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Global solutions of the Einstein–Maxwell equations in higher dimensions

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Abstract

We consider the Einstein–Maxwell equations in space-dimension n. We point out that the Lindblad–Rodnianski stability proof applies to those equations whatever the space-dimension $n \geqslant 3$. In even spacetime dimension $n + 1 \geqslant 6$, we use the standard conformal method on a Minkowski background to give a simple proof that the maximal globally hyperbolic development of initial data sets which are sufficiently close to the data for Minkowski spacetime and which are Schwarzschildian outside of a compact set lead to geodesically complete spacetimes, with a complete Scri, with a smooth conformal structure, and with the gravitational field approaching the Minkowski metric along null directions at least as fast as $r^{-(n-1)/2}$.

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1. Introduction

There is increasing interest in asymptotically flat solutions of Einstein equations in higher dimensions; see e.g. [2, 10, 14, 15]. The pioneering work of Christodoulou and Klainerman [4] proving the nonlinear stability of four-dimensional Minkowski spacetime uses the Bianchi equations, and therefore does not extend to dimensions larger than four in any obvious way. Now, global existence on $\mathbb{R}^{n+1}$ with $n \geqslant 4$ for small initial data of solutions of quasi-linear wave equations of the type of Einstein's equations in wave coordinates has been proved in [16, 20][4] (see also [3] for odd $n \geqslant 5$), but the analysis there assumes fall-off of initial data near spatial infinity incompatible with the Einstein constraints[5].

In this paper, we point out that the Lindblad–Rodnianski stability argument [21, 22] in space-dimension $n = 3$ can be repeated for all $n \geqslant 3$ for the Einstein–Maxwell system. Thus,

[4] Those works build upon [18, 19]; however, the structure conditions in [18, 19] are not compatible with the Einstein equations.

[5] In [16, 20] compactly supported data are considered. In the theorem for general quasilinear systems given in [3] the initial data are in a Sobolev space which requires fall-off at infinity faster than $r^{-n-3/2}$. This should be compared with a fall-off of $g_{\mu\nu} - \eta_{\mu\nu}$ *not faster* than r^{-n+2} required by the positive energy theorem.

Minkowski spacetime is indeed stable against electro-vacuum nonlinear perturbations in all dimensions $n + 1 \geqslant 4$.

Next, we point out that nonlinear electro-vacuum stability, for initial data which are Schwarzschildian outside of a compact set, can be proved by the standard conformal method on Minkowski spacetime for odd $n \geqslant 5$. As usual, the method gives detailed information on the asymptotic behaviour of the gravitational field, not directly available in the Lindblad–Rodnianski method.

It should be mentioned that in *vacuum*, and in even spacetime dimensions $n + 1 \geqslant 4$, existence of smooth conformal completions has been proved in [1] using the Fefferman-Graham obstruction tensor, for initial data which are stationary outside of a compact set. The argument there is simpler than the Lindblad–Rodnianski method, but less elementary than the standard conformal method presented here. Moreover, the direct conformal method here provides more information about the asymptotics of the fields; however, for hyperboloidal initial data our conditions are more restrictive. In any case, it is not clear whether the argument of [1] generalizes to Einstein–Maxwell equations. (Compare [11, 12] for a completely different approach when $n = 3$.)

2. Nonlinear stability in higher dimensions

Consider the Einstein–Maxwell equations, in spacetime dimension $n + 1$,

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}; \\ D_\mu \mathcal{F}^{\mu\nu} = 0, \end{cases} \tag{2.1}$$

with $T_{\mu\nu} = \frac{1}{4\pi}\big(\mathcal{F}_{\mu\lambda}\mathcal{F}_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu}\mathcal{F}^{\lambda\rho}\mathcal{F}_{\lambda\rho}\big)$ and $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is assumed throughout that $n \geqslant 3$.

We assume that we are given an n-dimensional Riemannian manifold $(\mathscr{S}, \bar{g})$, together with a symmetric tensor K, and initial data $(\bar{A} = \bar{A}_i \, \mathrm{d}x^i, \bar{E} = \bar{E}_i \, \mathrm{d}x^i)$ for the Maxwell field. (Throughout this work, quantities decorated with a bar are pull-backs to the initial data surface $\mathscr{S}$.) For the stability results we will assume that $\mathscr{S} = \mathbb{R}^n$, but this will not be needed for the local existence results. We seek a Lorentzian manifold $(\mathscr{M} = \mathbb{R} \times \mathscr{S}, g)$ with a 1-form field A, satisfying (2.1), such that $\bar{g}$ is the pull-back of g, K is the extrinsic curvature tensor of $\{0\} \times \mathscr{S}$, while $(\bar{A}, \bar{E})$ are the pull-backs to $\mathscr{S}$ of the vector potential $A_\mu \, \mathrm{d}x^\mu$ and of the electric field $\mathcal{F}_{\mu\nu} \, \mathrm{d}x^\mu n^\nu$, where n^μ is the field of unit normals to $\mathscr{S}$.

We assume the following fall-off behaviours as $r = |x|$ tends to infinity, for some constants $\alpha > 0, m$:

$$\forall\, i, j = 1, \ldots, n \begin{cases} \bar{g}_{ij} = \begin{cases} \big(1 + \frac{2m}{r}\big)\delta_{ij} + O(r^{-1-\alpha}), & \text{for} \quad n = 3, \\ \delta_{ij} + O\big(r^{-\frac{n-1}{2}-\alpha}\big), & \text{for} \quad n \geqslant 4, \end{cases} \\ \bar{A}_i = O\big(r^{-\frac{n-1}{2}-\alpha}\big), \\ K_{ij} = O\big(r^{-\frac{n+1}{2}-\alpha}\big), \\ \bar{E}_i = O\big(r^{-\frac{n+1}{2}-\alpha}\big). \end{cases} \tag{2.2}$$

We will, of course, assume that the constraint equations hold,

$$\forall\, i, j = 1, \ldots, n \begin{cases} \bar{R} - K^i_j K^j_i + K^i_i K^j_j = 2\mathcal{F}_{0i}\mathcal{F}_0{}^i + \mathcal{F}_{ij}\mathcal{F}^{ij}, \\ D^j K_{ij} - D_i K^j_j = \mathcal{F}_{0j}\mathcal{F}^j_i, \\ \nabla_i \mathcal{F}^{0i} = 0. \end{cases} \tag{2.3}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$, D is the covariant derivative operator associated with $\bar{g}$ while ∇ is the spacetime covariant derivative.

The following result can be proved by a repetition of the arguments in [21, 22]. We refer the reader to [23, 24] for details and for some information on the asymptotic behaviour of the fields.

Theorem 2.1. *Let $(\mathscr{S} = \mathbb{R}^n, \bar{g}, K, \bar{A}, \bar{E})$, $n \geqslant 3$, be initial data for the Einstein–Maxwell equations (2.1) satisfying (2.2) and (2.3), with ADM mass m, set*

$$N_n = 6 + 2\left[\frac{n+2}{2}\right]. \tag{2.4}$$

Write $\bar{g} = \delta + h_0^0 + h_0^1$ with

$$h^0_{0ij} = \begin{cases} \chi(r)\frac{2M}{r}\delta_{ij} & \text{for} \quad n = 3, \\ 0 & \text{for} \quad n \geqslant 4, \end{cases}$$

for a function $\chi \in C^\infty$ equal to 1 for $r \geqslant 3/4$ and to 0 for $r \leqslant 1/2$. Define[6]

$$\begin{aligned} E_{N_n,\gamma}(0) = \sum_{0\leqslant|I|\leqslant N_n} \big(&\big\|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I h_0^1\big\|^2_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I K\|^2_{L^2} \\ &+ \|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \bar{A}\|^2_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I \bar{E}\|^2_{L^2}\big). \end{aligned} \tag{2.5}$$

There exist constants $\varepsilon_0 > 0$ and $\gamma_0(\varepsilon_0)$, with $\gamma_0(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, such that, for all initial data satisfying

$$\sqrt{E_{N_n,\gamma}(0)} + m \leqslant \varepsilon_0, \tag{2.6}$$

for a certain $\gamma > \gamma_0$, the Cauchy problem described above has a global solution (g, A) defined on $\mathbb{R}^{n+1}$, with $(\mathbb{R}^{n+1}, g)$—geodesically complete. The solution is smooth if the initial data are.

The threshold value N_n in (2.4) arises, essentially, from the n-dimensional Klainerman–Sobolev inequalities [17].

For initial data which are polyhomogeneous, or conformally smooth, at i^0, one expects that the solutions will have a polyhomogeneous conformal completion at $\mathscr{I}$ (smooth, for conformally smooth initial data, in even spacetime dimensions). Unfortunately, the information about the asymptotic behaviour of the fields obtained in the course of the proof of theorem 2.1 does not establish this. In the remainder of our work we address this question, in odd space-dimension $n \geqslant 5$, for a restricted class of initial data. We will use a conformal transformation to both give an alternative, simpler proof of global existence for small data, and obtain information on the asymptotic behaviour.

3. Cauchy problem for the vacuum Einstein equations in wave coordinates

We first recall some well-known facts. We start with the Einstein equations without sources, the Einstein–Maxwell equations are considered in section 7.

The vacuum Einstein equations in wave coordinates on $\mathbb{R}^{n+1}$ constitute a set of quasi-diagonal quasi-linear wave equations for the components $g_{\mu\nu}$ of the spacetime metric g, which we write symbolically as

$$g^{\alpha\beta}\frac{\partial^2 g}{\partial x^\alpha \partial x^\beta} = F(g)\left(\frac{\partial g}{\partial x}\right)^2, \tag{3.1}$$

[6] We take this opportunity to note a misprint in the norm in [23, equation (9)].

where the right-hand side is a quadratic form in the first derivatives of the g with coefficients polynomials in the g and their contravariant associates. We take $\mathbb{R}^n$ as an initial manifold. The spacetime manifold will be a subset V of $\mathbb{R}^{n+1}$. The geometric initial data on $\mathbb{R}^n$ are a metric $\bar{g}$ and a symmetric 2-tensor K.

We suppose also given on $\mathbb{R}^n$ the lapse $\bar{N}$, shift $\bar{\beta}$ as well as their time derivatives, chosen so that the corresponding spacetime metric and its first derivatives satisfy on $\mathbb{R}^n$ the harmonicity conditions $\overline{F^\mu} = 0$. If the initial data $\bar{g}, K$ satisfy the constraints, a solution of the Einstein equations in wave coordinates on V is such that the harmonicity functions $F^\mu := \Box_g x^\mu$ satisfy an homogeneous linear system of wave equations on (V, g), with also $\overline{\partial_t F^\mu} = 0$; hence, $F^\mu = 0$ if (V, g) is globally hyperbolic.

4. Conformal mapping

4.1. Definition

To prove our global existence result we use a mapping $\phi : x \mapsto y$ from the future timelike cone with vertex 0, $I^+_{\eta,x}(0)$, of a Minkowski spacetime, which we denote as $\left(\mathbb{R}^{n+1}_x, \eta_x\right)$, into the past timelike cone with vertex 0 of another Minkowski spacetime, $\left(\mathbb{R}^{n+1}_y, \eta_y\right)$. This map is defined by, with η the diagonal quadratic form $(-1, 1, \ldots, 1)$,

$$\phi : I^+_{\eta,x}(0) \to \mathbb{R}^{n+1}_y \qquad \text{by} \quad x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu} x^\lambda x^\mu}. \tag{4.1}$$

It is easy to check that ϕ is a bijection from $I^+_{\eta,x}(0)$ onto $I^-_{y,\eta}(0)$ with inverse

$$\phi^{-1} : y \mapsto x \qquad \text{by} \quad x^\alpha := \frac{y^\alpha}{\eta_{\lambda\mu} y^\lambda y^\mu}. \tag{4.2}$$

Moreover ϕ is a conformal mapping between Minkowski metrics, it holds that

$$\eta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta = \Omega^{-2} \eta_{\alpha\beta}\, \mathrm{d}y^\alpha\, \mathrm{d}y^\beta \tag{4.3}$$

where Ω is a function defined on all $\mathbb{R}^{n+1}_y$, given by

$$\Omega := \eta_{\alpha\beta} y^\alpha y^\beta. \tag{4.4}$$

This conformal mapping appears to be better adapted to the context of the Einstein equations, which involve constraints, than the Penrose transform used in [3] for general quasi-linear wave equations.

4.2. Transformed equations

We set

$$f_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}. \tag{4.5}$$

We consider $f := (f_{\mu\nu})$ as a *set of scalar functions* on $\mathbb{R}^{n+1}_x$. The Einstein equations in wave coordinates, for the unknowns $f := (f_{\mu\nu})$, are then a quasi-diagonal set of quasi-linear wave equations of the form

$$\eta^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = -(g^{\alpha\beta} - \eta^{\alpha\beta}) \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + F(\eta + f) \left(\frac{\partial f}{\partial x}\right)^2. \tag{4.6}$$

The general relation between the wave operator on scalar functions in two conformal metrics transforms the left-hand side of (4.6) into the following partial differential operator (compare remark 4.1)

$$\eta^{\alpha\beta} \frac{\partial^2\left(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}\right)}{\partial y^\alpha \partial y^\beta} \equiv \Omega^{-\frac{n+3}{2}} \left(\eta^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\right) \circ \phi^{-1}. \tag{4.7}$$

We introduce the following new set of scalar functions on $\mathbb{R}^{n+1}_y$

$$\hat{f} := \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1} \qquad \text{i.e.} \quad \hat{f}_{\mu\nu} := \Omega^{-\frac{n-1}{2}} f_{\mu\nu} \circ \phi^{-1}. \tag{4.8}$$

With this notation, system (4.7) reads

$$\eta^{\alpha\beta} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} = -\Omega^{-\frac{n+3}{2}} \left\{ (g^{\alpha\beta} - \eta^{\alpha\beta}) \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} - F(\eta + f) \left(\frac{\partial f}{\partial x} \right)^2 \right\} \circ \phi^{-1}. \tag{4.9}$$

We note that $g^{\alpha\beta} - \eta^{\alpha\beta}$ is a rational function of $f \equiv (f_{\mu\nu})$ with numerator a linear function of f and with denominator bounded away from zero as long as $g_{\alpha\beta}$ is non-degenerate. Therefore we have

$$(g^{\alpha\beta} - \eta^{\alpha\beta}) \circ \phi^{-1} = \Omega^{\frac{n-1}{2}} \hat{h}^{\alpha\beta}, \tag{4.10}$$

with $\hat{h}^{\alpha\beta}$ a rational function of $\hat{f}$ and $\Omega^{\frac{n-1}{2}}$, with denominator bounded away from zero as long as $g_{\alpha\beta}$ is non-degenerate. Now,

$$\eta^{\alpha\beta} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} = -\Omega^{-2} \hat{h}^{\lambda\mu} \left\{ \frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \right\} \circ \phi^{-1} + \Omega^{-\frac{n+3}{2}} \left\{ F(\eta + f) \left(\frac{\partial f}{\partial x} \right)^2 \right\} \circ \phi^{-1}. \tag{4.11}$$

We use the definitions of Ω and of the mapping ϕ to compute the right-hand side of (4.9). It holds that

$$\frac{\partial \Omega}{\partial y^\alpha} = 2\eta_{\alpha\beta} y^\beta =: 2y_\alpha, \tag{4.12}$$

and

$$A^\alpha_\mu := \frac{\partial y^\alpha}{\partial x^\mu} \circ \phi^{-1} \equiv \Omega \delta^\alpha_\mu - 2y^\alpha y_\mu. \tag{4.13}$$

We see that A^α_μ is bounded on any bounded set of $\mathbb{R}^{n+1}_y$.

Elementary calculus gives for an arbitrary function f on $\mathbb{R}^{n+1}_x$ the identities

$$\frac{\partial f}{\partial x^\mu} \circ \phi^{-1} = A^\alpha_\mu \frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha}, \tag{4.14}$$

and

$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \circ \phi^{-1} = A^\beta_\lambda \frac{\partial}{\partial y^\beta} \left\{ A^\alpha_\mu \frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha} \right\}. \tag{4.15}$$

Hence, by straightforward calculation,

$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \circ \phi^{-1} = B^{\alpha\beta}_{\lambda\mu} \frac{\partial^2 (f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} + C^\alpha_{\lambda\mu} \frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha} \tag{4.16}$$

where the coefficients B and C are bounded on any bounded subset of $\mathbb{R}^{n+1}_y$. They are given by

$$B^{\alpha\beta}_{\lambda\mu} := A^{(\alpha}_\mu A^{\beta)}_\lambda, \tag{4.17}$$

i.e.

$$B^{\alpha\beta}_{\lambda\mu} \equiv \Omega^2 \delta^{(\beta}_\lambda \delta^{\alpha)}_\mu - 2\Omega \big(y^{(\beta} y_\lambda \delta^{\alpha)}_\mu + y^{(\alpha} y_\mu \delta^{\beta)}_\lambda \big) + 4y^\alpha y^\beta y_\lambda y_\mu \tag{4.18}$$

and

$$C^\alpha_{\lambda\mu} \equiv -2\Omega \big(y_\lambda \delta^\alpha_\mu + y_\mu \delta^\alpha_\lambda + y^\alpha \eta_{\lambda\mu} \big) + 8y^\alpha y_\lambda y_\mu \big). \tag{4.19}$$

If we now set $f \circ \phi^{-1} = \Omega^k \hat{f}$, we find

$$\frac{\partial(f \circ \phi^{-1})}{\partial y^\alpha} \equiv \frac{\partial(\Omega^k \hat{f})}{\partial y^\alpha} = \Omega^k \frac{\partial \hat{f}}{\partial y^\alpha} + 2k\Omega^{k-1} y_\alpha \hat{f} \tag{4.20}$$

and

$$\frac{\partial^2(f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} \equiv \Omega^k \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} + 2k\Omega^{k-1} \left(y_\beta \frac{\partial \hat{f}}{\partial y^\alpha} + y_\alpha \frac{\partial \hat{f}}{\partial y^\beta} \right) + k\Omega^{k-2} D_{\alpha\beta} \hat{f}, \tag{4.21}$$

with

$$D_{\alpha\beta} := 4(k-1) y_\alpha y_\beta + 2\eta_{\alpha\beta}\Omega. \tag{4.22}$$

The second term on the right-hand side of (4.9) is

$$\Omega^{-\frac{n+3}{2}} \left\{ F(\eta + f) \left(\frac{\partial f}{\partial x} \right)^2 \right\} \circ \phi^{-1} \equiv \Omega^{2k-2-\frac{n+3}{2}} F\left(\eta + \Omega^{\frac{n-1}{2}} \hat{f}\right) \left[A^\alpha_\mu \left(\Omega \frac{\partial \hat{f}}{\partial y^\alpha} + 2k y_\alpha \hat{f} \right) \right]^2. \tag{4.23}$$

Now, $A^\alpha_\mu y_\alpha = -\Omega y_\mu$, and recalling that $k = \frac{n-1}{2}$, the right-hand side of the last equation can be rewritten as

$$\Omega^{\frac{n-5}{2}} F\left(\eta + \Omega^{\frac{n-1}{2}} \hat{f}\right) \left[A^\alpha_\mu \frac{\partial \hat{f}}{\partial y^\alpha} - (n-1) y_\mu \hat{f} \right]^2. \tag{4.24}$$

This shows that this term extends smoothly, as long as the metric $\eta + \Omega^{\frac{n-1}{2}} \hat{f}$ is non-degenerate, to a smooth system on $\mathbb{R}^{n+1}_y$ if $n \geqslant 5$ and n is odd.

The first term on the right-hand side of (4.9) is

$$-\Omega^{-2} \hat{h}^{\lambda\mu} \left\{ B^{\alpha\beta}_{\lambda\mu} \frac{\partial^2(f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} + C^\alpha_{\lambda\mu} \frac{\partial(f \circ \phi^{-1})}{\partial y^\alpha} \right\} \equiv \Omega^{k-4} \hat{h}^{\lambda\mu} \left\{ \Omega^2 B^{\alpha\beta}_{\lambda\mu} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} \right.$$
$$\left. + \left[4k B^{\alpha\beta}_{\lambda\mu} \Omega y_\beta + \Omega^2 C^\alpha_{\lambda\mu} \right] \frac{\partial \hat{f}}{\partial y^\alpha} + \left[B^{\alpha\beta}_{\lambda\mu} k D_{\alpha\beta} + 2k\Omega C^\alpha_{\lambda\mu} y_\alpha \right] \hat{f} \right\}.$$

A similar analysis shows that this again extends smoothly if the metric $\eta + \Omega^{\frac{n-1}{2}} \hat{f}$ is non-degenerate, if $n \geqslant 5$, and if n is odd.

Remark 4.1. The particular case of the conformal covariance of the wave equation that we have used, the identity (4.7), results by a straightforward computation, when $k = \frac{n-1}{2}$, from the obtained identities, together with

$$\eta^{\alpha\beta} \frac{\partial \Omega}{\partial y^\beta} \frac{\partial \Omega}{\partial y^\alpha} \equiv 4\Omega, \qquad \text{and} \qquad \eta^{\alpha\beta} \frac{\partial^2 \Omega}{\partial y^\alpha \partial y^\beta} \equiv 2(n+1). \tag{4.25}$$

We have proved

Proposition 4.2. *The bijection ϕ transforms the Einstein equations in wave coordinates on $I^+_{\eta,x}(0)$ into a quasi-diagonal, quasi-linear system on $I^-_{\eta,y}(0)$ which extends smoothly to $\mathbb{R}^{n+1}_y$, as long as the metric $\eta + \Omega^{\frac{n-1}{2}} \hat{f}$, is non-degenerate, n is odd and satisfies $n \geqslant 5$.*

Proposition 4.2 immediately shows propagation of the decay rate $\Omega^{\frac{n-1}{2}}$ near $\mathscr{I}$ for hyperboloidal initial data which, in wave coordinates, admit a smooth compactification as described above with this decay rate. Further, hyperboloidal initial data in this class which are sufficiently close to the Minkowskian ones lead to solutions which are global to the future of a hyperboloidal surface by the usual stability results. In the remainder of this paper, we will show that proposition 4.2 can also be used to construct solutions which are global both to the past and to the past.

5. The local Cauchy problem in $\mathbb{R}_x^{n+1}$

5.1. Initial data

We consider the Cauchy problem on $\mathbb{R}_x^{n+1}$ for the Einstein equations in wave coordinates with initial data $(\bar{g}, K)$ on a manifold $\mathbb{R}^n$ embedded as a submanifold

$$M_x := \{x^0 = 2\lambda\}$$

of $\mathbb{R}_x^{n+1}$. We suppose these initial data satisfy the Einstein constraints, and we choose the initial lapse and shift so that the harmonicity conditions are everywhere initially zero. Any globally hyperbolic solution $\left(\mathcal{V}_x \subset \mathbb{R}_x^{n+1}, g\right)$ of the Einstein equations in wave coordinates taking these initial data is then a solution of the full Einstein equations. Such a solution exists if the initial data belong to $H_{s+1}^{\mathrm{loc}} \times H_s^{\mathrm{loc}}$, $s > \frac{n}{2}$.

For our global existence theorem we suppose that the initial data coincide with a time-symmetric Schwarzschild initial data set outside a ball B_{R_x} of radius $r = R_x$. Large families of such initial data sets have been constructed in [6, 8], arbitrarily close to those for Minkowski spacetime. (The construction of such data there is done for time-symmetric initial data sets, but there is no *a priori* reason known why all such initial data sets should have vanishing K.[7]) We will use this fact as in previous works [1, 5, 9], but here with the Schwarzschild metric in wave coordinates.

By the general uniqueness theorem for quasi-linear wave equations, the solution $(\mathcal{V}_x, g)$ coincides with the Schwarzschild spacetime, in wave coordinates, in the domain of dependence of the Schwarzschild region, $M_x \backslash B_{R_x}$.

5.2. The $(n+1)$-dimensional Schwarzschild metric in wave coordinates

The Schwarzschild metric g_m with mass parameter m, in any dimension $n \geqslant 3$ is in standard coordinates

$$g_m = -\left(1 - \frac{2m}{\bar{r}^{n-2}}\right)\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{1 - \frac{2m}{\bar{r}^{n-2}}} + \bar{r}^2\,\mathrm{d}\Omega^2 \tag{5.1}$$

where $\mathrm{d}\Omega^2$ is the round unit metric on S^{n-1}. Introduce a new coordinate system $x^i = r(\bar{r})n^i$, with $n^i \in S^n$; the requirement that $x^\mu = (t, x^i)$ be wave coordinates, $\Box_g x^\mu = 0$, is equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left[\bar{r}^{n-1}\left[1 - \frac{2m}{\bar{r}^{n-2}}\right]\frac{\mathrm{d}r}{\mathrm{d}\bar{r}}\right] = (n-1)\bar{r}^{n-3}r.$$

Setting $\rho = 1/\bar{r}$, one obtains an equation with a Fuchsian singularity at $\rho = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^{3-n}(1 - 2m\rho^{n-2})\frac{\mathrm{d}r}{\mathrm{d}\rho}\right] = (n-1)\rho^{1-n}r.$$

The characteristic exponents are -1 and $n-1$ so that, after matching a few leading coefficients, the standard theory of such equations provides solutions with the behaviour

$$r = \bar{r} - \frac{m}{(n-2)\bar{r}^{n-3}} + \begin{cases} \frac{m^2}{4}\bar{r}^{-3}\ln\bar{r} + O(\bar{r}^{-5}\ln\bar{r}), & n = 4 \\ O(\bar{r}^{5-2n}), & n \geqslant 5. \end{cases}$$

[7] A trivial example of initial data set with nonzero K is obtained by moving the initial data surface in a time-symmetric spacetime, Schwarzschildian outside of a compact set. This example raises the interesting question: is it true that vacuum, maximal globally hyperbolic spacetimes which contain a Cauchy surface which is Schwarzschildian at infinity necessarily contain a totally geodesic Cauchy surface?

Somewhat surprisingly, we find logarithms of $\bar{r}$ in an asymptotic expansion of r in dimension $n = 4$. However, for $n \geqslant 5$ there is a complete expansion of $r - \bar{r}$ in terms of inverse powers of $\bar{r}$, without any logarithmic terms: if we write g_m in the coordinates x^i, then

$$(g_m)_{\mu\nu} = \eta_{\mu\nu} + (f_m)_{\mu\nu} \tag{5.2}$$

with the functions $(f_m)_{\mu\nu}$ of the form

$$(f_m)_{\mu\nu} = \frac{1}{r^{n-2}} h_{\mu\nu}\left(\frac{1}{r}, \frac{\vec{x}}{r}\right), \qquad \vec{x} := (x^i), \tag{5.3}$$

with $h_{\mu\nu}(s, \vec{w})$ analytic functions of their arguments near $s = 0$. In fact, there exist functions $h_{00}(s)$, $h(s)$ and $\hat{h}(s)$, analytic near $s = 0$, such that

$$g_m = \left(-1 + \frac{h_{00}(r^{-1})}{r^{n-2}}\right)(\mathrm{d}x^0)^2 + \left[\left(1 + \frac{h(r^{-1})}{r^{n-2}}\right)\delta_{ij} + \frac{\hat{h}(r^{-1})}{r^{n-2}}\frac{x^i x^j}{r^2}\right]\mathrm{d}x^i\,\mathrm{d}x^j.$$

As $\mathbb{R}^n$ is spin, we will necessarily have $m \geqslant 0$ for the initial data that we consider.

5.3. Domain of dependence of the Schwarzschild initial data

The boundary $\dot{\mathcal{D}}^+_x\left(M_x \setminus B_{R_x}\right)$ of the future domain of dependence $\mathcal{D}^+_x\left(M_x \setminus B_{R_x}\right)$ is threaded by null radial outgoing geodesics of the Schwarzschild metric issued from $\dot{B}_{R_x}$, solutions the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \left(\frac{g_{m,rr}}{g_{m,tt}}\right)^{\frac{1}{2}} \tag{5.4}$$

such that $t(R_x) = 2\lambda$, i.e.

$$x^0 \equiv t = 2\lambda + \int_{R_x}^{r}\left(\frac{g_{m,rr}}{g_{m,tt}}\right)^{\frac{1}{2}}\mathrm{d}r. \tag{5.5}$$

For the Schwarzschild metric we have

$$\left|\left(\frac{g_{m,rr}}{g_{m,tt}}\right)^{\frac{1}{2}} - 1\right| = \left|\frac{2m}{r^{n-2}\left(1 - \frac{2m}{r^{n-2}}\right)}\right| \leqslant \frac{4m}{r^{n-2}} \tag{5.6}$$

for $r \geqslant R_x$ provided that

$$\frac{2m}{R_x^{n-2}} \leqslant 1/2; \tag{5.7}$$

at fixed R_x this can be achieved by requiring m to be sufficiently small; alternatively at given m we can increase R_x. For $n \geqslant 5$ we deduce from (5.5) that on $\dot{\mathcal{D}}^+_x\left(M_x \setminus B_{R_x}\right)$ we have

$$b \geqslant t - r \geqslant a, \tag{5.8}$$

with

$$b := 2\lambda - R_x + \frac{4m}{(n-3)R_x^{n-3}}, \qquad a := 2\lambda - R_x - \frac{4m}{(n-3)R_x^{n-3}}. \tag{5.9}$$

We can always choose λ large enough so that $a > 0$, requiring e.g. $2\lambda > R_x + 1$ for $R_x \geqslant 1$ (assuming (5.7)), which implies that $\dot{\mathcal{D}}^+\left(M_x \setminus B_{R_x}\right)$ is interior to $I^+_{\eta,x}(0)$.

6. The global Cauchy problem

The local Cauchy problem for the system (4.9) on $M_y := \{y^0 = -\frac{1}{2\lambda}\}$, with initial data in $H_{s+1} \times H_s$, $s > \frac{n}{2}$ in a ball of radius $R_y > \frac{1}{2\lambda}$, and $\eta + \hat{f}_{|M_y}$ a non-degenerate Lorentzian metric, has a solution $\hat{f}$ in a neighbourhood of

$$\mathcal{D}_y := I^-_{\eta,y}(0) \cap \left\{ y^0 \geqslant -\frac{1}{2\lambda} \right\}, \tag{6.1}$$

if the initial data are small enough. The Einstein equations in wave coordinates on $\mathbb{R}^{n+1}_x$ have then a solution on

$$\mathcal{D}_x := \phi^{-1}(\mathcal{D}_y) \equiv I^+_{\eta,x}(0) \cap x^0 \geqslant \lambda + \sqrt{\lambda^2 + r^2}, \tag{6.2}$$

$\mathcal{D}_x$ is the whole future of the hyperboloid $x^0 = \lambda + \sqrt{\lambda^2 + r^2}$, which is the image by ϕ^{-1} of the hyperplane $y^0 = -\frac{1}{2\lambda}$.

The initial data on M_y are deduced by the mapping ϕ from the values on the hyperboloid

$$H_x := \{x^0 = \lambda + \sqrt{r^2 + \lambda^2}\}$$

of the local solution in $\mathbb{R}^{n+1}_x$ and its first derivative, if H_x is included in the domain $\mathcal{V}_x$ of its existence.

The hyperboloid H_x is the union of two subsets

$$S_1 := H_x \cap \{x^0 - r \geqslant a\} \qquad \text{and} \qquad S_2 := H_x \cap \{x^0 - r \leqslant a\}. \tag{6.3}$$

The subset S_2 is included in the Schwarzschild spacetime region. On the subset S_1 it holds that

$$\lambda + \sqrt{r^2 + \lambda^2} \geqslant r + a. \tag{6.4}$$

A simple computation shows that r is bounded on S_1 if $\lambda > R_x$ and m is small, because, using the value of a, one finds that

$$r \leqslant \frac{R_x(2\lambda - R_x) + O(m^2)}{2(\lambda - R_x - O(m))}, \tag{6.5}$$

then x^0 is also bounded on S_1. This subset is therefore included in the existence domain $\mathcal{V}_x$ of the local solution with Cauchy data on M_x, for small enough Cauchy data.

We deduce from these results that, for small enough Cauchy data (including small m) on M_x the domain $\mathcal{V}_x$ of the solution contains the future of $x^0 = 2\lambda$, up to and including H_x.

On $\phi(H_x \cap S_2)$ the initial data $\hat{f}_2$ for $\hat{f}$ are deduced from the Schwarzschild metric in wave coordinates, which is static, we have

$$\hat{f}_2(\overrightarrow{y}) = \left[\Omega^{-\frac{n-1}{2}}(f_m \circ \phi^{-1})\right]\left(y^0 = -\tfrac{1}{2}, \overrightarrow{y}\right), \tag{6.6}$$

with, using the expression of f_m

$$f_m \circ \phi^{-1}(\overrightarrow{y}) = \frac{\Omega^{n-2}}{|\overrightarrow{y}|^{n-2}} h\left(\frac{\Omega}{|\overrightarrow{y}|}, \frac{\overrightarrow{y}}{|\overrightarrow{y}|}\right). \tag{6.7}$$

We see that $\hat{f}_2$ is a smooth function of y (since the origin $\overrightarrow{y} = 0$ does not belong to the domain $\phi(H_x \cap S_2)$), if $n \geqslant 3$. A simple calculation shows that the same is true for $\frac{\partial \hat{f}}{\partial y^0}$ on $\phi(H_x \cap S_2)$ as soon as $n \geqslant 5$.

On $\phi(H_x \cap S_1)$, the initial data $\hat{f}_1$, are deduced from the values on the uniformly spacelike submanifold $H_x \cap S_1$ of the solution in $\mathcal{V}_x$ and its derivative, by the restriction of ϕ to a neighbourhood of $H_x \cap S_1$, where ϕ is a smooth diffeomorphism.

6.1. Conclusions

By general methods used in [13] one can prove that our global solutions are causally geodesically complete. On the other hand, completeness of inextendible spatial geodesics can be established by adapting the arguments given in [22, Prop. 16.2], details can be found in [24]. We have therefore proved the following theorem.

Theorem 6.1. *Let n be odd and $n \geqslant 5$. Let be given on $\mathbb{R}^n$ gravitational data, perturbation of the Minkowski data given by sets of functions $\overline{f_{\mu\nu}}$ and $\overline{\frac{\partial}{\partial x^0} f_{\mu\nu}}$. Suppose that these data satisfy the Einstein constraints and the initial harmonicity conditions and coincide with the Schwarzschild data of mass m in the wave coordinates of section 5.2 outside a ball of finite Euclidean radius R_x. If these functions are small enough in $(H_{s+1} \times H_s)(B(R_x))$ norm, $s > n/2$, then the data admit a geodesically complete Einsteinian development $(\mathbb{R}^{n+1}, g)$.*

The smoothness of $\hat{f}$ as a function of y immediately provides a full asymptotic expansion of g in terms of inverse powers of $r = |\vec{x}|$. In fact, we obtain an asymptotic behaviour of the gravitational field as assumed in [14].

7. Einstein–Maxwell equations

Let us show that the conformal method extends easily to the electro-vacuum case.

7.1. The equations

The Einstein equations with electromagnetic sources (2.1) for the unknowns $f := g - \eta$ take in wave coordinates the form

$$g^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = F_E(g) \left(\frac{\partial f}{\partial x}, \frac{\partial A}{\partial x} \right), \tag{7.1}$$

where the right-hand side is a quadratic form in ∂f and ∂A, with coefficient polynomials in g and its contravariant associate. In the Lorenz gauge, $\nabla_\lambda A^\lambda = 0$, the Maxwell equations take also in wave coordinates the form

$$g^{\alpha\beta} \frac{\partial^2 A}{\partial x^\alpha \partial x^\beta} = F_M(g) \left(\frac{\partial f}{\partial x}, \frac{\partial A}{\partial x} \right), \tag{7.2}$$

where the right-hand side is bilinear in $\frac{\partial f}{\partial x}$ and in $\frac{\partial A}{\partial x}$, with coefficients polynomials in g and its contravariant associate. In fact, (7.2) reads in detail

$$g^{\alpha\beta} \frac{\partial^2 A_\sigma}{\partial x^\alpha \partial x^\beta} = -(\partial_\sigma g^{\mu\alpha}) \partial_\mu A_\alpha - g_{\nu\sigma} g^{\mu\alpha} (\partial_\mu g^{\nu\beta})(\partial_\alpha A_\beta - \partial_\beta A_\alpha); \tag{7.3}$$

this uses both the harmonic coordinates condition and the Lorenz gauge condition. One thus has a system of equations for (f, A) for which the previous analysis applies.

According to Corvino [7], the existence of a large set of non-vacuum electro-vacuum initial data which are Schwarzschildian[8] outside a compact set is also valid for the Einstein–Maxwell constraints; compare [9].

[8] It could appear that a natural generalization in this context is to consider initial data which coincide with those for the $(n + 1)$-dimensional Reissner–Nordström metric outside of a compact set. However, if the initial surface topology is $\mathbb{R}^n$, then the global electric charge is necessarily zero, and one is back in the Schwarzschild case.

7.2. Conclusions

We have obtained the following theorem.

Theorem 7.1. *Let n be odd and $n \geqslant 5$. Let be given on $\mathbb{R}^n$ Einstein–Maxwell data, perturbation of the Minkowski data given by sets of functions $\overline{f_{\mu\nu}}$ and $\overline{\frac{\partial}{\partial x^0} f_{\mu\nu}}$, $\overline{A_\lambda}$ and $\overline{\frac{\partial}{\partial x^0} A_\lambda}$, Suppose that these data satisfy the Einstein–Maxwell constraints and the initial harmonicity and Lorenz gauge conditions, and coincide with the Schwarzschild data with mass m in the wave coordinates of section 5.2 outside a ball of finite Euclidean radius R_x. If these functions are small enough in $(H_{s+1} \times H_s)(B(R_x))$ norm, $s > \frac{n}{2}$, then the data admit a geodesically complete electro-vacuum Einsteinian development $(\mathbb{R}^{n+1}, g, A)$.*

Similarly to theorem 6.1, one obtains a full asymptotic expansion both of the metric and of the Maxwell field in terms of inverse powers of r near Scri.

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