Nonlinear perturbations of the Kaluza-Klein monopole

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We consider the nonlinear stability of the Kaluza-Klein monopole viewed as the static solution of the five dimensional vacuum Einstein equations. Using both numerical and analytical methods we give evidence that the Kaluza-Klein monopole is asymptotically stable within the cohomogeneity-two biaxial Bianchi IX ansatz recently introduced in [3]. We also show that for sufficiently large perturbations the Kaluza-Klein monopole loses stability and collapses to a Kaluza-Klein black hole. The relevance of our results for the stability of BPS states in M/String theory is briefly discussed.

INTRODUCTION

The Taub-NUT instanton is the complete Riemannian Ricci-flat 4-manifold with the metric [1]

$$ds^2 = \left(1 + \frac{m}{\rho}\right) d\rho^2 + \left(1 + \frac{m}{\rho}\right) \rho^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ m^2 \left(1 + \frac{m}{\rho}\right)^{-1} (d\psi + \cos\theta d\phi)^2 , \qquad (1)$$

where $\rho \geq 0$, $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$ and $m > 0$ is a free parameter which sets the scale. The Taub-NUT instanton is topologically R^4 with the surfaces of constant ρ being the squashed three-spheres. Although complete, the metric (1) is not asymptotically Euclidean because the S^1 fibers approach a constant circumference at infinity (rather than growing like ρ). This kind of asymptotic behavior is called asymptotically locally flat.

By adding the trivial term $-dt^2$ one obtains from (1) the regular static solution of the five dimensional vacuum Einstein equations

$$ds^2 = -dt^2 + \left(1 + \frac{m}{\rho}\right) d\rho^2 + \left(1 + \frac{m}{\rho}\right) \rho^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ m^2 \left(1 + \frac{m}{\rho}\right)^{-1} (d\psi + \cos\theta d\phi)^2 . \qquad (2)$$

This metric has been used in the past to obtain the four-dimensional magnetic monopole via a Kaluza-Klein reduction along the coordinate ψ [2] and since then it is usually called the Kaluza-Klein (KK) monopole. To the best of our knowledge, the role of the KK monopole in the dynamics of non-asymptotically flat initial data, and in particular its nonlinear stability, has not been studied, probably because this problem appeared to require an analysis of the full five dimensional Einstein equations which is beyond currently available numerical tools, let alone the analytic ones. This situation has changed recently with the introduction of a new cohomogeneity-two symmetry reduction of the five dimensional vacuum Einstein equations (referred to below as the BCS ansatz [3]) which provided a framework for investigating the dynamics in a simple $1 + 1$ dimensional setting. Since the KK

monopole falls within the BCS ansatz, the question of its nonlinear stability now becomes tractable and is the subject of this paper.

The KK monopole plays an important role in M/String theory. For example, its metric product with six flat Euclidean dimensions gives a Ricci-flat eleven dimensional Lorentzian metric, which when reduced to ten spacetime dimensions may be interpreted as a D6-brane solution of Type IIA String theory [4]. As such, it is an example of what is called a supersymmetric or BPS solution. BPS solutions of supergravity theories are widely believed by many string theorists to be absolutely stable, both classically and quantum mechanically. However until now, this belief has never been tested in a fully non-linear setting. As we shall see, our results show that the classical stablity is not absolute since for sufficiently large perturbations a gravitational collapse to a black hole is possible. The implications of our results for BPS states will be discussed briefly in the conclusions.

PRELIMINARIES

The BCS ansatz has the form [3]

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \tfrac{1}{4}r^2e^{2B}(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ \tfrac{1}{4}r^2e^{-4B}(d\psi + \cos\theta d\phi)^2 , \qquad (3)$$

where A, δ, and B are functions of t and r. In contrast to spherical symmetry, this ansatz possesses a dynamical degree of freedom, the field B, which measures the deformation of sphericity of the angular part of the metric. Substituting the metric (3) into the vacuum Einstein equations one gets the following system of equations

$$A' = -\frac{2A}{r} + \frac{2}{3r}\left(4e^{-2B} - e^{-8B}\right) - 2r\left(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2\right) \quad (4a)$$

$$\dot{A} = -4rA\dot{B}B' , \qquad (4b)$$

$$\delta' = -2r\left(e^{2\delta}A^{-2}\dot{B}^2 + B'^2\right) , \qquad (4c)$$

$$\left(e^{\delta}A^{-1}r^3\dot{B}\right)^{\cdot} - \left(e^{-\delta}Ar^3B'\right)' + \frac{4}{3}e^{-\delta}r\left(e^{-2B} - e^{-8B}\right) = 0 , \qquad (4d)$$

where primes and dots denote derivatives with respect to r and t, respectively. The initial value problem for this system was investigated in [3] in the case of asymptotically flat initial data. Here we consider asymptotically locally flat solutions.

The KK monopole (2) expressed in terms of the ansatz (3) takes the form

$$B_0 = \frac{1}{3} \ln \left(1 + \frac{\rho}{m}\right), \; A_0 = \frac{(1 + \frac{4\rho}{3m})^2}{(1 + \frac{\rho}{m})^{8/3}}, \; e^{2\delta_0} = A_0 \,, \quad (5)$$

where

$$r = 2m^{1/2}\rho^{1/2} \left(1 + \frac{\rho}{m}\right)^{1/6} . \quad (6)$$

Note that scaling invariance excludes the existence of nontrivial solutions which are everywhere regular *and* asymptotically flat. The KK monopole is the unique nontrivial regular static solution of the system (4). To see this note that static solutions which are analytic at the origin form a one-parameter family with the following asymptotic behavior for $r \to 0$

$$B(r) \sim br^2, \quad A(r) \sim 1 - 4b^2r^4, \quad \delta \sim -2b^2r^4 \,, \quad (7)$$

where b is the shooting parameter. Since the system (4) is scale invariant, it follows from the uniqueness of solutions of ordinary differential equations that, up to scaling, there is only one static solution which is analytic at the origin. It is easy to verify that the solution (5) satisfies the regularity conditions (7) with $b = 1/(12m^2)$. For convenience, hereafter we choose units in which $m = 1$.

LINEAR STABILITY

Before discussing numerical results, we first demonstrate that the KK monopole is linearly stable within our ansatz. To this end, following the standard procedure we seek solutions of the system (4) in the form

$$\begin{aligned} B(t,r) &= B_0(\rho) + B_1(t,\rho), \quad A(t,r) = A_0(\rho) + A_1(t,\rho), \\ \delta(t,r) &= \delta_0(\rho) + \delta_1(t,\rho) \,, \end{aligned} \quad (8)$$

where ρ is a function of r determined implicitly by (6). Substituting the expansion (8) into the system (4), linearizing and separating the time dependence $B_1(t,\rho) = \exp(-i\lambda t)v_\lambda(\rho)$, after a straightforward but tedious calculation we get the eigenvalue equation for the spectrum of small perturbations around the KK monopole

$$\mathcal{A}v_\lambda = \lambda^2 v_\lambda, \quad (9)$$

$$\mathcal{A} = -\frac{1}{\rho(1+\rho)} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho}\right) + \frac{18}{\rho(1+\rho)(3+4\rho)^2} \,.$$

Although this equation cannot be solved analytically in general, there is an explicit solution for $\lambda = 0$

$$v_0 = \frac{\rho}{3+4\rho}. \quad (10)$$

This is the zero mode corresponding to the scaling freedom - it can be obtained by the action of the scaling generator $r\frac{d}{dr}$ on $B_0(\rho(r))$. The zero mode (10) has no zeros which implies by the standard Sturm-Liouville theory that the operator $\mathcal{A}$ has no negative eigenvalues. Thus, the KK monopole is linearly stable within our ansatz. Note that the zero mode is not a genuine eigenfunction because it is not square integrable. More precisely, on the Hilbert space $L^2([0,\infty), \rho(1+\rho)d\rho)$ the self-adjoint operator $\mathcal{A}$ has the purely continuous spectrum $\lambda^2 \in [0,\infty)$.

Nota bene, the Taub-NUT instanton (1) is known to be linearly stable against transverse traceless perturbations. This follows from the work of Hawking and Pope [5], who showed, using two linearly independent covariantly constant spinors which the Taub-NUT metric admits, that the spectrum of the Lichnerowicz operator, which governs the linearized transverse traceless perturbations, is the same (apart from zero modes) as for the scalar Laplacian, and therefore non-negative.

NUMERICAL RESULTS

Using a fourth-order accurate finite difference code we have solved the system (4) numerically for several families of regular initial data which represent various perturbations of the KK monopole. The overall picture does not depend on the specific choice of a family. The results shown below were produced for initial data of the form (using the momentum variable $P = e^\delta A^{-1}\dot{B}$)

$$B(0,r) = B_0(\rho(r)), \;\; P(0,r) = p\left(\frac{r}{R}\right)^4 e^{-\frac{(r-R)^4}{s^4}}, \quad (11)$$

where the amplitude p was varied and the parameters R and s were kept fixed. Note that although the perturbation (11) is exponentially localized, the induced perturbation of the function A has a $1/r^2$ tail as follows from the hamiltonian constraint (4a).

Small perturbations

We have found that for small perturbations, that is for small values of the control parameter p, the perturbation is scattered off to infinity and the solution returns to equilibrium, i.e., it settles down to the KK monopole with the *same* parameter m. This is shown in Fig. 1. One might wonder why the parameter m does not change under perturbation. The reason is simple: the perturbations considered by us have finite energy while a change of m would require infinite energy. Here by energy we mean the energy measured with respect to a given KK reference background as described by Deser and Soldate [6]. This energy is determined by the *next* to leading order term in the asymptotic expansion at spatial infinity.

The qualitative picture of pointwise convergence to the KK monopole is shown in Fig. 2. The details of this process, in particular the role of quasinormal modes and the rate of decay of a tail, will be pursued elsewhere.

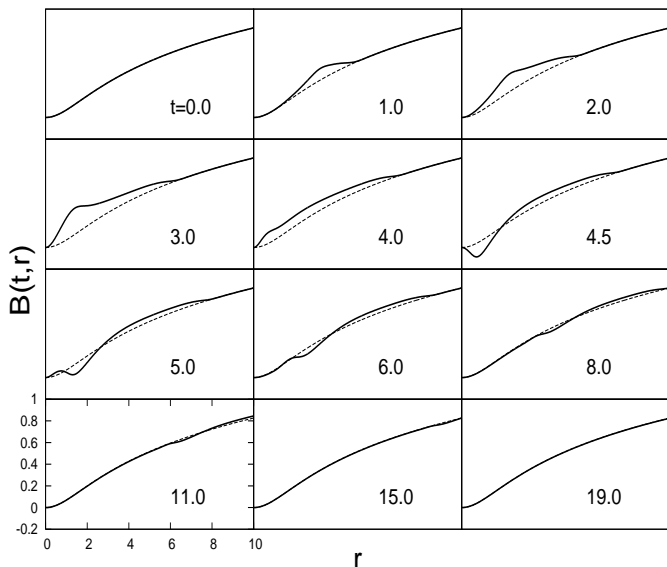We have found that a sufficiently strong kick may

As we pointed out above, the perturbations considered by us have finite energy and therefore they do not change the leading order asymptotic behavior (15). In this sense the magnetic charge P can be viewed as the constant of motion. For the KK monopole $P = 2M = m/2$, so all our configurations have the same magnetic charge $P = 1/2$ (recall that we use units in which $m = 1$) and the end-states of evolution differ only by M.
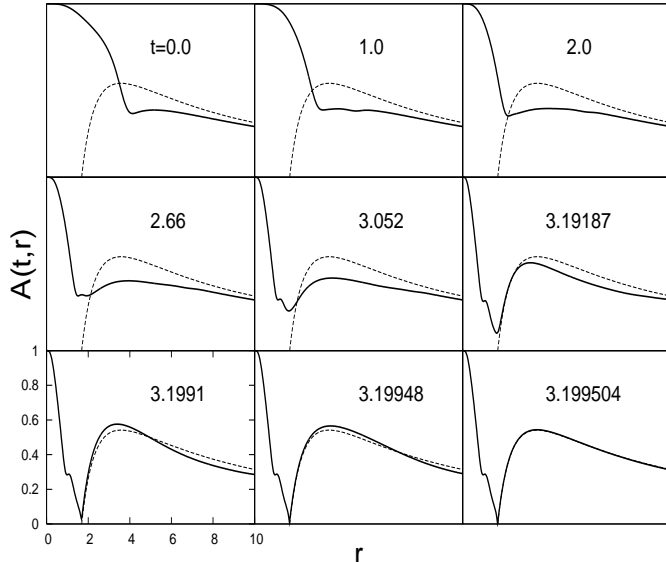

FIG. 3: Instability of the Kaluza-Klein monopole for large perturbations. For initial data (11) with a large amplitude ($p = 0.3, R = 3, s = 1$) we plot a series of snapshots of the function $A(t,r)$ (where t is central proper time). During the evolution $A(t,r)$ drops to zero at $r = r_H \approx 1.68$ which signals the formation of an apparent horizon there. Outside the horizon the solution relaxes to the Kaluza-Klein black hole (13) (shown by the dashed line) with $M = 0.502$.

As we approach the threshold of collapse from above, the parameter M decreases towards the limiting value $M = 1/4$ corresponding to the KK monopole. At the same time, as follows from (17), both β and r_H tend to zero. In this limit, close to the horizon the KK black hole is very well approximated by the Schwarzschild black hole. Near the threshold we observe the type II discretely self-similar critical behavior governed by the same critical solution as in the asymptotically flat situation [3] (see Fig. 4). This is not surprising in view of the fact that critical collapse is a local phenomenon, and, as such, does not depend on the far field behavior.

CONCLUSIONS

We have shown that although it is classically stable against small perturbations, the KK monopole is classically unstable against perturbations which are large enough to allow gravitational collapse to form a black hole. Classically, the black hole is probably stable. Quantum mechanically one expects it to emit Hawking radiation which can carry off energy but not magnetic charge. In the long run therefore one might expect the system to settle down to the original KK monopole. If this is so, then the common intuition about BPS states would

be correct, provided one bears in mind that the stability can only hold by virtue of quantum mechanical effects.
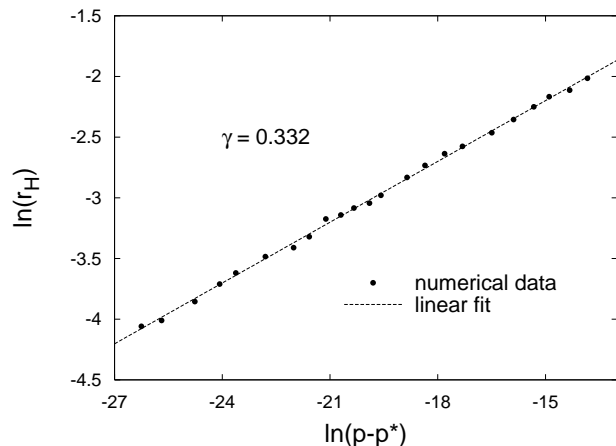

FIG. 4: Critical behavior. For supercritical solutions we plot the horizon radius vs. the distance to the threshold on the log-log scale. The fit of the power law $\ln(r_H) = (\gamma/2)\ln(p-p^*) + const$ yields $\gamma \approx 0.33$ which agrees (up to numerical errors) with the critical exponent obtained in [3]. The echoing period $\Delta \approx 0.47$ (read off from the period of wiggles around the linear fit above and computed independently from the spatial shift between the echoes) is also the same.

Finally, we remark that, besides the KK monopole, the BCS ansatz incorporates other non-asymptotically flat solutions. In particular, an exact regular time dependent solution is known [8]

$$ds^2 = -dt^2 + \left(\frac{t}{m} + \frac{m}{\rho}\right) d\rho^2 + \left(\frac{t}{m} + \frac{m}{\rho}\right) \rho^2 (d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ m^2 \left(\frac{t}{m} + \frac{m}{\rho}\right)^{-1} (d\psi + \cos\theta d\phi)^2 . \tag{18}$$

This solution can be viewed as the time evolving KK monopole whose S^1 fibers pinch off to a point for $t \to \infty$. It would be interesting to determine the role of this solution in dynamics.

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