# Global solutions of the Einstein-Maxwell equations in higher dimensions

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**Abstract**

We consider the Einstein-Maxwell equations in space-dimension $n$. We point out that the Lindblad-Rodnianski stability proof applies to those equations whatever the space-dimension $n \geq 3$. In even space-time dimension $n+1 \geq 6$ we use the standard conformal method on a Minkowski background to give a simple proof that the maximal globally hyperbolic development of initial data sets which are sufficiently close to the data for Minkowski space-time and which are Schwarzschildian outside of a compact set lead to geodesically complete space-times, with a complete Scri, with smooth conformal structure, and with the gravitational field approaching the Minkowski metric along null directions at least as fast as $r^{-(n-1)/2}$.

## 1 Introduction

There is increasing interest in asymptotically flat solutions of Einstein equations in higher dimensions, see e.g. [2, 10, 13, 14]. The pioneering work of Christodoulou and Klainerman [4] proving the nonlinear stability of four-dimensional Minkowski spacetime uses the Bianchi equations, and therefore does not extend to dimensions larger than four in any obvious way. Now, global existence on $\mathbb{R}^{n+1}$ with $n \geq 4$ for small initial data of solutions of quasi-linear wave equations of the type of Einstein's equations in wave coordinates has been proved in [15, 19][1], see also [3] for odd $n \geq 5$, but the analysis there assumes fall-off of initial data near spatial infinity incompatible with the Einstein constraints[2].

In this note we point out that the Lindblad-Rodnianski stability argument [20, 21] in space-dimension $n = 3$ can be repeated for all $n \geq 3$ for the Einstein-Maxwell system. Thus, Minkowski space-time is indeed stable against electro-vacuum non-linear perturbations in all dimensions $n+1 \geq 4$.

Next, we point out that non-linear electro-vacuum stability, for initial data which are Schwarzschildian outside of a compact set, can be proved by the standard conformal method on Minkowski space-time for odd $n \geq 5$. As usual, the method gives detailed information on the asymptotic behavior of the gravitational field, not directly available in the Lindblad-Rodnianski method. In retrospect, this result is a simple consequence of the work of Christodoulou [3], though our implementation of Christodoulou's argument does not seem to have been considered in the literature so far.

It should be mentioned that in *vacuum*, and in even space-time dimensions $n+1 \geq 4$, existence of smooth conformal completions has been proved in [1] using the Fefferman-Graham obstruction tensor, for initial data which are stationary outside of a compact set. The argument there is simpler than the Lindblad-Rodnianski method, but less elementary than the standard conformal method presented here. Moreover, the direct conformal method here provides more information about the asymptotics of the fields; however, for hyperboloidal initial data our conditions are more restrictive. In any case, it is not clear whether the argument of [1] generalizes to Einstein-Maxwell equations. (Compare [11, 12] for a completely different approach when $n = 3$.)

## 2 Nonlinear stability in higher dimensions

Consider the Einstein-Maxwell equations, in space-time dimension $n + 1$,

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \; ; \\ D_\mu \mathcal{F}^{\mu\nu} = 0 \; , \end{cases} \tag{2.1}$$

[1] Those works build upon [17, 18]; however the structure conditions in [17, 18] are not compatible with the Einstein equations.

[2] In [15, 19] compactly supported data are considered. In [3] the initial data are in a Sobolev space which requires fall-off at infinity faster than $r^{-n-3/2}$. This should be compared with a fall-off of $g_{\mu\nu} - \eta_{\mu\nu}$ *not faster* than $r^{-n+2}$ required by the positive energy theorem.

with $T_{\mu\nu} = \frac{1}{4\pi}(\mathcal{F}_{\mu\lambda}\mathcal{F}_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}\mathcal{F}^{\lambda\rho}\mathcal{F}_{\lambda\rho})$ and $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is assumed throughout that $n \geq 3$.

We assume that we are given an $n$–dimensional Riemannian manifold $(\mathscr{S}, \bar{g})$, together with a symmetric tensor $K$, and initial data $(\bar{A} = \bar{A}_i dx^i, \bar{E} = \bar{E}_i dx^i)$ for the Maxwell field. (Throughout this work, quantities decorated with a bar are pull-backs to the initial data surface $\mathscr{S}$.) For the stability results we will assume that $\mathscr{S} = \mathbb{R}^n$, but this will not be needed for the local existence results. We seek a Lorentzian manifold $(\mathscr{M} = \mathbb{R} \times \mathscr{S}, g)$ with a one-form field $A$, satisfying (2.1), such that $\bar{g}$ is the pull-back of $g$, $K$ is the extrinsic curvature tensor of $\{0\} \times \mathscr{S}$, while $(\bar{A}, \bar{E})$ are the pull-backs to $\mathscr{S}$ of the vector potential $A_\mu dx^\mu$ and of the electric field $\mathcal{F}_{\mu\nu} dx^\mu n^\nu$, where $n^\mu$ is the field of unit normals to $\mathscr{S}$.

We assume the following fall-off behaviors as $r = |x|$ tends to infinity, for some constants $\alpha > 0$, $m$:

$$\forall\, i,j = 1, ..., n \qquad \begin{cases} \bar{g}_{ij} = \begin{cases} (1 + \frac{2m}{r})\delta_{ij} + O(r^{-1-\alpha})\ , & \text{for } n = 3\ , \\ \delta_{ij} + O(r^{-\frac{n-1}{2}-\alpha})\ , & \text{for } n \geq 4\ , \end{cases} \\ \bar{A}_i = O(r^{-\frac{n-1}{2}-\alpha})\ , \\ K_{ij} = O(r^{-\frac{n+1}{2}-\alpha})\ , \\ \bar{E}_i = O(r^{-\frac{n+1}{2}-\alpha})\ . \end{cases} \tag{2.2}$$

We will, of course, assume that the constraint equations hold:

$$\forall\, i,j = 1, ..., n \qquad \begin{cases} \bar{R} - K^i_j K^j_i + K^i_i K^j_j = 2\mathcal{F}_{0i}\mathcal{F}_0{}^i + \mathcal{F}_{ij}\mathcal{F}^{ij}\ , \\ D^j K_{ij} - D_i K^j_j = \mathcal{F}_{0j}\mathcal{F}_i{}^j \quad , \\ \nabla_i \mathcal{F}^{0i} = 0\ . \end{cases} \tag{2.3}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$, $D$ is the covariant derivative operator associated with $\bar{g}$ while $\nabla$ is the space-time covariant derivative.

The following result can be proved by a repetition of the arguments in [20, 21]. We refer the reader to [22, 23] for details and for some information on the asymptotic behavior of the fields:

THEOREM 2.1 *Let $(\mathscr{S} = \mathbb{R}^n, \bar{g}, K, \bar{A}, \bar{E})$, $n \geq 3$, be initial data for the Einstein-Maxwell equations (2.1) satisfying (2.2) and (2.3), with ADM mass $m$, set*

$$N_n = 6 + 2\left[\frac{n+2}{2}\right]\ . \tag{2.4}$$

*Write $\bar{g} = \delta + h^0_0 + h^1_0$ with*

$$h^0_{0ij} = \begin{cases} \chi(r)\frac{2M}{r}\delta_{ij} & \textit{for } n = 3\ , \\ 0 & \textit{for } n \geq 4\ , \end{cases}$$

*for a function $\chi \in C^\infty$ equal to 1 for $r \geq 3/4$ and to 0 for $r \leq 1/2$. Define*[3]

$$E_{N_n,\gamma}(0) = \sum_{0\leq|I|\leq N_n} \Big( ||(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I h^1_0||^2_{L^2} + ||(1+r)^{1/2+\gamma+|I|}\nabla^I K||^2_{L^2} \tag{2.5}$$
$$+||(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \bar{A}||^2_{L^2} + ||(1+r)^{1/2+\gamma+|I|}\nabla^I \bar{E}||^2_{L^2}\Big) \ .$$

*There exist constants $\varepsilon_0 > 0$ and $\gamma_0(\varepsilon_0)$, with $\gamma_0(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, such that, for all initial data satisfying*

$$\sqrt{E_{N_n,\gamma}(0)} + m \leq \varepsilon_0 \ , \tag{2.6}$$

*for a certain $\gamma > \gamma_0$, the Cauchy problem described above has a global solution $(g, A)$ defined on $\mathbb{R}^{n+1}$, with $(\mathbb{R}^{n+1}, g)$ — geodesically complete. The solution is smooth if the initial data are.*

The threshold value $N_n$ in (2.4) arises, essentially, from the $n$-dimensional Klainerman-Sobolev inequalities [16].

For initial data which are polyhomogeneous, or conformally smooth, at $i^0$, one expects that the solutions will have a polyhomogeneous conformal completion at $\mathscr{I}$ (smooth, for conformally smooth initial data, in even space-time dimensions). Unfortunately, the information about the asymptotic behavior of the fields obtained in the course of the proof of Theorem 2.1 does not establish this. In the remainder of our work we address this question, in odd space-dimension $n \geq 5$, for a restricted class of initial data. We will use a simple conformal transformation to both prove global existence for small data, and obtain information on the asymptotic behavior.

# 3 Cauchy problem for the vacuum Einstein equations in wave coordinates

We first recall some well known facts. We start with the Einstein equations without sources, the Einstein-Maxwell equations are considered in Section 7 below.

The vacuum Einstein equations in wave coordinates on $\mathbb{R}^{n+1}$ constitute a set of quasi-diagonal quasi-linear wave equations for the components $g_{\mu\nu}$ of the spacetime metric $g$, which we write symbolically as

$$g^{\alpha\beta}\frac{\partial^2 g}{\partial x^\alpha \partial x^\beta} = F(g)(\frac{\partial g}{\partial x})^2 \ , \tag{3.1}$$

[3] We take this opportunity to note a misprint in the norm in [22, Eq. (9)].

where the right-hand-side is a quadratic form in the first derivatives of the $g$'s with coefficients polynomials in the $g$'s and their contravariant associates. We take $\mathbb{R}^n$ as an initial manifold. The spacetime manifold will be a subset $V$ of $\mathbb{R}^{n+1}$. The geometric initial data on $\mathbb{R}^n$ are a metric $\bar{g}$ and a symmetric 2-tensor $K$.

We suppose also given on $\mathbb{R}^n$ the lapse $\bar{N}$, shift $\bar{\beta}$ as well as their time derivatives, chosen so that the corresponding spacetime metric and its first derivatives satisfy on $\mathbb{R}^n$ the harmonicity conditions $\overline{F^\mu} = 0$. If the initial data $\bar{g}, K$ satisfy the constraints, a solution of the Einstein equations in wave coordinates on $V$ is such that the harmonicity functions $F^\mu := \Box_g x^\mu$ satisfy an homogeneous linear system of wave equations on $(V, g)$, with also $\overline{\partial_t F^\mu} = 0$, hence $F^\mu = 0$ if $(V, g)$ is globally hyperbolic.

# 4 Conformal mapping

## 4.1 Definition

To prove our global existence result we use a mapping $\phi : x \mapsto y$ from the future timelike cone with vertex 0, $I^+_{\eta,x}(0)$, of a Minkowski spacetime, which we denote $(\mathbb{R}^{n+1}_x, \eta_x)$, into the past timelike cone with vertex 0 of another Minkowski spacetime, $(\mathbb{R}^{n+1}_y, \eta_y)$. This map is defined by, with $\eta$ the diagonal quadratic form $(-1, 1, ..., 1)$,

$$\phi : \quad I^+_{\eta,x}(0) \to \mathbb{R}^{n+1}_y \quad \text{by} \quad x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu} x^\lambda x^\mu} \ . \tag{4.1}$$

It is easy to check that $\phi$ is a bijection from $I^+_{\eta,x}(0)$ onto $I^-_{y,\eta}(0)$ with inverse

$$\phi^{-1} : y \mapsto x \quad \text{by} \quad x^\alpha := \frac{y^\alpha}{\eta_{\lambda\mu} y^\lambda y^\mu} \ . \tag{4.2}$$

Moreover $\phi$ is a conformal mapping between Minkowski metrics, it holds that

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = \Omega^{-2} \eta_{\alpha\beta} dy^\alpha dy^\beta \tag{4.3}$$

where $\Omega$ is a function defined on all $\mathbb{R}^{n+1}_y$, given by

$$\Omega := \eta_{\alpha\beta} y^\alpha y^\beta \ . \tag{4.4}$$

This conformal mapping appears to be better adapted to the context of the Einstein equations, which involve constraints, than the Penrose transform used in [3] for general quasi-linear wave equations.

### 4.2 Transformed equations

We set

$$f_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu} \ . \tag{4.5}$$

We consider $f := (f_{\mu\nu})$ as a *set of scalar functions* on $\mathbb{R}^{n+1}_x$. The Einstein equations in wave coordinates, for the unknowns $f := (f_{\mu\nu})$, are then a quasi-diagonal set of quasi-linear wave equations of the form

$$\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = -(g^{\alpha\beta} - \eta^{\alpha\beta})\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + F(\eta + f)(\frac{\partial f}{\partial x})^2 \ . \tag{4.6}$$

The general relation between the wave operator on scalar functions in two conformal metrics transforms the left-hand-side of (4.6) into the following partial differential operator (compare Remark 4.1 below)

$$\eta^{\alpha\beta}\frac{\partial^2(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} \equiv \Omega^{-\frac{n+3}{2}}(\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}) \circ \phi^{-1} \ . \tag{4.7}$$

We introduce the following new set of scalar functions on $\mathbb{R}^{n+1}_y$

$$\hat{f} := \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1} \ , \quad \text{i.e.} \quad \hat{f}_{\mu\nu} := \Omega^{-\frac{n-1}{2}} f_{\mu\nu} \circ \phi^{-1} \ . \tag{4.8}$$

With this notation, the system (4.7) reads

$$\eta^{\alpha\beta}\frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} = -\Omega^{-\frac{n+3}{2}}\{(g^{\alpha\beta} - \eta^{\alpha\beta})\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} - F(\eta + f)(\frac{\partial f}{\partial x})^2\} \circ \phi^{-1} \ . \tag{4.9}$$

We note that $g^{\alpha\beta} - \eta^{\alpha\beta}$ is a rational function of $f \equiv (f_{\mu\nu})$ with numerator a linear function of $f$ and with denominator bounded away from zero as long as $g_{\alpha\beta}$ is non degenerate. Therefore we have

$$(g^{\alpha\beta} - \eta^{\alpha\beta}) \circ \phi^{-1} = \Omega^{\frac{n-1}{2}} \hat{h}^{\alpha\beta} \ , \tag{4.10}$$

with $\hat{h}^{\alpha\beta}$ a rational function of $\hat{f}$ and $\Omega^{\frac{n-1}{2}}$, with denominator bounded away from zero as long as $g_{\alpha\beta}$ is non degenerate. Now,

$$\eta^{\alpha\beta}\frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} = -\Omega^{-2}\hat{h}^{\lambda\mu}\{\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu}\} \circ \phi^{-1} + \Omega^{-\frac{n+3}{2}}\{F(\eta + f)(\frac{\partial f}{\partial x})^2\} \circ \phi^{-1} \ . \tag{4.11}$$

We use the definitions of $\Omega$ and of the mapping $\phi$ to compute the right-hand-side of (4.9). It holds that:

$$\frac{\partial \Omega}{\partial y^\alpha} = 2\eta_{\alpha\beta} y^\beta := 2y_\alpha \ , \tag{4.12}$$

and
$$A^\alpha_\mu := \frac{\partial y^\alpha}{\partial x^\mu}\circ\phi^{-1} \equiv \Omega\delta^\alpha_\mu - 2y^\alpha y_\mu \ . \tag{4.13}$$
We see that $A^\alpha_\mu$ is bounded on any bounded set of $\mathbb{R}^{n+1}_y$.

Elementary calculus gives for an arbitrary function $f$ on $\mathbb{R}^{n+1}_x$ the identities
$$\frac{\partial f}{\partial x^\mu}\circ\phi^{-1} = A^\alpha_\mu \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha} \ , \tag{4.14}$$
and
$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu}\circ\phi^{-1} = A^\beta_\lambda \frac{\partial}{\partial y^\beta}\{A^\alpha_\mu \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\} \ . \tag{4.15}$$
Hence, by straightforward calculation,
$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu}\circ\phi^{-1} = B^{\alpha\beta}_{\lambda\mu}\frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha\partial y^\beta} + C^\alpha_{\lambda\mu}\frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha} \tag{4.16}$$
where the coefficients $B$ and $C$ are bounded on any bounded subset of $\mathbb{R}^{n+1}_y$. They are given by
$$B^{\alpha\beta}_{\lambda\mu} := A^{(\alpha}_\mu A^{\beta)}_\lambda \ , \tag{4.17}$$
i.e.
$$B^{\alpha\beta}_{\lambda\mu} \equiv \Omega^2\delta^{(\beta}_\lambda\delta^{\alpha)}_\mu - 2\Omega(y^{(\beta}y_\lambda\delta^{\alpha)}_\mu + y^{(\alpha}y_\mu\delta^{\beta)}_\lambda) + 4y^\alpha y^\beta y_\lambda y_\mu \tag{4.18}$$
and
$$C^\alpha_{\lambda\mu} \equiv -2\Omega(y_\lambda\delta^\alpha_\mu + y_\mu\delta^\alpha_\lambda + y^\alpha\eta_{\lambda\mu}) + 8y^\alpha y_\lambda y_\mu) \ . \tag{4.19}$$
If we now set $f\circ\phi^{-1} = \Omega^k\hat f$, we find:
$$\frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha} \equiv \frac{\partial(\Omega^k\hat f)}{\partial y^\alpha} = \Omega^k\frac{\partial\hat f}{\partial y^\alpha} + 2k\Omega^{k-1}y_\alpha\hat f \tag{4.20}$$
and
$$\frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha\partial y^\beta} \equiv \Omega^k\frac{\partial^2\hat f}{\partial y^\alpha\partial y^\beta} + 2k\Omega^{k-1}(y_\beta\frac{\partial\hat f}{\partial y^\alpha} + y_\alpha\frac{\partial\hat f}{\partial y^\beta}) + k\Omega^{k-2}D_{\alpha\beta}\hat f \ , \tag{4.21}$$
with
$$D_{a\beta} := 4(k-1)y_\alpha y_\beta + 2\eta_{\alpha\beta}\Omega \ . \tag{4.22}$$
The second term on the right-hand-side of (4.9) is
$$\Omega^{-\frac{n+3}{2}}\left\{F(\eta+f)(\frac{\partial f}{\partial x})^2\right\}\circ\phi^{-1} \equiv \Omega^{2k-2-\frac{n+3}{2}}F(\eta+\Omega^{\frac{n-1}{2}}\hat f)\left[A^\alpha_\mu(\Omega\frac{\partial\hat f}{\partial y^\alpha} + 2ky_\alpha\hat f)\right]^2 \ . \tag{4.23}$$

Now, $A^\alpha_\mu y_\alpha = -\Omega y_\mu$, and recalling that $k=\frac{n-1}{2}$, the right-hand-side of the last equation can be rewritten as

$$\Omega^{\frac{n-5}{2}} F(\eta + \Omega^{\frac{n-1}{2}}\hat f)\Big[A^\alpha_\mu \frac{\partial \hat f}{\partial y^\alpha} - (n-1) y_\mu \hat f\Big]^2 . \tag{4.24}$$

This shows that this term extends smoothly, as long as the metric $\eta + \Omega^{\frac{n-1}{2}}\hat f$ is non degenerate, to a smooth system on $\mathbb{R}^{n+1}_y$ if $n \geq 5$ and $n$ is odd.

The first term at the right-hand-side of (4.9) is

$$-\Omega^{-2}\hat h^{\lambda\mu}\{B^{\alpha\beta}_{\lambda\mu}\frac{\partial^2 (f\circ\phi^{-1})}{\partial y^\alpha \partial y^\beta} + C^\alpha_{\lambda\mu}\frac{\partial (f\circ\phi^{-1})}{\partial y^\alpha}\} \equiv$$
$$\Omega^{k-4}\hat h^{\lambda\mu}\{\Omega^2 B^{\alpha\beta}_{\lambda\mu}\frac{\partial^2 \hat f}{\partial y^\alpha \partial y^\beta} + [4kB^{\alpha\beta}_{\lambda\mu}\Omega y_\beta + \Omega^2 C^\alpha_{\lambda\mu}]\frac{\partial \hat f}{\partial y^\alpha} + [B^{\alpha\beta}_{\lambda\mu} k D_{\alpha\beta} + 2k\Omega C^\alpha_{\lambda\mu} y_\alpha]\hat f \ .$$

A similar analysis shows that this again extends smoothly if the metric $\eta + \Omega^{\frac{n-1}{2}}\hat f$ is non degenerate, if $n\geq 5$, and if $n$ is odd.

**Remark 4.1** *The particular case of the conformal covariance of the wave equation that we have used, the identity (4.7), results by a straightforward computation, when $k=\frac{n-1}{2}$, from the obtained identities, together with*

$$\eta^{\alpha\beta}\frac{\partial\Omega}{\partial y^\beta}\frac{\partial\Omega}{\partial y^\alpha} \equiv 4\Omega, \quad \textit{and} \quad \eta^{\alpha\beta}\frac{\partial^2\Omega}{\partial y^\alpha\partial y^\beta} \equiv 2(n+1) \ . \tag{4.25}$$

We have proved

**Proposition 4.2** *The bijection $\phi$ transforms the Einstein equations in wave coordinates on $I^+_{\eta,x}(0)$ into a quasi-diagonal, quasi-linear system on $I^-_{\eta,y}(0)$ which extends smoothly to $\mathbb{R}^{n+1}_y$, as long as the metric $\eta + \Omega^{\frac{n-1}{2}}\hat f$, is non degenerate, $n$ is odd and satisfies $n \geq 5$.*

Proposition 4.2 immediately shows propagation of the decay rate $\Omega^{\frac{n-1}{2}}$ near $\mathscr{I}$ for hyperboloidal initial data with this decay rate. Further, hyperboloidal initial data in this class which are sufficiently close to the Minkowskian ones lead to solutions which are global to the future of a hyperboloidal surface by the usual stability results. In the remainder of this paper we will show that Proposition 4.2 can also be used to construct solutions which are global both to the future and to the past.

# 5 The local Cauchy problem in $\mathbb{R}_x^{n+1}$

## 5.1 Initial data

We consider the Cauchy problem on $\mathbb{R}_x^{n+1}$ for the Einstein equations in wave coordinates with initial data $(\bar{g}, K)$ on a manifold $\mathbb{R}^n$ embedded as a submanifold

$$M_x := \{x^0 = 2\lambda\}$$

of $\mathbb{R}_x^{n+1}$. We suppose these initial data satisfy the Einstein constraints, and we choose the initial lapse and shift so that the harmonicity conditions are everywhere initially zero. Any globally hyperbolic solution $(\mathcal{V}_x \subset \mathbb{R}_x^{n+1}, g)$ of the Einstein equations in wave coordinates taking these initial data is then a solution of the full Einstein equations. Such a solution exists if the initial data belong to $H_{s+1}^{\mathrm{loc}} \times H_s^{\mathrm{loc}}$, $s > \frac{n}{2}$.

For our global existence theorem we suppose that the initial data coincide with a time-symmetric Schwarzschild initial data set outside a ball $B_{R_x}$ of radius $r = R_x$. Large families of such initial data sets have been constructed in [6, 8], arbitrarily close to those for Minkowski space-time. (The construction of such data there is done for time-symmetric initial data sets, but there is no *a priori* reason known why all such initial data sets should have vanishing $K$.[4]) We will use this fact as in previous works [1, 5, 9], but here with the Schwarzschild metric in wave coordinates.

By the general uniqueness theorem for quasi-linear wave equations, the solution $(\mathcal{V}_x, g)$ coincides with the Schwarzschild spacetime, in wave coordinates, in the domain of dependence of the Schwarzschild region, $M_x \backslash B_{R_x}$.

## 5.2 The $(n+1)$–dimensional Schwarzschild metric in wave coordinates

The Schwarzschild metric $g_m$ with mass parameter $m$, in any dimension $n \geq 3$ is in standard coordinates

$$g_m = -\left(1 - \frac{2m}{\bar{r}^{n-2}}\right) dt^2 + \frac{d\bar{r}^2}{1 - \frac{2m}{\bar{r}^{n-2}}} + \bar{r}^2 d\Omega^2 \tag{5.1}$$

[4]A trivial example of initial data set with non-zero $K$ is obtained by moving the initial data surface in a time-symmetric space-time, Schwarzschildian outside of a compact set. This example raises the interesting question: is it true that vacuum, maximal globally hyperbolic space-times which contain a Cauchy surface which is Schwarzschildian at infinity necessarily contain a totally geodesic Cauchy surface?

where $d\Omega^2$ is the round unit metric on $S^{n-1}$. Introduce a new coordinate system $x^i = r(\bar r)n^i$, with $n^i \in S^n$; the requirement that $x^\mu = (t, x^i)$ be wave coordinates, $\Box_g x^\mu = 0$, is equivalent to the equation

$$\frac{d}{d\bar r}[\bar r^{n-1}[1 - \frac{2m}{\bar r^{n-2}}]\frac{dr}{d\bar r}] = (n-1)\bar r^{n-3} r \ .$$

Setting $\rho = 1/\bar r$, one obtains an equation with a Fuchsian singularity at $\rho = 0$:

$$\frac{d}{d\rho}[\rho^{3-n}(1 - 2m\rho^{n-2})\frac{dr}{d\rho}] = (n-1)\rho^{1-n} r \ .$$

The characteristic exponents are $-1$ and $n-1$ so that, after matching a few leading coefficients, the standard theory of such equations provides solutions with the behavior

$$r = \bar r - \frac{m}{(n-2)\bar r^{n-3}} + \begin{cases} \frac{m^2}{4}\bar r^{-3}\ln \bar r + O(\bar r^{-5}\ln \bar r), & n = 4 \\ O(\bar r^{5-2n}), & n \geq 5 \end{cases}$$

Somewhat surprisingly, we find logarithms of $\bar r$ in an asymptotic expansion of $r$ in dimension $n = 4$. However, for $n \geq 5$ there is a complete expansion of $r - \bar r$ in terms of inverse powers of $\bar r$, without any logarithmic terms: if we write $g_m$ in the coordinates $x^i$, then

$$(g_m)_{\mu\nu} = \eta_{\mu\nu} + (f_m)_{\mu\nu} \qquad (5.2)$$

with the functions $(f_m)_{\mu\nu}$ of the form

$$(f_m)_{\mu\nu} = \frac{1}{r^{n-2}} h_{\mu\nu}(\frac{1}{r}, \frac{\vec x}{r}), \quad \vec x := (x^i) \ , \qquad (5.3)$$

with $h_{\mu\nu}(s, \vec w)$ analytic functions of their arguments near $s = 0$. In fact, there exist functions $h_{00}(s)$, $h(s)$, and $\hat h(s)$, analytic near $s = 0$, such that

$$g_m = \Big(-1 + \frac{h_{00}(r^{-1})}{r^{n-2}}\Big)(dx^0)^2 + \Big[(1 + \frac{h(r^{-1})}{r^{n-2}})\delta_{ij} + \frac{\hat h(r^{-1})}{r^{n-2}}\frac{x^i x^j}{r^2}\Big]dx^i dx^j \ .$$

As $\mathbb{R}^n$ is spin, we will necessarily have $m \geq 0$ for the initial data that we consider.

### 5.3 Domain of dependence of the Schwarzschild initial data

The boundary $\dot{\mathcal{D}}^+_x(M_x \backslash B_{R_x})$ of the future domain of dependence $\mathcal{D}^+_x(M_x \backslash B_{R_x})$ is threaded by null radial outgoing geodesics of the Schwarzschild metric issued from $\dot B_{R_x}$, solutions the differential equation

$$\frac{dt}{dr} = \Big(\frac{g_{m,rr}}{g_{m,tt}}\Big)^{\frac{1}{2}} \qquad (5.4)$$

such that $t(R_x) = 2\lambda$, i.e.

$$x^0 \equiv t = 2\lambda + \int_{R_x}^{r} \Big(\frac{g_{m,rr}}{g_{m,tt}}\Big)^{\frac{1}{2}} dr \ . \tag{5.5}$$

For the Schwarzschild metric we have

$$\Big|\Big(\frac{g_{m,rr}}{g_{m,tt}}\Big)^{\frac{1}{2}} - 1\Big| = \Big|\frac{2m}{r^{n-2}(1-\frac{2m}{r^{n-2}})}\Big| \le \frac{4m}{r^{n-2}} \tag{5.6}$$

for $r \ge R_x$ provided that

$$\frac{2m}{R_x^{n-2}} \le 1/2 \ ; \tag{5.7}$$

at fixed $R_x$ this can be achieved by requiring $m$ to be sufficiently small; alternatively at given $m$ we can increase $R_x$. For $n \ge 5$ we deduce from (5.5) that on $\dot{\mathcal{D}}^+_x(M_x \backslash B_{R_x})$ we have

$$b \ge t - r \ge a \ . \tag{5.8}$$

with

$$b := 2\lambda - R_x + \frac{4m}{(n-3)R_x^{n-3}}, \qquad a := 2\lambda - R_x - \frac{4m}{(n-3)R_x^{n-3}} \ . \tag{5.9}$$

We can always choose $\lambda$ large enough so that $a > 0$, requiring e.g. $2\lambda > R_x + 1$ for $R_x \ge 1$ (assuming (5.7)), which implies that $\dot{\mathcal{D}}^+(M_x \backslash B_{R_x})$ is interior to $I^+_{\eta,x}(0)$.

## 6 The global Cauchy problem

The local Cauchy problem for the system (4.9) on $M_y := \{y^0 = -\frac{1}{2\lambda}\}$, with initial data in $H_{s+1} \times H_s$, $s > \frac{n}{2}$ in a ball of radius $R_y > \frac{1}{2\lambda}$, and $\eta + \hat{f}_{|M_y}$ a non degenerate Lorentzian metric, has a solution $\hat{f}$ in a neighborhood of

$$\mathcal{D}_y := I^-_{\eta,y}(0) \cap \{y^0 \ge -\frac{1}{2\lambda}\} \ , \tag{6.1}$$

if the initial data are small enough. The Einstein equations in wave coordinates on $\mathbb{R}^{n+1}_x$ have then a solution on

$$\mathcal{D}_x := \phi^{-1}(\mathcal{D}_y) \equiv I^+_{\eta,x}(0) \cap x^0 \ge \lambda + \sqrt{\lambda^2 + r^2} \ , \tag{6.2}$$

$\mathcal{D}_x$ is the whole future of the hyperboloid $x^0 = \lambda + \sqrt{\lambda^2 + r^2}$, which is the image by $\phi^{-1}$ of the hyperplane $y^0 = -\frac{1}{2\lambda}$.

The initial data on $M_y$ are deduced by the mapping $\phi$ from the values on the hyperboloid

$$H_x := \{x^0 = \lambda + \sqrt{r^2 + \lambda^2}\}$$

of the local solution in $\mathbb{R}^{n+1}_x$ and its first derivative, if $H_x$ is included in the domain $\mathcal{V}_x$ of its existence.

The hyperboloid $H_x$ is the union of two subsets

$$S_1 := H_x \cap \{x^0 - r \geq a\} \quad \text{and} \quad S_2 := H_x \cap \{x^0 - r \leq a\} \; . \tag{6.3}$$

The subset $S_2$ is included in the Schwarzschild spacetime region. On the subset $S_1$ it holds that

$$\lambda + \sqrt{r^2 + \lambda^2} \geq r + a \tag{6.4}$$

A simple computation shows that $r$ is bounded on $S_1$ if $\lambda > R_x$ and $m$ is small, because, using the value of $a$, one finds that:

$$r \leq \frac{R_x(2\lambda - R_x) + O(m^2)}{2(\lambda - R_x - O(m))} \; , \tag{6.5}$$

then $x^0$ is also bounded on $S_1$. This subset is therefore included in the existence domain $\mathcal{V}_x$ of the local solution with Cauchy data on $M_x$, for small enough Cauchy data.

We deduce from these results that, for small enough Cauchy data (including small $m$) on $M_x$ the domain $\mathcal{V}_x$ of the solution contains the future of $x^0 = 2\lambda$, up to and including $H_x$.

On $\phi(H_x \cap S_2)$ the initial data $\hat{f}_2$ for $\hat{f}$ is deduced from the Schwarzschild metric in wave coordinates, which is static, we have

$$\hat{f}_2(\overrightarrow{y}) = [\Omega^{-\frac{n-1}{2}}(f_m \circ \phi^{-1})](y^0 = -\frac{1}{2}, \overrightarrow{y}) \; , \tag{6.6}$$

with, using the expression of $f_m$

$$f_m \circ \phi^{-1}(\overrightarrow{y}) = \frac{\Omega^{n-2}}{|\overrightarrow{y}|^{n-2}} h(\frac{\Omega}{|\overrightarrow{y}|}, \frac{\overrightarrow{y}}{|\overrightarrow{y}|}) \; . \tag{6.7}$$

We see that $\hat{f}_2$ is a smooth function of $y$, except at the origin $\overrightarrow{y} = 0$, if $n \geq 3$. A simple calculation shows that the same is true of $\frac{\partial \hat{f}}{\partial y^0}$ on $\phi(H_x \cap S_2)$ as soon as $n \geq 5$.

On $\phi(H_x \cap S_1)$, the initial data $\hat{f}_1$, are deduced from the values on the uniformly spacelike submanifold $H_x \cap S_1$ of the solution in $\mathcal{V}_x$ and its derivative, by the restriction of $\phi$ to a neighborhood of $H_x \cap S_1$, where $\phi$ is a smooth diffeomorphism.

### 6.1 Conclusions

We have proved the following theorem.

**Theorem 6.1** *Let $n$ be odd and $n \geq 5$. Let be given on $\mathbb{R}^n$ gravitational data, perturbation of the Minkowski data given by sets of functions $\overline{f_{\mu\nu}}$ and $\overline{\frac{\partial}{\partial x^0} f_{\mu\nu}}$. Suppose that these data satisfy the Einstein constraints and the initial harmonicity conditions and coincide with the Schwarzschild data of mass $m$ in the wave coordinates of Section 5.2 outside a ball of finite Euclidean radius $R_x$. Then, if these functions are small enough in $\big(H_{s+1} \times H_s\big)\big(B(R_x)\big)$ norm, $s > n/2$, and if $m$ is small enough, then the data admit a complete Einsteinian development $(\mathbb{R}^{n+1}, g)$.*

The smoothness of $\hat{f}$ as a function of $y$ immediately provides a full asymptotic expansion of $g$ in terms of inverse powers of $r = |\vec{x}|$. In fact, we obtain an asymptotic behavior of the gravitational field as assumed in [13].

# 7 Einstein Maxwell equations

Let us show that the conformal method extends easily to the electro-vacuum case.

### 7.1 The equations

The Einstein equations with electromagnetic sources (2.1) for the unknowns $f := g - \eta$ take in wave coordinates the form

$$g^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = F_E(g)(\frac{\partial f}{\partial x}, \frac{\partial A}{\partial x}) \,, \tag{7.1}$$

where the right-hand-side is a quadratic form in $\partial f$ and $\partial A$, with coefficients polynomials in $g$ and its contravariant associate. In the Lorenz gauge, $\nabla_\lambda A^\lambda = 0$, the Maxwell equations take also in wave coordinates the form

$$g^{\alpha\beta} \frac{\partial^2 A}{\partial x^\alpha \partial x^\beta} = F_M(g)(\frac{\partial f}{\partial x}, \frac{\partial A}{\partial x}) \,, \tag{7.2}$$

where the right-hand-side is bilinear in $\frac{\partial f}{\partial x}$ and in $\frac{\partial A}{\partial x}$, with coefficients polynomials in $g$ and its contravariant associate. In fact, (7.2) reads in detail

$$g^{\alpha\beta} \frac{\partial^2 A_\sigma}{\partial x^\alpha \partial x^\beta} = -(\partial_\sigma g^{\mu\alpha})\partial_\mu A_\alpha - g_{\nu\sigma} g^{\mu\alpha} (\partial_\mu g^{\nu\beta})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \,; \tag{7.3}$$

this uses both the harmonic coordinates condition and the Lorenz gauge condition. One thus has a system of equations for $(f, A)$ for which the previous analysis applies. Note that it has been proved by Corvino [7] that the existence of a large set of initial data Schwarzschildian[5] outside a compact set is also valid for the Einstein-Maxwell constraints, compare [9].

### 7.2 Conclusions

We have obtained the following theorem:

**Theorem 7.1** *Let $n$ be odd and $n \geq 5$. Let be given on $\mathbb{R}^n$ Einstein-Maxwell data, perturbation of the Minkowski data given by sets of functions $\overline{f_{\mu\nu}}$ and $\overline{\frac{\partial}{\partial x^0} f_{\mu\nu}}$, $\overline{A_\lambda}$ and $\overline{\frac{\partial}{\partial x^0} A_\lambda}$, Suppose that these data satisfy the Einstein-Maxwell constraints and the initial harmonicity and Lorenz gauge conditions, and coincide with the Schwarzschild data with mass $m$ in the wave coordinates of Section 5.2 outside a ball of finite Euclidean radius $R_x$. Then, if these functions are small enough in $\big(H_{s+1} \times H_s\big)\big(B(R_x)\big)$ norm, $s > \frac{n}{2}$, and if $m$ is sufficiently small, then the data admit a complete electro-vacuum Einsteinian development $(\mathbb{R}^{n+1}, g, A)$.*

Similarly to Theorem 6.1, one obtains a full asymptotic expansion both of the metric and of the Maxwell field in terms of inverse powers of $r$ near Scri.

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