# Holographic currents in first order Gravity and finite Fefferman-Graham expansions

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ABSTRACT: We study the holographic currents associated to Chern-Simons theories. We start with an example in three dimensions and find the holographic representations of vector and chiral currents reproducing the correct expression for the chiral anomaly. In five dimensions, Chern-Simons theory for AdS group describes first order gravity and we show that there exists a gauge fixing leading to a finite Fefferman-Graham expansion. We derive the corresponding holographic currents, namely, the stress tensor and spin current which couple to the metric and torsional degrees of freedom at the boundary, respectively. We obtain the correct Ward identities for these currents by looking at the bulk constraint equations.

# Contents

## 1. Introduction

The AdS/CFT correspondence [1–3] has uncovered a deep and still somewhat mysterious relationship between fields propagating in $(d + 1)$-dimensional anti-de Sitter (AdS) space and correlators in a $d$-dimensional Conformal Field Theory (CFT). Scalars, vectors, spinors and tensor fields on AdS have been studied (see [4, 5] for reviews) and the expected results are obtained in all cases.

One of the most studied examples of this correspondence has been the gravitational field. This bulk field is dual to the CFT's energy momentum tensor, and solving the bulk Einstein equations in the presence of a cosmological term allows the computation of energy momentum tensor correlators. The calculation of the holographic anomalies performed in [6] has provided strong support for the validity of the correspondence.

In this paper we shall analyze the holographic structure of two unrelated, although similar, systems. In section 2, we study Abelian and non-Abelian Chern-Simons theories in three dimensions and work out the 1-point functions and anomalies associated to gauge symmetries of a dual theory on the boundary. The aim of section 2 is to put a simple Chern-Simons system in the AdS/CFT language and use this example as a warm-up exercise to deal with the more complicated case of five-dimensional first order Chern-Simons gravity, discussed in Sec. 3. Five dimensional Chern-Simons gravity has more degrees of freedom than standard gravity due to the fact that the spin connection is dynamical [7]. We carry out the holographic AdS/CFT prescription associated to this system and extract the vevs and corresponding Ward identities associated to the metric and spin connection degrees of freedom.

The systems discussed in sections 2 and 3 both have finite Fefferman-Graham (FG) [8] expansions. The first example of a finite FG expansions appeared in [9] in the context of three-dimensional gravity (see also [10, 11] for a Chern-Simons formulation). Incidentally, in three dimensions, the connection between anti-de Sitter space and the full 2d conformal group has been known for a long time [12]. This correspondence was later reformulated in terms of the Chern-Simons formulation [13, 14] of gravity, and associated WZW theories [15 – 17]. For a recent complete review of these issues and their applications, see [18].

## 2. Chiral anomaly in two dimensions

Before going into the more complicated case of first order gravity in five dimensions, let us review here the holographic description of the chiral anomaly in two dimensions, which is perhaps the simplest application of the AdS/CFT ideas. It also provides a simple example in which the FG expansion is finite. Most of the material of this section is well-known in different contexts.

In even dimensions, gauge fields can be coupled to a vector current or axial current. It is well known that only one of these gauge symmetries can be preserved at the quantum level. In the particular case of two dimensions, the vector $(J^i)$ and axial $(J_5^i)$ currents are related by

$$J_5^i = \varepsilon^{ij} J_j \,. \tag{2.1}$$

In the presence of a non-zero U(1) gauge field $A_i$, it follows that

$$\langle \partial_i J^i \rangle = 0 \,, \qquad \langle \partial_i J_5^i \rangle = F \,, \tag{2.2}$$

where $F$ is the field strength associated to $A$.

Chern-Simons theory in three dimensions provides a natural arena to describe (2.2) holographically. Since the AdS/CFT dual of a conserved current is a gauge field, one may naively think that a single Abelian Chern-Simons theory may be the right description leading to (2.2). This is not the case, and one needs two Abelian fields. We first explain why a single Chern-Simons theory is not enough to represent the chiral anomaly.

Consider an Abelian Chern-Simons action $I[B] = \kappa \int_{M_3} BdB$. (In the whole text we omit the wedge product symbol.) This action is not gauge invariant for parameters with non-zero support at the boundary $\partial M_3 = M_2$. In fact, under $\delta B = d\lambda$, the action changes as $\delta I = 2\kappa \int_{M_2} \lambda F$ with $F = dB$. Naively, one may conclude that this simple calculation represents the holographic version of the chiral anomaly, in analogy with the five-dimensional case discussed in [3]. This is however not correct because there is a mismatch between the number of sources.

According to AdS/CFT, the bulk fields boundary data (independent fields at the boundary) plays the role of sources in the CFT. For a single Chern-Simons theory, the boundary data is parameterized by only one component of the gauge field. In fact, let us choose local coordinates on $M_3$ as $x^\mu = (\rho, x^i)$, where the light-cone coordinates $x^i = (x^+, x^-)$ parameterize the boundary $M_2$ placed at $\rho = 0$. The gauge field has components $B_\mu = (B_\rho, B_+, B_-)$. In the gauge $B_\rho = 0$, for example, the fields $B_+, B_-$ become $\rho$-independent. The remaining equation of motion $F_{+-} = 0$ allows to solve $B_-$ as a function

of $B_+$. The field $B_+$ is thus the only "source". On the other hand, for a fermion coupled to a gauge field, the source is a vector $A_i$ with two components, thus it is clear that we need to double the number of degrees of freedom.[1]

## 2.1 Abelian chiral anomaly

Consider then two Abelian Chern-Simons actions with fields $B$ and $\bar{B}$,

$$I\left[B, \bar{B}\right] = \kappa \int B dB - \kappa \int \bar{B} d\bar{B} + \Omega\left[B, \bar{B}\right], \tag{2.3}$$

where $\Omega$ is a boundary term to be fixed. The boundary data of this action contains two fields, as desired.

The bulk local symmetries of this action are $U(1) \times U(1)$ gauge transformations acting independently on each field. Of course, these symmetries are broken at the boundary. The interesting point is that the boundary term $\Omega$ can be chosen such that *half* of the gauge symmetries of (2.3) are extended to the boundary. The unbroken part of the symmetries will be related to the vector current, and the broken half represents the axial current. Let us see how this is implemented in the language of AdS/CFT.

The crucial point is the choice of the boundary term $\Omega$. We choose

$$\Omega\left[B, \bar{B}\right] = 2\kappa \int\limits_{M_2} d^2x \left(B_+ B_- + \bar{B}_- \bar{B}_+ - 2B_+ \bar{B}_-\right). \tag{2.4}$$

The action $I$ in (2.3), with this choice of boundary term, has the following properties.

First, consider the variation of (2.3) under gauge transformations $\delta B = d\lambda$ and $\delta \bar{B} = d\bar{\lambda}$. One obtains

$$\delta I = 4\kappa \int\limits_{M_2} \left(\lambda - \bar{\lambda}\right)\left(\partial_+ \bar{B}_- - \partial_- B_+\right). \tag{2.5}$$

The action becomes strictly invariant — even at the boundary — under the diagonal subgroup with $\bar{\lambda} = \lambda$. Redefining $\lambda + \bar{\lambda} = \sigma_+$ and $\lambda - \bar{\lambda} = \sigma_-$, we conclude that half of the gauge symmetries, $\sigma_+$, of the action (2.3) are preserved at the boundary, as desired.

Second, under generic variations of the fields, the action (2.3) varies as

$$\delta I = 4\kappa \int\limits_{M_2} d^2x \left[\delta B_+ \left(B_- - \bar{B}_-\right) + \delta \bar{B}_- \left(\bar{B}_+ - B_+\right)\right] + \text{e.o.m.}, \tag{2.6}$$

where e.o.m. represents terms proportional to the equations of motion $dB = 0$ and $d\bar{B} = 0$. This shows that the bulk action has an extremum, provided $B_+$ and $\bar{B}_-$ are fixed at the boundary. The boundary fields $(B_+, \bar{B}_-)$ are thus the two "sources" in this theory. We

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[1]In fact, the $U(1)$ Chern-Simons action is dual to a single chiral fermion on the boundary and it reproduces the chiral anomaly. However, in this case the boundary theory does not have (preserved) symmetries and we are not interested in these cases here.

identify them with the 2d gauge field[2]

$$(A_+, A_-) = (B_+, \bar{B}_-) .\tag{2.7}$$

Note now that the variation of the action under the broken symmetry, eq. (2.5), becomes proportional to the gauge field curvature since $\partial_+\bar{B}_- - \partial_- B_+ = \partial_+ A_- - \partial_- A_+ = F$. This is consistent with the chiral anomaly.

Third, we can now write the on-shell action as a function of the sources, and compute vacuum expectation values of the currents. We can compute directly the vector current expectation value

$$\begin{aligned}
\langle J^i \rangle &= \frac{\delta I}{\delta A_i} \\
&= \left( \frac{\delta I}{\delta B_+}, \frac{\delta I}{\delta \bar{B}_-} \right) \\
&= 4\kappa \left( B_- - \bar{B}_-, \bar{B}_+ - B_+ \right) ,
\end{aligned}\tag{2.8}$$

where $I$ is the on-shell value of the action. Here, the first line is the AdS/CFT definition of the expectation value for the current. The second line follows from (2.7), and the last line from (2.6). $B_-$ and $\bar{B}_+$ are not independent of the sources $B_+$ and $\bar{B}_-$. From the Chern-Simons bulk equations of motion, $dB = 0$ and $d\bar{B} = 0$, one finds the non-local relations

$$B_- = \frac{\partial_-}{\partial_+} B_+ , \qquad \bar{B}_+ = \frac{\partial_+}{\partial_-} \bar{B}_- .\tag{2.9}$$

The final expression for the expectation value of the vector current in the presence of the source $A_i$ is

$$\langle J^i \rangle = 4\kappa \left( \frac{\partial_-}{\partial_+} A_+ - A_- , \frac{\partial_+}{\partial_-} A_- - A_+ \right) .\tag{2.10}$$

The corresponding formula for the axial current follows from (2.1),

$$\langle J_5^i \rangle = 4\kappa \left( \frac{\partial_-}{\partial_+} A_+ - A_- , -\frac{\partial_+}{\partial_-} A_- + A_+ \right) .\tag{2.11}$$

We can finally check explicitly the two relations

$$\partial_i \langle J^i \rangle = 0 ,\tag{2.12}$$

$$\partial_i \langle J_5^i \rangle = -8\kappa \left( \partial_+ A_- - \partial_- A_+ \right) = -4\kappa \, \varepsilon^{ij} F_{ij} ,\tag{2.13}$$

in full consistency with the Dirac fermion expectation values. Note that the Chern-Simons coupling then becomes related to the electric charge as $q/2\pi = 4\kappa$.

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[2]A similar phenomenon occurs in three-dimensional gravity and the corresponding AdS/CFT interpretation. In the Chern-Simons formulation, there are two $SO(2,1)$ gauge fields $A$ and $\bar{A}$. The induced 2-dimensional vielbein in the FG expansion is constructed with one leg in $A$ and the other in $\bar{A}$. For more details in this construction, see [10, 11].

## 2.2 Non-abelian anomaly

The above result can be generalized to a non-Abelian CS theory invariant under $G \times G$, with $G$ semi-simple for simplicity. Each of two sets of generators $G_a$ and $\bar{G}_a$ forms a Lie algebra with structure constants $f_{ab}{}^c$. The two isomorphic algebras commute with each other. The action with the fundamental fields $B = B^a G_a$, $\bar{B} = \bar{B}^a \bar{G}_a$ and the Cartan metric $g_{ab} = \langle G_a G_b \rangle$ (that raises and lowers group indices), is

$$I\left[B, \bar{B}\right] = I_{CS}\left[B\right] - I_{CS}\left[\bar{B}\right] + 2\kappa \int_{M_2} d^2 x \left(B_+^a B_{-a} + \bar{B}_+^a \bar{B}_{-a} - 2B_+^a \bar{B}_{-a}\right), \qquad (2.14)$$

where the CS action is

$$I_{CS}\left[B\right] = \kappa \int_{M_3} \left(B^a F_a - \frac{2}{3} f_{abc} B^a B^b B^c\right), \qquad (2.15)$$

and the boundary term is a covariant generalization of (2.4) [19]. The asymptotic conditions are

$$B_+^a\big|_{\rho=0} = A_+^a, \qquad \bar{B}_-^a\big|_{\rho=0} = A_-^a, \qquad (2.16)$$

and $A^a = B_+^a \, dx^+ + \bar{B}_-^a \, dx^-$ is a fixed vector field on $M_2$. The equations of motion

$$F^a \equiv dB^a + \frac{1}{2} f_{bc}{}^a B^b B^c = 0, \qquad \bar{F}^a \equiv d\bar{B}^a + \frac{1}{2} f_{bc}{}^a \bar{B}^b \bar{B}^c = 0, \qquad (2.17)$$

give $B_-^a$ and $\bar{B}_+^a$ as non-local functions of $A^a$. Similarly to the Abelian case, the action (2.14) is asymptotically invariant under the diagonal subgroup $G_D \subset G \times G$ generated by $G_a + \bar{G}_a$, while the other half of symmetries, generated by $G_a - \bar{G}_a$, are broken on the boundary. This asymptotic structure of the gauge theory maps holographically to the quantum structure of a field theory at the boundary, leading to one preserved quantum current and another current possessing a non-Abelian anomaly. Varying the action (2.14) on-shell, we find these currents as

$$\left\langle J_a^i \right\rangle = \frac{\delta I[A]}{\delta A_i^a} = 2\kappa \, g_{ab} \, \varepsilon^{ij} \left(B_j^b - \bar{B}_j^b\right), \qquad (2.18)$$

where $I$ is evaluated on-shell and $\bar{J}_a^i = \varepsilon^{ij} J_{ia}$. Using the field equations (2.17), the covariant derivatives of these currents are

$$D_i^{(0)} \left\langle J_a^i \right\rangle = 0, \qquad (2.19)$$

$$D_i^{(0)} \left\langle \bar{J}_a^i \right\rangle = -4\kappa \, g_{ab} \, \varepsilon^{ij} F_{ij}^{(0)b}, \qquad (2.20)$$

where here $D^{(0)}$ and $F^{(0)a} = dA^a + \frac{1}{2} f_{bc}{}^a A^b A^c$ are associated to the boundary field $A$. We conclude that $J_a^i$ is covariantly preserved, while $\bar{J}_a^i$ exhibits a covariant non-Abelian anomaly [20].

To end, note that the choice of the boundary term in (2.14) can be understood from the point of view of asymptotic symmetries. It was shown in [19] that, for a three-dimensional CS theory $I\left[B, \bar{B}\right] = \alpha \, I_{CS}\left[B\right] + \beta \, I_{CS}\left[\bar{B}\right] + \Omega\left[B, \bar{B}\right]$, the surface integral in (2.14)

is the unique boundary term (up to the chirality $x^+ \leftrightarrow x^-$) quadratic in gauge fields which preserves the maximal number of asymptotic symmetries $G_D$, and it exists only for $\alpha + \beta = 0$. In consequence, the asymptotic symmetry of three-dimensional gravity with torsion (which implies $\alpha + \beta \neq 0$) and gauge group $SO(1,2) \times SO(1,2)$ [21] is always smaller than $SO(1,2)$. Therefore, in this case the Lorentz symmetry cannot be preserved asymptotically, resulting in a Lorentz anomaly in the dual quantum theory.

## 3. First order gravitational theories

The motivation for this section is twofold. On the one hand, we are interested in studying the holographic renormalization method for a gravitational theory (Chern-Simons gravity in five dimensions [22]) which is quadratic in the curvature and has non-trivial torsional degrees of freedom. We would like to uncover the role of the spin connection in the AdS/CFT correspondence as the source for the spin current.

A different motivation to carry out this study follows by an analogy with three-dimensional Chern-Simons theory. It is well known that flat connections in three dimensions give rise to conformal structures in two dimensions [15]. The 3d Chern-Simons equations $g_{ab} F^b = 0$ are extended to five dimensions as $g_{abc} F^b F^c = 0$. In three dimensions on a manifold $\Re \times M_2$, it is direct to see that $F = 0$ projected to $M_2$ give rise to conformal Ward identities [23] (for recent discussions see [11, 24].) It is a natural question to ask whether the five-dimensional theory on $\Re \times M_4$ gives rise to conformal structures on $M_4$. (See [25] and [26] for earlier discussions of these issues.) We shall analyze these questions in the particular case of Chern-Simons gravity in five dimensions, and conclude that the correct Ward identities expected for a Lorentz and diff invariant theory on $M_4$ can in fact be derived from it.

### 3.1 First order gravitational sources and their Ward identities

In this section we consider CFT deformations defined by first order theories of gravity with non-trivial torsional degrees of freedom.

To fix the ideas, consider the example of a Dirac spinor field in four dimensions coupled to an external gravitational field. The action is

$$I_{e,\omega}[\psi] = \int d^4x \; |e| \; i\bar{\psi} \, \gamma^i \left( \partial_i + \frac{i}{2} \, \omega_i^{ab} \, \gamma_{ab} \right) \psi + \text{c.c.}, \qquad (3.1)$$

where $|e| = \sqrt{|g|}$ is the determinant of the vielbein field, and $\omega_i^{ab}$ is the spin connection. This action depends on both gravitational fields $e^a$ and $\omega^{ab}$.

In the standard situation of Riemannian geometry, the spin connection is regarded to be a function of the vielbein, $\omega = \omega(e)$, determined by solving the torsion equation. However, from the point of view of the action (3.1), these fields are not subject to variations and it is not mandatory to link them. In fact, if one could compute the effective action

$$e^{iW[e,\omega]} = \int D\psi \, e^{iI_{e,\omega}[\psi]}, \qquad (3.2)$$

for arbitrary values of $e^a$ and $\omega^{ab}$, the functional $W[e,\omega]$ would carry more information as it would have two independent arguments.

This is the calculation we attempt to do using holography. We shall consider a five-dimensional gravitational action having spin connection degrees of freedom. This means that its solutions are characterized by independent values of the (boundary) vielbein and spin connection, and the on-shell renormalized action is a four-dimensional functional $I_{ren}[e, \omega]$. Applying the rules of AdS/CFT correspondence in a classical gravity approximation, we shall identify[3] $I_{ren}[e, \omega] = W[e, \omega]$ and prove that the bulk constraints in the gravitational theory lead to the correct Ward identities for the 1-point functions derived from $W[e, \omega]$.

More concretely, we are interested in the 1-point functions

$$\tau^i{}_a(x) = \frac{1}{|e|} \frac{\delta W[e, \omega]}{\delta e^a_i(x)}, \qquad \sigma^i{}_{ab}(x) = \frac{1}{|e|} \frac{\delta W[e, \omega]}{\delta \omega^{ab}_i(x)}. \tag{3.3}$$

The tensor $\tau_a$ is related to the energy momentum tensor of the theory.[4] The tensor $\sigma_{ab}$ will be called "spin current". Note that $\tau_a$ is not equal to the second order energy-momentum tensor in which $\omega = \omega(e)$.

The Ward identities we expect to find are the following. First consider the invariance of the action (3.2) under Lorentz transformations

$$\delta_\lambda e^a_i = -\lambda^a{}_b e^b_i, \tag{3.4}$$

$$\delta_\lambda \omega^{ab}_i = D_i \lambda^{ab}. \tag{3.5}$$

If the measure is also Lorentz invariant then $W$ must be invariant as well. This implies

$$0 = \int d^4x \, |e| \left( -\lambda^{ab} \tau^i{}_a e_{ib} + \frac{1}{2} \sigma^i{}_{ab} D_i \lambda^{ab} \right) \tag{3.6}$$

$$= \frac{1}{2} \int d^4x \, \lambda^{ab} \left[ 2 |e| \, \tau_{ab} - D_i \left( |e| \, \sigma^i{}_{ab} \right) \right], \tag{3.7}$$

where $\tau_{ab} = e_{ia} \tau^i{}_b$. After antisymmetrization in $[ab]$, we arrive at the conservation law (Ward identity) for the Lorentz transformations

$$D_i \left( |e| \, \sigma^i{}_{ab} \right) = |e| \, (\tau_{ab} - \tau_{ba}). \tag{3.8}$$

When $\sigma$ vanishes, the energy momentum tensor is symmetric, as expected in the torsionless case.

Next, we consider diffeomorphisms. The action (3.1) coupled to $e$ and $\omega$ is also invariant under diffeomorphisms[5] with parameter $\xi^i$,

$$\delta_\xi e^a_i = D_i \left( \xi^j e^a_j \right) + \xi^j T^a{}_{ij}, \tag{3.9}$$

$$\delta_\xi \omega^{ab}_i = \xi^j R^{ab}{}_{ij}. \tag{3.10}$$

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[3]Of course, we do not claim that the dual of Chern-Simons gravity is a Dirac field. We have used the action (3.1) only to exhibit the structure of Ward identities we aim to derive.

[4]The energy momentum tensor is a Hodge dual to the 3-form $\tau_a$, and it has the components $\tau^i{}_a = \frac{1}{3!|e|} \varepsilon^{ijkl} \tau_{a\,jkl}$ (and similarly for $\sigma_{ab}$).

[5]This form of diff transformations is called improved diffeomorphisms [27] and it differs from a Lie derivative by a Lorentz transformation with parameter $\lambda^{ab} = \xi^i \omega^{ab}_i$.

Again assuming invariance of the measure, this symmetry gives rise to the equation

$$0 = \int d^d x\, \xi^j \left[ -D_i \left( |e|\, \tau^i{}_a \right) e^a_j + |e| \left( \tau^i{}_a T^a_{ij} + \frac{1}{2}\, \sigma^i{}_{ab} R^{ab}{}_{ij} \right) \right], \qquad (3.11)$$

or, eliminating the vector $\xi^i$, we get

$$D_i \left( |e|\, \tau^i{}_a \right) e^a_j = |e| \left( \tau^i{}_a T^a_{ij} + \frac{1}{2}\, \sigma^i{}_{ab} R^{ab}{}_{ij} \right). \qquad (3.12)$$

The meaning of this equation is clear after rewriting it in the form $\left( |e|\, \tau^i{}_j \right)_{;i} = \frac{|e|}{2}\, \sigma^i{}_{ab} R^{ab}{}_{ij}$. When $\sigma$ vanishes, the energy momentum tensor is covariantly preserved, as known in the torsionless case.

Equations (3.8) and (3.12) are the two Ward identities that we expect to derive by an AdS/CFT interpretation of five-dimensional Chern-Simons gravity.

## 3.2 First order actions

Our first goal is to find a gravitational action which has spin connection (or torsional) degrees of freedom. In other words, we seek a gravitational action whose equations of motion leave $\omega^{ab}$ as an independent field.

The Einstein-Hilbert action (with or without cosmological term and in any dimension) does not satisfy this condition because the torsion vanishes and spin connection is related to the vielbein. The Palatini formalism does not change this conclusion because one of the equations of motion of the Einstein-Hilbert action implies, precisely, that the torsion is zero.

There exist, however, gravitational actions for which the spin connection is an independent field. In fact, generically, all gravitational actions containing quadratic or higher powers of the curvature tensor will have this property. For actions of the form $R^n$ ($n > 1$ and $R$ represents the scalar, Ricci or Riemann curvature tensor with indices properly contracted), the equations of motion that follow by varying the metric and connection independently are not equivalent to those in which the torsion is assumed to vanish from the start. In other words, the Palatini and second order formalisms are not equivalent for these actions. To our knowledge, this phenomenon was first discussed in [7], in the context of Chern-Simons gravities. Within AdS/CFT, actions with quadratic powers in the curvature tensor and Ricci scalar have been considered in [28, 29].

The family of Lovelock actions of gravity is of particular interest because the equations of motion remain of second order (two derivatives) and that makes the analysis simpler. A particular case of the Lovelock action, which simplifies the analysis even further, is given by the Chern-Simons gravity action in five dimensions

$$I[e,\omega] = \kappa \int_{M_5} \varepsilon_{ABCDE} \left( \hat{R}^{AB} \hat{R}^{CD} \hat{e}^E + \frac{2}{3}\, \hat{R}^{AB} \hat{e}^C \hat{e}^D \hat{e}^E + \frac{1}{5}\, \hat{e}^A \cdots \hat{e}^E \right), \qquad (3.13)$$

where hatted fields are five-dimensional objects. Here $\hat{R}^{AB}$ is the curvature two form and the AdS radius $\ell$ is set to 1. Apart from diffeomorphism invariance, the action is also invariant under the gauge group AdS$_5$, or SO$(2,4)$. Varying $I$ with respect to $e^A$ and $\omega^{AB}$,

we obtain the equations of motion

$$\varepsilon_{ABCDE} \left( \hat{R}^{AB} + \hat{e}^A \hat{e}^B \right) \left( \hat{R}^{CD} + \hat{e}^C \hat{e}^D \right) = 0 , \tag{3.14}$$

$$\varepsilon_{ABCDE} \left( \hat{R}^{AB} + \hat{e}^A \hat{e}^B \right) \hat{T}^C = 0 . \tag{3.15}$$

Equation (3.15), associated to the variation of $\omega^{AB}$, can be solved by the torsion condition $\hat{T}^A = 0$. However, this is not the most general solution. In fact, equation (3.15) possesses a much larger space of solutions. See [7] for a detailed analysis of its dynamical structure.

For our purposes here, we only recall that the induced boundary values of the spin connection, $e_i^a$ and $\omega_i^{ab}$, can be taken as fully independent, and will be identified with the CFT sources discussed above.

The holographic description of dual anomalies for Chern-Simons gravity with no torsion has been studied in [30, 31]. The role of gravitational Chern-Simons forms for AdS$_3$/CFT$_2$ has recently been analyzed in [32], assuming also that the torsion vanishes.[6]

## 3.3 Chern-Simons theories and finite FG expansions

We now use the AdS/CFT prescription. The first goal is to solve the equations of motion (3.14) and (3.15) order by order in an asymptotic expansion near the boundary.

This analysis is actually remarkably simple, given the Chern-Simons structure of these equations. Recall that the action (3.13) belongs to the general set of five-dimensional Chern-Simons actions $\int AFF + \cdots$, for the particular case in which the Lie algebra is chosen as SO(4, 2) [22, 33]. The equations of motion (3.14) and (3.15) can be combined in the form

$$g_{abc}\, \varepsilon^{\mu\nu\lambda\rho\sigma} F^b_{\mu\nu}\, F^c_{\lambda\rho} = 0 , \tag{3.16}$$

where $F^a$ is the SO(4, 2) curvature, and the invariant tensor $g_{abc}$ is related to the Levi-Cevita form. (The components of the SO(4, 2) curvature $F^a$ are $\hat{R}^{AB} + \hat{e}^A \hat{e}^B$ and $\hat{T}^C$.) Note that the content of this paragraph is valid for any Lie algebra $\mathcal{G}$. Lower case Latin indices $a, b, c, \ldots$ refer to the corresponding adjoint representation. This notation should not be confused with that of next section in which Latin indices refer to 4-dimensional Lorentz indices.

We assume that the manifold has the asymptotic structure $\Re \times M_4$ and that it is parameterized by the local coordinates $x^\mu = (r, x^i)$. Here $r$ is related to the FG radial coordinate and $M_4$ is where the CFT lives. The gauge field is expanded as $A_\mu = (A_r, A_i)$ and the equations (3.16) split as

$$g_{abc}\, \varepsilon^{ijkl} F^b_{ij}\, F^c_{kl} = 0 , \tag{3.17}$$

$$g_{abc}\, \varepsilon^{ijkl} F^b_{ij}\, F^c_{kr} = 0 . \tag{3.18}$$

Note that (3.17) contains only derivatives with respect to $x^i$ and in that sense it is a "constraint" that must hold at all values of $r$. The Ward identities of the holographic

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[6] "Gravitational Chern-Simons terms" are Chern-Simon forms for the Lorentz group SO($D-1, 1$) and they depend only on the spin connection. They differ from "Chern-Simons Gravities" [13, 14] which are Chern-Simons forms for the AdS group SO($D-1, 2$) and depend on both the vielbein and spin connection.

CFT are contained in this set of equations. Equation (3.18) depends on $F_{i\,r}$ which does contain derivatives with respect to $r$. The structure of the FG expansion is controlled by this component of the curvature.

Equations (3.17) and (3.18) are entangled; the space of solutions of (3.18) depends on the solution of (3.17). To give just one example suppose we solve (3.17) by $F^a_{ij} = 0$. Then, (3.18) would imply that $F^a_{ir}$ is fully arbitrary. Of course (3.17) has other more interesting solutions for which $F_{ir}$ becomes constrained. A careful analysis of this point can be found in [7]. Conditions ensuring that all equations in (3.17,3.18) are functionally independent, have been discussed in [34, 35].

For our AdS/CFT applications, in which boundary fields play the role of sources, we need the most general solution of (3.17). For a generic solution of (3.17), it is shown in [7] that (3.18) implies

$$F^a_{ri} = F^a_{ij} N^j , \qquad (3.19)$$

where $N^j$ are arbitrary functions.

Now, recalling that $F^a_{ri} = \partial_r A^a_i - D_i A^a_r$, where $D_i$ is the covariant derivative in the connection $A_i$, equation (3.19) can be written as

$$\partial_r A^a_i = D_i A^a_r + F^a_{ij} N^j . \qquad (3.20)$$

The role of the functions $A_r$ and $N^i$ should now be clear. The first term on the r.h.s. of (3.20) represents a gauge transformation, while the second is a (improved) diffeomorphism with parameter $N^i$. This means that the value of $A_i(r + \delta r)$ is obtained from $A_i(r)$ by means of a gauge transformation plus a diffeomorphism. The functions $A_r$ and $N^i$ are the corresponding parameters of these transformations, and we can choose them at will.

The simplest gauge choice $A_r = 0$ and $N^i = 0$ would led to a trivial FG expansion in which the fields are independent of $r$. In the applications to Chern-Simons gravity, however, this is not allowed because setting $A_r = 0$ would lead to a degenerate vielbein and metric.

The simplest non-degenerate choice is

$$A_r = \text{constant Lie algebra element} , \qquad N^i = 0 . \qquad (3.21)$$

Equation (3.20) then becomes $\partial_r A_i = [A_i, A_r]$, whose general solution is

$$A_i(r, x^j) = e^{-rA_r} A_i(0, x^j) e^{rA_r} , \qquad (3.22)$$

where $A(0, x)$ does not depend on $r$.

The field $A_i(r, x)$ is therefore completely fixed once we give the initial condition $A_i(0, x)$. However, we must recall that the initial conditions are not all arbitrary because they must satisfy the constraints (3.17). These constraints can be easily translated into equations for $A_i(0, x)$ as follows. Since the $r$-dependence of $A_i(r, x)$ is a pure gauge transformation, we find $F_{ij}(r, x) = e^{-rA_r} F_{ij}(0, x) e^{rA_r}$, where $F_{ij}(0, x)$ is constructed with $A_i(0, x)$. The constraint on $A_i(0, x)$ is thus simply

$$g_{abc} \, \varepsilon^{ijkl} F^b_{ij}(0, x) \, F^c_{kl}(0, x) = 0 . \qquad (3.23)$$

As we shall see in the next subsection, these constraints contain the Ward identities (3.8) and (3.12) discussed above.

The expansion (3.22) can be put in a more explicit form by going to the Cartan-Weyl basis of the Lie algebra. Let $H_n, E_\alpha$ be this basis with the root system $\alpha_n$, so that $[H_n, E_\alpha] = \alpha_n E_\alpha$. Without loss of generality, we assume that $A_r$ lies along the Cartan subalgebra, $A_r = \sum_n c^n H_n$, where the coefficients $c^n$ are constant. The initial condition has the most general form $A_i(0, x) = \sum_n A_i^n(x) H_n + \sum_\alpha A_i^\alpha(x) E_\alpha$. The expression (3.22) can be computed by means of the identity $e^{cH_n} E_\alpha e^{-cH_n} = e^{c\alpha_n} E_\alpha$, and we find the expansion

$$A_i(r, x) = \sum_n A_i^n(x) H_n + \sum_\alpha e^{-r \sum_n c^n \alpha_n} A_i^\alpha(x) E_\alpha \,. \tag{3.24}$$

Defining $1/\rho = e^{2r}$, one obtains a finite FG expansion with powers determined by the roots $\alpha$. For 2+1 gravity this analysis was performed in detail in [10]. An expansion of this type was used in [24] in a derivation of W-algebras from $SL(2, N)$ Chern-Simons theories. We compute (3.24) for the particular case of 5d Chern-Simons gravity in the next paragraph.

We end this section with two comments. First, note that even though our FG expansions are finite, the solutions cannot be extended to the whole manifold. The reason is that the gauge choice (3.21) may not be extendible beyond the asymptotic region. Several patches may be necessary and this depends on global properties on the manifold and Lie group under consideration. In this manuscript, we restrict ourselves to the asymptotic analysis and relevant Ward identities. The global analysis is important to find the non-local dependence of the vevs (subleading terms in a FG expansion) on the sources. We hope to came back to this problem elsewhere.

Second, note that the solutions considered here, defined by the expansion (3.22), cannot be seen as fluctuations of AdS space, $F_{\mu\nu} = 0$. Our solutions have $F_{ri} = 0$, but $F_{ij}$ must be different from zero. This component of the curvature cannot be deformed continuously to zero keeping the expansion (3.22) valid. As we mentioned before, $F_{ij} = 0$ is a degenerate solution of (3.17) that trivializes (3.18).

## 3.4 Explicit FG structure of Chern-Simons gravity

We now go back to the particular case of Chern-Simons Gravity in five dimensions, for which the Lie algebra is SO(4, 2), and the gauge field is expanded in terms of the SO(4, 2) generators $P_A \equiv J_{A6}$, $J_{AB}$ as $(A, B = 1, \dots, 5)$

$$A_\mu = \hat{e}_\mu^A P_A + \frac{1}{2} \hat{\omega}_\mu^{AB} J_{AB} \,. \tag{3.25}$$

The fields $\hat{\omega}^{AB}$ and $\hat{e}^A$ satisfy the set of five-dimensional equations (3.14) and (3.15). As we have mentioned before, these equations are of the form $F \wedge F = 0$, and all the results of last paragraph apply to this case.

We start by imposing the analogues of (3.21) for this particular Lie algebra. This means fixing $\hat{e}_r^A$ and $\hat{\omega}_r^{AB}$. Splitting the Lorentz indices as $A = (a, 5)$, we set $A_r = P_5$ or

$$\hat{e}_r^5 = 1 \,, \qquad \hat{e}_r^a = 0 \,, \qquad \hat{\omega}_r^{AB} = 0 \,. \tag{3.26}$$

Having fixed the radial components of the gauge field, the radial dependence of the tangent components $\hat{e}_i^A(r, x)$ and $\hat{\omega}_i^{AB}(r, x)$ becomes completely determined by (3.22). Next, we shall impose one extra condition on the boundary vielbein, namely,

$$\hat{e}_i^5 = 0 \, . \tag{3.27}$$

This condition breaks the five-dimensional Lorentz symmetry down to a four-dimensional one. It also leaves a four-dimensional tetrad as a gravitational source.

The calculation of (3.22) for the SO(4, 2) algebra is a simple exercise. In four-dimensional notation the SO(4, 2) generators $P_A, J_{AB}$ are split as $P_5$, $P_a$, $J_{a5}$, $J_{ab}$. It will be convenient to define the combinations [7]

$$J_a^\pm \equiv P_a \pm J_{a5}. \tag{3.29}$$

Expanding the initial condition $A(0, x)$ in (3.22) as

$$A(0, x) = e^a(x) \, J_a^+ + k^a(x) \, J_a^- + \frac{1}{2} \, \omega^{ab}(x) \, J_{ab} \tag{3.30}$$

(from now on all forms are four-dimensional) and using the identity $e^{-\alpha P_5} J_a^\pm \, e^{\alpha P_5} = e^{\pm\alpha} J_a^\pm$, we obtain

$$A(\rho, x) = \frac{1}{\sqrt{\rho}} \, e^a \, J_a^+ + \sqrt{\rho} \, k^a(x) \, J_a^- + \frac{1}{2} \, \omega^{ab}(x) \, J_{ab} \, , \tag{3.31}$$

where we have returned to the usual FG radial coordinate $1/\rho = e^{2r}$.

From (3.25) and (3.29) we can also write the explicit relationship between the five-dimensional fields $\hat{e}^A, \hat{\omega}^{AB}$ and the components of (3.30),

$$\hat{e}^a(\rho, x) = \frac{1}{\sqrt{\rho}} \left[ e^a(x) + \rho \, k^a(x) \right] \, , \tag{3.32}$$

$$\hat{\omega}^{a5}(\rho, x) = \frac{1}{\sqrt{\rho}} \left[ e^a(x) - \rho \, k^a(x) \right] \, , \tag{3.33}$$

$$\hat{\omega}^{ab}(\rho, x) = \omega^{ab}(x) \, . \tag{3.34}$$

These expansions can be seen as a finite FG expansion. The five-dimensional line element $\hat{e}_\mu{}^A \, \hat{e}_{A\nu} \, dx^\mu dx^\nu$ has the finite FG form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \left[ g_{ij} + \rho(k_{ij} + k_{ji}) + \rho^2 k_i{}^l k_{lj} \right] dx^i dx^j \, , \tag{3.35}$$

where $g_{ij} = e_i{}^a \, e_{ja}$ and $k_{ij} = e_{ia} \, k_j{}^a$.

---

[7]The non-zero commutators of the full SO(4, 2) algebra are

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{ad} \, J_{bc} - \eta_{bd} \, J_{ac} - \eta_{ac} \, J_{bd} + \eta_{bc} \, J_{ad} \, , & \left[ J_a^+, J_b^- \right] &= 2 \left( -\eta_{ab} \, P_5 + J_{ab} \right) , \\
[J_{ab}, J_c^\pm] &= \eta_{bc} \, J_a^\pm - \eta_{ac} \, J_b^\pm \, , & \left[ J_a^\pm, P_5 \right] &= \pm J_a^\pm \, .
\end{aligned} \tag{3.28}$$

So far we have solved the equations (3.18) for the particular case of the $SO(4,2)$ Lie algebra. We now write the constraints (3.17) in terms of the basic fields $e^a, k^a, \omega^{ab}$. To this end, it is useful to have $\hat{T}^A$ and $F^{AB} = \hat{R}^{AB} + \hat{e}^A \hat{e}^B$ in terms of the fields $e, k, \omega$,

$$
\begin{aligned}
\hat{T}^5(\rho) &= -2\, e^a k_a\,, & F^{a5}(\rho) &= \frac{1}{\sqrt{\rho}}\left(T^a - \rho\, Dk^a\right), \\
\hat{T}^a(\rho) &= \frac{1}{\sqrt{\rho}}\left(T^a + \rho\, Dk^a\right), & F^{ab}(\rho) &= R^{ab} + 2\left(e^a k^b - e^b k^a\right),
\end{aligned}
\tag{3.36}
$$

where $R^{ab}$ and $T^a$ are four-dimensional curvature and torsion, respectively, and $D = D(\omega)$. Note that these are also finite series in $\rho$.

Inserting these expansions into the constraint equations (3.17), namely, the components of (3.14) and (3.15) tangent to the boundary, we obtain

$$
C \equiv \varepsilon_{abcd}\, F^{ab} F^{cd} = 0\,,
\tag{3.37}
$$

$$
C_a \equiv \varepsilon_{abcd}\, F^{bc} T^d = 0\,,
\tag{3.38}
$$

$$
\bar{C}_a \equiv \varepsilon_{abcd}\, F^{bc} Dk^d = 0\,,
\tag{3.39}
$$

$$
C_{ab} \equiv \varepsilon_{abcd}\left(F^{cd} e^e k_e + 2T^c Dk^d\right) = 0\,,
\tag{3.40}
$$

where $F^{ab} = R^{ab} + 2e^a k^b - 2e^b k^a$.

This set of $1 + 4 + 4 + 6 = 15$ equations gives relations between the fields $e^a, k^a$ and $\omega^{ab}$. We would like to argue now that (3.37)–(3.40) leave the fields $e_i^a$ and $\omega_i^{ab}$ arbitrary. They can then be interpreted as sources. Furthermore, the field $k$ will be associated to the vevs or 1-point functions, and equations (3.37)–(3.40) imply the correct Ward identities for them.

### 3.5 1-point functions and Ward identities

For orientation, it is useful at this point to recall the standard gravitational analysis of holographic renormalization [5, 6, 36], and compare it with our analysis. For the Einstein-Hilbert action, the asymptotic solution has the expansion

$$
ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(g_{(0)ij} + \rho\, g_{(1)ij} + \rho^2\left(g_{(2)ij} + \log \rho\, h_{(2)ij}\right) + \cdots\right) dx^i dx^j,
\tag{3.41}
$$

where the coefficients are determined by the Einstein equations. In a 4+1 decomposition, with $\rho$ playing the role of "time", these equations are separated into 10 "dynamical" (radial derivatives) equations plus 5 constraints. The 10 dynamical Einstein equations are of second order in the radial derivative and they require two initial conditions for their integration. In fact, plugging the series (3.41) into the dynamical equations one finds that $g_{(0)ij}$ and $g_{(2)ij}$ are left completely arbitrary while all other coefficients in the expansion (3.41) are fixed in terms of them. $g_{(0)ij}$ and $g_{(2)ij}$ are in that sense "conjugate" variables and, in fact, the variation of the (renormalized) action can be written as $\delta I_{ren} = 2\int \tau^{ij} \delta g_{(0)ij}$, where $\tau^{ij}$ is linear in $g_{(2)ij}$. The tensor $\tau^{ij}$ is identified with the holographic energy momentum tensor. Now, besides the dynamical equations, there are also 5 constraints which impose conditions on $g_{(2)ij}$, or $\tau^{ij}$. These constraints turn out to be exactly the Ward identities on the trace and divergence of $\tau^{ij}$ [5, 6]. (See [37] for a generic analysis of these relations.)

Our analysis follows a similar structure but with more fields. The initial conditions $A_i(0, x)$ appearing in (3.22) are the analogue of $g_{(0)ij}$ and $g_{(2)ij}$. The "dynamical" equations (3.18) impose no conditions on them. The constraints (3.17), on the other hand, impose conditions on $A_i(0, x)$, which should be related to the holographic Ward identities. For the particular Lie algebra SO(4, 2), these constraints are given by equations (3.37)–(3.40), where $A_i(0, x)$ has been expanded in terms of $e_i^a$, $k_i{}^a$ and $\omega_i^{ab}$ in (3.30).

The rest of the program proceeds just like for standard gravity, with minor variations. We shall prove that the variation of the renormalized bulk action can be expressed as ($M_4 = \partial M_5$)

$$\delta I_{ren} = \int_{M_4} \left( \delta e^a \, \tau_a + \frac{1}{2} \delta \omega^{ab} \, \sigma_{ab} \right). \tag{3.42}$$

The coefficients $\tau_a$ and $\sigma_{ab}$ both depend on $k$ and will be interpreted as the holographic energy momentum tensor and spin current. Finally, we show that the constraints (3.37)–(3.40) imply the Ward identities (3.8) and (3.12) for these tensors, in full analogy with the pure gravity case.

We start with the action (3.13) which yields the equations (3.14) and (3.15). This action will need boundary terms to be well defined for fixed $e^a$ and $\omega^{ab}$, and will also need to be renormalized to make it finite.

Varying (3.13) on-shell and keeping all boundary terms on-shell, one finds

$$\delta I = -2\kappa \int_{M_4} \varepsilon_{ABCDE} \left( \hat{R}^{AB} + \frac{1}{3} \hat{e}^A \hat{e}^B \right) \hat{e}^C \delta \hat{\omega}^{DE}. \tag{3.43}$$

Next, we evaluate this variation on the asymptotic solution described above. By a direct calculation this can be written as

$$\delta I = 4\kappa \int_{M_4} \varepsilon_{abcd} \left[ \left( R^{ab} + 2\, e^a k^b \right) \left( -k^c \delta e^d + e^c \delta k^d \right) + \left( D e^a k^b - D k^a e^b \right) \delta \omega^{cd} \right] + \delta V, \tag{3.44}$$

where

$$V = \frac{2\kappa}{3\rho^2} \int_{M_4} \varepsilon_{abcd} e^a e^b e^c e^d - \frac{2\kappa}{\rho} \int_{M_4} \varepsilon_{abcd} \left( R^{ab} + \frac{4}{3} e^a k^b \right) k^c e^d. \tag{3.45}$$

(We have omitted terms with positive powers of $\rho$ because they cancel in the limit $\rho \to 0$.)

Note that $V$ is divergent as $\rho \to 0$, while all other terms in (3.44) are finite (and $\rho$-independent). The divergencies thus appear in the form of a total variation which can be subtracted from $I$. We define the renormalized action $I'_{ren} = I - V$ whose variation is given by the first piece in (3.44). This completes the "renormalization" problem. But $I'_{ren}$ is not yet the correct action because we have to deal with the "Dirichlet problem", namely, we need to make sure that the action has well defined variations for $e^a$ and $\omega^{ab}$ fixed.

Having eliminated $V$ in (3.44), the remaining terms (in square bracket) contain variations of $e^a$, $k^a$ and $\omega^{ab}$. The variations of $k^a$ can be transformed into variations of $e^a$ and $\omega^{ab}$ by adding finite boundary terms,

$$4\kappa \, \varepsilon_{abcd} \left( R^{ab} + 2\, e^a k^b \right) e^c \delta k^d = 4\kappa \, \varepsilon_{abcd} \left[ -\left( R^{ab} + 2\, e^a k^b \right) k^c \delta e^d + D\left( e^a k^b \right) \delta \omega^{cd} \right] + \delta(\text{boundary term}), \tag{3.46}$$

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where a total derivative has been omitted since it will vanish under the integral. Discarding all total variations, we finally arrive at

$$\delta I_{ren} = -8\kappa \int\limits_{M_4} \varepsilon_{abcd} \left[ \left( R^{ab} + 2\,e^a k^b \right) k^c \delta e^d - T^a k^b \delta \omega^{cd} \right] , \tag{3.47}$$

from where we read off the holographic energy-momentum tensor and spin current,

$$\tau_a = -8\kappa\,\varepsilon_{abcd} \left( R^{bc} + 2\,e^b k^c \right) k^d , \tag{3.48}$$

$$\sigma_{ab} = -16\kappa\,\varepsilon_{abcd}\,T^c k^d . \tag{3.49}$$

(We omit writing expectation values $\langle \cdots \rangle$.) In components, these tensors are

$$\tau^i{}_a = -\frac{8\kappa}{|e|}\,\varepsilon^{ijkl}\varepsilon_{abcd} \left( \frac{1}{2}\,R^{bc}{}_{jk} + 2\,e^b_j\,k_k{}^c \right) k_l{}^d , \tag{3.50}$$

$$\sigma^i{}_{ab} = -\frac{8\kappa}{|e|}\,\varepsilon^{ijkl}\varepsilon_{abcd}\,T^c{}_{jk}k_l{}^d . \tag{3.51}$$

Our last step is to show that the tensors $\tau^i{}_a$ and $\sigma^i{}_{ab}$ satisfy the conservation laws, or Ward identities (3.8, 3.12), associated to Lorentz transformations and diffeomorphisms. To prove these identities we need to use the constraint equations (3.37)–(3.40).

To this end we consider the contraction operator $I_i$ which maps $p$-forms to $(p-1)$-forms.[8] Since any 5-form vanishes on $M_4$ and the contraction operator $I_i$ obeys the Leibnitz rule, from $I_i(\varepsilon_{abcd}\,e^a k^b k^c T^d) \equiv 0$ and $I_i\left( \varepsilon_{abcd}\,R^{ab}k^c T^d \right) \equiv 0$, we obtain the identities

$$\varepsilon_{abcd}\,I_i e^a k^b k^c T^d = \varepsilon_{abcd} \left( 2e^a I_i k^b k^c T^d + e^a k^b k^c I_i T^d \right) , \tag{3.52}$$

$$\varepsilon_{abcd}\,I_i R^{ab}k^c T^d = -\varepsilon_{abcd}\,R^{ab}I_i k^c T^d + \varepsilon_{abcd}\,R^{ab}k^c I_i T^d . \tag{3.53}$$

With the help of these identities and the expressions (3.38)–(3.40), it can be shown that

$$D\sigma_{ab} - (e_a\tau_b - e_b\tau_a) = -8\kappa\,C_{ab} , \tag{3.54}$$

$$D\tau_a - \left( I_a T^b\tau_b + \frac{1}{2}\,I_a R^{bc}\sigma_{bc} \right) = -8\kappa \left( \bar{C}_a - k_a{}^b C_b \right) . \tag{3.55}$$

Since all constraints vanish on $M_4$, the conservation laws (3.8, 3.12) are in fact satisfied.

## 3.6 Weyl anomaly

The energy-momentum tensor is also a generator of conformal transformations, where the conservation law requires that $\tau^i{}_a$ be traceless. With the help of eq. (3.37), one can show that

$$e^a \tau_a = \kappa\,\varepsilon_{abcd}\,R^{ab}R^{cd} - \kappa\,C . \tag{3.56}$$

The l.h.s. is the trace of the tensor $\tau^i{}_a$, and since $C = 0$, we obtain that its trace is

$$\tau^a{}_a = \frac{\kappa}{4}\,\varepsilon^{ijkl}\,\varepsilon_{nmpq}\,R^{nm}{}_{ij}\,R^{pq}{}_{kl} = \kappa E_4 . \tag{3.57}$$

---

[8]The contraction operator $I_i$ acts on the $p$-form $\Omega = \frac{1}{p!}\Omega_{i_1 \cdots i_p}\,dx^{i_1} \cdots dx^{i_p}$ as $I_i\Omega = \frac{1}{(p-1)!}\Omega_{i i_2 \cdots i_p}\,dx^{i_2} \cdots dx^{i_p}$.

We reproduce the result that the holographic Weyl anomaly is given by the Euler density, $E_4$ [6, 29, 38, 39]. The torsion does not enter the conformal anomaly explicitly, but only through the spin connection $\omega_i^{ab}$ in the curvature. This is not surprising since there is no four-dimensional topological invariant with even parity constructed from the torsion tensor $T_i^a$. As in torsion-less Chern-Simons gravity, only Type A anomaly emerges [30, 31].


## 4. Discussion

We would like to end with some comments on chiral anomalies for Chern-Simons gravity duals. In [3], Witten noted that the Chern-Simons term added to Type IIB supergravity on $\text{AdS}_5 \times S^5$ is responsible for the occurrence of the chiral anomaly in the corresponding CFT. It has been also known for some time that the fully antisymmetric part of the torsion is related to the chiral anomaly of spinors in Riemann-Cartan spaces [40, 41]. These facts suggest that the Chern-Simons gravity dual theory should exhibit a chiral anomaly, due to the coupling of the spin connection. The chiral anomaly on a four-dimensional Riemannian manifold is proportional to another topological invariant quadratic in curvature, the Pontryagin density $P_4 = R^{ab}R_{ab}$. The torsion on the other hand is seen as the field strength associated to the vielbein, and it is natural to ask whether a field theory coupled to both curvature and torsion would develop an anomaly which depends explicitly on torsion. A natural candidate to represent this generalized anomaly is the Pontryagin density for $\text{AdS}_4$

$$P_4 = R^{ab}R_{ab} + \frac{2}{\ell^2}\left(R^{ab}e_a e_b - T^a T_a\right),\tag{4.1}$$

where $\ell$ is the AdS radius. When $T^a$ vanishes, the term in parentheses also vanishes, since it can be locally written as $-d(T^a e_a)$. There is a controversy whether the second term in (4.1), which is by itself a topological invariant, should contribute to the chiral anomaly or not. See [42] and [43] for details. The AdS/CFT correspondence offers a rich ground to test the dependence of the chiral anomaly on torsion.

The explicit calculation could be performed as follows. Taking as a working example the Dirac field (3.1), we see that the chiral current is proportional to the fully antisymmetric part of the spin current,

$$J_{ch}^i = \frac{1}{3!}\,\varepsilon^{abcd}\,e_a^i \sigma_{bcd}.\tag{4.2}$$

In fact, the spin current for the Dirac field is $\sigma^i{}_{ab} = \frac{1}{|e|}\frac{\delta I_D}{\delta \omega_i^{ab}} = i\,\varepsilon_{abcd}\,e^{ic}\bar{\psi}\gamma_5\gamma^d\psi$, and its fully antisymmetric part yields $J_{ch}^a = i\,\bar{\psi}\gamma_5\gamma^a\psi$, as expected for the chiral current.

In our case, from (3.51), we obtain the chiral anomaly (defined as $\partial_i(|e|J_{ch}^i) = |e|\,\mathcal{A}_{ch}$)

$$\mathcal{A}_{ch} = \frac{8\kappa}{3\,|e|}\,\varepsilon^{jknm}\partial_i\left(k_{kj}T^i{}_{nm} - k_k{}^i T_{jnm}\right),\tag{4.3}$$

where $k_{ij}$ is a solution of the field equations (3.37)–(3.40). The question is whether this expression coincides with (4.1), or its Lorentz version $R^{ab}R_{ab}$, will be left open. This requires the general solution (or at least its local part) of $k_{ij}$ which has escaped us.

Finally, it is worth noticing that the whole analysis of holographic CFT associated to Chern-Simons gravities can be easily generalized to any odd dimension, for any extension (including supersymmetric ones) of the AdS group, since only the gravitational part requires non-trivial gauge fixing. In all cases, the Fefferman-Graham expansion is finite or truncated. In the gravitational sector, the only sources are $e$ and $\omega$ because the undetermined (on-shell) components of $k$ do not couple to the CFT gravitational current, corresponding therefore to the Fefferman-Graham ambiguity.

## Acknowledgments

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