A Gradient Flow for Worldsheet Nonlinear Sigma Models

T Oliynyk[†] [1], **V Suneeta**[♯‡] [2], **E Woolgar**[♭] [‡] [3]

[†] Max-Planck-Institut für Gravitationsphysik (Albert Einstein Institute), Am Mühlenberg 1, D-14476 Potsdam, Germany.
[♯] Dept of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3.
[‡] Theoretical Physics Institute, University of Alberta, Edmonton, AB, Canada T6G 2J1.
[♭] Dept of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1.

Abstract

We discuss certain recent mathematical advances, mainly due to Perelman, in the theory of Ricci flows and their relevance for renormalization group (RG) flows. We consider nonlinear sigma models with closed target manifolds supporting a Riemannian metric, dilaton, and 2-form B-field. By generalizing recent mathematical results to incorporate the B-field and by decoupling the dilaton, we are able to describe the 1-loop β-functions of the metric and B-field as the components of the gradient of a potential functional on the space of coupling constants. We emphasize a special choice of diffeomorphism gauge generated by the lowest eigenfunction of a certain Schrödinger operator whose potential and kinetic terms evolve along the flow. With this choice, the potential functional is the corresponding lowest eigenvalue, and gives the order α' correction to the Weyl anomaly at fixed points of $(g(t), B(t))$. The lowest eigenvalue is monotonic along the flow, and since it reproduces the Weyl anomaly at fixed points, it accords with the c-theorem for flows that remain always in the first-order regime. We compute the Hessian of the lowest eigenvalue functional and use it to discuss the linear stability of points where the 1-loop β-functions vanish, such as flat tori and K3 manifolds.

[1] todd.oliynyk@aie.mpg.de
[2] suneeta@math.unb.ca
[3] ewoolgar@math.ualberta.ca

I Introduction

A standard approach to renormalization group (RG) flow for a quantum field theory consists of deriving differential equations governing the behaviour of the coupling constants under changes in the renormalization scale ([1], [2], [3]). These RG flow equations are written in terms of β-functions which are components of a vector field tangent to the flow on the space of coupling constants of the theory. The β-functions can be computed in a loop expansion. For the worldsheet (i.e., 2-dimensional) bosonic sigma model, the loop expansion parameter is α', the square of the string scale.

A basic result is Zamolodchikov's c-theorem [4, 5], which implies that certain RG flows are irreversible. This theorem asserts the existence of a function on the space of coupling constants of certain 2-dimensional quantum field theories called the C-function, which decreases monotonically along any renormalization group trajectory from an unstable to a stable fixed point, and equals the Weyl anomaly (the central charge of the Virasoro algebra) at fixed points. The c-theorem was extended to nonlinear sigma models with compact target spaces by Tseytlin [6]. The C-function obeys the monotonicity formula $dC/dt = -\kappa(\beta, \beta)$ for t a parameter along the flow, κ a non-negative quadratic form on the space of coupling constants, and β the array of β-functions of the coupling constants. The c-theorem is not contingent on the loop expansion for β.

A longstanding question is whether RG flow is a gradient flow: Is the vector field defined by the β-functions orthogonal to level surfaces of a potential function on the space of coupling constants. Recent advances in mathematics have shed light on this matter. Consider the special case of a nonlinear sigma model whose target space is purely gravitational (in particular, the anti-symmetric B-field is absent), and with β replaced by its 1-loop (order α') approximation. Then the 1-loop RG flow for the target space metric is known in the mathematics literature as Ricci flow [7]. This flow was long known to be gradient on a space of coupling constant endowed with a metric of indefinite sign [8]. But the above discussion of the c-theorem and κ suggests there may be a positive-definite metric for 1-loop gradient flow, from which one could obtain a monotonicity formula. Perelman [9] has now shown that this flow on closed (target) manifolds of arbitrary dimension is in fact a gradient flow on a space of coupling constants with positive-definite metric.

Then what do these recent mathematical advances mean for RG flows of sigma models more generally, when B is not held to zero? This paper is intended to address this issue. We restrict our attention to the first of many potentially relevant results announced in [9], the gradient nature of the flow for g and related monotonicity. We endeavour to set out this result in some detail in language appropriate to the RG setting. Our first task is therefore to generalize it to incorporate not only the target manifold's Riemannian metric g but also the anti-symmetric 2-form field B and dilaton Φ (though the dilaton can often be ignored by decoupling it from the system using a suitably chosen diffeomorphism). We find that for sigma models whose target space is a closed manifold, the order α' RG flow of (g, B) is gradient on a space of coupling constants with positive-definite metric, and thus the flow is irreversible.[4] The irreversibility argument is easiest when we choose a certain diffeomorphism gauge along the flow, and then we call the potential function λ. We note that the full RG flow (i.e., not the order α' truncation) on closed target manifolds is known to be irreversible—this is a consequence of the c-theorem. However, in the absence of knowledge of the higher loop corrections to the β-function, it is not practically possible to compute the C-function on closed target manifolds. Our result can be considered valid when the 1-loop truncation of the β-function is a good approximation. In this case, we can explicitly compute the potential function λ that generates the 1-loop RG flow without requiring knowledge about regimes where "stringy" (higher order in α') corrections become important.

We find the value of the potential function λ at fixed points to be non-negative, and zero at any Ricci-flat fixed point. This raises the undesirable possibility that a Ricci-flat fixed point such as a flat torus might flow to a non-Ricci-flat one. To preclude this possibility, we compute the Hessian of the potential and use the resulting second variations to discuss the linear stability, as well as the rigidity (or isolation), of Ricci-flat fixed points. We confirm linear stability for the particular examples of flat tori and K3 manifolds. Our considerations lead us to briefly discuss the possible rigidity of Ricci-flat perturbative string vacua, at which the β-functions vanish to all orders in α'. This may be of interest when viewed in the light of investigations into the topology of the configuration space of string theory [13, 14].

In Section II, we discuss the sigma model under consideration, recalling its RG flow equations at order α' and decoupling the dilaton from the flow of g and B.

[4] Elsewhere, we have extended Perelman's work to noncompact asymptotically flat manifolds with somewhere-negative scalar curvature and zero B-field [12].

Section III is the heart of the paper, particularly the first subsection, wherein we describe and generalize Perelman's approach. Perelman's remarkable insight (see also [10, 11]) is that the Ricci flow of g, modified by the correct choice of diffeomorphism, becomes the gradient flow of a potential λ which is actually the lowest eigenvalue of a certain Schrödinger operator on the target manifold,[5] and we show that this carries over *mutatis mutandis* to the (g, B) flow as well. This leads to an easy proof of the absence of periodic or homoclinic behaviour (i.e., the "irreversibility") of the 1-loop flow. In the second subsection, we evaluate this eigenvalue at a fixed point. We show that it is ≥ 0 and is -4 times the order α' correction to the Weyl anomaly at fixed points. The final subsection contains a brief aside on the gradient nature of RG flow with an arbitrary diffeomorphism; i.e., not a diffeomorphism chosen as above to connect to the Schrödinger problem.

Section IV contains a derivation of the second variation of the potential function at a fixed point. This section builds on a similar result for Ricci flow presented in [15]. In Section V we apply the second variation formula to discuss linear stability of fixed points, including some particular Ricci-flat examples. We conclude Section V with some speculative remarks concerning the potential applicability of our results to an issue in string theory. Section VI contains remarks on higher-order flows and the C-theorem.

An Appendix contains some calculations related to Section III that we believe would unnecessarily clutter the main text. That said, we have tried to provide a certain level of calculational detail, especially when such detail has not been provided in the mathematics literature.

Throughout, the target manifold is closed (i.e., compact and without boundary) and is assumed to carry a positive-definite Riemannian metric. Our RG flow equations agree with those that appear in [16].

II RG Flow of the Worldsheet Nonlinear Sigma Model

The 2-dimensional, or worldsheet, nonlinear sigma model is a quantum field theory of maps X^i from a 2-dimensional Riemannian manifold (Σ, h), the

[5] In fact, the relevance of this eigenvalue problem for RG flow was proposed in [10], in the special case of zero B-field and a closed, 2-dimensional target manifold. It was noticed in [11] that the work of [9] validates this proposal.

worldsheet, to another Riemannian manifold (M, g), the *target manifold* or *target space*, which herein we take to be a closed manifold of dimension at least 3. We let ϵ denote the volume element on the worldsheet and let $R(h)$ be the worldsheet scalar curvature. The sigma model action is (using coordinates $\sigma^\alpha = (\sigma^1, \sigma^2)$ on Σ)

$$S = -\frac{1}{\alpha'} \int_\Sigma d^2\sigma \left[\sqrt{h} h^{\alpha\beta} g_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j - \alpha' \sqrt{h} \Phi R(h) \right] \,, \tag{II.1}$$

where g_{ij}, $B_{ij} = -B_{ji}$, and Φ are the target space metric, B-field, and dilaton, respectively. This model describes the motion of a bosonic string in a background wherein the massless string modes have acquired vacuum expectation values g_{ij}, B_{ij}, and Φ. The action is invariant under reparametrizations and conformal rescalings on the worldsheet and under the addition $B \mapsto B + d\omega$ of an exact form $d\omega$ to B, which is a target space 2-form. The gauge-invariant 3-form field strength for B is $H := dB$.

Cut-off independence of the regulated quantum theory leads to renormalization group flow equations (see, e.g., [16])

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' \left(R_{ij} + 2\nabla_i \nabla_j \Phi - \frac{1}{4} H_{ikl} H_j{}^{kl} \right) \,, \tag{II.2}$$

$$\frac{\partial H}{\partial t} = \alpha' \left(\frac{1}{2} \Delta_{\rm LB} H - d\langle H, {\rm grad}\Phi \rangle \right) \,, \tag{II.3}$$

$$\frac{\partial \Phi}{\partial t} = -A + \alpha' \left(\frac{1}{2} \Delta \Phi - |\nabla \Phi|^2 + \frac{1}{24} |H|^2 \right) \,, \tag{II.4}$$

where Δ is the Laplacian, $\Delta_{\rm LB} H := -(d\delta + \delta d)H$ is the Laplace-Beltrami operator acting on the 3-form $H := dB$, A is a constant whose value depends on the target manifold dimension (and the number of ghost fields when they are present), and t is the log of the renormalization scale.[6]

Equation (II.3) is actually derived by taking the curl of the flow equation

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} &= \alpha' \left(\frac{1}{2} \nabla^k H_{kij} - H_{kij} \nabla^k \Phi + \nabla_i \omega_j - \nabla_j \omega_i \right) \\ &= -\alpha' \left(\frac{1}{2} \delta H + \langle H, {\rm grad}\Phi \rangle + d\omega \right)_{ij} \,. \end{aligned} \tag{II.5}$$

[6] We do not presume t to be in any way related to physical time here, although it has been conjectured that RG flow could model real time evolution in certain situations.

The ω terms arise because B is determined only up to the exterior derivative of an arbitrary 1-form θ. Then $\frac{\partial B}{\partial t}$ acquires a contribution which, because exterior differentiation commutes with $\frac{\partial}{\partial t}$, can be written as the exterior derivative of the 1-form $\omega := \frac{\partial \theta}{\partial t}$.

These equations are written with respect to a coordinate basis fixed with respect to t. In a basis that changes with t, extra terms will be introduced into the evolution equations. We will exploit this now to decouple Φ, and later to demonstrate the monotonicity formula.

Pulling back by the t-dependent diffeomorphism φ_t generated by the vector field $\alpha' \text{grad}\Phi$ adds a Lie derivative term to each of the flow equations. For example, the left-hand side of the equation for g_{ij} becomes $\varphi_t^* \frac{\partial}{\partial t} g_{ij} = (\frac{\partial}{\partial t}\tilde{g}_{ij} - \pounds g_{ij}) \circ \varphi_t = (\frac{\partial}{\partial t}\tilde{g}_{ij} - 2\alpha' \nabla_i \nabla_j \Phi) \circ \varphi_t$, where $\tilde{g}_{ij} := \varphi_t^* g_{ij}$. The right-hand side is natural under diffeomorphisms and becomes, schematically, $\varphi_t^*(\text{RHS}) = \text{RHS} \circ \varphi_t$. The other evolution equations transform similarly, yielding

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -\alpha' \left(\tilde{R}_{ij} - \frac{1}{4} \tilde{H}_{ikl} \tilde{H}_j{}^{kl} \right) , \qquad \text{(II.6)}$$

$$\frac{\partial \tilde{H}}{\partial t} = \frac{\alpha'}{2} \Delta_{\text{LB}} \tilde{H} , \qquad \text{(II.7)}$$

$$\frac{\partial \tilde{\Phi}}{\partial t} = -A + \alpha' \left(\frac{1}{2} \Delta \tilde{\Phi} + \frac{1}{24} |\tilde{H}|^2 \right) . \qquad \text{(II.8)}$$

We say these equations are expressed in *Hamilton gauge* in recognition of the relationship of (II.6) to the Ricci flow equation of R Hamilton [7]. Notice that $\tilde{\Phi}$ has now decoupled from the evolution equations for $\tilde{g}_{ij}$ and $\tilde{H}_{ijk}$.

It will prove convenient to express these equations in an *arbitrary* t-dependent coordinate system; *i.e.*, to pull back by an arbitrary t-dependent diffeomorphism. To do so, we simply add Lie derivative terms to each equation. We are interested in particular in diffeomorphisms generated by the gradient of a scalar, so for later convenience we choose to write the generator as $-\alpha' \nabla \psi$ where ψ is arbitrary. Then the evolution equations become

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' \left(R_{ij} + \nabla_i \nabla_j \psi - \frac{1}{4} H_{ikl} H_j{}^{kl} \right) =: -\beta^g_{ij} , \qquad \text{(II.9)}$$

$$\frac{\partial H_{ijk}}{\partial t} = \frac{\alpha'}{2} \left(\Delta_{\text{LB}} H - d\langle H, \text{grad}\psi \rangle \right)_{ijk} =: -\beta^H_{ijk} , \qquad \text{(II.10)}$$

$$\frac{\partial \Phi}{\partial t} = -A + \frac{\alpha'}{2} \left(\Delta \Phi - \nabla \Phi \cdot \nabla \psi + \frac{1}{12} |H|^2 \right) =: -\beta^\Phi . \qquad \text{(II.11)}$$

The right-hand sides of these equations define what are called β-functions. We have dropped the tildes now, but note that there is of course a distinction between quantities such as g_{ij} appearing in equations (II.2–II.4) and those in (II.9–II.11). Namely, the former are obtained from the latter by choosing the gauge $\psi = 2\Phi$. Hamilton gauge (denoted by the tildes) is the choice $\psi = 0$.

In the arbitrary gauge, (II.5) becomes

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} &= \frac{\alpha'}{2}\left[\nabla^k H_{kij} - H_{kij}\nabla^k\psi\right] \\ &\qquad +2\alpha'\nabla_{[i}\left[\omega_{j]} - B_{j]k}\nabla^k\left(\Phi - \frac{1}{2}\psi\right)\right] \ , \end{aligned} \tag{II.12}$$

with square brackets on indices indicating anti-symmetrization. However, we will fix the evolution of the "internal gauge" by imposing the flow equation

$$\frac{\partial B_{ij}}{\partial t} = \frac{\alpha'}{2}\left[\nabla^k H_{kij} - H_{kij}\nabla^k\psi\right] =: -\beta^B_{ij} \ . \tag{II.13}$$

This forces

$$\sigma_i := \omega_j - B_{jk}\nabla^k\left(\Phi - \frac{1}{2}\psi\right) \tag{II.14}$$

to be a closed 1-form. We will now show that the flow given by (II.9, II.13) is a gradient flow.

III The Gradient Flow

III.1 The Monotonicity Formula

In this section we elucidate the first two sections of [9] (see also the detailed notes [17]) and generalize that work to sigma models with B-field.

The space of coupling constants $\mathcal{G} \ni (g_{ij}(x), B_{ij}(x), \Phi(x))$, $x \in M$, factors as $\mathcal{G} = G \times C^\infty(M)$, where $(g_{ij}(x), B_{ij}(x)) \in G$. The C^∞ factor[7] will be used to accommodate both the dilaton Φ and the diffeomorphism generating function ψ, but it is important not to equate these.[8]

[7] According to the way we have defined points of $\mathcal{G}$, strictly speaking it is not $C^\infty(M)$ that splits off from $\mathcal{G}$ but rather a trivial bundle whose sections belong to $C^\infty(M)$.

[8] That would be a gauge choice; e.g., the choice $\psi = 2\Phi$ produces Hamilton gauge (II.2–II.4).

Consider now a section (g_{ij}, B_{ij}, ψ) of $\mathcal{G}$, where we take g, B, and ψ to be related by (II.9) and (II.13) respectively, but the t-evolution of ψ is arbitrary. If a choice of t-evolution for ψ is made, equations (II.9) and (II.13) will then determine an evolving section $(g(t), B(t))$ in G. Because the dilaton Φ is decoupled, we do not need to compute its t-evolution simultaneously. Rather, we can compute $\Phi(t)$ *a posteriori* from (II.11), once the t-evolutions for $g(t)$ and $B(t)$ are determined. In this way, each choice of evolution $\psi(t)$ gives a distinct t-evolving parametrization of the coupling constants $(g_{ij}(t,x), B_{ij}(t,x), \Phi(t,x))$.

In Subsection III.3 we describe, for each choice ψ of parametrization of the coupling constants, a potential that generates a gradient flow for $(g(t), B(t))$ on the space $G \subseteq \mathcal{G}$ endowed with a natural choice of inner product. However, by choosing ψ in a certain very natural way (which we dub *Perelman gauge*), the resulting gradient flow is particularly useful, and it is that choice which we concern ourselves with first. Define the functional

$$F[g,B,\psi] := \int_M dV e^{-\psi} \left[R + |\nabla\psi|^2 - \frac{1}{12}|H|^2 \right] . \tag{III.1}$$

Integrating by parts, we can write

$$F[g,B,\psi] = \int_M dV e^{-\psi/2} \left[R - \frac{1}{12}|H|^2 - 4\Delta \right] e^{-\psi/2} . \tag{III.2}$$

Now let $u(t,x)$, $x \in M$, be the lowest eigenfunction of the Schrödinger operator $R - \frac{1}{12}|H|^2 - 4\Delta$. The operator depends on t through the flowing metric $g(t)$, and thus so do the eigenfunctions. Normalize u (and the other eigenfunctions) to unity:

$$\int_M u^2 dV = 1 . \tag{III.3}$$

Since the lowest eigenfunction u has no nodes, it has a well-defined logarithm. We use this to define a function P by

$$e^{-P(t,x)/2} := u(t,x) . \tag{III.4}$$

Then, for λ the eigenvalue belonging to u, we have

$$\begin{aligned} & \left[R - \frac{1}{12}|H|^2 - 4\Delta \right] u =: \lambda u \\ \Rightarrow \quad & R - \frac{1}{12}|H|^2 + 2\Delta P - |\nabla P|^2 = \lambda . \end{aligned} \tag{III.5}$$

Clearly, the choice $P = \psi$ minimizes the functional (III.2) over all $C^\infty(M)$ functions obeying $\int_M e^{-\psi} dV = 1$. Thus on G there is a new functional $\lambda[g, B]$, equal in value at $(g(t), B(t))$ to $\lambda(t)$, defined by

$$\begin{aligned} \lambda[g,B] &:= \inf_{\{\psi \mid \int_M e^{-\psi} dV = 1\}} F[g, H, \psi] \ , \\ \Rightarrow \lambda(t) &= \lambda[g(t), B(t)] \ . \end{aligned} \tag{III.6}$$

and the infimum is realized by the choice $\psi = P$.

We can compute the gradient of $\lambda[g, B]$ on G by evaluating the first variation of $F[g, B, \psi]$, requiring ψ to be a solution $\psi = P$ of (III.5) all along the variation. The first step is to compute the free variation in which ψ is not constrained. This variation is familiar in physics. If we momentarily think of ψ as the dilaton[9] for purposes of computing this variation, then F is the low energy effective action in string theory and its variational derivative is well-known. For completeness, we provide a derivation in the Appendix; cf. equation (A.9). The variation gives

$$\begin{aligned} \frac{dF}{ds} = & \int_M \bigg[\left(-R^{ij} - \nabla^i \nabla^j \psi + \frac{1}{4} H^i{}_{kl} H^{jkl} \right) \frac{\partial g_{ij}}{\partial s} \\ & + \left(R - \frac{1}{12} |H|^2 + 2\Delta\psi - |\nabla\psi|^2 \right) \left(\frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s} - \frac{\partial \psi}{\partial s} \right) \\ & + \frac{1}{2} \left(\nabla_k H^{kij} - H^{kij} \nabla_k \psi \right) \frac{\partial B_{ij}}{\partial s} \bigg] e^{-\psi} dV \ . \end{aligned} \tag{III.7}$$

Now impose the constraint that, for each value of s along the variation, ψ is not freely varied but rather is fixed to obey (III.4). That is, $\psi(s) = P(s) = -2\log u(s)$, where $u(s)$ is the lowest eigenfunction of the Schrödinger operator determined by the *varied* metric and B-field. Then (III.7) becomes the first variation formula for $\lambda[g, B]$. As well, on the right-hand side of (III.7) we use first (III.5) and then (III.3) and (III.4) to write

$$\begin{aligned} & \int_M \left(R - \frac{1}{12}|H|^2 + 2\Delta P - |\nabla P|^2 \right) \left(\frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s} - \frac{\partial P}{\partial s} \right) e^{-P} dV \\ = & \ \lambda \int_M \left(\frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s} - \frac{\partial P}{\partial s} \right) e^{-P} dV \end{aligned}$$

[9]keeping in mind of course that it is Φ, not ψ, that obeys the dilaton RG flow; indeed, when ψ is constrained in the manner of Subsection III.3, it can be shown to evolve in t according to a *backwards* parabolic evolution equation.

$$
\begin{aligned}
&= \lambda \frac{d}{ds} \int_M e^{-P} dV \\
&= 0 \ . \qquad \text{(III.8)}
\end{aligned}
$$

Thus, the middle line of (III.7) vanishes and we are left with

$$
\begin{aligned}
\frac{d\lambda}{ds} &= \int_M \Bigg[\left(-R^{ij} - \nabla^i \nabla^j P + \frac{1}{4} H^i{}_{kl} H^{jkl} \right) \frac{\partial g_{ij}}{\partial s} \\
&\quad + \frac{1}{2} \left(\nabla_k H^{kij} - H^{kij} \nabla_k P \right) \frac{\partial B_{ij}}{\partial s} \Bigg] e^{-P} dV \ . \qquad \text{(III.9)}
\end{aligned}
$$

Now we can consider the elements of G to be 2-index tensors with symmetric part g and skew part B. Then the inner product is given by

$$
\begin{aligned}
\langle T, T' \rangle &:= \int_M e^{-P} g^{ik} g^{jl} T_{ij} T'_{kl} dV \\
&= \int_M e^{-P} g^{ik} g^{jl} \left[S_{ij} S'_{kl} + A_{ij} A'_{kl} \right] dV \ , \qquad \text{(III.10)}
\end{aligned}
$$

where S is the symmetric part of $T \in TG$ and A is the anti-symmetric part. Then (III.9) is the directional derivative $\langle T, \text{Grad } \lambda \rangle$ of λ in the direction $T := \left(\frac{\partial g}{\partial s}, \frac{\partial B}{\partial s} \right)$. Then we can read off the gradient.

Proposition 3.1. *The RG flow for (g, B), with diffeomorphism gauge ψ fixed to be a solution P of (III.5), is the gradient flow in G generated by the potential $\lambda[g, B]$:*

$$
\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} g_{ij} \\ B_{ij} \end{pmatrix} &\equiv \begin{pmatrix} -\alpha' \left(R_{ij} + \nabla_i \nabla_j P - \frac{1}{4} H_{ikl} H_j{}^{kl} \right) \\ \frac{\alpha'}{2} \left[\nabla^k H_{kij} - H_{kij} \nabla^k P \right] \end{pmatrix} \\
&= \alpha' \ \text{Grad } \lambda[g, B] \ , \qquad \text{(III.11)}
\end{aligned}
$$

and $\lambda(t)$ is monotone increasing along the gradient flow:

$$
\begin{aligned}
\frac{d\lambda}{dt} &= \alpha' \int_M \Bigg[\left| R_{ij} + \nabla_i \nabla_j P - \frac{1}{4} H_{ikl} H_j{}^{kl} \right|^2 \\
&\quad + \frac{1}{4} \left| \nabla^k H_{kij} - H_{kij} \nabla^k P \right|^2 \Bigg] e^{-P} dV \ . \qquad \text{(III.12)}
\end{aligned}
$$

Furthermore, fixed points of (III.11) (where $H = dB$) are stationary points of λ.

Proof. The gradient formula (III.11) can be read off from (III.9). To obtain (III.12), consider the special case in (III.9) of a variation in (g, B) produced by evolving (g, B) along the RG flow. That is, let $\left(\frac{\partial g}{\partial s}, \frac{\partial B}{\partial s}\right)$ in (III.9) be given by the flow equations (II.9, II.13) with $s = t$. Finally, for fixed points of (III.11), the right-hand side of (III.12) vanishes. □

Corollary 3.2. *λ is monotonic along RG flow (II.9, II.13).*

Proof. Starting from the same initial data, the resulting solutions of the gradient flow (III.11) and the RG flow with arbitrary ψ (II.9, II.13) are related by at worst a time-dependent diffeomorphism. But $\lambda(t)$ is diffeomorphism-invariant, and monotonic along the gradient flow. □

We now discuss briefly the absence of periodic 1-loop RG flows. We will call a solution of the flow (II.9, II.13) a *breather* if it is periodic up to gauge and diffeomorphism; i.e., if there is a diffeomorphism φ, a 1-form ω, and parameter values $t_1 < t_2$ such that $(g(t_1), B(t_1)) = (\varphi^* g(t_2), \varphi^* B(t_2) + d\omega)$. A solution that is not a breather is a *homoclinic orbit* if it is eternal (i.e., defined for all $t \in (-\infty, \infty)$) with $(g(t), B(t))$ converging to (g_0, B_0) for $t \to -\infty$ and to $(\varphi^* g_0, \varphi^* B_0 + d\omega)$ for $t \to +\infty$.

Proposition 3.3. *There are no periodic or homoclinic orbits of the RG flow other than the fixed points of (III.11).*

Proof. For a periodic orbit of the flow (II.9, II.13), there will be some $t_1 < t_2$ such that $\lambda(t_1) = \lambda(t_2) =: \Lambda$. Then by monotonicity, $\lambda(t) = \Lambda$ for all $t \in [t_1, t_2]$. But by (III.12), this can only happen if the right-hand side of(III.11) vanishes throughout $[t_1, t_2]$. This is the condition for a fixed point (with $\psi = P$). For the homoclinic case, note that the sequence $\lambda(nT) - \lambda(-nT)$, $n \in \mathbb{Z}$, $T > 0$, is increasing. However, for a homoclinic orbit, this sequence must converge to zero. Therefore $\lambda(nT) - \lambda(-nT) = 0$ for all n. Setting $t_1 = -nT$, $t_2 = nT$, we see as before that the flow on $[t_1, t_2] = [-nT, nT]$ is at a fixed point. But we can take n arbitrarily large. □

Note that this result implies that if the flow equations with any potential ψ have a fixed point in which the right-hand sides of (II.9) and (II.13) vanish, *then the diffeomorphism generator ψ must be a solution P of* (III.5).

Perelman, in Section 2 of [9], was able to go farther. Using λ as we have defined it, but of course with $B = 0$, he was able to prove the non-existence of nontrivial expanding breathers, metrics that are equal at two different t-values up to a diffeomorphism and a homothety such that the metric with greater t-value has greater volume. However, the argument does not go through generally if the B-field is permitted to have nonzero field strength H at some time during the flow.

III.2 λ and the Weyl Anomaly at Fixed Points

We can now evaluate the eigenvalue λ at a stationary point, thus at a fixed point of the flow for which H is an exact 3-form. Since we consider a one-parameter family of flows with parameter s, we write λ_s for the eigenvalue and denote the stationary point by $s = 0$. At such a point, each term in (III.12) vanishes, and thus in particular we must have

$$R + \Delta P - \frac{1}{4}|H|^2 = 0 \ . \tag{III.13}$$

When this holds, (III.5) takes the form

$$\lambda_0 = \Delta P - |\nabla P|^2 + \frac{1}{6}|H|^2 \ . \tag{III.14}$$

Multiply this by e^{-P} and integrate. On the left-hand side, $\int_M \lambda_0 e^{-P} dV = \lambda_0 \int_M e^{-P} dV = \lambda_0$, and on the right-hand side the derivatives of P vanish upon integration by parts (since M is closed). This yields

$$\lambda_0 = \frac{1}{6} \int_M |H|^2 e^{-P} dV \ . \tag{III.15}$$

Notice now that, from the right-hand side of (II.11) with the diffeomorphism now chosen so that $\psi = P$ as must be the case at a fixed point, we can write

$$\int_M e^{-P} \beta^\Phi dV = A - \frac{\alpha'}{24} \int_M e^{-P} |H|^2 dV = A - \frac{\alpha'}{4} \lambda_0 \ . \tag{III.16}$$

(Recall that A is a constant depending on the dimension of the target manifold and the number of ghost fields, if any are present.) We compare this expression to Tseytlin [6] by considering the combination[10]

$$\tilde{\beta} := \beta^{\Phi} - \frac{1}{4} g^{ij} \beta^{g}_{ij} \ , \tag{III.17}$$

which equals the Weyl anomaly at fixed points of g and B [19], where of course β^g_{ij} vanishes and thus we obtain

$$\int_M e^{-P} \tilde{\beta} dV = A - \frac{\alpha'}{4} \lambda_0 \ . \tag{III.18}$$

But at fixed points, $\tilde{\beta}$ is constant on M ([19]; for the case where no B-field is present, this follows as an integrability condition for the fixed point equation $\beta^g_{ij} = 0$ as discussed in [20]; for the case with B-field, see the discussion in [21]). Then (III.18) reduces to the relation

$$\tilde{\beta} = A - \frac{\alpha'}{4} \lambda_0 \ . \tag{III.19}$$

An example is provided by a fixed point of $(g(t), B(t))$ with $|H|^2 := H_{ijk} H^{ijk}$ constant over the closed manifold M and with dilaton linear in the scale t. We must first note that (III.14) can now be written as

$$\Delta e^{-P} = \frac{1}{6} e^{-P} \left(|H|^2 - \int_M |H|^2 e^{-P} dV \right) \ . \tag{III.20}$$

Since $|H|^2$ is constant over M (and so by (III.15) $|H|^2 = 6\lambda$), the right-hand side of (III.20) vanishes and e^{-P} is harmonic. Then P is constant. (For the $B = 0$ case, the result was already known from work of Bourguignon [18].) Then (II.11) reduces to

$$\frac{\partial \Phi}{\partial t} = -A + \frac{\alpha'}{2} \left(\Delta \Phi + \frac{1}{2} \lambda \right) = -\beta^{\Phi} \ , \tag{III.21}$$

A solution is given by

$$\Phi = \left(\frac{\alpha'}{4} - A \right) t + \Phi_0 \ , \tag{III.22}$$

$$\frac{\partial \Phi_0}{\partial t} = \frac{\alpha'}{2} \Delta \Phi_0 \ , \tag{III.23}$$

[10]Tseytlin's definition of t differs from ours by a sign, but the signs in his RG equations are such that his β-functions agree with ours.

where the operator Δ has no time dependence, in consequence of our being at a fixed point with trivial diffeomorphism $\nabla P = 0$. Then if we choose that Φ_0 is constant in time, it also will be harmonic and thus spatially constant at all times, and we obtain, as claimed, that

$$\tilde{\beta} = \beta^{\Phi} = A - \frac{\alpha'}{4}\lambda \; . \tag{III.24}$$

III.3 A Gradient Flow with Arbitrary Diffeomorphism Term

The RG flow (II.9, II.13) with *arbitrary* diffeomorphism not necessarily determined by (III.5) is nonetheless a gradient flow as well. To see this, return to (III.7) but do not require now that $\psi = P = -2\log u$ at each s. Instead, to eliminate the middle line, choose any fiducial measure $dm := e^{-\psi}dV$ and hold it fixed pointwise along the variation. That is,

Proposition 3.4. *The RG flow (II.9, II.13) is the gradient flow of $F[g, B, \psi]$ along the surface in $\mathcal{G}$ determined by the fixed fiducial measure $dm := e^{-\psi}dV$ on M.*

Proof. If dm is fixed then $g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial \psi}{\partial s} = 0$ and the middle line on the right-hand side of (III.7) again vanishes. We obtain (III.9), from which the gradient can be read off. Since dm can be chosen arbitrarily, ψ is now arbitrary as well. □

While the approach of Proposition 3.4 may seem more general and perhaps simpler than the approach of Proposition 3.1, it is in fact far less powerful, because F depends on the arbitrary diffeomorphism potential ψ. To prove results such as Proposition 3.3, one must remove the ψ dependence by passing to λ which, in contrast to F, is geometrically meaningful—and has a clear physical interpretation.

IV The Second Variation of λ

In the case of Ricci flow ($B = 0$) the second variation formula for λ_s was written down in [15]. We generalize the formula here for arbitrary B-field. The next section applies this formula in a special case.

In an endeavour to minimize clutter, we establish a convention. If a quantity below is to be considered as a function of s, we write s explicitly as an argument or subscript. If s does not appear, the quantity is evaluated at $s = 0$ (possibly after s differentiation; this should be clear from context). We will use ∇_k to denote the covariant derivative compatible with $g_{ij}(s)$ and D_k for the covariant derivative compatible with $g_{ij} \equiv g_{ij}(0)$.

The second variation formula about an arbitrary point is complicated. Fortunately, for most purposes, we only need the second variation about a stationary point of λ_s. Thus we require the $s = 0$ fields (g, H) to obey

$$R_{ij} + D_i D_j P - \frac{1}{4} H_{ikl} H_j{}^{kl} = 0 , \tag{IV.1}$$

$$D^k H_{kij} - H_{kij} D^k P = 0 . \tag{IV.2}$$

Thus from (III.9) the second variation formula about a stationary point is

$$\left.\frac{d^2\lambda_s}{ds^2}\right|_{s=0} = \int_M \Big[h^{ij} \frac{\partial}{\partial s} \left(-R_{ij} - \nabla_i \nabla_j P + \frac{1}{4} H_{ikl} H_j{}^{kl} \right) + \frac{1}{2} \beta^{ij} \frac{\partial}{\partial s} \left(\nabla^k H_{kij} - H_{kij} \nabla^k P \right) \Big] e^{-P} dV , \tag{IV.3}$$

where we define

$$h_{kl} := \frac{\partial g_{kl}}{\partial s} , \quad h^{ij} := g^{ik} g^{jl} h_{kl} , \tag{IV.4}$$

$$\beta_{kl} := \frac{\partial B_{kl}}{\partial s} , \quad \beta^{ij} := g^{ik} g^{jl} \beta_{kl} , \tag{IV.5}$$

and for use below

$$Q := \frac{\partial P}{\partial s} . \tag{IV.6}$$

The standard formulas

$$\frac{\partial}{\partial s} R_{ij} = \nabla_k \frac{\partial}{\partial s} \Gamma^k_{ij} - \nabla_i \frac{\partial}{\partial s} \Gamma^k_{jk} \tag{IV.7}$$

$$\frac{\partial}{\partial s} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \right) \tag{IV.8}$$

yield the easy identity (writing D for $\nabla|_{s=0}$ whenever we can and using a standard result for the variation of a Christoffel symbol)

$$\frac{\partial}{\partial s} \left(-R_{ij} - D_i D_j P \right)$$

$$
\begin{aligned}
&= -e^P D_k \left(e^{-P} \frac{\partial}{\partial s} \Gamma^k_{ij} \right) - \left(D_i D_j \frac{\partial P}{\partial s} - D_i \frac{\partial}{\partial s} \Gamma^k_{jk} \right) \\
&= -e^P D_k \left(e^{-P} \frac{\partial}{\partial s} \Gamma^k_{ij} \right) + D_i D_j \left(\frac{1}{2} g^{kl} h_{kl} - Q \right) \\
&= -\frac{1}{2} e^P g^{kl} D_k \left[e^{-P} \left(D_i h_{jl} + D_j h_{il} - D_l h_{ij} \right) \right] \\
&\quad + D_i D_j \left(\frac{1}{2} g^{kl} h_{kl} - Q \right) \\
&= -\frac{1}{2} \Big(D_i D^k h_{jk} + D_j D^k h_{ik} - R_{kilj} h^{kl} - R_{kjli} h^{kl} \\
&\qquad + R^k_i h_{jk} + R^k_j h_{ik} - \Delta h_{ij} \Big) + D_i D_j \left(\frac{1}{2} g^{kl} h_{kl} - Q \right) \\
&= -\frac{1}{2} \Big(D_i D^k h_{jk} + D_j D^k h_{ik} - \Delta_L h_{ij} \Big) \\
&\qquad + D_i D_j \left(\frac{1}{2} g^{kl} h_{kl} - Q \right) \ , \qquad \text{(IV.9)}
\end{aligned}
$$

where Δ_L denotes the Lichnerowicz Laplacian

$$
\Delta_L h_{ij} := \Delta h_{ij} + R_{kilj} h^{kl} + R_{kjli} h^{kl} - R^k_i h_{jk} - R^k_j h_{ik} \ . \qquad \text{(IV.10)}
$$

Thus we obtain (round brackets around indices indicate symmetrization):

The Second Variation Formula:

$$
\begin{aligned}
\left. \frac{d^2 \lambda_s}{ds^2} \right|_{s=0} &= \frac{1}{2} \int_M e^{-P} h^{ij} \left(\Delta_L h_{ij} - H_{ikm} H_{jl}{}^m h^{kl} \right) dV \\
&\quad + \int_M e^{-P} h^{ij} \left[-D_{(i} D^k h_{j)k} + D_i D_j \left(\frac{1}{2} g^{kl} h_{kl} - Q \right) \right] dV \\
&\quad + \frac{1}{2} \int_M e^{-P} \beta^{ij} \frac{\partial}{\partial s} \left(\nabla^k H_{kij} - H_{kij} \nabla^k P \right) dV \ , \qquad \text{(IV.11)}
\end{aligned}
$$

for variations about a general stationary point.

As a check on our results, we restrict to variations $(h, 0)$ in which B is fixed. We determine Q in the second variation formula by noting that (III.5) must hold all along the variation (i.e., for all s). Differentiate it and evaluate the derivative at $s = 0$. On the left-hand side, $\frac{d}{ds} \lambda_s |_{s=0} = 0$ since $s = 0$ is a stationary point, while the right-hand side can be simplified by using (IV.7), etc, to obtain

$$D^i\left[e^{-P}D_i\left(g^{ij}h_{ij}-2Q\right)\right]=D^iD^j\left(e^{-P}h_{ij}\right)\ . \tag{IV.12}$$

We write the solution of this equation (one always exists and is unique modulo an inconsequential additive constant, since M is closed) as

$$v_h := g^{ij}h_{ij}-2Q\ , \tag{IV.13}$$

This determines Q, given h_{ij}. Now we further restrict to the case where $|H|^2$ is constant on the manifold. Then by (III.19), we can set $P=0$. Under these circumstances, the second variation formula becomes

$$\begin{aligned}\left.\frac{d^2\lambda_s}{ds^2}\right|_{s=0} &= \frac{1}{2}\int_M h^{ij}\left(\Delta_L h_{ij}-H_{ikm}H_{jl}{}^m h^{kl}\right)dV\ , \\ &+\int_M\left(|\text{div }(h)|^2-\frac{1}{2}|Dv_h|^2\right)dV\ ,\end{aligned} \tag{IV.14}$$

where $(\text{div }h)_i := D^k h_{ik}$, and this yields agreement with the Ricci flow result of [15] when $H_{ijk}=0$. For h_{ij} *transverse* (i.e., if $D^ih_{ij}=0$), *both* terms in the second integrand vanish.

V $H=0$ Fixed Points

Consider a family of solutions of the RG flow with t the parameter along each flow and s the family parameter. In the sequel, we always choose s such that $s=0$ is a stationary point of λ_s and thus a fixed point of the flow. If the second variation of λ_s about $s=0$ is positive at $t=0$, then $C:=\lambda_s(0)>\lambda_0(0)$ for some $s>0$. Since $\lambda_s(t)$ is monotonic in t and $\lambda_0(t)$ is constant, then $\lambda_s(t)\ge C>\lambda_0(0)$ for all t. Since λ_s is continuous on G, the coupling constants cannot approach the fixed point couplings along the flow. Thus, $\frac{d^2}{ds^2}\lambda_s|_{s=0}>0$ indicates an instability of the fixed point.

Now Ricci-flat fixed points have $\lambda=0$, which is the least possible value of λ at a fixed point. A particularly worrisome scenario would occur if certain Ricci-flat manifolds, such as flat tori and K3 manifolds, were unstable against small perturbations and could flow to other, non-Ricci-flat fixed points. To preclude this possibility, we must study the eigenvalues of the Hessian of λ_s about $s=0$. This notion of stability is called *linear stability*.

It is difficult to study general variations (h, β) in both g and B about an arbitrary fixed point, owing to the difficulty in diagonalizing the Hessian. However, Ricci-flat fixed points necessarily have $H = 0$, and for a fixed point with $H = 0$ things simplify considerably, making it possible. As well as the direct simplification of setting $H = 0$, the argument surrounding (III.19) then gives that P can be set to zero as well, and so the fixed point condition gives that $R_{ij} = 0$. We use (IV.11) to write

$$\frac{d^2\lambda_s}{ds^2}\Big|_{s=0} = \frac{1}{2}\int_M \left(h^{ij}\Delta_L h_{ij} + 2|\text{div }(h)|^2 - |Dv_h|^2 - \frac{1}{3}|d\beta|^2 \right) dV \ , \quad \text{(V.1)}$$

where now v_h solves

$$D^k D_k v_h \equiv D^k D_k \left(g^{ij}h_{ij} - 2Q \right) = D^i D^j h_{ij} \ . \quad \text{(V.2)}$$

Equation (V.1) has the form

$$\begin{aligned} \frac{d^2\lambda_s}{ds^2}\Big|_{s=0} &= \int_M \left(\langle h, \mathcal{L}h \rangle - \frac{1}{6}|d\beta|^2 \right) dV \\ &= \int_M \left(\langle h, \mathcal{L}h \rangle - \frac{1}{6}\langle \beta, d^* d\beta \rangle \right) dV \quad \text{(V.3)} \\ \mathcal{L}h &:= \frac{1}{2}\Delta_L h + \text{div}^* \text{div } h + DDv_h \ , \quad \text{(V.4)} \end{aligned}$$

where we have suppressed the indices, written $(\text{div } h)_i := D^k h_{ik}$, and let div^* denote the adjoint of the divergence and d^* denote the adjoint of d with respect to the inner product (III.10). It is known [24, 23] that eigenvectors of $\mathcal{L}$ belonging to positive eigenvalues, if any exist, are transverse and traceless, and moreover for h_{ij} transverse traceless then

$$\begin{aligned} \mathcal{L}|_N &= \frac{1}{2}\Delta_L|_N \ , \quad \text{(V.5)} \\ N &:= \{ h_{ij} \mid (\text{div}h)_i := D^k h_{ik} = 0, \ \text{tr}_g h := g^{ij}h_{ij} = 0 \} \ . \quad \text{(V.6)} \end{aligned}$$

Furthermore, on the complement of N, $\mathcal{L} < 0$ unless h_{ij} arises from the action of a diffeomorphism or homothetic rescaling on g_{ij}, in which case $\mathcal{L} = 0$.

The $|d\beta|^2$ term in (V.3) is obviously negative semi-definite, and negative if β is not closed. With that and the above comments concerning $\mathcal{L}$ in mind, we define a fixed point $(g, d\beta)$ with $R_{ij} = 0$, $H_{ijk} = 0$ to be *linearly stable* if

$\Delta_L|_N \leq 0$. If the tangent directions at $(g, d\beta)$ for which $\Delta_L|_N = 0$ can be exponentiated to give a smooth submanifold $\mathcal{U}$ of Ricci-flat fixed points in the space of coupling constants, then we say that the fixed point $(g, d\beta)$ is *integrable*. We will call this an *integral submanifold of Ricci-flat fixed points*. Since by construction $\Delta_L < 0$ in the complement in N of $T\mathcal{U}$ at each point along $\mathcal{U}$, we call $\mathcal{U}$ *strongly linearly stable*. If $\mathcal{U}$ is strongly linearly stable and if neither the spectrum of Δ_L nor that of $\delta d : \Lambda^2(M) \to \Lambda^2(M)$ accumulates at zero, we will call $\mathcal{U}$ *strictly linearly stable*.

Lemma 5.1. *If a strictly linearly stable integral submanifold $\mathcal{U}$ consists entirely of a disjoint set of one or more Ricci-flat fixed points, then those points are rigid (isolated): there are no neighbouring fixed points, Ricci-flat or not, except those obtained by diffeomorphism and/or homothety.*

Proof: At $p \in \mathcal{U}$, consider any submanifold S whose tangent space at p is contained in N. Because λ is zero and stationary at p and Δ_L is bounded below zero on TS, and δd is bounded below zero on $\Lambda^2(M)$ modulo closed forms, then $\lambda < 0$ on some neighbourhood of p in S. But fixed points have $\lambda \geq 0$ by (III.15). □

We have no examples of rigid fixed points. However, the proof generalizes to the case of submanifolds $\mathcal{U}$ of nonzero dimension provided what is meant by "isolated" is interpreted to mean that neighbouring fixed points must also belong to $\mathcal{U}$ are allowed. This brings us to the cases of greatest interest here:

Flat Tori:
For flat manifolds, it helps to write (V.3) as

$$\frac{d^2\lambda_s}{ds^2}\Big|_{s=0} = -\frac{1}{2}\int_M \left(3|D_{(i}h_{jk)}|^2 + \frac{1}{3}|d\beta|^2\right) dV \ , \tag{V.7}$$

for $h_{ij} \in N$. This is strictly negative unless β is closed and $D_{(i}h_{jk)} = 0$, and then h_{ij} is called a Killing tensor. For tori, the Killing tensors are always linear combinations of outer products of translation Killing vectors. These modes correspond to the relative rescaling of distinct cycles, holding the torus volume fixed. These relative rescalings give rise to the moduli space of flat structures on the torus, which is clearly a strongly linearly stable integral submanifold of Ricci-flat fixed points in the space of coupling constants. Moreover, it is evident from the triviality of the eigenvalue problem in this case that the moduli space is in fact strictly linearly stable.

K3 Manifolds:
The infinitesimal deformations (meaning in this situation the $h_{ij} \in N$ such that $\Delta_L h_{ij} = 0$) of Kähler Ricci-flat metrics on K3 manifolds are known to actually correspond to Ricci-flat metrics [25, 23]. Once again, these metrics form a submanifold $\mathcal{E}$ of Ricci-flat fixed points. It was shown in [24] that $\Delta_L < 0$ on the complement of $T\mathcal{E}$ in N. Thus $\mathcal{E}$ is a strongly linearly stable integral submanifold of Ricci-flat fixed points.

Although these results on linear stability are strongly suggestive, they do not demonstrate *dynamical stability* without further technical argument. In the case of Ricci flow (i.e., no B-field), Sesum [23] has found linear and dynamical stability to be equivalent when the fixed point satisfies the integrability condition. Thus flat tori and K3 manifolds are dynamically stable under Ricci flow. We expect similar results will hold when a B-field is present. The issue is presently under investigation. Nonetheless, the picture that emerges is one where flat tori and K3s are final and not initial endpoints of the flow (except of course for the trivial case of a flow that remains always at the fixed point). If flows that end at these points begin at unstable fixed points, then monotonicity of λ and the fact that $\lambda = 0$ at the final endpoint would imply that those initial fixed points would have $\lambda < 0$, contradicting (III.15). The resolution of this apparent paradox is simply that higher order terms α' are significant for such flows and cannot be neglected.

Finally, consider the stability of the subset of fixed points of the first-order flow that remain fixed points of the flow to all orders in α'. That is, they receive no 'stringy corrections' at any order in perturbation theory. These are called *perturbative string vacua*, and are described by conformal field theories. Questions concerning the topology of the space of such vacua, the dimension of the moduli space, etc., have been discussed in the string theory literature primarily in the language of CFTs and their operator content. Because our results for general order α' fixed points certainly obviously descend to perturbative string vacua, we have a complementary picture in which these questions can be phrased in the language of target manifold geometry. The clearest case would be that of a zero-dimensional integral submanifold of Ricci-flat perturbative string vacua. Then Lemma 5.1 would apply directly.[11] Checking stability would then be a matter of checking the

[11] Kahler-Ricci-flat K3 manifolds are perturbative string vacua. A version of Lemma 5 could be proved for the larger integral submanifolds that occur there. However, we do not know if these manifolds are strictly linearly stable, even though they are strongly linearly

eigenvalues of Δ_L. If this could be done and if stability were confirmed, we could infer that the related CFT should have no relevant or marginal operators. Now while we presently know of no specific example of such a zero-dimensional manifold of vacua, we have seen for flat tori that it need not be difficult to draw conclusions about relevant operators in the CFT even when the vacuum belongs to a nontrivial integral submanifold.

In [13], Vafa addresses the question of whether the C-function can serve as a Morse function for the configuration space of string theory, the hope being that this would shed light on the topology of this configuration space. In particular, he points out that the possible existence of nontrivial fixed points with no relevant or marginal directions—rigid perturbative string vacua in the language above—raises the question of whether the configuration space of string theory is connected.

VI Concluding Remarks

Throughout, we have limited attention to the most elementary of the monotone quantities for Ricci flow in [9] and to order α' β-functions. To do more would have made this article unwieldy. It is, however, interesting to ponder whether Perelman's W-entropy, reduced length, and reduced volume have useful analogues for the flows we have considered. An analogue of the scale invariant W-entropy would allow one to address and probably rule out the possibility that there may be solutions of the RG flow which are periodic except for an overall homothetic rescaling [26].

It is also natural to ask whether these techniques might show the RG flow to be gradient and have a monotonicity formula at higher order in α'. Higher-order RG flow equations have a significant difficulty, which is that nonlinear combinations of leading-order spatial derivative terms appear in the flow PDEs. Then the question may be vacuous, in that these PDEs might not admit any solutions at all. Physics does not require that they do, since the exact RG flow, by which we mean that the β-functions are not approximated using a truncated loop expansion, can still exist nonetheless. However, if a gradient flow is found for, say, the order α'^2 RG flow, then this would be an important step in showing existence of solutions of the PDEs (as streamlines of the gradient flow). Thus, in the absence of a separate existence proof for solutions there is no reason to expect to find a suitable gradient

stable.

flow, but it is the very absence of such a proof that makes the question more interesting.

Thinking beyond the loop expansion, we return to the C-theorem. In the present work, we utilized recent mathematical breakthroughs to say something about order α' physics. These advances occurred in the context of Ricci flow and attempts to prove conjectures concerning 3-manifold topology, but seem to echo the C-theorem (cf Section III.2 and [10]). It is intriguing to ask what may come from a reverse strategy. For example, might the C-theorem lead to a tower of geometric flows with monotonicity properties and interesting fixed points, order-by-order in α'?[12] Can the Ricci flow with surgery be given a physics interpretation and might the C-theorem have something to say about topology of manifolds? To realize this potential, it seems very important to understand more deeply the relationship between the C-function and λ, or perhaps the other monotonic quantities known for Ricci flow [9] but which we have not discussed herein.

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A Appendix

Here we compute the first variation in the functional $F[g, B, \psi]$ that results from a 1-parameter variation of g, B, and ψ. We denote the parameter by s and compute the first variation term-by-term, starting from the formula

$$\begin{aligned} \frac{dF}{ds} &= \int_M \left(\frac{\partial R}{\partial s} + \frac{\partial}{\partial s} |\nabla \psi|^2 - \frac{1}{12} \frac{\partial}{\partial s} |H|^2 \right) e^{\psi} dV \\ &\quad + \int_M \left(R + |\nabla \psi|^2 - \frac{1}{12} |H|^2 \right) \frac{\partial}{\partial s} \left(e^{-\psi} dV \right) \ . \end{aligned} \tag{A.1}$$

Lemma A.1. *For M compact, then*

$$\begin{aligned} \int_M \frac{\partial R}{\partial s} e^{-\psi} dV &= -\int_M \Big[R^{ij} + \nabla^i \nabla^j \psi - \nabla^i \psi \nabla^j \psi \\ &\qquad - g^{ij} \left(\Delta \psi - |\nabla \psi|^2 \right) \Big] \frac{\partial g_{ij}}{\partial s} e^{-\psi} dV \ , \end{aligned} \tag{A.2}$$

Proof. Use the standard formula

$$\frac{\partial R}{\partial s} = -R^{ij} \frac{\partial g_{ij}}{\partial s} + \nabla^i \left[\nabla^j \frac{\partial g_{ij}}{\partial s} - \nabla_i \left(g^{kl} \frac{\partial g_{kl}}{\partial s} \right) \right] \tag{A.3}$$

and integrate by parts twice. $\square$

Lemma A.2. *Assume that (III.1) holds. Then*

$$\begin{aligned}&\int_M \left(\frac{\partial}{\partial s}|\nabla\psi|^2\right) e^{-\psi} dV \\ &= \int_M \Big[2\left(|\nabla\psi|^2 - \Delta\psi\right)\frac{\partial\psi}{\partial s} - \frac{\partial g_{ij}}{\partial s}\nabla^i\psi\nabla^j\psi\Big] e^{-\psi} dV \ . \qquad \text{(A.4)}\end{aligned}$$

Proof.

$$\int_M \left(\frac{\partial}{\partial s}|\nabla\psi|^2\right) e^{-\psi} dV = \int_M \left[2\nabla^i\psi\nabla_i\frac{\partial\psi}{\partial s} - \frac{\partial g_{ij}}{\partial s}\nabla^i\psi\nabla^j\psi\right] e^{-\psi} dV \quad \text{(A.5)}$$

and integrate by parts. $\square$

Lemma A.3.

$$\begin{aligned}&-\frac{1}{12}\int_M \left(\frac{\partial}{\partial s}|H|^2\right) e^{-\psi} dV \\ &= \int_M \Big[\frac{1}{4} H^i{}_{kl} H^{jkl}\frac{\partial g_{ij}}{\partial s} + \left(\nabla_k H^{kij} - H^{kij}\nabla_k\psi\right)\frac{\partial B^{ij}}{\partial s}\Big] e^{-\psi} dV \ . \text{(A.6)}\end{aligned}$$

Proof. The first term on the right-hand side is obvious. The second term follows from the fact that $\frac{\partial}{\partial s}\partial_{[i}B_{jk]} = \partial_{[i}\frac{\partial}{\partial s}B_{jk]} = \nabla_{[i}\frac{\partial}{\partial s}B_{jk]}$. Then

$$\begin{aligned}&\int_M \left[-\frac{1}{6}H^{ijk}\frac{\partial}{\partial s}H_{ijk}\right] e^{-\psi} dV \\ =\ &\int_M \left[-\frac{1}{2}H_{ijk}g^{ip}g^{jq}g^{kl}\nabla_p\frac{\partial B_{ql}}{\partial s}\right] e^{-\psi} dV \\ =\ &\frac{1}{2}\int_M \left[\left(\nabla^k H_{kij} - H_{kij}\nabla^k\psi\right) g^{ip}g^{jq}\frac{\partial B_{pq}}{\partial s}\right] e^{-\psi} dV \qquad \text{(A.7)}\end{aligned}$$

$\square$

Lemma A.4.

$$\begin{aligned}&\int_M \left(R + |\nabla\psi|^2 - \frac{1}{12}|H|^2\right)\frac{\partial}{\partial s}\left(e^{-\psi} dV\right) \\ &= \int_M \left(R + |\nabla\psi|^2 - \frac{1}{12}|H|^2\right)\left(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial\psi}{\partial s}\right) e^{-\psi} dV \ .\text{(A.8)}\end{aligned}$$

Proof. Follows from the formula $\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial s} = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s}$ for the derivative of a determinant. □

Proposition A.5. *For any arbitrary smooth 1-parameter variation of g, B, and ψ, then*

$$\begin{aligned} \frac{dF}{ds} &= \int_M \Bigg[\left(-R^{ij} - \nabla^i\nabla^j\psi + \frac{1}{4}H^i{}_{kl}H^{jkl} \right) \frac{\partial g_{ij}}{\partial s} \\ &\quad + \left(R - \frac{1}{12}|H|^2 + 2\Delta\psi - |\nabla\psi|^2 \right) \left(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial s} - \frac{\partial\psi}{\partial s} \right) \\ &\quad + \frac{1}{2}\left(\nabla_k H^{kij} - H^{kij}\nabla_k\psi\right)\frac{\partial B_{ij}}{\partial s} \Bigg] e^{-\psi} dV \ . \end{aligned} \tag{A.9}$$

Proof: Follows immediately from Lemmata A.1–4. □