Problems which are well-posed in a generalized sense with applications to the Einstein equations

by

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Summary

In the harmonic description of general relativity, the principle part of Einstein equations reduces to a constrained system of 10 curved space wave equations for the components of the space-time metric. We use the pseudo-differential theory of systems which are strongly well-posed in the generalized sense to establish the well-posedness of constraint preserving boundary conditions for this system when treated in second order differential form. The boundary conditions are of a generalized Sommerfeld type that is benevolent for numerical calculation.

1. Introduction

Consider a first order symmetric system of partial differential equations

$$u_t = Au_x + Bu_y + F$$

on the half-space $x \geq 0,\ -\infty < y < \infty,\ t \geq 0$. Here u is a vector valued function with n components and $A = A^*,\ B = B^*$ are symmetric matrices which depend smoothly on x, y and t. Also, A is not singular at $x = 0$. At $t = 0$, we give the initial condition

$$u(x, y, 0) = f(x)$$

and, at $x = 0$, boundary conditions

$$Bu = g$$

which are strictly maximally dissipative. In this case one can use integration by parts to derive an energy estimate. The result can easily be extended to prove local existence of solutions of quasilinear systems because integration by parts allows us to estimate the derivatives.

This result depends heavily on integration by parts. If the boundary conditions are not maximally dissipative or the system is not symmetric hyperbolic, new techniques are needed. There is a rather comprehensive theory based on the principle of frozen coefficients, Fourier and Laplace transform and the theory of pseudo-differential operators which give necessary and sufficient conditions for well-posedness in the generalized sense [1],[2],[3],[4],[5, Chap.8],[6, Chap 10],[7]. This theory can also be extended to second order systems. In the context of pseudo-differential operators one can – in the same way as for ordinary differential equations – write a second order system as a first order system [8].

To make the results of the above theory more precise and how to apply it we have included an Appendix, which should make it easier to read the paper.

In this paper we shall demonstrate that flexibility of the permissible boundary conditions can be applied to solve the constraint problem for the harmonic Einstein equations. The pseudo-differential theory is applied here to the second order harmonic formulation, which was used to establish the first well-posed Cauchy problem for Einstein's equations [9].

The importance of a well-posed constraint preserving initial-boundary value problem (IBVP) to the simulation of Einstein equations has been recognized in numerous recent works. The pseudo-differential treatment of the IBVP presented here is most similar to treatments of first order formulations by Stewart [10], Reula and Sarbach [11] and Sarbach and Tiglio [12]. A well-posed IBVP for the nonlinear harmonic Einstein equations has been formulated for a combination of homogeneous Neumann and Dirichlet boundary conditions (or boundary data linearized off these homogeneous conditions) [13]. Here we consider strictly dissipative, Sommerfeld-type boundary conditions for these harmonic equations, which have proved to be more robust in numerical tests [14,15]. The only general treatment of the nonlinear case has been given by Friedrich and Nagy, based upon a quite different first order formulation of the Einstein equations [16]

In the next section we will explain our theory for the wave equation with boundary conditions which cannot be treated by integration by parts. We apply the results in the third section to the constraint problem of the linearized Einstein equation.

In the last section we will give a more physical interpretation of our technique and explain how it applies to the full Einstein theory.

2. Well-posed problems

Consider the half-plane problem for the wave equation

$$v_{tt} = v_{xx} + v_{yy} + F, \quad 0 \le x < \infty,\ -\infty < y < \infty,\ t \ge 0, \tag{1}$$

with smooth bounded initial data

$$v(x, y, 0) = f_1(x, y), \quad v_t(x, y, 0) = f_2(x, y), \tag{2}$$

and boundary condition

$$v_t = \alpha v_x + \beta v_y + g, \quad x = 0,\ -\infty < y < \infty,\ t \ge 0. \tag{3}$$

Here α, β are real constants. We assume always that $\alpha > 0$. Also, all data are $C^\infty-$smooth compatible functions with compact support.

The usual concept of well-posedness is based on the existence of an energy estimate which is often derived by integration by parts. Let $F \equiv g \equiv 0$ and

$$\|v(\cdot, \cdot, t)\|^2 = \int_0^\infty \int_{-\infty}^\infty |v(x, y, t)|^2 dx\, dy$$

denote the usual L_2-norm. Integration by parts gives us , for the energy E,

$$\frac{\partial}{\partial t} E =: \frac{\partial}{\partial t}(\|v_t\|^2 + \|v_x\|^2 + \|v_y\|^2) = -2 \int_{-\infty}^\infty v_t(0, y, t) v_x(0, y, t) dy$$

If in (3) $\beta \ne 0$, then there is no obvious way to estimate the boundary flux in terms of E. Of course, if $\beta = 0$ and $\alpha > 0$, there is an energy estimate. To be able to discuss the general case we will define well-posedness as in the Appendix.

We start with a very simple observation:

Lemma 1. *The problem is not well-posed if we can find a solution of the homogeneous equation (1) which satisfies the homogeneous boundary condition (3) and which is of the form*

$$v(x, y, t) = e^{st + i\omega y} \varphi(x), \quad |\varphi|_\infty < \infty. \tag{4}$$

Here $\varphi(x)$ is a smooth bounded function and ω is real and $s = i\xi + \eta$ a complex constant with $\eta =$ Re $s > 0$.

Proof. If (4) is a solution, then

$$\varphi_\gamma = e^{s\gamma t + i\gamma\omega y} \varphi(\gamma x)$$

is also a solution for any $\gamma > 0$. Thus we can find solutions which grow arbitrarily fast exponentially.

We shall now discuss how to determine whether such solutions exist. We introduce (4) into (1),(2) and obtain

$$\varphi_{xx} - (s^2 + \omega^2)\varphi = 0, \tag{5}$$

$$s\varphi(0) = \alpha\varphi_x(0) + i\beta\omega\varphi(0), \quad |\varphi|_\infty < \infty. \tag{6}$$

(5),(6) is an eigenvalue problem. We can phrase Lemma 1 also as

Lemma 1'. *The problem is not well-posed if (5),(6) has an eigenvalue with* $\operatorname{Re} s > 0$.

(5) is an ordinary differential equation with constant coefficients and its general solution is of the form

$$\hat{v} = \sigma_1 e^{\kappa_1 x} + \sigma_2 e^{\kappa_2 x}, \tag{7}$$

where $\kappa_1 = +\sqrt{s^2 + \omega^2}$, $\kappa_2 = -\sqrt{s^2 + \omega^2}$ are the solutions of the characteristic equation

$$\kappa^2 - (s^2 + \omega^2) = 0.$$

We define the $\sqrt{\ }$ by

$$-\pi < \arg(s^2 + \omega^2) \le \pi, \quad \arg\sqrt{s^2 + \omega^2} = \frac{1}{2}\arg(s^2 + \omega^2). \tag{8}$$

Thus

$$\operatorname{Re}\kappa_1 > 0 \quad \text{and} \quad \operatorname{Re}\kappa_2 < 0 \quad \text{for} \quad \operatorname{Re} s > 0, \text{ respectively.} \tag{9}$$

By assumption $\hat{v}$ is a bounded function and cannot contain exponentially growing components. Therefore, by (9), this is only possible if $\sigma_1 = 0$, i.e.

$$\varphi = \sigma_2 e^{\kappa_2 x}. \tag{10}$$

Introducing (10) into the boundary condition gives us

$$(s - \alpha\kappa_2 - i\beta\omega)\sigma_2 = 0. \tag{11}$$

$\operatorname{Re} s > 0$ and $\operatorname{Re}\kappa_2 < 0$ tell us that there are no solutions for $\operatorname{Re} s > 0$ since, by assumption, $\alpha > 0$.

For the purpose of proving well-posedness, from now on we assume that the initial data (2) vanish. This may always be achieved by the transformation

$$u = v - e^{-t} f_1 - te^{-t}(f_2 + f_1).$$

(We start the time evolution from 'rest'.) Then we can solve (1)–(3) by Fourier transform with respect to y and Laplace transform with respect to t and obtain the inhomogeneous versions of (5) and (6),

$$\begin{aligned} &\hat{u}_{xx} - (s^2 + \omega^2)\hat{u} = -\hat{F}, \quad \hat{u} = \hat{u}(x, \omega, s), \\ &(s - \alpha\kappa_2 - i\beta\omega)\hat{u}(0, \omega, s) = \hat{g}(\omega, s). \end{aligned} \tag{12}$$

Since (5) and (6) have no eigenvalues for $\operatorname{Re} s > 0$, we can solve (12). By inverting the Laplace and Fourier transform, we obtain a unique solution.

It is particularly simple to calculate the solution for $\hat{F} \equiv 0$. Corresponding to (10) and (11) we obtain

$$\hat{u} = e^{\kappa_2 x}\hat{u}(0, \omega, s) \tag{13}$$

where

$$(s - \alpha\kappa_2 - i\beta\omega)\hat{u}(0, \omega, s) = \hat{g}(\omega, s).$$

To obtain sharp estimates we need two lemmas.

Lemma 2. *There is a constant $\delta_1 > 0$ such that*

$$\operatorname{Re}\kappa = \operatorname{Re}\sqrt{\omega^2 + s^2} \geq \delta_1\eta, \quad s = i\xi + \eta, \quad \operatorname{Re} s = \eta. \tag{14}$$

Proof.

$$\kappa = \sqrt{\omega^2 - \xi^2 + 2i\xi\eta + \eta^2}.$$

Let

$$\kappa' = \frac{\kappa}{\sqrt{\omega^2 + \xi^2}}, \quad \xi' = \frac{\xi}{\sqrt{\omega^2 + \xi^2}}, \quad \omega' = \frac{\omega}{\sqrt{\omega^2 + \xi^2}}, \quad \eta' = \frac{\eta}{\sqrt{\omega^2 + \xi^2}}.$$

Then

$$\kappa' = \sqrt{(\omega')^2 - (\xi')^2 + 2i\xi'\eta' + (\eta')^2}, \quad (\omega')^2 + (\xi')^2 = 1.$$

If $\eta' >> 1$, then $\kappa' \approx \eta'$ and (14) holds. Thus we can assume that $(\omega')^2 + (\xi)^2 + (\eta')^2 \leq$ const. Assume that (14) is not true. Then there is a sequence

$$\omega' \to \omega_0', \quad \xi' \to \xi_0', \quad \eta' \to \eta_0'$$

such that

$$\operatorname{Re}\frac{\kappa'}{\eta'} \to 0. \tag{15}$$

This can only be true if $\eta_0' = 0$. If $(\omega_0')^2 > (\xi_0')^2$, then (15) cannot hold. If $(\omega_0')^2 < (\xi_0')^2$, then $(\xi')^2 \geq \frac{1}{2}$ and, for sufficiently small η', (8) gives

$$\begin{aligned} \kappa' &\approx i\sqrt{(\xi_0')^2 - (\omega_0')^2} + \frac{\xi_0'}{\sqrt{(\xi_0')^2 - (\omega_0')^2}}\eta', \quad \text{if} \quad \xi_0' > 0, \\ \kappa' &\approx -i\sqrt{(\xi_0')^2 - (\omega_0')^2} - \frac{\xi_0'}{\sqrt{(\xi_0')^2 - (\omega_0')^2}}\eta', \quad \text{if} \quad \xi_0' < 0, \end{aligned} \tag{16}$$

and (14) holds. The same is true if $(\xi_0')^2 = (\omega_0')^2$. This proves the lemma.

Lemma 3. *Assume that $\alpha > 0$ and $|\beta| < 1$. There is a constant $\delta_2 > 0$ such that, for all ω and s with* $\operatorname{Re} s \geq 0$,

$$|s - \alpha\kappa_2 - i\beta\omega| \geq \delta_2\sqrt{|s|^2 + |\omega|^2}. \tag{17}$$

We use the same normalization as in Lemma 2 and write (17) as

$$\begin{aligned} |L| =: &|i(\xi' - \beta\omega') + \eta' + \alpha\sqrt{(\omega')^2 - (\xi')^2 + 2i\xi'\eta' + (\eta')^2}| \\ &\geq \delta\sqrt{(\omega')^2 + (\xi')^2 + (\eta')^2}, \quad (\omega')^2 + (\xi')^2 = 1. \end{aligned} \tag{18}$$

Since $\alpha > 0$ and $\operatorname{Re}\sqrt{s^2 + \omega^2} \geq 0$, the inequality holds for $\eta' >> 1$. Thus we can assume that $|\eta'| \leq$ const. Assume now that there is no $\delta > 0$ such that (18) holds. Then there is a sequence $\xi' \to \xi_0',\ \omega' \to \omega_0',\ \eta' \to 0$ such that

$$L \to L_0 = 0. \tag{19}$$

Using (8) we obtain

$$L \to L_0 = \begin{cases} i(\xi_0' - \beta\omega_0') + \alpha\sqrt{(\omega_0')^2 - (\xi_0')^2} & \text{if } \omega_0^2 > \xi_0^2 \\ i(\xi_0 - \beta\omega_0) + \frac{\xi_0}{|\xi_0|} i\alpha\sqrt{(\xi_0')^2 - (\omega_0')^2} & \text{if } \xi_0^2 \geq \omega_0^2. \end{cases}$$

Clearly, $L_0 \neq 0$ if $|\beta| < 1$. Thus we arrive at a contradiction and (17) holds. This proves the lemma.

We can now prove

Theorem 1. *There is a constant K such that the solution (13) satisfies the estimates*

$$|\hat{u}_x(0,\omega,s)| \le K|\hat{g}(\omega,s)|,$$
$$\sqrt{|s|^2+\omega^2}\cdot|\hat{u}(0,\omega,s)| \le K|\hat{g}(\omega,s)|.$$

Therefore we can use the theory of pseudo-differential operators to obtain the estimates and results of the Appendix. In particular, the problem is strongly well-posed in the generalized sense.

Proof. By (13) and (17),

$$|\hat{u}_x(0,\omega,s)| \le |\kappa_2|\,|\hat{u}(0,\omega,s)| = |\sqrt{\omega^2+s^2}|\,|\hat{u}(0,\omega,s)| \le K|\hat{g}(\omega,s)|.$$

The estimates for the other derivatives follow directly from (17) and (14).

(A18) is the first order version of (12). For $F\equiv 0$, we have $\hat{u}_x = \sqrt{|s|^2+\omega^2}\,\hat{v}$ and therefore, for $x=0$,

$$\sqrt{|s|^2+|\omega|^2}\,|\hat{v}(0,\omega,s)| = |\hat{u}_x(0,\omega,s)| \le K|\hat{g}(\omega,s)|.$$

Thus the required estimate (A20) holds and we can apply the Main theorems A1 and A2 and obtain the estimates (A21) and (A22). This proves the theorem.

3. Linearized Einstein equations

We consider the half-plane problem for the linearized Einstein equations

$$(-\partial_t^2+\partial_x^2+\partial_y^2+\partial_z^2)\begin{pmatrix}\gamma^{tt} & \gamma^{tx} & \gamma^{ty} & \gamma^{tz}\\ \gamma^{tx} & \gamma^{xx} & \gamma^{xy} & \gamma^{xz}\\ \gamma^{ty} & \gamma^{yx} & \gamma^{yy} & \gamma^{yz}\\ \gamma^{tz} & \gamma^{zx} & \gamma^{zy} & \gamma^{zz}\end{pmatrix} = F \qquad (26)$$
$$x\ge 0,\ t\ge 0,\ -\infty<y<\infty,\ -\infty<z<\infty,$$

together with the constraints C^α,

$$\begin{aligned} C^t &= \partial_t\gamma^{tt}+\partial_x\gamma^{tx}+\partial_y\gamma^{ty}+\partial_z\gamma^{tz}=0,\\ C^x &= \partial_t\gamma^{tx}+\partial_x\gamma^{xx}+\partial_y\gamma^{xy}+\partial_z\gamma^{xz}=0,\\ C^y &= \partial_t\gamma^{ty}+\partial_x\gamma^{yx}+\partial_y\gamma^{yy}+\partial_z\gamma^{yz}=0,\\ C^z &= \partial_t\gamma^{tz}+\partial_x\gamma^{zx}+\partial_y\gamma^{zy}+\partial_z\gamma^{zz}=0.\end{aligned} \qquad (27)$$

The constraints are also solutions of the wave equation. We can guarantee that they remain zero at later times if $C^\alpha=0$, $\alpha=(t,x,y,z)$, are part of the boundary conditions for (26) at $x=0$.

A possibility is

$$
\partial_t \begin{pmatrix} \gamma^{tt} \\ \gamma^{tx} \\ \gamma^{xx} \\ \gamma^{ty} \\ \gamma^{xy} \\ \gamma^{tz} \\ \gamma^{xz} \\ \gamma^{yy} \\ \gamma^{yz} \\ \gamma^{zz} \end{pmatrix}
+ \partial_x \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_1 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_1 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix} \gamma^{tt} \\ \gamma^{tx} \\ \gamma^{xx} \\ \gamma^{ty} \\ \gamma^{xy} \\ \gamma^{tz} \\ \gamma^{xz} \\ \gamma^{yy} \\ \gamma^{yz} \\ \gamma^{zz} \end{pmatrix}
$$

$$
+ \partial_y \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} \gamma^{tt} \\ \gamma^{tx} \\ \gamma^{xx} \\ \gamma^{ty} \\ \gamma^{xy} \\ \gamma^{tz} \\ \gamma^{xz} \\ \gamma^{yy} \\ \gamma^{yz} \\ \gamma^{zz} \end{pmatrix} \tag{28}
$$

$$
+ \partial_z \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} \gamma^{tt} \\ \gamma^{tx} \\ \gamma^{xx} \\ \gamma^{ty} \\ \gamma^{xy} \\ \gamma^{tz} \\ \gamma^{xz} \\ \gamma^{yy} \\ \gamma^{yz} \\ \gamma^{zz} \end{pmatrix} = g.
$$

Here $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ and $c_1,\ c_2$ are real constants such that the eigenvalues λ_j of

$$
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ b_1 & b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix} \tag{29}
$$

are real and negative.

We want to prove

Theorem 3. *The half-plane problem for the system (26) with boundary conditions (28) is well-posed in the generalized sense if the eigenvalues of the matrix (29) are real and negative.*

Proof. We can assume that $F = 0$ and need only show that the estimate of Theorem 1 holds for every component $\gamma^{\mu\nu}$ with $|\hat{g}|$ denoting the Euclidean norm of all components of the forcing.

Fourier transform the problem with respect to the tangential variables and Laplace transform it with respect to time. For every component $\hat{\gamma}^{ij}$ we obtain

$$\left(\partial_x^2 - (s^2 + \omega_1^2 + \omega_2^2)\right)\hat{\gamma}^{\mu\nu} = 0, \quad |\hat{\gamma}|_\infty < \infty, \tag{30}$$

which are coupled through the corresponding transformed boundary condition. We start with the last component

$$\begin{aligned} &\left(\partial_x^2 - (s^2 + \omega_1^2 + \omega_2^2)\right)\hat{\gamma}^{zz} = 0, \quad x \geq 0, \\ &s\hat{\gamma}^{zz} = \partial_x \hat{\gamma}^{zz} + \hat{g}^{zz} \quad \text{for} \quad x = 0, \quad |\hat{\gamma}^{zz}|_\infty < \infty. \end{aligned} \tag{31}$$

This is a problem which we have treated in the last section. By Theorem 1, we gain one derivative on the boundary. For $\hat{\gamma}^{yz}$, $\hat{\gamma}^{yy}$ we have the same result.

The coupled boundary conditions for $\hat{\gamma}^{tz}$, $\hat{\gamma}^{xz}$

$$s\begin{pmatrix}\hat{\gamma}^{tz}\\ \hat{\gamma}^{xz}\end{pmatrix} = -\begin{pmatrix}0 & 1\\ c_1 & c_2\end{pmatrix}\partial_x\begin{pmatrix}\hat{\gamma}^{tz}\\ \hat{\gamma}^{xz}\end{pmatrix} + i\omega_1\begin{pmatrix}\hat{\gamma}^{yz}\\ 0\end{pmatrix} + i\omega_2\begin{pmatrix}\hat{\gamma}^{zz}\\ 0\end{pmatrix} + \begin{pmatrix}\hat{g}^{tz}\\ \hat{g}^{xz}\end{pmatrix} \tag{32}$$

can be decoupled. There is a unitary matrix U such that

$$U^*\begin{pmatrix}0 & 1\\ c_1 & c_2\end{pmatrix}U = \begin{pmatrix}-\lambda_1 & c_{12}\\ 0 & -\lambda_2\end{pmatrix}, \quad \lambda_1, \lambda_2 > 0.$$

Introducing new variables by

$$\begin{pmatrix}\tilde{\gamma}^{tz}\\ \tilde{\gamma}^{xz}\end{pmatrix} = U\begin{pmatrix}\hat{\gamma}^{tz}\\ \hat{\gamma}^{xz}\end{pmatrix}$$

gives us

$$s\begin{pmatrix}\tilde{\gamma}^{tz}\\ \tilde{\gamma}^{xz}\end{pmatrix} = \begin{pmatrix}\lambda_1 & c_{12}\\ 0 & \lambda_2\end{pmatrix}\partial_x\begin{pmatrix}\tilde{\gamma}^{tz}\\ \tilde{\gamma}^{xz}\end{pmatrix} + U\left(i\omega_1\begin{pmatrix}\hat{\gamma}^{yz}\\ 0\end{pmatrix} + i\omega_2\begin{pmatrix}\hat{\gamma}^{zz}\\ 0\end{pmatrix} + \begin{pmatrix}\hat{g}^{tz}\\ \hat{g}^{xz}\end{pmatrix}\right).$$

We start with the last component. By (31), we have gained a derivative which we loose by calculating $\partial_z \gamma^{zz} = i\omega_2 \gamma^{zz}$. But then we gain a derivative by solving for $\tilde{\gamma}^{xz}$. The same is true for $\tilde{\gamma}^{tz}$. The process can be continued and the theorem follows.

Clearly, Theorem 3 is also valid when the matrices of (28) for the tangential derivatives are strictly upper triangular, i.e., only terms above the diagonal are not zero. This allows the sequential argument associated with (31) and (32) above. One can also generalize the result to full matrices provided the elements are sufficiently small.

4. Constraint preserving boundary conditions in Sommerfeld form

The example in the preceding section illustrates how the pseudo-differential theory can used to establish a constraint-preserving IBVP for the linearized Einstein equations which is well-posed in the generalized sense. There are further boundary conditions not covered in the example that can be treated by the same technique because the boundary conditions can be written in the form (28) with the matrices for the tangential derivatives strictly upper triangular. Here we consider a simple hierarchy of Sommerfeld boundary conditions. The geometric nature of the construction is more transparent using standard tensor notation based upon spacetime coordinates $x^\alpha = (t, x, y, z)$. Our results center about systems whose components satisfy the scalar wave equation, now written in terms of the Minkowski metric $\eta^{\alpha\beta} = diag(-1, 1, 1, 1)$ as

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\Phi = 0,$$

in the half-space $x \geq 0,\ t \geq 0,\ -\infty < y < \infty,\ -\infty < z < \infty$.

For this IBVP, the energy

$$E = \frac{1}{2}\int \Big((\partial_t\Phi)^2 + (\partial_x\Phi)^2 + (\partial_y\Phi)^2 + (\partial_z\Phi)^2\Big)dxdydz$$

satisfies

$$\partial_t E = -\int_{x=0} \mathcal{F} dydz$$

where the energy flux through the boundary is

$$\mathcal{F} = (\partial_t\Phi)\partial_x\Phi.$$

This leads to a range of homogeneous, dissipative boundary conditions

$$A\partial_t + B\partial_x\Phi = 0,$$

subject to values of A and B such that $\mathcal{F} \geq 0$. The Dirichlet boundary condition $(A = 1, B = 0)$ corresponds to the case where Φ has an odd parity local reflection symmetry across the boundary; and the Neumann boundary condition $(A = 0, B = 1)$ corresponds to an even parity local reflection symmetry. Both the homogeneous Dirichlet and Neumann conditions are borderline dissipative cases for which $\mathcal{F} = 0$. The description of a traveling wave carrying energy across the boundary requires an inhomogeneous form of the Dirichlet or Neumann boundary condition, which provides the proper boundary data for the wave to pass through the boundary. However, in numerical simulations, such inhomogeneous Dirichlet or Neumann boundary data can only be prescribed for the signal and the numerical error is reflected by the boundary and accumulates in the grid. This can lead to poor performance in simulations of dynamical nonlinear systems such as Einstein's equations. For such computational purposes, it is more advantageous to use the strictly dissipative Sommerfeld condition given by $(A = 1, B = -1)$, i.e.

$$(\partial_t - \partial_x)\Phi = 0,$$

for which $\mathcal{F} = (\partial_t \Phi)^2$. The Sommerfeld condition is based upon the characteristic direction in the 2-space picked out by the outward normal to the boundary and a timelike direction (the evolution direction) tangent to the boundary.

Before formulating Sommerfeld boundary conditions for the constrained harmonic Einstein equations, it is instructive to consider the analogous case of Maxwell's equations for the electromagnetic field expressed in terms of a vector potential $A^\mu = (A^t, A^x, A^y, A^z)$ (see e.g. [17]). Subject to the Lorentz gauge condition

$$C := \partial_\mu A^\mu = 0, \tag{33}$$

the Maxwell equations reduce to the wave equations

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta A^\mu = 0. \tag{34}$$

The constraint C also satisfies the wave equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta C = 0$. Thus Cauchy data $A^\mu|_{t=0}$ and $\partial_t A^\mu|_{t=0}$ which satisfies $\mathcal{C}|_{t=0} = \partial_t \mathcal{C}|_{t=0} = 0$, i.e. for which C also has vanishing Cauchy data, leads to the well-known result that the constraint is preserved for the Cauchy problem.

In order to extend constraint preservation to the IBVP, the boundary condition for A^μ must imply a homogeneous boundary condition for $\mathcal{C}$. One way to accomplish this is to use locally reflection symmetric boundary data, as in the scalar case. For instance, for the above half-space problem, the even parity boundary conditions $\partial_x A^t|_{x=0} = \partial_x A^y|_{x=0} = \partial_x A^z|_{x=0} = A^x|_{x=0} = 0$ imply that $\partial_x C|_{x=0} = 0$. These boundary conditions are dissipative and lead to a well-posed IBVP for Maxwell's equations. The analogous approach has been used to formulate a well-posed constraint-preserving IBVP for the nonlinear Einstein equations [13]. However, this leads to a combination of Dirichlet and Neumann boundary conditions which is only borderline dissipative and the results of tests for the nonlinear Einstein problem show that a strictly dissipative Sommerfeld-type boundary condition gives better numerical accuracy [15].

The pseudo-differential theory offers alternative approaches. Consider the IBVP with Sommerfeld boundary conditions

$$(\partial_t - \partial_x)(A^t + A^x) = g, \tag{35}$$

$$(\partial_t - \partial_x)A^y = g^y, \tag{36}$$

$$(\partial_t - \partial_x)A^z = g^z. \tag{37}$$

$$\frac{1}{2}(\partial_t - \partial_x)(A^t - A^x) + \partial_t(A^t + A^x) + \partial_y A^y + \partial_z A^z = \frac{1}{2}g, \tag{38}$$

where g, g^y and g^z are free Sommerfeld data. The IBVP is well-posed in the generalized sense if the estimates of Theorem 1 are satisfied. Using the argument associated with (31) in the preceding section, it follows from (35)–(37) that $\hat{A}^t + \hat{A}^x$, $\hat{A}^y$, $\hat{A}^z$ and their derivatives satisfy these estimates. Next, consider the Sommerfeld boundary condition (38) for $A^t - A^x$. Using the argument associated with (32), the tangential derivatives of $(A^t + A^x)$, A^y and A^z introduce no problem, and the estimates extend to $\hat{A}^t - \hat{A}^x$ and its derivative. Thus the Sommerfeld boundary conditions (35)–(38) guarantee a well-posed IBVP for the system (34). But (35)–(38) also imply that the constraint satisfies the homogeneous boundary condition $C = 0$, as is evident by rewriting (33) in the form

$$C = \frac{1}{2}(\partial_t - \partial_x)(A^t - A^x) + \frac{1}{2}(\partial_t + \partial_x)(A^t + A^x) + \partial_y A^y + \partial_z A^z. \tag{39}$$

Although the boundary conditions (35)–(38) lead to a well-posed, constraint preserving IBVP for Maxwell's equations, they do not correspond to any physically familiar boundary conditions on the electric and magnetic field vectors **E** and **B**. However, there are numerous options in constructing boundary conditions by this approach. For instance, consider the choice

$$(\partial_t - \partial_x)(A^t + A^x) = 0, \tag{40}$$

$$(\partial_t - \partial_x)A^y + \partial_y(A^t + A^x) = 0, \tag{41}$$

$$(\partial_t - \partial_x)A^z + \partial_z(A^t + A^x) = 0. \tag{42}$$

$$\frac{1}{2}(\partial_t - \partial_x)(A^t - A^x) + \partial_t(A^t + A^x) + \partial_y A^y + \partial_z A^z = 0. \tag{43}$$

Again, the Sommerfeld condition (40) implies that $\hat{A}^t + \hat{A}^x = 0$ and its derivatives satisfy the estimates required for Theorem 1. In addition, the sequential manner in which the tangential derivatives of previously estimates quantities enter (41)–(43) ensures that all components of $\hat{A}^\mu$ satisfy the required estimates. Furthermore, (40)–(43) imply that the constraint satisfies the homogeneous boundary condition $C = 0$, as is again evident from (39). By using the wave equations (34) and the constraint (33), it can be verified that these boundary conditions give rise to the familiar plane wave relations $E^y = -B^z$ and $E^z = B^y$ on the components of the electric and magnetic fields. These relations imply that the Poynting flux $\mathbf{E} \times \mathbf{B}$ leads to a loss of electromagnetic energy from the system.

The half-space problem for the linearized harmonic Einstein equations can be treated in a similar manner. Given a metric tensor $g_{\mu\nu}$ with inverse $g^{\mu\nu}$ and determinant g, the linearized Einstein equations imply that the perturbations $\gamma^{\mu\nu} = \delta(\sqrt{-g}g^{\mu\nu})$ satisfy the 10 wave equations

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = 0 \tag{44}$$

provided the 4 constraints

$$\mathcal{C}^\mu := \partial_\nu\gamma^{\mu\nu} = 0 \tag{45}$$

are satisfied [17]. The constraints constitute the harmonic gauge conditions which reduce the Einstein equations to a symmetric hyperbolic system of wave equations.

A simple formulation of Sommerfeld boundary conditions for the half-space problem $0 \le x < \infty$ for the linearized gravitational field can be patterned after the above treatment of the electromagnetic problem. For convenience we write $x^A = (y, z)$. First we require the 6 Sommerfeld boundary conditions

$$(\partial_t - \partial_x)\ \gamma^{AB} = q^{AB}, \tag{46}$$

$$(\partial_t - \partial_x)(\gamma^{tA} + \gamma^{xA}) = q^A, \tag{47}$$

$$(\partial_t - \partial_x)(\gamma^{tt} + 2\gamma^{tx} + \gamma^{xx}) = q, \tag{48}$$

where q^{AB}, q^A and q are free Sommerfeld data. Then the constraints are used to supply 4 additional Sommerfeld-type boundary conditions in the hierarchical order

$$\mathcal{C}^A = \frac{1}{2}(\partial_t - \partial_x)(\gamma^{tA} - \gamma^{xA}) + \partial_t(\gamma^{tA} + \gamma^{xA}) + \partial_B\gamma^{AB} - \frac{1}{2}q^{AB} = 0, \tag{49}$$

$$\mathcal{C}^t + \mathcal{C}^x = \frac{1}{2}(\partial_t - \partial_x)(\gamma^{tt} - \gamma^{xx}) + \partial_t(\gamma^{tt} + 2\gamma^{tx} + \gamma^{xx}) + \partial_B(\gamma^{tB} + \gamma^{xB}) - \frac{1}{2}q = 0, \tag{50}$$

$$\mathcal{C}^t = \frac{1}{2}(\partial_t - \partial_x)(\gamma^{tt} + \gamma^{xx}) + \partial_t(\gamma^{tt} + \gamma^{tx}) + \partial_B \gamma^{tB} - \frac{1}{2} q = 0. \qquad (51)$$

In expressing the constraints in this form, we have used (47)–(49) and the prior constraints in the hierarchy.

The sequence of Sommerfeld conditions (46)–(51) for $(\partial_t - \partial_x)\gamma^{\mu\nu}$ have the property that all tangential derivatives of $\gamma^{\mu\nu}$ only involve prior components in the sequence. This allows us to again use the arguments associated with (31) and (32) to obtain the estimates for all components of $\hat{\gamma}^{\mu\nu}$ and their derivatives which are required for Theorem 1.

The boundary conditions (46)–(51) offer a simple and attractive scheme for numerical use. The pseudo-differential theory allows many more possibilities for a well-posed, constraint-preserving IBVP. It would be of value to find a version with a simple physical interpretation in the analogous way that (40)–(43) implies a positive Poynting flux in the electromagnetic case. However, this issue is complicated by the lack of a unique expression for the gravitational energy flux except in the asymptotic limit of null infinity. One practical alternative for a numerical scheme would be to use an external solution to provide the Sommerfeld data on the right hand sides of (46)–(48). This data can be obtained either by matching to an external linearized solution or, in the nonlinear case, by Cauchy-characteristic matching [18].

Our results generalize to the curved space linearize harmonic wave equation, whose principle part has the form

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu}, \qquad (52)$$

determined by a given space-time metric $g^{\alpha\beta}(t, x, y, z)$, i.e. a matrix which can be transformed at any point to diagonal Minkowski form. This variable coefficient problem is well-posed in the generalized sense if all frozen coefficient problems are well-posed [5]. This result is insensitive to lower order terms in the equations. This *principle of frozen coefficients* is an important result of the pseudo-differential theory. We can reduce the frozen coefficient problem to the above Minkowski space problem by adapting the harmonic coordinates so that the boundary is given by $x = 0$ and then introducing a orthonormal tetrad,

$$g_{\alpha\beta} = -T_\alpha T_\beta + X_\alpha X_\beta + Y_\alpha Y_\beta + Z_\alpha Z_\beta,$$

where X_α is in the direction $\nabla_\alpha x$ normal to the boundary. Freezing the tetrad at a boundary point, we can then introduce the linear coordinate transformation

$$\tilde{t} = T_\alpha x^\alpha,\ \tilde{x} = X_\alpha x^\alpha,\ \tilde{y} = Y_\alpha x^\alpha,\ \tilde{z} = Z_\alpha x^\alpha.$$

In the $\tilde{x}^\alpha$ coordinates, the frozen coefficient problem reduces to the Minkowski space problem, which we have treated.

The full treatment of the Einstein equations requires taking into account the relation $\gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$, which converts (52) into a quasi-linear operator. In this case the pseudo-differential theory outlined in the Appendix establishes that the IBVP is well-posed locally in time.

Appendix

In this section we shall give a short summary of the theory for first order systems. This includes also second order systems because our theory is based on pseudo-differential operators and therefore one can always write a second order system of differential equations in terms of a first order system of pseudo-differential operators. (See the references in the Introduction.)

Consider a first order system

$$u_t = P(\partial/\partial x)u + F, \quad P(\partial/\partial x) = A\,\partial/\partial x_1 + \sum_{j=2}^{m} B_j \partial/\partial x_j \tag{A1}$$

with constant coefficients on the half-space

$$t \geq 0, \quad x_1 \geq 0, \quad -\infty < x_j < \infty,\ j = 2, \ldots, m.$$

Here $u(x,t) = \big(u^{(1)}(x,t), \ldots, u^{(n)}(x,t)\big)$ is a vector valued function of the real variables $(x,t) = (x_1, \ldots, x_m, t)$ and A, B_j are constant $n \times n$ matrices.

We assume that the system is strictly hyperbolic, i.e., for all real $\omega = (\omega_1, \omega_-)$, $\omega_- = (\omega_2, \ldots, \omega_m)$ with $|\omega| = 1$, the eigenvalues of the symbol

$$P(i\omega) = iA\omega_1 + iB(\omega_-), \quad B(\omega_-) = \sum_{j=2}^{m} B_j \omega_j, \tag{A2}$$

are purely imaginary and distinct.

In [1] the theory has been extended to the case where the eigenvalues have constant multiplicity and there is a complete set of eigenvectors. In particular, the theory holds if the system consists of strictly hyperbolic subsystems which are only coupled through lower order terms and the boundary conditions.

We assume also that A is nonsingular and without restriction we can assume that it has the form

$$A = \begin{pmatrix} -\Lambda^I & 0 \\ 0 & \Lambda^{II} \end{pmatrix}. \qquad (A3)$$

Here Λ^I, Λ^{II} are real positive definite diagonal matrices of order r and $n-r$, respectively. For the singular case, see [7].

For $t = 0$, we give initial data

$$u(x, 0) = f(x) \qquad (A4)$$

and for, $x_1 = 0$, r boundary conditions

$$u^I(0, x_-, t) = Su^{II}(0, x_-, t) + g(x_-, t), \quad x_- = (x_2, \dots, x_m). \qquad (A5)$$

All data are smooth, compatible and have compact support.

The usual theory for well-posed problems depends strongly on the assumption that the system is symmetric and that the boundary conditions are maximally dissipative. If any of those two arguments is not satisfied, then the theory does not give anything. We want to discuss a concept of well-posedness for which we obtain necessary and sufficient conditions.

The main ingredient of a definition for well-posedness is the estimate of the solution in terms of the data. (See [5,Sec.7.3]). We will consider (A1),(A4),(A5) with homogeneous initial data $f \equiv 0$ and use

Definition A1. *Let $f(x) \equiv 0$. We call the problem strongly well-posed in the generalized sense if, for all smooth compatible data F, g, there is a unique solution u and in every time interval $0 \le t \le T$ there is a constant K_T which does not depend on F and g such that*

$$\int_0^t \|u(\cdot,\tau)\|^2 d\tau + \int_0^t \|u(\cdot,\tau)\|_-^2 d\tau \le K_T\{\int_0^t \|F(\cdot,\tau)\|^2 d\tau + \int_0^t \|g(\cdot,\tau)\|_-^2 d\tau\}. \qquad (A6)$$

Here $\|\cdot\|, \|\cdot\|_-$ denote the L_2-norm with respect to the half-space and the boundary space, respectively.

We start with a simple observation. For $F = g = 0$, we construct simple wave solutions

$$u(x_1, x_-, t) = e^{st + i\langle\omega, x\rangle_-}\varphi(x_1), \quad \langle\omega, x\rangle_- = \sum_{j=2}^m \omega_j x_j, \qquad (A7)$$

satisfying the boundary conditions

$$\varphi^I(0, x_-) = S\varphi^{II}(0, x_-), \quad |\varphi|_\infty < \infty. \qquad (A8)$$

We have

Lemma A1. *The half-plane problem is not well-posed if, for some* ω_0 *and complex* s_0 *with* $\operatorname{Re} s_0 > 0$, *there is a solution (A7) which satisfies (A8).*

Proof. If there is a solution then, by homogeneity,

$$u(\gamma x_1, \gamma x_-, \gamma t) = e^{\gamma(s_0 t + i\langle\omega_0, x\rangle_-)}\varphi(\gamma x_1),$$

is also a solution for any $\gamma > 0$. Thus there are solutions which grow arbitrarily fast exponentially. This proves the lemma.

We shall now derive algebraic conditions so that we can decide whether these simple wave solutions exist.

Introducing (A7) into (A1) and (A5), we obtain

$$\begin{aligned} s\varphi &= A\varphi_x + iB(\omega_-)\varphi, \quad x \geq 0, \\ \varphi^I(0) &= S\varphi^{II}(0), \quad |\varphi|_\infty < \infty. \end{aligned} \tag{A9}$$

(A9) is an eigenvalue problem for a system of ordinary differential equations which can be solved in the usual way. Let κ denote the solutions of the characteristic equation

$$\operatorname{Det}|A\kappa - (sI - iB(\omega_-))\,| = 0. \tag{A10}$$

One can prove

Lemma A2.

1) For $\operatorname{Re} s > 0$, *there are no* κ *with* $\operatorname{Re}\kappa = 0$.

2) There are exactly r *eigenvalues with* $\operatorname{Re}\kappa < 0$ *and* $n - r$ *eigenvalues with* $\operatorname{Re}\kappa > 0$.

3) There is a constant $\delta > 0$ *such that, for all* $s = i\xi + \eta,\ \xi, \eta > 0$, *and all* ω_-,

$$|\operatorname{Re}\kappa| > \delta\eta, \quad s = i\xi + \eta,\ \xi, \eta > 0 \text{ real}.$$

(See [2]).

We can now write down the general solution of (A9). If all κ_j are distinct, the solution is of the form

$$\varphi = \sum_{\operatorname{Re}\kappa_j < 0} \sigma_j e^{\kappa_j x} h_j + \sum_{\operatorname{Re}\kappa_j > 0} \sigma_j e^{\kappa_j x} h_j. \tag{A11}$$

Here h_j are the corresponding eigenvectors. (If the eigenvalues are not distinct, the usual modifications apply.)

Since we are only interested in bounded solutions, all σ_j in the second term are zero. Introducing φ into the boundary conditions at $x = 0$ gives us a linear system of r equations for r unknowns $(\sigma_1, \ldots, \sigma_r) = \underline{\sigma}$ which we write as

$$C(\omega_-, s)\underline{\sigma} = 0. \tag{A12}$$

The problem is not well-posed if for some ω_-, there is an eigenvalue s_0 with $\text{Re } s_0 > 0$, i.e., $\text{Det}\, C(\omega_-, s_0) = 0$. Then the linear system of equations (A12) and therefore also (A9) has a nontrivial solution.

From now on we shall assume that $\text{Det}\, C \neq 0$ for $\text{Re } s > 0$. Then we can solve the initial boundary value problem by Laplace transform in time and Fourier transform in the tangential variables. For convenience, we start the solution from 'rest', i.e., $u(x, 0) = f(x) \equiv 0$. Then we obtain

$$\begin{aligned} s\hat{u} &= A\hat{u}_x + iB(\omega_-)\hat{u} + \hat{F}, \\ \hat{u}^I(0) &= S\hat{u}^{II}(0) + \hat{g}. \end{aligned} \tag{A13}$$

Since, by assumption, (A9) has only the trivial solution for $\text{Re } s > 0$ and $|\text{Re}\, \kappa| > \delta\eta$, (A12) has a unique solution. Inverting the Fourier and Laplace transforms we obtain the solution in physical space.

It is particularly simple to solve (A13) if $F \equiv 0$. By (A11) and (A12),

$$\hat{u}(0, \omega_-, s) = \sum_{\text{Re}\, \kappa_j < 0} \sigma_j e^{\kappa_j x} h_j,$$

where the σ_j are determined by

$$C(\omega_-, s)\underline{\sigma} = \hat{g}.$$

Definition A2. *Consider (A13) with* $\hat{F} \equiv 0$. *The problem is called boundary stable if, for all* ω, s *with* $\eta = \text{Re } s > 0$, *there is a constant* K *which does not depend on* ω, s *and* $\hat{g}$ *such that*

$$|\hat{u}(0, \omega, s)| \leq K|\hat{g}(\omega, s)|. \tag{A14}$$

One can also phrase the condition as: The eigenvalue problem (A9) has no eigenvalues for $\text{Re } s \geq 0$ or $\text{Det}\, C(\omega_-, s) \neq 0$ for $\text{Re } s \geq 0$.

The estimate (A14) is crucial to the theory. It allows us to construct a symmetrizer to obtain an energy estimate in the generalized sense for the full problem.

We introduce normalized variables

$$s' = s/\sqrt{|s|^2 + |\omega_-|^2} = i\xi' + \eta', \quad \omega'_- = \omega_-/\sqrt{|s|^2 + |\omega_-|^2}$$

and write (A13) as

$$\begin{aligned} -A\hat{u}_x + \sqrt{|s|^2 + |\omega_-|^2}\left(s'I - iB(\omega'_-)\right)\hat{u} = \hat{F}, \\ \hat{u}^I(0) - S\hat{u}^{II}(0) = \hat{g}. \end{aligned} \tag{A15}$$

Main theorem A1. *Assume that the half-plane problem is boundary stable. Then there exists a symmetrizer $\hat{R} = \hat{R}(s', \omega'_-)$ with the following properties.*

1) $\hat{R}$ is a smooth bounded function of s', ω'_- and the coefficients of A, B_j and S.

2) $\hat{R}A$ is Hermitian and for all vectors y which satisfy the boundary conditions

$$\langle y, \hat{R}Ay \rangle \geq \delta_1|y|^2 - c|g|^2.$$

3) $\sqrt{|s|^2 + |\omega_-|^2}\,\mathrm{Re}\left\{\hat{R}\left(s'I - iB(\omega'_-)\right)\right\} \geq \delta_2\eta I.$

Here $\delta_1, \delta_2 > 0,\ c > 0$ are constants independent of s', ω'_-.

We can now prove

Main theorem A2. *Assume that the half-plane problem is boundary stable. Then it is well-posed in the sense of Definition A1.*

Proof. Multiplying (A15) by $\hat{R}$ we obtain, for the L_2 scalar product with respect to x_1,

$$\begin{aligned} \mathrm{Re}(\hat{u}, \hat{R}\hat{F}) &= \mathrm{Re}\{-(\hat{u}, \hat{R}A\partial\hat{u}/\partial x_1) + \left(\hat{u}, \sqrt{|s|^2 + |\omega_-|^2}\cdot\hat{R}(s'I - iB(\omega'_-)\hat{u}\right)\} \\ &= \mathrm{Re}\{-\frac{1}{2}\langle\hat{u}, \hat{R}A\hat{u}\rangle\big|_{x_1=0}^{\infty} + \left(\hat{u}, \hat{R}(sI - iB(\omega_-)\hat{u}\right)\} \\ &\geq \delta_1|\hat{u}(0, s, \omega_-)|^2 + \delta_2\eta\|\hat{u}(x_1, \omega_-)\|^2 - c|\hat{g}|^2. \end{aligned}$$

Thus we obtain

$$\eta\|\hat{u}(x_1, s, \omega)\|^2 + |\hat{u}(0, s, \omega)|^2 \leq \text{ const. } \left(\frac{1}{\eta}\|\hat{F}\|^2 + c|\hat{g}|^2\right). \tag{A16}$$

Inverting the Laplace and Fourier transform proves the theorem.

Remark. The estimate (A14) and the properties of the symmetrizer and therefore also the estimate (A16) need only be valid for $\eta \geq \eta_0 > 0$. This is important if lower order terms or variable coefficients are present.

One could have derived the estimate (A16) by directly calculating the solution of (A9). However, the importance of the symmetrizer is that we can consider $\hat{R}$ as a symbol and (A9) as an equation for pseudo-differential operators. The theory of pseudo-differential operators has far reaching consequences. In particular, the computational rules for pseudo-differential operators show:

1) The estimate is also valid if the symbols depend smoothly on x, t, provided we assume that $\eta > \eta_0$, η_0 sufficiently large. Here η_0 depends on a finite number of x, t-derivatives of the symbols. Therefore we extend the estimate to systems with variable coefficients.

2) Well-posedness will not be destroyed by lower order terms. Therefore one can localize the problem, and well-posedness in general domains can be reduced to the study of the Cauchy problem and the half-plane problems.

3) The principle of frozen coefficients holds.

4) The properties of the pseudo-differential operators allow us to estimate derivatives in the same way as for standard partial differential equations. Therefore we obtain well-posedness in the generalized sense for linear systems with variable coefficients which gives us the corresponding local results for quasi-linear problems.

Since pseudo-differential operators are much more flexible than standard differential operators, we can always write second order systems as first order systems. Consider, for example, the problem we discuss in Section 2.

$$\begin{aligned} &u_{tt} = u_{xx} + u_{yy} + F \quad \text{for} \quad x \geq 0,\ -\infty < y < \infty,\ t \geq 0, \\ &u_t - \alpha u_x - \beta u_y = g, \quad x = 0,\ -\infty < y < \infty,\ t \geq 0. \end{aligned} \tag{A17}$$

After Laplace-Fourier transform it becomes

$$\begin{aligned} &\hat{u}_{xx} = (s^2 + \omega^2)\hat{u} - \hat{F}, \\ &s\hat{u} - \alpha\hat{u}_x - \beta i\omega\hat{u} = \hat{g}. \end{aligned}$$

Introducing a new variable $\hat{u}_x = \sqrt{|s|^2 + \omega^2}\,\hat{v}$ gives us the first order system

$$\begin{aligned} &\hat{\mathbf{u}}_x = \sqrt{|s|^2 + \omega^2} \begin{pmatrix} 0 & 1 \\ {s'}^2 + {\omega'}^2 & 0 \end{pmatrix} \hat{\mathbf{u}} - \tilde{F}, \quad \hat{\mathbf{u}} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \\ &s'\hat{u} - \alpha\hat{v} - \beta i\omega'\hat{u} = \tilde{g}, \end{aligned} \tag{A18}$$

where

$$\tilde{F} = \frac{1}{\sqrt{|s|^2+\omega^2}} \begin{pmatrix} 0 \\ \hat{F} \end{pmatrix}, \quad \tilde{g} = \frac{1}{\sqrt{|s|^2+\omega^2}} \hat{g}. \tag{A19}$$

Definition A3. *We call the problem (A17) strongly well-posed in the generalized sense if the corresponding first order problem (A18) with general data $\tilde{F}, \tilde{g}$ has this property.*

If (A18) is boundary stable, i.e., if the estimate (A14) holds with $\hat{g}$ replaced by $\tilde{g}$, then we can use the same technique as in [2, Sec.4] to construct the symmetrizer. Therefore Main Theorems 1 and 2 are valid and we obtain the estimate (A16) with $\hat{F}, \hat{g}$ replaced by $\tilde{F}, \tilde{g}$.

Starting from (A17), $\tilde{F}, \tilde{g}$ satisfy (A19) and therefore (A14) becomes

$$|\hat{u}(0,s,\omega)| + |\hat{v}(0,s,\omega)| \leq \frac{K}{\sqrt{|s|^2+\omega^2}} \, |\hat{g}(\omega,s)|. \tag{A20}$$

In Section 2 we prove that (A20) holds by directly calculating the solution of (A17) for $\hat{F} = 0$.

Similarly, in terms of $\hat{F}$ and $\hat{g}$ for the second order system, the estimate (A16) becomes

$$(|s|^2+\omega^2)\,(\eta\|\hat{\mathbf{u}}(\cdot,\omega,s)\|^2 + |\hat{\mathbf{u}}(0,s,\omega)|^2) \leq \text{ const. } \left(\frac{1}{\eta}\|\hat{F}\|^2 + c|\hat{g}|^2\right). \tag{A21}$$

We have also

$$\|\hat{u}_x\|^2 = (|s|^2+\omega^2)\,\|\hat{v}\|^2 \leq (|s|^2+\omega^2)\|\hat{\mathbf{u}}\|^2. \tag{A22}$$

Therefore, by inverting the Laplace and Fourier transform, we can estimate the L_2-norm of all first derivatives in terms of the L_2-norm of the data. Thus we gain one derivative.

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