

# Models of mass segregation at the Galactic centre

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## Abstract.

We study the process of mass segregation through 2-body relaxation in galactic nuclei with a central massive black hole (MBH). This study has bearing on a variety of astrophysical questions, from the distribution of X-ray binaries at the Galactic centre, to tidal disruptions of main-sequence and giant stars, to inspirals of compact objects into the MBH, an important category of events for the future space borne gravitational wave interferometer LISA. In relatively small galactic nuclei, typical hosts of MBHs with masses in the range  $10^4 - 10^7 M_\odot$ , the relaxation induces the formation of a steep density cusp around the MBH and strong mass segregation. Using a spherical stellar dynamical Monte-Carlo code, we simulate the long-term relaxational evolution of galactic nucleus models with a spectrum of stellar masses. Our focus is the concentration of stellar black holes to the immediate vicinity of the MBH. Special attention is given to models developed to match the conditions in the Milky Way nucleus.

## 1. Method, physics and initial conditions

This contribution summarises our recent series of simulations of galactic nuclei aimed at the investigation of mass segregation [1].

We used a stellar dynamics code based on the Monte Carlo (MC) scheme pioneered by Hénon [2, 3]. This method offers a unique compromise between physical realism and computational efficiency. Like in direct  $N$ -body codes, the stellar system is represented as a set of particles; this allows the implementation of a rich physics: self-gravity, general stellar mass spectrum and velocity distribution, 2-body relaxation, collisions between stars, stellar evolution, binaries (not included in the present code), tidal disruptions and captures by the central MBH ([4, 5, 6, 7, 8, 9] for a description of the code and previous applications). Unlike  $N$ -body, the MC method assumes spherical symmetry and dynamical equilibrium. This makes it much faster; while the CPU time per relaxation time increases like  $N^3$  for  $N$ -body simulations, the MC code has a  $N \ln N$  scaling.

We performed about 90 different simulations, to investigate the effects of various physical ingredients, assumptions about their treatment, of the initial nucleus structure and to perform some limited parameter-space exploration. For most models,  $4 \times 10^6$  particles were used, requiring a few days of computing time on a single-CPU PC. A few cases were computed with  $10^6$  or  $8 \times 10^6$  particles to establish that our results are not strongly affected by the limitations in numerical resolution.

All runs included the effects of the gravity of a central MBH, of the self gravity of the stars, of 2-body relaxation, treated in the Chandrasekhar (diffusive) approximation, and of the tidal

disruption of main-sequence (MS) stars at the Roche limit around the MBH as well as direct coalescence with the MBH for stars too compact to be tidally disrupted. In most cases, stellar evolution was not included explicitly; instead the stellar population consists, from the beginning of the simulation, of a mixture of MS and compact remnants corresponding to a single star formation episode that took place 10 Gyr ago. In a few models, explicit stellar evolution was included with all stars starting on the MS and turning into compact remnants at the end of their MS life time. For simplicity, giants were not considered because, as far as mass segregation is concerned, only the mass of the star matters and the evolution of the stellar distribution, being a relaxational process requires timescales much longer than the duration of the giant phase. Stellar collisions and large-angle gravitational deflections (not accounted for in the diffusive treatment of relaxation) were considered in a small number of models. We made no attempt to determine whether a given star-MBH coalescence would occur as a gradual “extreme-mass ratio inspiral” (EMRI) detectable by LISA or a direct plunge through the horizon of the MBH (see [10, 11] for background information and references on EMRIs and [12, 13] for the importance of the distinction between EMRIs and plunges).

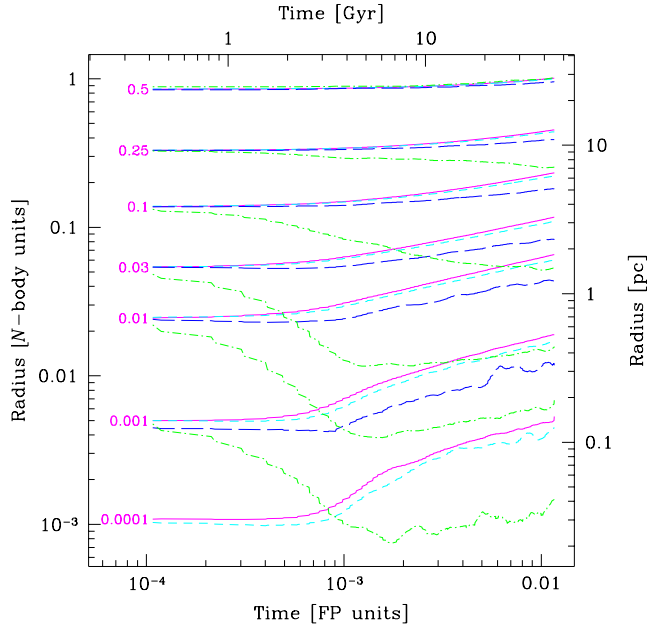
In all our runs the galactic nucleus is started as an  $\eta$ -model [14, 15], with a central power-law density cusp,  $\rho \propto R^{\eta-3}$  and steeper “cut-off” at large radii,  $\rho \propto R^{-4}$ . In most cases, we used parameters (mass of the MBH, stellar density around it, etc) corresponding to the stellar cluster around Sgr A\* at the centre of our Galaxy [16, 17]. We did not try to reproduce the very peculiar spatial and age distribution of the bright IR stars observed within 1 pc of Sgr A\*. In this work, we adopt the position that these stars, useful as they are as probes of the gravitational potential, are not representative of the overall stellar population at the Galactic centre, assumed to be much older and therefore amenable to our treatment. This defines a well-posed problem which constitutes an interesting limiting case. Clearly other situations have to be considered in future studies.

In addition to the Sgr A\* models, we followed the evolution of galactic nuclei hosting an MBH with a mass  $M_{\bullet}$  in the range  $10^4 - 10^7 M_{\odot}$ . Based on a somewhat naive application of the  $M - \sigma$  relation [18], we scaled the size of the stellar cluster according to  $R_{\text{nucl}} \propto M_{\bullet}^{0.5}$  where  $R_{\text{nucl}}$  is any characteristic length of the stellar distribution. We have considered two families of models; one with  $\eta = 2.0$ , the other with  $\eta = 1.5$ . The interval in  $M_{\bullet}$  was chosen mostly to cover the values that should yield gravitational wave signals in LISA band when a compact star inspiral into the MBH. The present study is a first step towards more robust determinations of the rate and characteristic of such EMRIs. This range of models also covers systems that are both large enough (in terms of the number of stars) to be amenable to treatment with the MC method and small enough for relaxational effects to play a significant role over some 10 Gyr.

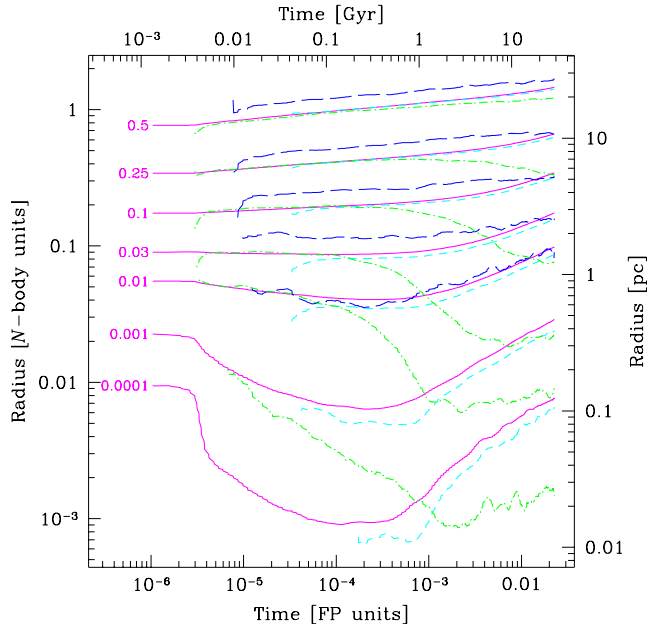
To ensure that the MC code, based as it is on a number of simplifying assumptions, yield correct results, we carried out a number of comparisons with simulations performed with the highly accurate (but much more computationally demanding) direct-summation NBODY4 code [19, 20]. In particular, we compared with results obtained by Baumgardt et al. [21] and performed a new NBODY4 simulation of a two-component model with a central massive object using 64 000 particles. On the GRAPE hardware at disposal not more than  $\sim 10^5$  particles can be used; hence it is not yet possible to simulate a system with a realistic mass function using direct  $N$ -body but this 2-component toy model demonstrated for the first time in a direct fashion that the MC code treats mass segregation around a MBH very satisfactorily.

## 2. Results

For the Sgr A\* models, our main results are the following. In all cases, the stellar BHs, being the most massive objects (with a fixed mass of  $10 M_{\odot}$  or a range of masses, depending on the model), segregate to the central regions. This segregation takes about 5 Gyr to complete. The nucleus then enters a second evolutionary phase which is characterised by the overall expansion



**Figure 1.** Evolution of Lagrange radii for a “standard” Sgr A\* nucleus model. We plot the evolution of the radii of spheres that enclose the indicated fractions of the mass of various stellar species. Solid lines are for MS stars, short-dashed lines for white dwarfs, long-dashed lines for neutron stars and dash-dotted lines for stellar BHs. For this model, a 10 Gyr-old, non-evolving stellar population was used and the central MBH has an initial mass of  $M_{\bullet} = 3.6 \times 10^6 M_{\odot}$ . It grows only very little through tidal disruptions and stellar coalescences.



**Figure 2.** Same as Fig. 1 but for a Sgr A\* model with all stars on the MS and  $M_{\bullet} = 1000 M_{\odot}$  at start. The MBH grows by accreting an *ad hoc* fraction of the gas emitted by stellar evolution to reach  $M_{\bullet} \simeq 3.9 \times 10^6 M_{\odot}$  after 10 Gyr. The structure of the nucleus after  $\sim 5$  Gyr is similar to that of the case without stellar evolution. The initial size of the cluster was adjusted to obtain a reasonably good fit to the observed stellar mass profile around Sgr A\* at  $t = 10$  Gyr.

of the central regions, powered by the accretion of stellar mass (of very negative energy) on to the MBH. Although all species participate in the expansion, mass segregation continues in a relative fashion, as the system of BHs expands slower than the other components. The structure of the nucleus at distances from the centre larger than  $\sim 10$  pc is unaffected by relaxation over a Hubble time. The evolution of two Sgr A\* models is shown in Figures 1 and 2.

BHs dominate the mass density within  $\sim 0.2$  pc of the MBH but we do not find them to be more numerous than MS stars in any region we can resolve (down to a few mpc, at  $t = 10$  Gyr). Estimating the exponent for the density cusp the BHs form,  $\rho \propto R^{-\gamma}$ , is difficult because of numerical noise but, in most cases,  $\gamma$  is compatible with the Bahcall-Wolf value  $\gamma = 1.75$  [22, 23]. In contrast, the less massive objects, such as MS stars, form a cusp with  $\gamma$  generally in the range 1.3–1.4 which is significantly lower than the value of 1.5 predicted by [23]. This is also found in

the 2-component  $N$ -body simulation. After 5-10 Gyr of evolution, we find of order  $2 - 3 \times 10^3$ ,  $6 - 8 \times 10^3$  and  $2 \times 10^4$  stellar BHs within 0.1, 0.3 and 1 pc of the centre, respectively. About  $10^4$  BHs coalesce with the MBH during a Hubble time. Using the formalism of the dynamical friction for objects on circular orbits in a fixed stellar background is an easy alternative for estimating the concentration of massive objects in the central regions. However, for stellar BHs, although this approach offers a qualitatively correct picture, it overpredicts the effectiveness of mass segregation. In the case of a model with  $\eta = 1.5$  (whose relaxation time does not increase towards the centre), this yields too large a number of BHs accreted by the MBH and *too few* (by a factor 3-5) being present within the inner 1 pc after some 10 Gyr.

All types of objects lighter than the BHs, including the neutron stars, are pushed away from the central regions. Using the observed distribution of these objects to infer the presence of segregated BHs [24] does not seem to be possible, though because, in the absence of BHs, it would take the neutron stars more than 10 Gyr to form a Bahcall-Wolf Cusp of their own, even without natal kick.

These results are not significantly affected by stellar collisions, large-angle scatterings or the initial  $\eta$  value. We also considered three different prescriptions for the masses (and types) of compact remnants and found no strong variations in the simulation outcomes. Most interestingly, an alternative model in which stellar evolution was included and the central MBH was grown from an IMBH seed (by accretion of an *ad hoc* fraction of the mass lost by stars when they turn into compact objects) yields basically the same structure of mass segregation (and same rates of coalescences and tidal disruptions) at  $t \simeq 10$  Gyr (see Fig. 2). These findings suggest that our main results are not very sensitive to the special “initial conditions” used, as long as they are fine-tuned to produce at  $t = 10$  Gyr a given MBH mass and stellar mass within  $\sim 1$  pc of the MBH. However, it would be instructive to consider a larger variety of models in future work, including some with extended period of stellar formation. Our present assumption of a single burst of stellar formation maximises the number fraction of stellar BHs and the time available for mass segregation.

We find rates of tidal disruptions and coalescences with the MBH of  $\sim 4 \times 10^{-5} \text{yr}^{-1}$  and  $\sim 3 \times 10^{-5} \text{yr}^{-1}$ , respectively, at  $t = 10$  Gyr. When stellar collisions are allowed, they occur at a rate about 10 times lower. On average, each MS-MS collision releases about  $0.01 M_{\odot}$  of gas; if collisions between MS stars and compact remnants lead to complete disruption of the MS star, collisions (of all types) yield an average of  $0.05 - 0.06 M_{\odot}$  per event.

When large-angle scatterings are explicitly included (essentially as a special case of collisions), they are found to have only little impact on the rate of tidal disruptions or coalescences with the MBH. A stellar BH is about 10 times more likely to be swallowed by the MBH than to be ejected from the nucleus. In contrast with this, in their multi-mass  $N$ -body simulations, Baumgardt et al. [25] find that all stellar BHs except one are ejected from the cluster and ascribe this result to strong interactions with objects (generally another stellar BH) deeply bound to the IMBH. These interactions are likely to be “resonant”, i.e., the three objects (including the IMBH) form a strongly interacting, chaotic configuration for many orbital times until one of the lighter objects is ejected (Baumgardt, personal communication; see e.g., [26]). In principle, this mechanism can be included into the MC code by extending the loss-cone treatment used for tidal disruptions and coalescences to interactions with the binary consisting of the MBH and the most bound stellar object and resorting to explicit integration of 3-body motion for close interactions between the binary and a third object. However a simple analysis based on extrapolation of the cross sections for single-binary encounters [27] suggests that, in galactic nuclei, such ejections are considerably less likely to occur than a direct plunges through the MBH’s horizon.

To first order, the evolution of galactic nuclei of different masses (but same initial  $\mu = M_{\bullet}/M_{\text{stars}}$  and  $\eta$  values, where  $M_{\text{stars}}$  is the total stellar mass), can be obtained by rescaling the mass, the size and the time units if stellar evolution is negligible. The evolution timescale is

set by the relaxation time which increases approximately like  $t_{\text{rlx}} \propto M_{\bullet}^{5/4}$  for our  $R_{\text{nucl}} \propto M_{\bullet}^{0.5}$  assumption. This means that small nuclei are affected by faster segregation and MBH-driven expansion and may have been significantly more compact in the past than observed in the local universe. On the other hand, only very little relaxational mass-segregation can have occurred in nuclei with MBHs more massive than  $\sim 10^7 M_{\odot}$  over a Hubble time. The evolution of the region very close to the MBH cannot be scaled from one nucleus to another with different mass and size, however because the processes of tidal disruptions and coalescences with the MBH introduce physical scales. In particular, we find that stellar BHs in more massive nuclei experience less segregation than in smaller nuclei (at a given value of  $t/t_{\text{rlx}}$ ). Also the rate of coalescences (in units of  $N_*/t_{\text{rlx}}$ , where  $N_*$  is the total number of stars) is higher in more massive nuclei. This is probably an effect of a larger critical radius<sup>1</sup>, requiring a larger accretion rate to yield the same amount of energy to power the expansion of the nucleus.

### 3. Observational consequences and future developments

Although we have not attempted a realistic modelling of the Galactic centre, it is tempting to apply our results to one specific observation of the Sgr A\* region. Using the *Chandra* X-ray space-born observatory, Munro et al. [28] have detected 7 transient sources which appear to be much more concentrated around Sgr A\* than the overall stellar population. Here we examine whether this may be a direct consequence of mass segregation, if these sources are all stellar BHs accreting from a lower-mass companion. We make the strong assumption that these binaries are not formed or affected by interactions with other stars such as 3-body binary formation, partner exchange, ionisation, etc. Instead we consider that they just react to 2-body relaxation as point objects with a total mass approximated by the mass of the stellar BH. If we pick up at random 7 sources with projected distance from the centre smaller or equal to 23 pc according to the profile for stellar BHs found in our simulations, we find that their distribution is at least as concentrated as the observed one in 15% of the cases. It is therefore at this point not possible to exclude that the transients owe their peaked profile purely to mass segregation but this seems somewhat unlikely.

As pointed out by Munro et al., the rate of binary interactions should also increase steeply towards the centre and this probably combines with mass segregation to produce the observed distribution. The problem of binary dynamics in the vicinity of a MBH is complicated by the fact that there is no clear-cut definition of the hard-soft transition [29]. The Keplerian velocity dispersion increases virtually without bound when one approaches the centre. This may affect a binary on an orbit of relatively large semimajor axis  $a$  around the MBH because 2-body relaxation will cause the orbit to reach down to a value  $R_{\text{peri}} = (1 - e)a \ll a$  over a timescale of order  $t_{\text{rlx}} \ln(1/(1 - e))$  [30].

The most extreme type of dynamical interaction a binary can experience is the tidal separation of its members if its orbit brings it within  $\sim a_{\text{bin}}(M_{\bullet}/m_{\text{bin}})^{1/3}$  of the MBH. Here  $a_{\text{bin}}$  is the semimajor axis of the binary itself and  $m_{\text{bin}}$  its mass. This process is of great interest by itself both as a way to create “hypervelocity stars” and to deposit a star on a tight orbit around the MBH [31, 32, 33, 34, 35].

To our knowledge, the complex question of binary dynamics in a galactic centre has not been investigated yet. This is an ideal subject for self-consistent stellar dynamical simulations of the sort presented here but including binary processes. MC codes are particularly well suited for following the evolution of large systems with a significant fraction of binaries whose interaction can be computed accurately by direct 3- and 4-body integrations [36, 37, 38].

The stellar BHs at the Galactic centre may have detectable consequences even if they are single. If the motions of the S-stars can be tracked with high enough a precision, an extended

<sup>1</sup> The critical radius is essentially the typical value of the semimajor axis for stars swallowed by the MBH.

distribution of non-luminous matter around the Galactic MBH should signal itself through its effect on their orbits. Present-day observations are insufficient to detect the slight Newtonian retrograde precession induced by an extended “dark cusp” [39], but future 30 – 100 m telescopes may allow to realise such a measurement and to witness  $\sim 3$  trajectory deflections caused by gravitational encounters per year between any of  $\sim 100$  monitored S-stars and a stellar BH [40]. The compact stars can also collide and merge with MS stars and giants, hence creating unusual objects, once suggested to be the S-stars themselves [41], or increase the rotation rate of extended stars through multiple tidal interactions [42]. In this context, we note that our models predict about  $2 - 3 \times 10^4$  collisions between MS stars and stellar BHs and a similar number of MS–white-dwarf events to occur over a Hubble time in a Sgr A\*-type nucleus. Collisions with neutron stars are at least 10 times rarer. Finally, it seems relatively probable that at any given time some stellar BHs in the Galactic centre are bright X-ray sources as they accrete from the clumpy interstellar gas (poster by Patrick Deegan in this conference; Nayakshin & Sunyaev, in preparation).

As for EMRI rates and properties, the determination of how 2-body relaxation shapes the stellar distribution around the MBH is only a first –crucial– step. A robust estimate of the fraction of stars that eventually inspiral into LISA band, rather than plunge directly through the horizon while still on a wide orbit [12, 13] will probably require the development of a specific code. For stars on very eccentric orbits, one needs to follow the combined effects of GW emission and relaxation on a timescale significantly shorter than allowed by the present MC code. Recently Hopman & Alexander ([43] and these proceedings) have considered, for the first time in the study of EMRIs, the role of “resonant relaxation”, i.e., of the random changes in eccentricity and orientation of the orbital planes due to the non-vanishing but fluctuating torque exerted on an orbit by the other orbits, each considered as an elliptical mass wire [44]. These authors find that resonant relaxation can increase the EMRI rate by of order a factor 10, an exciting result which is calling for confirmation by other computation techniques. Resonant relaxation can be included in the MC algorithm using the same approximate formulae as Hopman & Alexander. Unfortunately, this treatment relies on the calibration realised through a few very low- $N$   $N$ -body simulations [44] and more accurate numerical studies of this potentially important effect are called for (see Touma, these proceedings for a possible high-efficiency method).

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