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The $\mathcal{N} = 8$ Supergravity Hamiltonian as a Quadratic Form

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Abstract

We conjecture that the light-cone Hamiltonian of $\mathcal{N} = 8$ Supergravity can be expressed as a quadratic form. We explain why this rewriting is *unique* to maximally supersymmetric theories. The $\mathcal{N} = 8$ quartic interaction vertex is constructed and used to verify that this conjecture holds to order κ^2.

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1 Introduction

Eleven-dimensional supergravity [1] on reduction to four dimensions yields the maximally supersymmetric $\mathcal{N} = 8$ theory. ($\mathcal{N} = 8, d = 4$) Supergravity is similar in many ways to the other maximally supersymmetric theory in four dimensions, $\mathcal{N} = 4$ Yang-Mills. In reference [2], it was shown that the light-cone Hamiltonian of $\mathcal{N} = 4$ Yang-Mills could be expressed as a quadratic form. The key to this rewriting was the maximal supersymmetry present in the theory. Since this is true in $\mathcal{N} = 8$ as well, we conjecture that a similar rewriting must be possible. To start with, we show this explicitly to order κ. We explain why this quadratic-form structure is *unique* to maximally supersymmetric theories and simply does not apply to other cases.

A light-cone superspace formulation of $\mathcal{N} = 8$ Supergravity was first achieved in [3, 4] wherein the authors constructed the action to order κ. The formulation has not been extended since. In this paper, we extend the action to order κ^2 by constructing the quartic interaction vertex. We will then see that the Hamiltonian is a quadratic form at order κ^2 as well.

When formulating a maximally supersymmetric theory (like $\mathcal{N} = 8$ supergravity) in light-cone superspace, it suffices to show that the superspace action correctly reproduces the component action for any one component field (our focus will be on the graviton). Once this is done, the remaining component terms in the action will follow from supersymmetry transformations. We will therefore construct the quartic interaction vertex in $\mathcal{N} = 8$ Supergravity by requiring that it correctly reproduce pure gravity in light-cone gauge.

The long-term aim of this program of research [2, 5] is aimed at understanding the divergent behavior of $\mathcal{N} = 8$ Supergravity. Curtright [6] conjectured that any divergences occurring in the $\mathcal{N} = 8$ theory, were attributible to the incomplete cancellation of Dynkin indices of $SO(9)$ representations. The light-cone offers an ideal framework to study this conjecture since it highlights the role of the spacetime little group. In addition, working on the light-cone ensures that we deal exclusively with the theory's physical degrees of freedom.

2 $\mathcal{N}=8$ Supergravity in light-cone superspace

With the space-time metric $(-,+,+,+)$, the light-cone coordinates and their derivatives are

$$
\begin{aligned}
x^{\pm} &= \frac{1}{\sqrt{2}}\,(\,x^0 \pm x^3\,)\;; \qquad & \partial^{\pm} &= \frac{1}{\sqrt{2}}\,(\,-\,\partial_0 \pm \partial_3\,)\;, \\
x &= \frac{1}{\sqrt{2}}\,(\,x_1\,+\,i\,x_2\,)\;; \qquad & \bar{\partial} &= \frac{1}{\sqrt{2}}\,(\,\partial_1\,-\,i\,\partial_2\,)\;, \\
\bar{x} &= \frac{1}{\sqrt{2}}\,(\,x_1\,-\,i\,x_2\,)\;; \qquad & \partial &= \frac{1}{\sqrt{2}}\,(\,\partial_1\,+\,i\,\partial_2\,)\;.
\end{aligned}
\tag{1}
$$

The degrees of freedom of $\mathcal{N}=8$ Supergravity theory may be captured in a single superfield [3]. In terms of Grassmann variables θ^m, which transform as the 8 of $SU(8)$ $(m = 1\,\ldots\,8)$, we define[1]

$$
\begin{aligned}
\phi\,(\,y\,) \;=\; & \frac{1}{\partial^{+2}}\,h\,(y)\,+\,i\,\theta^m\,\frac{1}{\partial^{+2}}\,\bar{\psi}_m\,(y)\,+\,\frac{i}{2}\,\theta^m\,\theta^n\,\frac{1}{\partial^{+}}\,\bar{A}_{mn}\,(y)\;, \\
& -\,\frac{1}{3!}\,\theta^m\,\theta^n\,\theta^p\,\frac{1}{\partial^{+}}\,\bar{\chi}_{mnp}\,(y)\,-\,\frac{1}{4!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\bar{C}_{mnpq}\,(y)\;, \\
& +\,\frac{i}{5!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\epsilon_{mnpqrstu}\,\chi^{stu}\,(y)\;, \\
& +\,\frac{i}{6!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\epsilon_{mnpqrstu}\,\partial^{+}\,A^{tu}\,(y)\;, \\
& +\,\frac{1}{7!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\epsilon_{mnpqrstu}\,\partial^{+}\,\psi^u\,(y)\;, \\
& +\,\frac{4}{8!}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\theta^r\,\theta^s\,\theta^t\,\theta^u\,\epsilon_{mnpqrstu}\,\partial^{+2}\,\bar{h}\,(y)\;,
\end{aligned}
\tag{2}
$$

In this notation, the two-component graviton is represented by

$$
h \;=\; \frac{1}{\sqrt{2}}\,(\,h_{11}\,+\,i\,h_{12}\,)\;; \qquad \bar{h} \;=\; \frac{1}{\sqrt{2}}\,(\,h_{11}\,-\,i\,h_{12}\,)\;. \tag{3}
$$

The $\bar{\psi}_m$ represent the spin-$\frac{3}{2}$ gravitinos in the theory. The 28 gauge fields are $\bar{A}_{mn}$ and the corresponding gauginos are the $\bar{\chi}_{mnp}$. The $\bar{C}_{mnpq}$ are the 70 scalar fields in the spectrum. Complex conjugation of the fields is denoted by a bar.

All fields are local in the modified light-cone coordinates

$$
y \;=\; (\,x,\,\bar{x},\,x^{+},\,y^{-}\equiv x^{-}\,-\,\frac{i}{\sqrt{2}}\,\theta^{\alpha}\,\bar{\theta}_{\alpha}\,)\;. \tag{4}
$$

[1] The factor of 4 in the final term was missed in reference [3]

In this LC_2 (two-component light-cone) form, all the unphysical degrees of freedom have been integrated out. The superfield ϕ and its complex conjugate $\bar{\phi}$ satisfy chiral constraints,

$$d^m \, \phi \,(\, y \,) \;=\; 0 \quad ; \qquad \bar{d}_n \, \bar{\phi} \,(\, y \,) \;=\; 0 \; , \tag{5}$$

where

$$d^m \;=\; -\,\frac{\partial}{\partial\,\bar{\theta}_m} \;-\; \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ \quad ; \qquad \bar{d}_n \;=\; \frac{\partial}{\partial\,\theta^n} \;+\; \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \; , \tag{6}$$

The superfield and its conjugate are furthermore related via the "inside-out" constraint

$$\phi \;=\; \frac{1}{4}\,\frac{(\,d\,)^8}{\partial^{+4}}\,\bar{\phi} \; , \tag{7}$$

where $(\,d\,)^8 \;=\; d^1\,d^2\,\ldots\,d^8$. This constraint is unique to maximally supersymmetric theories.

On the light-cone, supersymmetry splits into two varieties [4], the kinematical ones

$$q^{\,m}_{\,+} \;=\; -\,\frac{\partial}{\partial\,\bar{\theta}_m} \;+\; \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ ; \qquad \bar{q}_{\,+\,n} = \;\; \frac{\partial}{\partial\,\theta^n} \;-\; \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \; , \tag{8}$$

and the dynamical supersymmetries

$$\begin{aligned} q^{\,m}_{\,-} \;&\equiv i\,[\,\bar{j}^{\,-}\,,\, q^{\,m}_{\,+}\,] \;=\; \frac{\bar{\partial}}{\partial^+}\,q^{\,m}_{\,+} \;+\; \mathcal{O}(\kappa) \; , \\ \bar{q}_{\,-\,n} \;&\equiv i\,[\,j^{\,-}\,,\, \bar{q}_{\,+\,n}\,] \;=\; \frac{\partial}{\partial^+}\,\bar{q}_{\,+\,n} \;+\; \mathcal{O}(\kappa) \; . \end{aligned} \tag{9}$$

2.1 The action to order κ

The $\mathcal{N}\,=\,8$ supergravity action to order κ was derived using purely algebraic methods in reference [4] and further simplified in reference [5]. For a detailed listing of all the superPoincaré generators and their commutation relations, see references [2, 4, 5]. The light-cone superspace action for $\mathcal{N}\,=\,8$ supergravity reads

$$\beta \int d^4x \int d^8\theta \, d^8\bar{\theta} \, \mathcal{L} \; , \tag{10}$$

where $\beta \;=\; -\,\frac{1}{64}$ and

$$\mathcal{L} \;\;=\;\; -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; 2\,\kappa\,(\,\frac{1}{\partial^{+2}}\,\overline{\phi}\;\;\bar{\partial}\,\phi\;\bar{\partial}\,\phi \,+\, \frac{1}{\partial^{+2}}\,\phi\,\partial\,\overline{\phi}\,\partial\,\overline{\phi}) \; . \tag{11}$$

The d'Alembertian is

$$\Box \;=\; 2\,(\,\partial\,\bar\partial\,-\,\partial^+\,\partial^-\,)\ , \tag{12}$$

$\kappa\;=\;\sqrt{8\,\pi\,G}$ and Grassmann integration is normalized such that $\int d^8\theta\,(\theta)^8=1$.

3 The $\mathcal{N}=8$ Hamiltonian as a quadratic form

Maximally supersymmetric theories (like $\mathcal{N}=4$ Yang-Mills and $\mathcal{N}=8$ Supergravity) are special for various reasons. Specifically, the superfield governing these theories satisfies the inside-out constraint (equation (7)). This constraint allows us to express the Hamiltonian of the theory as a quadratic form. In this section, we illustrate this at lowest order and then at order κ.

3.1 Lowest order Hamiltonian

Based on comparison to $\mathcal{N}=4$ Yang-Mills [2], we conjecture that the light-cone Hamiltonian of $\mathcal{N}=8$ Supergravity can be expressed as

$$\mathcal{H}\;=\;\frac{1}{4\,\sqrt{2}}\,(\,\mathcal{W}_m\,,\,\mathcal{W}_m\,)\ , \tag{13}$$

with

$$\mathcal{W}_m\;=\;\bar q_{-\,m}\,\phi\ , \tag{14}$$

and the inner product defined as

$$(\,\phi\,,\,\xi\,)\;\equiv\;2i\int d^4x\,d^8\theta\,d^8\,\bar\theta\;\bar\phi\,\frac{1}{\partial^{+3}}\xi\ . \tag{15}$$

This relation (13), which we will prove presently (to order κ^2), is *unique* to maximally supersymmetric theories. It is unrelated to the fact that the Hamiltonian is the anticommutator of two supersymmetries (this will become obvious in a few steps).

We start by verifying this conjecture at the lowest order

$$\begin{aligned}\mathcal{H}^0\;&=\;\frac{1}{4\,\sqrt{2}}\,(\,\mathcal{W}^0_m\,,\,\mathcal{W}^0_m\,)\ ,\\ &=\;\frac{2i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar\theta\;\;q_-^{\;m}\,\bar\phi\,\frac{1}{\partial^{+3}}\,\bar q_{-\,m}\,\phi\ ,\end{aligned} \tag{16}$$

and rewrite this as two terms

$$\mathcal{H}^0 \;=\; \frac{i}{4\sqrt{2}} \int d^4x\, d^8\theta\, d^8\bar{\theta}\, \left(q_-^{\;m}\, \bar{\phi}\, \frac{1}{\partial^{+3}}\, \bar{q}_{-\,m}\, \phi \;+\; q_-^{\;m}\, \bar{\phi}\, \frac{1}{\partial^{+3}}\, \bar{q}_{-\,m}\, \phi \right) . \qquad (17)$$

In a non-maximally supersymmetric theory, this expression does not simplify further.

3.1.1 Maximally supersymmetric theories

In the special case, where the supersymmetric theory under consideration is maximally supersymmetric, equation (17) can be further simplified. This is because such theories are always described by constrained superfields. Equation (7) is the relevant constraint in the case of the $\mathcal{N}=8$ superfield.

We apply this constraint to the second term in equation (17) to obtain

$$\mathcal{H}^0 \;=\; \frac{i}{4\sqrt{2}} \int d^4x\, d^8\theta\, d^8\bar{\theta}\, \left(q_-^{\;m}\, \bar{\phi}\, \frac{1}{\partial^{+3}}\, \bar{q}_{-\,m}\, \phi \;+\; \frac{1}{\partial^{+4}}\, q_-^{\;m}\, \phi\, \partial^+\, \bar{q}_{-\,m}\, \bar{\phi} \right) . \quad (18)$$

Using the explicit expressions for the dynamical supersymmetries from equation (9), we get

$$\mathcal{H}^0 \;=\; \frac{i}{4\sqrt{2}} \int d^4x\, d^8\theta\, d^8\bar{\theta}\, \left(\frac{\bar{\partial}}{\partial^+}\, q_+^{m}\, \bar{\phi}\, \frac{\partial}{\partial^{+4}}\, \bar{q}_{+\,m}\, \phi \;+\; \frac{\bar{\partial}}{\partial^{+5}}\, q_+^{m}\, \phi\, \partial\, \bar{q}_{+\,m}\, \bar{\phi} \right) . \,(19)$$

After some integration by parts, this may be reexpressed as

$$\mathcal{H}^0 \;=\; \frac{i}{4\sqrt{2}} \int d^4x\, d^8\theta\, d^8\bar{\theta}\, \frac{\partial\,\bar{\partial}}{\partial^{+5}}\, \bar{\phi}\, \{\, q_+^{m} \,,\, \bar{q}_{+\,m} \,\}\, \phi \;. \qquad (20)$$

Since $\{\, q_+^{m} \,,\, \bar{q}_{+\,m} \,\}\, \phi \;=\; i\, 8\, \sqrt{2}\, \partial^+\, \phi$, we have

$$\mathcal{H}^0 \;=\; \int d^4x\, d^8\theta\, d^8\bar{\theta}\, \bar{\phi}\, \frac{2\,\partial\bar{\partial}}{\partial^{+4}}\, \phi \;, \qquad (21)$$

which is indeed the correct kinetic term in the Hamiltonian describing the $\mathcal{N}=8$ theory [4].

3.2 Order κ

We now extend this to order κ. The dynamical supersymmetry generators are known to this order [4] so we can write down the expressions for $\mathcal{W}$ and its complex conjugate,

$$\mathcal{W}_m \;=\; \frac{\partial}{\partial^+} \bar{q}_{+\,m}\, \phi \;+\; \kappa\, \frac{1}{\partial^+} \left(\bar{\partial}\, \bar{d}_m\, \phi\, \partial^{+2}\, \phi \;-\; \partial^+\, \bar{d}_m\, \phi\, \partial^+\, \bar{\partial}\, \phi \right) \;+\; \mathcal{O}(\kappa^2) \;, \quad (22)$$

$$\overline{\mathcal{W}}^m \,=\, \frac{\bar\partial}{\partial^+}\, q_+^m\, \bar\phi \,+\, \kappa\, \frac{1}{\partial^+}\Big(\partial\, d^m\, \bar\phi\, {\partial^+}^2\, \bar\phi \,-\, \partial^+\, d^m\, \bar\phi\, \partial^+\, \partial\, \bar\phi\Big) \,+\, \mathcal{O}(\kappa^2)\ . \quad (23)$$

With these, we directly compute the quadratic form

$$\frac{1}{4\sqrt{2}}\,(\,\mathcal{W}\,,\,\mathcal{W}\,) \,=\, \frac{2\,i}{4\sqrt{2}}\int d^4x\, d^8\theta\, d^8\,\bar\theta\, \overline{\mathcal{W}}\, \frac{1}{{\partial^+}^3}\, \mathcal{W}\ . \quad (24)$$

The calculation is straightforward but fairly lengthy so details are relegated to Appendix **A**. After simplification, the cubic interaction vertex occuring in the $\mathcal{N}=8$ Hamiltonian is

$$\int\, d^4x\, d^8\theta\, d^8\bar\theta\ \ 2\,\kappa\,\Big(\frac{1}{{\partial^+}^2}\,\overline{\phi}\ \ \bar\partial\,\phi\,\bar\partial\,\phi + \frac{1}{{\partial^+}^2}\,\phi\,\partial\,\overline{\phi}\,\partial\,\overline{\phi}\Big)\ , \quad (25)$$

which is exactly equal to the equivalent term in the Lagrangian (obtained using different means) in equation (11).

4 The Hamiltonian to order κ^2 : lessons from the quadratic form

The quadratic form studied so far will not immediately tell us the Hamiltonian to order κ^2. This is because the dynamical supersymmetry (and hence $\mathcal{W}$) is known only to order κ. However the quadratic form still offers a lot of insight into possible forms the quartic interaction may take.

Our plan then is as follows. In this section, we will collect information from the quadratic form, dimensional analysis and helicity considerations. Based on these pointers, the general structure of the quartic interaction vertex will become much clearer. We will then make a guess at such an interaction vertex and check that the guess is correct.

In order to check our Ansatz, we will set all terms of the superfield (in the superspace vertex) except the graviton to zero. The resulting vertex (in components) must reproduce pure gravity in the light-cone gauge. Once the graviton vertex is fixed, the supersymmetry transformations (which are manifest within the superfield) will produce the remaining components of the action.

4.1 Information from the quadratic form

With this general plan in mind, we first draw information from the fact that the Hamiltonian is expressible as a quadratic form. Notice that the $\mathcal{N}=8$

Hamiltonian (at order κ^2) has two contributions from the quadratic form. The first is

$$(\,\mathcal{W}^{\kappa}\,,\,\mathcal{W}^{\kappa}\,)\ . \tag{26}$$

This is straightforward to compute and reads

$$\begin{aligned} -\frac{i}{2\sqrt{2}}\,\kappa^2\int d^4x\,d^8\theta\,d^8\,\bar\theta\,\frac{1}{\partial^{+5}}\Big(\partial^{+2}\phi\,\bar d\,\bar\partial\,\phi\,-\,\partial^{+}\bar d\,\phi\,\partial^{+}\bar\partial\,\phi\Big)& \\ \times\Big(\partial^{+2}\bar\phi\,d\,\partial\,\bar\phi\,-\,\partial^{+}d\,\bar\phi\,\partial^{+}\partial\,\bar\phi\Big).& \end{aligned} \tag{27}$$

The second contribution is

$$(\,\mathcal{W}^{0}\,,\,\mathcal{W}^{\kappa^2}\,)\,+\,(\,\mathcal{W}^{\kappa^2}\,,\,\mathcal{W}^{0}\,)\ . \tag{28}$$

Even though $\mathcal{W}^{\kappa^2}$ is unknown, there is still structural information in this expression. The first of the two terms in (28) is

$$(\,\mathcal{W}^{0}\,,\,\mathcal{W}^{\kappa^2}\,)\,=\,2i\int d^4x\,d^8\theta\,d^8\,\bar\theta\,\frac{\bar\partial}{\partial^{+4}}\,q^{+}\,\bar\phi\,\mathcal{W}^{\kappa^2}\ , \tag{29}$$

where the measure factor $\frac{1}{\partial^{+3}}$ has been integrated on to the $\mathcal{W}^{0}$ term. This tells us that the quartic interaction has the form

$$\kappa^2\,\frac{\bar\partial}{\partial^{+4}}\,q^{+}\,\bar\phi\,\cdot\,\aleph \tag{30}$$

We know (from comparison to equation (27)) that $\aleph$ involves two chiral superfields and one anti-chiral superfield.

4.2 Dimensional analysis and helicity considerations

We list below the helicities and length-dimensions (with $\hbar\,=\,1$) of some quantities at our disposal.

Variable	Helicity (h)	Dimension (D)
ϕ	$+2$	$+1$
$\bar\phi$	-2	$+1$
∂	$+1$	-1
$\bar\partial$	-1	-1
∂^{+}	0	-1
$d^m\,,\,q^m$	$+\,1/2$	$-\,1/2$
$\bar d_n\,,\,\bar q_n$	$-\,1/2$	$-\,1/2$

The quartic interaction term at order κ^2 presently reads (up to a constant),

$$\int\,d^4x\,d^8\theta\,d^8\bar\theta\,\kappa^2\,(\,\frac{\bar\partial}{\partial^{+4}}\,q^{+}\,\bar\phi\,)\,\cdot\,\aleph\ . \tag{31}$$

The measure has dimension -4, κ^2 has dimension $+2$. It is then easy to see that $\aleph$ has dimension $-\frac{3}{2}$ and helicity $\frac{5}{2}$.

From equation (27) and the table we see that $\aleph$ must contain: two ϕs, one $\bar{\phi}$, one $\bar{\partial}$ and either a $\bar{d}$ or a $\bar{q}$. The dimensions can be taken care of by introducing the necessary number of ∂^+s and $\frac{1}{\partial^+}$s.

The issue of chiralization

$\mathcal{W}^{\kappa^2}$ must commute with both d and $\bar{d}$. This is necessary to ensure that the dynamical supersymmetry generator respects the chirality of the superfield it acts on. Thus

$$\mathcal{W}^{\kappa^2} = \mathcal{C}(\aleph) \ , \tag{32}$$

where $\mathcal{C}$ represents chiralization and is explained in Appendix **B**. In the quartic vertex, this appears as

$$\int \frac{1}{\partial^{+3}} \overline{\mathcal{W}}^0 \, \mathcal{W}^{\kappa^2} = \kappa^2 \int \frac{1}{\partial^{+3}} \overline{\mathcal{W}}^0 \cdot \left[\aleph + \bar{d}_{m_1 \dots m_a} \{ \dots \} \right] . \tag{33}$$

However, the $\bar{d}$s that appear in the chiralized expression (on partial integration) annihilate the $\overline{\mathcal{W}}^0$ (since it is anti-chiral). Hence the chiralizing terms are not relevant to the calculation at this order.

4.3 The Ansatz for the quartic interaction

Our Ansatz for the quartic interaction vertex is simply $(\, \mathcal{W}^\kappa \, , \, \mathcal{W}^\kappa \,)$ plus

$$\begin{aligned} & c^A_{abcd} \, \frac{1}{\partial^{+a}} \left(\partial^{+b} \phi \, \partial^{+c} \, \bar{q}_+ \, \phi \right) \partial^{+d} \, \partial \, \bar{\phi} \cdot \frac{1}{\partial^{+3}} \overline{\mathcal{W}}^0 \\ + \; & c^B_{abcd} \, \frac{1}{\partial^{+a}} \left(\partial^{+b} \, \bar{q}_+ \, \phi \, \partial^{+c} \partial \, \phi \right) \partial^{+d} \, \bar{\phi} \cdot \frac{1}{\partial^{+3}} \overline{\mathcal{W}}^0 \\ + \; & c^C_{abcd} \, \frac{1}{\partial^{+a}} \left(\partial^{+b} \phi \, \partial^{+c} \, \bar{q}_+ \, \partial \, \phi \right) \partial^{+d} \, \bar{\phi} \cdot \frac{1}{\partial^{+3}} \overline{\mathcal{W}}^0 \ , \end{aligned} \tag{34}$$

subject to the constraint

$$a + 3 = b + c + d \ . \tag{35}$$

The constants c^A_{abcd}, c^B_{abcd} and c^C_{abcd} are to be fixed by comparison with the gravity action.

It is such a general Ansatz that we expand using Mathematica (details in Appendix **D**). We only focus on the graviton components and require that the resulting expression reproduce the quartic vertex for pure gravity in the light-cone gauge.

Gravity on the light-cone has been studied previously by numerous authors [7, 8, 9, 10, 11]. A brief but self contained review of the formalism is presented in Appendix **C**.

5 The $\mathcal{N} = 8$ Supergravity action to order κ^2

By explicit comparison with the gravity vertex, we find that the $\mathcal{N} = 8$ Supergravity quartic interaction vertex is

$$
\begin{aligned}
-\kappa^2 \frac{i}{\sqrt{2}} \Bigg[\; X + \overline{X} + \frac{1}{2}\frac{1}{\partial^{+5}} \left(\partial^{+2}\phi\, \bar{d}\,\bar{\partial}\,\phi - \partial^{+}\bar{d}\,\phi\,\partial^{+}\bar{\partial}\,\phi\right) \\
\times \left(\partial^{+2}\bar{\phi}\, d\,\partial\,\bar{\phi} - \partial^{+} d\,\bar{\phi}\,\partial^{+}\partial\,\bar{\phi}\right) \;\Bigg],
\end{aligned}
\tag{36}
$$

where the explicit form of X is given in Appendix **D**. The final answer (as explained in the appendix) confirms our conjecture regarding the quadratic form, to order κ^2.

The answer does not seem to simplify very much based on the usual tricks (partial integrations, inside-out constraints and so on). Our result is surprisingly lengthy for a superspace expression, however it is important to remember that the entire component action for the $\mathcal{N} = 8$ theory in light-cone gauge involves many thousand terms. Also, unlike $\mathcal{N} = 4$ Yang-Mills, where the dynamical supersymmetry generator stops at order κ, the $\mathcal{N} = 8$ supersymmetry generator extends to all orders.

It is worth noting that there is no freedom arising from superfield redefinitions. This is because any shift of the form

$$
\phi \;\rightarrow\; \phi + \phi^3 \tag{37}
$$

in the kinetic term will result in time-derivatives ∂^- reappearing in the action to order κ^2. These time derivatives (as explained in the gravity section) have been pushed to higher orders and no longer occur at κ^2.

6 Summary

The quartic interaction governing $\mathcal{N} = 8$ supergravity has been constructed. In addition, we have shown (to order κ^2) that the Hamiltonian is expressible as a quadratic form. We believe this quadratic structure of the Hamiltonian will hold to higher orders as well. We expect to return to this issue and explore its consequences further.

The general structure of the four-point interaction term is quite simple. The complication stems from the many ways the various derivatives enter. However, when computing the Feynman rules, momentum conservation will help us to reduce the number of terms, especially if we use some specific frame for the momenta. This should lead to tractable forms for the simpler diagrams. The fact remains that supergraph computations, essential to understanding the divergent nature of this theory, are going to prove extremely tedious. It would be very satisfying if the quartic interaction could be recast in an elegant manner. The light-cone formalism was essential in the proof of finiteness of $\mathcal{N} = 4$ Yang-Mills [12] and one hope is that the techniques used in that paper will prove equally useful when studying $\mathcal{N} = 8$ Supergravity.

Finally, it will be interesting to see if the quartic interaction simplifies considerably on reduction to three dimensions. Such a simplification is likely to occur due to the large $SO(16)$ symmetry that the $d = 3$ theory is known to possess [13].

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Appendix A

Detailed computation of the Hamiltonian to order κ

In this section, we drop the spinor indices for convenience. Start by considering

$$\mathcal{H} \,=\, \frac{i}{2\,\sqrt{2}} \int \overline{\mathcal{W}}\, \frac{1}{{\partial^+}^3}\, \mathcal{W}\ . \tag{A-1}$$

We will not carry the constant $\frac{i}{2\,\sqrt{2}}$ any further. Instead we will reintroduce it into the expression at the end. Substituting the expression for $\mathcal{W}^\kappa$ from equation (22) we obtain

$$\mathcal{H}^\kappa \,=\, \kappa \int \frac{\bar\partial}{{\partial^+}^5}\, q_+\, \bar\phi\, \{\bar\partial\, \bar d\, \phi\, {\partial^+}^2\, \phi \,-\, \partial^+\, \bar d\, \phi\, \partial^+\, \bar\partial\, \phi\} \,+\, \text{c.c}\ . \tag{A-2}$$

We will ignore the complex conjugate in this section. For any chiral combination X, $q_+\, X \,=\, i\,\sqrt{2}\,\partial^+\,\theta\, X$. This yields

$$\begin{aligned} \mathcal{H}^\kappa \,&=\, i\,\sqrt{2}\,\kappa \int \frac{\bar\partial}{{\partial^+}^4}\, \theta\, \bar\phi\, \{\bar\partial\, \bar d\, \phi\, {\partial^+}^2\, \phi \,-\, \partial^+\, \bar d\, \phi\, \partial^+\, \bar\partial\, \phi\}\ , \\ &=\, \qquad\qquad\qquad\qquad Y \qquad\quad + \qquad\quad X \end{aligned} \tag{A-3}$$

We now focus on Y and rewrite it using the fact that $\theta\, \bar d \,=\, 8 \,-\, \bar d\, \theta$ to obtain

$$\begin{aligned} Y \,&=\, -\,i\, 8\,\sqrt{2}\,\kappa \int \frac{\bar\partial}{{\partial^+}^4}\, \bar\phi\, \bar\partial\, \phi\, {\partial^+}^2\, \phi \,-\, i\,\sqrt{2}\,\kappa \int \frac{\bar\partial}{{\partial^+}^4}\, \bar\phi\, \theta\, \bar\partial\, \phi\, {\partial^+}^2\, \bar d\, \phi\ , \\ &=\, \qquad\qquad\qquad\qquad A \qquad\quad + \qquad\quad B \end{aligned} \tag{A-4}$$

Now partially integrate B with respect to ∂^+ to get

$$\begin{aligned} B \,&=\, -\,i\,\sqrt{2}\,\kappa \int \frac{\bar\partial}{{\partial^+}^3}\, \bar\phi\, \bar\partial\, \phi\, \partial^+\, \theta\, \bar d\, \phi \,-\, X\ , \\ &=\, \qquad\qquad\qquad G \qquad\quad - \qquad\quad X \end{aligned} \tag{A-5}$$

and we simply cancel the second term against the X that appears in equation (A-3).

We now focus on term G. To simplify calculations, rather than track $\theta\,\cdot\,\bar d$, we will focus on $\theta^1\, \bar d_1$ for the moment and then multiply by a factor of 8 at the end (although we are missing this factor of 8, we will continue to refer to term G as G). In term G, we now impose the inside-out constraint (equation (7)) on the middle superfield to obtain

$$G \,=\, -\,i\,\sqrt{2}\,\kappa \int \frac{\bar\partial}{{\partial^+}^3}\, \bar\phi\, \bar\partial\, \frac{d^{1\ldots8}}{4\,{\partial^+}^4}\, \bar\phi\, \partial^+\, \theta^1\, \bar d_1\, \phi \tag{A-6}$$

We then integrate the chiral derivatives away from the middle superfield to obtain

$$
\begin{aligned}
G \,=\, & -\, i\,\sqrt{2}\,\kappa \int\, \bar{\partial}\partial^{+}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{4}}\,\bar{\phi}\,\partial^{+}\,\theta^{1}\,\bar{d}_{1}\,\phi \\
& +\, i\,\sqrt{2}\,\kappa \int\, \frac{\bar{\partial}}{{\partial^{+}}^{3}}\,\bar{\phi}\,\bar{\partial}\,\frac{d^{2\ldots 8}}{4\,{\partial^{+}}^{4}}\,\bar{\phi}\,\partial^{+}\,\theta^{1}\,(\,-\,i\,\sqrt{2}\,\partial^{+}\,)\,\phi \\
=\, & \qquad H \\
& +\qquad I
\end{aligned}
\tag{A-7}
$$

Term I has two ∂^{+}s acting on the last superfield so we integrate one of these out and obtain

$$
\begin{aligned}
I \,=\, & i\,\sqrt{2}\,\kappa \int\, \bar{\partial}\partial^{+}\,\bar{d}_{1}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{4}}\,\bar{\phi}\,\partial^{+}\,\theta^{1}\,\phi \,+\, i\,\sqrt{2}\,\kappa \int\, \bar{\partial}\,\bar{d}_{1}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{3}}\,\bar{\phi}\,\partial^{+}\,\theta^{1}\,\phi \\
=\, & \qquad J \qquad + \qquad K
\end{aligned}
\tag{A-8}
$$

Term J is rewritten as

$$
\begin{aligned}
J \,=\, & -\,i\,\sqrt{2}\,\kappa \int\, \bar{\partial}\partial^{+}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{4}}\,\bar{\phi}\,\partial^{+}\,\phi \,-\, H\ , \\
=\, & \qquad L \quad - \quad H\ ,
\end{aligned}
\tag{A-9}
$$

with the second term canceling against the H that occurs in equation (A-7). Now term K also simplifies to

$$
K \,=\, -\,i\,\sqrt{2}\,\kappa \int\, \bar{\partial}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{3}}\,\bar{\phi}\,\partial^{+}\,\phi \,-\, G\ . \tag{A-10}
$$

This leads to (after putting back the factor of 8)

$$
G \,=\, i\,4\,\sqrt{2}\,\kappa \int\, \bar{\partial}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{4}}\,\bar{\phi}\,{\partial^{+}}^{2}\,\phi\ . \tag{A-11}
$$

After taking into account, the constant from equation (A-1), we obtain the final answer,

$$
\mathcal{H}^{\kappa} \,=\, A \,+\, G \,=\, 2\,\kappa \int\, \bar{\partial}\,\phi\, \frac{\bar{\partial}}{{\partial^{+}}^{4}}\,\bar{\phi}\,{\partial^{+}}^{2}\,\phi\ . \tag{A-12}
$$

Imposing the inside-out constraint on the third superfield yields

$$
\mathcal{H}^{\kappa} \,=\, 2\,\kappa \int\, \frac{1}{{\partial^{+}}^{2}}\,\bar{\phi}\,\bar{\partial}\,\phi\,\bar{\partial}\,\phi\ , \tag{A-13}
$$

and proves the quadratic form structure of the Hamiltonian at order κ.

Appendix B

Chiralization

The object $\aleph$ is simply the dynamical supersymmetry generator at order κ^2. All generators must respect the chirality of the superfields they act on and hence must commute with both d^m and $\bar{d}_m$. $\aleph$ contains two chiral superfields and one anti-chiral superfield. In addition it contains either a $\bar{d}$ or a $\bar{q}$. We choose the $\bar{q}$ since it commutes with both chiral derivatives.

Now, $\aleph$ needs to be chiralized. This can be accomplished through a descent procedure. There seem to be two non-equivalent ways of doing this.
For any general non-chiral expression of the form, $A\,\bar{B}$ (where A is any compound chiral function and $\bar{B}$ a compound anti-chiral function), we define a "chiral product"

Scheme I

$$\mathcal{C}\,(\,A\bar{B}\,)\;=\;A\bar{B}\,+\,\sum_{a=1}^{8}\frac{(-1)^a}{a!}\,\frac{\bar{d}_{m_1\,\dots\,m_a}}{(-\,i\,\sqrt{2}\,\partial^+)^a}\,(\,A\,d^{m_a\,\dots\,m_1}\,\bar{B}\,)\;, \tag{B-1}$$

or

Scheme II

$$\mathcal{C}\,(\,A\bar{B}\,)\;=A\bar{B}\,+\,\sum_{a=1}^{8}\frac{(-1)^a}{a!}\,\frac{\bar{d}_{m_1\,\dots\,m_a}}{(-\,i\,\sqrt{2}\,\partial^+)^a}\,A\,d^{m_a\,\dots\,m_1}\,\bar{B}\;. \tag{B-2}$$

in both cases,

$$\bar{d}_{m_1\,\dots\,m_a}\;=\;\bar{d}_{m_1}\,\dots\,\bar{d}_{m_a}\;;\qquad d^{m_a\,\dots\,m_1}\;=\;d^{m_a}\,\dots\,d^{m_1}\;. \tag{B-3}$$

$\mathcal{C}\,(\,A\bar{B}\,)$ via either scheme is now a chiral function and satisfies $d\,\mathcal{C}\;=\;0$.

Notice that the quartic interaction from equation (30) is of the form

$$\kappa^2\,\frac{\bar{\partial}}{{\partial^+}^4}\,q^+\,\bar{\phi}\,\cdot\,\aleph\;. \tag{B-4}$$

Adopting Scheme **I** for chiralizing $\aleph$ is more convenient because all the "chiralizing" terms simply vanish. This is because $\aleph$ is multiplied by an antichiral field. The relevant expression is

$$\kappa^2 \, \frac{\bar\partial}{{\partial^+}^4} \, q^+ \, \bar\phi \cdot \left[\aleph + \bar d_{m_1 \, \dots \, m_a} \, \{ \dots \} \right] . \tag{B-5}$$

A simple partial integration of the overall $\bar d$ onto the antichiral superfield on the left, kills all the chiralizing terms. Hence **at this order**, chiralization of $\aleph$ is unimportant.

It is possible that Scheme **II** may yield more interesting results as far as the quartic interaction is concerned.

Appendix C

Einstein gravity in the LC_2 formalism

This section offers a quick review of pure gravity in light-cone gauge. This section follows closely, the treatment in reference [10]. Other references dealing with light-cone gravity include [7, 8, 9, 11].

The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x\, L \;=\; \frac{1}{2\,\kappa^2}\int d^4x\,\sqrt{-g}\,R \;, \tag{C-1}$$

where $g = \det g_{\mu\nu}$ and R is the curvature scalar.

Gauge choices

Light-cone gauge is chosen by setting

$$g_{--} \;=\; g_{-i} \;=\; 0 \;. \tag{C-2}$$

A fourth gauge choice will be made shortly. The metric is parametrized as follows

$$\begin{aligned} g_{+-} \;&=\; -\,e^{\phi} \;, \\ g_{i\,j} \;&=\; e^{\psi}\,\gamma_{i\,j} \;. \end{aligned} \tag{C-3}$$

The fields $\phi\,,\ \psi$ are real while γ_{ij} is a 2×2 real, symmetric, unimodular matrix.

We us the equations of motion

$$R_{\mu\nu} \;=\; 0 \;, \tag{C-4}$$

to eliminate the unphysical degrees of freedom. The $R_{-i} \;=\; 0$ relation implies that

$$\begin{aligned} g^{-i} \;=\; -\mathrm{e}^{-\,\phi}\,\frac{1}{\partial^+}\Bigg[\;\gamma^{ij}\,\mathrm{e}^{\phi\,-\,2\,\psi}\,\frac{1}{\partial^+}\bigg\{\,\mathrm{e}^{-\,\psi}\,\Big(\,\frac{1}{2}\,\partial^+\,\gamma^{kl}\,\partial_j\,\gamma_{kl} \;-\; \partial^+\,\partial_j\,\phi \\ -\;\partial^+\,\partial_j\,\psi \;+\; \partial_j\phi\,\partial^+\,\psi\Big) \;+\; \partial_l\,\Big(\,\mathrm{e}^{-\,\psi}\,\gamma^{kl}\,\partial^+\,\gamma_{jk}\,\Big)\,\bigg\}\,\Bigg]\;, \end{aligned} \tag{C-5}$$

while $R_{--} = 0$ gives

$$2\,\partial^+\,\phi\,\partial^+\,\psi \;-\; 2\,{\partial^+}^2\,\psi \;-\; \left(\partial^+\,\psi\right)^2 \;+\; \frac{1}{2}\,\partial^+\,\gamma^{kl}\,\partial^+\,\gamma_{kl} \;=\; 0 \;. \tag{C-6}$$

We solve equation (C-6) by choosing

$$\phi \;=\; \frac{1}{2}\,\psi \;. \tag{C-7}$$

This constitutes our fourth gauge choice. Equation (C-6) yields

$$\psi = \frac{1}{4}\frac{1}{\partial^{+2}}\left[\partial^+\gamma^{ij}\partial^+\gamma_{ij}\right]. \tag{C-8}$$

The action in light cone gauge

The LC_2 Lagrangian for gravity is

$$L = \frac{1}{2\kappa^2}\sqrt{-g}\left(2g^{+-}R_{+-} + g^{ij}R_{ij}\right) \ . \tag{C-9}$$

Based on the metric choices, this becomes

$$\begin{aligned} L \ = \ & \frac{1}{2\kappa^2}\Bigg(\mathrm{e}^\psi\Big(\partial^-\partial^+\phi + \partial^+\partial^-\psi - \frac{1}{2}\partial^-\gamma^{ij}\partial^+\gamma_{ij}\Big) \\ & -\mathrm{e}^\phi\gamma^{ij}\Big(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\Big) \\ & -\frac{1}{2}\mathrm{e}^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial^+}R_i\frac{1}{\partial^+}R_j\Bigg), \end{aligned} \tag{C-10}$$

where

$$\begin{aligned} R_i \ = \ & \mathrm{e}^\psi\left(\frac{1}{2}\partial^+\gamma^{jk}\partial_i\gamma_{jk} - \partial^+\partial_i\phi - \partial_i\partial^+\psi + \partial_i\phi\partial^+\psi\right) \\ & +\partial_k\left(\mathrm{e}^\psi\gamma^{jk}\partial^+\gamma_{ij}\right). \end{aligned} \tag{C-11}$$

Perturbative expansion

In order to obtain a perturbative expansion of the metric, we set

$$\gamma_{ij} \ = \ \left(\mathrm{e}^{\kappa H}\right)_{ij} \ , \tag{C-12}$$

$$H \ = \ \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix} \ . \tag{C-13}$$

The physical graviton is represented by h and $\bar{h}$ which were defined in equation (3). In terms of these fields,

$$\psi = -\frac{1}{\partial^{+2}}\left[\partial^+h\partial^+\bar{h}\right] + \mathcal{O}(h^4) \ . \tag{C-14}$$

At this order, we ignore terms of order $\mathcal{O}(h^4)$ and beyond in ψ. The Lagrangian (C-10) is then

$$L = \frac{1}{2} h \,\Box\, \bar{h} \tag{C-15}$$

$$+\kappa\, \bar{h}\, {\partial^+}^2 \left[-h \frac{\bar{\partial}^2}{{\partial^+}^2} h + \frac{\bar{\partial}}{\partial^+} h \frac{\bar{\partial}}{\partial^+} h \right] + c.c. \tag{C-16}$$

$$\begin{aligned}
&+\kappa^2 \Bigg[\frac{1}{{\partial^+}^2} \left[\partial^+ h \partial^+ \bar{h}\right] \, \frac{\partial\bar{\partial} + 2\,\partial^-\partial^+}{{\partial^+}^2} \left[\partial^+ h \partial^+ \bar{h}\right] \\
&\quad - \frac{1}{{\partial^+}^2} \left[\partial^+ h \partial^+ \bar{h}\right] \left(\partial^- h \partial^+ \bar{h} + \partial^+ h \partial^- \bar{h}\right) \\
&\quad + \frac{1}{3} \partial^- \bar{h} \left(h\, \bar{h} \partial^+ h - h^2 \partial^+ \bar{h}\right) + \frac{1}{3} \partial^- h \left(h\, \bar{h} \partial^+ \bar{h} - \bar{h}^2 \partial^+ h\right) \\
&\quad - \frac{1}{{\partial^+}^2} \left[\partial^+ h \partial^+ \bar{h}\right] \Big(2\partial\bar{\partial} h\, \bar{h} + 2 h \partial\bar{\partial}\bar{h} + 9 \bar{\partial} h \partial \bar{h} + \partial h \bar{\partial} \bar{h} \\
&\quad -2 \frac{\partial\bar{\partial}}{\partial^+} h\, \partial^+ \bar{h} - 2\partial^+ h \frac{\partial\bar{\partial}}{\partial^+} \bar{h} + 3 \frac{1}{\partial^+} \left[\partial\bar{\partial} h\, \partial^+ \bar{h} + \partial^+ h \partial\bar{\partial}\bar{h}\right] \Big) \\
&\quad - h\, \bar{h} \left(\frac{4}{3} \partial\bar{\partial} h\, \bar{h} + \frac{4}{3} h \partial\bar{\partial}\bar{h} + 2\, \bar{\partial} h \partial \bar{h} \bar{\partial} \bar{h} + 4 \frac{\partial\bar{\partial}}{\partial^+} h\, \partial^+ \bar{h} + 4 \partial^+ h \frac{\partial\bar{\partial}}{\partial^+} \bar{h} \right) \\
&\quad - 2 \frac{1}{\partial^+} \left[2\, \bar{\partial} h\, \partial^+ \bar{h} + h \partial^+ \bar{\partial} \bar{h} - \partial^+ \bar{\partial} h \bar{h}\right] \, h\, \partial \bar{h} \\
&\quad - 2 \frac{1}{\partial^+} \left[2\, \partial^+ h\, \partial \bar{h} + \partial^+ \partial h\, \bar{h} - h \partial^+ \partial \bar{h}\right] \, \bar{\partial} h\, \bar{h} \qquad\qquad \text{(C-17)} \\
&\quad - \frac{1}{\partial^+} \left[2 \bar{\partial} h\, \partial^+ \bar{h} + h \partial^+ \bar{\partial} \bar{h} - \partial^+ \bar{\partial} h \bar{h}\right] \frac{1}{\partial^+} \left[2 \partial^+ h\, \partial \bar{h} + \partial^+ \partial h\, \bar{h} - h \partial^+ \partial \bar{h}\right] \Bigg] \, .
\end{aligned}$$

We then shift all the light-cone time-derivatives ∂^- that occur in this action to higher orders in κ. This is achieved by means of the field redefinition

$$\begin{aligned}
h \quad \rightarrow \quad & h - \kappa^2 \frac{1}{\partial^+} \Bigg[2\, {\partial^+}^2 h\, \frac{1}{{\partial^+}^3} \left[\partial^+ h \partial^+ \bar{h}\right] + \partial^+ h\, \frac{1}{{\partial^+}^2} \left[\partial^+ h \partial^+ \bar{h}\right] \\
& + \frac{1}{3} \partial^- \bar{h} \left(h\, \bar{h} \partial^+ h - h^2 \partial^+ \bar{h}\right) \Bigg] \, .
\end{aligned} \tag{C-18}$$

The shifted Lagrangian

After the shift, the LC_2 Lagrangian for pure gravity reads

$$
\begin{aligned}
L^{\kappa^2} = \; & \frac{1}{{\partial^+}^2}\left[\partial^+ h \partial^+ \bar{h}\right] \frac{\partial\bar{\partial}}{{\partial^+}^2}\left[\partial^+ h \partial^+ \bar{h}\right] \\
& + \frac{1}{{\partial^+}^3}\left[\partial^+ h \partial^+ \bar{h}\right]\left(\partial\bar{\partial} h\, \partial^+ \bar{h} + \partial^+ h \partial\bar{\partial}\bar{h}\right) \\
& - \frac{1}{{\partial^+}^2}\left[\partial^+ h \partial^+ \bar{h}\right]\, \Big(2\,\partial\bar{\partial} h\, \bar{h} + 2\, h \partial\bar{\partial}\bar{h} + 9\,\bar{\partial} h \partial \bar{h} + \partial h \bar{\partial}\bar{h} \\
& \qquad\qquad\qquad\qquad - \frac{\partial\bar{\partial}}{\partial^+} h\, \partial^+ \bar{h} - \partial^+ h \frac{\partial\bar{\partial}}{\partial^+}\bar{h}\Big) \\
& - 2\frac{1}{\partial^+}\left[2\bar{\partial} h\, \partial^+ \bar{h} + h \partial^+ \bar{\partial}\bar{h} - \partial^+ \bar{\partial} h \bar{h}\right]\, h\, \partial\bar{h} \\
& - 2\frac{1}{\partial^+}\left[2\partial^+ h\, \partial\bar{h} + \partial^+ \partial h\, \bar{h} - h \partial^+ \partial\bar{h}\right]\, \bar{\partial} h\, \bar{h} \\
& - \frac{1}{\partial^+}\left[2\bar{\partial} h\, \partial^+ \bar{h} + h \partial^+ \bar{\partial}\bar{h} - \partial^+ \bar{\partial} h \bar{h}\right] \frac{1}{\partial^+}\left[2\partial^+ h\, \partial\bar{h} + \partial^+ \partial h\, \bar{h} - h \partial^+ \partial\bar{h}\right] \\
& - h\, \bar{h}\, \left(\partial\bar{\partial} h\, \bar{h} + h \partial\bar{\partial}\bar{h} + 2\,\bar{\partial} h \partial\bar{h} + 3\frac{\partial\bar{\partial}}{\partial^+} h\, \partial^+ \bar{h} + 3\partial^+ h \frac{\partial\bar{\partial}}{\partial^+}\bar{h}\right)\;. \qquad \text{(C-19)}
\end{aligned}
$$

Notice that the shift is unique. Any additional shift in h will reintroduce the ∂^- (from the kinetic term) into the action.

Appendix D

The quartic interaction

Assuming that our conjecture is valid to order κ^2, the form of X (from Section **5**) ought to be

$$\int \overline{\mathcal{W}}^0 \, \frac{1}{\partial^{+3}} \, \mathcal{W}^{\kappa^2} \ . \tag{D-1}$$

This is indeed what we find. The explicit expression for X is shown below and proves that the conjecture holds to this order. The Mathematica code used in these computations is available for download from this site:

http://www.aei.mpg.de/∼harald/supergravity.html

$$X \;=\; L_A \;+\; L_B \;+\; L_C \ . \tag{D-2}$$

We use A, B and C to denote the three classes of terms that occur in the general ansatz (34). Subscripts $_1$ and $_2$ differentiate between terms that have an overall $\frac{1}{\partial^{+3}}$ and terms that have an overall $\frac{1}{\partial^{+}}$ (acting on the first two superfields).

$$L_{A,1} = \frac{3}{4} \, \frac{1}{\partial^{+3}} \Big(\partial^+ \phi \, \partial^+ \bar{q}_+ \phi \Big) \partial^{+4} \partial \bar{\phi} \, \frac{1}{\partial^{+4}} q_+ \bar{\partial} \bar{\phi} \tag{D-3}$$

$$L_{A,2} = \frac{53}{12}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ \frac{1}{\partial^+}\bar{q}_+\phi\Big)\ {\partial^+}^6\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-4}$$

$$+\frac{265}{12}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ \bar{q}_+\phi\Big)\ {\partial^+}^5\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-5}$$

$$+\frac{265}{6}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ \partial^+\bar{q}_+\phi\Big)\ {\partial^+}^4\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-6}$$

$$+\frac{265}{6}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ {\partial^+}^2\bar{q}_+\phi\Big)\ {\partial^+}^3\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-7}$$

$$+\frac{265}{12}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ {\partial^+}^3\bar{q}_+\phi\Big)\ {\partial^+}^2\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-8}$$

$$+\frac{53}{12}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\phi\ {\partial^+}^4\bar{q}_+\phi\Big)\ \partial^+\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-9}$$

$$+\frac{119}{12}\ \frac{1}{\partial^+}\Big(\phi\ \frac{1}{\partial^+}\bar{q}_+\phi\Big)\ {\partial^+}^5\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-10}$$

$$+\frac{358}{9}\ \frac{1}{\partial^+}\Big(\phi\ \bar{q}_+\phi\Big)\ {\partial^+}^4\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-11}$$

$$+\frac{553}{9}\ \frac{1}{\partial^+}\Big(\phi\ \partial^+\bar{q}_+\phi\Big)\ {\partial^+}^3\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-12}$$

$$+\frac{121}{3}\ \frac{1}{\partial^+}\Big(\phi\ {\partial^+}^2\bar{q}_+\phi\Big)\ {\partial^+}^2\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-13}$$

$$+\frac{373}{36}\ \frac{1}{\partial^+}\Big(\phi\ {\partial^+}^3\bar{q}_+\phi\Big)\ \partial^+\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-14}$$

$$+\frac{1}{9}\ \frac{1}{\partial^+}\Big(\phi\ {\partial^+}^4\bar{q}_+\phi\Big)\ \partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-15}$$

$$+\frac{193}{36}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ \frac{1}{\partial^+}\bar{q}_+\phi\Big)\ {\partial^+}^4\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-16}$$

$$+\frac{305}{18}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ \bar{q}_+\phi\Big)\ {\partial^+}^3\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-17}$$

$$+\frac{67}{4}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ \partial^+\bar{q}_+\phi\Big)\ {\partial^+}^2\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-18}$$

$$+\frac{107}{18}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ {\partial^+}^2\bar{q}_+\phi\Big)\ \partial^+\partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-19}$$

$$+\frac{1}{9}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ {\partial^+}^3\bar{q}_+\phi\Big)\ \partial\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-20}$$

$$L_{B,1} = -\frac{3}{2}\ \frac{1}{{\partial^+}^3}\Big(\partial^+\bar{q}_+\phi\ \partial\phi\Big){\partial^+}^5\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-21}$$

$$-\frac{1}{{\partial^+}^3}\Big(\bar{q}_+\phi\ \partial^+\partial\phi\Big){\partial^+}^5\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-22}$$

$$-\frac{1}{2}\ \frac{1}{{\partial^+}^3}\Big(\partial^+\bar{q}_+\phi\ \partial^+\partial\phi\Big){\partial^+}^4\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-23}$$

$$-\frac{1}{{\partial^+}^3}\Big({\partial^+}^2\bar{q}_+\phi\ \partial^+\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-24}$$

$$-\frac{1}{{\partial^+}^3}\Big(\partial^+\bar{q}_+\phi\ {\partial^+}^2\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-25}$$

$$L_{B,2} = -\frac{1}{2}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\bar{q}_+\phi\ {\partial^+}^2\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-26}$$

$$-\frac{1}{2}\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\bar{q}_+\phi\ {\partial^+}^3\partial\phi\Big){\partial^+}^2\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-27}$$

$$+\frac{3}{2}\ \frac{1}{\partial^+}\Big(\bar{q}_+\phi\ \partial^+\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-28}$$

$$-\frac{1}{2}\ \frac{1}{\partial^+}\Big(\bar{q}_+\phi\ {\partial^+}^2\partial\phi\Big){\partial^+}^2\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-29}$$

$$-\frac{1}{2}\ \frac{1}{\partial^+}\Big(\partial^+\bar{q}_+\phi\ \partial^+\partial\phi\Big){\partial^+}^2\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-30}$$

$$+\ \frac{1}{\partial^+}\Big(\frac{1}{\partial^+}\bar{q}_+\phi\ \partial\phi\Big){\partial^+}^5\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-31}$$

$$L_{C,1} = -\frac{3}{2}\ \frac{1}{{\partial^+}^3}\Big(\phi\ \partial^+\bar{q}_+\partial\phi\Big){\partial^+}^5\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-32}$$

$$-\frac{11}{4}\ \frac{1}{{\partial^+}^3}\Big(\partial^+\phi\ \partial\bar{q}_+\phi\Big){\partial^+}^5\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-33}$$

$$-\frac{1}{2}\ \frac{1}{{\partial^+}^3}\Big(\partial^+\phi\ \partial^+\bar{q}_+\partial\phi\Big){\partial^+}^4\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-34}$$

$$-\ \frac{1}{{\partial^+}^3}\Big(\partial^+\phi\ {\partial^+}^2\bar{q}_+\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-35}$$

$$-\ \frac{1}{{\partial^+}^3}\Big({\partial^+}^2\phi\ \partial^+\bar{q}_+\partial\phi\Big){\partial^+}^3\bar{\phi}\ \frac{1}{{\partial^+}^4}q_+\bar{\partial}\bar{\phi} \tag{D-36}$$

$$
\begin{aligned}
L_{C,2} = &-\frac{1}{6}\ \frac{1}{\partial^+}\Big(\partial^+\phi\ \partial^+\bar{q}_+\partial\phi\Big)\partial^{+2}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-37)}\\
&-\frac{13}{6}\ \frac{1}{\partial^+}\Big(\partial^{+2}\phi\ \frac{1}{\partial^+}\bar{q}_+\partial\phi\Big)\partial^{+3}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-38)}\\
&+\frac{1}{6}\ \frac{1}{\partial^+}\Big(\partial^{+2}\phi\ \bar{q}_+\partial\phi\Big)\partial^{+2}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-39)}\\
&-\frac{1}{6}\ \frac{1}{\partial^+}\Big(\partial^{+2}\phi\ \partial^+\bar{q}_+\partial\phi\Big)\partial^+\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-40)}\\
&-\frac{179}{18}\ \frac{1}{\partial^+}\Big(\partial^{+3}\phi\ \frac{1}{\partial^+}\bar{q}_+\partial\phi\Big)\partial^{+2}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-41)}\\
&-\frac{407}{18}\ \frac{1}{\partial^+}\Big(\partial^{+3}\phi\ \bar{q}_+\partial\phi\Big)\partial^+\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-42)}\\
&-\frac{199}{12}\ \frac{1}{\partial^+}\Big(\partial^{+3}\phi\ \partial^+\bar{q}_+\partial\phi\Big)\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-43)}\\
&-\frac{53}{12}\ \frac{1}{\partial^+}\Big(\partial^{+3}\phi\ \partial^{+2}\bar{q}_+\partial\phi\Big)\frac{1}{\partial^+}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-44)}\\
&-\frac{191}{18}\ \frac{1}{\partial^+}\Big(\partial^{+4}\phi\ \frac{1}{\partial^+}\bar{q}_+\partial\phi\Big)\partial^+\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-45)}\\
&-\frac{221}{18}\ \frac{1}{\partial^+}\Big(\partial^{+4}\phi\ \bar{q}_+\partial\phi\Big)\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-46)}\\
&-\frac{53}{12}\ \frac{1}{\partial^+}\Big(\partial^{+4}\phi\ \partial^+\bar{q}_+\partial\phi\Big)\frac{1}{\partial^+}\bar{\phi}\ \frac{1}{\partial^{+4}}q_+\bar{\partial}\bar{\phi} && \text{(D-47)}
\end{aligned}
$$

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