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$K(E_9)$ from $K(E_{10})$

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Abstract

We analyse the M-theoretic generalisation of the tangent space structure group after reduction of the $D = 11$ supergravity theory to two space-time dimensions in the context of hidden Kac–Moody symmetries. The action of the resulting infinite-dimensional 'R symmetry' group $K(E_9)$ on certain unfaithful, finite-dimensional spinor representations inherited from $K(E_{10})$ is studied. We explain in detail how these representations are related to certain finite codimension ideals within $K(E_9)$, which we exhibit explicitly, and how the known, as well as new finite-dimensional 'generalised holonomy groups' arise as quotients of $K(E_9)$ by these ideals. In terms of the loop algebra realisations of E_9 and $K(E_9)$ on the fields of maximal supergravity in two space-time dimensions, these quotients are shown to correspond to (generalised) evaluation maps, in agreement with previous results of [1]. The outstanding question is now whether the related unfaithful representations of $K(E_{10})$ can be understood in a similar way.

1 Introduction

The study of a one-dimensional bosonic geodesic σ-model based on the the Kac–Moody coset $E_{10}/K(E_{10})$ has revealed a tantalizing dynamical link to the bosonic dynamics of maximal $D = 11$ supergravity in the vicinity of a space-like singularity [2] (see also [3]).[1] Though striking, this link is limited to truncations on both the Kac–Moody side and the supergravity side. Further progress is partly inhibited by a lack of understanding of the structure of E_{10} and of its maximal compact subgroup $K(E_{10})$ which is not even of Kac–Moody type [7]. The extension of the partial results in the bosonic sector to fermionic fields requires the representation theory of the infinite-dimensional $K(E_{10})$. As an important first step it was shown in [8, 9, 10, 11] that $K(E_{10})$ admits (unfaithful) spinor representations of

[1] An alternative covariant approach to the bosonic dynamics of $D = 11$ supergravity based on E_{11} and the conformal group was initiated in [4, 5]. See also [6] for a proposal combining some features of [5] and [2].

dimensions 320 and 32 with the correct properties to parallel the promising features of the bosonic model. In particular, it was shown there that the fermionic field equations of maximal supergravity (with appropriate truncations) take the form of a $K(E_{10})$ covariant 'Dirac equation'. Furthermore, the decomposition of these spinor representations under those subgroups of $K(E_{10})$ which are known to lead to the massive type IIA and type IIB theories were shown to result in precisely the right (respectively, vector-like and chiral) fermionic field representations of type IIA and type IIB supergravity [12] (the corresponding embeddings of the bosonic sectors had already been studied previously in [13, 14] for E_{10}, and [15, 16, 17] for E_{11}). In this way the E_{10} and $K(E_{10})$ structures incorporate kinematically and dynamically the known duality relations between the maximal supergravity theories for bosons and fermions alike.[2]

In this paper we extend the analysis of the unfaithful $K(E_{10})$ representations to a decomposition under its $K(E_9)$ subgroup. The latter is the maximal compact subgroup of the affine E_9 which is known to be a symmetry of the field equations of maximal $N = 16$ supergravity in $D = 2$ [20, 21, 22, 23].[3] While the finite-dimensional exceptional 'hidden symmetries' E_n of maximal supergravity in $D = 11-n$ for $n \leq 8$ can be directly realised on the supergravity fields [26, 27], the infinite-dimensional affine symmetries of the $D = 2$ theory are realised via a linear system whose integrability condition yields the equations of motion. The fermionic fields (as well as the supercharges) form linear representations of the maximal compact subgroup $K(E_n)$ for $n \leq 9$. Here we will show how, using $K(E_{10})$ and its representations, the $K(E_9)$ transformation rules for the fermions in two space-time dimensions can be derived purely algebraically from the reduction. This constitutes the first direct proof of the $K(E_9)$ properties of $D = 2$ supergravity that does not resort to the linear system. Moreover, we will show that our algebraic action is equivalent to the analytic description of the $K(E_9)$ action in terms of a spectral parameter via a 'generalised evaluation map' [1]. The equivalence of the latter with the algebraic construction of the present work suggests that $K(E_{10})$ may admit a similar realisation – a tantalizing opportunity for future research, since it may also lead to a new realisation of the hyperbolic Kac–Moody algebra E_{10} itself!

A major tool in our investigation is the so-called level decomposition of the global hidden symmetries E_n. In fig. 1 below, we display the Dynkin diagram of E_{10} with our labelling conventions; the lower rank exceptional algebras are obtained by removing nodes from the left. The level decomposition with regard to the $A_{n-1} \equiv \mathfrak{sl}(n)$ subalgebras of E_n allows one to identify the physical fields from the adjoint representation of E_n in terms of $SL(n)$ tensors. More specifically, these level decompositions follow the scheme presented in table 1 for $n = 5, \dots, 9$,

[2] The correct structure for the non-maximal type I supergravity theory in $D = 10$ is $DE_{10} \subset E_{10}$ [18]. The **32** and **320** representations of $K(E_{10})$ decompose into the correct spinors of $K(DE_{10})$. The bosonic sector of this theory was previously studied in relation to DE_{11} in [19].

[3] See also [24, 25] for similar infinite-dimensional symmetries in pure Einstein gravity.

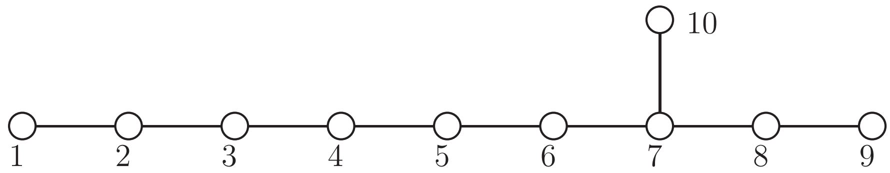

Figure 1: *Dynkin diagram of E_{10} with numbering of nodes.*

where we label the relevant $SL(n)$ representations by bold face letters in the usual way, noting that the entries of the columns $\ell = 1, 2$ always correspond to the three- and six-form representations of $SL(n)$, respectively (and the columns $\ell = -1, -2$ to their contragredient representations). Naturally, E_6 in five dimensions is the first time the six-forms appear in the scalar coset.

$E_n \backslash \ell$	-4	-3	-2	-1	0	1	2	3	4
E_5				$\mathbf{10} \oplus$	$(\mathbf{24}\oplus\mathbf{1})$	$\oplus\, \mathbf{10}$			
E_6			$\mathbf{1} \oplus$	$\mathbf{20} \oplus$	$(\mathbf{35}\oplus\mathbf{1})$	$\oplus\, \mathbf{20}$	$\oplus\, \mathbf{1}$		
E_7			$\mathbf{7} \oplus$	$\mathbf{35} \oplus$	$(\mathbf{48}\oplus\mathbf{1})$	$\oplus\, \mathbf{35}$	$\oplus\, \mathbf{7}$		
E_8		$\overline{\mathbf{8}} \oplus$	$\mathbf{28} \oplus$	$\overline{\mathbf{56}} \oplus$	$(\mathbf{63}\oplus\mathbf{1})$	$\oplus\, \mathbf{56}$	$\oplus\, \overline{\mathbf{28}}$	$\oplus\, \mathbf{8}$	
E_9	$\cdots \oplus$	$\mathbf{80} \oplus$	$\mathbf{84} \oplus$	$\overline{\mathbf{84}} \oplus$	$(\mathbf{80}\oplus\mathbf{1}\oplus\mathbf{1})$	$\oplus\, \mathbf{84}$	$\oplus\, \overline{\mathbf{84}}$	$\oplus\, \mathbf{80}$	$\oplus \cdots$

Table 1: *The level decompositions of the global E_n hidden symmetries in $D = 11 - n$ dimensions under the gravity $SL(n)$ subgroup. The column headings ℓ refer to the level in this level decomposition. For $\ell = 0$, the adjoint of $SL(n)$ always combines with the singlet into the adjoint of $GL(n)$, in the affine case also extended by the derivation d. The central element c of $\mathfrak{e}_9$ is part of $\mathfrak{gl}(9)$.*

For the finite-dimensional algebras in this series (that is, for $n \leq 8$) these results have been known for a long time (for a systematic analysis, see [27]). For $n = 9$, the triple of representations $\overline{\mathbf{84}} \oplus \mathbf{80} \oplus \mathbf{84}$ is repeated an infinite number of times, giving rise to the affine extension of E_8 in the standard way (the two singlets appearing in the middle column for E_9 are the central charge c and the derivation d). For $n = 10$ and $n = 11$, we can no longer display the representations in such a simple fashion, as the number of representations 'explodes'; but see [28] for the tables up to levels $\ell = 18$ and $\ell = 10$, respectively, which were obtained by computer algebra,[4] and also [2] and [29] for earlier results on very low levels of E_{10} and E_{11}, respectively.

We conclude this introduction with some comments on the link between the mathematical structures (ideals, and unfaithful representations of infinite-dimensional compact subgroups of hidden symmetries) exhibited in the main

[4]In fact, for E_{10}, the tables are available up to A_9 level $\ell = 28$ with a total of 4 400 752 653 representations [28].

part of this paper, and the so-called 'generalised holonomies' discussed in the recent literature. *Quite generally, the latter should be identified with quotients of the infinite-dimensional algebras $K(E_9)$ and $K(E_{10})$ by certain finite codimension ideals.* Given any Lie algebra $\mathfrak{k}$ and a linear representation space V, the subspace

$$\mathfrak{i}_V := \left\{ x \in \mathfrak{k} \,|\, x \cdot v = 0 \quad \forall v \in V \right\} \subset \mathfrak{k} \tag{1.1}$$

defines an *ideal* in $\mathfrak{k}$. The representation is unfaithful if $\mathfrak{i}_V \neq 0$. The existence of non-trivial ideals implies in particular that the Lie algebra $\mathfrak{k}$ is not simple. For any $\mathfrak{i}_V$, we can define the quotient algebra

$$\mathfrak{q}_V := \mathfrak{k}/\mathfrak{i}_V \subset \mathfrak{gl}(V). \tag{1.2}$$

The unfaithful *finite-dimensional* spinorial representations of $K(E_9)$ and $K(E_{10})$ discovered in [1, 8, 9, 10, 11] are directly related to the Dirac- and vector (gravitino) spinors appearing in maximal supergravities. For instance, the relevant representations for $K(E_{10})$ are the **32** and the **320** [9, 10]. These representations are inherited by $K(E_9) \subset K(E_{10})$, such that the **32** decomposes into two inequivalent 16-dimensional Dirac-type representations of $K(E_9)$. *As one of our main results we are able to present the associated ideals in $K(E_9)$ in complete detail*, cf. section 3. Because a single ideal may be associated to more than one (and sometimes infinitely many) representations, the description of these structures in terms of ideals appears to be the most economical way to study them.

It is perhaps worth stressing that the quotient group $SO(16)_+ \times SO(16)_-$ associated to the $\mathbf{16}_+ \oplus \mathbf{16}_-$ representation of $K(E_9)$ is *not* a subgroup of $K(E_9)$, because the would-be $SO(16)_+ \times SO(16)_-$ generators are *distributional objects*, as we will explain (see also [11]). The latter group has been proposed as a 'generalised holonomy group' of M-theory [30, 31], generalising the $SO(9)$ Lorentz structure group of the tangent space of the nine torus on which the $D = 11$ theory was reduced. By studying its subgroups and the branching of the **32** representation under these, supersymmetric solutions can be studied and classified [32, 30, 31, 33]. On the other hand, it is known that neither this generalised holonomy group, nor its extensions $SO(32)$ and $SL(32)$, can extend to symmetries of the full equations because of global obstructions [34]. In addition, the generalised holonomies proposed so far do not admit acceptable vector-spinor representations, and as such are restricted to the Killing spinor equation instead of the full supergravity system (in particular, the Rarita Schwinger equation). Our results strengthen the case for $K(E_9)$ and for $K(E_{10})$ as the correct generalised holonomy (and 'R symmetry') groups since both groups do allow for all the required spinor representations. Moreover, $K(E_9)$ is a genuine local symmetry of the reduced theory.

This article is organised as follows. Section 2 summarizes some (largely known) results on the embedding of E_8 and E_9 in a notation adapted to the level decomposition, and goes on to derive their embedding into E_{10}. Informed readers may skip the bulk of this section and proceed directly to section 3, where we derive the branching of the unfaithful $K(E_{10})$ spinors under the $K(E_9)$ subalgebra. The resulting $K(E_9)$ transformation rules are compared to those of the linear system in section 4 where we establish complete agreement with previous results of [1].

2 E_8, E_9 and E_{10}

We here study the chain of embeddings $E_8 \subset E_9 \subset E_{10}$ in $A_7 \subset A_8 \subset A_9$ level decompositions and fix necessary notation for our analysis of the spinor representations in the next section. For the algebras E_8, E_9 and E_{10}, we adopt throughout this paper the following indexing conventions for the $SL(n)$ tensors arising in the decomposition of E_n:

$$\begin{array}{cccc} E_{10} & \leftrightarrow & a,b,\dots & \in \{1,\dots,10\} \\ E_9 & \leftrightarrow & \alpha,\beta,\dots & \in \{2,\dots,10\} \\ E_8 & \leftrightarrow & i,j,\dots & \in \{3,\dots,10\} \end{array} \tag{2.1}$$

2.1 E_8 via A_7

The E_8 subalgebra of E_{10} is generated by nodes 3 through to 10 of fig. 1 and can be written in terms of irreducible tensors of its $A_7 \cong \mathfrak{sl}(8)$ subalgebra (corresponding to nodes 3 through to 9). By adjoining the eigth Cartan generator, this $\mathfrak{sl}(8)$ subalgebra can be extended to a $\mathfrak{gl}(8)$ subalgebra generated by

$$G^i{}_j \quad , \quad \text{with} \quad \left[G^i{}_j, G^k{}_l\right] = \delta^k_j G^i{}_l - \delta^i_l G^k{}_j, \tag{2.2}$$

where the indices take values $i,j = 3,\dots,10$. The A_7 decomposition of E_8 gives the $\mathfrak{sl}(8)$ tensors displayed in table 2 [27].

A_7 level ℓ in E_8	Generator	$SL(8)$ representation
-3	Z_i	$\overline{\mathbf{8}}$
-2	$Z_{i_1\dots i_6}$	$\mathbf{28}$
-1	$Z_{i_1i_2i_3}$	$\overline{\mathbf{56}}$
0	$G^i{}_j$	$\mathbf{63}\oplus\mathbf{1}$
1	$Z^{i_1i_2i_3}$	$\mathbf{56}$
2	$Z^{i_1\dots i_6}$	$\overline{\mathbf{28}}$
3	Z^i	$\mathbf{8}$

Table 2: *A_7 decomposition of E_8.*

In the left column we have indicated the $\mathfrak{sl}(8)$ level, that is the number of times the exceptional simple root α_{10} occurs in the associated roots. All indices $i, j, \dots$ run from $3, \dots, 10$ and all tensors, except for $G^i{}_j$, are totally anti-symmetric in their $SL(8)$ (co-)vector indices. The Chevalley transposition $(\cdot)^T$ acts by $(G^i{}_j)^T = G^j{}_i$ and $(Z_{i_1i_2i_3})^T = Z^{i_1i_2i_3}$, etc. The $\mathfrak{gl}(8)$ tensors in the table with upper (lower) indices correspond to positive (negative) roots. In E_{10} language, the former correspond to the 'E-type' generators, while the latter transform in the contragredient representations and correspond to the 'F-type' generators in the notation of [3].

The commutation relations between $G^i{}_j$ and the positive and negative $\mathfrak{gl}(8)$ level 'step operators' are

$$\begin{aligned}
\left[G^i{}_j, Z^{k_1k_2k_3}\right] &= 3\delta_j^{[k_1} Z^{k_2k_3]i}, \\
\left[G^i{}_j, Z^{k_1\dots k_6}\right] &= -6\delta_j^{[k_1} Z^{k_2\dots k_6]i}, \\
\left[G^i{}_j, Z^{k}\right] &= \delta_j^k Z^i + \delta_j^i Z^k, \\
\left[G^i{}_j, Z_{k_1k_2k_3}\right] &= -3\delta^i_{[k_1} Z_{k_2k_3]j}, \\
\left[G^i{}_j, Z_{k_1\dots k_6}\right] &= 6\delta^i_{[k_1} Z_{k_2\dots k_6]j}, \\
\left[G^i{}_j, Z_{k}\right] &= -\delta^i_k Z_j - \delta^i_j Z_k.
\end{aligned} \tag{2.3}$$

Note the trace terms in the commutation relations involving the $\mathfrak{gl}(8)$ vectors Z^i and Z_i which are needed for the correct transformation under the trace of $\mathfrak{gl}(8)$, and for the consistency of the first two relations with the second relation in (2.4) below. Furthermore,

$$\begin{aligned}
\left[Z^{i_1i_2i_3}, Z^{i_4i_5i_6}\right] &= Z^{i_1\dots i_6}, \\
\left[Z^{i_1i_2i_3}, Z^{i_4\dots i_9}\right] &= 3Z^{[i_1}\epsilon^{i_2i_3]i_4\dots i_9},
\end{aligned} \tag{2.4}$$

where $\epsilon^{i_1\dots i_8}$ is the $SL(8)$ totally anti-symmetric tensor. Similar expressions are obtained for the negative level generators by applying the Chevalley transposition.

The mixed commutation relations are

$$\begin{aligned}
\left[Z^{i_1i_2i_3}, Z_{j_1j_2j_3}\right] &= -2\delta^{i_1i_2i_3}_{j_1j_2j_3} G + 18\delta^{[i_1i_2}_{[j_1j_2} G^{i_3]}{}_{j_3]}, \\
\left[Z^{i_1i_2i_3}, Z_{j_1\dots j_6}\right] &= -5!\, \delta^{i_1i_2i_3}_{[j_1j_2j_3} Z_{j_4j_5j_6]}, \\
\left[Z^{i_1\dots i_6}, Z_{j_1\dots j_6}\right] &= -\frac{2}{3}\cdot 6!\, \delta^{i_1\dots i_6}_{j_1\dots j_6} G + 6\cdot 6!\, \delta^{[i_1\dots i_5}_{[j_1\dots j_5} G^{i_6]}{}_{j_6]}, \\
\left[Z^{i_1i_2i_3}, Z_j\right] &= \frac{1}{5!}\epsilon^{i_1i_2i_3k_1\dots k_5} Z_{k_1\dots k_5 j}, \\
\left[Z^{i_1\dots i_6}, Z_j\right] &= \frac{1}{2}\epsilon^{i_1\dots i_6k_1k_2} Z_{k_1k_2j}, \\
\left[Z^{i}, Z_j\right] &= G^i{}_j.
\end{aligned} \tag{2.5}$$

Here, $G \equiv \sum_{k=3}^{10} G^k{}_k$. Equations (2.3), (2.4) and (2.5), together with their Chevalley transposes, constitute a complete set of E_8 commutation relations. The normalisations of the generators are

$$\begin{aligned}\left\langle G^i{}_j \middle| G^k{}_l \right\rangle &= \delta^i_l \delta^k_j + \delta^i_j \delta^k_l , \\ \left\langle Z^{i_1 i_2 i_3} \middle| Z_{j_1 j_2 j_3} \right\rangle &= 3!\, \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} , \\ \left\langle Z^{i_1 \dots i_6} \middle| Z_{j_1 \dots j_6} \right\rangle &= 6!\, \delta^{i_1 \dots i_6}_{j_1 \dots j_6} , \\ \left\langle Z^i \middle| Z_j \right\rangle &= \delta^i_j . \end{aligned} \tag{2.6}$$

Modulo normalisation factors, the same relations have been given for example in [27, 35]. In comparison with the notation of [35] the tensors on levels $\ell = \pm 2$ have been dualised using the ϵ-tensor of $SL(8)$ and some of the normalisations have changed.

2.2 E_9 as extended current algebra

As is well known (see e.g. [36]), the affine Lie algebra $E_9 \equiv E_8^{(1)}$ is obtained from E_8 by 'affinization', that is by embedding E_8 in its current algebra (parametrized by the spectral parameter t), and by adjoining two more Lie algebra elements, the central charge c and the derivation d: $E_9 = E_8[[t, t^{-1}]] \oplus \mathbb{R}c \oplus \mathbb{R}d$ (as always, we restrict attention to the split real forms of these Lie algebras). By writing $X^{(m)} \equiv X \otimes t^m$ (for $m \in \mathbb{Z}$) the E_9 commutation relations are

$$\begin{aligned}\left[X^{(m)}, Y^{(n)} \right] &= \left[X \otimes t^m, Y \otimes t^n \right] = [X, Y] \otimes t^{m+n} + m \delta_{m+n,0} \langle X | Y \rangle\, c , \\ \left[d, X^{(m)} \right] &= m X^{(m)} , \\ \left[c, X^{(m)} \right] &= 0, \qquad [c, d] = 0. \end{aligned} \tag{2.7}$$

They can thus be read off directly from the E_8 commutation relations above in the standard fashion. The inner product between c and d is $\langle c | d \rangle = 1$. The 'horizontal' E_8 at affine level 0 is isomorphic to E_8 and we will often write $X \equiv X^{(0)}$ for any E_8 generator X, for example

$$G^i{}_j \equiv G^{(0)i}{}_j \ , \ Z_i \equiv Z_i^{(0)} \ , \quad etc. \tag{2.8}$$

Next, we will study how the current algebra generators emerge from E_{10}, that is how they are obtained from the latter algebra by truncation and by 'dimensional reduction'.

2.3 Embedding of E_9 in E_{10}

With regard to the E_{10} Dynkin diagram, the E_9 subalgebra of E_{10} is obtained by deleting node 1 from the diagram 1, or equivalently by restricting to level zero in an E_9 level decomposition[5] which counts the number of occurrences of the simple

[5] In comparison to the A_9 level decomposition of E_{10} which can be thought of as a space-like foliation of the Lorentzian root lattice, the E_9 decomposition foliates the root lattice by *null* planes.

root α_1. However, one does keep the Cartan generator h_1 which is needed to 'desingularize' the metric on the root lattice (so the Cartan subalgebra of E_9 can be identified with the one of E_{10}, h_1 appears only in d). Using the $\mathfrak{gl}(10)$ basis of E_{10}, where small Latin indices take values $a = 1, \ldots, 10$,

$$K^a{}_b \quad , \quad \text{with} \quad [K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{2.9}$$

the E_{10} Cartan generators are

$$\begin{aligned} h_a &= K^a{}_a - K^{a+1}{}_{a+1} \qquad (a = 1, \ldots, 9), \\ h_{10} &= -\frac{1}{3}\sum_{a=1}^{10} K^a{}_a + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}. \end{aligned} \tag{2.10}$$

The invariant inner product of these generators is given by

$$\langle K^a{}_b | K^c{}_d \rangle = \delta^a_d \delta^c_b - \delta^a_b \delta^c_d. \tag{2.11}$$

The coefficient of the second term is not fixed by invariance but by requiring that $\langle h_{10} | h_{10} \rangle = 2$, where h_{10} in (2.10) was fixed by requiring the right $\mathfrak{gl}(10)$ commutation relation with the A_9 level $\ell = 1$ generator E^{abc}.[6] We follow the conventions of [3] except for two differences: Firstly, we take e_{10} to be $E^{8\,9\,10}$ since the exceptional node is attached at the other end. Secondly, we rescale the A_9 level $\ell = \pm 3$ generators by a factor 1/3.

In terms of the A_9 level decomposition of E_{10} the E_9 elements are precisely those contained in the 'gradient representations' of [2] where indices are restricted to take values $2, \ldots, 10$. As shown there (see also [28]), every nth order gradient generator contains n sets of nine anti-symmetric indices, and thus has A_9 Dynkin labels $[n * * * * * * * *]$. For instance, at A_9 level $\ell = 3n+1$, we have the following gradient generators

$$E^{a_1^{(1)} \cdots a_9^{(1)} | \cdots | a_1^{(n)} \cdots a_9^{(n)} | bcd} \qquad \text{with } a_i^{(j)}, b, c, d \in \{1, \ldots, 10\}.$$

which are antisymmetric in all *9-tuples* $(a_1^{(j)} \cdots a_9^{(j)})$. Restricting all indices on the above element to the values $2, \ldots, 10$, we see that, up to permutations, there is only one choice of filling indices into these *9-tuples*, and we thus only need to remember that there were n such sets. In fact, this restriction is physically motivated since E_9 arises in the reduction to two dimensions with one left-over non-compact spatial direction 1 (obviously, there are nine alternative choices for this residual spatial dimension, corresponding to ten distinguished E_9 subgroups

[6] This is the reason for the minus sign in the bilinear form (2.11), resulting in the indefiniteness of the inner product (2.11). By contrast, (2.6) has a plus sign in the corresponding formula, whence the inner product is positive definite for E_8.

A_9 level in E_{10}	(Restricted) gradient generator	$\mathfrak{sl}(9)$ irrep
$\ell = 3n+3$	$\overset{(n)}{E}{}^{\alpha_0\vert\alpha_1\ldots\alpha_8}$	**80**
$\ell = 3n+2$	$\overset{(n)}{E}{}^{\alpha_1\ldots\alpha_6}$	$\overline{\mathbf{84}}$
$\ell = 3n+1$	$\overset{(n)}{E}{}^{\alpha_1\alpha_2\alpha_3}$	**84**
$\ell = -3n-1$	$\overset{(n)}{F}{}_{\alpha_1\alpha_2\alpha_3}$	$\overline{\mathbf{84}}$
$\ell = -3n-2$	$\overset{(n)}{F}{}_{\alpha_1\ldots\alpha_6}$	**84**
$\ell = -3n-3$	$\overset{(n)}{F}{}_{\alpha_0\vert\alpha_1\ldots\alpha_8}$	**80**

Table 3: *Identification of the E_9 generators in terms of E_{10} gradient generators.*

in E_{10}). Accordingly, we introduce the following shorthand notation for the gradient generators

$$\begin{aligned} E^{\overbrace{2\ldots10|\cdots|2\ldots10|}^{n\text{ times}}\alpha_1\alpha_2\alpha_3} &\equiv \overset{(n)}{E}{}^{\alpha_1\alpha_2\alpha_3} \\ E^{\overbrace{2\ldots10|\cdots|2\ldots10|}^{n\text{ times}}\alpha_1\ldots\alpha_6} &\equiv \overset{(n)}{E}{}^{\alpha_1\ldots\alpha_6} \\ E^{\overbrace{2\ldots10|\cdots|2\ldots10|}^{n\text{ times}}\alpha_0|\alpha_1\ldots\alpha_8} &\equiv \overset{(n)}{E}{}^{\alpha_0|\alpha_1\ldots\alpha_8} \end{aligned} \tag{2.12}$$

where $\alpha_0, \alpha_1, \alpha_2, \cdots = 2, \ldots, 10$. The 'F-type' gradient generators are defined analogously. Our notation is summarized in table 3: the indices here take values $\alpha = 2, \ldots, 10$, and together with $K^\alpha{}_\beta$ and $K^1{}_1$ from A_9 level $\ell = 0$ they constitute *all* E_9 generators expressed in E_{10} variables. As will be seen below, the central charge c of E_9 in terms of E_{10} generators is proportional to $K^1{}_1$ and commutes with all elements of E_9 whence the restriction of indices to $\alpha = 2, \ldots, 10$ is the correct restriction to E_9. That the suppression of the blocks of nine indices is justified will be shown below. Now we want to relate these generators to the E_9 generators of section 2.

The generators of E_8 are embedded regularly in E_{10} and, away from the Cartan subalgebra, are identical to those of E_{10} for levels $|\ell| \leq 3$ if the indices are restricted to the range $\{3, \ldots, 10\}$. Therefore we find immediately that

$$\begin{aligned} Z^{(0)i_1i_2i_3} &= \overset{(0)}{E}{}^{i_1i_2i_3}, & Z^{(0)}_{i_1i_2i_3} &= \overset{(0)}{F}{}_{i_1i_2i_3}, \\ Z^{(0)i_1\ldots i_6} &= \overset{(0)}{E}{}^{i_1\ldots i_6}, & Z^{(0)}_{i_1\ldots i_6} &= \overset{(0)}{F}{}_{i_1\ldots i_6}, \\ \epsilon^{k_1\ldots k_8} Z^{(0)i} &= \overset{(0)}{E}{}^{i|k_1\ldots k_8}, & \epsilon_{k_1\ldots k_8} Z^{(0)}_i &= \overset{(0)}{F}{}_{i|k_1\ldots k_8}, \end{aligned} \tag{2.13}$$

where the superscript on the l.h.s. denotes the affine level, whereas the superscript on the r.h.s. denotes the 'gradient' level as explained in (2.12). As a mnemonic and notational device to distinguish between these two kinds of levels we place the superscripts slightly differently, as evident from the preceding equation. The objects on the r.h.s. are $GL(8)$ tensors, and we recall that, for the comparison between E_8 and E_{10} we must restrict the indices on the $SL(10)$ tensors appearing in the A_9 decomposition of E_{10} to the values $i = 3, \ldots, 10$. To identify the $GL(8)$ generators in terms of the Cartan generators we note that the only difference can be in the diagonal part of $G^{(0)i}{}_j$ since the off-diagonal elements correspond to step operators. A simple calculation shows that the correct identification between $G^i{}_j \in E_8$ and $K^i{}_j \in E_{10}$ is[7]

$$G^i{}_j \equiv G^{(0)i}{}_j = K^i{}_j + \delta^i_j(c - d), \tag{2.14}$$

where the central element c and derivation d of E_9 in terms of the $\mathfrak{gl}(10)$ generators are given by

$$d = K^2{}_2, \qquad c = -K^1{}_1. \tag{2.15}$$

It is easy to see that c indeed commutes with all elements of E_9 and has inner product $+1$ with d. Furthermore, d commutes with E_8 of (2.13) as it should. Evidently, the affine level operator d counts the number of tensor indices taking the value 2 (with $(+1)$ for upper and (-1) for lower indices). The extra terms in (2.14) also induce the relative change in sign between (2.6) and (2.11).

Using the relation of the general linear subalgebras (2.14) we can show that the blocks of nine anti-symmetric indices suppressed in the gradient generators are not 'seen' by the $\mathfrak{gl}(8)$ generators, as we already claimed above. Consider a generator $X^{2k_1 \ldots k_8}$ which is totally anti-symmetric in its nine indices and $k = 3, \ldots, 10$. Then

$$\left[G^{(0)i}{}_j, X^{2k_1 \ldots k_8}\right] = 8\delta_j^{[k_1} X^{k_2 \ldots k_8]2i} - \delta^i_j X^{2k_1 \ldots k_8} = -9\delta_j^{[i} X^{k_1 \ldots k_8]2} = 0 \tag{2.16}$$

by Schouten's identity; the last term in the middle expression is due to the correction term with d in (2.14), which is thus crucial for the vanishing of the above commutator. This confirms that we can indeed replace each *9-tuple* of indices by a label indicating the number of such *9-tuples* and assume that the *9-tuples* are filled in some fixed way by $2, \ldots, 10$.

From the form of d in (2.15) we see that the number of upper indices equal to 2 on a positive step generators is the affine level and similarly for negative step operators. It is not hard to identify the following affine level $+1$ generators

[7] One way to see the necessity of this redefinition is to compute $[Z^{(0)8\,9\,10}, Z^{(0)}_{8\,9\,10}]$ both in E_8 and E_{10}, and to demand that the central charge c and the derivation d drop out from this commutator for E_8.

among the E_{10} generators

$$\begin{aligned} Z^{(1)}_j &= K^2{}_j, \\ Z^{(1)}_{j_1\dots j_6} &= \frac{1}{2}\epsilon_{j_1\dots j_6 k_1 k_2}\overset{(0)}{E}{}^{k_1k_2 2}, \\ Z^{(1)}_{j_1j_2j_3} &= \frac{1}{5!}\epsilon_{j_1j_2j_3k_1\dots k_5}\overset{(0)}{E}{}^{k_1\dots k_5 2}, \\ G^{(1)i}{}_j &= -\frac{1}{7!}\epsilon_{jk_1\dots k_7}\overset{(0)}{E}{}^{i|2k_1\dots k_7} - \frac{1}{8!}\delta^i_j\epsilon_{k_1\dots k_8}\overset{(0)}{E}{}^{2|k_1\dots k_8}. \end{aligned} \tag{2.17}$$

This involves only generators with A_9 level $\ell = 0, \dots, 3$ in the E_{10} decomposition. Similarly, at affine level -1 we have

$$\begin{aligned} Z^{(-1)i} &= K^i{}_2, \\ Z^{(-1)i_1\dots i_6} &= \frac{1}{2}\epsilon^{i_1\dots i_6k_1k_2}\overset{(0)}{F}{}_{k_1k_2 2}, \\ Z^{(-1)i_1i_2i_3} &= \frac{1}{5!}\epsilon^{i_1i_2i_3k_1\dots k_5}\overset{(0)}{F}{}_{k_1\dots k_5 2}, \\ G^{(-1)i}{}_j &= -\frac{1}{7!}\epsilon^{ik_1\dots k_7}\overset{(0)}{F}{}_{j|2k_1\dots k_7} - \frac{1}{8!}\delta^i_j\epsilon^{k_1\dots k_8}\overset{(0)}{F}{}_{2|k_1\dots k_8}. \end{aligned} \tag{2.18}$$

Again, the redefinition (2.14) is crucial for the correct E_9 transformation rules, e.g.

$$\begin{aligned} \left[G^{(0)i}{}_j, Z^{(1)}_{k_1\dots k_6}\right] &= \frac{1}{2}\epsilon_{k_1\dots k_6l_1l_2}\left[K^i{}_j - \delta^i_j d\,,\ \overset{(0)}{E}{}^{l_1l_2 2}\right] \\ &= \frac{1}{2}\epsilon_{k_1\dots k_6l_1l_2}\left(2\delta^{l_1}_j\overset{(0)}{E}{}^{il_2 2} - \delta^i_j\overset{(0)}{E}{}^{l_1l_2 2}\right) \\ &= \frac{1}{2\cdot 6!}\epsilon_{k_1\dots k_6l_1l_2}\left(2\delta^{l_1}_j\epsilon^{il_2m_1\dots m_6} - \delta^i_j\epsilon^{l_1l_2m_1\dots m_6}\right)Z^{(1)}_{m_1\dots m_6} \\ &= 6\delta^i_{[k_1}Z^{(1)}_{k_2\dots k_6]j}, \end{aligned} \tag{2.19}$$

in agreement with (2.3) for affine level $+1$. We identify also the following elements at affine level ± 2

$$\begin{aligned} Z^{(2)}_j &= -\frac{1}{7!}\epsilon_{jk_1\dots k_7}\overset{(0)}{E}{}^{2|2k_1\dots k_7}, \\ Z^{(-2)i} &= -\frac{1}{7!}\epsilon^{ik_1\dots k_7}\overset{(0)}{F}{}_{2|2k_1\dots k_7}. \end{aligned} \tag{2.20}$$

Indeed, one can check from these relations that

$$\left[Z^{(-2)i}, Z^{(2)}_j\right] = G^{(0)i}{}_j - 2\delta^i_j\, c \tag{2.21}$$

as it should be for this affine commutator. Again we see, that the affine level is equal to the difference between the number of upper and lower indices equalling 2.

With relations (2.13), (2.14), (2.15), (2.17), (2.18) and (2.20) we have identified all E_9 generators appearing on A_9 levels $-3 \leq \ell \leq 3$ in E_{10}. It should now be clear how to obtain the higher affine levels: the scheme repeats itself after shifting $\ell \to \ell + 3$, as illustrated in figure 2. As evident from these formulæ, the affine level and the A_9 level are 'oblique' w.r.t. each other: The elements of affine level n are spread over the A_9 levels $3n-3 \leq \ell \leq 3n+3$. This is also shown in figure 2.

For completeness, we write the general formulæ for $n > 1$

$$
\begin{aligned}
Z_i^{(n)} &= -\frac{1}{7!}\epsilon_{ik_1\dots k_7} \overset{(n-2)}{E}{}^{2|2k_1\dots k_7}, \\
Z_{i_1\dots i_6}^{(n)} &= \frac{1}{2}\epsilon_{i_1\dots i_6 k_1 k_2} \overset{(n-1)}{E}{}^{k_1k_2 2}, \\
Z_{i_1 i_2 i_3}^{(n)} &= \frac{1}{5!}\epsilon_{i_1 i_2 i_3 k_1\dots k_5} \overset{(n-1)}{E}{}^{k_1\dots k_5 2}, \\
G^{(n)i}{}_j &= -\frac{1}{7!}\epsilon_{jk_1\dots k_7} \overset{(n-1)}{E}{}^{i|2k_1\dots k_7} - \frac{1}{8!}\delta^i_j \epsilon_{k_1\dots k_8} \overset{(n-1)}{E}{}^{2|k_1\dots k_8}, \\
Z^{(n)i_1i_2i_3} &= \overset{(n)}{E}{}^{i_1i_2i_3}, \\
Z^{(n)i_1\dots i_6} &= \overset{(n)}{E}{}^{i_1\dots i_6}, \\
Z^{(n)i} &= \frac{1}{8!}\epsilon_{k_1\dots k_8} \overset{(n)}{E}{}^{i|k_1\dots k_8}.
\end{aligned} \tag{2.22}
$$

for the positive current modes and

$$
\begin{aligned}
Z_i^{(-n)} &= \frac{1}{8!}\epsilon^{k_1\dots k_8} \overset{(n)}{F}{}_{i|k_1\dots k_8}, \\
Z_{i_1\dots i_6}^{(-n)} &= \overset{(n)}{F}{}_{i_1\dots i_6}, \\
Z_{i_1 i_2 i_3}^{(-n)} &= \overset{(n)}{F}{}_{i_1 i_2 i_3}, \\
G^{(-n)i}{}_j &= -\frac{1}{7!}\epsilon^{ik_1\dots k_7} \overset{(n-1)}{F}{}_{j|2k_1\dots k_7} - \frac{1}{8!}\delta^i_j \epsilon_{k_1\dots k_8} \overset{(n-1)}{F}{}_{2|k_1\dots k_8}, \\
Z^{(-n)i_1i_2i_3} &= \frac{1}{5!}\epsilon^{i_1i_2i_3k_1\dots k_5} \overset{(n-1)}{F}{}_{k_1\dots k_5 2}, \\
Z^{(-n)i_1\dots i_6} &= \frac{1}{2}\epsilon^{i_1\dots i_6 k_1k_2} \overset{(n-1)}{F}{}_{k_1k_2 2}, \\
Z^{(-n)i} &= -\frac{1}{7!}\epsilon^{ik_1\dots k_7} \overset{(n-2)}{F}{}_{2|2k_1\dots k_7}.
\end{aligned} \tag{2.23}
$$

for the negative current modes with $n > 1$. Observe that the $\mathfrak{sl}(8)$ representations appearing in the vertical lines in fig. 2 combine 'sideways' into the required $\mathfrak{sl}(9)$ representations in accordance with the decompositions

$$
\begin{aligned}
\mathbf{80} &\to \mathbf{8} \oplus (\mathbf{63} \oplus \mathbf{1}) \oplus \overline{\mathbf{8}} \\
\mathbf{84} &\to \mathbf{56} \oplus \mathbf{28} \\
\overline{\mathbf{84}} &\to \overline{\mathbf{56}} \oplus \overline{\mathbf{28}}.
\end{aligned} \tag{2.24}
$$

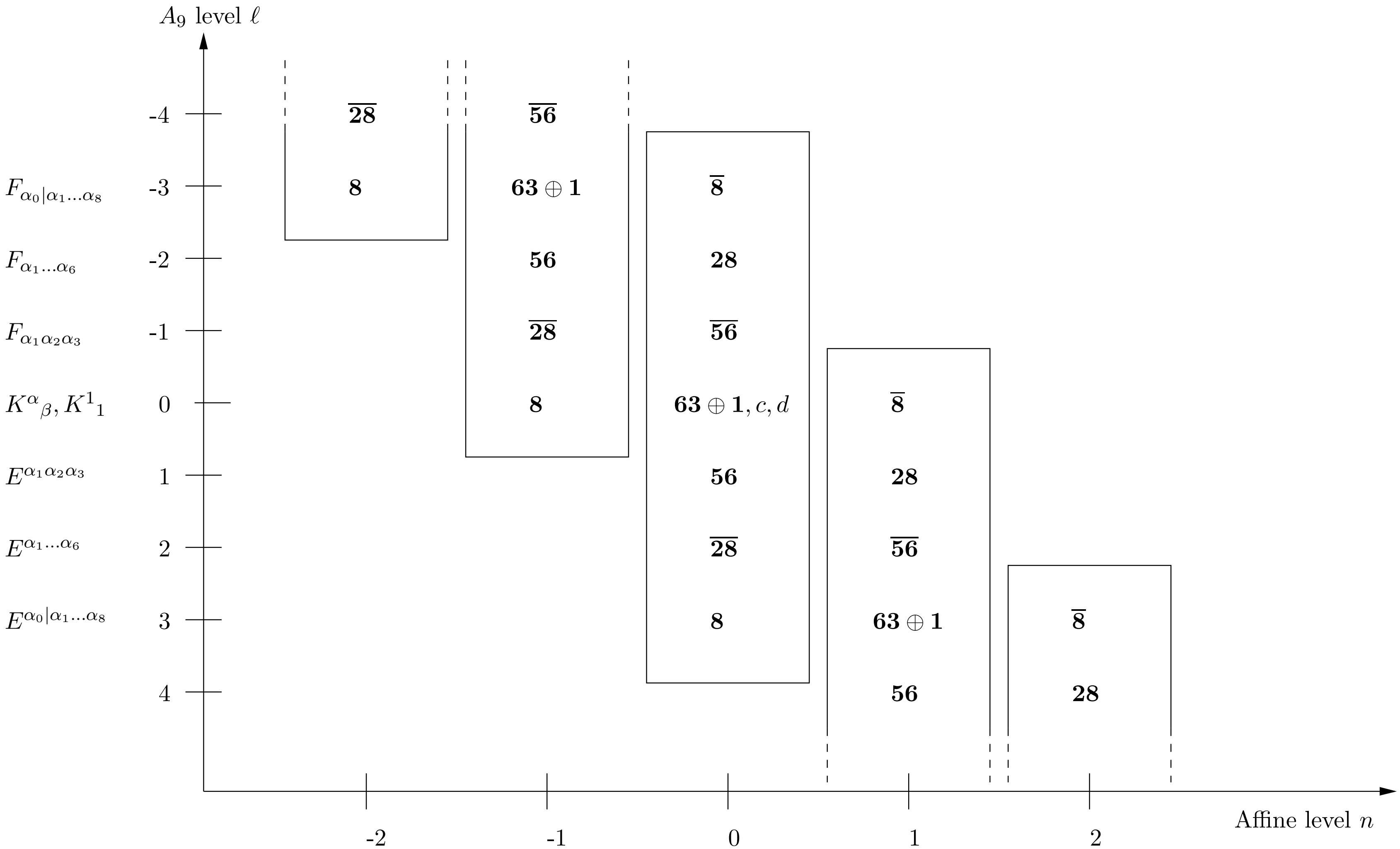

Figure 2: *Diagram illustrating the distribution of A_9 levels and affine levels in E_{10}. The affine level n is given by the number of upper 2s minus the number of lower 2s on an E_{10} generator. The indices on the A_9 level $\ell \neq 0$ generators range over $\alpha = 2, \ldots, 10$. The boxed set of generators correspond to copies of E_8, at affine level 0, the central charge and derivation are also included.*

3 $K(E_9)$ spinor representations from $K(E_{10})$

The generators of $K(E_{10})$ are the anti-symmetric elements under the Chevalley transposition (see e.g. [11]). Therefore, we can construct a $K(E_{10})$ generators for any positive root step operator E by taking $J = E - E^T \equiv E - F$. The restriction to $K(E_9)$ is then obtained by considering only those positive step operators of table 3. As mentioned in the introduction $K(E_9)$ is not of Kac–Moody type (nor is $K(E_{10})$). The reason for this is that the invariant inner product

$$(x|y) := -\langle x|y\rangle \qquad \text{for all } x,y \in K(E_9) \text{ (or } K(E_{10})), \tag{3.1}$$

inherited from the invariant bilinear form on E_9 (E_{10}), is positive definite on the compact subalgebras.

Despite this complication, finite-dimensional, hence *unfaithful*, representations corresponding to Dirac-spinor and vector-spinor (gravitino) representations of $K(E_{10})$ have been constructed in [8, 9, 10, 11]. We now study the branching of these representations to $K(E_9) \subset K(E_{10})$. Before doing so we derive the complete $K(E_9)$ commutation relations in a form convenient for this computation.

The $K(E_{10})$ generators at 'A_9 levels' $\ell = 0, \ldots, 3$ are defined by

$$\begin{aligned}
J^{ab}_{(0)} &= K^a{}_b - K^b{}_a, \\
J^{a_1a_2a_3}_{(1)} &= \overset{(0)}{E}{}^{a_1a_2a_3} - \overset{(0)}{F}{}_{a_1a_2a_3}, \\
J^{a_1\ldots a_6}_{(2)} &= \overset{(0)}{E}{}^{a_1\ldots a_6} - \overset{(0)}{F}{}_{a_1\ldots a_6}, \\
J^{a_0|a_1\ldots a_8}_{(3)} &= \overset{(0)}{E}{}^{a_0|a_1\ldots a_8} - \overset{(0)}{F}{}_{a_0|a_1\ldots a_8},
\end{aligned} \tag{3.2}$$

for $a, b = 1, \ldots, 10$. Observe that on the l.h.s. the position of indices no longer matters, as these tensors transform only under the $SO(10)$ subgroup of $K(E_{10})$ and indices can be raised and lowered with the invariant δ^{ab}. The lower indices in parentheses on the l.h.s. indicate the A_9 level in E_{10} (or A_8 level in E_9), where as the indices placed above the generators on the r.h.s. indicate the gradient level of (2.12). As before, the $K(E_9)$ generators are obtained from these by 'dimensional reduction', that is by restricting the indices to $\alpha, \beta = 2, \ldots, 10$, corresponding to the A_8 level decomposition of E_9. The relation between the A_8 decomposition and the current algebra decomposition of E_9 was explained in the preceding section.

In the remainder we will make use of the following notation for the $K(E_9)$ generators in $K(E_{10})$ for $k \geq 0$

$$\begin{aligned}
J^{\alpha\beta}_{(0)} &= K^\alpha{}_\beta - K^\beta{}_\alpha, \\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \overset{(k)}{E}{}^{\alpha_1\alpha_2\alpha_3} - \overset{(k)}{F}{}_{\alpha_1\alpha_2\alpha_3}, \\
J^{\alpha_1\ldots\alpha_6}_{(3k+2)} &= \overset{(k)}{E}{}^{\alpha_1\ldots\alpha_6} - \overset{(k)}{F}{}_{\alpha_1\ldots\alpha_6}, \\
J^{\alpha_0|\alpha_1\ldots\alpha_8}_{(3k+3)} &= \overset{(k)}{E}{}^{\alpha_0|\alpha_1\ldots\alpha_8} - \overset{(k)}{F}{}_{\alpha_0|\alpha_1\ldots\alpha_8},
\end{aligned} \tag{3.3}$$

using the notation of (2.12) and table 3. The generator at A_8 level $(3k+3)$ is $\mathfrak{so}(9)$ reducible and decomposes after dualisation into

$$J^{\beta|\alpha_1\ldots\alpha_8}_{(3k+3)} = \left(J^{\beta\gamma}_{(3k+3)} + S^{\beta\gamma}_{(3k+3)}\right)\epsilon^{\gamma\alpha_1\ldots\alpha_8} \qquad \text{for } k \geq 0 \tag{3.4}$$

Here, the anti-symmetric tensor $J^{\alpha\beta}_{(3k+3)} = -J^{\beta\alpha}_{(3k+3)}$ is the trace part of the orginal tensor $J^{\alpha_0|\alpha_1\ldots\alpha_8}_{(3k+3)}$, and the symmetric $S^{\alpha\beta}_{(3k+3)} = +S^{\beta\alpha}_{(3k+3)}$ is traceless, $S^{\gamma\gamma}_{(3k+3)} = 0$, according to the original Young symmetry. The anti-symmetric part has the same representation structure as $J^{\alpha\beta}_{(0)}$; by contrast, the symmetric generators $S^{\alpha\beta}_{(3n)}$ have no zero mode part, and exist only for $n \geq 1$.

From (2.22) and (2.23) we deduce the following $K(E_9)$ relations (for $m \geq n$)

$$\begin{aligned}
\left[J^{\alpha\beta}_{(3m)}, J^{\gamma\delta}_{(3n)}\right] &= 2\delta^{\beta\gamma}J^{\alpha\delta}_{(3(m+n))} + 2\delta^{\beta\gamma}J^{\alpha\delta}_{(3(m-n))},\\
\left[J^{\alpha\beta}_{(3m)}, S^{\gamma\delta}_{(3n)}\right] &= 2\delta^{\beta\gamma}S^{\alpha\delta}_{(3(m+n))} - 2\delta^{\beta\gamma}S^{\alpha\delta}_{(3(m-n))},\\
\left[S^{\alpha\beta}_{(3m)}, J^{\gamma\delta}_{(3n)}\right] &= 2\delta^{\beta\gamma}S^{\alpha\delta}_{(3(m+n))} + 2\delta^{\beta\gamma}S^{\alpha\delta}_{(3(m-n))},\\
\left[S^{\alpha\beta}_{(3m)}, S^{\gamma\delta}_{(3n)}\right] &= 2\delta^{\beta\gamma}J^{\alpha\delta}_{(3(m+n))} - 2\delta^{\beta\gamma}J^{\alpha\delta}_{(3(m-n))},\\
\left[J^{\alpha\beta}_{(3m)}, J^{\gamma_1\gamma_2\gamma_3}_{(3n+1)}\right] &= 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m+n)+1)} - \frac{3}{6!}\delta^{\beta\gamma_1}\epsilon^{\alpha\gamma_2\gamma_3\delta_1\ldots\delta_6}J^{\delta_1\ldots\delta_6}_{(3(m-n)-1)},\\
\left[J^{\alpha\beta}_{(3n)}, J^{\gamma_1\gamma_2\gamma_3}_{(3m+1)}\right] &= 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m+n)+1)} + 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m-n)+1)},\\
\left[S^{\alpha\beta}_{(3m)}, J^{\gamma_1\gamma_2\gamma_3}_{(3n+1)}\right] &= 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m+n)+1)} + \frac{3}{6!}\delta^{\beta\gamma_1}\epsilon^{\alpha\gamma_2\gamma_3\delta_1\ldots\delta_6}J^{\delta_1\ldots\delta_6}_{(3(m-n)-1)}\\
&\quad -\frac{1}{3}\delta^{\alpha\beta}J^{\gamma_1\gamma_2\gamma_3}_{(3(m+n)+1)} - \frac{1}{3\cdot 6!}\delta^{\alpha\beta}\epsilon^{\gamma_1\gamma_2\gamma_3\delta_1\ldots\delta_6}J^{\delta_1\ldots\delta_6}_{(3(m-n)-1)},\\
\left[S^{\alpha\beta}_{(3n)}, J^{\gamma_1\gamma_2\gamma_3}_{(3m+1)}\right] &= 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m+n)+1)} - 3\delta^{\beta\gamma_1}J^{\alpha\gamma_2\gamma_3}_{(3(m-n)-1)}\\
&\quad -\frac{1}{3}\delta^{\alpha\beta}J^{\gamma_1\gamma_2\gamma_3}_{(3(m+n)+1)} + \frac{1}{3}\delta^{\alpha\beta}J^{\gamma_1\gamma_2\gamma_3}_{(3(m-n)-1)},\\
\left[J^{\alpha\beta}_{(3m)}, J^{\gamma_1\ldots\gamma_6}_{(3n+2)}\right] &= 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m+n)+2)} - \delta^{\beta\gamma_1}\epsilon^{\alpha\gamma_2\ldots\gamma_6\delta_1\delta_2\delta_3}J^{\delta_1\delta_2\delta_3}_{(3(m-n)-2)},\\
\left[J^{\alpha\beta}_{(3n)}, J^{\gamma_1\ldots\gamma_6}_{(3m+2)}\right] &= 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m+n)+2)} + 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m-n)+2)},\\
\left[S^{\alpha\beta}_{(3m)}, J^{\gamma_1\ldots\gamma_6}_{(3n+2)}\right] &= 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m+n)+2)} + \delta^{\beta\gamma_1}\epsilon^{\alpha\gamma_2\ldots\gamma_6\delta_1\delta_2\delta_3}J^{\delta_1\delta_2\delta_3}_{(3(m-n)-2)}\\
&\quad -\frac{2}{3}\delta^{\alpha\beta}J^{\gamma_1\ldots\gamma_6}_{(3(m+n)+2)} - \frac{1}{9}\delta^{\alpha\beta}\epsilon^{\gamma_1\ldots\gamma_6\delta_1\delta_2\delta_3}J^{\delta_1\delta_2\delta_3}_{(3(m-n)-2)},\\
\left[S^{\alpha\beta}_{(3n)}, J^{\gamma_1\ldots\gamma_6}_{(3m+2)}\right] &= 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m+n)+2)} - 6\delta^{\beta\gamma_1}J^{\alpha\gamma_2\ldots\gamma_6}_{(3(m-n)+2)}\\
&\quad -\frac{2}{3}\delta^{\alpha\beta}J^{\gamma_1\ldots\gamma_6}_{(3(m+n)+2)} + \frac{2}{3}\delta^{\alpha\beta}J^{\gamma_1\ldots\gamma_6}_{(3(m-n)+2)},\\
\left[J^{\alpha_1\alpha_2\alpha_3}_{(3m+1)}, J^{\beta_1\beta_2\beta_3}_{(3n+1)}\right] &= J^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}_{(3(m+n)+2)} - 18\delta^{\alpha_1\beta_1}\delta^{\alpha_2\beta_2}\left(J^{\alpha_3\beta_3}_{(3(m-n))} + S^{\alpha_3\beta_3}_{(3(m-n))}\right),
\end{aligned}$$

$$
\begin{aligned}
\left[J^{\alpha_1\alpha_2\alpha_3}_{(3m+1)}, J^{\beta_1\ldots\beta_6}_{(3n+2)}\right] &= 3\epsilon^{\gamma\beta_1\ldots\beta_6\alpha_1\alpha_2}\left(J^{\alpha_3\gamma}_{(3(m+n)+3)} + S^{\alpha_3\gamma}_{(3(m+n)+3)}\right) \\
&\quad +\frac{1}{6}\delta^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3}\epsilon^{\beta_4\beta_5\beta_6\gamma_1\ldots\gamma_6}J^{\gamma_1\ldots\gamma_6}_{(3(m-n)-1)}, \\
\left[J^{\alpha_1\alpha_2\alpha_3}_{(3n+1)}, J^{\beta_1\ldots\beta_6}_{(3m+2)}\right] &= 3\epsilon^{\gamma\beta_1\ldots\beta_6\alpha_1\alpha_2}\left(J^{\alpha_3\gamma}_{(3(m+n)+3)} + S^{\alpha_3\gamma}_{(3(m+n)+3)}\right) \\
&\quad -120\delta^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3}J^{\beta_4\beta_5\beta_6}_{(3(m-n)+1)}, \\
\left[J^{\alpha_1\ldots\alpha_6}_{(3m+2)}, J^{\beta_1\ldots\beta_6}_{(3n+2)}\right] &= -6\cdot 6!\delta^{\alpha_1\beta_1}\cdots\delta^{\alpha_5\beta_5}\left(J^{\alpha_6\beta_6}_{(3(m-n))} + S^{\alpha_6\beta_6}_{(3(m-n))}\right) \\
&\quad -400\delta^{\alpha_1\beta_1}\cdots\delta^{\alpha_3\beta_3}\epsilon^{\alpha_4\ldots\alpha_6\beta_4\ldots\beta_5\gamma_1\gamma_2\gamma_3}J^{\gamma_1\gamma_2\gamma_3}_{(3(m+n)+4)}\,, \qquad (3.5)
\end{aligned}
$$

with implicit (anti-)symmetrizations on the r.h.s. according to the symmetries of the l.h.s. and with the understanding that the level zero symmetric piece vanishes: $S^{\alpha\beta}_{(0)} = 0$. Note that in some relations a level index become negative for $m = n$; in those cases one has to use the formula in the next row for which this does not happen. Let us emphasize once more that these formulas were deduced by making use of the identifications found in the previous section, and by exploiting the fact that the affine E_9 commutators are known *for all levels*, whereas we have no complete knowledge of the higher level commutation relations for E_{10}. From the above commutation relations, one readily verifies that the Lie algebra $K(E_9)$ indeed possesses a 'filtered' structure, with

$$
[J_{(k)}\,,\,J_{(l)}] = J_{(k+l)} + J_{(|k-l|)} \qquad (k,l\geq 0). \qquad (3.6)
$$

3.1 Dirac-spinor ideal

Under $K(E_{10})$ the 32-dimensional Dirac-spinor ε transforms as follows on the first four levels [8, 9, 11]

$$
\begin{aligned}
J^{ab}_{(0)}\varepsilon &= \frac{1}{2}\Gamma^{ab}\varepsilon, \\
J^{a_1a_2a_3}_{(1)}\varepsilon &= \frac{1}{2}\Gamma^{a_1a_2a_3}\varepsilon, \\
J^{a_1\ldots a_6}_{(2)}\varepsilon &= \frac{1}{2}\Gamma^{a_1\ldots a_6}\varepsilon, \\
J^{a_0|a_1\ldots a_8}_{(3)}\varepsilon &= 4\delta^{a_0[a_1}\Gamma^{a_2\ldots a_8]}\varepsilon, \qquad (3.7)
\end{aligned}
$$

where Γ^a are the ten real, symmetric (32×32) Γ-matrices of $SO(10)\subset GL(10)$ and $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$ etc. denote their anti-symmetrised products. Note that only the $SO(10)$ trace part of $J^{a_0|a_1\ldots a_8}_{(3)}$ is realised non-trivially, in accordance with the fact that no Young tableaux other than fully antisymmetric ones can be built with Γ-matrices. Furthermore, we have rescaled the 'level' 3 generator by a factor 1/3 relative to [3, 9, 11]. As emphasized in [9, 10, 11], the above representation is *unfaithful* as the infinite-dimensional group is realized on a finite number of spinor components.

Before proceeding it is useful to define the matrix

$$\Gamma^* := \Gamma^1\Gamma^0, \tag{3.8}$$

in terms of which the following relation holds for the (32×32) Γ-matrices

$$\Gamma^{\alpha_1\dots\alpha_9} = \epsilon^{\alpha_1\dots\alpha_9}\Gamma^* \quad \Rightarrow \quad \Gamma^{\alpha_1\dots\alpha_k} = \frac{(-1)^{k(k-1)/2}}{(9-k)!}\epsilon^{\alpha_1\dots\alpha_k\beta_{k+1}\dots\beta_9}\Gamma_{\beta_{k+1}\dots\beta_9}\Gamma^* \tag{3.9}$$

with the $SO(9)$ invariant tensor $\epsilon^{\alpha_1\dots\alpha_9}$. The matrix Γ^* satisfies $(\Gamma^*)^2=1$ and commutes with all Γ^α for $\alpha=2,\dots,10$, but anticommutes with Γ^0 and Γ^1, and hence should be identified with the chirality (helicity) matrix in $(1+1)$ space-time dimensions. By defining $\chi_\pm = \frac{1}{2}(1\pm\Gamma^*)\chi$ for any 32-component spinor, it therefore serves to split any such χ into two sets of 16-component objects, which can be viewed as the right- and left-handed components, respectively, of a spinor in (1+1) dimensions, and whose 16 'internal' components transform as spinors under $SO(9)=K(SL(9))\subset K(E_9)$.

The (unfaithful) action of $K(E_9)$ on a Dirac-spinor ε is obtained from (3.7) by restricting the range of the indices, as described before. From the construction of the consistent representation we can in this case derive a closed formula for the action of *all* $K(E_9)$ generators by repeated commutation of the low level elements (3.7) and use of (3.9), and finally comparison with (3.5). The result is[8]

$$\begin{aligned}
J^{\alpha\beta}_{(3k)} &= \frac{1}{2}\Gamma^{\alpha\beta}(\Gamma^*)^k,\\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}(\Gamma^*)^k,\\
J^{\alpha_1\dots\alpha_6}_{(3k+2)} &= \frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}(\Gamma^*)^k,\\
S^{\alpha\beta}_{(3k+3)} &= 0,
\end{aligned} \tag{3.10}$$

where, of course, $k\geq 0$. It follows from (3.10) in particular that $S^{\alpha\beta}_{(3k+3)}$ is represented trivially on the Dirac spinor, and likewise that the relations involving $S^{\alpha\beta}_{(3k+3)}$ all trivialise, as it should be. For the (reducible) Dirac representation, we thus read off the relations (again for $k\geq 0$)

$$\begin{aligned}
J^{\alpha\beta}_{(3k)} &= J^{\alpha\beta}_{(3k+6)}\ , \qquad S^{\alpha\beta}_{(3k+3)} = 0\ ,\\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= -\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}J^{\beta_1\dots\beta_6}_{(3k+5)},\\
J^{\alpha_1\dots\alpha_6}_{(3k+2)} &= -\frac{1}{3!}\epsilon^{\alpha_1\dots\alpha_6\beta_1\beta_2\beta_3}J^{\beta_1\beta_2\beta_3}_{(3k+4)}.
\end{aligned} \tag{3.11}$$

The existence of a 32-dimensional unfaithful representation of $K(E_9)$ (derived from the 32-dimensional irreducible Dirac spinor of $K(E_{10})$) is thus reflected

[8]The rescaling of the level $\ell=3$ generators by $1/3$ in comparison with [3] is needed to ensure that the level $(3k)$ generators are uniformly normalised.

by the existence of a non-trivial ideal within the Lie algebra $K(E_9)$, via (1.1). For obvious reasons, we will refer to this ideal as the *Dirac ideal* and designate it by $\mathfrak{i}_{\rm Dirac}$. To be completely precise, the latter is defined as the linear span within $K(E_9)$ of the relations (3.11). It is straightforward to check that $\mathfrak{i}_{\rm Dirac}$ is indeed an ideal, *i.e.* $[K(E_9),\mathfrak{i}_{\rm Dirac}]\subset\mathfrak{i}_{\rm Dirac}$. Furthermore, since by (3.11) all generators of level greater than three can be expressed in terms of lower level generators, the codimension of this ideal within $K(E_9)$ is finite, and equal to the number of independent non-zero elements up to level three, which is $2\times(36+84)$. The resulting quotient is a finite-dimensional subalgebra of $\mathfrak{gl}(32)$ and has the structure

$$\mathfrak{q}_{\rm Dirac}=K(E_9)/\mathfrak{i}_{\rm Dirac}=\mathfrak{so}(16)_+\oplus\mathfrak{so}(16)_-. \tag{3.12}$$

To see that the Lie algebra on the r.h.s. has been correctly identified, recall from [9, 11] that the quotient algebra associated with the unfaithful Dirac-spinor in $K(E_{10})$ is $\mathfrak{so}(32)$; according to (3.12) this splits into $\mathfrak{so}(16)_+\oplus\mathfrak{so}(16)_-$, since all anti-symmetric (16×16) matrices are contained in the list (3.10).

Since Γ^* commutes with all these representation matrices, we can decompose the 32-dimensional $K(E_9)$ representation space further into eigenspaces of Γ^* which are invariant under the $K(E_9)$ action. These are projected out by $\frac{1}{2}(1\pm\Gamma^*)$, and we have the branching

$$\mathbf{32}\quad\to\quad\mathbf{16}_+\oplus\mathbf{16}_- \tag{3.13}$$

into two inequivalent spinor representations of $K(E_9)$. On the $\mathbf{16}_\pm$ representations of $K(E_9)$, one can thus replace Γ^* by ±1. This allows us to enlarge the Dirac ideal (3.11) in two possible ways by replacing the relations (3.11) by

$$\begin{aligned}
J^{\alpha\beta}_{(3k)} &= \pm J^{\alpha\beta}_{(3k+3)},\\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \mp\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\ldots\beta_6}J^{\beta_1\ldots\beta_6}_{(3k+2)},\\
S^{\alpha\beta}_{(3k)} &= 0,
\end{aligned} \tag{3.14}$$

for the $\mathbf{16}_\pm$ representations, thereby defining two new ideals $\mathfrak{i}^\pm_{\rm Dirac}\supset\mathfrak{i}_{\rm Dirac}$. The quotient algebras are easily seen to be

$$\mathfrak{q}^\pm_{\rm Dirac}=K(E_9)/\mathfrak{i}^\pm_{\rm Dirac}=\mathfrak{so}(16)_\pm. \tag{3.15}$$

Let us now study in a bit more detail the ideal associated with the $\mathbf{16}_\pm$ Dirac-spinors of $K(E_9)$ determined by (3.14) and, in particular, its orthogonal complement with respect to the $K(E_9)$ (and E_9 [36]) invariant form $\langle\cdot|\cdot\rangle$ under which

$$
\begin{aligned}
\left\langle J^{\alpha\beta}_{(3k)} \middle| J^{\gamma\delta}_{(3k)} \right\rangle &= -2\cdot 2!\,\delta^{\alpha\beta}_{\gamma\delta} && \left[= \frac{1}{16}\mathrm{Tr}\left(\Gamma^{\alpha\beta}\Gamma^{\gamma\delta}\right) \right], \\
\left\langle J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} \middle| J^{\beta_1\beta_2\beta_3}_{(3k+1)} \right\rangle &= -2\cdot 3!\,\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} && \left[= \frac{1}{16}\mathrm{Tr}\left(\Gamma^{\alpha_1\alpha_2\alpha_3}\Gamma^{\beta_1\beta_2\beta_3}\right) \right], \\
\left\langle J^{\alpha_1\ldots\alpha_6}_{(3k+2)} \middle| J^{\beta_1\ldots\beta_6}_{(3k+2)} \right\rangle &= -2\cdot 6!\,\delta^{\alpha_1\ldots\alpha_6}_{\beta_1\ldots\beta_6} && \left[= \frac{1}{16}\mathrm{Tr}\left(\Gamma^{\alpha_1\ldots\alpha_6}\Gamma^{\beta_1\ldots\beta_6}\right) \right].
\end{aligned}
\tag{3.16}
$$

We also have the consistency of orthogonality relations

$$
\left\langle J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} \middle| J^{\beta_1\ldots\beta_6}_{(3k+2)} \right\rangle = 0 \qquad \left[= \frac{1}{16}\mathrm{Tr}\left(\Gamma^{\alpha_1\alpha_2\alpha_3}\Gamma^{\beta_1\ldots\beta_6}\right) \right]. \tag{3.17}
$$

Note that the invariant inner product Tr on the 32-dimensional representation agrees with the one on the algebra for the $J_{(m)}$ generators. Evaluated on $S^{\alpha\beta}_{(3k)}$ it vanishes in contrast with the non-vanishing inner product in $K(E_9)$. This is no contradiction since we are dealing with an unfaithful representation.

Defining the infinite linear combinations

$$
\begin{aligned}
\mathcal{J}^{\alpha\beta}_{\pm} &= \sum_{n\geq 0} (\pm 1)^n J^{\alpha\beta}_{(3n)} \\
\mathcal{J}^{\alpha\beta\gamma}_{\pm} &= \sum_{n\geq 0} (\pm 1)^n \left(J^{\alpha\beta\gamma}_{(3n+1)} \pm \epsilon^{\alpha\beta\gamma\delta_1\ldots\delta_6} J^{\delta_1\ldots\delta_6}_{(3n+2)} \right)
\end{aligned}
\tag{3.18}
$$

one checks that w.r.t. (3.1)

$$
\left(\mathcal{J}^{\alpha\beta}_{\pm} \middle| Z \right) = \left(\mathcal{J}^{\alpha\beta\gamma}_{\pm} \middle| Z \right) = 0 \qquad \text{for all } Z \in \mathfrak{i}^{\pm}_{\mathrm{Dirac}} \tag{3.19}
$$

and so $\mathcal{J}^{\alpha\beta}_{\pm}$ and $\mathcal{J}^{\alpha\beta\gamma}_{\pm}$ formally belong to the orthogonal complement of $\mathfrak{i}^{\pm}_{\mathrm{Dirac}}$.[9] Thus, the elements (3.18) are not proper elements of the vector space underlying the Lie algebra $K(E_9)$ because the infinite series (3.18) do not converge in the (Hilbert space) completion of $K(E_9)$ w.r.t. the norm (3.1). However, they do exist as *distributions*, that is, as elements of the dual of the space of finite linear combinations of basis elements (3.10) (which is dense in the Hilbert space completion of $K(E_9)$). This is also the reason why the elements $\{\mathcal{J}^{\alpha\beta}_{\pm}, \mathcal{J}^{\alpha\beta\gamma}_{\pm}\}$ do not close into a proper subalgebra of $K(E_9)$, as would be the case for the orthogonal complement of an ideal in a finite-dimensional Lie algebra. Nevertheless, as we saw above, there is a way to make sense of (3.18) as defining a Lie algebra by passing to the quotient algebras (3.12) and (3.15). In section 4.2 we will see that these quotient algebras correspond to *generalised evaluation maps* in terms of a loop algebra description. The *distributional nature* of these objects is also

[9] Where the elements of $Z \in \mathfrak{i}^{\pm}_{\mathrm{Dirac}}$ are understood to be *finite* linear combinations of (3.14).

evident from the fact that formal commmutation of the elements (3.18) leads to infinite factors $\delta(0) \sim \sum_{k=1}^{\infty} 1$. Whereas for $K(E_9)$ the distributional nature can be made precise in terms of usual δ-functions on the spectral parameter plane (see section 4.2), such a description is not readily available for $K(E_{10})$. Giving a more precise definition of the space of distributions for $K(E_{10})$ could prove helpful in understanding the $K(E_{10})$ structure better.

It may seem surprising that $K(E_9)$ admits ideals, whereas E_9 does not. One reason that E_9 (or any other simple affine Lie algebra) does not admit ideals is the presence of the derivation d as an element of E_9 (or any other affine) Lie algebra: because relations such as (3.11) and (3.14) involve *different* affine levels (even within generators $J_{(n)}$ of fixed n, as we saw), commutation with d will change the relative coefficients between the terms defining the ideal by (2.7), hence will force the individual terms to vanish also, thus leading to the trivial ideal $\mathfrak{i} = 0$. The existence of non-trivial ideals in $K(E_9)$ is thus due in particular to the fact that d is *not* an element of $K(E_9)$. In the section 4.2 we shall give a loop algebra interpretation of this result.

3.2 Vector-spinor ideal

The $K(E_{10})$ transformation of the 320-component vector-spinor ψ_a can also be written in terms of $SO(10)$ Γ-matrices [9, 10]. For the first three $SO(10)$ 'levels' the $K(E_{10})$ expressions are[10]

$$\begin{aligned}
(J^{ab}_{(0)} \cdot \psi)_c &= \frac{1}{2}\Gamma^{ab}\psi_c + 2\delta_c^{[a}\psi^{b]}, \\
(J^{a_1a_2a_3}_{(1)} \cdot \psi)_b &= \frac{1}{2}\Gamma^{a_1a_2a_3}\psi_b + 4\delta_b^{[a_1}\Gamma^{a_2}\psi^{a_3]} - \Gamma_b{}^{[a_1a_2}\psi^{a_3]}, \\
(J^{a_1\dots a_6}_{(2)} \cdot \psi)_b &= \frac{1}{2}\Gamma^{a_1\dots a_6}\psi_b - 10\delta_b^{[a_1}\Gamma^{a_2\dots a_5}\psi^{a_6]} + 4\Gamma_b{}^{[a_1\dots a_5}\psi^{a_6]}, \\
(J^{a_0|a_1\dots a_8}_{(3)} \cdot \psi)_b &= \frac{16}{9}\left(\Gamma_b{}^{a_1\dots a_8}\psi^{a_0} - \Gamma_b{}^{a_0[a_1\dots a_7}\psi^{a_8]}\right) \\
&\quad +4\delta^{a_0[a_1}\Gamma^{a_2\dots a_8]}\psi_b - 56\delta^{a_0[a_1}\Gamma_b{}^{a_2\dots a_7}\psi^{a_8]} \\
&\quad +\frac{16}{9}\left(8\delta_b^{a_0}\Gamma^{[a_1\dots a_7}\psi^{a_8]} - \delta_b^{[a_1}\Gamma^{a_2\dots a_8]}\psi^{a_0} + 7\delta_b^{[a_1}\Gamma_{a_0}{}^{a_2\dots a_7}\psi^{a_8]}\right).
\end{aligned} \tag{3.20}$$

Reducing these transformations to $K(E_9)$ one decomposes ψ_a into ψ_α and ψ_1, and the former field further into a traceless part and a trace; we can then impose the tracelessness condition

$$\Gamma^\alpha\psi_\alpha = 0. \tag{3.21}$$

As shown in [12] and [11], cf. eq. (2.29), this condition is compatible with $K(E_n)$ only for $n = 9$, as required. The 2×16 components of this trace appear as direct

[10] When comparing these expressions to [9] we recall once more that we re-scaled $J_{(3)}$ by 1/3 as for the Dirac-spinor.

summands in the branching of the vector-spinor to $K(E_9)$. We are then left with 2×128 fermionic components in ψ_α, which is indeed the number of physical fermionic degrees of freedom of maximal $N = 16$ supergravity in two dimensions. In addition to the $SO(9)$ vector spinor ψ_α, the gravitino field ψ_a, when viewed from the reduction to two space-time dimensions, gives rise to

$$\eta := \Gamma^1 \psi_1 \tag{3.22}$$

where the extra factor of Γ^1 is required for η to transform as a proper Dirac-spinor (and not the first component of a vector-spinor) under the two-dimensional Lorentz group $SO(1,1)$. With regard to $SO(9) \subset K(E_9)$, η carries 2×16 components, transforming in the spinor representation of $SO(9)$. The correspondence of the fields ψ_α and η with the fermionic fields used in [1] will be explained in section 4.2.

Computing the $K(E_9)$ transformations for 'levels' 0 up to 3 on the components ψ_α one obtains

$$\begin{aligned}
(J^{\alpha\beta}_{(0)}\psi)_\gamma &= \frac{1}{2}\Gamma^{\alpha\beta}\psi_\gamma + 2\delta^{[\alpha}_\gamma\psi^{\beta]},\\
(J^{\alpha_1\alpha_2\alpha_3}_{(1)}\psi)_\beta &= \frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\psi_\beta + 4\delta^{[\alpha_1}_\beta\Gamma^{\alpha_2}\psi^{\alpha_3]} - \Gamma_\beta{}^{[\alpha_1\alpha_2}\psi^{\alpha_3]},\\
(J^{\alpha_1\dots\alpha_6}_{(2)}\psi)_\beta &= \frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}\psi_\beta - 10\delta^{[\alpha_1}_\beta\Gamma^{\alpha_2\dots\alpha_5}\psi^{\alpha_6]} + 4\Gamma_\beta{}^{[\alpha_1\dots\alpha_5}\psi^{\alpha_6]},\\
(J^{\alpha\beta}_{(3)}\psi)_\gamma &= -\Gamma^*\left[\frac{1}{2}\Gamma^{\alpha\beta}\psi_\gamma + 2\delta^{[\alpha}_\gamma\psi^{\beta]}\right],\\
(S^{\alpha\beta}_{(3)}\psi)_\gamma &= 0,
\end{aligned} \tag{3.23}$$

where (3.21) must be used repeatedly, for instance, to show that $S^{\alpha\beta}_{(3)}$ acts trivially. Note that the transformations on ψ_α close on themselves, in agreement with the existence of a 128-dimensional representation of $K(E_9)$ [12]. Extending the action (3.23) by the commutation relations (3.5) we deduce the general action on the ψ_α components:

$$\begin{aligned}
(J^{\alpha\beta}_{(3k)}\psi)_\gamma &= (-\Gamma^*)^k\left[\frac{1}{2}\Gamma^{\alpha\beta}\psi_\gamma + 2\delta^{[\alpha}_\gamma\psi^{\beta]}\right] = (J^{\alpha\beta}_{(3k+6)}\psi)_\gamma,\\
(J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)}\psi)_\beta &= (-\Gamma^*)^k\left[\frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\psi_\beta + 4\delta^{[\alpha_1}_\beta\Gamma^{\alpha_2}\psi^{\alpha_3]} - \Gamma_\beta{}^{[\alpha_1\alpha_2}\psi^{\alpha_3]}\right]\\
&= -\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\gamma_1\dots\gamma_6}(J^{\gamma_1\dots\gamma_6}_{(3k+5)}\psi)_\beta,\\
(J^{\alpha_1\dots\alpha_6}_{(3k+2)}\psi)_\beta &= (-\Gamma^*)^k\left[\frac{1}{2}\Gamma^{\alpha_1\dots\alpha_6}\psi_\beta - 10\delta^{[\alpha_1}_\beta\Gamma^{\alpha_2\dots\alpha_5}\psi^{\alpha_6]} + 4\Gamma_\beta{}^{[\alpha_1\dots\alpha_5}\psi^{\alpha_6]}\right]\\
&= -\frac{1}{3!}\epsilon^{\alpha_1\dots\alpha_6\gamma_1\gamma_2\gamma_3}(J^{\gamma_1\gamma_2\gamma_3}_{(3k+4)}\psi)_\beta,\\
(S^{\alpha\beta}_{(3k)}\psi)_\gamma &= 0.
\end{aligned} \tag{3.24}$$

where we have already written out the relations defining the ideal, which are the same as in (3.11). That is, we have the same $SO(16)_+ \times SO(16)_-$ acting on this part of the gravitino. The action for $J^{\alpha_1\ldots\alpha_6}_{(3k+2)}$ can be written in a dual form as shown above. Moreover, we see that we can again specialise to the $\Gamma^* = \pm 1$ subspaces. There it is easiest to deduce the following ideal relations for these vector-spinor components

$$\begin{aligned} J^{\alpha\beta}_{(3k+3)} &= \mp J^{\alpha\beta}_{(3k)}, \\ J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \pm\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\ldots\beta_6} J^{\beta_1\ldots\beta_6}_{(3k+2)}, \\ S^{\alpha\beta}_{(3k)} &= 0 \end{aligned} \tag{3.25}$$

in analogy with (3.14) (except that Γ^* is replaced by $(-\Gamma^*)$). By the arguments of the preceding sections the relevant ideal on the components ψ_α gives a quotient isomorphic to $\mathfrak{so}(16)_\pm$. However, whereas the ψ_α transform among themselves as a $K(E_9)$ subrepresentation they do not form a direct summand in the decomposition into $K(E_9)$ representations since they mix into the η component as we will see presently. Therefore the resulting quotient algebra will be different from $\mathfrak{so}(16)_\pm$.

The transformation properties of the $\eta = \Gamma^1\psi_1$ component of the original 320-dimensional $K(E_{10})$ representation are more complicated. At the first three levels, they read

$$\begin{aligned} J^{\alpha\beta}_{(0)}\eta &= \frac{1}{2}\Gamma^{\alpha\beta}\eta, \\ J^{\alpha_1\alpha_2\alpha_3}_{(1)}\eta &= -\frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\eta - \Gamma^{[\alpha_1\alpha_2}\psi^{\alpha_3]}, \\ J^{\alpha_1\ldots\alpha_6}_{(2)}\eta &= \frac{1}{2}\Gamma^{\alpha_1\ldots\alpha_6}\eta + 4\Gamma^{[\alpha_1\ldots\alpha_5}\psi^{\alpha_6]}, \\ J^{\alpha\beta}_{(3)}\eta &= -\frac{1}{2}\Gamma^*\Gamma^{\alpha\beta}\eta, \\ S^{\alpha\beta}_{(3)}\eta &= 2\Gamma^*\Gamma^{(\alpha}\psi^{\beta)}. \end{aligned} \tag{3.26}$$

The mixing of ψ_α into η under $K(E_9)$ transformations is manifest in these relations. We can again use the $K(E_9)$ commutation relations (3.5) to deduce the action for all generators from (3.26)

$$\begin{aligned} J^{\alpha\beta}_{(3k)}\eta &= (-\Gamma^*)^k\left[\frac{1}{2}\Gamma^{\alpha\beta}\eta\right], \\ J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)}\eta &= (-\Gamma^*)^k\left[-\frac{1}{2}\Gamma^{\alpha_1\alpha_2\alpha_3}\eta - (3k+1)\Gamma^{[\alpha_1\alpha_2}\psi^{\alpha_3]}\right], \\ J^{\alpha_1\ldots\alpha_6}_{(3k+2)}\eta &= (-\Gamma^*)^k\left[\frac{1}{2}\Gamma^{\alpha_1\ldots\alpha_6}\eta + 2(3k+2)\Gamma^{[\alpha_1\ldots\alpha_5}\psi^{\alpha_6]}\right], \\ S^{\alpha\beta}_{(3k)}\eta &= -(-\Gamma^*)^k\left[2k\Gamma^{(\alpha}\psi^{\beta)}\right]. \end{aligned} \tag{3.27}$$

Similar to (3.11) we can immediately deduce the relations defining the vector-spinor ideal $\mathfrak{i}_{\rm vs}$ in $K(E_9)$

$$\begin{aligned}
J^{\alpha\beta}_{(3k)} &= J^{\alpha\beta}_{(3k+6)},\\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+7)} - J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}\left(J^{\beta_1\dots\beta_6}_{(3k+5)} - J^{\beta_1\dots\beta_6}_{(3k-1)}\right),\\
(3k+1)J^{\alpha_1\alpha_2\alpha_3}_{(3k+7)} - (3k+7)J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= -\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}\\
&\quad \times\left((3k-1)J^{\beta_1\dots\beta_6}_{(3k+5)} - (3k+5)J^{\beta_1\dots\beta_6}_{(3k-1)}\right),\\
(k+2)S^{\alpha\beta}_{(3k)} &= k\, S^{\alpha\beta}_{(3k+6)}.
\end{aligned}\tag{3.28}$$

These are valid both on the ψ_α and η components. The first two relations arise from considering the η parts of the transformed spinor (3.27), the latter two can be derived by focussing on the ψ_α pieces in the transformed spinor (3.27) and are evidently k-dependent. Note also that the relations in the middle involve *four* different $\mathfrak{sl}(9)$ levels.

Just as in the Dirac case it follows immediately from the form of the transformations (3.24) and (3.27) that Γ^* commutes with all representation matrices and therefore one can restrict to the $\Gamma^* = \pm\mathbf{1}$ eigenspaces. Hence, on the $\Gamma^* = \pm\mathbf{1}$ eigenspaces the vector-spinor ideal relations (3.28) simplify in analogy with (3.14)

$$\begin{aligned}
J^{\alpha\beta}_{(3k)} &= \mp J^{\alpha\beta}_{(3k+3)},\\
J^{\alpha_1\alpha_2\alpha_3}_{(3k+4)} \pm J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \mp\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}\left(J^{\beta_1\dots\beta_6}_{(3k+5)} \pm J^{\beta_1\dots\beta_6}_{(3k+2)}\right),\\
(3k+1)J^{\alpha_1\alpha_2\alpha_3}_{(3k+4)} \pm (3k+4)J^{\alpha_1\alpha_2\alpha_3}_{(3k+1)} &= \pm\frac{1}{6!}\epsilon^{\alpha_1\alpha_2\alpha_3\beta_1\dots\beta_6}\\
&\quad \times\left((3k+2)J^{\beta_1\dots\beta_6}_{(3k+5)} \pm (3k+5)J^{\beta_1\dots\beta_6}_{(3k+2)}\right),\\
S^{\alpha\beta}_{(3k)} &= (\pm1)^{k+1}\, kS^{\alpha\beta}_{(3)}.
\end{aligned}\tag{3.29}$$

Again these relations are valid both for the ψ_α and for the η components. On the not completely reducible representation encountered here, they leave more freedom: Counting the number of independent generators on the $\mathbf{144}_\pm$ representation we find that there are $(40+80)+(80+48) = 120+128$ independent generators. It follows from (3.29) that the action of all $K(E_9)$ generators in the vector spinor representation can be reduced to that of $J^{\alpha\beta}_{(0)}$, $J^{\alpha_1\alpha_2\alpha_3}_{(1)}$, $J^{\alpha_1\dots\alpha_6}_{(2)}$ and $S^{\alpha\beta}_{(3)}$. Via the above relations, all higher level generators can thus be expressed as linear combinations of these 248 basic ones.

Let us now summarize our findings and write out the branching of the $\mathbf{320}$ representation of $K(E_{10})$ into representations of its $K(E_9)$ subalgebra. In comparison with the Dirac representation, the vector-spinor representation exhibits a curious new feature in the branching. Namely, the transformations on η contain contributions also involving ψ_α. On the other hand the ψ_α components transform solely among themselves. This means that the ψ_a representation of $K(E_{10})$ does not completely reduce into irreducible representations of $K(E_9)$ as one might have expected, rather we have

$$\mathbf{320} \rightarrow \mathbf{144}_+ \oplus \mathbf{144}_- \big(\oplus \mathbf{16}'_+ \oplus \mathbf{16}'_- \big) \tag{3.30}$$

where the primed representations in parantheses are the trace components $\Gamma^\alpha \psi_\alpha$ which we set to zero in our analysis. The remaining chiral 144-dimensional $K(E_9)$ representations have as subrepresentations the 128-dimensional irreducible representations of $K(E_9)$ [12] described by (3.23), but the $\mathbf{144}_\pm$ representations are not completely reducible because the $\mathbf{16}_\pm$ representations $\eta_\pm$ do not only transform among themselves under $K(E_9)$, but their variations contain an admixture of the $\mathbf{128}_\pm$. These results are structurally in accordance with the results of [1], see eqns. (5.12) there, as we will discuss in more detail below.

4 Relation to current algebra realisation

In previous work [1], $K(E_9)$ transformations of unfaithful fermion representations were derived starting from the linear system description of $N = 16$ supergravity in $D = 2$ [21, 23]. In the present section we will show that the transformations (3.24) and (3.27) we deduced from the dimensionally reduced theory above are completely equivalent to those in the linear system.

4.1 $\mathfrak{so}(16) \subset E_{8(8)}$

Since the linear system transformations are written using the spectral parameter presentation of $K(E_9)$ in the $K(E_8) \equiv \mathfrak{so}(16)$ decomposition of E_8 we first need to briefly recall some notation necessary for the comparison; in particular, we require the E_8 commutation relations adapted to the compact $\mathfrak{so}(16)$ subalgebra. In this basis, $E_{8(8)}$ decomposes into the adjoint $\mathbf{120}$ of $\mathfrak{so}(16)$ (corresponding to the anti-symmetric compact generators) and the $\mathfrak{so}(16)$ spinor representation $\mathbf{128_s}$ (corresponding to the symmetric non-compact generators) which can be further decomposed as

$$\begin{aligned} X^{IJ} \in \mathbf{120} \;\rightarrow\quad & (\mathbf{28}, \overline{\mathbf{1}}) \oplus (\mathbf{1}, \overline{\mathbf{28}}) \oplus (\mathbf{8_s}, \overline{\mathbf{8}}_\mathbf{c}) \rightarrow \quad \mathbf{28} \oplus \mathbf{28} \oplus \mathbf{56_v} \oplus \mathbf{8_v}, \\ Y^A \in \mathbf{128_s} \rightarrow \quad & (\mathbf{8_v}, \overline{\mathbf{8}}\mathbf{v}) \oplus (\mathbf{8_s}, \overline{\mathbf{8}}_\mathbf{c}) \rightarrow \quad \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \oplus \mathbf{8_v} \oplus \mathbf{56_v}, \end{aligned} \tag{4.1}$$

with the chain of embeddings $\mathfrak{so}(16) \supset \mathfrak{so}(8) \oplus \mathfrak{so}(8) \supset \mathfrak{so}(8)$, where the indices v, s, c (= vector, spinor, and conjugate spinor) label the three inequivalent eight-dimensional representations of the various $SO(8)$ groups. The diagonal subalgebra $\mathfrak{so}(8)$ is to be identified with the $\mathfrak{so}(8) \subset \mathfrak{sl}(8)$ of the preceding sections. Furthermore, we here take over the notation from [1]: $I, J = 1, \ldots, 16$ are $SO(16)$ vector indices and $A = 1, \ldots, 128$ labels the components of a chiral $SO(16)$ spinor.

Evidently, the first line in (4.1) corresponds to the $\mathfrak{so}(8)$ representations inherited from table 2. However, we will not spell out the formulas relating the $SO(9)$ and $SO(16)$ bases in detail here, but rather refer to the appendix of [35] for further details (note, however, that those formulas correspond to a different normalisation of the $SL(8)$ generators). From (4.1) we also recover the decompositions of $SO(16)$ under its $SO(9)$ subgroup, viz.

$$\mathbf{120} \;\rightarrow\; \mathbf{36} \oplus \mathbf{84} \quad , \quad \mathbf{128_s} \;\rightarrow\; \mathbf{44} \oplus \mathbf{84} \tag{4.2}$$

In the conventions of [35]), the E_8 commutation relations read

$$\begin{aligned} [X^{IJ}, X^{KL}] &= 2\delta^{I[K} X^{L]J} - 2\delta^{J[K} X^{L]I}, \\ [X^{IJ}, Y^{A}] &= -\frac{1}{2}\Gamma^{IJ}_{AB} Y^B \quad , \qquad [Y^A, Y^B] = \frac{1}{4}\Gamma^{IJ}_{AB} X^{IJ}. \end{aligned} \tag{4.3}$$

With the current algebra generators (for $m \in \mathbb{Z}$)

$$X^{(m)IJ} \equiv X^{IJ} \otimes t^m \quad , \qquad Y^{(m)A} \equiv Y^A \otimes t^m \tag{4.4}$$

the $K(E_9)$ generators can be represented in the form (for $m \geq 0$)

$$A^{(m)IJ} := \frac{1}{2}\left(X^{(m)IJ} + X^{(-m)IJ}\right) \quad , \qquad S^{(m)A} := \frac{1}{2}\left(Y^{(m)A} - Y^{(-m)A}\right) \tag{4.5}$$

implying $S^{(0)A} \equiv 0$. The $K(E_9)$ commutation relations then read

$$\begin{aligned} [A^{(m)IJ}, A^{(n)KL}] &= 2\delta^{[I[K}\Big(A^{(m+n)L]J]} + A^{(|m-n|)L]J]}\Big) \\ [A^{(m)IJ}, S^{(n)A}] &= -\frac{1}{4}\Gamma^{IJ}_{AB}\Big(S^{(m+n)B} - \mathrm{sgn}(m-n) S^{(|m-n|)B}\Big) \\ [S^{(m)A}, S^{(n)B}] &= \frac{1}{8}\Gamma^{IJ}_{AB}\Big(A^{(m+n)IJ} - A^{(|m-n|)IJ}\Big) \end{aligned} \tag{4.6}$$

for $m, n \geq 0$ (recall that the central term drops out).

In the formulation (4.6) we can immediately look for ideals of $K(E_9)$. The Dirac ideals $\mathfrak{i}^{\pm}_{\text{Dirac}}$ are now defined by the relations

$$A^{(m)IJ} - (\pm 1)^m A^{(0)IJ} = 0 \quad , \qquad S^{(m)A} = 0. \tag{4.7}$$

That is, the ideals are defined as the linear span of the expressions on the l.h.s., and it is then straightforward to verify the ideal property, namely that these subspaces are mapped onto themselves under the adjoint action of $K(E_9)$. The quotient algebras obtained by division of $K(E_9)$ by these ideals are obviously isomorphic to $\mathfrak{so}(16)$ for both choices of signs.

The vector-spinor ideals $\mathfrak{i}^{\pm}_{\text{vs}}$, on the other hand, can be defined by the relations (for $m \geq 1$)

$$A^{(m)IJ} - (\pm 1)^m A^{(0)IJ} = 0 \quad , \qquad S^{(m)A} \mp (\pm 1)^m m S^{(1)A} = 0. \tag{4.8}$$

They define smaller ideals of codimension 248 since everything is determined by $A^{(0)IJ}$ and $S^{(1)A}$. The part of the above relations involving $A^{(m)IJ}$ is identical to that of the Dirac-spinor (4.7) indicating that there is some relation of the associated quotient to $\mathfrak{so}(16)$ with an additional part arising from the $S^{(m)A}$ relations. We will see this in more detail below.

The vector-spinor ideals $\mathfrak{i}^{\pm}_{\mathrm{vs}}$ can be generated from $A^{(1)IJ} \mp A^{(0)IJ} = 0$ since for example

$$\left[A^{(1)IJ} \mp A^{(0)IJ}, S^{(1)A}\right] = -\frac{1}{4}\Gamma^{IJ}_{AB}\left(S^{(2)B} \mp 2S^{(1)B}\right) \tag{4.9}$$

implies by the ideal property that $S^{(2)B} \mp 2S^{(1)B}$ has to vanish. Similar calculations show that $A^{(1)IJ} \mp A^{(0)IJ} = 0$ generates all ideal relations.

In this basis it is not hard to construct further ideals. One example is obtained by starting from the relation $(S^{(2)A} \mp 2S^{(1)A}) = 0$, without requiring that $(A^{(1)IJ} \mp A^{(0)IJ}) = 0$. Commuting with $S^{(1)B}$ and demanding that the resulting expression also belongs to the ideal leads to

$$A^{(3)IJ} - A^{(1)IJ} \mp 2A^{(1)IJ} \pm A^{(0)IJ} = 0 \tag{4.10}$$

a relation involving four affine levels. In the case of the vector-spinor these vanish by taking pairwise combinations, here they define a new ideal which is strictly smaller than the vector-spinor ideal.

In section 3.1 we explained that the absence of ideals in E_9 can be interpreted as a consequence of the presence of the derivation d. In the current algebra realization, d acts by differentiation: $d \equiv \frac{d}{dt}$. Setting $X(t_0) = 0$ for some fixed t_0 would then force all higher repeated commutators of this element with d to vanish at $t = t_0$ by consistency. This, in turn would imply the vanishing of all derivatives $X^{(n)}(t_0)$, hence would force $X(t) = 0$ (assuming analyticity in t). This confirms again that the existence of non-trivial ideals in $K(E_9)$ is thus due in particular to the fact that d is not an element of $K(E_9)$. The orthogonal complement of the ideal, given formally by (3.18), corresponds to distributions $X(t) = X_0\,\delta(t - t_0)$ where, as we will see presently, $t_0 = \pm 1$. The associated ideal then consists of all elements of the loop algebra which vanish at those points. We stress that this requires studying a distribution space outside of $K(E_9)$ and that this could prove a useful strategy also for further investigations of $K(E_{10})$.

4.2 Current algebra fermion transformations

In [1] it was realised that in the linear systems approach to two-dimensional $N = 16$ supergravity the transformation rules for the fermions (4.11) can be written succinctly in terms of a current algebra description with a current parameter t. In order to exhibit the relation of those results with the ones derived in the foregoing sections, we need to spell out the correspondence between the $SO(9)$

spinors used in the foregoing sections and the $SO(16)$ representations used in [1]. Keeping track of the Γ-trace (which we set to zero previously), we have

$$\begin{aligned} \chi^{\dot{A}} \quad &\leftrightarrow \quad \psi_\alpha - \frac{1}{9}\Gamma_\alpha \Gamma^\beta \psi_\beta \\ \psi_2^I \quad &\leftrightarrow \quad \Gamma^\alpha \psi_\alpha \\ \psi^I \quad &\leftrightarrow \quad \eta \end{aligned} \tag{4.11}$$

with $SO(16)$ vector and chiral[11] spinor indices $I = 1, \ldots, 16$ and $\dot{A} = 1, \ldots, 128$, respectively. On the r.h.s. we have not written out the $SO(9)$ spinor indices (of which there are 2×16), and furthermore, we have suppressed the two-dimensional space-time indices on both sides. Thus, the spinor components η and $\Gamma^\alpha\psi_\alpha$ correspond to the (non-propagating) gravitino and dilatino degrees of freedom in two dimensions, while the traceless part accommodates the 128 propagating fermionic degrees of freedom. The dilatino component ψ_2^I can be gauged away by use of local supersymmetry [1], in correspondence with our choice $\Gamma^\alpha\psi_\alpha = 0$.

The most general $K(E_9)$ Lie algebra element can be written in the form [1] [12]

$$\begin{aligned} h(t) &= \frac{1}{2}h_0^{IJ}X^{IJ} + \sum_{n=1}^{\infty}\left[\frac{1}{2}h_n^{IJ}X^{IJ}\otimes(t^{-n}+t^n) + h_n^A Y^A \otimes (t^{-n}-t^n)\right] \\ &\equiv \frac{1}{2}h^{IJ}(t)X^{IJ} + h^A(t)Y^A, \end{aligned} \tag{4.12}$$

It can then be shown that $K(E_9)$ acts on the chiral components of the fermions via evaluation at the points $t = \pm 1$ in the spectral parameter plane (cf. eqn. (5.12) of [1])[13]

$$\begin{aligned} \delta_h \psi_{2\pm}^I &= \psi_{2\pm}^J h^{IJ}|_{t=\mp 1}, \\ \delta_h \chi_\pm^{\dot{A}} &= \frac{1}{4}\Gamma^{IJ}_{\dot{A}\dot{B}}\chi_\pm^{\dot{B}} h^{IJ}|_{t=\mp 1} - \Gamma^I_{A\dot{A}}\psi^I_{2\pm}\partial_t h^A|_{t=\mp 1}, \\ \delta_h \psi_\pm^I &= \psi_\pm^J h^{IJ}|_{t=\mp 1} \pm \Gamma^I_{A\dot{B}}\chi_\pm^{\dot{B}}\partial_t h^A|_{t=\mp 1} \mp 2\psi^J_{2\pm}(\partial_t^2 h^{IJ} \mp \partial_t h^{IJ})|_{t=\mp 1}. \end{aligned} \tag{4.13}$$

where $\Gamma^I_{A\dot{A}}$ now denote chiral $SO(16)$ gamma matrices of dimension 128×128. Thus, from the point of view [1] the action of $K(E_9)$ on the fermions can be viewed as an *evaluation map* of the $K(E_9)$ elements, not at the origin in spectral parameter space $t = 0$ but at $t = \pm 1$. In fact, we are dealing with a *generalised* evaluation map in that the transformations depend on up to second derivatives in the spectral parameter at the points $t = \pm 1$.

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[11] The 128 physical fermions $\chi^{\dot{A}}$ transform in the conjugate spinor representation of $SO(16)$.

[12] Since we are interested for the moment in the purely algebraic aspects of the transformation we suppress the space-time dependence throughout. (The spectral parameter t also depends on two-dimensional space-time.)

[13] We note that in [1] it was also shown that, considering only induced $K(E_9)$ transformations, there is non-linear combination of the fermionic and bosonic fields that reduces this action to an action of $SO(16)_+ \times SO(16)_-$.

To relate these results to our findings of the foregoing section, let us adopt the gauge $\psi^I_{2\pm} = 0$, which corresponds to the traceless gauge (3.21). We define

$$h^{IJ}_\pm \equiv h(t)^{IJ}|_{t=\mp 1} \ , \quad h^A_\pm \equiv \partial_t h(t)^A|_{t=\mp 1}, \tag{4.14}$$

so that for example for $t = +1$

$$h^{IJ}_+ = \frac{1}{2}\left(h^{IJ}_0 + 2\sum_{n=1}^{\infty} h^{IJ}_n\right) X^{IJ}, \tag{4.15}$$

$$h^A_+ = -2\sum_{n\geq 1} n h^A_n Y^A. \tag{4.16}$$

Note that (4.16) contains an explicit factor proportional to the affine level n. In terms of these parameters the transformations (4.13) assume the form

$$\begin{aligned} \delta_h \chi^{\dot{A}}_\pm &= \frac{1}{4}\Gamma^{IJ}_{\dot{A}\dot{B}}\chi^{\dot{B}}_\pm h^{IJ}_\pm, \\ \delta_h \psi^I_\pm &= \psi^J_\pm h^{IJ}_\pm \pm \Gamma^I_{A\dot{B}}\chi^{\dot{B}}_\pm h^A_\pm. \end{aligned} \tag{4.17}$$

Now we compare (4.17) to (3.23). The infinite linear combination (4.15) suggests the structure of an $\mathfrak{so}(16)$ transformation as evident in (4.17). This infinite linear combination should indeed be identified with the formal infinite sum in (3.18). For the vector spinor, this structure gets enlarged by another set of 128 parameters h^A_+, so that the associated quotient group in this situation has the structure of a semi-direct product, with non-semi-simple quotient Lie algebra

$$\mathfrak{q}^\pm_{\text{vs}} = \mathfrak{so}(16)_\pm \oplus \mathfrak{t}^\pm_{128} \subset \mathfrak{gl}(248) \tag{4.18}$$

where the abelian translations $\mathfrak{t}^\pm_{128}$ transform in the spinor representation of $\mathfrak{so}(16)_\pm$. We can now see the structural agreement between the transformations (4.13) with those of the vector-spinor (ψ_α, η) in section 3.2. The $K(E_9)$ transformations acting solely on ψ_α were seen to correspond to ideal relations resembling those of the Dirac-spinor with quotients $\mathfrak{so}(16)_\pm$ where the explicit construction of the $\mathfrak{so}(16)_\pm$ generators involved taking infinite sums, cf. (3.18). At the same time, by eq. (4.15) $K(E_9)$ acts solely as $\mathfrak{so}(16)_\pm$ rotation on $\chi^{\dot{A}} \leftrightarrow \psi_\alpha$. The transformation (3.27) on η also includes a linearly k-dependent piece which mixes ψ_α back into η; this is the counterpart of the derivative term (4.16) in (4.13).

Taking the quotient of the loop algebra by this ideal is therefore completely equivalent to the evaluation map (4.13) which assigns to a loop group element $X(t)$ its value at $t = \pm 1$ (which effectively reduces to an $SO(16)_\pm$ matrix). In this fashion, we have recovered precisely the results of [1]. It should now be clear how to extend the above results to the situation where $\psi^I_{2\pm} \neq 0$, and to work out the corresponding formulas in the algebraic setting of the previous section. We leave this computation as an exercise for the reader.

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