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MAXIMAL SUPERGRAVITIES AND THE E_{10} COSET MODEL*

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The maximal rank hyperbolic Kac–Moody algebra $\mathfrak{e}_{10}$ has been conjectured to play a prominent role in the unification of duality symmetries in string and M-theory. We review some recent developments supporting this conjecture.

Keywords: Kac–Moody symmetries; supergravity; supersymmetry.

1. Introduction

Hidden symmetries of exceptional type in the reduction of supergravity theories were first discovered in maximal $N = 8$ supergravity in $D = 4$.[1,2] The unexpected emergence of the coset $E_7/\mathrm{SU}(8)$ describing the scalar sector of the theory was soon generalized to other dimensions and other theories.[3] The most prominent example remains the chain of hidden symmetries occurring in the dimensional reduction of $D = 11$ maximal supergravity on a torus T^n. For $1 \leq n \leq 8$ the resulting scalars, after maximal dualization of forms to scalars at every step, always appear in group cosets which have become known as $E_n/K(E_n)$ where we use $K(E_n)$ to designate the maximal compact subgroup of E_n. For $n > 8$ it was soon conjectured that the resulting symmetry groups become infinite-dimensional,[4] and formulations using the centrally extended loop group E_9 (Ref. 5) and partial results on E_{10} (Ref. 6) have since been obtained.

In an initially unrelated development, the study of the asymptotic behavior of $D = 11$ supergravity (and IIA and IIB supergravity) near a space-like singularity also revealed evidence for infinite-dimensional symmetries, and the hyperbolic Kac–Moody group E_{10}, in particular. Namely, in this limit, the dynamics can be described aymptotically by a cosmological billiard taking place in the fundamental Weyl chamber of E_{10}.[7,8] The dynamical variables in this case are the spatial

scale factors. This led the authors of Ref. 9 to propose a one-dimensional non-linear σ-model based on a coset $E_{10}/K(E_{10})$, and Damour *et al.*[9,10] uncovered a remarkable dynamical equivalence between a truncation of the bosonic $D = 11$ supergravity equations and a truncated version of this infinite-dimensional coset model. It is the aim of the present contribution to review this correspondence and similar correspondences to the $D = 10$ maximal supergravities which were derived in Refs. 11 and 12. We will not present the relation of higher derivative corrections to the E_{10} model which can be found in Ref. 13. Related works in Refs. 14 and 15 discuss the role of imaginary roots of E_{10} from a brane point of view and orbifolds.

In yet another development, it was proposed in Refs. 16 and 17 that $D = 11$ supergravity is a non-linear realization of the bigger group E_{11} (and the conformal group via a Borisov–Ogievetsky-type construction[18]). The non-linear E_{11} model is thus supposed to operate directly in 11 (or even more) dimensions, and is therefore very different from the one-dimensional E_{10} model which we will present below. In addition, as will be discussed in Sec. 5.3, space–time is thought to emerge in the present scheme from the E_{10} model itself, whereas for E_{11} space–time is realized through an additional E_{11} invariant structure.[19] These two space–time concepts, and a third one based on a one-dimensional E_{11} model proposed in Ref. 20, were discussed in Ref. 21. The relation of that latter E_{11} model to the model considered here[a] was studied in Ref. 25. It was already shown in Refs. 26 and 27 that E_{11} can unify the symmetries of the bosonic sectors type IIB and massive type IIA supergravity by use of the same techniques as in Ref. 17, which is therefore a feature shared by the E_{10} model and the E_{11} proposal.

A most remarkable feature of the results obtained so far is that E_{10} implies several results on the bosonic sectors of maximal supergravity theories which were heretofore thought to require (maximal) local supersymmetry, to wit:

- the correct bosonic multiplets of all maximal supergravities in 11 and all lower dimensions, in particular for both (massive) type IIA and type IIB supergravity,[11,12] as first shown for the embedding of these theories into E_{11}[26,27];
- the self-duality of the 5-form field strength in IIB supergravity[12];
- the correct bosonic self-couplings for all these theories, in particular, of the $D = 11$ Chern–Simons term[9];
- the vanishing of the cosmological constant in $D = 11$ supergravity,[10] originally shown in Ref. 28;
- (possibly) restrictions on the form of the higher order corrections in M theory.[13]

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[a]There exists a related "brane version" of the E_{10} model for which the denominator is non-compact and has non-unique space–time signatures[22,23] which are identical to those of the exotic M-theories of Ref. 24.

This casts some doubt on widely held expectations concerning the role of (local) supersymmetry as a fundamental symmetry, and may indicate that the concept of supersymmetry may have to be replaced by yet another, and in some sense, even more "fundamental" symmetry concept[b] (possibly also involving quantization). Moreover, in a scheme where space(–time) is treated as an "emergent" phenomenon, the distinction between bosons and fermions may well disappear, too, and only "emerge" together with space(–time) itself.

The structure of this contribution is as follows. Section 2 reviews some basic facts about the hyperbolic Kac–Moody algebra $\mathfrak{e}_{10}$ underlying E_{10}, leading in particular to a spectral analysis of the $E_{10}/K(E_{10})$ model. The model itself is defined in Sec. 3. The correspondences to the various maximal supergravity theories in $D = 10$ and $D = 11$ are derived or reviewed in Sec. 4. In Sec. 5 we discuss some cosmological applications and extensions of the model presented here and end with a few open problems.

2. The $\mathfrak{e}_{10}$ Kac–Moody Algebra

In this section we recall the definition of the hyperbolic Kac–Moody Lie algebra $\mathfrak{e}_{10} = \text{Lie}(E_{10})$ and present its basic properties required for the coset model which will be defined in Sec. 3.

2.1. *Definition of $\mathfrak{e}_{10}$*

We use the Chevalley–Serre presentation also employed in Refs. 29 and 30. This definition starts from the generalized Cartan matrix or, equivalently, the $\mathfrak{e}_{10}$ Dynkin diagram given in Fig. 1.

The *rank* of $\mathfrak{e}_{10}$ is 10 since there are 10 nodes in Fig. 1. The Cartan matrix $A = (A_{ij})$ encoded in the $\mathfrak{e}_{10}$ Dynkin diagram is given by $(i, j = 1, \ldots, 10)$

$$
A_{ij} = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{if there is a link between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}
$$

For the case of $\mathfrak{e}_{10}$ the Cartan matrix is non-degenerate and indefinite (with nine positive eigenvalues and one negative eigenvalue). According to the general theory of Kac–Moody Lie algebras,[29] the Chevalley–Serre presentation therefore starts from 30 *Chevalley generators*

$$
e_i, f_i, h_i \qquad (i = 1, \ldots, 10) \tag{1}
$$

subject to the relations

$$
[h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0. \tag{2}
$$

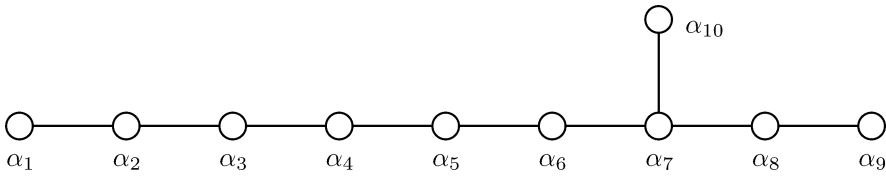[b]Of course, we will have to await what LHC has to say on this issue!

Fig. 1.   Dynkin diagram of $\mathfrak{e}_{10}$.

On top of the 30 generators of (1) one now considers multiple commutators of the *simple positive generators* e_i and of the *simple negative generators* f_i of the form

$$[e_{i_1}, [\cdots [e_{i_{k-1}}, e_{i_k}]]] \quad \text{and} \quad [f_{i_1}, [\cdots [f_{i_{k-1}}, f_{i_k}]]], \tag{3}$$

spanning free Lie algebras on $\{e_i\}$ and $\{f_i\}$ and then subject to the *Serre relations* (for $i \neq j$)

$$(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0 \quad \text{and} \quad (\operatorname{ad} f_i)^{1-A_{ij}} f_j = 0, \tag{4}$$

and, of course, to the Jacobi and anti-symmetry relations of the Lie bracket. The number of Chevalley generators in such a nested multiple commutator is called the *height* of the element.

The Lie algebra consisting of (1) and (Serre) non-trivial elements (3) is the infinite-dimensional Kac–Moody algebra $\mathfrak{e}_{10}$, which we will also sometimes denote by $\mathfrak{g} \equiv \mathfrak{e}_{10}$. We consider $\mathfrak{e}_{10}$ in its *split real form* obtained by taking only real linear combinations of the basis elements. Like every Kac–Moody algebra, $\mathfrak{e}_{10}$ possesses a triangular structure

$$\mathfrak{e}_{10} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{5}$$

where $\mathfrak{h}$ is the *Cartan subalgebra* (CSA) spanned by the elements h_i, and $\mathfrak{n}_\pm$ have bases consisting of the positive simple generators e_i together with their multiple commutators and the negative simple generators f_i together with their multiple commutators, respectively. The positive half $\mathfrak{n}_+$ and the negative half $\mathfrak{n}_-$ are exchanged by the *Chevalley involution* θ which acts on the Chevalley generators (1) by

$$\theta(e_i) = -f_i, \quad \theta(f_i) = -e_i, \quad \theta(h_i) = -h_i \tag{6}$$

and extends to all of $\mathfrak{e}_{10}$; sometimes it is convenient to define a *generalized transposition* by $x^{\mathrm{T}} := -\theta(x)$. The fixed point set of θ

$$\mathfrak{k}_{10} := K(\mathfrak{e}_{10}) := \{x \in \mathfrak{e}_{10} : \theta(x) = x\} \tag{7}$$

will be called the *maximal compact subalgebra* in analogy with the finite-dimensional theory. It consists of all anti-symmetric elements. The Lie algebra $\mathfrak{k}_{10}$ is *not* a Kac–Moody algebra.[31]

We will furthermore make use of the *root space decomposition* of $\mathfrak{g} \equiv \mathfrak{e}_{10}$ defined through

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{8}$$

with

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}. \tag{9}$$

The non-trivial linear maps $\alpha : \mathfrak{h} \to \mathbb{R}$ are called *roots*. The *simple roots* are denoted by α_i; their associated root spaces $\mathfrak{g}_{\alpha_i} = \langle e_i \rangle$ are thus one-dimensional. The *root lattice Q* is obtained as the integer lattice over the simple roots. All roots are linear combinations of simple roots with either all non-negative or all non-positive coefficients; the roots are then called "positive" or "negative," respectively. We thus write an arbitrary root $\alpha \in \Delta$ as

$$\alpha = \sum_{i=1}^{10} m_i \alpha_i, \tag{10}$$

with either all $m_i \geq 0$ or all $m_i \leq 0$. The *height of α* is $\mathrm{ht}(\alpha) = \sum_i m_i$. The set of all roots $\Delta \subset Q$ hence decomposes as $\Delta = \Delta_+ \cup \Delta_-$ into positive roots Δ_+, and negative roots Δ_-.

For the hyperbolic[c] Kac–Moody algebra $\mathfrak{e}_{10}$ one can furthermore show[29] that the set of all roots $\Delta \subset Q$ is given by

$$\Delta = \{ \alpha \in Q : \alpha^2 \leq 2 \} \setminus \{0\}. \tag{11}$$

Here, the norm $\alpha^2 = \langle \alpha | \alpha \rangle$ is computed using the Cartan matrix as inner product (on both $\mathfrak{h}$ and $\mathfrak{h}^*$), such that for all simple roots $\alpha_i^2 = \langle \alpha_i | \alpha_i \rangle = 2$ and for an arbitrary root (10) we have $\alpha^2 = \sum_{i,j} m_i A_{ij} m_j$. The symmetric form $\langle \cdot | \cdot \rangle$ on $\mathfrak{h}$ can be extended to a symmetric invariant form on all of $\mathfrak{e}_{10}$ by letting $\langle e_i | f_j \rangle = \delta_{ij}$ and then using the invariance to define $\langle \cdot | \cdot \rangle$ on multiple commutators. A final piece of terminology we require from the theory of Kac–Moody algebras is the notion of *real* and *imaginary* roots. A root α of $\mathfrak{e}_{10}$ is said to be real if $\alpha^2 = 2$ and imaginary otherwise; in the latter case, one further distinguishes light-like (null) and time-like roots, for which $\alpha^2 = 0$ and $\alpha^2 < 0$, respectively.

Further details can be found in Ref. 29.

2.2. *Spectral analysis: Level decomposition*

Among the basic quantities of interest of a Kac–Moody algebra are the *root multiplicities* $\mathrm{mult}(\alpha) = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ for roots $\alpha \in \Delta$. In contradistinction to the finite-dimensional Lie algebras and affine Kac–Moody algebras there is no closed formula determining the root multiplicity of an arbitrary root α. Presently, only recursive

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[c]The adjective "hyperbolic" means that upon deletion of any single node from the Dynkin diagram, the remaining diagram consists only of diagrams of finite-dimensional or affine diagrams.[29]
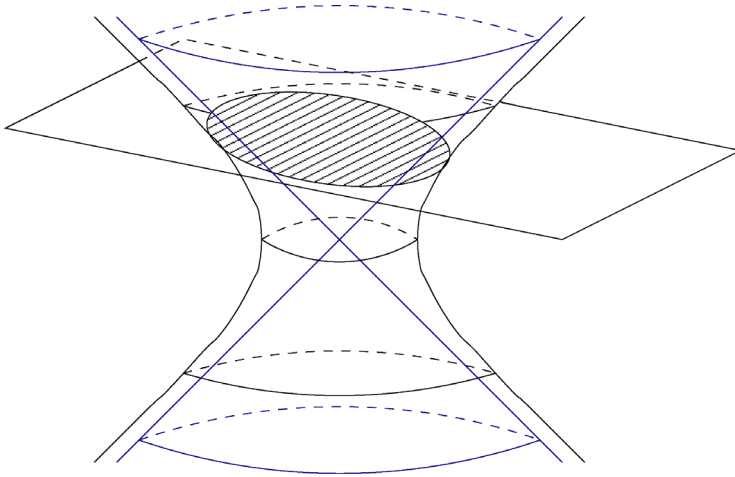
Fig. 2.   A sketch of the solid hyperboloid in root space with an elliptic slice corresponding to a finite-dimensional subalgebra of $\mathfrak{e}_{10}$. For the pictorial presentation the number of dimensions has been lowered and the discrete lattice structure has been smoothened.

techniques such as the Peterson formula can be used to calculate the multiplicities[d] by working up in height, starting from the fact that for the simple roots one has $\text{mult}(\alpha_i) = 1$. By performing such calculations on a computer one can obtain a good picture of the root structure of a Kac–Moody algebra.[32,33] Due to the Lorentzian structure of the Cartan matrix of $\mathfrak{e}_{10}$ and the condition (11) we end up with the following picture: all lattice points α inside the solid hyperboloid $\{\alpha^2 \leq 2\} \subset Q$ in the Lorentzian space $\mathfrak{h}^*$ are roots of $\mathfrak{e}_{10}$, and each such point represents the vector space $\mathfrak{g}_\alpha$ associated with the corresponding root α. We imagine the lattice points as being labelled in addition by the dimension (= multiplicity) of the root space.

An economical and physically motivated way to present the algebraic data is the *level decomposition* under a finite-dimensional regular subalgebra[e] introduced in Ref. 9. In the above picture this corresponds to an elliptic slicing of the solid hyperboloid. From this it is obvious that any given slice contains only a finite number of points representing a root space, see Fig. 2. In this language, the algebra can be described by a stack of slices each of which contains a finite number of irreducible representations of the subalgebra (acting via the adjoint action). The *level ℓ* of such a slice is given as the vector of numbers of times the simple generators of the deleted nodes appears in the elements on the slice, and ℓ provides a grading of the algebra. Instead of giving the details of how this decomposition is done, we will give the relevant examples below and refer the reader to Refs. 9, 32 and 34–36 for expositions of the general technique. We will only give levels $\ell \geq 0$ since

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[d]It is known that the root multiplicities grow exponentially in $-\alpha^2$.

[e]By this we mean a subalgebra obtained by deleting nodes from the Dynkin diagram. Examples will be given below. This terminology is slightly non-standard.

Table 1.   Levels $0 \leq \ell \leq 3$ in the A_9 decomposition of $\mathfrak{e}_{10}$.

ℓ	A_9 representation	$\mathfrak{e}_{10}$ root α	Generator
0	[100000001]	$(1,1,1,1,1,1,1,1,1,0)$	$K^a{}_b$ (traceless)
	[000000000]	$(0,0,0,0,0,0,0,0,0,0)$	$K := \sum_{a=1}^{10} K^a{}_a$
1	[000000100]	$(0,0,0,0,0,0,0,0,0,1)$	$E^{a_1 a_2 a_3}$
2	[000100000]	$(0,0,0,0,1,2,3,2,1,2)$	$E^{a_1 \cdots a_6}$
3	[010000001]	$(0,0,1,2,3,4,5,3,1,3)$	$E^{a_0|a_1 \cdots a_8}$

the negative levels are contragradient to the positive levels due to the Chevalley involution θ.[f]

2.2.1. *Decomposition under A_9*

We first single out the "exceptional" node 10 in Fig. 1 and delete it together with its link to node 7, leaving the subalgebra $A_9 \equiv \mathfrak{sl}(10)$. The level ℓ in this decomposition is the last entry m_{10} of a root $\alpha = \sum_{i=1}^{10} m_i \alpha_i$. The spectrum on levels $0 \leq \ell \leq 3$ in the A_9 decomposition was computed in Ref. 9 and is given in Table 1.

The notation in Table 1 is as follows. Irreducible $\mathfrak{sl}(10)$ representations are given in terms of their Dynkin labels, such that the entry in the first row of the table is the adjoint representation, whereas the second representation for $\ell = 0$ is the trivial one. They combine to give the generators $K^a{}_b$ of $\mathfrak{gl}(10)$. Indices a, b etc. are $\mathfrak{sl}(10)$ vector indices and hence take values $1, \ldots, 10$. Similarly, the representations on levels $\ell = 1$ and $\ell = 2$ are totally anti-symmetric of rank 3 and 6, respectively, and we have already introduced the symbols we will use below when referring to these representations. The $\ell = 3$ generator satisfies the Young irreducibility constraints

$$E^{a_0|[a_1 \cdots a_8]} = E^{a_0|a_1 \cdots a_8}, \quad E^{[a_0|a_1 \cdots a_8]} = 0. \tag{12}$$

These $0 \leq \ell \leq 3$ tensors will play a role below when we relate an E_{10} symmetric coset model to $D = 11$ supergravity in Sec. 4.1. Let us also note that up to level $\ell \leq 3$, the decomposition is essentially the same for all E_n; in particular, for E_{11}, the representations are analogous.[35] Beyond level $\ell = 3$, however, they differ by A_{10} representations which have no counterparts in E_{10}.[32]

2.2.2. *Decomposition under D_9*

Another possible slicing of the hyperboloid is obtained by deleting the node 9 in Fig. 1. The remaining subalgebra is $D_9 \equiv \mathfrak{so}(9,9)$. The level decomposition was carried out in Ref. 11 with result for $\ell = 0, 1, 2$ reproduced in Table 2. Here, the indices $I, J, K = 1, \ldots, 18$ are vector indices of $\mathfrak{so}(9,9)$, whereas $A = 1, \ldots, 256$ is a spinor index. The representations on level $\ell = 0$ are the adjoint and the scalar

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[f]Another reason for restricting the spectral analysis to $\ell \geq 0$ is that in Sec. 3 we will use a triangular gauge which only employs $\ell \geq 0$ generators.

Table 2.   Levels $0 \leq \ell \leq 2$ in the D_9 decomposition of $\mathfrak{e}_{10}$.

ℓ	D_9 representation	$\mathfrak{e}_{10}$ root α	Generator
0	[010000000]	$(1, 2, 2, 2, 2, 2, 2, 1, 0, 1)$	M^{IJ}
	[000000000]	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	T
1	[000000010]	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$	E_A
2	[001000000]	$(0, 0, 0, 1, 2, 3, 4, 3, 2, 2)$	E^{IJK}

Table 3.   Levels $0 \leq \ell \leq 4$ in the $A_8 \oplus A_1$ decomposition of $\mathfrak{e}_{10}$.

ℓ	$A_8 \oplus A_1$ representation	$\mathfrak{e}_{10}$ root α	Generator
0	([10000001], $\mathbf{1}$)	$(1, 1, 1, 1, 1, 1, 1, 0, 0, 1)$	$K^a{}_b$ (traceless)
	([00000000], $\mathbf{1}$)	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	K
	([00000000], $\mathbf{3}$)	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$	J_i
1	([00000010], $\mathbf{2}$)	$(0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$	$E_\alpha^{a_1 a_2}$
2	([00001000], $\mathbf{1}$)	$(0, 0, 0, 0, 0, 1, 2, 2, 1, 1)$	$E^{a_1 \dots a_4}$
3	([00100000], $\mathbf{2}$)	$(0, 0, 0, 1, 2, 3, 4, 3, 1, 2)$	$E_\alpha^{a_1 \dots a_6}$
4	([01000001], $\mathbf{1}$)	$(0, 0, 1, 2, 3, 4, 5, 4, 2, 2)$	$E^{a_0 | a_1 \dots a_7}$
	([10000000], $\mathbf{3}$)	$(0, 1, 2, 3, 4, 5, 6, 4, 1, 3)$	$E_i^{a_1 \dots a_8}$

representation, respectively. Level $\ell = 1$ contains the Dirac spinor, and $\ell = 2$ an anti-symmetric three form.[g] In Sec. 4.2 we will relate the corresponding tensors to quantities appearing in (massive) type IIA supergravity.

2.2.3. *Decomposition under $A_8 \oplus A_1$*

The final choice of subalgebra we consider is obtained by deleting node 8 from Fig. 1 (this analysis follows an earlier one on the embedding of A_9 into E_{11}[26,36]). The remaining subalgebra is now $A_8 \oplus A_1 \equiv \mathfrak{sl}(9) \oplus \mathfrak{sl}(2)$. The decomposition was carried out in Ref. 12 with the result for $\ell = 0, 1, 2, 3, 4$ reproduced in Table 3. Here, the $\mathfrak{sl}(9)$ vector indices a, b, etc. range from 1 to 9, the indices $i = 1, 2, 3$ are $\mathfrak{so}(2,1) \equiv \mathfrak{sl}(2)$ vector indices and $\alpha = 1, 2$ are $\mathfrak{so}(2,1)$ spinor indices. These tensors will be related to IIB supergravity quantities in Sec. 4.3. Beyond the levels displayed in the table, there again appear differences between E_{10} and E_{11}: one example are the ten-forms studied in Ref. 37 which have no analog in E_{10}, see also Refs. 12, 38–40.

2.3. *Commutation relations*

Once the representation content (spectrum) of the algebra has been determined to the required level one then needs to work out the commutation relations between the Lie algebra elements. Here the power of the level decomposition becomes evident since only a few structure constants appear between the representations. The

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[g]Generally, the representations occurring on even levels are tensor representations of $\mathfrak{so}(9,9)$ and the odd level representations are spinor representations of $\mathfrak{so}(9,9)$.

number of $\mathfrak{e}_{10}$ elements thus covered can still be quite large if the representations are large but covariance under the subalgebra action fixes all the structure constants within a representation. We again illustrate these facts in examples. A more general discussion on how to compute the commutation relations can be found in Ref. 40.

We consider the A_9 decomposition of $\mathfrak{e}_{10}$ in some detail. As discussed in Sec. 2.2.1, the $\ell = 0$ generators are $K^a{}_b$ $(a, b = 1, \ldots, 10)$ with standard $\mathfrak{gl}(10)$ commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{13}$$

and normalization

$$\langle K^a{}_b | K^c{}_d \rangle = \delta^c_b \delta^a_d - \delta^a_b \delta^c_d. \tag{14}$$

Their identification with the Chevalley generators of $\mathfrak{e}_{10}$ is

$$e_i = K^i{}_{i+1}, \quad f_i = K^{i+1}{}_i, \quad h_i = K^i{}_i - K^{i+1}{}_{i+1} \qquad (i = 1, \ldots, 9), \tag{15}$$

and

$$h_{10} = -\frac{1}{3}K + K^8{}_8 + K^9{}_9 + K^{10}{}_{10}. \tag{16}$$

This follows from demanding $[h_{10}, e_i] = A_{10i}e_i$. The "exceptional" generators e_{10} and f_{10} belong to levels $\ell = 1$ and $\ell = -1$, respectively and will be identified below.

On $\ell = 1$, the generators $K^a{}_b$ act by $\mathfrak{gl}(10)$ rotations[h]

$$[K^a{}_b, E^{c_1 c_2 c_3}] = \delta^{c_1}_b E^{a c_2 c_3} + \delta^{c_2}_b E^{c_1 a c_3} + \delta^{c_3}_b E^{c_1 c_2 a} \equiv 3\delta^{[c_1}_b E^{c_2 c_3]a}. \tag{17}$$

The next relation to be worked out involves the transposed $\ell = -1$ generator

$$F_{a_1 a_2 a_3} := (E^{a_1 a_2 a_3})^{\mathrm{T}} := -\theta(E^{a_1 a_2 a_3}), \tag{18}$$

which transforms contragradiently under $K^a{}_b$

$$[K^a{}_b, F_{c_1 c_2 c_3}] = -3\delta^a_{[c_1} F_{c_2 c_3]b}. \tag{19}$$

Due to the grading property of the level the commutator $[(\ell = 1), (\ell = -1)]$ has to be contained in $(\ell = 0)$. The relations (17) and (19) are insensitive to the normalizations of $E^{a_1 a_2 a_3}$ (and hence $F_{a_1 a_2 a_3}$) but $[E^{a_1 a_2 a_3}, F_{b_1 b_2 b_3}]$ is not. We fix

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[h]We use (anti-)symmetrizers of strength one.

the norm in terms of the invariant bilinear form to be

$$\langle E^{a_1 a_2 a_3} | F_{b_1 b_2 b_3} \rangle = 3! \, \delta^{a_1 a_2 a_3}_{b_1 b_2 b_3}. \tag{20}$$

The lowest element on $\ell = 1$ is $E^{8\,9\,10}$ and in view of $\langle e_{10} | f_{10} \rangle = 1$ and (20) we can identify

$$e_{10} = E^{8\,9\,10}, \qquad f_{10} = F_{8\,9\,10}. \tag{21}$$

This also fixes the coefficients in the commutation relation

$$[E^{a_1 a_2 a_3}, F_{b_1 b_2 b_3}] = -2\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} K + 18\delta^{[a_1 a_2}_{[b_1 b_2} K^{a_3]}{}_{b_3]}, \tag{22}$$

which completes the full set of commutation relations on levels $\ell = -1, 0, 1$. Thus, in the above relation, GL(10) covariance fixes all structure constants in terms of only *two* coefficients. More precisely, these two coefficients are fixed by invariance of the bilinear form and the normalization (20). Similar remarks apply to higher level commutators.

Proceeding to $\ell = 2$, we again have to fix the normalization of the relevant generator $E^{a_1 \cdots a_6}$, which we do by *defining*

$$E^{a_1 \cdots a_6} := [E^{a_1 a_2 a_3}, E^{a_4 a_5 a_6}], \tag{23}$$

which, using the invariance of the bilinear form, leads to

$$\langle E^{a_1 \cdots a_6} | F_{b_1 \ldots b_6} \rangle = 6! \, \delta^{a_1 \cdots a_6}_{b_1 \ldots b_6}. \tag{24}$$

Using Jacobi identities the remaining commutators for $|\ell| \leq 2$ can be worked out and are listed in Ref. 10.

For $\ell = 3$ we define

$$E^{[a_0 | a_1 a_2] a_3 \cdots a_8} := [E^{a_0 a_1 a_2}, E^{a_3 \cdots a_8}], \tag{25}$$

which is equivalent to

$$E^{a_0 | a_1 \cdots a_8} = 4[E^{a_0 [a_1 a_2}, E^{a_3 \cdots a_8]}], \tag{26}$$

with normalization

$$\langle E^{a_0 | a_1 \cdots a_8} | F_{b_0 | b_1 \ldots b_8} \rangle = 8 \cdot 8! \left(\delta^{a_0}_{b_0} \delta^{a_1 \cdots a_8}_{b_1 \ldots b_8} - \delta^{[a_1}_{b_0} \delta^{a_2 \cdots a_8]}_{[b_1 \ldots b_7} \delta^{a_0}_{b_8]} \right). \tag{27}$$

Again, Jacobi identities can be used to determine the remaining commutation relations. These were given in Ref. 10.

Similar considerations can be used for the D_9 and $A_8 \oplus A_1$ decompositions of Secs. 2.2.2 and 2.2.3. The results can be found in Refs. 11 and 12, respectively.

3. The E_{10} Coset Model

In this section, we present, following Ref. 9, a coset model with manifest E_{10} symmetry. This model is a null geodesic model on the coset space $E_{10}/K(E_{10})$, where E_{10} is the Kac–Moody group with Lie algebra $\mathfrak{e}_{10}$ and $K(E_{10})$ its "maximal compact subgroup" with Lie algebra $\mathfrak{k}_{10} \subset \mathfrak{e}_{10}$ fixed by the Chevalley involution θ from (6). We take a time-dependent coset element $\mathcal{V}(t) \in E_{10}/K(E_{10})$ of the form

$$\mathcal{V}(t) = \exp\left(\sum_{i=1}^{10} \phi_i(t) h_i\right) \exp\left(\sum_{\alpha \in \Delta_+} \sum_{s=1}^{\text{mult}(\alpha)} A_\alpha^{(s)}(t) E_\alpha^{(s)}\right), \tag{28}$$

which is in the so-called Borel gauge, where only the CSA $\mathfrak{h}$ and the upper triangular part $\mathfrak{n}_+$ are used. $E_\alpha^{(s)}$ label the independent generators in the root space of α. In the finite-dimensional situation the parametrization (28) can be reached due to the Iwasawa decomposition. Here, we simply take (28) as definition for the coset $E_{10}/K(E_{10})$. We will sometimes also use a slight modification of this parametrization where only levels $\ell \geq 0$ are used and the exponentials are separated differently, which is a non-linear change of coordinates on the coset.

The $\mathfrak{e}_{10}$-valued velocity (Cartan form) associated with the coset element (28) is

$$\partial_t \mathcal{V} \mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}, \quad \mathcal{Q} \in \mathfrak{k}_{10}, \ \mathcal{P} \in \mathfrak{e}_{10} \ominus \mathfrak{k}_{10}, \tag{29}$$

such that $\theta(\mathcal{Q}) = \mathcal{Q}$ and $\theta(\mathcal{P}) = -\mathcal{P}$. We define the generators along the coset and the subgroup by

$$S_\alpha = E_\alpha + F_\alpha, \qquad J_\alpha = E_\alpha - F_\alpha, \tag{30}$$

where $F_\alpha := (E_\alpha)^{\text{T}} := -\theta(E_\alpha)$ denotes the transposed generator to E_α.[i] Therefore, $\mathfrak{k}_{10}$ consists of the anti-symmetric elements and the remaining symmetric elements belong to $\mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$.

The time reparametrization invariant Lagrange function defining the dynamics of the geodesic model is given by

$$\mathcal{L} = \mathcal{L}(t) = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle, \tag{31}$$

where $n(t)$ is a Lagrange multiplier ("einbein") needed for reparametrization invariance and $\langle \cdot | \cdot \rangle$ is the $\mathfrak{e}_{10}$ invariant bilinear form discussed in Sec. 2.1. The equations of motion following from this Lagrange function are

$$\partial_t(n^{-1}\mathcal{P}) = [\mathcal{Q}, n^{-1}\mathcal{P}], \qquad \langle \mathcal{P} | \mathcal{P} \rangle = 0. \tag{32}$$

We will refer to the second equation (obtained by varying n) as the Hamiltonian constraint. It expresses the light-like orientation of the geodesic.

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[i]The multiplicity index will be suppressed for clarity of notation and is summed over implicitly.

The system is formally *integrable*, as is already evident from the Lax formulation of the geodesic equation in (32). It is easy to write down infinitely many conserved charges through the $\mathfrak{e}_{10}$-valued current

$$\mathcal{J} = n^{-1}\mathcal{V}^{-1}\mathcal{P}\mathcal{V}. \tag{33}$$

This is the Noether current associated with the global E_{10} invariance of (31). The transformation

$$\mathcal{V}(t) \to k(t)\mathcal{V}(t)g^{-1} \tag{34}$$

for constant $g \in E_{10}$ (and compensating $k(t) \in K(E_{10})$ to maintain the Borel triangular gauge) induces the transformations

$$\mathcal{P} \to k\mathcal{P}k^{-1}, \qquad \mathcal{Q} \to k\mathcal{Q}k^{-1} + \partial_t k k^{-1}, \tag{35}$$

so that $\mathcal{P}$ transforms covariantly and $\mathcal{Q}$ like a gauge connection. This is as expected since $\mathcal{Q}$ is associated with the unbroken $K(E_{10})$ gauge invariance of the coset $E_{10}/K(E_{10})$. We define a $K(E_{10})$ covariant derivative by

$$\mathcal{D} := \partial_t - \mathcal{Q}, \tag{36}$$

in terms of which the dynamical equation (32) becomes $\mathcal{D}(n^{-1}\mathcal{P}) = 0$.

The set of equations of motion (32) now can be expanded in the basis consisting of the various S_α. We will do this in a form adapted to a level decomposition, where we expand

$$\mathcal{P} = \sum_{\ell \geq 0} P^{(\ell)} * S^{(\ell)}, \qquad \mathcal{Q} = \sum_{\ell \geq 0} Q^{(\ell)} * J^{(\ell)}, \tag{37}$$

where the asterisk indicates a contraction of the various irreducible generators (like $S^{(\ell)}$) occurring on a given level with coefficients (like $P^{(\ell)}$). It is an important result that we can consistently truncate the equations of motion by demanding

$$P^{(\ell)} = 0, \quad \text{for } \ell > \ell_0, \tag{38}$$

where ℓ_0 is some fixed but arbitrary level. The consistency of this truncation was established in Ref. 10.

Although the equations (32) are written in first order form by use of $\mathcal{Q}$ and $\mathcal{P}$, the equations of motion are, of course, second order because the components of $\mathcal{P}$ must be ultimately expressed as first derivatives of coordinates (ϕ_i, A_α) on the coset space $E_{10}/K(E_{10})$[j]; in terms of such an explicit parametrization, the equation of motion then indeed takes the form of the standard geodesic equation. Equation (38) then implies that the higher level coordinates are non-trivial but evolve in just the right manner prescribed by the first order equation (38). When such an explicit parametrization is not required, it is convenient to work with the "tangent space" objects $\mathcal{P}$ rather than the coordinates because one

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[j]In triangular gauge there is no obstruction to working out $\mathcal{P}$ to arbitrarily high orders by use of the Baker–Campbell–Haussdorff equalities.

always deals with Lie algebra valued quantities transforming as tangent vectors under $K(E_{10})$ (the use of tangent space quantities is furthermore indispensable once one introduces fermions). The choice of fields parametrizing the triangular gauge corresponds to a choice of local coordinates on the coset manifold, and as such is subject to a huge variety of coordinate reparametrizations (field redefinitions).

4. Correspondence to Maximal Supergravities

We will now construct correspondences between the abstract E_{10} σ-model defined in the preceding section on the one hand, and the bosonic sectors of the maximal supergravity theories in 10 and 11 space–time dimensions on the other hand. These correspondences consist of a *dictionary* between the coset fields and supergravity fields under which the null geodesic equation (32) become equivalent to the field equations of the bosonic fields of the supergravity theories in a truncation to first spatial gradients only.

It is important that in all cases we make use of the *same E_{10}* invariant model and the correspondences arise through reading that very same model in terms of different subalgebras used for writing the model by means of a level decomposition, see Sec. 2.2. In other words, one and the same E_{10} model gives rise to the different (and suitably truncated) maximal supergravity equations of motion, depending on how one slices the lightcone in the space of E_{10} roots. Therefore the E_{10} σ-model could realize one of the central *desiderata* of M-theory: to explain the known maximal theories (hence, all the maximal supergravity theories) in terms of a single "Ur"-theory.[41,42] We refer to this aspect of the E_{10} model as *versatility*.

In order to establish the correspondences it is necessary to make a few common gauge choices in the supergravity theories. These choices are[k]

- A pseudo-Gaussian gauge for the vielbein where there are no mixed space–time components:

$$E_M{}^A = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & e_m{}^a \end{array} \right).$$

Here, N is the lapse function, and we denote $g^{1/2} = \det(e_m{}^a)$. In the cosmological applications of Sec. 5.1 we will choose the convenient gauge $N = \det(e_m{}^a)$.

---

[k]Our index conventions for gravity in $D = d + 1$ dimensions are such that capital Latin indices run from 0 to d, with 0 corresponding to the time direction, whereas lower case Latin indices run only over the spatial directions 1 to d. Letters from the beginning of the alphabet are flat (tangent space) indices and from the middle of the alphabet are curved (world) indices. We use the "mostly plus" convention for the metric.

- The spatial coefficients of anholonomy

$$\Omega_{ab\,c} = -\Omega_{ba\,c} = 2e_{[a}{}^{m}e_{b]}{}^{n}\partial_m e_{n\,c}$$

are fixed, by the subgroup SO(10) of spatial rotations of the local Lorentz group, to have vanishing trace $\Omega_{ab\,b} = 0$. Following Refs. 10 and 12 we will denote the remaining traceless part as $\tilde{\Omega}_{ab\,c}$. The spin connection $\omega_{a\,bc}$ is defined in terms of $\Omega_{ab\,c}$ by

$$\omega_{a\,bc} = -\omega_{a\,cb} = \frac{1}{2}\left(\Omega_{ab\,c} + \Omega_{ca\,b} - \Omega_{bc\,a}\right),$$

with $\Omega_{ab\,b} = 0$ implying $\omega_{b\,ba} = 0$, and *vice versa*.
- Spatial frame derivatives of the lapse N, of ω and of the gauge invariant field strengths are neglected.
- The coset model time parameter t is chosen such that, near the cosmological singularity, we have $t \to \infty$ as $T \to 0$, where T is the proper time.
- All fermionic terms are set to zero.

The necessity of fixing these gauges on the supergravity side in order for the correspondence to work is well known. The precise nature of these choices will be discussed in detail in Ref. 43 where again the residual gauge freedom will be identified with the invariances of the one-dimensional σ-model of Sec. 3.

Under these assumptions the spatial components of the Ricci tensor (in flat indices) splits as

$$R_{ab} = R_{ab}^{\text{temp}} + R_{ab}^{\text{spat}},$$

where we defined, following Ref. 10,

$$R_{ab}^{\text{temp}} := \partial_0\omega_{a\,b0} + \omega_{c\,c0}\omega_{a\,b0} - 2\omega_{0\,c(a}\omega_{b)\,c0}, \tag{39}$$

$$R_{ab}^{\text{spat}} := \frac{1}{4}\tilde{\Omega}_{cd\,a}\tilde{\Omega}_{cd\,b} - \frac{1}{2}\tilde{\Omega}_{ac\,d}\tilde{\Omega}_{bc\,d} - \frac{1}{2}\tilde{\Omega}_{ac\,d}\tilde{\Omega}_{bd\,c} - \partial_c\tilde{\Omega}_{c(a\,b)}. \tag{40}$$

In standard normalization of the kinetic term for a p-form gauge field $A_{M_1\cdots M_p}$ with $(p+1)$-form field strength $F_{M_1\cdots M_{p+1}} = (p+1)\,\partial_{[M_1}A_{M_2\cdots M_{p+1}]}$ the space–space components of the Einstein equation reads in flat indices[l]

$$R_{ab}^{\text{temp}} = -T_{ab}^{\text{el}} + T_{ab}^{\text{magn}} - R_{ab}^{\text{spat}}, \tag{41}$$

where the stress energy contribution of the p-form was split into "electric" and "magnetic" contributions according to

$$T_{ab}^{\text{el}} = \frac{p}{2p!}F_{0ac_1\cdots c_{p-1}}F_{0bc_1\cdots c_{p-1}} - \frac{p}{2p!(D-1)}\delta_{ab}F_{0c_1\cdots c_p}F_{0c_1\cdots c_p}, \tag{42}$$

$$T_{ab}^{\text{magn}} = \frac{1}{2p!}F_{ac_1\cdots c_p}F_{bc_1\cdots c_p} - \frac{p}{2(p+1)!(D-1)}\delta_{ab}F_{c_1\cdots c_{p+1}}F_{c_1\cdots c_{p+1}}. \tag{43}$$

---

[l]The remaining components are the Hamiltonian and diffeomorphism constraints of the theory. While the Hamiltonian constraint is linked to the constraint of the E_{10} σ-model, there is currently no full understanding of the diffeomorphism constraint in general. Some progress on this issue was reported recently in Ref. 44.

4.1. $D = 11$ *supergravity*

Using the results of Secs. 2.2.1 and 2.3, we first write the coset model equations (32) in A_9 decomposition up to $\ell = 3$. With

$$\mathcal{P} = \frac{1}{2} P_{ab}^{(0)} S^{ab} + \frac{1}{3!} P_{a_1 a_2 a_3}^{(1)} S^{a_1 a_2 a_3} + \frac{1}{6!} P_{a_1 \cdots a_6}^{(2)} S^{a_1 \cdots a_6} + \frac{1}{9!} P_{a_0 | a_1 \cdots a_8}^{(3)} S^{a_0 | a_1 \cdots a_8}$$

(44)

they are[9,10]

$$n \mathcal{D}^{(0)}(n^{-1} P_{ab}^{(0)}) = P_{acd}^{(1)} P_{bcd}^{(1)} - \frac{1}{9} \delta_{ab} P_{c_1 \cdots c_3}^{(1)} P_{c_1 \cdots c_3}^{(1)} + \frac{2}{5!} \left(P_{ac_1 \cdots c_5}^{(2)} P_{bc_1 \cdots c_5}^{(2)} \right.$$

$$- \frac{1}{9} \delta_{ab} P_{c_1 \cdots c_6}^{(2)} P_{c_1 \cdots c_6}^{(2)} \right) + \frac{16}{9!} \left(P_{c_1 | c_2 \cdots c_8 a}^{(3)} P_{c_1 | c_2 \cdots c_8 b}^{(3)} \right.$$

$$+ \frac{1}{8} P_{a | c_1 \cdots c_8}^{(3)} P_{b | c_1 \cdots c_8}^{(3)} - \frac{1}{8} \delta_{ab} P_{c_0 | c_1 \cdots c_8}^{(3)} P_{c_0 | c_1 \cdots c_8}^{(3)} \right), \quad (45)$$

$$n \mathcal{D}^{(0)}(n^{-1} P_{abc}^{(3)}) = \frac{1}{3} P_{abcd_1 \cdots d_3}^{(2)} P_{d_1 \cdots d_3}^{(1)} - \frac{2}{3 \cdot 5!} P_{d_1 | d_2 \cdots d_6 abc}^{(3)} P_{d_1 \cdots d_6}^{(2)}, \quad (46)$$

$$n \mathcal{D}^{(0)}(n^{-1} P_{a_1 \cdots a_6}^{(2)}) = -\frac{1}{3} P_{c_1 | c_2 c_3 a_1 \cdots a_6}^{(3)} P_{c_1 c_2 c_3}^{(1)}, \quad (47)$$

$$n \mathcal{D}^{(0)}(n^{-1} P_{a_0 | a_1 \cdots a_8}^{(3)}) = 0. \quad (48)$$

These equations are $SO(10) = K(GL(10))$ covariant by construction, and $SO(10)$ is the spatial Lorentz group of an 11-dimensional theory. We have put the A_9 level on the components $P^{(\ell)}$ to make the structure more transparent. The derivative operator $\mathcal{D}^{(0)}$ appearing here is only partly covariantized and defined by

$$\mathcal{D}^{(0)}(P^{(\ell)} * S^{(\ell)}) = \partial_t (P^{(\ell)} * S^{(\ell)}) - [Q^{(0)} * J^{(0)}, P^{(\ell)} * S^{(\ell)}]$$

$$+ [P^{(0)} * S^{(0)}, Q^{(\ell)} * J^{(\ell)}]. \quad (49)$$

Maximal supergravity in $d = (1 + 10)$ dimensions[1] has as bosonic fields gravity and a three-form gauge potential $A_{M_1 M_2 M_3}$ with field strength $F_{M_1 \cdots M_4} = 4 \partial_{[M_1} A_{M_2 M_3 M_4]}$. Since in this case $D - 1 = 10 - 1 = 9$ and $p = 3$ we see that the structure of the $\ell = 0$ equation (45) is very similar to the Einstein equation (41) given that we map

$$n \mathcal{D}^{(0)}(n^{-1} P_{ab}^{(0)})(t) \leftrightarrow N^2 R_{ab}^{\text{temp}}(t, \mathbf{x}_0), \quad (50)$$

where we chose a fixed but arbitrary spatial point $\mathbf{x}_0$. The map can be rewritten in terms of maps for $P_{ab}^{(0)}$, $Q_{ab}^{(0)}$ and n which we present together with corresponding

maps for the other fields[m]

$$n(t) \leftrightarrow N g^{-\frac{1}{2}}(t, \mathbf{x}_0),$$

$$Q_{ab}^{(0)}(t) \leftrightarrow -N \omega_{0\,ab}(t, \mathbf{x}_0),$$

$$P_{ab}^{(0)}(t) \leftrightarrow -N \omega_{a\,b0}(t, \mathbf{x}_0),$$

$$P_{abc}^{(1)}(t) \leftrightarrow \frac{1}{2} N F_{0abc}(t, \mathbf{x}_0), \tag{51}$$

$$P_{a_1 \cdots a_6}^{(2)}(t) \leftrightarrow -\frac{N}{2 \cdot 4!} \epsilon_{a_1 \cdots a_6 b_1 \cdots b_4} F_{b_1 \cdots b_4}(t, \mathbf{x}_0),$$

$$P_{a_0|a_1 \cdots a_8}^{(3)}(t) \leftrightarrow \frac{3}{4} N \epsilon_{a_1 \cdots a_8 b_1 b_2} \tilde{\Omega}_{b_1 b_2 a_0}(t, \mathbf{x}_0).$$

Using this correspondence Eq. (45) is mapped to the relevant Einstein equation,[n] Eq. (46) is mapped to the suitably truncated equation of motion for the gauge field,[o] Eq. (47) is the Bianchi identity for the gauge potential, and Eq. (48) expresses the existence of a factorization of the space–time-dependent vielbein into a space-dependent times a time-dependent factor.[10]

The correspondence (51) provides a map between the dynamics of two seemingly different systems: the E_{10} invariant geodesic model (31) (truncated beyond $\ell = 3$) and the bosonic sector of $d = 11$ supergravity with the gauge choices and truncations detailed above.

4.2. *$D = 10$ massive type IIA supergravity*

The analysis for $d = 11$ can be repeated for massive type IIA supergravity in $d = 10$.[45] This was carried out in Ref. 11 in a formalism which uses the $D_9 = \mathfrak{so}(9,9)$ subalgebra of $\mathfrak{e}_{10}$. The spatial Lorentz group is now SO(9) and is the diagonal of $SO(9) \times SO(9) = K(SO(9,9))$. In order to find the right representations under the Lorentz group one has to further decompose the SO(9, 9) representations of Sec. 2.2.2.

The $(SO(9,9) \times GL(1))/(SO(9) \times SO(9))$ coset model which is the restriction of the $E_{10}/K(E_{10})$ model to $\ell = 0$ in the D_9 decomposition can be shown to be equivalent to the reduction of the bosonic sector of $d = 10$ type I supergravity

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[m]Note that the position of the flat indices does not matter any more since they are SO(10) $\subset$ GL(10) indices and we use the Euclidean flat δ_{ab} to raise and lower them. Similarly, the 10 index ϵ-symbol appearing in Eq. (51) is the invariant tensor of SO(10).

[n]One term in (40) does not fully match when plugging in the correspondence for the mixed symmetry generator: although one recovers the first and second term in (40) and the last term vanishes in the truncation, the third term with "crossed index contraction" is not reproduced. In the cosmological billiard picture,[8] this term is subdominant.

[o]Which, in our conventions, and using flat indices reads

$$D_A F^{ABCD} = -\frac{1}{8 \cdot 144} \epsilon^{BCDE_1 \cdots E_4 F_1 \cdots F_4} F_{E_1 \cdots E_4} F_{F_1 \cdots F_4}.$$

by extending arguments of Maharana and Schwarz.[46] As supergravity the bosonic sector of type I is identical to the NSNS sector of type IIA supergravity.

Turning to the $\ell = 1$ contributions in the $E_{10}/K(E_{10})$ σ-model, we first decompose the **256**-component SO(9,9) spinor under the diagonal SO(9) Lorentz group with the result

$$\mathbf{256} \rightarrow \mathbf{16} \otimes \mathbf{16} = \mathbf{9} + \mathbf{84} + \mathbf{126} + \mathbf{36} + \mathbf{1}. \tag{52}$$

These representations can be seen as anti-symmetric tensors of rank p for $p = 1, 3, 5, 7, 9$, which are exactly the RR potentials of massive type IIA supergravity.[47]

The suitably truncated sector of massive type IIA supergravity was rewritten in terms of the underlying SO(9,9) symmetry in Ref. 11 and shown to be dynamically equivalent to the truncated E_{10} model under a correspondence similar to (51). The precise correspondence requires a number of redefinitions which we do not spell out here but instead refer the reader to Ref. 11 for details. Reference 11 also contains a partial treatment of fermions and supersymmetry which were used to derive some of the redefinitions required to make SO(9) × SO(9) manifest. The relation between the bosonic fields of massive IIA supergravity transforming under GL(10) and E_{11} was analyzed earlier in Ref. 27.

4.3. $D = 10$ *type IIB supergravity*

The final example we consider is chiral $d = 10$ IIB supergravity.[48,49] The corresponding level decomposition of $\mathfrak{e}_{10}$ is the $A_8 \oplus A_1$ decomposition discussed in Sec. 2.2.3. The explicit factor of $A_1 \equiv \mathfrak{sl}(2)$ together with tentative spatial Lorentz group $\mathfrak{so}(9) = K(A_8) = K(\mathfrak{sl}(9))$ already hints at a type IIB interpretation. That this is indeed true was shown in Ref. 12. We briefly recall the correspondence and highlight the interesting new features.

The field content of Table 3 is interpreted in terms of IIB quantities. On $\ell = 0$ we find the spatial vielbein coset GL(9)/SO(9) and the axion–dilaton coset SL(2)/SO(2). Level $\ell = 1$ contains the SL(2) doublet of electric field strengths of the two-form potentials which correspond to the F- and the D-string.[50] Level $\ell = 2$ contains an SL(2) singlet electric field strength of the four-form potential. Level $\ell = 3$ carries the SL(2) doublet of magnetic field strengths of the two-form potentials. Finally, level $\ell = 4$ contains the dual graviton mixed symmetry tensor and the magnetic field strengths of the axion–dilaton pair, together with a gauge potential. As shown in Ref. 12 there is again a dynamical match of the truncated IIB equations of motion and the truncated E_{10} σ-model. We refer readers there for details of the correspondence. An important point here concerns the self-duality constraint of the five-form field strength in IIB supergravity. As explained in Ref. 12, the dynamical correspondence is only valid if this self-duality is used on the supergravity side, eliminating all magnetic field strength components in favor of electric ones. The relation between IIB supergravity and E_{11} was analyzed earlier in Ref. 26.

As also discussed in Ref. 12 there is a qualitative difference between E_{11} and E_{10} which manifests itself in the present analysis. The corresponding type IIB
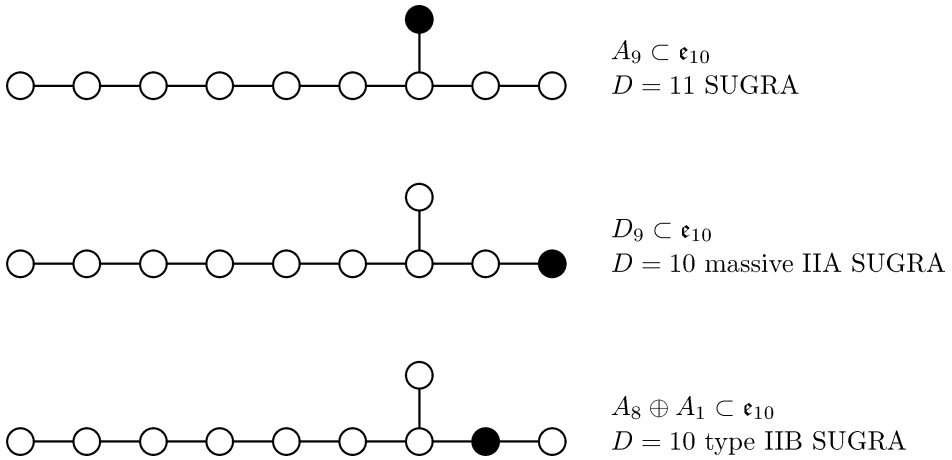
Fig. 3. E_{10} versatility summarized. The filled nodes indicate the deleted nodes in the corresponding level decompositions.

decomposition of E_{11} is under $A_9 = \mathfrak{sl}(10)$ which allows for anti-symmetric 10-form potentials, which in string theory are thought to be the source for nine-branes.[51] The analysis of Refs. 12 and 36 revealed the existence of an SL(2) *quadruplet* and a *doublet* in the E_{11} spectrum under A_9. This somewhat unexpected SL(2) structure was later confirmed to be compatible kinematically with an analysis of the IIB supersymmetry algebra.[38,39]

4.4. *Versatility summary*

The results presented so far can be summarized as follows. Different level decompositions of E_{10} produce the correct spectra of the maximal supergravity theories, see Sec. 2.2. These identifications provide a *dynamical* match between the geodesic $E_{10}/K(E_{10})$ model defined in Sec. 3, and written according to these decompositions and truncated at prescribed levels, and the appropriately truncated bosonic equations of motion of all maximal supergravities. The relevant decompositions are summarized in Fig. 3.

5. Discussion and Outlook

Finally, we briefly discuss a few related topics and open problems in the E_{10} approach.

5.1. *Cosmological solutions*

Given that there is correspondence between two dynamical systems, as detailed in the map (51) for the A_9 decomposition and the D_9 and $A_8 \oplus A_1$ correspondences of Refs. 11 and 12, one can map solutions of one system to solutions of the corresponding other system. Besides the fact that the E_{10} model might be simpler to solve in

subsectors since it is integrable (as explained in Sec. 3), constructing solutions of the E_{10} model has the additional advantage that one can map *one* solution of the E_{10} model to *three different* solutions of the maximal supergravity theories due to the versatility of the model.

The simplest way to obtain solutions to the E_{10} σ-model is by restricting to a submanifold which is a coset of a finite-dimensional subgroup of E_{10}. On such spaces the geodesic motion is known to be integrable of Toda type (see for example Refs. 52 and 53 for overviews). The simplest example is $\mathrm{SL}(2)/\mathrm{SO}(2)$ where the problem reduces to solving the one-dimensional Liouville equation. In the E_{10} context this was studied in Ref. 54. Using the $\mathrm{SL}(2)$ generated by the $\mathfrak{e}_{10}$ generators e_{10}, h_{10}, f_{10}, and one additional (orthogonal) CSA element to satisfy the Hamiltonian constraint of (32), the following solution to the bosonic sector of $d = 11$ supergravity was found in Ref. 54

$$ds^2 = -e^{14\tilde{\phi}-2\phi/3}dt^2 + e^{4\phi/3}(dx^2 + dy^2 + dz^2) + e^{2\tilde{\phi}-2\phi/3}(dw_1^2 + \cdots + dw_7^2),$$

$$F_{txyz} = \frac{E}{a\cosh^2(\sqrt{E}t)} = ae^{4\phi}, \tag{53}$$

with

$$\phi(t) = -\frac{1}{2}\ln\left[\frac{a}{\sqrt{E}}\cosh(\sqrt{E}(t-t_0))\right],$$

$$\tilde{\phi}(t) = \frac{1}{42}\sqrt{\tilde{E}}(t-t_1). \tag{54}$$

Here, x, y, z correspond to directions $8, 9, 10$, and $w_1, \ldots, w_7$ are seven transverse directions. The "energies" E and $\tilde{E}$ are related by $\tilde{E} = 21E$ as follows from the Hamiltonian constraint of the E_{10} model. Furthermore, the gauge $n = 1$ was chosen. This solution is known to have phases of accelerated and decelerated expansion,[55,56] and is identical in form to the SM2-brane solution, see Ref. 57 and references therein. Similar BPS solutions and their properties in the context of E_{11} were discussed earlier in Refs. 58–63. References to papers containing cosmological solutions to supergravity can be found in Ref. 54.

5.2. E_{10}, fermions and generalized holonomy

Recently progress was made on the important question of how to bring fermions into the picture.[64,65] This amounts to adding spin to the massless particle moving on a geodesic trajectory in the $E_{10}/K(E_{10})$ coset space. In analogy with what happens for smaller hidden symmetries E_n ($n \leq 8$) one expects the fermions to transform under the denominator group $K(E_n)$. In the case of E_{10} this poses an algebraic problem since $K(\mathfrak{e}_{10})$ is *not* a Kac–Moody algebra as mentioned in Sec. 2.1. In fact, hardly anything is known about the representation theory of this infinite-dimensional algebra.

The idea of Damour *et al.*[64] was to bring the fermionic equation of motion into play *and* fix a supersymmetric gauge since for the bosonic correspondences one also needs to fix all gauges. The gauge chosen was $\psi_0 - \Gamma_0\Gamma^a\psi_a = 0$ and to write the remaining **320** equations of motion for the $d = 11$ gravitino as[p] $\mathcal{E}_a = 0 \leftrightarrow \mathcal{D}\Psi = 0$, where Ψ is a $K(E_{10})$ spinor on which the $K(E_{10})$-covariant derivative acts. The spinor Ψ is thought to be infinite-dimensional components and, if decomposed under the spatial Lorentz SO(10), to contain the gravitino ψ_a. The analysis of Damour *et al.*[63] showed that by evaluating the equations of motion and using the dictionary (51) one can deduce the action of the $K(E_{10})$ generators up to "level"[q] $\ell = 3$ and show consistency with the $K(E_{10})$ commutation relations. In fact, it can be proven that the **320** components of the supergravity fermion furnish a representation of $K(\mathfrak{e}_{10})$ by themselves. This representation is necessarily *unfaithful*.

Repeating the same analysis for Dirac fermions shows that the **32** representation of SO(10) is also an unfaithful representation of $K(\mathfrak{e}_{10})$.[r] The action on the Dirac spinor is in terms of anti-symmetric 32×32 matrices, hence fundamental SO(32)-matrices. This is somewhat along the lines of proposals for a generalized holonomy of M-theory[67–69] but with a very important difference. The Dirac spinor representation of SO(1, 10) is *not* turned into a representation of SO(32) but into an unfaithful representation of $K(E_{10})$. This way one circumvents global problems with SO(32) as a symmetry group rather than a generalized holonomy pointed out in Ref. 70.[s] Similar remarks apply to SL(32) which was discovered in the M5-brane equations[71] and discussed in the context of E_{11} in Ref. 19, as has been shown in Ref. 72.

5.3. *Open problems*

Despite all the encouraging results presented in this contribution, there remain a number of very important open problems with the E_{10} model.

- The most pressing question is probably the following: *What role do the higher levels play?* When establishing the correspondences to the supergravity theories in Sec. 4, we truncated the σ-model equations of motion after a fixed level. However, there are infinitely many higher levels whose contributions to the dynamics can be determined in principle but whose physical interpretation is not clear. The original paper[9] made the conjecture that their effect could be to re-introduce the full space dependence from supergravity and thereby turn space–time into an emergent concept. This so-called *gradient conjecture* is based on the observation that in the infinite list of representations in the A_9 level decomposition there exist the following representations for $k \geq 0$.

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[p]The index a runs over the spatial directions and hence takes 10 values, the 32 component spinor index has been suppressed.
[q]Level here is not meant as a grading of $K(\mathfrak{e}_{10})$ which does not exist.
[r]See also Ref. 66 for low level results on Dirac fermions and E_{10}.
[s]The gravitino representation could never have been formed an SO(32) representation.

ℓ	A_9 Dynkin	Generator	Interpretation
$3k+1$	$[k00000100]$	$E_{a_1 \cdots a_k}{}^{b_1 b_2 b_3}$	$\partial_{a_1} \cdots \partial_{a_k} F_{t b_1 b_2 b_3}$
$3k+2$	$[k00100000]$	$E_{a_1 \cdots a_k}{}^{b_1 \cdots b_6}$	$\partial_{a_1} \cdots \partial_{a_k} \epsilon_{b_1 \cdots b_6 c_1 \cdots c_4} F_{c_1 \cdots c_4}$
$3k+3$	$[k10000001]$	$E_{a_1 \cdots a_k}{}^{b_0 | b_1 \cdots b_8}$	$\partial_{a_1} \cdots \partial_{a_k} \epsilon_{b_1 \cdots b_8 c_1 c_2} \tilde{\Omega}_{c_1 c_2 b_0}$

The first Dynkin label entry k translates into k symmetric sets of nine anti-symmetric indices which each have been lowered using the invariant $\mathfrak{sl}(10)$ ϵ-symbol.[t] The generators are therefore symmetric in the lower indices, which makes the suggested interpretation possible at least in principle.[u] Of course, the term "Taylor expansion" here must be interpreted *cum grano salis*, since it is very well known from previous results on the cascades of dual potentials appearing for the affine Geroch group (see e.g. Ref. 73) that these higher order spatial gradients involve *non-local* relations between the fields in the coset model, see also Ref. 31. There is an infinity of additional representations besides the gradient representations and these have been conjectured to be associated partly with new M-theoretic degrees of freedom.

- There are very few results concerning involutory subgroups of infinite dimensional Kac–Moody groups, an $K(E_{10})$ in particular. For instance, is it possible to construct *faithful* spinor representations of $K(E_{10})$ which could also accommodate the spatial dependence of the fermionic fields? This hinges largely on a better understanding of the relevant representation theory.[44]
- Can one construct similar correspondences for other Kac–Moody cosets and other theories? This seems likely and has already been done for the hyperbolic extension of G_2 in Ref. 74 and for pure type I supergravity in Ref. 75.

Acknowledgments

References

1. E. Cremmer, B. Julia and J. Scherk, Supergravity theory in 11 dimensions, *Phys. Lett. B* **76**, 409–412 (1978).
2. E. Cremmer and B. Julia, The $SO(8)$ supergravity, *Nucl. Phys. B* **159**, 141 (1979).

[t]As $\mathfrak{gl}(10)$ representations this requires a compensating term in the transformation rules.[31]
[u]There is an additional tracelessness constraint on the generators following from the Young symmetries of the $\mathfrak{sl}(10)$ representations.

3. B. Julia, Application of supergravity to gravitation theories, in *Unified Field Theories of More Than 4 Dimensions*, eds. V. De Sabbata and E. Schmutzer (World Scientific, Singapore, 1983).

4. B. Julia, in Lectures in Applied Mathematics, Vol. 21 (1985), AMS-SIAM, p. 335 [preprint LPTENS 80/16].

5. H. Nicolai, The integrability of $N = 16$ supergravity, *Phys. Lett. B* **194**, 402 (1987).

6. S. Mizoguchi, E_{10} symmetry in one-dimensional supergravity, *Nucl. Phys. B* **528**, 238–264 (1998) [hep-th/9703160].

7. T. Damour and M. Henneaux, E_{10}, BE_{10} and arithmetical chaos in superstring cosmology, *Phys. Rev. Lett.* **86**, 4749 (2001) [hep-th/0012172].

8. T. Damour, M. Henneaux and H. Nicolai, Cosmological billiards, *Class. Quant. Grav.* **20**, R145–R200 (2003) [hep-th/0212256].

9. T. Damour, M. Henneaux and H. Nicolai, E_{10} and a "small tension expansion" of M-theory, *Phys. Rev. Lett.* **89**, 221601 (2002) [hep-th/0207267].

10. T. Damour and H. Nicolai, Eleven dimensional supergravity and the $E_{10}/K(E_{10})$ σ-model at low A_9 levels, in *Group Theoretical Methods in Physics*, Institute of Physics Conference Series No. 185 (IoP Publishing, 2005) [hep-th/0410245].

11. A. Kleinschmidt and H. Nicolai, E_{10} and SO(9, 9) invariant supergravity, *J. High Energy Phys.* **0407**, 041 (2004) [hep-th/0407101].

12. A. Kleinschmidt and H. Nicolai, IIB supergravity and E_{10}, *Phys. Lett. B* **606**, 391 (2005) [hep-th/0411225].

13. T. Damour and H. Nicolai, Higher order M-theory corrections and the Kac–Moody algebra E_{10}, *Class. Quant. Grav.* **22**, 2849–2879 (2005) hep-th/0504153.

14. J. Brown, O. J. Ganor and C. Helfgott, M-theory and E_{10}: billiards, branes and imaginary roots, hep-th/0401053.

15. J. Brown, S. Ganguli, O. J. Ganor and C. Helfgott, E_{10} orbifolds, *J. High Energy Phys.* **0506**, 057 (2005) [hep-th/0409037].

16. P. C. West, Hidden superconformal symmetry in M-theory, *J. High Energy Phys.* **0008**, 007 (2000) [hep-th/0005270].

17. P. C. West, E_{11} and M-theory, *Class. Quant. Grav.* **18**, 4443–4460 (2001) [hep-th/0104081].

18. A. B. Borisov and V. I. Ogievetsky, Theory of dynamical affine and conformal symmetries as gravity theory, *Theor. Math. Phys.* **23**, 1179–1188 (1976).

19. P. West, E_{11}, $SL(32)$ and central charges, *Phys. Lett. B* **575**, 333–342 (2003) [hep-th/0307098].

20. F. Englert and L. Houart, $\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: exact BPS solutions, *J. High Energy Phys.* **0401**, 002 (2004) [hep-th/0311255].

21. A. Kleinschmidt and P. West, Representations of $\mathcal{G}^{+++}$ and the role of space–time, *J. High Energy Phys.* **0402**, 033 (2004) [hep-th/0312247].

22. A. Keurentjes, E_{11}: sign of the times, hep-th/0402090.

23. S. de Buyl, L. Houart and N. Tabti, Dualities and signatures of $\mathcal{G}^{++}$ invariant theories, *J. High Energy Phys.* **0506**, 084 (2005) [hep-th/0505199].

24. C. M. Hull, Duality and the signature of space–time, *J. High Energy Phys.* **9811**, 017 (1998) [hep-th/9807127].

25. F. Englert, M. Henneaux and L. Houart, From very-extended to overextended gravity and M-theories, *J. High Energy Phys.* **0502**, 070 (2005) [hep-th/0412184].

26. I. Schnakenburg and P. C. West, Kac–Moody symmetries of IIB supergravity, *Phys. Lett. B* **517**, 421–428 (2001) [hep-th/0107081].

27. I. Schnakenburg and P. C. West, Massive IIA supergravity as a nonlinear realization, *Phys. Lett. B* **540**, 137–145 (2002) [hep-th/0204207].

28. K. Bautier, S. Deser, M. Henneaux and D. Seminara, *Phys. Lett. B* **406**, 49–53 (1997) [hep-th/9704131].
29. V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd edn. (Cambridge University Press, 1990).
30. R. V. Moody and A. Pianzola, *Lie Algebras with Triangular Decomposition* (John Wiley & Sons, New York, 1995).
31. A. Kleinschmidt and H. Nicolai, Gradient representations and affine structures in AE_n, *Class. Quant. Grav.* **22**, 4457 (2005) [hep-th/0506238].
32. H. Nicolai and T. Fischbacher, Low level representations of E_{10} and E_{11}, in *Proceedings of the Ramanujan International Symposium on Kac–Moody Algebras and Applications, ISKMAA-2002*, Chennai, India, 28–31 January [hep-th/0301017].
33. A. Kleinschmidt, E_{11} as E_{10} representation at low levels, *Nucl. Phys. B* **677**, 553–586 (2004) [hep-th/0304246].
34. M. R. Gaberdiel, D. I. Olive and P. West, A class of Lorentzian Kac–Moody algebras, *Nucl. Phys. B* **645**, 403–437 (2002) [hep-th/0205068].
35. P. West, Very extended E_8 and A_8 at low levels, gravity and supergravity, *Class. Quant. Grav.* **20**, 2393–2406 (2003) [hep-th/0212291].
36. A. Kleinschmidt, I. Schnakenburg and P. West, Very-extended Kac–Moody algebras and their interpretation at low levels, *Class. Quant. Grav.* **21**, 2493 (2004) [hep-th/0309198].
37. P. West, E_{11}, ten forms and supergravity, hep-th/0511153.
38. E. A. Bergshoeff, M. de Roo, S. F. Kerstan and F. Riccioni, IIB supergravity revisited, *J. High Energy Phys.* **0508**, 098 (2005) [hep-th/0506013].
39. E. A. Bergshoeff, M. de Roo, S. F. Kerstan, T. Ortin and F. Riccioni, IIB nine-branes, hep-th/0601128.
40. T. Fischbacher, The structure of E_{10} at higher A_9 levels: a first algorithmic approach, *J. High Energy Phys.* **0508**, 012 (2005) [hep-th/0504230].
41. C. M. Hull and P. K. Townsend, Unity of superstring dualities, *Nucl. Phys. B* **438**, 109–137 (1995) [hep-th/9410167].
42. E. Witten, String theory dynamics in various dimensions, *Nucl. Phys. B* **443**, 85–126 (1995) [hep-th/9503124].
43. C. Hillman, A. Kleinschmidt and H. Nicolai, A note on gauge fixing in supergravity/ Kac–Moody correspondences, hep-th/0603156.
44. T. Damour, A. Kleinschmidt and H. Nicolai, $K(E_{10})$, Supergravity and Fermions, *J. High Energy Phys.* **0608**, 046 (2006) [hep-th/0606105].
45. L. J. Romans, Massive $N = 2a$ supergravity in ten dimensions, *Phys. Lett. B* **169**, 374–380 (1986).
46. J. Maharana and J. H. Schwarz, Noncompact symmetries in string theory, *Nucl. Phys. B* **390**, 3–32 (1993) [hep-th/9207016].
47. E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, Duality of type II 7-branes and 8-branes, *Nucl. Phys. B* **470**, 113 (1996) [hep-th/9601150].
48. J. H. Schwarz and P. C. West, Symmetries and transformations of chiral $N = 2$, $D = 10$ supergravity, *Phys. Lett. B* **126**, 301 (1983).
49. J. H. Schwarz, Covariant field equations of chiral $N = 2$, $D = 10$ supergravity, *Nucl. Phys. B* **226**, 269 (1983).
50. J. H. Schwarz, An $SL(2, \mathbb{Z})$ multiplet of type IIB superstrings, *Phys. Lett. B* **360**, 13 (1995) [Erratum-ibid. **B364**, 252 (1995)] [hep-th/9508143].
51. J. Polchinski, Dirichlet-branes and Ramond-Ramond charges, *Phys. Rev. Lett.* **75**, 4724 (1995) [hep-th/9510017].

52. A. V. Razumov and M. V. Saveliev, *Lie Algebras, Geometry and Toda-Type Systems*, Cambridge Lecture Notes in Physics (Cambridge University Press, 1997).
53. O. Babelon, D. Bernard and M. Talon, *Introduction to Classical Integrable Systems* (Cambridge University Press, 2003).
54. A. Kleinschmidt and H. Nicolai, E_{10} cosmology, *J. High Energy Phys.* **0601**, 137 (2006) [hep-th/0511290].
55. N. Ohta, Accelerating cosmologies from S-branes, *Phys. Rev. Lett.* **91**, 061303 (2003) [hep-th/0303238].
56. M. N. R. Wohlfarth, Accelerating cosmologies and a phase transition in M-theory, *Phys. Lett. B* **563**, 1 (2003) [hep-th/0304089].
57. N. Ohta, Accelerating cosmologies and inflation from M/superstring theories, *Int. J. Mod. Phys. A* **20** (2005) [hep-th/0411230].
58. F. Englert, L. Houart, A. Taormina and P. C. West, The symmetry of M-theories, *J. High Energy Phys.* **0309**, 020 (2003) [hep-th/0304206].
59. F. Englert, L. Houart and P. C. West, Intersection rules, dynamics and symmetries, *J. High Energy Phys.* **0308**, 025 (2003) [hep-th/0307024].
60. P. C. West, The IIA, IIB and eleven-dimensional theory and their common E_{11} origin, hep-th/0402140.
61. F. Englert and L. Houart, $\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: exact intersecting brane solutions, *J. High Energy Phys.* **0405**, 059 (2004) [hep-th/0405082].
62. P. P. Cook and P. C. West, $\mathcal{G}^{+++}$ and brane solutions, hep-th/0405149.
63. L. Houart, Kac–Moody algebras in gravity and M-theories, hep-th/0511009.
64. T. Damour, A. Kleinschmidt and H. Nicolai, Hidden symmetries and the fermionic sector of eleven-dimensional supergravity, hep-th/0512163.
65. S. de Buyl, M. Henneaux and L. Paulot, Extended E_8 invariance of 11-dimensional supergravity, hep-th/0512292.
66. S. de Buyl, M. Henneaux and L. Paulot, Hidden symmetries and Dirac fermions, *Class. Quant. Grav.* **22**, 3595 (2005) [hep-th/0506009].
67. M. J. Duff and K. S. Stelle, Multimembrane solutions of $D = 11$ supergravity, *Phys. Lett. B* **253**, 113–118 (1991).
68. M. J. Duff and J. T. Liu, Hidden space–time symmetries and generalized holonomy in M-theory, *Nucl. Phys. B* **674**, 217–230 (2003) [hep-th/0303140].
69. C. Hull, Holonomy and symmetry in M-theory, hep-th/0305039.
70. A. Keurentjes, The topology of U Duality (sub)groups, *Class. Quant. Grav.* **21**, 1695–1708 (2004) [hep-th/0309106].
71. O. Bärwald and P. C. West, Brane rotating symmetries and the five-brane equations of motion, *Phys. Lett. B* **476**, 157–164 (2000) [hep-th/9912226].
72. A. Kleinschmidt and H. Nicolai, IIA and IIB spinors from $K(E_{10})$, *Phys. Lett. B* **637**, 107–112 (2006) [hep-th/0603205].
73. P. Breitenlohner and D. Maison, On the Geroch group, *Ann. Poincaré Phys. Theor.* **46**, 215–246 (1987).
74. S. Mizoguchi, K. Mohri and Y. Yamada, Five-dimensional supergravity and hyperbolic Kac–Moody algebra G_2^H, hep-th/0512092.
75. C. Hillmann and A. Kleinschmidt, Pure type I supergravity and DE_{10}, hep-th/0608092.