encoded into a set of certain integral (Bethe type) equations [11] supported on the corresponding algebraic curves [12]. Finally, the (quantum) gauge and (classical) string theories reveal a certain interesting similarity between their integrable structures that can be manifested either through the study of infinite towers of conserved charges [13 − 15] or, equivalently, by comparing the corresponding Bethe equations [11, 16]. A complementary approach to quantum string based on the knowledge of a quantum integrable sigma-model in the infinite volume was suggested in [17].

Recently, the knowledge of the classical string Bethe equations [11] together with the asymptotic gauge theory Bethe ansatz [16] allowed us to conjecture a novel Bethe ansatz [18] which is supposed to capture the leading *quantum dynamics* of strings in $AdS_5 \times S^5$. We will refer to the corresponding construction as the quantum string Bethe ansatz. Conjectured originally for the so-called $\mathfrak{su}(2)$ sector, it has been generalized to other sectors [19] and, finally, to the whole superstring sigma-model [20]. Classical spinning strings, the $1/J$ corrections to energies of the plane-wave states, the famous $\lambda^{1/4}$ strong coupling asymptotics, all these limiting cases can be derived from the quantum string Bethe ansatz. As was shown very recently [21, 22] the ansatz seems to be capable to incorporate the $1/J$ corrections to classical spinning strings. Quite intriguing, it also admits interpretation in terms of integrable long-range spin chains [23] and naturally emerges in the study of the plane-wave matrix models [24]. However, in spite of all these remarkable developments it remains unclear how the quantum string Bethe ansatz could arise upon quantization of strings beyond the semi-classical approximation.[1]

On the other hand, it is known that string theory admits consistent truncations to smaller sectors which contain in particular string states dual to operators from the corresponding closed sectors of gauge theory. In gauge theory a closed sector is an invariant subspace of composite operators on which the action of the dilatation operator closes. Studying the mixing problem within a closed sector provides certain simplifications, e.g., in formulating the corresponding Bethe ansatz, etc. One can try to apply a similar idea to string theory. Instead of dealing with the complicated dynamics of the whole model one can *consistently truncate* the classical string equations to a smaller set of fields and further study their dynamical properties. One can also try to construct the quantum theory of a truncated sector although it is not a priori guaranteed that this theory will have a certain relation to the actual quantum string: the procedures of truncation and quantization are not expected to commute. Thus it is of interest to look at this problem: It might help to understand the origin and the range of validity of the quantum string Bethe ansatz as well as interrelation between truncation and quantization procedures.

From all varieties of the closed sectors [4] on the gauge theory side the so-called $\mathfrak{su}(1|1)$ sector seems particularly attractive. In the $\mathcal{N} = 1$ language it contains composite operators made of two Yang-Mills elementary fields, Z and Ψ, where Z is the complex scalar from a scalar supermultiplet and Ψ is the Weyl fermion from the gaugino supermultiplet. The

[1] Possible sources of the corrections to the gauge/string Bethe ansäte have been recently discussed in [25].

$\mathfrak{su}(1|1)$ symmetry group transforms Z and Ψ into each other. In this sector the dilatation operator [4] and the corresponding asymptotic Bethe ansatz [19] are known up to three-loop order of perturbation theory; at one loop the dilatation operator just coincides with the Hamiltonian of the free lattice fermion [26]. The coherent state description of the $\mathfrak{su}(1|1)$ sector with its further comparison to string theory was considered in [27].

In our previous work [28] we have found the consistent truncation of the classical superstring theory to the $\mathfrak{su}(1|1)$ sector. We have further removed all unphysical degrees of freedom by fixing the so-called uniform gauge [29, 30]: The world-sheet time τ was identified with the global AdS time t, while the momentum of an angle variable ϕ of S^5 was declared to be equal to the Noether charge J corresponding to translations of ϕ. The space-time energy E of the string coincides in this approach with the world-sheet Hamiltonian H which is a function of the charge J:

$$E = H(J)\,.$$

The physical degrees of freedom are two complex fermions which can be organized in a single world-sheet Dirac fermion. The resulting theory appears to be a new *non-trivial interacting theory of the 2-dim massive Dirac fermion.* It is integrable because the consistent reduction can be carried over for the Lax representation of the original sigma-model. Finally, we used the corresponding Hamiltonian to derive the $1/J$ correction to the energies of the plane-wave states and found a perfect agreement with the results by [31, 32].

To proceed with quantization one has to first identify the action and angle variables for the classical model. This is rather non-trivial in our present setting because the Lagrangian of the reduced theory is apparently complicated and involves terms up to six order in fermions and their derivatives.

In this paper we will solve and find the semi-classical spectrum $E(J)$ of our interacting theory in a way which bypasses direct diagonalization of the interacting Hamiltonian. The basic idea is to fix reparametrization invariance by choosing a gauge most suitable for computing the spectrum of E. Quite remarkably, there exists a gauge choice which linearizes equations of motion! In the following we will refer to this gauge as the *uniform light-cone gauge.* In the light-cone coordinates $x_\pm = \frac{1}{2}(\phi \pm t)$ this gauge consists in fixing $x_+ = \tau$ and $p_+ = P_+ = \text{const}$, where p_+ is the momentum conjugate to x_-. The world-sheet Hamiltonian H and the parameter P_+ are now related to the global charges of the model E and J as

$$H = E - J\,, \qquad P_+ = E + J\,.$$

Since H itself is a certain function of P_+ we get an equation

$$E = J + H(E+J)\,,$$

which can be solved for the energy $E \equiv E(J)$. It appears that the Hamiltonian corresponding to this gauge choice is just the quadratic Hamiltonian for two complex fermions! Thus, in the uniform light-cone gauge the theory becomes free and the spectrum of H is trivially computed.

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It is worth emphasizing that in this way we solved our reduced model exactly, i.e., without assuming anything about the range of J. Taking J to be large we can easily construct the $1/J$ expansion of the energy. The leading order in this expansion is the energy of a plane-wave state, while at the subleading order we again reproduce the $1/J$ correction found in [31, 32]. Remarkably, to obtain this correction in our approach we need only free fermions. The subleading $1/J^2$ correction turns out to be an analytic function of $\lambda' = \lambda/J^2$, while the recent studies [22, 33] of the $1/J$ correction to energies of classical spinning strings suggest the appearance at this order of non-analytic terms. This possible mismatch can be attributed to the fact that the $\mathfrak{su}(1|1)$ sector is not closed in quantum string theory.

It is also easy to analyze the strong coupling expansion, $\lambda \to \infty$, of our exact result. Again we reproduce the leading asymptotic $\lambda^{1/4}$, and find a disagreement of the subleading $\lambda^{-1/4}$ correction with the quantum string Bethe ansatz predictions.

The paper is organized as follows. In section 2 we explain the uniform gauge approach which is then applied in section 3 to the $\mathfrak{su}(1|1)$ sector. We show that the gauge-fixed Hamiltonian is that for free massive fermions. In section 4 we compute the spectrum of the model and analyze the near-plane wave and strong coupling expansions. We also discuss generalization to the case of non-vanishing winding. In section 5 we establish a relation between the energy spectrum we found and the quantum string Bethe ansatz. In Conclusion we discuss the consequences of our results and open problems. Finally, in the appendix A we compute the $1/J^2$ and $1/\sqrt{\lambda}$ corrections by using the quantum string Bethe ansatz and in appendix B we discuss the Lax pair which arises upon fixing the uniform light-cone gauge.

2. Uniform light-cone gauge

In this section we introduce the uniform light-cone gauge for strings propagating on a target manifold. This gauge generalizes the standard phase-space light-cone gauge of [35] to a curved background [36]. It belongs to the class of gauges used to study the dynamics of strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [29, 30].

We denote the time coordinate of the manifold by t, and assume that the manifold possesses a U(1) isometry realized by shifts of an angle variable ϕ. To impose the uniform light-cone gauge we also assume that the string sigma-model action is invariant under shifts of the time coordinate t and the angle variable ϕ, with all the other bosonic and fermionic fields being invariant under the shifts. This means that the string action does not have an explicit dependence on t and ϕ and depends only on the derivatives of the fields. An example of such a string action is provided by the Green-Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ where the metric can be written in the form

$$ds^2 = f_a(z)dt^2 + f_s(y)d\phi^2 + g^a_{ij}(z)dz^i dz^j + g^s_{ij}(y)dy^i dy^j \ .$$

Here t is the global time coordinate of AdS_5, ϕ is an angle of S^5, and z^i and y^i are the remaining coordinates of AdS_5 and S^5, respectively. Strictly speaking, the original Green-Schwarz action presented in [5] contains fermions which are charged under the U(1) transformations generated by the shifts of t and ϕ. However, it is possible to redefine the fermions and make them neutral in the same way as it was done in [28] for fermions from the $\mathfrak{su}(1|1)$ sector.

The invariance of the string action under the shifts leads to the existence of two conserved currents, E^α and J^α, and two conserved charges

$$E = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, E^0\ ; \qquad J = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, J^0\ .$$

It is clear that the charge E is the target space-time energy, and J is the total U(1) charge of the string. It is well-known that the time components, E^0 and J^0, of the abelian charges are just equal to the momenta conjugate to the coordinates t and ϕ:[2]

$$p_t \equiv E^0 \qquad\qquad p_\phi \equiv J^0\ .$$

To impose the uniform light-cone gauge we introduce the light-cone coordinates:

$$\begin{aligned}
&t = x_+ - x_-\ , \quad \phi = x_+ + x_-\ , \quad p_t = \frac{1}{2}(p_+ + p_-)\ , \quad p_\phi = \frac{1}{2}(p_+ - p_-) \\
&x_+ = \frac{1}{2}(\phi + t)\ , \quad x_- = \frac{1}{2}(\phi - t)\ , \quad p_+ = p_\phi + p_t\ , \quad p_- = p_t - p_\phi\ .
\end{aligned} \tag{2.1}$$

In terms of the light-cone coordinates the kinetic term takes the form

$$-p_t\dot{t} + p_\phi\dot{\phi} = -p_-\dot{x}_+ + p_+\dot{x}_-\ . \tag{2.2}$$

Then we fix the uniform light-cone gauge by the conditions

$$x_+ = \tau + \frac{m}{2}\sigma\ , \quad p_+ = P_+ = E + J \ \text{ is a constant}\ . \tag{2.3}$$

The integer number m is the winding number which appears because the coordinate ϕ is an angle variable with the range $0 \le \phi \le 2\pi$. It is clear from (2.2) that in this gauge the 2-dim Hamiltonian is identified with the integral over σ of the momenta p_-:

$$H = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, p_- = E - J\ . \tag{2.4}$$

In the AdS/CFT correspondence the space-time energy E of a string state is identified with the conformal dimension Δ of the dual CFT operator: $E \equiv \Delta$. Since the Hamiltonian H is a function of $P_+ = E + J$, the relation (2.4) gives us a nontrivial equation on the energy E. Computing the spectrum of H and solving the equation (2.4) would allow us to find conformal dimensions of dual CFT operators.

[2] We assume that the kinetic term in the Hamiltonian form of the string action has the form $-p_t\dot{t} + p_\phi\dot{\phi} + \cdots$, with the negative sign in front of p_t.

3. The $\mathfrak{su}(1|1)$ sector

In this section we use the uniform light-cone gauge to analyze the string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ reduced to the $\mathfrak{su}(1|1)$ sector. It was shown in [28] that the sector contains the scalars t and ϕ, and two complex fermions ϑ_3 and ϑ_8. The Lagrangian[3] of the reduced model can be written in the form

$$\begin{aligned}\mathscr{L} =& \frac{\sqrt{\lambda}}{2}\gamma^{\tau\tau}\left(\dot{t}^2-\dot{\phi}^2+\frac{i}{2}(\dot{t}+\dot{\phi})\zeta_\tau-\frac{1}{2}(\dot{t}+\dot{\phi})^2\Lambda\right)\\ &+\frac{\sqrt{\lambda}}{2}\gamma^{\sigma\sigma}\left(t'^2-\phi'^2+\frac{i}{2}(t'+\phi')\zeta_\sigma-\frac{1}{2}(t'+\phi')^2\Lambda\right)\\ &+\sqrt{\lambda}\gamma^{\tau\sigma}\left(\dot{t}t'-\dot{\phi}\phi'+\frac{i}{4}(\dot{t}+\dot{\phi})\zeta_\sigma+\frac{i}{4}(t'+\phi')\zeta_\tau-\frac{1}{2}(\dot{t}+\dot{\phi})(t'+\phi')\Lambda\right)+\mathscr{L}_{\rm wz}\,.\end{aligned} \tag{3.1}$$

Here the Wess-Zumino term has a remarkably simple form ($\kappa=\pm\sqrt{\lambda}/2$)

$$\mathscr{L}_{\rm wz}=\frac{\kappa}{2}\Omega_\tau(t'+\phi')-\frac{\kappa}{2}\Omega_\sigma(\dot{t}+\dot{\phi})\,, \tag{3.2}$$

and for various fermionic contributions we use the concise notations

$$\begin{aligned}&\zeta_\tau=\vartheta_i\dot{\vartheta}^i+\vartheta^i\dot{\vartheta}_i\,, \qquad \Omega_\tau=\vartheta_3\dot{\vartheta}_8+\vartheta_8\dot{\vartheta}_3-\vartheta^3\dot{\vartheta}^8-\vartheta^8\dot{\vartheta}^3\,, \qquad \Lambda=\vartheta_i\vartheta^i\,,\\ &\zeta_\sigma=\vartheta_i\vartheta'^i+\vartheta^i\vartheta'_i\,, \qquad \Omega_\sigma=\vartheta_3\vartheta'_8+\vartheta_8\vartheta'_3-\vartheta^3\vartheta'^8-\vartheta^8\vartheta'^3\,.\end{aligned} \tag{3.3}$$

It is important to mention that the periodicity condition for the fermions ϑ_3 and ϑ_8 depends on the winding number m. If m is even the fermions are periodic, and if m is odd they are anti-periodic. The dependence appears because one makes the original periodic fermions neutral under the shifts of t and ϕ by means of a field redefinition, and this induces the change in the periodic condition, see [28] for details.[4]

In [28] the action (3.1) was studied by imposing the phase-space uniform gauge $t=\tau$, $p_\phi=J$, where p_ϕ is the canonical momentum conjugate to the angle variable ϕ. It was shown that the gauge-fixed action arising in this way defines an integrable model of a massive *interacting* Dirac fermion described by the following Lagrangian

$$\begin{aligned}\mathscr{L} =& J\Big[-1-\frac{1}{2}\left(i\bar{\psi}\rho^\alpha\partial_\alpha\psi-i\partial_\alpha\bar{\psi}\rho^\alpha\psi\right)+\bar{\psi}\psi\\ &-\frac{1}{4}\epsilon^{\alpha\beta}(\bar{\psi}\partial_\alpha\psi\ \bar{\psi}\rho^5\partial_\beta\psi-\partial_\alpha\bar{\psi}\psi\ \partial_\beta\bar{\psi}\rho^5\psi)+\frac{1}{8}\epsilon^{\alpha\beta}(\bar{\psi}\psi)^2\partial_\alpha\bar{\psi}\rho^5\partial_\beta\psi\Big]\,,\end{aligned} \tag{3.4}$$

where ψ is the fermion, and ρ^α are 2-dim Dirac matrices. Here we will reanalyze (3.1) by imposing the uniform light-cone gauge, and show that in this gauge the gauge-fixed action is a free action for the fermions ϑ_3 and ϑ_8. This allows us to solve the integrable system described by the Lagrangian (3.4) in the semi-classical approximation. We do not know however if the change of gauge leads to quantum-equivalent models because the reduction of the string theory to the $\mathfrak{su}(1|1)$ sector breaks the conformal invariance that is necessary

[3] We are using the notations from [28]. The reader can consult [28] for details of the reduction.

[4] An equivalent change of boundary conditions was also found in the analysis of the spectrum of fluctuations around a multi-spin circular string [34].

for quantum equivalence of different gauges. It would be interesting to quantize (3.4) directly, and compare its spectrum with the spectrum we find from the light-cone gauge free fermion action.

Introducing the light-cone coordinates (2.1) we rewrite (3.1) in the Hamiltonian form

$$\begin{aligned}\mathscr{L} =& - p_-\dot{x}_+ + p_+\dot{x}_- - \frac{i}{4}p_+\zeta_\tau + \kappa x_+'\Omega_\tau \\ &- \frac{1}{\gamma^{\tau\tau}\sqrt{\lambda}}\Big[\frac{1}{2}p_+p_- + \frac{1}{4}p_+^2\Lambda - \frac{\kappa}{2}p_+\Omega_\sigma + \frac{i}{2}\lambda x_+'\zeta_\sigma - 2\lambda x_+'x_-' - \lambda {x_+'}^2\Lambda\Big] \\ &+ \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big[- \frac{i}{4}p_+\zeta_\sigma + p_+x_-' - p_-x_+' + \kappa x_+'\Omega_\sigma\Big] . \end{aligned} \tag{3.5}$$

As is usual in string theory with two-dimensional reparametrization invariance, the components of the world-sheet metric $\gamma^{\alpha\beta}$ enter the phase-space Lagrangian in the form of the Lagrangian multipliers. Imposing the uniform light-cone gauge (2.3) and solving equations of motion for the components $\gamma^{\tau\tau}$ and $\gamma^{\tau\sigma}$ we find

$$p_- = \kappa\Omega_\sigma - \frac{1}{2}P_+\Lambda\,, \tag{3.6}$$

$$x_-' = \frac{i}{4}(\zeta_\sigma + im\Lambda)\ . \tag{3.7}$$

Integrating (3.7) over σ, we get the level-matching condition

$$\mathcal{V} = \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\,\frac{i}{2}(\zeta_\sigma + im\Lambda) = m\ . \tag{3.8}$$

As usual the condition should be imposed on the physical states of the model. In fact the field x_- is unphysical and varying the Lagrangian w.r.t. p_+ we find that it evolves according to a first-order equation

$$\dot{x}_- = \frac{i}{4}(\zeta_\tau + 2i\Lambda)\,. \tag{3.9}$$

The components of the world-sheet metric can be found from the equations of motion for p_- and x_- and they are given by

$$\gamma^{\tau\tau} = \frac{1}{2}\Big(\frac{\sqrt{\lambda}}{P_+}m^2 - \frac{P_+}{\sqrt{\lambda}}\Big)\,, \qquad \gamma^{\tau\sigma} = -\frac{m\sqrt{\lambda}}{P_+}\,. \tag{3.10}$$

Since the unitarity of the model requires $\gamma^{\tau\tau} < 0$ we get that $P_+ = E + J > \sqrt{\lambda}|m|$. The origin of this condition is easy to understand. If $m \neq 0$ the string winds around a circle, and has the length equal to $2\pi|m|$. The energy of such a string must be greater than the product of the string tension and its length, and this leads to the condition. The condition shows that the energy of a long winding string always scales as $\sqrt{\lambda}$ [30] as opposite to the usual $\lambda^{1/4}$ scaling of a short string with $m = 0$.

Finally, substituting the solutions of the Virasoro constraints to the Lagrangian (3.5), we get the gauge-fixed Lagrangian for strings in the $\mathfrak{su}(1|1)$ sector

$$\mathscr{L} = -\frac{i}{4}P_+\zeta_\tau + \frac{1}{2}\kappa m\Omega_\tau - \kappa\Omega_\sigma + \frac{1}{2}P_+\Lambda\,. \tag{3.11}$$

Recalling the definitions (3.3), we see that the action is a *free* action for two complex fermions! If the winding number $m = 0$ then rescaling σ it can be cast to the form of the action for a free massive Dirac fermion [28].

4. Spectrum

In this section we discuss the spectrum of the model. We start with the simplest case of the vanishing winding number $m = 0$. In this case the Lagrangian coincides with the quadratic part of the Lagrangian obtained in [28], and, therefore, we can just use the results from [28]. It was shown there that after a proper change of the fermionic variables (see section 7 of [28]) the action (3.11) takes the form

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \left[-i\left(a_n^+ \dot{a}_n^- + b_n^+ \dot{b}_n^-\right) - \omega_n \left(a_n^+ a_n^- + b_n^+ b_n^-\right)\right] , \tag{4.1}$$

where $\omega_n = \sqrt{1 + \tilde{\lambda} n^2}$, and we define $\tilde{\lambda}$ by the formula

$$\tilde{\lambda} = \frac{4\lambda}{P_+^2} = \frac{4\lambda}{(E+J)^2} .$$

In terms of the oscillators $a^\pm, b^\pm$ the level matching condition has the usual form

$$\mathcal{V} = \frac{2}{P_+} \sum_{n=-\infty}^{\infty} \left(n\, a_n^+ a_n^- - n\, b_n^+ b_n^-\right) = 0 , \tag{4.2}$$

and therefore the sum of a-modes should be equal to the sum of b-modes. As was discussed in [28], the SYM operators from the $\mathfrak{su}(1|1)$ subsector are dual to states obtained by acting by operators a_n^+ on the vacuum. A general M-impurity state with $M = M_a + M_b$ obtained by acting by M_a operators a_n^+ and M_b operators b_n^+ is

$$|M_a, M_b\rangle = b_{j_1}^+ \dots b_{j_{M_b}}^+\, a_{i_1}^+ \dots a_{i_{M_a}}^+ |0\rangle . \tag{4.3}$$

It is obvious that the 2-dim energy of this state is equal to

$$H|M_a, M_b\rangle = \left(\sum_{i=1}^{M} \omega_{n_i}\right) |M_a, M_b\rangle . \tag{4.4}$$

As was discussed in section 2, the 2-dim energy of a string state is related to the space-time energy by the formula $H = E - J$. Taking into account (4.4), we get the following equation for the space-time spectrum of string states

$$E - J = \sum_{i=1}^{M} \sqrt{1 + \frac{4\lambda n_i^2}{(E+J)^2}} . \tag{4.5}$$

Since all fermions are neutral under the U(1) subgroup shifting the bosonic field ϕ, the state (4.3) carries the same J units of the corresponding charge for any number of excitations M. That means that an M-impurity string state should be dual to the field theory operator of the form

$$\mathrm{tr}\left(\Psi_+^{M_a} \Psi_-^{M_b} Z^{J-\frac{M}{2}}\right) + \dots , \tag{4.6}$$

where $\Psi_\pm$ are the two fermions from the gaugino multiplet of $\mathcal{N} = 4$ SYM, carrying the Lorentz charge $\frac{1}{2}$ and $-\frac{1}{2}$ under one of the $\mathfrak{su}(2)$'s from the Lorentz algebra $\mathfrak{su}(2,2)$. According to the AdS/CFT correspondence, the space-time energy E of a string state is equal to the conformal dimension of the dual CFT operator, and, therefore, solutions of eq. (4.5) give us dimensions of the operators (4.6).

In what follows we restrict our attention to the states dual to the closed $\mathfrak{su}(1|1)$ subsector of gauge theory. For such states $M_b = 0$, and the sum of the modes vanishes.

4.1 Near-plane wave correction to the energy

It is very simple to use eq. (4.5) to compute the $1/J$ correction to the energy of the plane-wave states from the $\mathfrak{su}(1|1)$ sector. All one needs to do is to introduce the effective coupling constant $\lambda' = \lambda/J^2$, and solve (4.5) in powers of $1/J$ keeping λ' and M finite. After a simple algebra we find

$$E - J = \sum_{i=1}^{M} \omega_i \left(1 - \frac{\lambda'}{2J}\sum_{j=1}^{M}\frac{n_j^2}{\omega_j}\right) + \mathcal{O}\left(\frac{1}{J^2}\right), \tag{4.7}$$

where $\omega_i = \sqrt{1+\lambda' n_i^2}$. Taking into account the level-matching condition this formula can be rewritten in the form

$$E = J + \sum_{i=1}^{M}\omega_i - \frac{\lambda'}{4J}\sum_{i\neq j}^{M}\frac{n_i^2 + n_j^2 + 2n_i^2 n_j^2\lambda' - 2n_i n_j \omega_i\omega_j}{\omega_i\omega_j}\,. \tag{4.8}$$

This *precisely* reproduces the $1/J$ correction to the M-impurity plane-wave states obtained in [32, 28] by using nontrivial interacting Hamiltonian for fermions. Here we reproduced the spectrum by using *free* fermions!

It is clear that eq. (4.5) can be used to compute the $1/J^2$ and higher corrections. For the $1/J^2$ correction we find

$$E_2 = \frac{\lambda'}{8}\sum_{i,j,k=1}^{M}\omega_i\omega_j\frac{n_k^2(3+2\lambda' n_k^2)}{\omega_k^3} + \frac{\lambda'^2}{4}\sum_{i,j,k=1}^{M}\omega_k\frac{n_i^2 n_j^2}{\omega_i\omega_j}\ , \tag{4.9}$$

where $E - J = \sum_{i=1}^{M}\omega_i + \frac{E_1}{J} + \frac{E_2}{J^2} + \cdots$. However, since the $\mathfrak{su}(1|1)$ sector is not closed in quantum string theory one should also take into account contributions from the fields which were set to zero in the reduction to the sector. In particular, we do not see the term $\lambda'^{5/2}/J^2$ recently predicted from the analysis of the $1/J$ correction to spinning strings [22]. Moreover, as follows from the analysis in next section, eq. (4.9) does not reproduce correctly even all terms analytic in λ. The $1/J^2$ corrections in the $\mathfrak{su}(2)$ sector were recently studied in [37] by using a properly adjusted fast-string action. Our results suggest that to derive the action one would have to take into account the contribution of fields that are not from the $\mathfrak{su}(2)$ sector.

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4.2 Strong coupling limit

In our derivation of eq. (4.5) we never assumed that J, and therefore E must be of order $\sqrt{\lambda}$. That means that we can also consider the strong coupling limit when $\lambda \to \infty$ and $E \sim \lambda^{1/4}$. We need to consider the two cases: *i)* $J \sim 1$, *ii)* $J \sim E \sim \lambda^{1/4}$. To simplify the notations it is convenient to introduce the radius of S^5: $R = \sqrt{\lambda}$. Then, in the first case, $J \sim 1$, we find

$$E = 2\sqrt{nR}\left(1 + \frac{1}{8R}\left(\frac{J^2}{n} + \sum_i \frac{1}{|n_i|}\right) + \frac{1}{8\sqrt{n}R^{3/2}}\sum_i \frac{1}{|n_i|} + \cdots\right) , \qquad (4.10)$$

where $n \equiv \frac{1}{2}\sum_i |n_i|$ is the level of the string state. To understand the physical meaning of the formula it is useful to find E^2:

$$E^2 = 4nR + J^2 + n\sum_i \frac{1}{|n_i|} + \frac{\sqrt{n}}{\sqrt{R}}\sum_i \frac{1}{|n_i|} + \cdots \ . \qquad (4.11)$$

It is clear from the formula that the first two terms give the usual dispersion relation for a string state of level n moving in the ϕ-direction with the momenta J. The remaining terms are the leading $1/\sqrt{R}$ corrections. Let us also note that the expansion goes in powers of $1/\sqrt{R} = 1/\lambda^{1/4}$.

In the second case, $J \sim E \sim \lambda^{1/4}$, the eq. (4.5) gives

$$E = \sqrt{4nR + J^2}\left(1 + \frac{1}{2R}\left(n + \frac{1}{2}j(j + \sqrt{4n + j^2})\right)\sum_i \frac{1}{|n_i|} + \mathcal{O}(1/R^2)\right) , \qquad (4.12)$$

where $j = J/\sqrt{R}$ is kept finite in the expansion. In this case the expansion goes in powers of $1/R = 1/\sqrt{\lambda}$.

We do not expect that our formulas reproduce correctly the leading strong coupling corrections because as we will discuss in the next section and in the appendix A they do not match the predictions of the quantum string Bethe ansatz.

4.3 Non-vanishing winding number

Here we discuss the spectrum of the model for the case of non-vanishing winding number $m \neq 0$. In this case one can show that there is a change of the fermions such that the action (3.11) takes the form

$$\mathcal{L} = \sum_{n=-\infty}^{\infty}\left[-i\left(a_n^+\dot{a}_n^- + b_n^+\dot{b}_n^-\right) - \omega_n^+\, a_n^+a_n^- - \omega_n^-\, b_n^+b_n^-\right] , \qquad (4.13)$$

where

$$\omega_n^{\pm} = \frac{P_+\sqrt{P_+^2 - \lambda m^2 + 4\lambda n^2} \pm 2\lambda mn}{P_+^2 - \lambda m^2} \ . \qquad (4.14)$$

Since the fermions are periodic if the winding number m is even and anti-periodic if m is odd, the mode numbers n in (4.13) are integer or half-integer, respectively.

The space-time energy of a generic string state (4.3) can be again found from the equation (2.4) that takes the form

$$E - J = \sum_{i=1}^{M_a} \omega_{n_i}^+ + \sum_{i=1}^{M_b} \omega_{k_i}^- \ . \tag{4.15}$$

The string states must satisfy the level matching condition (3.8). In terms of the oscillators $a^\pm, b^\pm$ it has a rather unusual form

$$\mathcal{V} = \sum_{n=-\infty}^{\infty} \left(c_n^+ \, a_n^+ a_n^- - c_n^- \, b_n^+ b_n^- \right) = m \ , \tag{4.16}$$

where

$$c_n^\pm = \frac{2nP_+ \pm m\sqrt{P_+^2 - \lambda m^2 + 4\lambda n^2}}{P_+^2 - \lambda m^2} \ . \tag{4.17}$$

Acting by the level-matching condition on a string state (4.3), we get the following condition on the mode numbers

$$\sum_{i=1}^{M_a} c_{n_i}^+ - \sum_{i=1}^{M_b} c_{k_i}^- = m \ . \tag{4.18}$$

It is not difficult to show by using (4.15) and (4.17) that the condition just says that the sum of a-modes minus the sum of b-modes is equal to mJ:

$$\sum_{i=1}^{M_a} n_i - \sum_{i=1}^{M_b} k_i = mJ \ . \tag{4.19}$$

For states from the $\mathfrak{su}(1|1)$ sector we have $M_b = 0$. The simplest state is created by acting by the operator a_{mJ}^+ on the vacuum. In this case eq. (4.15) can be solved exactly and we get for the energy of the state $\psi = a_{mJ}^+|0\rangle$

$$E_{mJ} = J + \sqrt{1 + \lambda m^2} \ .$$

This formula demonstrates explicitly that the energy of a long string with non-vanishing winding number scales as $\sqrt{\lambda}$ contrary to the usual $\lambda^{1/4}$ scaling of a short string.

Eq. (4.15) cannot be solved exactly for other states but it can be readily used to compute $1/\sqrt{\lambda}$ corrections to the energy of a long string. The form of the correction depends on what scaling we assume for J, and mode numbers n_i. To illustrate the J and mode number dependence we present below a formula for the energy of a state obtained by acting by M creation operators on the vacuum:

$$|m_1 J, \ldots, m_M J\rangle = a_{m_1 J}^+ \cdots a_{m_M J}^+ |0\rangle \ ,$$

where $m_1 + m_2 + \cdots + m_M = m$. We assume that J is kept fixed in the large λ expansion. Note also that m_i can be positive and/or negative, and are not required to be integer. Then, by using (4.15) we find the energy of the state

$$E_{m_1 J,\ldots,m_M J} = m\sqrt{\lambda} + J\sum_{i=1}^{M}\frac{|m_i|}{m} + \frac{J^2}{2m\sqrt{\lambda}}\left(1-\left(\sum_{i=1}^{M}\frac{|m_i|}{m}\right)^2\right) + \frac{1}{4\sqrt{\lambda}}\left(1+\sum_{i=1}^{M}\frac{|m_i|}{m}\right)\sum_{j=1}^{M}\frac{1}{|m_j|} + \cdots$$

We see that for all these states the large λ behavior is the same: $m\sqrt{\lambda}$. It is interesting that if some of m_i are negative the constant term in the expansion is not equal to J. If all m_i are positive the formula simplifies and takes the form

$$E_{m_1 J,\ldots,m_M J} = m\sqrt{\lambda} + J + \frac{1}{2\sqrt{\lambda}}\sum_{j=1}^{M}\frac{1}{m_j} + \cdots\ , \qquad m_i > 0\ .$$

5. Relation to quantum string Bethe ansatz

In this section we discuss the relation of the string spectrum (4.15) we obtained in the previous section with the spectrum that can be derived by using the quantum string Bethe ansatz for the $\mathfrak{su}(1|1)$ sector.

We start by showing that eqs.(4.5) and (4.15) can be derived from the following set of Bethe ansatz type equations

$$\exp\left(ip_k L + \frac{i}{2}\sum_{j=1}^{M}\left(p_k\left(e_j-1\right)-\left(e_k-1\right)p_j\right)\right) = 1\ . \tag{5.1}$$

Here p_k are to be interpreted as the momenta of excitations of a spin chain of length $L = J + M/2$ with M excitations, and

$$e_k = \sqrt{1+\frac{\lambda p_k^2}{4\pi^2}} \tag{5.2}$$

is the energy of an elementary excitation, and the spectrum is determined by the equation

$$E - J = \sum_{k=1}^{M} e_k \tag{5.3}$$

We have written eq. (5.3) in such a form to make obvious its similarity with eq. (4.15).

The sum over j in eq. (5.1) can be easily taken by using (5.3), and we get

$$\exp\left(ip_k\left(J+\frac{1}{2}M\right) + \frac{i}{2}\left(p_k\left(E-J-M\right)-2\pi m\left(e_k-1\right)\right)\right) = 1\,, \tag{5.4}$$

where

$$m = \frac{1}{2\pi} \sum_{j=1}^{M} p_j$$

is the winding number which we suppose to be an integer.

Collecting the terms with p_k together, we get

$$\exp\left(\frac{i}{2} p_k (E+J) - i\pi m e_k + i\pi m\right) = 1 \ . \tag{5.5}$$

Finally, taking the logarithm of both sides of the equation and recalling that $P_+ = E + J$, we obtain

$$\frac{1}{2} p_k P_+ - \pi m e_k = 2\pi n_k \ , \tag{5.6}$$

where the mode numbers n_k are integer if the winding number m is even, and half-integer if m is odd. Note that it is in complete agreement with our consideration in the previous section. The mode numbers n_k cannot be arbitrary, they must satisfy a consistency condition which should be equivalent to the level-matching condition. To find the condition we take the sum over k of both sides of (5.6) and get

$$\sum_{k=1}^{M} n_k = mJ \ . \tag{5.7}$$

It is exactly the same relation we obtained in the previous section from the level-matching condition.

To derive eqs.(4.5) and (4.15) let us first consider the simplest case of the vanishing winding number. Then we get from (5.6)

$$p_k = \frac{4\pi n_k}{E+J} \ , \tag{5.8}$$

and substituting the formula into (5.2) and (5.3), we immediately obtain eq. (4.5).

The consideration can be easily generalized to the case of the non-vanishing winding number. Expressing now p_k as a function of e_k, and solving eq. (5.6) for e_k, we get that the energy of an elementary excitation e_k with the mode number n_k coincides with the frequency $\omega_{n_k}^{+}$ (4.14), and therefore eq. (5.3) just takes the form of eq. (4.15).

Thus, we have shown that the space-time energy spectrum of strings in the $\mathfrak{su}(1|1)$ sector and in the uniform light-cone gauge follows from the Bethe type equations (5.1-5.3). It is not difficult to see that the equations (5.1) in fact coincide with the Bethe ansatz equations for strings in the $\mathfrak{su}(1|1)$ sector derived in [19] by analyzing the near BMN spectrum.[5] The only difference is that in [19] the energy of an elementary excitation was supposed to be

$$e_k = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} \ . \tag{5.9}$$

[5] Let us note that in [19] the winding number m was set to zero. Our consideration here is a generalization of [19] to the general m case.

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Comparing this formula with eq. (5.2), we see that the set of eqs.(5.1-5.3) is an approximation of the quantum string Bethe ansatz [19] valid in the regime of small momenta p_k. According to (5.8), this is also a high-energy regime where $n_i/(E+J) \ll 1$. It is not difficult to see that in the large J limit the approximation is valid up to the $1/J$ order (see appendix A for details) and in the strong coupling limit only up to the leading $\lambda^{1/4}$ order. At higher orders in $1/J$ and $1/\lambda^{1/4}$ the $\sin^2\frac{p}{2}$ begins to give additional contributions to the space-time energy. That means that computing $1/J^2$ and $1/\lambda^{1/4}$ corrections should provide nontrivial tests of the dispersion relation (5.9). Let us stress out that in [18] the choice of the dispersion relation (5.9) was motivated by the asymptotic Bethe ansatz in gauge theory [16]. A priori there is no reason why it should not be corrected at large λ. Recent tests of the quantum string Bethe ansatz for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors performed in [21, 37] indicate, however, that $1/J^2$ corrections are compatible with the dispersion relation. It would be very interesting to compute the leading $1/\lambda^{1/4}$ corrections.

6. Conclusion

In this paper we have introduced the uniform light-cone gauge for superstrings in $\mathrm{AdS}_5\times\mathrm{S}^5$, and applied it to analyze the classically-consistent reduction of the Green-Schwarz action to the $\mathfrak{su}(1|1)$ sector.

It appears that in this gauge the reduced model is described by a free action of two complex fermions, and therefore the spectrum of the model can be easily found. We have explained how the spectrum is related to the space-time energy spectrum of strings that by the AdS/CFT correspondence coincides with the spectrum of scaling dimensions of $\mathcal{N}=4$ SYM.

The space-time energy spectrum appeared to reproduce correctly the leading $1/J$ correction in the large J limit, and the leading $\lambda^{1/4}$ behavior in the strong coupling limit. We have shown that the space-time energy equation (4.15) can be reproduced from the low-momentum approximation to the quantum string Bethe ansatz for the $\mathfrak{su}(1|1)$ sector.

We have noted however that the naive $1/J^2$ and $1/\sqrt{\lambda}$ corrections found by using the free fermion action differ from the predictions of the quantum string Bethe ansatz, and computing them by using the whole string sigma-model would provide nontrivial tests of the ansatz.

Calculating the $1/J^2$ corrections and even the leading $1/\sqrt{\lambda}$ corrections would require using the second-order perturbation theory. The uniform light-cone gauge and $\mathfrak{su}(1|1)$ sector seem to be the most suitable ones for such a computation because in this gauge the sector is described by a free theory, and therefore only the fields that were set to zero in the reduction to the $\mathfrak{su}(1|1)$ sector would contribute to the corrections.

Our results also show explicitly that quantizing a classically-closed sector of superstrings in $\mathrm{AdS}_5\times\mathrm{S}^5$ cannot lead to results correct for finite J and λ. Moreover, quantizing such a sector in different gauges might lead to contradictory results. The reason for that is that the quantum gauge equivalence requires the conformal invariance of the string theory that is broken when we reduce the theory to a classically-closed sector. It would be interesting to study the gauge dependence of the $\mathfrak{su}(1|1)$ sector spectrum by quantizing the integrable model of a massive Dirac fermion [28] that describes strings in the $\mathfrak{su}(1|1)$ sector in the uniform gauge $p_\phi = J$.

We conclude therefore that the results derived within a closed sector must be taken with great care since the correct quantization of superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ would require taking into account all bosonic and fermionic fields of the superstring.

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A. Quantum string Bethe ansatz in the $\mathfrak{su}(1|1)$ sector

Here we outline the derivation of the $1/J^2$ correction to the energy of a plane-wave state by using the conjectured quantum string Bethe ansatz in the $\mathfrak{su}(1|1)$ sector [19, 20]. We will also analyze the strong coupling expansion $\lambda \to \infty$.

A.1 The $1/J$ expansion

To formulate the Bethe equations one introduces the function

$$x(u) = \frac{1}{2}u + \frac{1}{2}u\sqrt{1 - \frac{2g^2}{u^2}}\,, \qquad g^2 = \frac{\lambda' J^2}{8\pi^2}$$

and the notation $x^\pm = x(u \pm \frac{i}{2})$. The variable u is related to the momentum p of an elementary excitation through the formula

$$ip = \log\frac{x^+(u)}{x^-(u)}\,. \tag{A.1}$$

The (logarithm of) Bethe equations are the set of M equations for the momenta p_k, $k = 1, \ldots, M$. In the $\mathfrak{su}(1|1)$ sector they read [20]

$$\begin{aligned} iLp_k = 2\pi i n_k + \sum_{j\neq k}^{M} \log\big[1 - \frac{g^2}{2x_k^- x_j^+}\big] - \log\big[1 - \frac{g^2}{2x_k^+ x_j^-}\big] \\ + iu_{kj}\Big[\log\big[1 - \frac{g^2}{2x_k^- x_j^+}\big] + \log\big[1 - \frac{g^2}{2x_k^+ x_j^-}\big] - \log\big[1 - \frac{g^2}{2x_k^+ x_j^+}\big] - \log\big[1 - \frac{g^2}{2x_k^- x_j^-}\big]\Big]. \end{aligned} \tag{A.2}$$

Here n_k are the excitation numbers, $u_{kj} \equiv u_k - u_j$ and L is the *length* which in our present case is $L = J + \frac{1}{2}M$. As soon as momenta p_k are found the energy can be computed by using the formula

$$E^{\mathrm{BA}} = ig^2 \sum_{k=1}^{M} \left(\frac{1}{x_k^+} - \frac{1}{x_k^-} \right) = \sum_{k=1}^{M} \Big[-1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} \Big] . \tag{A.3}$$

Assuming the following expansion for momentum p_k in the large J limit

$$p_k = \frac{2\pi n_k}{J} + \frac{p_k^{(2)}}{J^2} + \frac{p_k^{(3)}}{J^3} + \dots \tag{A.4}$$

we determine the leading behavior of $u_k \equiv u(p_k)$ by using the formula (A.1). We find

$$u_k = \frac{J\omega_k}{2\pi n_k} - \frac{p_k^{(2)}}{4\pi^2 \omega_k n_k^2} + \frac{(3\omega_k^2 - 1)(-4\pi^4 n_k^2 \omega_k^2 + 3(p_k^{(2)})^2) - 12\pi n_k \omega_k^2 p_k^{(3)}}{48\pi^3 n_k^3 \omega_k^3} + \dots$$

The Bethe equations generate then the perturbative solution for p_k

$$\frac{p_k^{(2)}}{\pi} = -Mn_k + \sum_{j \neq k}^{M} n_k(1 - \omega_j) - n_j(1 - \omega_k) \tag{A.5}$$

and

$$p_k^{(3)} = -\frac{1}{2} M p_k^{(2)} + \frac{1}{2} \sum_{j \neq k}^{M} p_j^{(2)} \left(\omega_k - 1 - \frac{\lambda' n_j n_k}{\omega_j} \right) - p_k^{(2)} \left(\omega_j - 1 - \frac{\lambda' n_j n_k}{\omega_k} \right) . \tag{A.6}$$

Now by using eq. (A.3) we obtain the first few leading terms in the large J expansion of the energy

$$E^{\mathrm{BA}} = \sum_{k=1}^{M} (\omega_k - 1) + \frac{E_1^{\mathrm{BA}}}{J} + \frac{E_2^{\mathrm{BA}}}{J^2} + \dots ,$$

where

$$E_1^{\mathrm{BA}} = \lambda' \sum_{k=1}^{M} \frac{n_k}{\omega_k} \frac{p_k^{(2)}}{2\pi} \tag{A.7}$$

and

$$E_2^{\mathrm{BA}} = \frac{\lambda'}{24} \sum_{k=1}^{M} \frac{1}{\omega_k^3} \left[3 \left(\frac{p_k^{(2)}}{\pi} \right)^2 + 12 n_k \omega_k^2 \left(\frac{p_k^{(3)}}{\pi} \right) - 4\pi^2 n_k^4 \omega_k^2 \right] . \tag{A.8}$$

Thus, we have computed the $1/J^2$ correction E_2 to the energy of the plane-wave M-impurity state. On the other hand, the theory of the free Dirac fermion leads to the corrections (4.7) and (4.9) which we repeat here for convenience

$$E_1 = -\frac{\lambda'}{2} \sum_{i,j=1}^{M} \omega_i \frac{n_j^2}{\omega_j} , \tag{A.9}$$

$$E_2 = \frac{\lambda'}{8} \sum_{i,j,k=1}^{M} \omega_i \omega_j \frac{n_k^2 (3 + 2\lambda' n_k^2)}{\omega_k^3} + \frac{\lambda'^2}{4} \sum_{i,j,k=1}^{M} \omega_k \frac{n_i^2 n_j^2}{\omega_i \omega_j} . \tag{A.10}$$

Upon substituting in eqs.(A.7),(A.8) the momenta (A.5) and (A.6) one can show that the first correction to the energy is the one and the same, while eq. (A.8) coincides with eq. (A.10) except for the term which explicitly depends on π^2:

$$\begin{aligned} E_1^{\mathrm{BA}} &= E_1 \,, \\ E_2^{\mathrm{BA}} &= E_2 - \pi^2 \frac{\lambda'}{6} \sum_{k=1}^{M} \frac{n_k^4}{\omega_k} \,. \end{aligned} \tag{A.11}$$

Thus, the formula for E_2^{BA} *disagrees* with eq. (4.9) which provides the $1/J^2$ correction from the theory of the free Dirac fermion. In fact, the additional term proportional to π^2 in the expression for the Bethe ansatz energy E_2^{BA} occurs due to the term p^4 of $\sin^2 \frac{p}{2}$ in the large J expansion of the elementary excitation charge e_k in eq. (5.9).

A.2 The strong coupling expansion

Now we analyze the corrections to the strong coupling limit $\lambda \to \infty$, $L \sim 1$. Our discussion is very close to that of [18]. To investigate the strong coupling expansion it is convenient to express the $x_\pm$ as functions of the momentum p.

$$x_\pm(p) = \frac{e^{\pm i \frac{p}{2}}}{4 \sin \frac{p}{2}} \Big(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \,\Big) \,. \tag{A.12}$$

The function $u(p)$ is then

$$u(p) = \frac{1}{2} \cot(\frac{p}{2}) \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \,. \tag{A.13}$$

We assume that in the strong coupling regime the momentum p admits the following expansion

$$p = \frac{p^{(1)}}{\sqrt[4]{\lambda}} + \frac{p^{(2)}}{\sqrt{\lambda}} + \ldots \,, \tag{A.14}$$

where the coefficients $p^{(i)}$ should be determined from the Bethe ansatz equations. The roots p_k (generically complex) obey the conservation law $\sum_{k=1}^{M} p_k = 0$. We find it convenient to group the leading momenta $p_k^{(1)}$ into two sets: p_k^+ with $\mathrm{Re} p_k^{(1)} > 0$, $k = 1, \ldots, m$ and p_k^- with $\mathrm{Re} p_k^{(1)} < 0$, $k = m+1, \ldots, M$.

Expanding the Bethe equations in the limit $\lambda \to \infty$ we obtain at the first three leading orders the following equations

$$\lambda^0 : \quad 2\pi n_k - \frac{1}{2\pi} \sum_{j=m+1}^{M} p_k^+ p_j^- + \sum_{j=1}^{m} \chi_{kj}^{(1)} = 0 \,, \tag{A.15}$$

$$\frac{1}{\sqrt[4]{\lambda}} : \quad L p_k^+ = \frac{1}{2\pi} \sum_{j=m+1}^{M} \Big[\pi (p_k^+ - p_j^-) + p_k^+ p_j^{(2)} + p_j^- p_k^{(2)} \Big] + \sum_{j=1}^{m} \chi_{kj}^{(2)} \,, \tag{A.16}$$

$$\begin{aligned} \frac{1}{\sqrt{\lambda}} : \quad L p_k^{(2)} = \sum_{j=m+1}^{M} \Big[& \frac{\pi}{2} \Big(\frac{p_j^-}{p_k^+} + \frac{p_k^+}{p_j^-} \Big) - \frac{1}{48\pi} (p_j^{-3} p_k^+ + p_j^- p_k^{+3}) + \frac{1}{2} (p_k^{(2)} - p_j^{(2)}) + \\ & + \frac{1}{2\pi} p_j^{(2)} p_k^{(2)} + \frac{1}{2\pi} (p_k^+ p_j^{(3)} + p_j^- p_k^{(3)}) \Big] + \sum_{j=1}^{m} \chi_{kj}^{(3)} \,. \end{aligned} \tag{A.17}$$

Here we presented equations for $k = 1, \ldots, m$ only as they are enough for our further discussion. The functions $\chi^{(1)}_{kj}$ are rather complicated, e.g.,

$$\chi^{(1)}_{kj} = \frac{1}{i} \log \frac{\frac{1}{p_j^+} + \frac{1}{p_k^+} + \frac{i}{4\pi}(p_j^+ - p_k^+)}{\frac{1}{p_j^+} + \frac{1}{p_k^+} - \frac{i}{4\pi}(p_j^+ - p_k^+)} + \tag{A.18}$$

$$+ 2\pi \Big[\frac{1}{p_k^{+2}} - \frac{1}{p_j^{+2}} - \frac{1}{16\pi^2}(p_k^{+2} - p_j^{+2}) \Big] \log \frac{\Big(\frac{1}{p_j^+} + \frac{1}{p_k^+}\Big)^2 + \frac{1}{16\pi^2}(p_j^+ - p_k^+)^2}{\Big(\frac{1}{p_j^+} + \frac{1}{p_k^+}\Big)^2 + \frac{1}{16\pi^2}(p_j^+ + p_k^+)^2} .$$

Fortunately, the only property of $\chi^{(i)}_{kj}$ we need, is that they are antisymmetric w.r.t. the change $i \leftrightarrow j$: $\chi^{(i)}_{kj} = -\chi^{(i)}_{jk}$.

Summing up the Bethe equations (A.15) over k and using the antisymmetry property of $\chi^{(1)}_{kj}$ we first find

$$\sum_{k=1}^{m} p_k^+ = - \sum_{k=m+1}^{M} p_k^- = 2\pi\sqrt{n} , \qquad n \equiv \sum_{k=1}^{m} n_k . \tag{A.19}$$

Second, summing up eqs.(A.16) and using the momentum conservation we obtain

$$L - \frac{M}{2} = -\frac{1}{2\pi} \left(\sum_{k=1}^{m} p_k^{(2)} - \sum_{j=m+1}^{M} p_j^{(2)} \right) . \tag{A.20}$$

Third, summing up eqs.(A.17) and using eqs.(A.19),(A.20) we get the relation

$$\frac{\Big(L - \frac{M}{2}\Big)^2}{4\sqrt{n}} = \sum_{k=1}^{m} \frac{24\pi^2 - p_k^{+4} + 24 p_k^+ p_k^{(3)}}{48\pi p_k^+} - \sum_{j=m+1}^{M} \frac{24\pi^2 - p_j^{-4} + 24 p_j^- p_j^{(3)}}{48\pi p_j^-} .$$

Expanding the energy in the large λ limit we get

$$E^{\rm BA} = -M + \frac{\sqrt[4]{\lambda}}{2\pi} \left(\sum_{k=1}^{m} p_k^+ - \sum_{k=m+1}^{M} p_k^- \right) + \frac{1}{2\pi} \left(\sum_{k=1}^{m} p_k^{(2)} - \sum_{j=m+1}^{M} p_j^{(2)} \right) +$$

$$+ \frac{1}{\sqrt[4]{\lambda}} \left(\sum_{k=1}^{m} \frac{48\pi^2 - p_k^{+4} + 24 p_k^+ p_k^{(3)}}{48\pi p_k^+} - \sum_{j=m+1}^{M} \frac{48\pi^2 - p_j^{-4} + 24 p_j^- p_j^{(3)}}{48\pi p_j^-} \right) + \ldots$$

Substituting here our findings we obtain

$$E^{\rm BA} = 2\left(n^2\lambda\right)^{\frac{1}{4}} - \Big(L + \frac{1}{2}M\Big) + \frac{1}{\sqrt[4]{\lambda}} \left[\frac{\Big(L - \frac{M}{2}\Big)^2}{4\sqrt{n}} + \frac{\pi}{2}\Big(\sum_{k=1}^{m} \frac{1}{p_k^+} - \sum_{j=m+1}^{M} \frac{1}{p_j^-} \Big) \right] + \ldots$$

It is rather remarkable that the subleading term in the strong coupling expansion of the Bethe ansatz energy appears to coincide with the canonical dimension of the gauge theory operator taken with the negative sign. Thus, at strong coupling the total conformal dimension, $\Delta = J + M + E^{\mathrm{BA}}$, of the dual gauge theory operator has an expansion

$$\Delta = 2\left(n^2\lambda\right)^{\frac{1}{4}} + \frac{1}{\sqrt[4]{\lambda}}\left[\frac{J^2}{4\sqrt{n}} + \frac{\pi}{2}\Big(\sum_{k=1}^{m}\frac{1}{p_k^+} - \sum_{j=m+1}^{M}\frac{1}{p_j^-}\Big)\right] + \dots \,. \tag{A.21}$$

Thus, as in [18], we found that the gauge theory operators are dual to string modes with masses $m^2 = 4n\sqrt{\lambda}$, where the level n is determined by the mode numbers of the roots with a positive real part $n = \sum_{k=1}^{m} n_k$. We also see that the constant term in the large λ expansion *cancels out*. We were not able to write this formula for general M in terms of the excitation numbers n_k because the individual momenta p_k, which are solutions of the complicated equation (A.15), are not explicitly known.

The formula (A.21) can be confronted with eq. (4.10) describing the strong coupling asymptotics of our reduced model. For the reader convenience we repeat this equation here

$$E = 2\left(n^2\lambda\right)^{\frac{1}{4}} + \frac{1}{\sqrt[4]{\lambda}}\left[\frac{J^2}{4\sqrt{n}} + \frac{\sqrt{n}}{4}\sum_i \frac{1}{|n_i|}\right] + \dots \,. \tag{A.22}$$

The leading terms in eqs.(A.21) and (A.22) coincide. In both equations the constant (subleading) piece is absent. However, the terms of order $1/\sqrt[4]{\lambda}$ are different. They coincide only for the special case of two impurities, $M = 2$.

The results about the subleading behavior of conformal dimensions should be taken with caution. Indeed, in general the quantum string Bethe ansatz [18] involves an infinite number interpolating functions $c_r(\lambda)$ with the property $c_r(\lambda) \to 1$ as $\lambda \to \infty$. The results by [22] suggest that beyond the semiclassical limit these functions become non-trivial. In our computation above we assumed that all $c_r = 1$. Taking into account the actual functions c_r (yet to be determined) might change the conclusion about the strong coupling expansion beyond the leading order.

B. Lax representation

Integrability of the classical superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was demonstrated in [6] by means of constructing the Lax (zero-curvature) representation for the superstring equations of motion. In [28] we have shown that this connection admits a consistent reduction to the fields describing excitations from the $\mathfrak{su}(1|1)$ sector. Thus, the non-trivial interacting Dirac Hamiltonian [28] which governs the dynamics in this sector is integrable, but its integrable properties are not transparent rather they are hidden in the highly non-trivial Lax pair. This pair can be formulated in terms of two 4×4 matrices, $\mathscr{L}_\sigma$ and $\mathscr{L}_\tau$, depending on a spectral parameter z and satisfying the condition of zero curvature

$$\partial_\sigma \mathscr{L}_\tau - \partial_\tau \mathscr{L}_\sigma - [\mathscr{L}_\sigma, \mathscr{L}_\tau] = 0 \tag{B.1}$$

as a consequence of the dynamical equations.

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In opposite, in our present approach based on the uniform light-cone gauge integrability of the reduced model is manifest as it is the theory of a free 2-dim massive Dirac fermion. In fact the proper choice of the gauge resulted into linearization of the dynamical equations! Non-triviality of the original theory is now hidden in the new dispersion formula relating the energy E to the charges M and J.

In spite of the manifest integrability of the model it is still interesting to know what is the reduction of the general Lax connection to the $\mathfrak{su}(1|1)$ sector in the uniform light-cone gauge. For the sake of clarity we restrict our further discussion to the case of zero winding number,[6] $m = 0$, and fix $\kappa = \frac{\sqrt{\lambda}}{2}$. We also introduce the concise notation for the original fermionic variables ϑ of [28]

$$\psi_1 = \vartheta_3\,, \qquad \psi_2 = \vartheta^8\,, \qquad \overset{*}{\psi}_1 = \vartheta^3\,, \qquad \overset{*}{\psi}_2 = \vartheta_8 \tag{B.2}$$

as well as two even quantities

$$\varsigma = \psi_1\,\overset{*}{\psi}{}'_1 + \overset{*}{\psi}_1\,\psi'_1 - \overset{*}{\psi}_2\,\psi'_2 - \psi_2\,\overset{*}{\psi}{}'_2\,, \tag{B.3}$$

$$\varrho = (\psi_1\,\overset{*}{\psi}_2 - \overset{*}{\psi}_1\,\psi_2)'\,. \tag{B.4}$$

One can further show that the minimal Lax connection for the $\mathfrak{su}(1|1)$ sector in the uniform light-cone gauge is given in terms of 2×2 matrices of the form

$$\mathscr{L}_\sigma = \begin{pmatrix} -\frac{i}{\sqrt{\tilde\lambda}}\frac{z}{1-z^2} + \frac{1}{4}\varsigma & \frac{-\overset{*}{\psi}{}'_1 - iz\overset{*}{\psi}{}'_2}{\sqrt{1-z^2}} \\ \frac{-\psi'_1 + iz\psi'_2}{\sqrt{1-z^2}} & \frac{i}{\sqrt{\tilde\lambda}}\frac{z}{1-z^2} + \frac{1}{4}\varsigma \end{pmatrix}, \tag{B.5}$$

$$\mathscr{L}_\tau = \begin{pmatrix} -\frac{i}{2}\frac{1+z^2}{1-z^2} + \frac{i\sqrt{\tilde\lambda}}{4}\varrho & -\sqrt{\tilde\lambda}\frac{z\overset{*}{\psi}{}'_1 + i\overset{*}{\psi}{}'_2}{\sqrt{1-z^2}} \\ -\sqrt{\tilde\lambda}\frac{z\psi'_1 - i\psi'_2}{\sqrt{1-z^2}} & \frac{i}{2}\frac{1+z^2}{1-z^2} + \frac{i\sqrt{\tilde\lambda}}{4}\varrho \end{pmatrix}. \tag{B.6}$$

Here z is the spectral parameter and we recall the definition $\tilde\lambda = \frac{4\lambda}{P_+^2}$. One can easily check that the Lax connection above has zero curvature (B.1) by virtue of the fermionic equations of motion followed from eq. (3.11)

$$\dot\psi_1 = i\psi_1 - i\sqrt{\tilde\lambda}\,\psi'_2\,, \qquad \dot\psi_2 = -i\psi_2 + i\sqrt{\tilde\lambda}\,\psi'_1\,. \tag{B.7}$$

Thus, the Lax connection for the whole superstring sigma-model [6] boils down under the reduction to the $\mathfrak{su}(1|1)$ sector in the uniform light-cone gauge to that of the free massive Dirac fermion. In writing component $\mathscr{L}_\tau$ we also used the evolution equations (B.7) to trade the τ-derivatives of fermions for their σ-derivatives.

[6]Of course, for $m \neq 0$ the reduced Lax connection also exists.

Some comments are in order. Upon choosing the minimal reduction to 2×2 matrices the corresponding Lax connection has the non-vanishing supertrace (its usual trace is also non-zero). This is not problematic since the only requirement that matters is the fulfillment of the equations of motion which is indeed the case. One can also see that the off-diagonal part of $\mathscr{L}_\sigma$ contains the σ-derivatives of the fermions rather than the fermions themselves. This problem can be cured by means of a certain gauge transformation which removes in $\mathscr{L}_\sigma$ the off-diagonal derivatives in favor of the fields. Then the off-diagonal elements of eq. (B.1) will directly generate the fermionic equations (B.7) rather than their σ-derivatives. This gauge transformation leads however to a slightly more complicated form of the diagonal matrix elements in $\mathscr{L}_\sigma$ and $\mathscr{L}_\tau$ and, therefore, we have not attempted to discuss it here.

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