Hidden symmetries and the fermionic sector of eleven-dimensional supergravity

Thibault Damour[1], Axel Kleinschmidt[2] and Hermann Nicolai[2]

[1]*Institut des Hautes Etudes Scientifiques*
35, Route de Chartres, F-91440 Bures-sur-Yvette, France

[2]*Max-Planck-Institut für Gravitationsphysik*
Albert-Einstein-Institut
Mühlenberg 1, D-14476 Potsdam, Germany

Abstract: We study the hidden symmetries of the fermionic sector of $D = 11$ supergravity, and the role of $K(E_{10})$ as a generalised 'R symmetry'. We find a consistent model of a massless spinning particle on an $E_{10}/K(E_{10})$ coset manifold whose dynamics can be mapped onto the fermionic and bosonic dynamics of $D = 11$ supergravity in the near space-like singularity limit. This E_{10}-invariant superparticle dynamics might provide the basis of a new definition of M-theory, and might describe the 'de-emergence' of space-time near a cosmological singularity.

Eleven-dimensional supergravity (SUGRA$_{11}$) [1] is believed to be the low-energy limit of the elusive 'M-theory', which is, hopefully, a unified framework encompassing the various known string theories. Understanding the symmetries of SUGRA$_{11}$ is therefore important for reaching a satisfactory formulation of M-theory. Many years ago it was found that the toroidal dimensional reduction of SUGRA$_{11}$ to lower dimensions leads to the emergence of unexpected ('hidden') symmetry groups, notably E_7 in the reduction to four non-compactified spacetime dimensions [2], E_8 in the reduction to $D = 3$ [2, 3, 4, 5], and the affine Kac–Moody group E_9 in the reduction to $D = 2$

[6]. It was also conjectured [7] that the hyperbolic Kac–Moody group E_{10} might appear when reducing SUGRA$_{11}$ to only *one* (time-like) dimension.

Recently, the consideration, *à la* Belinskii, Khalatnikov and Lifshitz [8], of the *near space-like singularity limit*[1] of generic *inhomogeneous* bosonic eleven-dimensional supergravity solutions has uncovered some striking evidence for the hidden rôle of E_{10} [9, 10]. Ref. [10] related the gradient expansion $(\partial_x \ll \partial_t)$, which organises the near space-like singularity limit [11], to an algebraic expansion in the height of positive roots of E_{10}. A main conjecture of [10] was the existence of a *correspondence* between the time evolution, around any given spatial point $\mathbf{x}$, of the supergravity bosonic fields $g_{MN}^{(11)}(t, \mathbf{x})$, $A_{MNP}^{(11)}(t, \mathbf{x})$, together with their infinite towers of spatial gradients, on the one hand, and the dynamics of a structureless massless particle on the infinite-dimensional coset space $E_{10}/K(E_{10})$ on the other hand. Here, $K(E_{10})$ is the maximal compact subgroup of E_{10}. Further evidence for the rôle of the one-dimensional non-linear sigma model $E_{10}/K(E_{10})$ in M-theory was provided in [12, 13, 14, 15]. For alternative coset model based proposals aiming at capturing hidden symmetries of M-theory see [16, 17].

In this letter, we extend the bosonic coset construction of [10] to the full supergravity theory by including fermionic variables; more specifically, we provide evidence for the existence of a correspondence between the time evolution of the coupled supergravity fields $g_{MN}^{(11)}(t, \mathbf{x})$, $A_{MNP}^{(11)}(t, \mathbf{x})$, $\psi_M^{(11)}(t, \mathbf{x})$ and the dynamics of a *spinning massless particle* on $E_{10}/K(E_{10})$. Previous work on E_{10} which included fermions can be found in [12, 18].[2]

To motivate our construction of a fermionic extension of the bosonic one-dimensional $E_{10}/K(E_{10})$ coset model we consider the equation of motion of the gravitino in $D = 11$ supergravity [1].[3] Projecting all coordinate indices on

---

[1]This limit can also be viewed as a small tension limit, $\alpha' \to \infty$.

[2]Results similar to some of the ones reported here have been obtained in [19].

[3]We use the mostly plus signature; $M, N, \ldots = 0, \ldots, 10$ denote spacetime coordinate (world) indices; $m, n, p, \ldots = 1, \ldots, 10$ denote spatial coordinate indices, and the indices $i, j, k, l = 1, \ldots, 10$ label the non-orthonormal frame components $\theta^i{}_m dx^m$. Spacetime Lorentz (flat) indices are denoted $A, B, C, \ldots, F = 0, \ldots, 10$, while $a, b, \ldots, f = 1, \ldots, 10$

an elfbein $E_{(11)}^A = E_{(11)}^{\ A}{}_M dx^M$, the equation of motion for $\psi_A^{(11)} = E_{(11)A}^{\ M}\psi_M^{(11)}$ are (neglecting quartic fermion terms)

$$0 = \hat{\mathcal{E}}_A := \Gamma^B \left[(D_A(\omega) + \mathcal{F}_A)\,\psi_B^{(11)} - (D_B(\omega) + \mathcal{F}_B)\,\psi_A^{(11)} \right], \qquad (1)$$

where $D_A(\omega) = E_{(11)A}^{\ M}D_M$ denotes the moving-frame covariant derivative $D_A(\omega)\psi_B^{(11)} = \partial_A\psi_B^{(11)} + \omega_{ABC}^{(11)}\psi^{(11)C} + \frac{1}{4}\omega_{ACD}^{(11)}\Gamma^{CD}\psi_B^{(11)}$, and where $\mathcal{F}_A := +\frac{1}{144}(\Gamma_A{}^{BCDE} - 8\delta_A^B\Gamma^{CDE})F_{BCDE}^{(11)}$ denotes the terms depending on the 4-form field strength $F_{MNPQ}^{(11)} = 4\partial_{[M}A_{NPQ]}^{(11)}$. Here $\omega_{ABC}^{(11)} = -\omega_{ACB}^{(11)} = E_{(11)A}^{\ M}\omega_{M\,BC}^{(11)}$ denotes the moving frame components of the spin connection, with $\omega_{ABC}^{(11)} = \frac{1}{2}(\Omega_{ABC}^{(11)} + \Omega_{CAB}^{(11)} - \Omega_{BCA}^{(11)})$, where $\Omega_{ABC}^{(11)} = -\Omega_{BAC}^{(11)}$ are the coefficients of anholonomicity. Following [10, 14] we use a pseudo-Gaussian (zero-shift) coordinate system t, x^m and we accordingly decompose the elfbein $E_{(11)}^A$ in separate time and space parts as $E_{(11)}^0 = Ndt$, $E_{(11)}^a = e_{(10)m}^a dx^m$. We note that the zehnbein $E_{(11)}^a = e_{(10)}^a$ is related to the non-orthogonal, time-independent spatial frame $\theta^i(\mathbf{x}) = \theta^i{}_m(\mathbf{x})dx^m$ used in [10] via $e_{(10)}^a = S^a{}_i\theta^i$ [14].

Using the $D = 11$ local supersymmetry to impose the relation $\psi_0^{(11)} = \Gamma_0\Gamma^a\psi_a^{(11)}$, and defining $\mathcal{E}_a := Ng^{1/4}\Gamma^0\hat{\mathcal{E}}_a$ (with $g^{1/2} = \det(e_{(10)m}^a)$), we find that the spatial components of the gravitino equation of motion (1), when expressed in terms of a rescaled $\psi_a^{(10)} := g^{1/4}\psi_a^{(11)}$, take the following form

$$
\begin{aligned}
\mathcal{E}_a =\ & \partial_t\psi_a^{(10)} + \omega_{t\,ab}^{(11)}\psi^{(10)b} + \frac{1}{4}\omega_{t\,cd}^{(11)}\Gamma^{cd}\psi_a^{(10)} \qquad\qquad (2)\\
& - \frac{1}{12}F_{tbcd}^{(11)}\Gamma^{bcd}\psi_a^{(10)} - \frac{2}{3}F_{tabc}^{(11)}\Gamma^b\psi^{(10)c} + \frac{1}{6}F_{tbcd}^{(11)}\Gamma_a{}^{bc}\psi^{(10)d}\\
& + \frac{N}{144}F_{bcde}^{(11)}\Gamma^0\Gamma^{bcde}\psi_a^{(10)} + \frac{N}{9}F_{abcd}^{(11)}\Gamma^0\Gamma^{bcde}\psi_e^{(10)} - \frac{N}{72}F_{bcde}^{(11)}\Gamma^0\Gamma_{abcdef}\psi^{(10)f}\\
& + N(\omega_{a\,bc}^{(11)} - \omega_{b\,ac}^{(11)})\Gamma^0\Gamma^b\psi^{(10)c} + \frac{N}{2}\omega_{a\,bc}^{(11)}\Gamma^0\Gamma^{bcd}\psi_d^{(10)} - \frac{N}{4}\omega_{b\,cd}^{(11)}\Gamma^0\Gamma^{bcd}\psi_a^{(10)}\\
& + Ng^{1/4}\Gamma^0\Gamma^b\left(2\partial_a\psi_b^{(11)} - \partial_b\psi_a^{(11)} - \frac{1}{2}\omega_{c\,cb}^{(11)}\psi_a^{(11)} - \omega_{0\,0a}^{(11)}\psi_b^{(11)} + \frac{1}{2}\omega_{0\,0b}^{(11)}\psi_a^{(11)}\right).
\end{aligned}
$$

denote purely spatial Lorentz indices. We use the conventions of [1, 2] except for the replacement $\Gamma_{\mathrm{CJS}}^M = +i\Gamma_{\mathrm{here}}^M$ (linked to the mostly plus signature) which allows us to use real gamma matrices and real (Majorana) spinors. The definition of the Dirac conjugate is $\bar{\psi} := \psi^T\Gamma_{\mathrm{here}}^0$, and thus differs from [1] by a factor of i. The field strength F_{MNPQ}^{here} used in this letter is equal to $+1/2$ the one used in [10].

3

Refs. [10, 14] defined a *dictionary* between the temporal-gauge bosonic super-gravity fields $g_{mn}^{(11)}(t, \mathbf{x})$, $A_{mnp}^{(11)}(t, \mathbf{x})$ (and their first spatial gradients: spatial connection and magnetic 4-form) and the four lowest levels $h^i{}_a(t)$, $A_{ijk}(t)$, $A_{i_1...i_6}(t)$, $A_{i_0|i_1...i_8}(t)$ of the infinite tower of coordinates parametrising the coset manifold $E_{10}/K(E_{10})$. Here, we extend this dictionary to fermionic variables by showing that the rescaled, SUSY gauge-fixed gravitino field $\psi_a^{(10)}$ can be identified with the first rung of a 'vector-spinor-type' representation of $K(E_{10})$, whose Grassmann-valued representation vector will be denoted by $\Psi = (\psi_a, \psi_{...}, ...)$.[4] We envisage Ψ to be an infinite-dimensional representation of $K(E_{10})$ which is decomposed into a tower of $SO(10)$ representations, starting with a vector-spinor one ψ_a. Our labelling convention is that coset quantities, such as A_{ijk} or Ψ do not carry sub- or superscripts, whereas supergravity quantities carry an explicit dimension label.

We shall give several pieces of evidence in favour of this identification and of the consistency of this $K(E_{10})$ representation. As in the bosonic case, the correspondence $\psi_a^{(10)}(t, \mathbf{x}) \leftrightarrow \psi_a^{\mathrm{coset}}(t) \equiv \psi_a(t)$ is defined at a fixed, but arbitrary, spatial point $\mathbf{x}$. A dynamical system governing a 'massless spinning particle' on $E_{10}/K(E_{10})$ will be presented as an extension of the coset dynamics of [10] and we will demonstrate the consistency of this dynamical system with the supergravity model under this correspondence. More precisely, we will first show how to consistently identify the Rarita–Schwinger equation (2) with a $K(E_{10})$-covariant equation

$$0 = \overset{\mathrm{vs}}{\mathcal{D}}\, \Psi(t) := \left(\partial_t - \overset{\mathrm{vs}}{\mathcal{Q}}(t)\right)\Psi(t). \tag{3}$$

This equation expresses the parallel propagation of the vector-spinor-type '$K(E_{10})$ polarisation' $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle. Our notation here is as follows. A one-parameter dependent generic group element of E_{10} is denoted by $\mathcal{V}(t)$. The Lie algebra valued 'velocity' of $\mathcal{V}(t)$, namely $v(t) = \partial_t \mathcal{V} \mathcal{V}^{-1} \in \mathfrak{e}_{10} \equiv \mathrm{Lie}(E_{10})$ is decomposed into its 'symmetric' and 'antisymmetric' parts according to $\mathcal{P}(t) := v_{\mathrm{sym}}(t) :=$

---

[4]By contrast, [18] considered 'Dirac-spinor-type' representations of $K(E_{10})$.

$\frac{1}{2}(v(t) + v^T(t))$, $\mathcal{Q}(t) := v_{\text{anti}}(t) := \frac{1}{2}(v(t) - v^T(t))$, where the transposition $(\cdot)^T$ is the generalised transpose of an $\mathfrak{e}_{10}$ Lie algebra element $x^T := -\omega(x)$ defined by the Chevalley involution ω [20]. $K(E_{10})$ is defined as the set of 'orthogonal elements' $k^{-1} = k^T$. Its Lie algebra $\mathfrak{k}_{10} = \text{Lie}(K(E_{10}))$ is made of all the antisymmetric elements of $\mathfrak{e}_{10}$, such as $\mathcal{Q}$.

The bosonic coset model of [10] is invariant under a global E_{10} right action and a local $K(E_{10})$ left action $\mathcal{V}(t) \longrightarrow k(t)\mathcal{V}(t)g_0$. Under the local $K(E_{10})$ action, $\mathcal{P}$ varies covariantly as $\mathcal{P} \longrightarrow k\mathcal{P}k^{-1}$, while $\mathcal{Q}$ varies as a $K(E_{10})$ connection $\mathcal{Q} \longrightarrow k\mathcal{Q}k^{-1} + \partial_t k\, k^{-1}$, with $\partial_t k\, k^{-1} \in \mathfrak{k}_{10}$ following from the orthogonality condition. The coset equation (3) will therefore be $K(E_{10})$ covariant if Ψ varies, under a local $K(E_{10})$ left action, as a certain ('vector-spinor') linear representation

$$\Psi \longrightarrow \overset{\text{vs}}{R}(k) \cdot \Psi \tag{4}$$

and if $\overset{\text{vs}}{\mathcal{Q}}$ in (3) is the value of $\mathcal{Q} \in \mathfrak{k}_{10}$ in the same representation $\overset{\text{vs}}{R}$. In order to determine the concrete form of $\overset{\text{vs}}{\mathcal{Q}}$ in the vector-spinor representation we need an explicit parametrisation of the coset manifold $E_{10}/K(E_{10})$.

Following [10, 14] we decompose the E_{10} group w.r.t. its $GL(10)$ subgroup. Then the $\ell = 0$ generators of $\mathfrak{e}_{10}$ are $\mathfrak{gl}(10)$ generators $K^a{}_b$ satisfying the standard commutation relations $[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b$. The $\mathfrak{e}_{10}$ generators at levels $\ell = 1, 2, 3$ as $GL(10)$ tensors are, respectively, $E^{a_1 a_2 a_3} = E^{[a_1 a_2 a_3]}$, $E^{a_1 \ldots a_6} = E^{[a_1 \ldots a_6]}$, and $E^{a_0|a_1 \ldots a_8} = E^{a_0|[a_1 \ldots a_8]}$, where the $\ell = 3$ generator is also subject to $E^{[a_0|a_1 \ldots a_8]} = 0$. In a suitable (Borel) gauge, a generic coset element $\mathcal{V} \in E_{10}/K(E_{10})$ can be written as $\mathcal{V} = \exp(X_h)\exp(X_A)$ with

$$X_h = h^b{}_a K^a{}_b, \tag{5}$$
$$X_A = \frac{1}{3!}A_{a_1 a_2 a_3}E^{a_1 a_2 a_3} + \frac{1}{6!}A_{a_1 \ldots a_6}E^{a_1 \ldots a_6} + \frac{1}{9!}A_{a_0|a_1 \ldots a_8}E^{a_0|a_1 \ldots a_8} + \ldots.$$

Defining $e^i{}_a := (\exp h)^i{}_a = \delta^i{}_a + h^i{}_a + \frac{1}{2!}h^i{}_s h^s{}_a + \ldots$ and $\bar{e}^a{}_i := (e^{-1})^a{}_i$ one

finds that the velocity $v \in \mathfrak{e}_{10}$ reads, expanded up to $\ell = 3$,

$$
\begin{aligned}
v \;=\; & \bar{e}^b{}_i\, \partial_t e^i{}_a K^a{}_b + \frac{1}{3!} e^{i_1}{}_{a_1} e^{i_2}{}_{a_2} e^{i_3}{}_{a_3} D A_{i_1 i_2 i_3} E^{a_1 a_2 a_3} \\
& + \frac{1}{6!} e^{i_1}{}_{a_1} \cdots e^{i_6}{}_{a_6} D A_{i_1 \ldots i_6} E^{a_1 \ldots a_6} + \frac{1}{9!} e^{i_0}{}_{a_0} \cdots e^{i_8}{}_{a_8} D A_{i_0 | i_1 \ldots i_8} E^{a_0 | a_1 \ldots a_8}.
\end{aligned}
\tag{6}
$$

Here, $D A_{i_1 i_2 i_3} = \partial_t A_{i_1 i_2 i_3}$, and the more complicated expressions for $D A_{i_1 \ldots i_6}$ and $D A_{i_0 | i_1 \ldots i_8}$ were given in [10]. In the expansion (6) of v one can think of the indices on the generators $K^a{}_b$ etc. as *flat* (Euclidean) indices. As for the indices on $D A_{i_1 i_2 i_3}$ etc. the dictionary of [10, 14] shows that they correspond to a time-independent non-orthonormal frame $\theta^i = \theta^i{}_m dx^m$. The object $e^i{}_a = (\exp h)^i{}_a$ (which is the 'square root' of the contravariant 'coset metric' $g^{ij} = \sum_a e^i{}_a e^j{}_a$) relates the two types of indices, and corresponds to the inverse of the matrix $S^a{}_i$ mentioned above. The parametrization (5) corresponds to a special choice of coordinates on the coset manifold $E_{10}/K(E_{10})$.

We introduce the $\mathfrak{k}_{10}$ generators through

$$
\begin{aligned}
J^{ab} &= K^a{}_b - K^b{}_a , & J^{a_1 a_2 a_3} &= E^{a_1 a_2 a_3} - F_{a_1 a_2 a_3} , \\
J^{a_1 \ldots a_6} &= E^{a_1 \ldots a_6} - F_{a_1 \ldots a_6} , & J^{a_0 | a_1 \ldots a_8} &= E^{a_0 | a_1 \ldots a_8} - F_{a_0 | a_1 \ldots a_8} ,
\end{aligned}
\tag{7}
$$

where $F_{a_1 a_2 a_3} = (E^{a_1 a_2 a_3})^T$ etc., that is, with the general normalization $J = E - F$. Henceforth, we shall refer to J^{ab}, $J^{a_1 a_2 a_3}$, $J^{a_1 \ldots a_6}$, and $J^{a_0 | a_1 \ldots a_8}$ as being of 'levels' $\ell = 0, 1, 2, 3$, respectively. However, this 'level' is not a grading of $\mathfrak{k}_{10}$; rather one finds for commutators that $\left[\mathfrak{k}^{(\ell)}, \mathfrak{k}^{(\ell')} \right] \subset \mathfrak{k}^{(\ell + \ell')} \oplus \mathfrak{k}^{(|\ell - \ell'|)}$ (in fact, $\mathfrak{k}_{10}$ is neither a graded nor a Kac–Moody algebra). Computing the antisymmetric piece $\mathcal{Q}$ of the velocity v we conclude that the explicit form of the fermionic equation of motion (3) is

$$
\begin{aligned}
\Bigg(\partial_t &- \frac{1}{2} \bar{e}^b{}_i \partial_t e^i{}_a \overset{\text{vs}}{J}{}^{ab} - \frac{1}{2} \cdot \frac{1}{3!} e^{i_1}{}_{a_1} \cdots e^{i_3}{}_{a_3} D A_{i_1 \ldots i_3} \overset{\text{vs}}{J}{}^{a_1 a_2 a_3} \\
&- \frac{1}{2} \cdot \frac{1}{6!} e^{i_1}{}_{a_1} \cdots e^{i_6}{}_{a_6} D A_{i_1 \ldots i_6} \overset{\text{vs}}{J}{}^{a_1 \ldots a_6} \\
&- \frac{1}{2} \cdot \frac{1}{9!} e^{i_0}{}_{a_0} \cdots e^{i_8}{}_{a_8} D A_{i_0 | i_1 \ldots i_8} \overset{\text{vs}}{J}{}^{a_0 | a_1 \ldots a_8} + \ldots \Bigg) \Psi = 0.
\end{aligned}
\tag{8}
$$

Here, $\overset{\text{vs}}{J}{}^{ab} := \overset{\text{vs}}{R}(J^{ab})$ etc. are the form the $\mathfrak{k}_{10}$ generators take in the sought-for vector-spinor representation Ψ. The crucial consistency condition for Ψ to be a linear representation is that the generators $\overset{\text{vs}}{J}{}^{ab}$ etc. (to be deduced below) should satisfy the abstract $\mathfrak{k}_{10}$ commutation relations

$$
\begin{aligned}
\left[J^{ab}, J^{cd}\right] &= \delta^{bc}J^{ad} + \delta^{ad}J^{bc} - \delta^{ac}J^{bd} - \delta^{bd}J^{ac} \equiv 4\delta^{bc}J^{ad} \\
\left[J^{a_1a_2a_3}, J^{b_1b_2b_3}\right] &= J^{a_1a_2a_3b_1b_2b_3} - 18\delta^{a_1b_1}\delta^{a_2b_2}J^{a_3b_3} \\
\left[J^{a_1a_2a_3}, J^{b_1...b_6}\right] &= J^{[a_1|a_2a_3]b_1...b_6} - 5!\,\delta^{a_1b_1}\delta^{a_2b_2}\delta^{a_3b_3}J^{b_4b_5b_6} \\
\left[J^{a_1...a_6}, J^{b_1...b_6}\right] &= -6\cdot 6!\,\delta^{a_1b_1}\cdots\delta^{a_5b_5}J^{a_6b_6} + \ldots \\
\left[J^{a_1a_2a_3}, J^{b_0|b_1...b_8}\right] &= -336\left(\delta^{b_0b_1b_2}_{a_1a_2a_3}J^{b_3...b_8} - \delta^{b_1b_2b_3}_{a_1a_2a_3}J^{b_4...b_8b_0}\right) + \ldots \\
\left[J^{a_1...a_6}, J^{b_0|b_1...b_8}\right] &= -8!\left(\delta^{b_0b_1...b_5}_{a_1...a_6}J^{b_6b_7b_8} - \delta^{b_1...b_6}_{a_1...a_6}J^{b_7b_8b_0}\right) + \ldots \\
\left[J^{a_0|a_1...a_8}, J^{b_0|b_1...b_8}\right] &= -8\cdot 8!\left(\delta^{a_1...a_8}_{b_1...b_8}J^{a_0b_0} - \delta^{a_1...a_8}_{b_0b_1...b_7}J^{a_0b_8} - \delta^{a_0a_1...a_7}_{b_1...b_8}J^{a_8b_0}\right. \\
&\qquad\left. +8\,\delta^{a_0}_{b_0}\delta^{a_1...a_7}_{b_1...b_7}J^{a_8b_8} + 7\delta^{a_1}_{b_0}\delta^{a_0a_2...a_7}_{b_1...b_7}J^{a_8b_8}\right) + \ldots \quad (9)
\end{aligned}
$$

computed up to $\ell = 3$ in the basis for $\mathfrak{e}_{10}$ used in [14]. Here we have used a shorthand notation where the terms on the r.h.s. should be antisymmetrised (with weight one) according to the antisymmetries on the l.h.s., as written out for the $SO(10)$ generators J^{ab} in the first line. For the mixed symmetry generator $J^{a_0|a_1...a_8}$ this includes only antisymmetrisation over $[a_1\ldots a_8]$. Under $SO(10)$ the tensors on the higher levels rotate in the standard fashion.[5]

To compare eqs. (2) and (8) we now use the bosonic dictionary obtained in [10, 14]. In terms of our present conventions, and in terms of 'flat' indices on both sides[6] this dictionary consists of asserting the correspondences

$$
e^i{}_a \leftrightarrow \theta^i{}_m e_{(10)a}^{\ m}, \qquad\qquad DA_{a_1a_2a_3} \leftrightarrow 2F^{(11)}_{ta_1a_2a_3} = 2NF^{(11)}_{0a_1a_2a_3},
$$

$$
DA_{a_1...a_6} \leftrightarrow -\frac{2}{4!}N\epsilon_{a_1...a_6b_1...b_4}F^{(11)}_{b_1...b_4}, \quad DA_{a_0|a_1...a_8} \leftrightarrow \frac{3}{2}N\epsilon_{a_1...a_8b_1b_2}\tilde{\Omega}^{(10)}_{b_1b_2\,a_0}.
$$
$$(10)$$

---

[5] We use the flat Euclidean δ^{ab} of $SO(10)$ to raise and lower indices. As $SO(10)$ representation the generator $J^{a_0|a_1...a_8}$ is reducible with irreducible components $\bar{J}$ and $\hat{J}$ defined by $\bar{J}^{a_1|a_2...a_9} = J^{a_1|a_2...a_9} - \frac{8}{3}\delta^{a_1[a_2}\hat{J}^{a_3...a_9]}$ and $\hat{J}^{a_3...a_9} = \delta_{a_1a_2}J^{a_1|a_2a_3...a_9}$.

[6] To convert 'frame' indices $i, j, k, \ldots$ into 'flat' ones $a, b, c, \ldots$, one uses $e^i{}_a$ on the coset side, and $e_{(10)a}^{\ i} := \theta^i{}_m e_{(10)a}^{\ m} \equiv (S^{-1})^i{}_a$ on the SUGRA side

Here, as in [14], $\tilde{\Omega}_{abc}^{(10)} = \Omega_{abc}^{(10)} - \frac{2}{9}\delta_{c[a}\Omega_{b]d\,d}^{(10)}$ denotes the tracefree part of the spatial anholonomy coefficient $\Omega_{abc}^{(10)} = 2e_{(10)[a}^{\,m}e_{(10)b]}^{\,n}\partial_m e_{(10)n}^{\,c}$.

Using the correspondences (10), as well as their consequence $-\frac{1}{2}(\bar{e}^b{}_i\partial_t e^i{}_a - \bar{e}^a{}_i\partial_t e^i{}_b) \leftrightarrow +\omega_{t\,ab}^{(11)} = N\omega_{0\,ab}^{(11)}$, we can tentatively re-interpret most terms in the supergravity equation (2) as terms in the putatively $K(E_{10})$ covariant equation (8). Using, as is always locally possible, a spatial frame such that the trace $\omega_{b\,bc}^{(11)} = 0$ (and therefore $\tilde{\Omega}_{abc}^{(10)} = \Omega_{abc}^{(10)}$), and neglecting, as in the bosonic case [10], the *frame* spatial derivatives $\partial_a\psi_b^{(10)}$ and $\partial_a N = -N\omega_{0\,0a}^{(11)}$, we can identify eq. (2) with eq. (8) if we *define* the action of $K(E_{10})$ generators in the vector-spinor representation by

$$
\begin{aligned}
\left(\overset{\text{vs}}{J}{}_\Lambda^{(0)}\Psi\right)_a &:= \Lambda_{ab}\psi^b + \frac{1}{4}\Lambda_{bc}\Gamma^{bc}\psi_a, \\
\left(\overset{\text{vs}}{J}{}_\Lambda^{(1)}\Psi\right)_a &:= \frac{1}{12}\Lambda_{bcd}\Gamma^{bcd}\psi_a + \frac{2}{3}\Lambda_{abc}\Gamma^b\psi^c - \frac{1}{6}\Lambda_{bcd}\Gamma_{abc}\psi^d, \\
\left(\overset{\text{vs}}{J}{}_\Lambda^{(2)}\Psi\right)_a &:= \frac{1}{1440}\Lambda_{bcdefg}\Gamma^{bcdefg}\psi_a + \frac{1}{180}\Lambda^{bcdefg}\Gamma_{abcdef}\psi_g \\
&\qquad -\frac{1}{72}\Lambda_{abcdef}\Gamma^{bcde}\psi^f, \\
\left(\overset{\text{vs}}{J}{}_\Lambda^{(3)}\Psi\right)_a &:= \frac{2}{3}\cdot\frac{1}{8!}\left(\Lambda_{b|c_1\dots c_8}\Gamma_a{}^{c_1\dots c_8}\psi^b + 8\Lambda_{a|c_1\dots c_8}\Gamma^{c_1\dots c_7}\psi^{c_8}\right. \\
&\qquad \left. +2\Lambda_{b|bc_1\dots c_7}\Gamma^{c_1\dots c_7}\psi_a - 28\Lambda_{b|bc_1\dots c_7}\Gamma_a{}^{c_1\dots c_6}\psi^{c_7}\right) \quad (11)
\end{aligned}
$$

Here, we have used a shorthand notation for the action of $\overset{\text{vs}}{J}$ by absorbing the transformation parameters into the generators according to $\overset{\text{vs}}{J}{}_\Lambda^{(0)} \equiv \frac{1}{2}\Lambda_{a_1 a_2}\overset{\text{vs}}{J}{}^{a_1 a_2}$, $\overset{\text{vs}}{J}{}_\Lambda^{(1)} \equiv \frac{1}{3!}\Lambda_{a_1 a_2 a_3}\overset{\text{vs}}{J}{}^{a_1 a_2 a_3}$, $\overset{\text{vs}}{J}{}_\Lambda^{(2)} \equiv \frac{1}{6!}\Lambda_{a_1\dots a_6}\overset{\text{vs}}{J}{}^{a_1\dots a_6}$, and $\overset{\text{vs}}{J}{}_\Lambda^{(3)} \equiv \frac{1}{9!}\Lambda_{a_0|a_1\dots a_8}\overset{\text{vs}}{J}{}^{a_0|a_1\dots a_8}$. The last parameter $\Lambda_{a_0|a_1\dots a_8}$ has two irreducible pieces analogous to $\overset{\text{vs}}{J}{}^{(3)}$ (see footnote 5) and the trace appears explicitly in (11).

Proving the $K(E_{10})$ covariance of the coset fermionic equation (8) now reduces to proving that the generators $\overset{\text{vs}}{J}{}^{(\ell)}$ defined by (11) do satisfy the $K(E_{10})$ relations which were given in (9). It is easy to see that the commutators of the level-zero generators $\overset{\text{vs}}{J}{}^{(0)}$ with themselves, as well as with any other $\overset{\text{vs}}{J}{}^{(\ell)}$ for $\ell > 0$, produce the required $SO(10)$ rotations of (9). The other

commutators require some tedious calculations using the gamma algebra. The result of this computation is

$$\left(\left[\overset{\text{vs}}{J}{}^{(1)}_\Lambda, \overset{\text{vs}}{J}{}^{(1)}_{\Lambda'}\right]\Psi\right)_a = 20\left(\overset{\text{vs}}{J}{}^{(2)}_\Sigma\Psi\right)_a - \left(\overset{\text{vs}}{J}{}^{(0)}_\Sigma\Psi\right)_a,$$

$$\left(\left[\overset{\text{vs}}{J}{}^{(1)}_\Lambda, \overset{\text{vs}}{J}{}^{(2)}_{\Lambda'}\right]\Psi\right)_a = 56\left(\overset{\text{vs}}{J}{}^{(3)}_\Sigma\Psi\right)_a - \frac{1}{6}\left(\overset{\text{vs}}{J}{}^{(1)}_\Sigma\Psi\right)_a, \qquad (12)$$

where the $\overset{\text{vs}}{J}{}^{(\ell)}_\Sigma$ are defined as above, but now with new parameters given by $\Sigma^{(0)}_{ab} = \Lambda_{d_1d_2[a}\Lambda'_{b]}{}^{d_1d_2}$, $\Sigma^{(2)}_{b_1...b_6} = \Lambda_{[b_1b_2b_3}\Lambda'_{b_4b_5b_6]}$, $\Sigma^{(1)}_{a_1a_2a_3} = \Lambda^{b_1b_2b_3}\Lambda'_{b_1b_2b_3a_1a_2a_3}$, and $\Sigma^{(3)}_{a_0|a_1...a_8} = \Lambda_{a_0[a_1a_2}\Lambda'_{a_3...a_8]} - \Lambda_{[a_1a_2a_3}\Lambda'_{a_4...a_8]a_0}$. One can now check that the relations (12) are consistent with the $K(E_{10})$ commutators (9). All other commutators have to produce terms on the r.h.s. which have contributions of 'level' $\ell > 3$ and therefore cannot be checked fully. However, we have verified, where possible, that the expected contributions of the lower levels appear with the correct normalisation required by the structure constants of (9). Therefore we find that the vector-spinor representation $\overset{\text{vs}}{J}{}^{(\ell)}$ of $K(E_{10})$ which we deduced from comparing (2) and (8) is a good linear representation up to the level we have supergravity data to define the commutation relations.

Using arguments from the the general representation theory of Lie algebras one can actually show that the checks we have carried out are sufficient to guarantee the existence of an extension of the vector-spinor representation $\overset{\text{vs}}{J}{}^{(\ell)}$ to 'levels' $\ell > 3$ on the same components ψ_a. That is, we can define on ψ_a alone an *unfaithful, irreducible* **320**-dimensional representation of $K(E_{10})$ on which infinitely many $K(E_{10})$ generators are realised non-trivially. For this definition it is sufficient to define the action of $\overset{\text{vs}}{J}{}^{(0)}$ and $\overset{\text{vs}}{J}{}^{(1)}$ on ψ_a and check some compatibility conditions. We view the fact that the $\overset{\text{vs}}{J}{}^{(2)}$ and $\overset{\text{vs}}{J}{}^{(3)}$ transformations deduced from the supergravity correspondence above agree with this general construction as strong evidence for the relevance of the vector-spinor component of the *infinite-dimensional $K(E_{10})$* spinor $\Psi = (\psi_a, \ldots)$ we have in mind. If one repeats the same analysis for the Dirac spinor, where the representation matrices on this **32**-dimensional space are given in terms of anti-symmetric Γ-matrices (see (16) below), one finds that one can consistently realise $K(E_{10})$ on a **32**-component spinor of $SO(10)$.

The fact that the anti-symmetric Γ-matrices together with Γ^0 span the fundamental representation of $SO(32)$ has led a number of authors to propose $SO(32)$ as a 'generalised holonomy' for M-theory [21, 22]. Global problems with this proposal (and with an analogous $SL(32)$ proposal [23, 24, 25]) were subsequently pointed out in [26] where it was shown that no suitable *spinor* (i.e. double valued) representation with the correct number of components of these generalised holonomy groups exist. Our approach is radically different, since we have an action not of $SO(32)$ but of $K(E_{10})$, with infinitely many generators acting in a non-trivial manner, on a *bona fide* spinor representation of $SO(10)$. We therefore evade the conclusions of [26].[7] The appearance of an unfaithful representation for the fermions was already noted and studied in the affine case for $K(E_9)$, which shows very similar features consistent with our present findings [27]. One possibility to construct a faithful representation of $K(E_{10})$ already pointed out there might be to consider the tensor product of such unfaithful representations with a faithful representation, like the adjoint $\mathfrak{k}_{10}$ or the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$. Let us also note that the **320**-dimensional representation of $K(E_{10})$ is compatible with the fermionic representations studied in [12]. More details on these aspects will be given in a future publication [28].

A deeper confirmation of the hidden $K(E_{10})$ symmetry of SUGRA$_{11}$ is obtained by writing down a $K(E_{10})$ invariant action functional describing a massless spinning particle on $E_{10}/K(E_{10})$. We will be brief and defer the details to [28]. The bosonic part of the action is the one of [10]

$$S_{\text{bos}} = \int dt \, \frac{1}{2n} \langle \mathcal{P}(t) | \mathcal{P}(t) \rangle \tag{13}$$

where $\langle \cdot | \cdot \rangle$ is the standard invariant bilinear form on $\mathfrak{e}_{10}$ [20] and where the coset 'lapse' function n can be identified with the rescaled supergravity lapse $Ng^{-1/2}$ (denoted (denoted $\tilde{N}$ in [11]).

---

[7]Similar arguments would apply to $K(E_{11})$ and $SL(32)$ although it is by no means clear if our construction of the vector-spinor representation can be lifted to $K(E_{11})$.

The fermionic term we add to this action reads

$$S_{\text{ferm}} = -\frac{i}{2} \int dt \, \left(\Psi(t) | \overset{\text{vs}}{\mathcal{D}} \Psi(t) \right)_{\text{vs}}, \tag{14}$$

where $(\cdot|\cdot)_{\text{vs}}$ is a $K(E_{10})$ invariant symmetric form on the vector-spinor representation space.[8] On the lowest component of $\Psi = (\psi_a, \ldots)$ it is explicitly given by $(\Psi|\Phi)_{\text{vs}} = \psi_a^T \Gamma^{ab} \phi_b$. The invariance of this form under the generators $\overset{\text{vs}}{J}{}^{(\ell)}$ defined in (11) is a quite restrictive condition. We have verified that invariance holds, but only since we are working over a *ten-dimensional* Clifford algebra. By using induction arguments we find that $(\Psi|\Phi)_{\text{vs}}$ is invariant not only under (11) but under the (unfaithful) extension to the *full* $K(E_{10})$ transformations mentioned above. We expect that the form $(\Psi|\Phi)_{\text{vs}}$ will extend to an invariant symmetric form on a faithful representation $\Psi = (\psi_a, \ldots)$.

Further important hints of a hidden $K(E_{10})$ symmetry come from considering the local SUSY constraint $\mathcal{S}^{(11)} = 0$ which is proportional to the time component of the Rarita Schwinger equation (1). First, we find that, under the dictionary of [10, 14], $\mathcal{S}^{(11)}$ is mapped into a $K(E_{10})$ covariant constraint of the form $\mathcal{P} \odot \Psi = 0$, when neglecting frame gradients $\partial_a \psi_b$ as we have done in the derivation of (11). The product $\odot$ symbolises a map from the tensor product of $\mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$ with Ψ onto a Dirac-spinor-type representation space of $\mathfrak{k}_{10}$. The coset constraint $\mathcal{P} \odot \Psi = 0$ suggests to augment the action $S_{\text{bos}} + S_{\text{ferm}}$ by a 'Noether' term of the form

$$S_{\text{Noether}} = \int dt \, (\chi(t)|\mathcal{P}(t) \odot \Psi(t))_{\text{s}}, \tag{15}$$

with a local Dirac-spinor $\chi(t)$ Lagrange multiplier (that is, a one-dimensional 'gravitino'). As will be discussed elsewhere [28], the total action $S_{\text{bos}} + S_{\text{ferm}} + S_{\text{Noether}}$ then turns out to be both $K(E_{10})$ invariant (disregarding Ψ^4 terms) and to be invariant under *quasi-rigid*, time-dependent supersymmetry transformations which involve a Dirac-spinor-type $K(E_{10})$ representation $\epsilon(t)$

---

[8]Observe that this symmetric form is actually *anti*-symmetric when eveluated on *anti*-commuting (Grassmann valued) fermionic variables $\Psi(t)$, such that e.g. $(\Psi(t)|\Psi(t))_{\text{vs}} = 0$.

constrained to satisfy $\overset{s}{\mathcal{D}}\, \epsilon(t) \equiv (\partial_t - \overset{s}{\mathcal{Q}})\epsilon = 0$. This equation is formally the same as (3) and (8) but now the generators are found to be (cf. [18])

$$\overset{s}{J}{}^{ab} = \frac{1}{2}\Gamma^{ab}, \qquad\qquad \overset{s}{J}{}^{a_1 a_2 a_3} = \frac{1}{2}\Gamma^{a_1 a_2 a_3},$$

$$\overset{s}{J}{}^{a_1...a_6} = \frac{1}{2}\Gamma^{a_1...a_6}, \qquad \overset{s}{J}{}^{a_0|a_1...a_8} = 12\,\delta^{a_1...a_8}_{a_0 b_1...b_7}\Gamma^{b_1...b_7}. \qquad (16)$$

The particular form of the Dirac-spinor representation on $\ell = 3$ implies that the irreducible component $\overset{s}{\bar{J}}{}^{a_0|a_1...a_8}$ is mapped to zero under this correspondence. This is in contrast to the vector-spinor representation: there is no way to represent a non-trivial Young tableau purely in terms of gamma matrices.

In summary, we have given evidence for the following generalisation of the correspondence conjectured in [10]: The time evolution of the eleven-dimensional supergravity fields $g^{(11)}_{MN}(t, \mathbf{x}), A^{(11)}_{MNP}(t, \mathbf{x}), \psi^{(11)}_M(t, \mathbf{x})$ and their spatial gradients (considered around any given spatial point $\mathbf{x}$, in temporal gauge and with fixed SUSY gauge) can be mapped onto the dynamics of a (supersymmetric) *spinning massless particle* $(\mathcal{V}(t), \Psi(t))$ on $E_{10}/K(E_{10})$. The E_{10}-invariant quantum dynamics of this superparticle might provide the basis of a new definition of M-theory. Much work remains to be done to extend the evidence indicated here, for instance by proving the existence of irreducible faithful (and hence infinite-dimensional) 'vector-spinor-type' and 'Dirac-spinor-type' representations of $K(E_{10})$.

Let us finally note on the physical side, that we deem it probable that the proposed correspondence between M-theory and the coset model is such that the two sides do not have a common range of physical validity: Indeed, the coset model description emerges in the near space-like singularity limit $T \to 0$, where T denotes the proper time[9], which indicates that the coset description might be well defined only when $T \ll T_{\text{Planck}}$, i.e. in a strong curvature regime where the spacetime description 'de-emerges'.

Acknowledgements

---

[9]Note that the coordinate and 'coset time' t used above is (in the gauge $n = 1$) roughly proportional to $-\log T$, and actually goes to $+\infty$ near the space-like singularity.

References

[1] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. B **76** (1978) 409–412

[2] E. Cremmer and B. Julia, *The SO(8) Supergravity*, Nucl. Phys. B **159** (1979) 141

[3] B. Julia, *Application of supergravity to gravitation theories*, in: Unified Field Theories of more than 4 Dimensions, eds. V. De Sabbata and E. Schmutzer, World Scientific (Singapore, 1983)

[4] N. Marcus and J. H. Schwarz, *Three-Dimensional Supergravity Theories*, Nucl. Phys. B **228** (1983) 145

[5] H. Nicolai, $D = 11$ *Supergravity With Local SO(16) Invariance*, Phys. Lett. B **187** (1987) 316

[6] H. Nicolai, *The Integrability Of N = 16 Supergravity*, Phys. Lett. B **194** (1987) 402

[7] B. Julia, in: Lectures in Applied Mathematics, Vol. 21 (1985), AMS-SIAM, p. 335; preprint LPTENS 80/16

[8] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, *Oscillatory Approach To A Singular Point In The Relativistic Cosmology*, Adv. Phys. **19** (1970) 525

[9] T. Damour and M. Henneaux, E_{10}, BE_{10} *and arithmetical chaos in superstring cosmology*, Phys. Rev. Lett. **86** (2001) 4749, `hep-th/0012172`

[10] T. Damour, M. Henneaux and H. Nicolai, E_{10} *and a "small tension expansion" of M-theory*, Phys. Rev. Lett. **89** (2002) 221601, `hep-th/0207267`

[11] T. Damour, M. Henneaux and H. Nicolai, *Cosmological Billiards*, Class. Quant. Grav. **20** (2003) R145–R200, `hep-th/0212256`

[12] A. Kleinschmidt and H. Nicolai, E_{10} *and* $SO(9,9)$ *invariant supergravity*, JHEP **0407** (2004) 041, `hep-th/0407101`

[13] A. Kleinschmidt and H. Nicolai, *IIB supergravity and* E_{10}, Phys. Lett. B **606** (2005) 391, `hep-th/0411225`

[14] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the* $E_{10}/K(E_{10})$ σ-*model at low* A_9 *levels*, in: Group Theoretical Methods in Physics, Institute of Physics Conference Series No. 185, IoP Publishing, 2005, `hep-th/0410245`

[15] T. Damour and H. Nicolai, *Higher order M theory corrections and the Kac-Moody algebra* E_{10}, Class. Quant. Grav. **22** (2005) 2849, `hep-th/0504153`

[16] P. C. West, E_{11} *and M theory*, Class. Quant. Grav. **18** (2001) 4443–4460, `hep-th/0104081`

[17] F. Englert and L. Houart, $\mathcal{G}^{+++}$ *invariant formulation of gravity and M-theories: exact BPS solutions*, JHEP **0401** (2004) 002, `hep-th/0311255`

[18] S. de Buyl, M. Henneaux and L. Paulot, *Hidden symmetries and Dirac fermions*, Class. Quant. Grav. **22** (2005) 3595, `hep-th/0506009`

[19] S. de Buyl, M. Henneaux and L. Paulot, in preparation.

[20] V. G. Kac, *Infinite dimensional Lie algebras*, 3rd edition, Cambridge University Press (Cambridge, 1990)

[21] M. J. Duff and K. S. Stelle, *Multimembrane solutions of* $D = 11$ *supergravity*, Phys. Lett. B **253** (1991) 113–118

[22] M. Duff and J. T. Liu, *Hidden space-time symmetries and generalized holonomy in M theory*, Nucl. Phys. B **674** (2003) 217–230, `hep-th/0303140`

[23] O. Bärwald and P. C. West, *Brane rotating symmetries and the five-brane equations of motion*, Phys. Lett. B **476** (2000) 157–164, `hep-th/9912226`

[24] C. Hull, *Holonomy and symmetry in M theory*, `hep-th/0305039`

[25] P. West, E_{11}, $SL(32)$ *and central charges*, Phys. Lett. B **575** (2003) 333–342, `hep-th/0307098`

[26] A. Keurentjes, *The topology of U Duality (sub)groups*, Class. Quantum. Grav. **21** (2004) 1695-1708, `hep-th/0309106`

[27] H. Nicolai and H. Samtleben, *On $K(E_9)$*, Q. J. Pure Appl. Math. **1** (2005) 180, `hep-th/0407055`

[28] T. Damour, A. Kleinschmidt and H. Nicolai, in preparation.