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Green-Schwarz Strings in TsT-transformed backgrounds

Luis F. Alday,[a] **Gleb Arutyunov**[a*] **and Sergey Frolov**[b*]

[a] *Institute for Theoretical Physics and Spinoza Institute, Utrecht University 3508 TD Utrecht, The Netherlands*
[b] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut Am Mühlenberg 1, D-14476 Potsdam, Germany*
E-mail: L.F.Alday@phys.uu.nl, G.Arutyunov@phys.uu.nl, frolovs@aei.mpg.de

Abstract: We consider classical strings propagating in a background generated by a sequence of TsT transformations. We describe a general procedure to derive the Green-Schwarz action for strings. We show that the U(1) isometry variables of the TsT-transformed background are related to the isometry variables of the initial background in a universal way independent of the details of the background. This allows us to prove that strings in the TsT-transformed background are described by the Green-Schwarz action for strings in the initial background subject to twisted boundary conditions. Our construction implies that a TsT transformation preserves integrability properties of the string sigma model. We discuss in detail type IIB strings propagating in the γ_i-deformed $AdS_5 \times S^5$ space-time, find the twisted boundary conditions for bosons and fermions, and use them to write down an explicit expression for the monodromy matrix. We also discuss string zero modes whose dynamics is governed by a fermionic generalization of the integrable Neumann model.

Keywords: String Duality, AdS-CFT Correspondence.

[*] Also at Steklov Mathematical Institute, Moscow.

Contents

1. Introduction

It is well-known that a T-duality transformation applied to a circle which could contract to zero size produces a singular geometry from a regular one. Recently, it was noticed in [1] that in a situation when the initial geometry contains a two-torus a regular background may be generated by using a combination of a T-duality transformation on one angle variable, a shift of another isometry variable, followed by the second T-duality on the first angle. We will refer to the chain of these transformations producing a one-parameter deformation of the initial background as a TsT transformation. The observation of [1] can be easily generalized to construct regular multi-parameter deformations of gravity backgrounds if they contain a higher-dimensional torus [2] by using a chain of TsT transformations.

A TsT transformation appears to be very useful in a search of new less supersymmetric examples of the AdS/CFT correspondence [3]. In particular, it was successfully used in [1] to obtain a deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry which was conjectured to be dual to a supersymmetric marginal deformation of $\mathcal{N} = 4$ SYM sometimes called a β deformation [4]–[6]. Various aspects of the deformed gauge and string theories, and the conjectured duality have been studied in [7, 8] by using the ideas and methods developed to test the duality between the undeformed models [9, 10].

Strings in the more general three-parameter deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [2], and the dual nonsupersymmetric deformation of $\mathcal{N} = 4$ SYM have been studied in [11]–[13]. It is unclear, however, if the nonsupersymmetric string background is stable,[1] and the double-trace operators are not generated in the deformed gauge theory, thus, breaking conformal invariance as it happens for instance in nonsupersymmetric orbifold models [15].

TsT transformations have been also used to deform other interesting string backgrounds [16]. Further related results can be found in [17, 18].

A nice property of a TsT transformation is that it can be implemented on the string sigma model level leading to simple relations between string coordinates of the initial and TsT-transformed background [2]. The relations have been used to show that classical solutions of string theory equations of motion in a deformed background are in one-to-one correspondence with those in the initial background with twisted boundary conditions imposed on the U(1) isometry fields parametrizing the torus. An interesting property of the twist is that it depends on the conserved U(1) charges of the model.

The consideration in [2] was restricted to the bosonic part of type IIB Green-Schwarz superstring action on the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$. Dealing with the Green-Schwarz superstring we face a new problem of how to define the TsT transformation for fermionic variables. The answer is not immediately clear, because the operation of T-duality must include a change of the fermionic chirality. The TsT transformation involves the angle variables which transform under the commuting isometries of the five-sphere. Generically, fermions of the Green-Schwarz superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ also transform under the same isometries. A key idea which allows us to solve the problem is to redefine the original fermions in such a way that they become neutral under the isometries in question. After this redefinition is found we can perform the TsT-transformations on the angle variables with fermions being just the spectators. The very existence of such a redefinition is non-trivial and will be established in section 3.

The aim of the current paper is to extend the discussion in [2] to the most general case of a fermionic string propagating in an arbitrary background possessing several U(1) isometries. We analyze a TsT transformation and show that if fermions are neutral under the isometries then the relations are universal and do not depend on the details of the background in complete accord with the expectations in [2]. In the case of Green-Schwarz strings in the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background our consideration implies the existence of a Lax pair representation, and, therefore, classical integrability of the model.

The plan of the paper is as follows. In section 2 we consider a general sigma model action for fermionic strings propagating in a curved background. We assume that the action is invariant under at least two U(1) isometry transformations. Each U(1) transformation is realized as a shift of an angle variables with all other bosonic and fermionic fields being neutral under the shift. We then perform a TsT transformation on a torus parametrized by any two of the angles, and find a TsT-transformed action. We show that the TsT trans-

[1] It is known that the spectrum of string theory in the TsT-transformed flat space contains tachyons [14]. It does not imply that string theory on the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ is unstable because the TsT-transformed flat space is singular at space infinity while the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ is regular everywhere. In fact, it seems that a TsT-transformation produces a nonsingular background only if the two-torus is of a finite size.

formation preserves the U(1) currents corresponding to the angles, and, moreover, the TsT-transformed angles are related to the original angles by exactly the same formulas as the ones derived for the pure bosonic case in [2] leading to the same twisted boundary conditions for the angle variables. This implies that strings in the TsT-transformed background are described by the Green-Schwarz action for strings in the initial background subject to the twisted boundary conditions. We point out that if the original Green-Schwarz string action is *classically* integrable then the TsT-transformed action is also integrable extending the consideration of [2] to the general case. Further we discuss the chains of TsT transformations applied to a background containing a d-dimensional torus, and show that the most general deformation is parametrized by a skew-symmetric $d \times d$-dimensional matrix which determines twisted boundary conditions for the U(1) isometry variables. The results obtained in section 2 have a partial intersection with those of [18] where a general bosonic string background was considered.

In section 3 we apply a sequence of TsT transformations to the Green-Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [19] to generate the Green-Schwarz action for nonsupersymmetric strings in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. We explain how to redefine the bosonic and fermionic fields so that the U(1) isometry transformations would be realized as shifts of the angle variables. We then use the considerations in section 2 to find the twisted boundary conditions for bosons and fermions, and conclude that the integrability of superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [20] implies the integrability of the fermionic string in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. We use the Lax pair for Green-Schwarz superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the twisted boundary conditions to derive the monodromy matrix for strings in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$. The monodromy matrix can be used to analyze the spectrum of classical strings in the deformed background.

In section 4 we discuss the zero-mode part of the Green-Schwarz action for nonsupersymmetric strings in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. It describes a particle with fermionic degrees of freedom moving in the deformed background. The particle action is integrable, and generalizes the well-known Neumann model to the fermionic case. The Lax pair for the model is induced by the Lax pair for strings in the deformed background. Quantization of the fermionic Neumann model should describe the spectrum of type IIB supergravity on the nonsupersymmetric γ_i-deformed background.

In Conclusion we summarize the results obtained and discuss open problems. In appendices we collect some useful formulae.

2. The γ-deformed action

We start with the following general sigma model action describing propagation of a fermionic closed string in a background with several U(1) isometries

$$
\begin{aligned}
S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\ \Big[&\gamma^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, G^0_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, B^0_{ij} \\
&+2\partial_\alpha\phi^i\left(\gamma^{\alpha\beta}U^0_{\beta,i} - \epsilon^{\alpha\beta}V^0_{\beta,i}\right) + \mathcal{L}^0_{\rm rest}\Big]\,.
\end{aligned}
\tag{2.1}
$$

Here $\frac{\sqrt{\lambda}}{2\pi}$ is the effective string tension which is identified with the 't Hooft coupling in the AdS/CFT correspondence, $\epsilon^{01} \equiv \epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} \equiv \sqrt{-h}\, h^{\alpha\beta}$, where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. In the conformal gauge $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$ although in the following we will not attempt to fix any gauge. We assume that the action is invariant under U(1) isometry transformations geometrically realized as shifts of the angle variables ϕ_i, $i = 1, 2, \ldots, d$. That means that the string background contains a d-dimensional torus T^d. We show explicitly the dependence of the action on ϕ_i, and their coupling to the background fields G^0_{ij}, B^0_{ij} and $U^0_{\beta,i}, V^0_{\beta,i}$ which generalizes the usual coupling of bosons to the target space metric and B-field. These background fields are independent of ϕ^i but can depend on other bosonic and fermionic string coordinates which are neutral under the U(1) isometry transformations. By $\mathcal{L}^0_{\text{rest}}$ we denote the part of the Lagrangian which depends on these other fields of the theory. We will see in the next section that the Green-Schwarz action for superstrings on $\text{AdS}_5 \times \text{S}^5$ [19] can be cast to the form (2.1).

The action has d global symmetries corresponding to constant shifts of $\phi's$. The corresponding Noether currents are

$$J_i^\alpha(\phi) = -\sqrt{\lambda}\Big(\gamma^{\alpha\beta}\partial_\beta\phi^j\, G^0_{ij} - \epsilon^{\alpha\beta}\partial_\beta\phi^j\, B^0_{ij} + \gamma^{\alpha\beta}U^0_{\beta,i} - \epsilon^{\alpha\beta}V^0_{\beta,i}\Big)\,, \tag{2.2}$$

and they are conserved, $\partial_\alpha J_i^\alpha = 0$, as the consequence of the dynamical equations.

Now we perform a TsT transformation of the angle variables. To this end we pick up a two-torus, for instance, the one, generated by ϕ_1 and ϕ_2. The TsT transformation consists in dualizing the variable ϕ_1 with the further shift $\phi_2 \to \phi_2 + \hat{\gamma}\phi_1$ and dualizing ϕ_1 back. Application of the TsT transformation can be symbolically expressed as the change of variables

$$(\phi_1, \phi_2) \overset{\text{TsT}}{\to} (\tilde{\phi}_1, \tilde{\phi}_2)\,. \tag{2.3}$$

The procedure to construct the TsT-transformed action is explained in appendices A and B. The corresponding action can be written in the same fashion as the original one

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\ \Big[&\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\, G_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\, B_{ij} \\ &+2\partial_\alpha\tilde{\phi}^i\left(\gamma^{\alpha\beta}U_{\beta,i} - \epsilon^{\alpha\beta}V_{\beta,i}\right) + \mathcal{L}_{\text{rest}}\Big] \end{aligned} \tag{2.4}$$

with the new fields G_{ij}, etc given in terms of the original ones. The explicit relations are listed in appendix B. Clearly, the new action also has the same number of symmetries related to the constant shifts of the variables $\tilde{\phi}^i$. The conserved Noether currents have now the form

$$\tilde{J}_i^\alpha(\tilde{\phi}) = -\sqrt{\lambda}\Big(\gamma^{\alpha\beta}\partial_\beta\tilde{\phi}^j\, G_{ij} - \epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}^j\, B_{ij} + \gamma^{\alpha\beta}U_{\beta,i} - \epsilon^{\alpha\beta}V_{\beta,i}\Big)\,. \tag{2.5}$$

The relation between the dual variables $\tilde{\phi}$ and the original ones ϕ can be found by using the formulas from appendices A and B, and is given by

$$\partial_\alpha\tilde{\phi}^1 \,= \partial_\alpha\phi^1 - \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}\partial_{\tilde{\beta}}\phi^i G_{i2} + \hat{\gamma}\partial_\alpha\phi^i B_{i2} - \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}U_{\tilde{\beta}2} - \hat{\gamma}V_{\alpha 2}$$

$$
\begin{aligned}
\partial_\alpha\tilde{\phi}^2 &= \partial_\alpha\phi^2 + \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}\partial_{\tilde{\beta}}\phi^i G_{i1} - \hat{\gamma}\partial_\alpha\phi^i B_{i1} + \hat{\gamma}\epsilon_{\alpha\beta}\gamma^{\beta\tilde{\beta}}U_{\tilde{\beta}1} + \hat{\gamma}V_{\alpha 1} \\
\partial_\alpha\tilde{\phi}^i &= \partial_\alpha\phi^i\,, \quad i \geq 3
\end{aligned}
\tag{2.6}
$$

Using these transformation rules, one can check that the following relation holds

$$
\tilde{J}_i^\alpha(\tilde{\phi}) = J_i^\alpha(\phi)\,. \tag{2.7}
$$

It shows that independently of the form of the action (2.1) and the presence of fermions the TsT transformation preserves the U(1) isometry currents corresponding to the angles ϕ_i, thus, generalizing and proving the considerations in [2] (see, also [18] where an arbitrary bosonic background was analyzed).

The equality (2.7) of the original and the TsT-transformed currents also shows that the TsT-transformation is a particular example of the *Bäcklund transformations*. Indeed, in full generality the Bäcklund transformation is defined as follows [21]

$$
\tilde{J}^\alpha - J^\alpha = \epsilon^{\alpha\beta}\partial_\beta\chi \tag{2.8}
$$

for some function χ. Here J^α and $\tilde{J}^\alpha$ correspond to the global Noether currents computed on the original and on the Bäcklund transformed solutions respectively. Eq. (2.8) states that the difference between two currents conserved dynamically, the original and the Bäcklund transformed, is proportional to the trivially conserved topological current. The TsT-transformation simply corresponds to taking $\chi = 0$. However, in our present situation we do not require that the Bäcklund transformations should preserve the boundary conditions for the fundamental fields of the theory.[2]

The relation (2.7) allows one to find a relation between the σ-derivatives of the original and transformed angles

$$
\begin{aligned}
\tilde{\phi}_1' - \phi_1' &= -\gamma J_2^\tau\,, \qquad \hat{\gamma} = \sqrt{\lambda}\gamma \\
\tilde{\phi}_2' - \phi_2' &= \gamma J_1^\tau\,, \\
\tilde{\phi}_i' - \phi_i' &= 0\,, \quad i \geq 3\,.
\end{aligned}
\tag{2.9}
$$

Here J^τ means the τ-component of the conserved current. This is the *same* relation as was found in the bosonic case [2].

Since we consider the closed strings on the γ-deformed background the angles $\tilde{\phi}_i$ have the following periodicity conditions

$$
\tilde{\phi}_i(2\pi) - \tilde{\phi}_i(0) = 2\pi n_i\,, \qquad n_i \in \mathbb{Z}\,. \tag{2.10}
$$

Then integrating eqs. (2.9) we obtain the twisted boundary conditions for the original angles ϕ_1 and ϕ_2, and the usual periodicity conditions (2.10) for the other $d-2$ angles

$$
\phi_1(2\pi) - \phi_1(0) = 2\pi(n_1 + \gamma J_2)\,, \tag{2.11}
$$

[2]It would be interesting to study the general Bäcklund transformation with a non-trivial function χ but without imposing the same boundary conditions on the original and transformed fields. This should lead to an alternative proof of integrability of strings in the γ-deformed background, in the spirit of [22, 21].

$$\phi_2(2\pi) - \phi_2(0) = 2\pi(n_2 - \gamma J_1)\,,$$

where

$$J_i = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} J_i^\tau$$

is the corresponding Noether charge. We see that the twisted boundary conditions are universal and do not depend on the details of the background and the presence of fermions. They depend only on the angles involved in the TsT transformation, and the total U(1) charges.

To understand better the meaning of the relations (2.7) and (2.9) we notice that the time components of the U(1) currents coincide with the momenta canonically conjugated to the angles ϕ_i: $J_i^\tau = p_i = \delta S/\delta\dot{\phi}_i$. Therefore, (2.7) and (2.9) can be written in the form

$$\tilde{p}_i = p_i\,, \qquad \tilde{\phi}_i' = \phi_i' - \gamma_{ij} p_j\,, \qquad i,j = 1,2,\dots,d\,, \tag{2.12}$$

where we take summation over j, and γ_{ij} is skew-symmetric, $\gamma_{ij} = -\gamma_{ji}$, with just one nonvanishing component equal to the deformation parameter: $\gamma_{12} = \gamma$.

It is obvious from the relations (2.12) that up to the twisted boundary conditions a TsT transformation is just a simple linear canonical transformation of the U(1) isometry variables. It is the twist that makes the original and TsT-transformed theories inequivalent. It is also clear that the most general multi-parameter TsT-transformed background obtained by applying TsT transformations successively, many times, each time picking up a new torus and a new deformation parameter, is completely characterized by the relations (2.12) with an arbitrary skew-symmetric matrix γ_{ij}. Therefore, a background containing a d-dimensional torus admits a $d(d-1)/2$-parameter TsT deformation. In particular, the most general TsT-transformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with TsT transformations applied only to the five-sphere S^5 (to preserve the isometry group of AdS_5) has three independent parameters, and, therefore, is the one found in [2].[3] The twisted boundary conditions for the original angles ϕ_i in the case of the most general deformation take the form

$$\phi_i(2\pi) - \phi_i(0) = 2\pi\,(n_i - \nu_i)\,, \qquad \nu_i = -\gamma_{ik}\,J_k\,. \tag{2.13}$$

Notice, that the twists ν_i always satisfy the restriction $\nu_i\,J_i = 0$.

In the next section we will discuss the most general three-parameter deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. For reader's convenience below we specialize our formulae to this case.

The general three-parameter γ-deformed background is obtained by applying the TsT transformation three times. We express the corresponding procedure as

$$(\phi_1,\phi_2,\phi_3) \xrightarrow{\gamma_3} (\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3) \xrightarrow{\gamma_1} (\tilde{\tilde{\phi}}_1,\tilde{\tilde{\phi}}_2,\tilde{\tilde{\phi}}_3) \xrightarrow{\gamma_2} (\check{\phi}_1,\check{\phi}_2,\check{\phi}_3)\,. \tag{2.14}$$

[3] Let us note that a Ts…sT transformation discussed in [18] is just a sequence of TsT transformations applied to the tori (ϕ_1,ϕ_i). The two-parameter deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$ they considered is, therefore, a particular case of the general three-parameter deformation.

Since under every step the corresponding Noether currents remain the same we can summarize relation between the angles in the following table

$$\begin{array}{lll} \tilde{\phi}'_1 - \phi'_1 = -\gamma_3 J_2^\tau & \tilde{\tilde{\phi}}'_1 - \tilde{\phi}'_1 = 0 & \check{\phi}_1 - \tilde{\tilde{\phi}}'_1 = \gamma_2 J_3^\tau \\ \tilde{\phi}'_2 - \phi'_2 = \gamma_3 J_1^\tau & \tilde{\tilde{\phi}}'_2 - \tilde{\phi}'_2 = -\gamma_1 J_3^\tau & \check{\phi}'_2 - \tilde{\tilde{\phi}}'_2 = 0 \\ \tilde{\phi}'_3 - \phi'_3 = 0 & \tilde{\tilde{\phi}}'_3 - \tilde{\phi}'_3 = \gamma_1 J_2^\tau & \check{\phi}'_3 - \tilde{\tilde{\phi}}'_3 = -\gamma_2 J_1^\tau \end{array} \tag{2.15}$$

From here we straightforwardly find the relation between the derivatives of the angles ϕ_i and the derivatives of $\check{\phi}_i$, the latter being attributed to string on the γ-deformed background:

$$\check{\phi}'_i - \phi'_i = \epsilon_{ijk}\gamma_j J_k^\tau \,. \tag{2.16}$$

We see from the formula that $\gamma_{ik} = -\epsilon_{ijk}\gamma_j$. Integrating eq. (2.16) and taking into account that $\check{\phi}_i(2\pi) - \check{\phi}_i(0) = 2\pi n_i \,, n_i \in \mathbb{Z}$, we obtain the twisted boundary conditions for the original angles

$$\phi_i(2\pi) - \phi_i(0) = 2\pi(n_i - \nu_i)\,, \qquad \nu_i = \epsilon_{ijk}\gamma_j J_k \,. \tag{2.17}$$

3. Green-Schwarz strings in γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this section we apply TsT transformations to the Green-Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [19] to generate nonsupersymmetric Green-Schwarz action for strings in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. To this end we need to redefine the bosonic and fermionic fields so that the U(1) isometry transformations would be realized as shifts of the angle variables. We then use the considerations in section 2 to find the twisted boundary conditions for bosons and fermions, and conclude that the integrability of superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [20] implies the integrability of the fermionic string in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. We use the Lax pair for Green-Schwarz superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the twisted boundary conditions to derive the monodromy matrix for strings in the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$.

3.1 Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ as the coset sigma-model

The Green-Schwarz superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be described as the sigma model whose target-space is the coset [19]

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}\,,$$

where $\mathrm{PSU}(2,2|4)$ is supergroup of the superconformal algebra $\mathfrak{psu}(2,2|4)$. In what follows we will use the convention of [23].[4]

Consider a group element g belonging to $\mathrm{PSU}(2,2|4)$ and construct the following current

$$\mathbf{A} = -g^{-1}\mathrm{d}g = \underbrace{\mathbf{A}^{(0)} + \mathbf{A}^{(2)}}_{\text{even}} + \underbrace{\mathbf{A}^{(1)} + \mathbf{A}^{(3)}}_{\text{odd}} \,. \tag{3.1}$$

[4] See also [24] and [23] for the introduction into the theory of the superalgebra $\mathfrak{psu}(2,2|4)$.

We recall that $\mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$-grading automorphism with respect to which it decomposes as the vector space into the direct sum of four components: two of them are even (bosons) and two are odd (fermions). In eq. (3.1) $\mathbf{A}^{(0,2)}$ are bosonic elements, and $\mathbf{A}^{(1,3)}$ are the fermionic ones. By construction the current $\mathbf{A}$ is flat, i.e. it has the vanishing curvature. Then the Lagrangian density for superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be written in the form [19, 25]

$$\mathscr{L} = -\frac{1}{2}\sqrt{\lambda}\,\mathrm{str}\Big(\gamma^{\alpha\beta}\mathbf{A}^{(2)}_\alpha \mathbf{A}^{(2)}_\beta + \kappa\epsilon^{\alpha\beta}\mathbf{A}^{(1)}_\alpha \mathbf{A}^{(3)}_\beta\Big)\,, \tag{3.2}$$

which is the sum of the kinetic and the Wess-Zumino terms, and κ-symmetry requires $\kappa = \pm 1$.

The next step is related to an explicit choice of the coset representative g. As was shown in [23] a convenient parametrization is provided by choosing

$$g = g(\theta)g(z). \tag{3.3}$$

Here $g(\theta) \equiv \exp(\theta)$, where θ is an odd element of $\mathfrak{psu}(2,2|4)$ which comprises 32 fermionic degrees of freedom. The element $g(z)$ belongs to SU(2,2)$\times$ SU(4). The coordinates $z \equiv (x_a, y_a)$ with $a = 1,\ldots,5$ parametrize the five-sphere and AdS_5 respectively.
With parametrization (3.3) we get for the flat current the following representation

$$\mathbf{A} = -g^{-1}dg = -g^{-1}(z)g^{-1}(\theta)dg(\theta)g(z) - g^{-1}(z)dg(z)\,. \tag{3.4}$$

Since

$$g(\theta) = \cosh\theta + \sinh\theta\,, \qquad g^{-1}(\theta) = \cosh\theta - \sinh\theta$$

we see that

$$g^{-1}(\theta)dg(\theta) = \mathrm{F} + \mathrm{B}\,, \tag{3.5}$$

where

$$\begin{aligned}\mathrm{B} &\equiv \cosh\theta\, d\cosh\theta - \sinh\theta\, d\sinh\theta\,,\\ \mathrm{F} &\equiv \cosh\theta\, d\sinh\theta - \sinh\theta\, d\cosh\theta\end{aligned} \tag{3.6}$$

are the even (boson) and odd (fermion) elements respectively. Thus, the even component of $\mathbf{A}$ is

$$\mathbf{A}_{\mathrm{even}} = -g^{-1}(z)\mathrm{B}g(z) - g^{-1}(z)dg(z)\,, \tag{3.7}$$

while the odd component is

$$\mathbf{A}_{\mathrm{odd}} = -g^{-1}(z)\mathrm{F}g(z)\,. \tag{3.8}$$

It is interesting to note that for such a parametrization of the coset the even component of the flat current is a gauge transform of the even element B, while the odd component is conjugate to F with the bosonic matrix $g(z)$.

To write down the final Lagrangian we have to find the projections $\mathbf{A}^{(i)}$. This can be easily done by using an explicit representation for the action of the $\mathbb{Z}_4$-grading automorphism and we refer the reader to [23] for the corresponding discussion. To present further results we introduce two 8×8 matrices

$$K_8 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \qquad \widetilde{K}_8 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix},$$

where K is a 4×4 matrix obeying the condition $K^2 = -\mathbb{I}$. These matrices are used to define

$$\mathrm{G} = g(z) K_8 g(z)^t \equiv \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix}, \quad \tilde{\mathrm{G}} = g(z) \widetilde{K}_8 g(z)^t \equiv \begin{pmatrix} g_a & 0 \\ 0 & -g_s \end{pmatrix}.$$

As was discussed in [23], the 4×4 matrices $g_a \in \mathrm{SU}(2,2)$ and $g_s \in \mathrm{SU}(4)$ provide another parametrization of the five-sphere and the AdS space. On coordinates z the global symmetry algebra $\mathfrak{psu}(2,2|4)$ is realized non-linearly. In opposite, g_a and g_s carry a linear representation of the superconformal algebra. Such realization of symmetries makes an identification of string states with operators of the dual gauge theory more transparent. We further find

$$\begin{aligned}
2\,\mathbf{A}^{(0)} &= \mathbf{A}_{\rm even} + K_8 \mathbf{A}^t_{\rm even} K_8 = -2g^{-1}dg - g^{-1}\left(\mathrm{B} - \mathrm{G}\mathrm{B}^t\mathrm{G}^{-1} - d\mathrm{G}\mathrm{G}^{-1}\right) g\,, \\
2\,\mathbf{A}^{(2)} &= \mathbf{A}_{\rm even} - K_8 \mathbf{A}^t_{\rm even} K_8 = -g^{-1}\left(\mathrm{B} + \mathrm{G}\mathrm{B}^t\mathrm{G}^{-1} + d\mathrm{G}\mathrm{G}^{-1}\right) g\,, \\
2\,\mathbf{A}^{(1)} &= \mathbf{A}_{\rm odd} + i\widetilde{K}_8 \mathbf{A}^t_{\rm odd} K_8 = -g^{-1}\left(\mathrm{F} - i\tilde{\mathrm{G}}\mathrm{F}^{\rm t}\mathrm{G}^{-1}\right) g\,, \\
2\,\mathbf{A}^{(3)} &= \mathbf{A}_{\rm odd} - i\widetilde{K}_8 \mathbf{A}^t_{\rm odd} K_8 = -g^{-1}\left(\mathrm{F} + i\tilde{\mathrm{G}}\mathrm{F}^{\rm t}\mathrm{G}^{-1}\right) g\,.
\end{aligned} \tag{3.9}$$

Substituting these projections into the string Lagrangian (3.2) we obtain[5]

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2}\sqrt{\lambda}\,\mathrm{str}\Big[\gamma^{\alpha\beta}(\mathrm{B}_\alpha + \mathrm{G}\mathrm{B}^{\rm t}_\alpha\mathrm{G}^{-1} + \partial_\alpha\mathrm{G}\mathrm{G}^{-1})(\mathrm{B}_\beta + \mathrm{G}\mathrm{B}^{\rm t}_\beta\mathrm{G}^{-1} + \partial_\beta\mathrm{G}\mathrm{G}^{-1}) \\
&+ \kappa\epsilon^{\alpha\beta}(\mathrm{F}_\alpha - i\tilde{\mathrm{G}}\mathrm{F}^t_\alpha\mathrm{G}^{-1})(\mathrm{F}_\beta + i\tilde{\mathrm{G}}\mathrm{F}^t_\beta\mathrm{G}^{-1})\Big]\,.
\end{aligned}$$

By using the cyclic property of the supertrace the Wess-Zumino term can be further simplified and we get

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2}\sqrt{\lambda}\,\mathrm{str}\Big[\gamma^{\alpha\beta}(\mathrm{B}_\alpha + \mathrm{G}\mathrm{B}^{\rm t}_\alpha\mathrm{G}^{-1} + \partial_\alpha\mathrm{G}\mathrm{G}^{-1})(\mathrm{B}_\beta + \mathrm{G}\mathrm{B}^{\rm t}_\beta\mathrm{G}^{-1} + \partial_\beta\mathrm{G}\mathrm{G}^{-1}) \\
&+ 2i\kappa\epsilon^{\alpha\beta}\mathrm{F}_\alpha\tilde{\mathrm{G}}\mathrm{F}^t_\beta\mathrm{G}^{-1}\Big]\,.
\end{aligned} \tag{3.10}$$

The nice feature of this Lagrangian is that it depends only on fields which carry linear representation of the superconformal group. In particular, we have three linearly realized U(1) isometries which are used to construct the Green-Schwarz superstring on the γ-deformed background.

[5] For convenience we rescaled the whole Lagrangian by the factor of 4.

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With a certain choice of the matrix K the matrix g_s parametrizing S^5 can be written as follows (see, *e.g.* [26]):

$$g_s = \begin{pmatrix} 0 & u_3 & u_1 & u_2 \\ -u_3 & 0 & u_2^* & -u_1^* \\ -u_1 & -u_2^* & 0 & u_3^* \\ -u_2 & u_1^* & -u_3^* & 0 \end{pmatrix}, \tag{3.11}$$

This is the unitary matrix $g_s^\dagger g_s = \mathbb{I}$ provided the three complex coordinates u_i obey the constraint $|u_1|^2 + |u_2|^2 + |u_3|^2 = 1$. A similar parametrization of the AdS_5 space is given by

$$g_a = \begin{pmatrix} 0 & v_3 & v_1 & v_2 \\ -v_3 & 0 & -v_2^* & v_1^* \\ -v_1 & v_2^* & 0 & v_3^* \\ -v_2 & -v_1^* & -v_3^* & 0 \end{pmatrix}. \tag{3.12}$$

Here $g_a \in SU(2,2)$, i.e. it obeys $g_a^\dagger E g_a = E$ with $E = \mathrm{diag}(1,1,-1,-1)$ provided the complex numbers v_i satisfy the constraint: $|v_1|^2 + |v_2|^2 - |v_3|^2 = -1$.

3.2 Fermions twisting

The original fermions appearing in the Lagrangian (3.10) transform under the commuting isometries of the five-sphere. To apply the consideration in section 2 to Green-Schwarz superstrings in $AdS_5 \times S^5$ we need to redefine these fermions in such a way that they become neutral under the isometries in question. After this redefinition is found we can perform the TsT-transformations on the angle variables with fermions being just the spectators, and use the general formulas derived in section 2. The twisted boundary conditions (2.17) for the original angles of $AdS_5 \times S^5$ then induce twisted boundary conditions for the original charged fermions of $AdS_5 \times S^5$.

Let us explore in more detail the invariance of the Lagrangian under the abelian subalgebra of the superconformal group. In full generality the bosonic symmetry algebra $SO(4,2) \times SO(6)$ has six Cartan generators: three for $SO(4,2)$ and three for $SO(6)$. If we introduce the polar representation

$$u_i = r_i\, e^{i\,\phi_i}\,, \qquad v_i = \rho_i\, e^{i\,\psi_i}\,,$$

with r_i, ρ_i being real, then the six commuting isometries are realized as constant shifts of the angle variables

$$\phi \to \phi + \epsilon\,, \qquad \psi \to \psi + \epsilon\,.$$

Remarkably, it turns out that the matrices g_s and g_a enjoy the following factorization property [2] (see also [26])

$$g_s(r,\phi) = M(\phi)\, \hat{g}_s(r)\, M(\phi)\,, \tag{3.13}$$

$$g_a(\rho,\psi) = M(\psi)\, \hat{g}_a(\rho)\, M(\psi)\,, \tag{3.14}$$

where

$$\hat{g}_s(r) = \begin{pmatrix} 0 & r_3 & r_1 & r_2 \\ -r_3 & 0 & r_2 & -r_1 \\ -r_1 & -r_2 & 0 & -r_3 \\ -r_2 & r_1 & r_3 & 0 \end{pmatrix}, \qquad \hat{g}_a(\rho) = \begin{pmatrix} 0 & \rho_3 & \rho_1 & \rho_2 \\ -\rho_3 & 0 & \rho_2 & -\rho_1 \\ -\rho_1 & -\rho_2 & 0 & \rho_3 \\ -\rho_2 & \rho_1 & -\rho_3 & 0 \end{pmatrix}. \tag{3.15}$$

Here also $M(\phi) = e^{\frac{i}{2}\Phi(\phi)}$, where $\Phi(\phi) = \mathrm{diag}(\Phi_1, \dots, \Phi_4)$ with

$$\begin{aligned} \Phi_1 &= \phi_1 + \phi_2 + \phi_3 \\ \Phi_2 &= -\phi_1 - \phi_2 + \phi_3 \\ \Phi_3 &= \phi_1 - \phi_2 - \phi_3 \\ \Phi_4 &= -\phi_1 + \phi_2 - \phi_3 \end{aligned} \tag{3.16}$$

The simplest way to see that all fermions are charged under the six commuting isometries is to notice that any fermionic term in the Lagrangian (3.10) explicitly depends on all the angle variable ϕ_i and ψ_i. To find the fermion redefinition that makes them neutral we represent the odd matrix θ as

$$\theta = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \tag{3.17}$$

Then it is clear that to uncharge the fermions under all U(1)'s we have to make the following rescaling

$$X = M(\psi_i)\hat{X}M(\phi_i)^{-1} \tag{3.18}$$

$$Y = M(\phi_i)\hat{Y}M(\psi_i)^{-1} \tag{3.19}$$

This leads to the following transformation formula

$$g(\theta) = \begin{pmatrix} M(\psi_i) & 0 \\ 0 & M(\phi_i) \end{pmatrix} g(\hat{\theta}) \begin{pmatrix} M(\psi_i)^{-1} & 0 \\ 0 & M(\phi_i)^{-1} \end{pmatrix}, \tag{3.20}$$

where the fermions $\hat{\theta}$ are uncharged under all U(1)s.

In what follows we restrict our attention to TsT transformations applied to the five-sphere, and, therefore, we do not need to make fermions neutral under the isometries of AdS_5. The corresponding redefinition of fermions simplifies and takes the following form

$$X = \hat{X}M(\phi_i)^{-1}, \quad Y = M(\phi_i)\hat{Y} \tag{3.21}$$

$$g(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & M(\phi_i) \end{pmatrix} g(\hat{\theta}) \begin{pmatrix} 1 & 0 \\ 0 & M(\phi_i)^{-1} \end{pmatrix}. \tag{3.22}$$

Let us mention, however, that the fermions do have to be neutral under some isometries of AdS_5, in particular, shifts of the global AdS time $t \equiv \psi_3$, if one wants to impose the uniform light-cone gauge that was recently used to solve the $\mathfrak{su}(1|1)$ sector of superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [27].

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Now, to determine the twisted boundary conditions for fermions we just need to take into account that the redefined neutral fermions do not transform under the TsT transformations. Therefore, the original charged fermions in $AdS_5 \times S^5$ satisfy twisted boundary conditions which can be easily found by using (3.21), and the twisted boundary conditions (2.17) for the angles ϕ_i:

$$X(2\pi) = X(0)e^{i\pi\Lambda}\,, \quad Y(2\pi) = e^{-i\pi\Lambda}Y(0)\,, \tag{3.23}$$

$$g(\theta)(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\Lambda} \end{pmatrix} g(\theta)(0) \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,, \tag{3.24}$$

where Λ is the following diagonal matrix $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_4)$ with

$$\begin{aligned}
\Lambda_1 &= \gamma_1(J_2 - J_3) + \gamma_2(J_3 - J_1) + \gamma_3(J_1 - J_2) = \nu_1 + \nu_2 + \nu_3 \\
\Lambda_2 &= \gamma_1(J_2 + J_3) - \gamma_2(J_1 + J_3) - \gamma_3(J_1 - J_2) = -\nu_1 - \nu_2 + \nu_3 \\
\Lambda_3 &= -\gamma_1(J_2 - J_3) + \gamma_2(J_1 + J_3) - \gamma_3(J_1 + J_2) = \nu_1 - \nu_2 - \nu_3 \\
\Lambda_4 &= -\gamma_1(J_2 + J_3) - \gamma_2(J_3 - J_1) + \gamma_3(J_1 + J_2) = -\nu_1 + \nu_2 - \nu_3
\end{aligned} \tag{3.25}$$

Obviously, the four variables Λ_k depend on three ν_i's precisely in the same fashion as Φ_k depend on ϕ_i's, c.f. eqs. (3.16). The formulas (2.17) and (3.23) allow us to analyze strings in the deformed background by using twisted strings in $AdS_5 \times S^5$.

3.3 Lax pair and monodromy matrix

As was discussed in detail in [2], the relations (2.6) can be used to find a local periodic Lax pair for strings in a TsT-transformed background if an isometry invariant Lax pair for strings in the initial background is known. The twisted boundary conditions (2.17) then can be used to get a simple expression for the TsT-transformed monodromy matrix in terms of the initial monodromy matrix and the twist matrix.

We begin by recalling the structure of the Lax pair found in [20]. It is based on the two-dimensional Lax connection $\mathscr{L}$ with components

$$\mathscr{L}_\alpha = \ell_0 \mathbf{A}^{(0)}_\alpha + \ell_1 \mathbf{A}^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho}\mathbf{A}^{(2)}_\rho + \ell_3 Q^+_\alpha + \ell_4 Q^-_\alpha\,, \tag{3.26}$$

where ℓ_i are functions of a spectral parameter, and $Q^\pm = \mathbf{A}^{(1)} \pm \mathbf{A}^{(3)}$. The zero curvature condition for the connection $\mathscr{L}$,

$$\partial_\alpha \mathscr{L}_\beta - \partial_\beta \mathscr{L}_\alpha - [\mathscr{L}_\alpha, \mathscr{L}_\beta] = 0\,, \tag{3.27}$$

follows from the dynamical equations and the flatness of $\mathbf{A}_\alpha$ if ℓ_i are chosen in the form

$$\ell_0 = 1, \quad \ell_1 = \frac{1+x^2}{1-x^2}\,, \quad \ell_2 = s_1\frac{2x}{1-x^2}\,, \quad \ell_3 = s_2\frac{1}{\sqrt{1-x^2}}\,, \quad \ell_4 = s_3\frac{x}{\sqrt{1-x^2}}\,,$$

where x is the spectral parameter, and the constants s_i satisfy

$$s_2^2 = s_3^2 = 1$$

$$s_1 + \kappa s_2 s_3 = 0\,.$$

Thus, for every choice of κ we have four different solutions for ℓ_i specified by the choice of $s_2 = \pm 1$ and $s_3 = \pm 1$. By using eqs. (3.4) for $\mathbf{A}_\alpha$, the Lax connection (3.26) can be explicitly realized in terms of 8×8 supermatrices from the Lie algebra $\mathfrak{su}(2,2|4)$. However, as was explained in [24], in the algebra $\mathfrak{su}(2,2|4)$ the curvature (3.27) of $\mathscr{L}_\alpha$ is not exactly zero, rather it is proportional to the identity matrix (anomaly) with a coefficient depending on fermionic variables. However, since $\mathfrak{psu}(2,2|4)$ is the factor-algebra of $\mathfrak{su}(2,2|4)$ over its central element proportional to the identity matrix, the curvature is regarded to be zero [24, 28] in the algebra $\mathfrak{psu}(2,2|4)$.

The Lax connection (3.26) cannot be used to derive a Lax pair for strings in the deformed background because $\mathbf{A}_\alpha$ explicitly depends on ϕ_i, and, therefore, $\mathscr{L}_\alpha$ is not isometry invariant. To get a proper Lax connection we need to make a gauge transformation of $\mathscr{L}_\alpha$ similar to the one used in [2] for the bosonic case.

The necessary gauge transformation can be found in two steps. First, we use the group element g and formulas (3.9) to derive a Lax connection $\widetilde{\mathscr{L}_\alpha}$ which depends only on the coset element G. The transformed Lax connection still has an explicit dependence on the angles ϕ_i, but it can be easily gauged away by using the factorization property (3.13) of G, and making the fermions neutral under the U(1) isometries of S^5 by using (3.21). The resulting gauge transformation that converts the Lax connection (3.26) to an isometry invariant form, therefore, is

$$h = M^{-1} g\,, \quad \partial_\alpha - \mathscr{L}_\alpha \to \partial_a - \hat{\mathscr{L}_\alpha} = M^{-1}\left(\partial_\alpha - \widetilde{\mathscr{L}_\alpha}\right) M = h\left(\partial_\alpha - \mathscr{L}_\alpha\right) h^{-1}\,, \tag{3.28}$$

where $\widetilde{\mathscr{L}_\alpha} = g\mathscr{L}_\alpha g^{-1} + \partial_\alpha g g^{-1}$ can be easily found by using (3.9) and (3.26), and the 8 by 8 matrix M is

$$M = \begin{pmatrix} 1 & 0 \\ 0 & M(\phi_i) \end{pmatrix}\,.$$

The Lax connection $\hat{\mathscr{L}_\alpha}$ depends only on the derivatives of ϕ_i, and, as was explained in [2], to get a Lax connection for strings in the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ all one needs to do is to express $\partial_\alpha \phi_i$ in terms of $\partial_\alpha \tilde{\phi}_i$ by using the relations (2.6). The resulting expression for the Lax connection $\hat{\mathscr{L}_\alpha}$ is rather complicated, and it is difficult to write down its explicit form.

The gauged transformed Lax connection $\hat{\mathscr{L}_\alpha}$ is, obviously, flat, and is invariant under the U(1) isometries, and is periodic in σ. It can be used to compute the monodromy matrix $\mathrm{T}(x)$ which is defined as the path-ordered exponential of the spatial component $\hat{\mathscr{L}_\sigma}(x)$ of the Lax connection [29]

$$\mathrm{T}(x) = \mathcal{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\ \hat{\mathscr{L}_\sigma}(x)\,, \tag{3.29}$$

The key property of the monodromy matrix is the time conservation of all its spectral invariants. In particular, any eigenvalue of $\mathrm{T}(x)$, $\exp(ip_k(x))$ where $p_k(x)$ is called a quasi-momentum, generates an infinite set of integrals of motion.

In the context of the AdS/CFT correspondence the monodromy matrix of the Lax connection $\mathscr{L}_\alpha$ of superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ was used in [30, 28] to derive finite-gap integral

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equations which describe the spectrum of classical spinning strings in the scaling limit of [10].

The derivation of the equations requires a careful analysis of various asymptotic properties of the monodromy matrix $\mathrm{T}(x)$ and the quasi-momenta $p(x)$ at small and large values of the spectral parameter x. An important distinction of $\hat{\mathscr{L}}_\alpha$ from $\mathscr{L}_\alpha$ is that it does not vanish at large values of x, and that makes more difficult to study the large x asymptotic properties of the monodromy matrix.

To analyze the asymptotics it is more convenient to use the nonlocal and nonperiodic Lax connection $\widetilde{\mathscr{L}}_\alpha$ explicitly depending on the angles ϕ_i which satisfy the twisted boundary conditions (2.17). In terms of the Lax connection the monodromy matrix $\mathrm{T}(x)$ takes the form

$$\mathrm{T}(x) = M^{-1}(2\pi) \cdot \mathcal{P} \exp \int_0^{2\pi} \mathrm{d}\sigma \; \widetilde{\mathscr{L}}_\sigma(x) \cdot M(0)\,. \tag{3.30}$$

It is clear that the monodromy matrix is not similar to the path-ordered exponential of the Lax connection $\widetilde{\mathscr{L}}_\alpha$ because the matrix M is not periodic.

The quasi-momenta p_k can be expressed through eigenvalues of

$$\widetilde{\mathrm{T}}(x) = M(0)M^{-1}(2\pi) \cdot \mathcal{P} \exp \int_0^{2\pi} \mathrm{d}\sigma \; \widetilde{\mathscr{L}}_\sigma(x)\,. \tag{3.31}$$

It is not difficult to check that

$$M(0)M^{-1}(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\Lambda} \end{pmatrix}\,,$$

where Λ is given in (3.25).

It would be interesting to analyze the properties of the monodromy matrix and derive finite-gap integral equations for the deformed model analogous to those derived for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in [30, 28]. It was done for the simplest $\mathfrak{su}(2)$ sector in [7].

4. Spinning particle and Neumann model

In section 3 we established equivalence between strings on the γ_i-deformed background and strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ with twisted boundary conditions. This equivalence allows one to construct an action for the "γ_i-deformed" spinning particle. Further quantization of this action should lead to determination of the spectrum of IIB supergravity compactified on the corresponding (generically non-supersymmetric) background.

A spinning particle is the string zero mode. To obtain the spinning particle in the γ-deformed background we have to assume that all the embedding fields describing this background depend on the world-sheet time τ only. Correspondingly, from the point of view of the string on $\mathrm{AdS}_5 \times \mathrm{S}^5$, this means that the embedding bosonic fields must have the following τ, σ-dependence

$$u_i = r_i(\tau) e^{i\phi_i(\tau) - i\sigma\nu_i}\,, \qquad v_i = \rho_i(\tau) e^{i\psi_i(\tau)}\,. \tag{4.1}$$

Here $\phi_i(\tau)$ and $\psi_i(\tau)$ are the time-dependent phases and the σ-dependence of u_i reflects the twisted boundary conditions. For the matrix G (and similar for $\tilde{\mathrm{G}}$) this implies the following structure

$$\mathrm{G}(\tau,\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}\Lambda\sigma} \end{pmatrix} \mathrm{G}(\tau) \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}\Lambda\sigma} \end{pmatrix} . \tag{4.2}$$

The zero modes of the fermionic fields are described in an analogous manner

$$X(\tau,\sigma) = X(\tau) e^{\frac{i}{2}\Lambda\sigma} , \qquad Y(\tau,\sigma) = e^{-\frac{i}{2}\Lambda\sigma} Y(\tau) , \tag{4.3}$$

which is equivalent to

$$\theta(\tau,\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}\Lambda\sigma} \end{pmatrix} \theta(\tau) \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}\Lambda\sigma} \end{pmatrix} . \tag{4.4}$$

Upon substituting these formulae into the general string action (3.10) one can see that the σ-dependence cancels out leaving behind the dependence on the deformation parameters γ_i. As the result we obtain an action for the spinning particle in the γ-deformed background. Since the corresponding bosonic action is known [12] to be the same as the action for the so-called Neumann-Rosochatius (NR) integrable model, we therefore obtain the fermionic generalization of the NR model.

If we restrict for the moment our attention to the purely bosonic case and introduce the diagonal metric $\eta = \mathrm{diag}(1,1,-1)$ we find the following action

$$\begin{aligned} \mathscr{L}_{\rm bos} &= -2\sqrt{\lambda}\gamma^{\tau\tau}\left(\sum_{i=1}^{3} \dot{r}_i^2 + r_i^2\dot{\phi}^2 + \sum_{ij}^{3} \eta_{ij}\dot{\rho}_i\dot{\rho}_j + \eta_{ij}\rho_i^2\dot{\psi}_j^2\right) + \frac{2\sqrt{\lambda}}{\gamma^{\tau\tau}}\sum_{i=1}^{3}\nu_i^2 r_i^2 \\ &\quad -2\sqrt{\lambda}\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\sum_{i=1}^{3}\nu_i r_i^2(\nu_i\gamma^{\tau\sigma} - 2\gamma^{\tau\tau}\dot{\phi}_i) . \end{aligned} \tag{4.5}$$

As usual components of the world-sheet metric are non-dynamical and play the role of the Lagrangian multipliers. In particular, equation of motion for $\gamma^{\tau\sigma}$ is equivalent to the following Virasoro constraint

$$\sum_{i=1}^{3} \nu_i r_i^2 (\gamma^{\tau\sigma}\nu_i - \gamma^{\tau\tau}\dot{\phi}_i) = 0 . \tag{4.6}$$

Assume now that our particle rotates both in five-sphere with angular momenta J_i and also in AdS_5 with spins S_i.[6] Fixing J_i and S_i we can integrate all the time-dependent phases $\phi_i(\tau)$, $\psi_i(\tau)$ out by using their equations of motion. Indeed, we have

$$\dot{\psi}_i = -\frac{\eta_{ij}S_j}{4\sqrt{\lambda}\gamma^{\tau\tau}\rho_i^2} , \qquad \dot{\phi}_i = -\frac{J_i}{4\sqrt{\lambda}\gamma^{\tau\tau}r_i^2} + \nu_i\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} .$$

[6] The spin S_3 coincides with the space-time energy of the particle.

Upon substituting this solution for all six angle variables we obtain the following bosonic action

$$\begin{aligned}\mathscr{L}_{\rm bos} &= -2\sqrt{\lambda}\gamma^{\tau\tau}(\dot{r}_1^2+\dot{r}_2^2+\dot{r}_3^2+\dot{\rho}_1^2+\dot{\rho}_2^2-\dot{\rho}_3^2)\\ &\quad -\frac{1}{8\sqrt{\lambda}\gamma^{\tau\tau}}\left(\frac{J_1^2}{r_1^2}+\frac{J_2^2}{r_2^2}+\frac{J_3^2}{r_3^2}+\frac{S_1^2}{\rho_1^2}+\frac{S_2^2}{\rho_2^2}-\frac{S_3^2}{\rho_3^2}-16\lambda\sum_{i=1}^{3}\nu_i^2 r_i^2\right). \end{aligned} \tag{4.7}$$

This is an action of the integrable NR system written in an arbitrary world-line metric density $\gamma^{\tau\tau}$. Notice that the second independent component of the metric $\gamma^{\tau\sigma}$ cancels out from the action. On the other hand, the Virasoro constraint (4.6) reduces to

$$\sum_i \nu_i J_i = 0 \tag{4.8}$$

with the general solution $\nu_i = \epsilon_{ijk}\gamma_j J_k$. Thus, if we would start with arbitrary parameters ν_i defining the twisted boundary conditions (4.1), compatibility of the dynamics with the Virasoro constraints would require that $\nu_i = \epsilon_{ijk}\gamma_j J_k$. This provides a new interesting interpretation of the equation (4.8).

In the general fermionic case it is also possible to integrate out the angle variables provided the fermions are redefined to be neutral under all U(1) isometries. This redefinition has been already discussed in the previous section and therefore we will not repeat it here. Introducing the 16 complex uncharged fermions $\theta = \{\theta_\alpha\}_{\alpha=1,\dots,16}$ we integrate out the angle variables and obtain the fermionic generalization of the NR model. Due to the complexity of the explicit answer, below we indicate the structure of the quadratic fermionic action only. It reads

$$\begin{aligned}\mathscr{L}_{\rm 2ferm} &= \sqrt{\lambda}\gamma^{\tau\tau}\Big(\epsilon_{ijk}r_j\dot{r}_k(\theta^*\Upsilon^i_r\dot{\theta}-\dot{\theta}^*\Upsilon^i_r\theta)+\epsilon_{ijk}\rho_j\dot{\rho}_k(\theta^*\Upsilon^i_\rho\dot{\theta}-\dot{\theta}^*\Upsilon^i_\rho\theta)\Big)\\ &\quad +\sqrt{\lambda}\kappa\Big(r_i\rho_j\theta\Omega^{ij}\dot{\theta}+r_i\rho_j\theta^*\Omega^{ij}\dot{\theta}^*\Big)+\frac{\sqrt{\lambda}}{\gamma^{\tau\tau}}r_ir_j\theta^*\Sigma^{ij}\theta\\ &\quad +\frac{1}{8\sqrt{\lambda}\gamma^{\tau\tau}}\theta^*(T_1+T_2)\theta\,. \end{aligned} \tag{4.9}$$

Here the matrices $\Upsilon^i_{r,\rho}$ are constant 16×16 anti-symmetric matrices. Matrices Ω^{ij} and Σ^{ij} are symmetric under $i\leftrightarrow j$ and they depend on the deformation parameters ν_i; they vanish if $\nu_i\to0$. The explicit formulas for the matrices T_1 and T_2 can be found in appendix C. These matrices depend non-trivially on all the spins as well as on coordinates r_i and ρ_i but they are independent of ν_i.

Since we have not attempted to fix the κ-symmetry the action (4.9) still depends on 32 fermionic degrees of freedom and the kinetic term for fermions appears to be degenerate reflecting thereby the presence of the κ-symmetry. Finally we note that the fermionic NR model remains classically integrable, because the Lax connection for the string on the γ-deformed background admits further reduction to zero modes. It would be very interesting to further investigate the integrable properties of the fermionic NR model and ultimately to quantize it.

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5. Conclusion

In this paper we have discussed classical strings propagating in a background obtained from an arbitrary string theory background by a sequence of TsT transformations.

Assuming that the initial background is invariant under d U(1) isometries, we have described a procedure to derive the most general $d(d-1)$ parameter deformation of the background, and the Green-Schwarz action governing the dynamics of the strings.

We have shown that angle variables of a TsT-transformed background are related to angle variables of the initial background in a universal way independent of the particular form of the background metric and other fields. This has allowed us to prove that strings in the TsT-transformed background are described by the Green-Schwarz action for strings in the initial background with bosonic and fermionic fields subject to twisted boundary conditions. Due to this relation for many purposes it is not necessary to know the explicit Green-Schwarz action for strings in a TsT-transformed background. These strings can be analyze by mapping them to twisted strings in the initial background. We have stressed that our construction implies that a TsT transformation preserves integrability properties of string sigma model.

We have discussed in detail type IIB strings propagating in γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time and found the twisted boundary conditions for bosons and fermions. We then have used a known Lax pair for superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$, and the relation between the angles to derive a local and periodic Lax representation for the γ_i-deformed model. The existence of the Lax pair implies the integrability of the fermionic string sigma model on the deformed background generalizing the construction of [2]. The twisted boundary conditions for string coordinates have been used to write down an explicit expression for the TsT-transformed monodromy matrix in terms of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ monodromy matrix, and the twist matrix.

It would be interesting to use the Lax representation and the monodromy matrix to derive finite-gap integral equations for the deformed model analogous to those derived for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in [30]. These equations could be then compared with the thermodynamic limit of the Bethe equations for the deformed $\mathcal{N} = 4$ SYM theory [5, 6, 11]. It has been already done for the simplest $\mathfrak{su}(2)_\gamma$ case in [7].

We have also discussed string zero modes and shown that their dynamics is governed by a new fermionic generalization of the integrable Neumann-Rosochatius model. Quantization of the model should give the spectrum of type IIB supergravity on the γ_i-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. The knowledge of the spectrum is important for checking if the nonsupersymmetric TsT-transformed background is perturbatively stable.

The twisted boundary conditions for string coordinates may be also used to find $1/J$ corrections to the spectrum of strings in near-plane-wave backgrounds generalizing the computation done in [31] for the undeformed case. The Hamiltonian formulation, and the uniform gauge of [32] seem to be very useful to handle the problem. It should be also straightforward to compute the spectrum of fluctuations around simple spinning circular strings, and analyze $1/J$ corrections to their energies generalizing the consideration of [33]. In particular, it would be interesting to determine the γ_i-dependence of the terms nonanalytic in λ recently found in [34], and their influence on the dressing factor of the quantum string Bethe ansatz [35].

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A. T-duality transformation and rules

In this appendix we present the T-duality transformation [36] for the most general Green-Schwarz action in the form used in the paper. Our way of deriving a T-dual Green-Schwarz action is very similar to the one used in [37] where the part of the Green-Schwarz action quadratic in fermions was also given in an explicit form, and shown that the quadratic fermionic term couples to background fluxes through generalized covariant derivative.

We start with the following Green-Schwarz action

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\ \Big[\gamma^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, G^0_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j\, B^0_{ij} \\ +2\partial_\alpha\phi^i\left(\gamma^{\alpha\beta}U^0_{\beta,i} - \epsilon^{\alpha\beta}V^0_{\beta,i}\right) + \mathcal{L}^0_{\rm rest}\Big]\,. \tag{A.1}$$

Here $\epsilon^{01}\equiv\epsilon^{\tau\sigma}=1$ and $\gamma^{\alpha\beta}\equiv\sqrt{-h}\,h^{\alpha\beta}$, where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. The action is invariant under U(1) isometries realized as shifts of the angle variables ϕ_i, $i=1,2,\dots,d$. We show explicitly the dependence of the action on ϕ_i, and their coupling to the background fields G^0_{ij}, B^0_{ij} and $U^0_{\beta,i}, V^0_{\beta,i}$. These background fields are independent of ϕ^i but depend on other bosonic and fermionic string coordinates which are neutral under the U(1) isometries. By $\mathcal{L}^0_{\rm rest}$ we denote the part of the Lagrangian which depends on these other fields of the model.

We perform a T-duality on a circle parametrized by ϕ_1. To find the T-duality rules it is useful to represent the action (A.1) in the following equivalent form

$$S = -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\left[p^\alpha\left(\partial_\alpha\phi_1 + \frac{U^0_{\alpha,1}}{G^0_{11}} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\frac{V^0_{\rho,1}}{G^0_{11}}\right) - \frac{1}{2\,G^0_{11}}\gamma_{\alpha\beta}\,p^\alpha p^\beta \right. \\ \left. -\frac{1}{2}\gamma^{\alpha\beta}\frac{U^0_{\alpha,1}U^0_{\beta,1} - V^0_{\alpha,1}V^0_{\beta,1}}{G^0_{11}} + \frac{1}{2}\epsilon^{\alpha\beta}\frac{U^0_{\alpha,1}V^0_{\beta,1} - U^0_{\beta,1}V^0_{\alpha,1}}{G^0_{11}} + \mathcal{L}'_{\rm rest}\right]\,, \tag{A.2}$$

where $\mathcal{L}'_{\rm rest}$ denotes the part of the Lagrangian (A.1) which does not depends on ϕ_1. Indeed, varying with respect to p^α, one gets the following equation of motion for p^α

$$p^\alpha = \gamma^{\alpha\beta}\partial_\beta\phi_1 G^0_{11} + \gamma^{\alpha\beta}U^0_{\beta,1} - \epsilon^{\alpha\beta}V^0_{\beta,1}\,. \tag{A.3}$$

Substituting (A.3) into (A.2) and using the identity $\epsilon^{\alpha\gamma}\gamma_{\gamma\rho}\epsilon^{\rho\beta} = \gamma^{\alpha\beta}$, we reproduce the action (A.1). Let us also mention that up to an unessential multiplier p^α coincides with the U(1) current corresponding to shifts of ϕ_1:

$$p^\alpha \sim J_1^\alpha \,.$$

On the other hand, varying (A.2) with respect to ϕ_1 gives

$$\partial_\alpha \, p^\alpha = 0 \,. \tag{A.4}$$

The general solution to this equation can be written in the form

$$p^\alpha = \epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}_1 \,, \tag{A.5}$$

where $\tilde{\phi}_1$ is the scalar T-dual to ϕ_1. Substituting (A.5) into the action (A.2), we obtain the following T-dual action

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\ \Big[&\gamma^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\,\widetilde{G}_{ij} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\,\widetilde{B}_{ij} \\ &+2\partial_\alpha\tilde{\phi}^i\left(\gamma^{\alpha\beta}\widetilde{U}_{\beta,i} - \epsilon^{\alpha\beta}\widetilde{V}_{\beta,i}\right) + \widetilde{\mathcal{L}}_{\rm rest}\Big] \,. \end{aligned} \tag{A.6}$$

with the new fields $\widetilde{G}_{ij}$, etc given in terms of the original ones.

$$\begin{aligned}
&\widetilde{G}_{11} = \frac{1}{G^0_{11}}\,, \quad \widetilde{G}_{ij} = G^0_{ij} - \frac{G^0_{1i}G^0_{1j} - B^0_{1i}B^0_{1j}}{G^0_{11}}\,, \quad \widetilde{G}_{1i} = \frac{B^0_{1i}}{G^0_{11}}\,, \qquad (A.7)\\
&\widetilde{B}_{ij} = B^0_{ij} - \frac{G^0_{1i}B^0_{1j} - B^0_{1i}G^0_{1j}}{G^0_{11}}\,, \quad \widetilde{B}_{1i} = \frac{G^0_{1i}}{G^0_{11}}\,,\\
&\widetilde{U}_{\alpha,1} = \frac{V^0_{\alpha,1}}{G^0_{11}}\,, \quad \widetilde{V}_{\alpha,1} = \frac{U^0_{\alpha,1}}{G^0_{11}}\,,\\
&\widetilde{U}_{\alpha,i} = U^0_{\alpha,i} - \frac{G^0_{1i}U^0_{\alpha,1} - B^0_{1i}V^0_{\alpha,1}}{G^0_{11}}\,,\\
&V_{\alpha,i} = V^0_{\alpha,i} - \frac{G^0_{1i}V^0_{\alpha,1} - B^0_{1i}U^0_{\alpha,1}}{G^0_{11}}\,,\\
&\widetilde{\mathcal{L}}_{\rm rest} = \mathcal{L}^0_{\rm rest} - \gamma^{\alpha\beta}\frac{U^0_{\alpha,1}U^0_{\beta,1} - V^0_{\alpha,1}V^0_{\beta,1}}{G^0_{11}} + \epsilon^{\alpha\beta}\frac{U^0_{\alpha,1}V^0_{\beta,1} - V^0_{\alpha,1}U^0_{\beta,1}}{G^0_{11}}\,,\\
&\epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}^1 = \gamma^{\alpha\beta}\partial_\beta\phi^i G^0_{1i} - \epsilon^{\alpha\beta}\partial_\beta\phi^i B^0_{1i} + \gamma^{\alpha\beta}U^0_{\beta,1} - \epsilon^{\alpha\beta}V^0_{\beta,1}\,, \qquad (A.8)\\
&\tilde{\phi}^i = \phi^i\,, \quad i \geq 2\,.
\end{aligned}$$

In principle these formulas can be used to find the T-duality transformed NS-NS and RR fields (see e.g. [38]) of the background in which strings propagate.

B. The background after TsT transformation

By using the formulas obtained in appendix A, and performing a TsT transformation one finds the TsT-transformed background fields G_{ij}, etc

$$G_{ij} = \frac{G^0_{ij}}{D}, \qquad G_{i3} = \frac{G^0_{i3}}{D} + \hat{\gamma}\frac{B^0_{23}G^0_{1i} - B^0_{13}G^0_{2i} + B^0_{12}G^0_{i3}}{D} \tag{B.1}$$

$$
\begin{aligned}
G_{33} &= G^0_{33} + \frac{\hat\gamma + \hat\gamma^2 B^0_{12}}{D} 2(B^0_{23}G^0_{13} - B^0_{13}G^0_{23}) + \\
&\quad \frac{\hat\gamma^2}{D}\left(G^0_{11}((B^0_{23})^2 - (G^0_{23})^2) + G^0_{22}((B^0_{13})^2 - (G^0_{13})^2) + 2G^0_{12}(G^0_{23}G^0_{13} - B^0_{13}B^0_{23})\right) && \text{(B.2)}\\
B_{12} &= \frac{B^0_{12}}{D} + \frac{\hat\gamma}{D}\left((B^0_{12})^2 - (G^0_{12})^2 + G^0_{11}G^0_{22}\right) && \text{(B.3)}\\
B_{i3} &= \frac{B^0_{i3}}{D} + \frac{\hat\gamma}{D}\left(B^0_{12}B^0_{i3} - G^0_{13}G^0_{i2} + G^0_{23}G^0_{i1}\right) && \text{(B.4)}\\
U_{\alpha,i} &= \frac{U^0_{\alpha,i}}{D} + \frac{\hat\gamma}{D}\left(B^0_{12}U^0_{\alpha,i} + G^0_{1i}V^0_{\alpha,2} - G^0_{2i}V^0_{\alpha,1}\right) && \text{(B.5)}\\
V_{\alpha,i} &= \frac{V^0_{\alpha,i}}{D} + \frac{\hat\gamma}{D}\left(B^0_{12}V^0_{\alpha,i} + G^0_{1i}U^0_{\alpha,2} - G^0_{2i}U^0_{\alpha,1}\right) && \text{(B.6)}\\
U_{\alpha,3} &= U^0_{\alpha,3} + \frac{(\hat\gamma + \hat\gamma^2 B^0_{12})}{D}\left(\epsilon^{ij}G^0_{i3}V^0_{\alpha,j} - \epsilon^{ij}B^0_{i3}U^0_{\alpha,j}\right) + && \text{(B.7)}\\
&\quad + \frac{\hat\gamma^2}{D}\left(\epsilon^{ij}U^0_{\alpha,i}\left(G^0_{23}G^0_{1j} - G^0_{13}G^0_{2j}\right) + \epsilon^{ij}V^0_{\alpha,i}\left(-B^0_{23}G^0_{1j} + B^0_{13}G^0_{2j}\right)\right)\\
V_{\alpha,3} &= V^0_{\alpha,3} + \frac{(\hat\gamma + \hat\gamma^2 B^0_{12})}{D}\left(\epsilon^{ij}G^0_{i3}U^0_{\alpha,j} - \epsilon^{ij}B^0_{i3}V^0_{\alpha,j}\right) + && \text{(B.8)}\\
&\quad + \frac{\hat\gamma^2}{D}\left(\epsilon^{ij}V^0_{\alpha,i}\left(G^0_{23}G^0_{1j} - G^0_{13}G^0_{2j}\right) + \epsilon^{ij}U^0_{\alpha,i}\left(-B^0_{23}G^0_{1j} + B^0_{13}G^0_{2j}\right)\right)\\
\mathscr{L}_{\text{rest}} &= \mathscr{L}^0_{\text{rest}} + \frac{(\hat\gamma + \hat\gamma^2 B^0_{12})}{D}\left(2\epsilon^{ij}(V^0_{0,i}V^0_{1,j} - U^0_{0,i}U^0_{1,j} + \gamma^{\alpha\beta}U^0_{\alpha,i}V^0_{\beta,j})\right) + && \text{(B.9)}\\
&\quad + \frac{\hat\gamma^2}{D}\left(G^0_{ij}\epsilon^{i\tilde i}\epsilon^{j\tilde j}\gamma^{\alpha\beta}\left(V^0_{\alpha,\tilde i}V^0_{\beta,\tilde j} - U^0_{\alpha,\tilde i}U^0_{\beta,\tilde j}\right) + G^0_{ij}\epsilon^{i\tilde i}\epsilon^{j\tilde j}\epsilon^{\alpha\beta}U^0_{\alpha,\tilde i}V^0_{\beta,\tilde j}\right)
\end{aligned}
$$

Here the indices $i, j = 1, 2$ define the directions of a two-torus, while the index 3 is singled out (in case we are dealing with more than three indices, 3 should be replaced by a generic index I different from 1 and 2.) . The element D is given by

$$D = 1 + 2\hat\gamma B^0_{12} + \hat\gamma^2(G^0_{11}G^0_{22} - (G^0_{12})^2 + (B^0_{12})^2)\,, \qquad \hat\gamma = \sqrt{\lambda}\gamma\,.$$

C. The matrices

We choose the following parametrization of the fermionic element

$$g(\theta,\eta) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\ 0 & 0 & 0 & 0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\ 0 & 0 & 0 & 0 & \theta^1 & \theta^2 & \theta^3 & \theta^4 \\ 0 & 0 & 0 & 0 & \theta^5 & \theta^6 & \theta^7 & \theta^8 \\ \eta_5 & \eta_1 & -\theta_1 & -\theta_5 & 0 & 0 & 0 & 0 \\ \eta_6 & \eta_2 & -\theta_2 & -\theta_6 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_3 & -\theta_3 & -\theta_7 & 0 & 0 & 0 & 0 \\ \eta_8 & \eta_4 & -\theta_4 & -\theta_8 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.1}$$

Here θ^α and η^α are $8+8$ complex fermions obeying the following conjugation rule $\theta^{\alpha\,*} = \theta_\alpha$ and $\eta^{\alpha\,*} = \eta_\alpha$. Under dilatation the fermions η^α and θ^α have charges $\frac{1}{2}$ and $-\frac{1}{2}$ respectively [23]. This explains the notational distinction we made for the fermions η's and θ's. In what follows it is useful to introduce the unifying notation θ_α for fermionic variables. We therefore identify $\eta_\alpha \equiv \theta_{\alpha+8}$ with $\alpha = 1, \dots, 8$.

In section 4 to describe the fermionic Neumann-Rosochatius model we have used the following matrices

$$
\begin{aligned}
\Upsilon_r^1 &= \sigma_3 \otimes \mathbb{I}_2 \otimes (-i\sigma_2) \otimes \sigma_3 & \qquad \Upsilon_\rho^1 &= (-i\sigma_2) \otimes \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \\
\Upsilon_r^2 &= \sigma_3 \otimes \mathbb{I}_2 \otimes (-i\sigma_2) \otimes \sigma_1 & \qquad \Upsilon_\rho^2 &= i\sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \\
\Upsilon_r^3 &= \sigma_3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes (-i\sigma_2) & \qquad \Upsilon_\rho^3 &= \sigma_3 \otimes i\sigma_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2
\end{aligned}
$$

To present the matrices Ω^{ij} we introduce the three auxiliary 4×4 matrices Δ_i:

$$
\Delta_1 = \begin{pmatrix} & \Lambda_1 & & \\ & & & -\Lambda_2 \\ -\Lambda_3 & & & \\ & & \Lambda_4 & \end{pmatrix}, \quad
\Delta_2 = \begin{pmatrix} & & & \Lambda_1 \\ & & \Lambda_2 & \\ & -\Lambda_3 & & \\ -\Lambda_4 & & & \end{pmatrix}, \quad
\Delta_3 = \begin{pmatrix} & \Lambda_1 & & \\ -\Lambda_2 & & & \\ & & & \Lambda_3 \\ & & -\Lambda_4 & \end{pmatrix}.
$$

With this definition the matrices Ω^{ij} can be written as

$$
\Omega^{i1} = \begin{pmatrix} & & & -\Delta_i \\ & & -\Delta_i & \\ & \Delta_i & & \\ \Delta_i & & & \end{pmatrix}, \quad
\Omega^{i2} = \begin{pmatrix} & & \Delta_i & \\ & & & -\Delta_i \\ -\Delta_i & & & \\ & \Delta_i & & \end{pmatrix}, \quad
\Omega^{i3} = \begin{pmatrix} & -\Delta_i & & \\ \Delta_i & & & \\ & & & \Delta_i \\ & & -\Delta_i & \end{pmatrix}.
$$

Next we describe the structure of the matrix Σ^{ij} which depends on the deformation parameters γ_i and is symmetric under $i \leftrightarrow j$. We find that Σ is block-diagonal, $\Sigma^{ij} = (-\omega^{ij}, -\omega^{ij}, \omega^{ij}, \omega^{ij})$, where the symmetric 4×4 matrices ω^{ij} read as

$$
\begin{aligned}
\omega^{11} &= 2\nu_1 \begin{pmatrix} \nu_1+\nu_2+\nu_3 & & & \\ & \nu_1+\nu_2-\nu_3 & & \\ & & \nu_1-\nu_2-\nu_3 & \\ & & & \nu_1-\nu_2+\nu_3 \end{pmatrix}, \\
\omega^{22} &= 2\nu_2 \begin{pmatrix} \nu_1+\nu_2+\nu_3 & & & \\ & \nu_1+\nu_2-\nu_3 & & \\ & & -\nu_1+\nu_2+\nu_3 & \\ & & & -\nu_1+\nu_2-\nu_3 \end{pmatrix}, \\
\omega^{33} &= 2\nu_3 \begin{pmatrix} \nu_1+\nu_2+\nu_3 & & & \\ & -\nu_1-\nu_2+\nu_3 & & \\ & & -\nu_1+\nu_2+\nu_3 & \\ & & & \nu_1-\nu_2+\nu_3 \end{pmatrix}, \\
\omega^{12} &= \nu_3 \begin{pmatrix} & \nu_1-\nu_2 & & \\ \nu_1-\nu_2 & & & \\ & & & -\nu_1-\nu_2 \\ & & -\nu_1-\nu_2 & \end{pmatrix}, \\
\omega^{13} &= \nu_2 \begin{pmatrix} & & & \nu_3-\nu_1 \\ & & \nu_1+\nu_3 & \\ & \nu_1+\nu_3 & & \\ \nu_3-\nu_1 & & & \end{pmatrix}, \\
\omega^{23} &= \nu_1 \begin{pmatrix} & & \nu_2-\nu_3 & \\ & & & -\nu_2-\nu_3 \\ \nu_2-\nu_3 & & & \\ & -\nu_2-\nu_3 & & \end{pmatrix}.
\end{aligned}
$$

Finally we collect the 16 by 16 matrices, T_1 and T_2:

$$
\begin{aligned}
T_1 = & \left(-\frac{S_1^2}{\rho_1^2} - \frac{S_2^2}{\rho_2^2} + \frac{S_3^2}{\rho_3^2} + \frac{J_1^2}{r_1^2} + \frac{J_2^2}{r_2^2} + \frac{J_3^2}{r_3^2} \right) M^1 \otimes M^0 + S_1 S_3 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_3^2} \right) M^3 \otimes M^0 \\
& + S_1 S_2 \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) M^0 \otimes M^0 + S_1 J_1 \left(\frac{1}{r_1^2} - \frac{1}{\rho_1^2} \right) M^2 \otimes M^2 + S_1 J_2 \left(\frac{1}{r_2^2} - \frac{1}{\rho_1^2} \right) M^2 \otimes M^3
\end{aligned}
$$

$$
\begin{aligned}
&+S_1J_3\left(\frac{1}{r_3^2}-\frac{1}{\rho_1^2}\right)M^2\otimes M^1+S_2S_3\left(\frac{1}{\rho_3^2}-\frac{1}{\rho_2^2}\right)M^2\otimes M^0+S_2J_1\left(\frac{1}{\rho_2^2}-\frac{1}{r_1^2}\right)M^3\otimes M^2\\
&+S_2J_2\left(\frac{1}{\rho_2^2}-\frac{1}{r_2^2}\right)M^3\otimes M^3+S_2J_3\left(\frac{1}{\rho_2^2}-\frac{1}{r_3^2}\right)M^3\otimes M^1+S_3J_1\left(-\frac{1}{\rho_3^2}-\frac{1}{r_1^2}\right)M^0\otimes M^2\\
&+S_3J_2\left(-\frac{1}{\rho_3^2}-\frac{1}{r_2^2}\right)M^0\otimes M^3+S_3J_3\left(-\frac{1}{\rho_3^2}-\frac{1}{r_3^2}\right)M^0\otimes M^1-J_1J_2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)M^1\otimes M^1\\
&+J_1J_3\left(\frac{1}{r_1^2}+\frac{1}{r_3^2}\right)M^1\otimes M^3+J_2J_3\left(\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)M^1\otimes M^2\,, \qquad \text{(C.2)}
\end{aligned}
$$

$$
T_2 = G^0\otimes M^0+\cdots+G^3M^3+M^0\otimes\tilde{G}^0+\cdots+M^3\otimes\tilde{G}^3\,. \qquad \text{(C.3)}
$$

Here the M's are the following diagonal 4 by 4 matrices

$$
M^0=\mathrm{diag}(1,1,1,1), \qquad M^1=\mathrm{diag}(1,1,-1,-1), \qquad \text{(C.4)}
$$

$$
M^2=\mathrm{diag}(1,-1,1,-1), \qquad M^3=\mathrm{diag}(1,-1,-1,1) \qquad \text{(C.5)}
$$

and G and $\tilde{G}$ are 4 by 4, symmetric matrices, with zeros in the diagonal. Decomposing them in terms of the following orthogonal "basis"

$$
\begin{aligned}
O_1&=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \quad
O_2=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}, \quad
O_3=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\\
O_4&=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}, \quad
O_5=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \quad
O_6=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix} \qquad \text{(C.6)}
\end{aligned}
$$

we can show their explicit dependence on the coordinates and the currents

$$
\begin{aligned}
G^0&=-\frac{S_2S_3\rho_1}{\rho_2\rho_3^2}O_1-\frac{S_1S_3\rho_2}{\rho_1\rho_3^2}O_2+\frac{S_2S_3\rho_1}{\rho_3\rho_2^2}O_3-\frac{S_1S_2\rho_3}{\rho_1\rho_2^2}O_4-\frac{S_1S_2\rho_3}{\rho_2\rho_1^2}O_5-\frac{S_1S_3\rho_2}{\rho_3\rho_1^2}O_1\\
G^1&=\frac{S_1J_3\rho_2}{\rho_1r_3^2}O_1+\frac{S_2J_3\rho_1}{\rho_2r_3^2}O_2-\frac{S_1J_3\rho_3}{\rho_1r_3^2}O_3+\frac{S_3J_3\rho_1}{\rho_3r_3^2}O_4+\frac{S_3J_3\rho_2}{\rho_3r_3^2}O_5+\frac{S_2J_3\rho_3}{\rho_2r_3^2}O_6\\
G^2&=\frac{S_1J_1\rho_2}{\rho_1r_1^2}O_1+\frac{S_2J_1\rho_1}{\rho_2r_1^2}O_2-\frac{S_1J_1\rho_3}{\rho_1r_1^2}O_3+\frac{S_3J_1\rho_1}{\rho_3r_1^2}O_4+\frac{S_3J_1\rho_2}{\rho_3r_1^2}O_5+\frac{S_2J_1\rho_3}{\rho_2r_1^2}O_6\\
G^3&=\frac{S_1J_2\rho_2}{\rho_1r_2^2}O_1+\frac{S_2J_2\rho_1}{\rho_2r_2^2}O_2-\frac{S_1J_2\rho_3}{\rho_1r_2^2}O_3+\frac{S_3J_2\rho_1}{\rho_3r_2^2}O_4+\frac{S_3J_2\rho_2}{\rho_3r_2^2}O_5+\frac{S_2J_2\rho_3}{\rho_2r_2^2}O_6\\
\tilde{G}^0&=-\frac{S_3J_1r_2}{r_1\rho_3^2}O_1+\frac{S_3J_2r_1}{r_2\rho_3^2}O_2-\frac{S_3J_2r_3}{r_2\rho_3^2}O_3+\frac{S_3J_3r_2}{r_3\rho_3^2}O_4-\frac{S_3J_3r_1}{r_3\rho_3^2}O_5+\frac{S_3J_1r_3}{r_1\rho_3^2}O_6\\
\tilde{G}^1&=-\frac{J_2J_3r_1}{r_2r_3^2}O_1+\frac{J_1J_3r_2}{r_1r_3^2}O_2-\frac{J_1J_3r_2}{r_3r_1^2}O_3+\frac{J_1J_2r_3}{r_2r_1^2}O_4-\frac{J_1J_2r_3}{r_1r_2^2}O_5+\frac{J_2J_3r_1}{r_3r_2^2}O_6\\
\tilde{G}^2&=-\frac{S_1J_1r_2}{r_1\rho_1^2}O_1+\frac{S_1J_2r_1}{r_2\rho_1^2}O_2-\frac{S_1J_2r_3}{r_2\rho_1^2}O_3+\frac{S_1J_3r_1}{r_3\rho_1^2}O_4-\frac{S_1J_3r_1}{r_3\rho_1^2}O_5+\frac{S_1J_1r_3}{r_1\rho_1^2}O_6\\
\tilde{G}^3&=\frac{S_2J_1r_2}{r_1\rho_2^2}O_1-\frac{S_2J_2r_1}{r_2\rho_2^2}O_2+\frac{S_2J_2r_3}{r_2\rho_2^2}O_3-\frac{S_2J_3r_2}{r_3\rho_2^2}O_4+\frac{S_2J_3r_1}{r_3\rho_2^2}O_5-\frac{S_2J_1r_3}{r_1\rho_2^2}O_6
\end{aligned}
$$

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