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Non-perturbative effects in the BMN limit of $\mathcal{N}=4$ supersymmetric Yang-Mills

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Abstract: One-instanton contributions to the correlation functions of two gauge-invariant single-trace operators in $\mathcal{N} = 4$ SU(N) Yang-Mills theory are studied in semi-classical approximation in the BMN limit. The most straightforward examples involve operators with four bosonic impurities (whereas examples with two-impurity operators pose technical problems). The explicit form for the correlation functions, which determine the anomalous dimensions, follows after integration over the large number of bosonic and fermionic moduli. Our results demonstrate that the instanton contributions scale appropriately in the BMN limit. We find impressive agreement with the D-instanton contributions to mass matrix elements of the dual plane-wave IIB superstring theory, obtained in a previous paper. Not only does the dependence on the scaled coupling constants match, but the dependence on the mode numbers of the states is also in striking agreement.

Keywords: Nonperturbative Effects, AdS-CFT Correspondence, Solitons Monopoles and Instantons.

Contents

1. Introduction

According to [1] there is a very interesting limit of the AdS/CFT correspondence that relates a special sector of the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory to type-IIB string theory in a maximally supersymmetric plane-wave background. A notable feature of this proposal is that it provides the first example of a gauge/gravity duality which can be studied in a quantitative way beyond the supergravity approximation. This is possible because string theory in the relevant background, which is obtained as a Penrose limit of $\mathrm{AdS}_5 \times S^5$ [2], can be quantised in the light cone gauge [3, 4] and moreover there exists a regime in which both the string and the gauge theory are weakly coupled. This has allowed very precise comparisons between perturbative corrections on the two sides [5, 6].

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The duality relates the string mass spectrum to the spectrum of scaling dimensions of gauge theory operators in the so called BMN sector of $\mathcal{N}=4$ SYM. This consists of gauge invariant operators of large conformal dimension, Δ, and large charge, J, with respect to a U(1) subgroup of the SU(4) R-symmetry group. The duality involves the double limit $\Delta \to \infty$, $J \to \infty$. The combination $\Delta - J$, which is kept finite, is related to the string theory hamiltonian,

$$\Delta - J = \frac{1}{\mu} H^{(2)}, \tag{1.1}$$

where μ is the background value of the R⊗R five-form and is related to the mass parameter, m, which appears in the light cone string action by $m = \mu p_- \alpha'$ [3, 4], where p_- is the light cone momentum.

The correspondence between the spectra of the two theories is thus the statement that the eigenvalues of the operators on the two sides of the equality (1.1) coincide. A quantitative comparison is possible if one considers the large-N limit in the gauge theory focusing on operators in the BMN sector. As a result of combining the large-N limit with the limit of large Δ and J, new effective parameters arise [7, 8], which are related to the ordinary 't Hooft parameters, λ and $1/N$, by a rescaling,

$$\lambda' = \frac{g_{\rm YM}^2 N}{J^2}, \qquad g_2 = \frac{J^2}{N}. \tag{1.2}$$

The correspondence relates these effective gauge theory couplings to string theory parameters in the plane-wave background,

$$m^2 = (\mu p_- \alpha')^2 = \frac{1}{\lambda'}, \qquad 4\pi g_{\rm s} m^2 = g_2. \tag{1.3}$$

The double scaling limit, $N \to \infty$, $J \to \infty$ with J^2/N fixed, connects the weak coupling regime of the gauge theory to string theory at small g_s and large m.

In this limit the leading perturbative corrections to the scaling dimensions of BMN operators have been successfully compared to the leading quantum corrections to the masses of the dual plane-wave string states [7 − 10, 5, 6, 11], see also the reviews [12] for further references. In the present paper we will study one-instanton effects in the $\mathcal{N}=4$ Yang-Mills theory. These will be compared with D-instanton [13] induced corrections to the plane-wave string mass spectrum that were computed in [14] in order to check the validity of the BMN conjecture in non-perturbative sectors. In the original formulation of the AdS/CFT correspondence very good agreement was found between the effects of instantons in the $\mathcal{N}=4$ Yang-Mills theory and of D-instantons in type-IIB string theory in $\mathrm{AdS}_5 \times S^5$ [15 − 17]. It is therefore of interest to see if similar agreement can be established in the BMN limit and whether the results of [14] can be reproduced from the study of instanton contributions to the anomalous dimensions of BMN operators.

The possibility of testing the correspondence at the non-perturbative level is especially relevant since several aspects of the perturbative tests of the duality are only partially understood. A precise holographic formulation of the duality connecting the dynamics of the two theories beyond the identification of the spectra is still lacking and even the explicit tests at the level of the spectrum are not comprehensive. A limited class of states/operators has been studied and agreement has been explicitly verified only at leading order in g_2 (the planar limit). This is the limit in which the string is free and the λ' perturbation series on the gauge side reproduces the free string spectrum. The first non-planar contributions, of order $\lambda' g_2^2$ or one-loop on the string side, have also been compared successfully (although in this case simplifying assumptions were made about the one-loop string theory calculation, which have since been questioned [18]). The systematics of the perturbative expansion beyond these leading order contributions has not been studied and the fact that the double scaling parameters (1.2) that arise at low orders indeed represent the correct expansion parameters at all orders remains a conjecture.

Results obtained in different but related limits of the AdS/CFT duality, both in string theory [19] and on the gauge theory side [20], suggest the possibility that BMN scaling, i.e. the order by order reorganisation of the perturbative expansion into a double series in λ' and g_2, might break down at higher orders. In the strict BMN sector the scaling (1.2) has been verified to three loops in perturbation theory [21], but a deviation was observed in a related matrix model calculation [22] at four loops.

In this paper we will show that instanton contributions to the conformal dimensions of BMN operators display BMN scaling. Two-point functions of BMN operators computed in the semi-classical approximation will be shown to be in striking agreement with the D-instanton induced two-point amplitudes computed in [14]. The agreement includes not only the dependence on the parameters λ' and g_2, but also the dependence on the mode numbers characterising the states. In particular the agreement with [14] in the mode number dependence is highly non-trivial and requires dramatic cancellations. These results combined with the three loop perturbative result provide substantial evidence indicating that BMN scaling should persist at all orders.

Instanton contributions to the anomalous dimensions of BMN operators are extracted from two-point correlation functions computed in the semi-classical approximation. Conformal invariance determines the form of two-point functions of primary operators, $\mathscr{O}$ and $\bar{\mathscr{O}}$, to be

$$\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = \frac{c}{(x_1-x_2)^{2\Delta}}\,, \tag{1.4}$$

where Δ is the scaling dimension. In general in the quantum theory Δ acquires an anomalous term, $\Delta(g_{\rm YM}) = \Delta_0 + \gamma(g_{\rm YM})$. At weak coupling the anomalous dimension $\gamma(g_{\rm YM})$ is small and substituting in (1.4) gives

$$\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = \frac{c\,\Lambda^{2\gamma(g_{\rm YM})}}{(x_1-x_2)^{2\Delta_0}}\left(1-\gamma(g_{\rm YM})\log\left[\Lambda^2(x_1-x_2)^2\right]+\cdots\right)\,, \tag{1.5}$$

where Λ is an arbitrary renormalisation scale. As a function of the coupling constant the anomalous dimension admits an expansion consisting of a perturbative series plus non-perturbative corrections. The generic two-point function at weak coupling takes the form

$$\begin{aligned}\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = \frac{c(g_{\rm YM})}{(x_1-x_2)^{2\Delta_0}}\big(1 - g_{\rm YM}^2\gamma^{(1)}\log\left[\Lambda^2(x_1-x_2)^2\right] + \\ + \cdots - {\rm e}^{2\pi i\tau}\gamma^{(\rm inst)}\log\left[\Lambda^2(x_1-x_2)^2\right]+\cdots\big).\end{aligned} \tag{1.6}$$

Therefore perturbative and instanton contributions to the anomalous dimension are extracted from the coefficients of the logarithmically divergent terms in a two-point function. When there is more than one operator with the same quantum numbers operator mixing occurs. In this case the resulting set of two-point functions determines a matrix of anomalous dimensions and the eigenvalues of this matrix are the physical anomalous dimensions. The issue of operator mixing was first discussed in the context of the BMN limit in [23].

The procedure for calculating the instanton-induced contribution to the anomalous dimensions in semi-classical approximation is as follows. The gauge-invariant operators in the BMN sector are defined by colour traces involving a large number of elementary scalar fields together with a finite number of bosonic or fermionic 'impurities'. In the semi-classical approximation correlation functions of such operators are computed by replacing the fields by the solution to the corresponding field equations in the presence of an instanton, expressed in terms of the fermionic and bosonic moduli, and integrating the resulting profiles over these moduli. These moduli encode the broken superconformal symmetries together with the (super)symmetries associated with the orientation of a SU(2) instanton within SU(N). For large-N integration over these moduli is carried out by a saddle point procedure (as in [24]).

The general structure of the anomalous dimensions of gauge invariant operators in the $\mathcal{N}=4$ Yang-Mills theory with SU(N) gauge group is an expansion of the form

$$\gamma(g_{\rm YM},\theta,N)=\sum_{n=1}^{\infty}\gamma_n^{\rm pert}(N)\,g_{\rm YM}^{2n}+\sum_{K>0}\sum_{m=0}^{\infty}\left[\gamma_m^{(K)}(N)\,g_{\rm YM}^{2m}\,{\rm e}^{2\pi i\tau K}+{\rm c.c.}\right], \tag{1.7}$$

where $\tau=\frac{\theta}{2\pi}+i\frac{4\pi}{g_{\rm YM}^2}$. The double series in the second term in (1.7) contains the contributions of multi-instanton sectors as well as the perturbative fluctuations in each such sector. One reason for studying instanton effects even though they are exponentially suppressed in the small coupling limit, is that they determine the dependence on the θ-angle in $\mathcal{N}=4$ SYM. They therefore play an essential rôle in implementing S-duality which is a symmetry of the theory, just as D-instantons are crucial for the S-duality in type-IIB string theory.

If the BMN sector of the gauge theory scales appropriately (1.7) becomes a series in the scaled couplings λ' and g_2. In particular, we will show that the leading one-instanton contribution to the two-point functions of a class of four impurity BMN operators scales as it should in the BMN limit and has the form $(1/n_1n_2)^2\,g_2^{7/2}\exp\left(-8\pi^2/g_2\lambda'+i\theta\right)$, where the integers n_1 and n_2 correspond to the mode numbers of the dual string state. This result is in striking agreement with the corresponding D-instanton induced mass matrix on the string side found in [14].

For certain other classes of operators the leading one-instanton contribution vanishes and the first non-zero correction is of higher order in λ' (or a lower power of m in the string calculation). In such cases the calculation requires knowledge of a non-leading term in the scalar solution — a term involving six fermionic moduli (whereas the leading term is quadratic in fermionic moduli). We have not evaluated the precise form of this contribution and so have not determined the precise form of the matrix elements in these cases. However, there is strong evidence that these also match the string calculations. For example, for two impurity operators, with some mild assumptions about the manner in which the fermion moduli are distributed in the profile of the operators, we will find a contribution to the two-point function of the form ${\lambda'}^2 g_2^{7/2} \exp\left(-8\pi^2/g_2\lambda' + i\theta\right)$, in accord with expectations from the string side. Later we will comment on the systematics of the expansion in the one-instanton sector and on how the higher order corrections can give rise to a double series in λ' and g_2.

This paper is organised as follows. In section 2 we review some general aspects of the $\mathcal{N}=4$ Yang-Mills theory and the BMN limit. The general method for evaluating instanton-induced contributions to two-point functions of BMN operators in terms of zero modes and the integration over super-moduli is described in section 3. The manner in which the profiles of the fields depend on these moduli is presented in section 4. In section 5 we consider some specific examples of two-point functions of BMN operators and derive expressions for the anomalous dimensions that arise after integration over all the moduli. We first consider the case of two-impurity operators (which presents the technical difficulty alluded to above) and then four-impurity operators. We conclude with a discussion in section 6, which includes a comparison with the string results in [14]. Some technical details of the calculations are presented in the appendices.

2. Fields and operators in the BMN limit

The purpose of this section is mainly to present the notation used in the paper and to define the dictionary to be used for the comparison with string theory in the plane-wave background. We will only consider a small set of BMN operators with scalar impurities which are dual to the string states studied in [14]. A more detailed discussion of the various types of operators relevant for the comparison with string theory in the plane-wave background can be found in the review papers [12].

2.1 Fields in $\mathcal{N}=4$ SYM

The $\mathcal{N}=4$ multiplet comprises six real scalars, $\hat{\varphi}^i$, $i=1,\ldots,6$, four Weyl fermions, λ^A_α, $A=1,\ldots,4$, and a vector, A_μ, with field strength $F_{\mu\nu}$, all transforming in the adjoint representation of the gauge group. These are the building blocks used to construct gauge invariant composite operators which are classified according to the irreducible representations of the superconformal group, $\mathrm{SU}(2,2|4)$. The latter are identified by the quantum numbers $(\Delta, j_1, j_2; a, b, c)$ of the maximal bosonic subgroup $\mathrm{SO}(2,4)\times\mathrm{SO}(6)$, where Δ is the scaling dimension, j_1 and j_2 the Lorentz spins and $[a,b,c]$ the $\mathrm{SU}(4)\sim\mathrm{SO}(6)$ Dynkin labels.

Under the SU(4) R-symmetry group the scalars transform in the **6**, the fermions in the **4** (and their conjugates in the $\overline{\mathbf{4}}$) and the gauge field is a singlet. It is often convenient to label the scalars by an antisymmetric pair of indices in the **4**, $\varphi^{[AB]}$, subject to the reality condition

$$\bar{\varphi}_{AB} \equiv \left(\varphi^{AB}\right)^* = \frac{1}{2}\varepsilon_{ABCD}\varphi^{CD}\,. \tag{2.1}$$

The two parametrisations of the $\mathscr{N}$=4 scalars, $\hat{\varphi}^i$ and φ^{AB}, are related by

$$\hat{\varphi}^i = \frac{1}{\sqrt{2}} \Sigma^i_{AB} \varphi^{AB} , \qquad \varphi^{AB} = \frac{1}{\sqrt{8}} \bar{\Sigma}^{AB}_i \hat{\varphi}^i , \tag{2.2}$$

where Σ^i_{AB} ($\bar{\Sigma}^{AB}_i$) are Clebsch-Gordan coefficients projecting the product of two $\mathbf{4}$'s ($\overline{\mathbf{4}}$'s) onto the $\mathbf{6}$. They are defined in appendix A. The representation of the scalars in terms of the φ^{AB} fields is the most convenient for instanton calculations since, as we shall see, it makes manifest which fermion zero modes in a correlation function can be soaked up by each scalar field.

In the limit relevant for the comparison with string theory in the plane-wave background the symmetry group is a contraction of the original group and the operators are classified according to representations of the bosonic subgroup $\mathrm{SO}(4)\times\mathrm{SO}(4)\times\mathrm{U}(1)\times\mathrm{U}(1)$. We shall denote by $\mathcal{D}$ the dilation operator and by $\mathcal{J}$ the U(1) generator selected by the Penrose limit in the dual AdS background. The $\mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathrm{U}(1) \times \mathrm{U}(1)$ quantum numbers are $(s_1, s_2; s'_1, s'_2; \Delta, J)$, where Δ and J refer to $\mathcal{D}$ and $\mathcal{J}$ and the spins s_i and s'_i refer to the two SO(4) factors. These can be considered to be respectively subgroups of the original SO(6) and SO(2, 4) groups. This identification is not completely correct. The generators of the two SO(4)'s corresponding to the isometries of the dual string background, $\widetilde{G}_i$, are related to the generators of the euclidean Lorentz group and to those of an SO(4) subgroup of the R-symmetry group, G_i, $i = 1, 2$, by a similarity transformation, $\widetilde{G}_i = TG_iT^{-1}$. This distinction, however, will not be relevant for our analysis.

Since a precise formulation of the gauge theory dual to the plane wave string theory is not known, the rules for the decomposition of the $\mathscr{N}$=4 fields according to representations of $\mathrm{SO}(4)\times\mathrm{SO}(4)\times\mathrm{U}(1)\times\mathrm{U}(1)$ are determined by the quantum numbers of the dual string excitations. The gauge invariant operators corresponding to states in the string spectrum will be discussed in the next subsection. String excitations created by bosonic and fermionic oscillators are associated respectively with the insertion of bosonic and fermionic elementary fields ("impurities") in composite operators.

Bosonic excitations in the plane wave string theory originate from the vector of SO(8) which decomposes under $\mathrm{SO}(4) \times \mathrm{SO}(4)$ as

$$\mathbf{8}_{\mathrm{v}} = \left[\left(\frac{1}{2}, \frac{1}{2}\right); (0,0)\right] \oplus \left[(0,0); \left(\frac{1}{2}, \frac{1}{2}\right)\right] , \tag{2.3}$$

i.e. they are vectors of one SO(4) and singlets of the second or vice versa. Correspondingly in the $\mathscr{N}$=4 theory the six real scalars are reorganised into a complex field, Z, and its conjugate, $\bar{Z}$, which are singlets of $\mathrm{SO}(4)\times\mathrm{SO}(4)$ and have $\Delta = 1$ and $J = \pm 1$ respectively, and four real fields which transform in the $\mathbf{4} = (\frac{1}{2}, \frac{1}{2})$ of the first SO(4) and are singlets with respect to the second and have $J = 0$ and $\Delta = 1$. The insertion of the four real scalars in a composite operator corresponds to the insertion of bosonic creation operators with an index in one of the two SO(4) factors in the dual string state. States created by bosonic oscillators which are vectors of the second SO(4) correspond to operators involving insertions of $D_\mu Z$. The fields $D_\mu Z$ are in the $\mathbf{4} = (\frac{1}{2}, \frac{1}{2})$ of the second SO(4) and have

$J = 1$ and $\Delta = 2$. Explicitly the scalar fields are

$$
\begin{aligned}
&Z = \phi^1 = 2\varphi^{14}\,, && \bar{Z} = \phi_1^\dagger = 2\varphi^{23}\,,\\
&\varphi^1 = \hat{\varphi}^2 = \frac{1}{\sqrt{2}}\left(-\varphi^{13} + \varphi^{24}\right), && \varphi^2 = \hat{\varphi}^3 = \frac{1}{\sqrt{2}}\left(\varphi^{12} + \varphi^{34}\right),\\
&\varphi^3 = \hat{\varphi}^5 = \frac{i}{\sqrt{2}}\left(-\varphi^{13} - \varphi^{24}\right), && \varphi^4 = \hat{\varphi}^6 = \frac{i}{\sqrt{2}}\left(\varphi^{12} - \varphi^{34}\right),
\end{aligned}
\tag{2.4}
$$

Here, for convenience of notation, we have introduced the scalars φ^i, $i = 1, \ldots, 4$, related to four of the $\hat{\varphi}^i$'s by a relabelling. This should not cause any confusion since in the following we shall only work with (2.4) and we shall not need the SO(6) fields (A.5).

Unlike the scalar fields the fermions transform non trivially with respect to both SO(4)'s. The four Weyl fermions of the $\mathcal{N}{=}4$ SYM theory, λ_α^A, transform in the $\mathbf{4}$ of $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ and their conjugates, $\bar{\lambda}_A^{\dot{\alpha}}$, transform in the $\overline{\mathbf{4}}$. Their decomposition with respect to $\mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathrm{U}(1)$ is dictated by the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ decomposition of the SO(8) fermions of the light-cone string. The type IIB fermions transform in the $\mathbf{8}_\mathrm{s}$ of SO(8), which under $\mathrm{SO}(4) \times \mathrm{SO}(4)$ decomposes as

$$
\mathbf{8}_\mathrm{s} = \left[\left(0, \frac{1}{2}\right); \left(0, \frac{1}{2}\right)\right] \oplus \left[\left(\frac{1}{2}, 0\right); \left(\frac{1}{2}, 0\right)\right]. \tag{2.5}
$$

In terms of the $\mathbf{8}_\mathrm{s}$ fermions S^a and $\widetilde{S}^a$ the decomposition is achieved via a projector [3], $\frac{1}{2}(1 \pm \Pi)$,

$$
\begin{aligned}
S^a &\to (S^-)^a_\alpha \oplus (S^+)^{\dot{\alpha}}_{\dot{a}}\\
\widetilde{S}^a &\to (\widetilde{S}^-)^a_\alpha \oplus (\widetilde{S}^+)^{\dot{\alpha}}_{\dot{a}}\,.
\end{aligned}
\tag{2.6}
$$

The Yang-Mills fermions, λ_α^A, have $\Delta = \frac{3}{2}$ and U(1) charge $\frac{1}{2}$ for $A = 1, 4$ and $-\frac{1}{2}$ for $A = 2, 3$. Similarly their conjugates, $\bar{\lambda}_A^{\dot{\alpha}}$, have $\Delta = \frac{3}{2}$ and U(1) charge $\frac{1}{2}$ for $A = 2, 3$ and $-\frac{1}{2}$ for $A = 1, 4$. To match the quantum numbers of the string oscillators we choose the following decomposition

$$
\lambda_\alpha^A \to \psi^{-\,a}_{+\frac{1}{2};\alpha} \oplus \bar{\psi}^{+}_{-\frac{1}{2};\alpha a}\,, \qquad a = 1, 4\,, \tag{2.7}
$$

where the fermions $\bar{\psi}^+_{\alpha a}$ are defined as

$$
\bar{\psi}^+_{-\frac{1}{2};\alpha a} = \left(M^+ \lambda\right)_{-\frac{1}{2};\alpha a}\,, \tag{2.8}
$$

where the matrix M^+ is related to the matrix Π used in the plane-wave string theory to project the SO(8) fermions onto chiral $\mathrm{SO}(4)\times\mathrm{SO}(4)$ spinors. The spinors $\psi^{-\,a}_{+\frac{1}{2};\alpha}$ and $\bar{\psi}^{+}_{-\frac{1}{2};\alpha a}$ transform under $\mathrm{SO}(4) \times \mathrm{SO}(4)$ in the $(\mathbf{2}_L; \mathbf{2}_L) = \left[\left(\frac{1}{2}, 0\right); \left(\frac{1}{2}, 0\right)\right]$ and have $(J = \frac{1}{2}, \Delta = \frac{3}{2})$ and $(J = -\frac{1}{2}, \Delta = \frac{3}{2})$ respectively.

Similarly

$$
\bar{\lambda}_A^{\dot{\alpha}} \to \psi^{+\,\dot{\alpha}}_{+\frac{1}{2};\dot{a}} \oplus \bar{\psi}^{-\,\dot{\alpha}\dot{a}}_{-\frac{1}{2}}\,, \qquad \dot{a} = 2, 3 \tag{2.9}
$$

Field	Δ	J	$\Delta - J$	$\Delta + J$	$\mathrm{SO}(4)_R$	$\mathrm{SO}(4)_C$
Z	1	1	0	2	$(0,0)$	$(0,0)$
$\bar{Z}$	1	-1	2	0	$(0,0)$	$(0,0)$
φ^i	1	0	1	1	$\left(\frac{1}{2},\frac{1}{2}\right)$	$(0,0)$
$D_\mu Z$	2	1	1	3	$(0,0)$	$\left(\frac{1}{2},\frac{1}{2}\right)$
$\psi^{-\,a}_{\alpha}$	$\frac{3}{2}$	$\frac{1}{2}$	1	2	$\left(\frac{1}{2},0\right)$	$\left(\frac{1}{2},0\right)$
$\bar{\psi}^{+}_{\alpha a}$	$\frac{3}{2}$	$-\frac{1}{2}$	2	1	$\left(\frac{1}{2},0\right)$	$\left(\frac{1}{2},0\right)$
$\psi^{+\,\dot{\alpha}}_{\dot{a}}$	$\frac{3}{2}$	$\frac{1}{2}$	1	2	$\left(0,\frac{1}{2}\right)$	$\left(0,\frac{1}{2}\right)$
$\bar{\psi}^{-\,\dot{\alpha}\dot{a}}$	$\frac{3}{2}$	$-\frac{1}{2}$	2	1	$\left(0,\frac{1}{2}\right)$	$\left(0,\frac{1}{2}\right)$
$F^{-}_{\mu\nu}$	2	0	2	2	$(0,0)$	$(1,0)$
$F^{+}_{\mu\nu}$	2	0	2	2	$(0,0)$	$(0,1)$

Table 1: $\mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathrm{U}(1) \times \mathrm{U}(1)$ quantum numbers of the $\mathcal{N}=4$ elementary fields.

where

$$\bar{\psi}^{-\,\dot{\alpha}\dot{a}}_{-\frac{1}{2}} = \left(M^{-}\bar{\lambda}\right)^{\dot{\alpha}\dot{a}}_{-\frac{1}{2}} \tag{2.10}$$

and M^- is also related to the projector used to define the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ spinors in the dual plane-wave string theory. The fermions $\psi^{+\,\dot{\alpha}}_{+\frac{1}{2};\dot{a}}$ and $\bar{\psi}^{-\,\dot{\alpha}\dot{a}}_{-\frac{1}{2}}$ transform in the $(\mathbf{2}_R;\mathbf{2}_R) = \left[\left(0,\frac{1}{2}\right);\left(0,\frac{1}{2}\right)\right]$ representation and have respectively $(J=\frac{1}{2}, \Delta=\frac{3}{2})$ and $(J=-\frac{1}{2}, \Delta=\frac{3}{2})$. Some aspects of instanton contributions to BMN operators involving fermionic impurities will be discussed in [25].

The field strength, $F_{\mu\nu}$, is a singlet with respect to the first $\mathrm{SO}(4)$ and decomposes into $F^{\pm}_{\mu\nu}$ transforming in the $\mathbf{3}^- = (1,0)$ and $\mathbf{3}^+ = (0,1)$ with respect to the second. $F^{\pm}_{\mu\nu}$ both have $J=0$ and $\Delta = 2$.

In the string amplitudes in the plane-wave background P_+ and P_- are conserved, so the operators in the gauge theory are conveniently classified according to the dual quantities, i.e. $\Delta - J$ and $\Delta + J$ respectively ($\Delta + J$ is actually infinite in the limit; it is proportional to P_-, but the proportionality constant diverges). Table 1 summarises the Δ, J and $\mathrm{SO}(4) \times \mathrm{SO}(4)$ quantum numbers for the $\mathcal{N}=4$ elementary fields. The notation $\mathrm{SO}(4)_R$ and $\mathrm{SO}(4)_C$ has been introduced to denote the $\mathrm{SO}(4)$ groups descending from the original $\mathrm{SO}(6)$ (R-symmetry) and $\mathrm{SO}(2,4)$ (conformal) groups respectively.

2.2 BMN operators

The composite operators dual to states in the spectrum of string theory in the plane wave background are also classified in terms of the same quantum numbers. In particular, $\Delta - J$, which is dual to the light-cone hamiltonian, measures the number of "impurities" and will be used to classify the operators

At finite J and Δ the selection rules of the $\mathscr{N}=4$ theory, implied by the superconformal symmetry, apply. So only two-point functions of (primary) operators of the same dimension can be non-zero. More precisely the two-point functions that are relevant for the calculation of anomalous dimensions are

$$\langle \bar{\mathscr{O}}^{i}_{\bar{\mathbf{r}}_i,\Delta_i}(x)\mathscr{O}^{j}_{\mathbf{r}_j,\Delta_j}(y)\rangle\,, \tag{2.11}$$

where the subscripts denote the SU(4) representation and the scaling dimension. The SU(2, 2|4) symmetry imposes $\Delta_i = \Delta_j$ and $\mathbf{r}_i = \mathbf{r}_j$ so that both Δ and J are conserved in two-point functions. In the case of the U(1) charge J this means that the two operators in a non-zero two-point function must have equal and opposite charges.

Gauge invariant composite operators which correspond to physical string states are of the form

$$\begin{aligned}
\mathscr{O}^{i_1\ldots i_k}_{J;n_1\ldots n_k} &= \frac{1}{\sqrt{J^{\Delta-J+1}\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+k}}} \sum_{\substack{p_1,\ldots,p_k=0\\ p_1\le p_2\le\cdots\le p_k}}^{J} \mathrm{e}^{2\pi i(p_1n_1+p_2n_2+\cdots+p_kn_k)/J}\times\\
&\qquad\times \mathrm{Tr}\left(Z^{p_1}X_{\ell_1}Z^{p_2-p_1}X_{\ell_2}\cdots Z^{p_k-p_{k-1}}X_{\ell_k}Z^{J-p_k}\right)\\
&= \frac{1}{\sqrt{J^{\Delta-J-1}\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+k}}} \sum_{\substack{q_2,\ldots,q_k=0\\ q_2+\cdots+q_k\le J}}^{J} \mathrm{e}^{2\pi i[(n_2+\cdots+n_k)q_2+(n_3+\cdots+n_k)q_3+\cdots+n_kq_k]/J}\times\\
&\qquad\times \mathrm{Tr}\left(Z^{J-(q_2+\cdots+q_k)}X_{\ell_1}Z^{q_2}X_{\ell_2}\cdots Z^{q_k}X_{\ell_k}\right)\,,
\end{aligned} \tag{2.12}$$

where $\Delta - J$ denotes the total number of impurities. Here the cyclicity of the trace has been used and, after the change of variables, $p_1 \to q_1$, $p_i \to q_i - q_{i-1}$ $(i = 2, \ldots, k)$, the sum over q_1 has been performed resulting in the condition

$$n_1 = -(n_2 + \cdots + n_k)\,. \tag{2.13}$$

In (2.12) the X_ℓ's denote generic impurities, i.e. any of the elementary fields discussed in the previous subsection. String states dual to these operators are created acting on the vacuum with bosonic and fermionic oscillators. The integers $n_1, \ldots, n_k$ in (2.12) are identified with the mode numbers in the dual string state and the relation (2.13) corresponds to the level matching condition obeyed by the physical string states.

String creation operators are in one to one correspondence with $\Delta - J = 1$ impurities, see table 1. Bosonic oscillators α^i_{-n} and α^μ_{-n} in the $(\mathbf{4};\mathbf{1})$ and $(\mathbf{1};\mathbf{4})$ of SO(4) $\times$ SO(4) correspond to the insertion of φ^i and $D^\mu Z$ impurities respectively [1]. Fermionic oscillators, S^+_{-n} and S^-_{-n}, in the $(\mathbf{2}_L;\mathbf{2}_L)$ and $(\mathbf{2}_R;\mathbf{2}_R)$ correspond to the insertion of $\psi^{-\,a}_{+\frac{1}{2},\alpha}$ and $\psi^{+\,\dot{a}}_{+\frac{1}{2},\dot{\alpha}}$ impurities respectively. In string theory for each type of excitation one must consider left-

[1]Here we are using a different notation with respect to [14], where the oscillators α^μ_{-n} were denoted by $\alpha^{i'}_{-n}$.

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and right-moving oscillators. These correspond to the insertion of the same field, but with the associated n_i in the phase factor in (2.12) being respectively positive or negative.

The normalisation of operators of the form (2.12) involving only $\Delta - J = 1$ impurities is such that their tree level two-point functions are of order 1 in the BMN limit, $N \to \infty$, $J \to \infty$ with J^2/N finite. Operators involving $\Delta - J = 2$ impurities have vanishing two-point functions at tree level because for equal total $\Delta - J$ they are normalised by the same prefactor but their definition involves fewer sums. Therefore these operators do not correspond to independent degrees of freedom in the BMN limit. In some cases, however, the insertion of $\Delta - J = 2$ impurities is necessary in order to construct combinations which are well behaved in the double limit $N \to \infty$, $J \to \infty$ at higher orders in perturbation theory. For instance it is necessary to consider terms in which pairs of φ^i impurities are replaced by a $Z\bar{Z}$ insertion in order to cancel divergences which arise at the level of the leading non planar perturbative corrections [9].

Operators with $\Delta - J = 0, 1$ are protected and so their two-point functions do not receive instanton contributions. At the level of two and more impurities the situation is more interesting. The spectrum is significantly richer and more importantly there appear unprotected operators. In the following we shall only discuss a small selection of gauge invariant composite operators with scalar impurities which are dual to the string states analysed in [14]. A complete analysis would require computing the two-point functions involving all the operators in each sector and then diagonalising the resulting matrix of anomalous dimensions. We shall not carry out this program in this paper, but we shall concentrate on a few specific cases which illustrate the striking agreement with the corrections to the string mass spectrum calculated in [14]. The generic operator with k scalar impurities is of the form

$$\mathscr{O}^{i_1\ldots i_k}_{J;n_1\ldots n_k} = \frac{1}{\sqrt{J^{k-1}\left(\frac{g_{\rm YM}^2 N}{8\pi^2}\right)^{J+k}}} \sum_{\substack{p_1,\ldots,p_{k-1}=0\\ p_1+\cdots+p_{k-1}\le J}}^{J} \mathrm{e}^{2\pi i[(n_1+\cdots+n_{k-1})p_1+(n_2+\cdots+n_{k-1})p_2+\cdots+n_{k-1}p_{k-1}]/J}$$
$$\times \operatorname{Tr}\left(Z^{J-(p_1+\cdots+p_{k-1})}\varphi^{i_1}Z^{p_1}\varphi^{i_2}\cdots Z^{p_{k-1}}\varphi^{i_k}\right), \quad (2.14)$$

In our discussion we shall only consider operators with an even number of impurities. This is because operators with odd $\Delta - J$, which receive perturbative corrections [26], are expected not to receive contributions in the one-instanton sector. This is a prediction following from the calculation of string amplitudes in [14]. In the plane-wave string theory the absence of instanton contributions to two-point amplitudes of states with an odd number of non-zero mode excitations is a straightforward consequence of the structure of the D-instanton boundary state. In the $\mathscr{N}=4$ theory, however, the corresponding statement is far from obvious.

2.2.1 Two impurity operators

No field in the $\mathscr{N}=4$ multiplet has negative $\Delta - J$ hence the two impurity operators are obtained with the insertion of either two $\Delta - J = 1$ fields or of a single field with

$\Delta - J = 2$. Because of the normalisation (2.12) only operators with two $\Delta - J = 1$ insertions are relevant in the BMN limit.

Even restricting the attention to $\mathrm{SO}(4)_C$ singlets, already at the two impurity level there is a rather rich spectrum of operators, which becomes even richer when multi-trace operators with the same quantum numbers are taken into account. In the $\mathrm{SO}(4)_R$ singlet sector one can construct gauge invariant operators in the representations $(0,0) \equiv \mathbf{1}$, $(1,0) \equiv \mathbf{3}^+$, $(0,1) \equiv \mathbf{3}^-$ and $(1,1) \equiv \mathbf{9}$ of $\mathrm{SO}(4)_R$. The singlet can be realised with the insertion of two scalars, two gauge fields (through covariant derivatives) or two fermions of the same chirality. Operators in the $\mathbf{3}^\pm$ include combinations of two scalar or two fermionic impurities. The $\mathbf{9}$ can only be obtained with the insertion of two scalar impurities.

The operators with two φ^i insertions are[2]

$$\mathscr{O}_{\mathbf{1};J;n} = \frac{1}{\sqrt{J\left(\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\right)^{J+2}}} \left[\sum_{p=0}^{J} \mathrm{e}^{2\pi i p n/J}\ \mathrm{Tr}\left(Z^{J-p}\varphi^i Z^p \varphi^i\right) - \mathrm{Tr}\left(Z^{J+1}\bar{Z}\right)\right] \tag{2.15}$$

$$\mathscr{O}^{[ij]}_{\mathbf{3}^\pm;J;n} = \frac{1}{\sqrt{J\left(\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\right)^{J+2}}} \sum_{p=0}^{J} \mathrm{e}^{2\pi i p n/J}\ \varGamma_\pm^{ijkl}\ \mathrm{Tr}\left(Z^{J-p}\varphi^{[k} Z^p \varphi^{l]}\right) \tag{2.16}$$

$$\begin{aligned}\mathscr{O}^{\{ij\}}_{\mathbf{9};J;n} &= \frac{1}{\sqrt{J\left(\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\right)^{J+2}}} \sum_{p=0}^{J} \mathrm{e}^{2\pi i p n/J}\ \mathrm{Tr}\left(Z^{J-p}\varphi^{\{i} Z^p \varphi^{j\}}\right) \qquad (2.17)\\ &\equiv \frac{1}{\sqrt{J\left(\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\right)^{J+2}}} \sum_{p=0}^{J} \mathrm{e}^{2\pi i p n/J} \left[\mathrm{Tr}\left(Z^{J-p}\varphi^{(i} Z^p \varphi^{j)}\right) - \frac{\delta^{ij}}{2}\ \mathrm{Tr}\left(Z^{J-p}\varphi^{k} Z^p \varphi^{k}\right)\right],\end{aligned}$$

where the projectors onto the $\mathbf{3}^+$ and $\mathbf{3}^-$ are defined as $\varGamma_\pm^{ijkl} = \frac{1}{4}\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} \pm \varepsilon^{ijkl}\right)$. The singlet operator (2.15) provides an example of what mentioned earlier about $\bar{Z}$ insertions. In order to define a well behaved BMN operator it is necessary to consider the combination in (2.15). The second term is needed to cancel a divergent contribution to the two-point function of $\mathscr{O}_{\mathbf{1};J;n}$ arising at the leading non planar level [9].

The operators (2.15)-(2.17) are dual to string states in the plane-wave background of the form

$$\alpha^i_{-n}\widetilde{\alpha}^i_{-n}|0\rangle_h \tag{2.18}$$

$$\varGamma_\pm^{ijkl}\alpha^k_{-n}\widetilde{\alpha}^l_{-n}|0\rangle_h \tag{2.19}$$

$$\alpha^{\{i}_{-n}\widetilde{\alpha}^{j\}}_{-n}|0\rangle_h\,, \tag{2.20}$$

where $|0\rangle_h$ denotes the BMN ground state and the indices $i, j, \ldots$ are taken to be in one of the two SO(4) factors (to be identified with $\mathrm{SO}(4)_R$). The integer n in (2.15)-(2.17)

[2]Here and in the following we use square brackets to denote antisymmetrisation, curly brackets to denote symmetrisation and subtraction of the trace and parentheses to indicate symmetrisation without subtraction of the trace part.

$(i)\ \ \varphi^i\varphi^j\varphi^k\varphi^l$	$(ii)\ \ t_{\mu_1\mu_2\mu_3\mu_4}D^{\mu_1}ZD^{\mu_2}ZD^{\mu_3}ZD^{\mu_4}Z$	$(iii)\ \ \varphi^i\varphi^j D_\mu ZD^\mu Z$
$(iv)\ \ \varphi^i\varphi^j\psi^{-\,\alpha a}\psi_\alpha^{-\,b}$	$(v)\ \ \varphi^i\varphi^j\psi^+_{\dot\alpha\dot a}\psi_{\dot b}^{+\,\dot\alpha}$	$(vi)\ \ D_\mu ZD^\mu Z\psi^{-\,\alpha a}\psi_\alpha^{-\,b}$
$(vii)\ \ D_\mu ZD^\mu Z\psi^+_{\dot\alpha\dot a}\psi_{\dot b}^{+\,\dot\alpha}$	$(viii)\ \ \varphi^i D_\mu Z\psi_\alpha^{-\,a}\psi_{\dot a}^{+\,\dot\alpha}$	$(ix)\ \ \psi^{-\,\alpha a}\psi_\alpha^{-\,b}\psi^{-\,\beta c}\psi_\beta^{-\,d}$
$(x)\ \ \psi^+_{\dot\alpha\dot a}\psi_{\dot b}^{+\,\dot\alpha}\psi^+_{\dot\beta\dot c}\psi_{\dot d}^{+\,\dot\beta}$	$(xi)\ \ \psi^{-\,\alpha a}\psi_\alpha^{-\,b}\psi^+_{\dot\alpha\dot a}\psi_{\dot b}^{+\,\dot\alpha}$	

Table 2: $\Delta - J = 4$ combinations of impurities

corresponds to the level of the dual string excitation. The $n = 0$ operators are protected and correspond to supergravity states.

As already remarked, in order to compute the instanton induced anomalous dimensions of the various operators in each sector it is in principle necessary to diagonalise the appropriate matrix. In the following we shall not consider the problem of mixing between single- and multi-trace operators, since it is a subleading effect in the large N limit. In general, however, at the instanton level mixing occurs at leading order among all the single trace operators in each sector [27]. In the case of the two impurity operators it has been shown [28] that all the operators in different sectors have the same anomalous dimension as a consequence of superconformal invariance. Therefore in the following we shall only analyse the single operator in the **9** for which there is no mixing to resolve. Superconformal symmetry guarantees that the results apply to operators in others sectors as well.

In general the problem of resolving the operator mixing and computing anomalous dimensions can be drastically simplified using the constraints imposed by superconformal invariance, in particular, the fact that all the operators in a multiplet have the same anomalous dimension as the superconformal primary operator.

2.2.2 Four impurity operators

The number of independent BMN operators grows very rapidly with the number of impurities and at the four impurity level the spectrum is already extremely rich. Bosonic operators in the $\mathrm{SO}(4)_C$ singlet sector exist in the following representations of $\mathrm{SO}(4)_R$

$$\begin{aligned} &(0,0)=\mathbf{1}\,, \quad (0,1)=\mathbf{3}^+\,, \quad (1,0)=\mathbf{3}^-\,, \quad (0,2)=\mathbf{5}^+\,, \quad (2,0)=\mathbf{5}^-\,, \\ &(1,1)=\mathbf{9}\,, \quad (1,2)=\mathbf{15}^+\,, \quad (2,1)=\mathbf{15}^-\,, \quad (2,2)=\mathbf{25}\,. \end{aligned} \tag{2.21}$$

Operators relevant in the BMN limit involve four $\Delta - J = 1$ impurities. The combinations which contribute to $\mathrm{SO}(4)_C$ scalars are listed (up to permutations of the four fields) in table 2 [3].

The $\mathrm{SO}(4)_C$ singlet sector contains the largest number of operators, involving all the combinations (i)-(xi) in table 2. Operators in the $\mathbf{3}^+$ can contain $(i), (ii), (iv)$-$(vi), (viii)$ and (ix) and those in the $\mathbf{3}^-$ $(i), (ii), (iv), (vii), (viii)$ and (x). The **9** involves $(i), (ii), (iv)$, (v) and $(viii)$. Operators in the $\mathbf{5}^+$ and $\mathbf{5}^-$ can be obtained from $(i), (iv)$ and (ix) and

[3] $t_{\mu_1\mu_2\mu_3\mu_4}$ is a projector onto the singlet, i.e. $\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}$, $\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}$ or $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$.

from $(i), (v)$ and (x) respectively, those in the $\mathbf{15}^+$ and $\mathbf{15}^-$ from (i) and (iv) and from (i) and (v) respectively. In the $\mathbf{25}$ there is only one operator corresponding to the combination (i) with indices fully symmetrised.

In the following we shall concentrate on a few specific two-point functions corresponding to the amplitudes computed in [14]. This will be sufficient to show how instanton contributions to gauge theory correlation functions precisely reproduce various features observed in string theory amplitudes. The operators we study in detail are those containing four scalar impurities. These are of the form

$$\mathscr{O}_{\mathbf{r};J;n_1,n_2,n_3} = \frac{t^{\mathbf{r}}_{ijkl}}{\sqrt{J^3\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+4}}} \sum_{\substack{p_1,p_2,p_3=0\\ p_1+p_2+p_3\leq J}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)p_1+(n_2+n_3)p_2+n_3p_3]/J} \times {\rm Tr}\left(Z^{J-(p_1+p_2+p_3)}\varphi^i Z^{p_1}\varphi^j Z^{p_2}\varphi^k Z^{p_3}\varphi^l\right), \quad (2.22)$$

where $t^{\mathbf{r}}_{ijkl}$ is a projector onto the representation $\mathbf{r}$ of ${\rm SO}(4)_R$. In particular in the singlet sector there are three independent operators in this class, corresponding to the three tensors

$$t^{(1)}_{ijkl} = \varepsilon_{ijkl}\,, \qquad t^{(2)}_{ijkl} = \delta_{ij}\delta_{kl}\,, \qquad t^{(3)}_{ijkl} = \delta_{ik}\delta_{jl}\,. \quad (2.23)$$

In section 5.2 we study instanton contributions to two-point functions of operators of the type (2.22). We discuss in detail the case of the singlet corresponding the choice of the $t^{(1)}_{ijkl}$ projector. We will show that the dependence on both the coupling constants, λ' and g_2, and the mode numbers, n_i, is in exact agreement with the results of [14]. We also find that for all the operators with four scalar impurities in sectors other than the singlet instanton contributions to the matrix of anomalous dimensions are suppressed by powers of λ'. This result is also in agreement with the string prediction.

3. Instanton contributions to two-point functions

In this section we recall some general aspects of the calculation of instanton contributions to correlation functions, in particular two-point functions, in $\mathscr{N}=4$ SYM.

In the semi-classical limit correlation functions are evaluated using a saddle point approximation around the classical instanton configuration. In this limit the computation of expectation values reduces to an integration over the finite dimensional instanton moduli space as parametrised in the ADHM construction [29]. Before focusing on operators of large dimension, Δ, and R-charge, J, in the following sections, we briefly discuss the general formalism for extracting instanton contributions to the anomalous dimensions of gauge invariant local operators. Comprehensive reviews of instanton calculus in supersymmetric gauge theories can be found in [30, 24] and instanton contributions to anomalous dimensions of scalar operators in $\mathscr{N}=4$ SYM were studied in detail in [27].

Contributions to the (matrix of) anomalous dimensions are extracted from the logarithmically divergent terms in two-point functions. In the semi-classical approximation in the background of an instanton the two-point function of a generic local operator, $\mathscr{O}(x)$,

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and its conjugate takes the form

$$\langle\bar{\mathscr{O}}(x_1)\mathscr{O}(x_2)\rangle_{\rm inst} = \int {\rm d}\mu_{\rm inst}(m_{\rm b}, m_{\rm f})\, {\rm e}^{-S_{\rm inst}}\, \hat{\bar{\mathscr{O}}}(x_1; m_{\rm b}, m_{\rm f})\hat{\mathscr{O}}(x_2; m_{\rm b}, m_{\rm f})\,, \tag{3.1}$$

where we have denoted the bosonic and fermionic collective coordinates by $m_{\rm b}$ and $m_{\rm f}$ respectively. In (3.1) ${\rm d}\mu_{\rm inst}(m_{\rm b}, m_{\rm f})$ is the integration measure on the instanton moduli space, $S_{\rm inst}$ is the classical action evaluated on the instanton solution and $\hat{\mathscr{O}}$ and$\hat{\bar{\mathscr{O}}}$ denote the classical expressions for the operators $\mathscr{O}$ and $\bar{\mathscr{O}}$ computed in the instanton background.

A one-instanton configuration in SU(N) Yang-Mills theory is characterised by $4N$ bosonic collective coordinates. With a particular choice of parametrisation these bosonic moduli can be identified with the size, ρ, and position, x_0, of the instanton as well as its global gauge orientation. The latter can be described by three angles identifying the iso-orientation of a SU(2) instanton and $4N$ additional constrained variables, $w_{u\dot\alpha}$ and $\bar{w}^{\dot\alpha u}$ (where $u = 1, \ldots, N$ is a colour index), in the coset SU(N)/(SU($N-2$) × U(1)) describing the embedding of the SU(2) configuration into SU(N). In the one-instanton sector in the $\mathscr{N}$=4 theory there are additionally $8N$ fermionic collective coordinates corresponding to zero modes of the Dirac operator in the background of an instanton. They comprise the 16 moduli associated with Poincaré and special supersymmetries broken by the instanton and denoted respectively by η^A_α and $\bar\xi^{\dot\alpha A}$ (where A is an index in the fundamental of the SU(4) R-symmetry group) and $8N$ additional parameters, ν^A_u and $\bar\nu^{Au}$, which can be considered as the fermionic superpartners of the gauge orientation parameters. The sixteen superconformal moduli are exact, i.e. they enter the expectation values (3.1) only through the classical profiles of the operators. The other fermion modes, ν^A_u and $\bar\nu^{Au}$, appear explicitly in the integration measure via the classical action, $S_{\rm inst}$. This distinction plays a crucial rôle in the calculation of correlation functions. The ν^A_u and $\bar\nu^{Au}$ modes satisfy the fermionic ADHM constraints

$$\bar{w}^{\dot\alpha u}\nu^A_u = 0\,, \qquad \bar\nu^{Au}w_{u\dot\alpha} = 0\,, \tag{3.2}$$

which effectively reduce their number to $8(N-2)$.

In the one-instanton sector the gauge-invariant measure on the instanton moduli space takes the form

$$\begin{aligned}\int {\rm d}\mu_{\rm phys}\, {\rm e}^{-S_{\rm inst}} &= \\ &= \frac{\pi^{-4N} g_{\rm YM}^{4N} {\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int {\rm d}\rho\, {\rm d}^4x_0 \prod_{A=1}^{4} {\rm d}^2\eta^A {\rm d}^2\bar\xi^A\, {\rm d}^{N-2}\nu^A {\rm d}^{N-2}\bar\nu^A\, \rho^{4N-13}{\rm e}^{-S_{4F}}\,,\end{aligned} \tag{3.3}$$

where the instanton action is

$$S_{\rm inst} = -2\pi i\tau + S_{4F} = -2\pi i\tau + \frac{\pi^2}{2g_{\rm YM}^2\rho^2}\varepsilon_{ABCD}\mathscr{F}^{AB}\mathscr{F}^{CD} \tag{3.4}$$

with

$$\tau = \frac{4\pi i}{g_{\rm YM}^2} + \frac{\theta}{2\pi}\,, \qquad \mathscr{F}^{AB} = \frac{1}{2\sqrt{2}}(\bar\nu^{Au}\nu^B_u - \bar\nu^{Bu}\nu^A_u)\,. \tag{3.5}$$

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In (3.3) we have omitted an overall (N-independent) numerical constant that will be reinstated in the final expression.

The two-point function (3.1) thus becomes

$$\begin{aligned}\langle \bar{\mathscr{O}}(x_1)\mathscr{O}(x_2)\rangle &= \frac{\pi^{-4N} g_{\rm YM}^{4N} {\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\times \\ &\times \int {\rm d}\rho\, {\rm d}^4x_0 \prod_{A=1}^{4} {\rm d}^2\eta^A {\rm d}^2\bar{\xi}^A\, {\rm d}^{N-2}\nu^A {\rm d}^{N-2}\bar{\nu}^A\, \rho^{4N-13}\times \\ &\times {\rm e}^{\frac{\pi^2}{16 g_{\rm YM}^2\rho^2}\varepsilon_{ABCD}(\bar{\nu}^{[A}\nu^{B]})(\bar{\nu}^{[C}\nu^{D]})}\ \hat{\bar{\mathscr{O}}}\big(x_1;x_0,\rho,\eta,\bar{\xi},\nu,\bar{\nu}\big)\ \hat{\mathscr{O}}\big(x_2;x_0,\rho,\eta,\bar{\xi},\nu,\bar{\nu}\big)\,.\end{aligned}\tag{3.6}$$

Following [16] the integration over the non-exact fermion modes can be reduced to a gaussian form introducing auxiliary bosonic coordinates, χ^i, $i=1,\dots,6$, to rewrite the gauge invariant measure as

$$\begin{aligned}&\frac{\pi^{-4N} g_{\rm YM}^{4N} {\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int {\rm d}\rho\, {\rm d}^4x_0\, {\rm d}^6\chi \prod_{A=1}^{4} {\rm d}^2\eta^A {\rm d}^2\bar{\xi}^A\, {\rm d}^{N-2}\nu^A {\rm d}^{N-2}\bar{\nu}^A\times \\ &\qquad\times\rho^{4N-7}\exp\left[-2\rho^2\chi^i\chi^i + \frac{4\pi i}{g_{\rm YM}}\chi_{AB}\mathscr{F}^{AB}\right],\end{aligned}\tag{3.7}$$

where $\chi_{AB}=\frac{1}{\sqrt{8}}\Sigma^i_{AB}\chi^i$ and the symbols Σ^i_{AB} are defined in (A.1).

The fermion modes ν^A_u and $\bar{\nu}^{Bu}$ enter explicitly in the classical profiles of the operators in the instanton background as well as in the measure through the instanton action. It is thus convenient to construct a generating function as in [17], which allows to deal easily with the otherwise complicated combinatorics associated with the integration over ν^A_u and $\bar{\nu}^{Au}$. We introduce sources, $\bar{\vartheta}^u_A$ and ϑ_{Au}, coupled to ν^A_u and $\bar{\nu}^{Au}$ and define

$$\begin{aligned}Z[\vartheta,\bar{\vartheta}]=&\frac{\pi^{-4N} g_{\rm YM}^{4N} {\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int {\rm d}\rho\, {\rm d}^4x_0\, {\rm d}^6\chi \prod_{A=1}^{4} {\rm d}^2\eta^A\, {\rm d}^2\bar{\xi}^A\, {\rm d}^{N-2}\bar{\nu}^A\, {\rm d}^{N-2}\nu^A\times \\ &\times\rho^{4N-7}\exp\left[-2\rho^2\chi^i\chi^i + \frac{\sqrt{8}\pi i}{g_{\rm YM}}\bar{\nu}^{Au}\chi_{AB}\nu^B_u + \bar{\vartheta}^u_A\nu^A_u + \vartheta_{Au}\bar{\nu}^{Au}\right].\end{aligned}\tag{3.8}$$

Performing the gaussian integrals over $\bar{\nu}$ and ν and introducing polar coordinates,

$$\chi^i \to (r,\Omega)\,, \qquad \sum_{i=1}^{6}(\chi^i)^2 = r^2\,,\tag{3.9}$$

we find

$$\begin{aligned}Z[\vartheta,\bar{\vartheta}] &= \frac{2^{-29}\pi^{-13}\, g_{\rm YM}^{8}\, {\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\times \\ &\times\int {\rm d}\rho\, {\rm d}^4x_0\, {\rm d}^5\Omega \prod_{A=1}^{4} {\rm d}^2\eta^A\, {\rm d}^2\bar{\xi}^A\, \rho^{4N-7}\int_0^\infty {\rm d}r\, r^{4N-3}{\rm e}^{-2\rho^2 r^2}\mathscr{Z}(\vartheta,\bar{\vartheta};\Omega,r)\,,\end{aligned}\tag{3.10}$$

where all the numerical coefficients have been reinstated. In (3.10) we have introduced the density

$$\mathscr{Z}(\vartheta,\bar\vartheta;\Omega,r) = \exp\left[-\frac{ig_{\rm YM}}{\pi r}\,\bar\vartheta^u_A\Omega^{AB}\vartheta_{Bu}\right], \tag{3.11}$$

where the symplectic form Ω^{AB} is given by

$$\Omega^{AB} = \bar\Sigma^{AB}_i\Omega^i\,, \qquad \sum_{i=1}^{6}\left(\Omega^i\right)^2 = 1\,. \tag{3.12}$$

Gauge invariant operators depend on the ν^A_u and $\bar\nu^{Au}$ variables via colour singlet bilinears. These arise in symmetric or anti-symmetric combinations transforming respectively in the **10** and **6** dimensional representations of the SU(4) R-symmetry

$$(\bar\nu^A\nu^B)_{\mathbf{10}} \equiv \bar\nu^{u(A}\nu^{B)}_u = (\bar\nu^{Au}\nu^B_u + \bar\nu^{Bu}\nu^A_u)\,, \tag{3.13}$$

$$(\bar\nu^A\nu^B)_{\mathbf{6}} \equiv \bar\nu^{u[A}\nu^{B]}_u = (\bar\nu^{Au}\nu^B_u - \bar\nu^{Bu}\nu^A_u)\,. \tag{3.14}$$

Using the generating function defined in (3.10) the $\bar\nu^A\nu^B$ bilinears in the operators $\mathscr{O}$ and $\bar{\mathscr{O}}$ in (3.7) can be rewritten in terms of derivatives of $\mathscr{Z}(\vartheta,\bar\vartheta;\Omega,r)$ with respect to the sources, ϑ_A and $\bar\vartheta_B$. The result for a two-point function in which the operator insertions contain a total of p $(\bar\nu\nu)_{\mathbf{6}}$ and q $(\bar\nu\nu)_{\mathbf{10}}$ bilinears is of the form

$$\langle\bar{\mathscr{O}}(x_1)\mathscr{O}(x_2)\rangle = \frac{g^8_{\rm YM}\,{\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int {\rm d}\rho\,{\rm d}^4x_0\,{\rm d}^5\Omega\prod_{A=1}^4{\rm d}^2\eta^A\,{\rm d}^2\bar\xi^A\,\rho^{4N-7} \tag{3.15}$$

$$\int {\rm d}r\,r^{4N-3}{\rm e}^{-2\rho^2r^2}\;\left.\frac{\delta^{2p+2q}\mathscr{Z}[\vartheta,\bar\vartheta;\Omega,r]}{\delta\vartheta_{u_1[A_1}\delta\bar\vartheta^{u_1}_{B_1]}\delta\vartheta_{v_1(C_1}\delta\bar\vartheta^{v_1}_{D_1)}\cdots}\right|_{\vartheta=\bar\vartheta=0}\widetilde{\bar{\mathscr{O}}}\big(x_1;x_0,\rho,\eta,\bar\xi\big)\,\widetilde{\mathscr{O}}\big(x_2;x_0,\rho,\eta,\bar\xi\big)\,,$$

where $\widetilde{\mathscr{O}}$ and $\widetilde{\bar{\mathscr{O}}}$ contain the dependence on the exact moduli, η^A and $\bar\xi^A$, and on the bosonic collective coordinates after extracting the $\bar\nu^A\nu^B$ bilinears. Computing the r integral gives

$$\langle\bar{\mathscr{O}}(x_1)\mathscr{O}(x_2)\rangle\sim\alpha(p,q;N)\,g^{8+p+q}_{\rm YM}\,{\rm e}^{2\pi i\tau}\int{\rm d}\rho\,{\rm d}^4x_0\,{\rm d}^5\Omega\prod_{A=1}^4{\rm d}^2\eta^A\,{\rm d}^2\bar\xi^A\,\rho^{p+q-5}\,\times$$

$$\times\, f(\Omega)\,\widetilde{\bar{\mathscr{O}}}\big(x_1;\rho,x_0;\eta,\bar\xi\big)\,\widetilde{\mathscr{O}}\big(x_2;\rho,x_0;\eta,\bar\xi\big)\,, \tag{3.16}$$

where $f(\Omega)$ contains the dependence on the Ω^{AB} variables obtained from the derivatives of $\mathscr{Z}(\vartheta,\bar\vartheta;\Omega,r)$. The coefficient $\alpha(p,q;N)$ contains the N dependence and in the large-N limit we find

$$\begin{aligned}\alpha(p,q;N) &= \frac{2^{-2N+\frac12(p+q)}\,\pi^{-(p+q)}\,\Gamma\left(2N-1-\frac12(p+q)\right)}{(N-1)!(N-2)!}\left(N^{p+\frac{q}{2}}+O(N^{p+\frac{q}{2}-1})\right)\\ &= \frac{N^{\frac12(p+1)}}{4\pi^{p+q+\frac12}}\left(1+O(1/N)\right).\end{aligned} \tag{3.17}$$

From (3.16) and (3.17) it follows that the insertion of a $(\bar\nu\nu)_{\mathbf{10}}$ or $(\bar\nu\nu)_{\mathbf{6}}$ bilinear in a correlation function produces a factor of $g_{\rm YM}$ or $g_{\rm YM}\sqrt{N}$ respectively.

In computing the moduli space integrations in expressions for two-point functions of the type (3.16) it will prove convenient to calculate first the fermionic integrals over η^A and $\bar{\xi}^A$ and the angular integration over the five-sphere. These give rise to selection rules that determine which operators receive instanton contributions to their scaling dimensions. In particular since the superconformal modes are exact a correlation function can only receive instanton contribution if the operator expressions contain exactly sixteen such modes in the combination

$$\prod_{A=1}^{4} \left(\eta^{\alpha A}\eta^{A}_{\alpha}\right)\left(\bar{\xi}^{A}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}A}\right). \tag{3.18}$$

The integration over the five-sphere parametrised by the angular variables Ω^{AB} factorises and gives rise to further selection rules. It gives a non-vanishing result only if the SU(4) indices carried by the Ω's in the two operators can be combined to form a SU(4) singlet. The SU(4) indices are originally carried by the fermion modes which are all in the **4**, so the only possible singlet combinations correspond to products of ε^{ABCD} tensors. The generic five-sphere integral is of the form

$$\int \mathrm{d}^5\Omega\, \Omega^{A_1B_1}\ldots\Omega^{A_{2n}B_{2n}} = c(n)\left(\varepsilon^{A_1B_1A_2B_2}\ldots\varepsilon^{A_{2n-1}B_{2n-1}A_{2n}B_{2n}} + \text{permutations}\right), \tag{3.19}$$

where the normalisation constant $c(n)$ is

$$c(n) = \frac{\pi^{5/2}\,\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(n+4)}. \tag{3.20}$$

Equations (3.18) and (3.19) imply that a two-point function can receive a non-zero contribution only if the combined profiles of the two operators contain fermion modes of the four flavours with the same multiplicity.

The bosonic integrations over the position and size of the instanton are left as a last step. In the case of two-point functions these integrals are logarithmically divergent, signalling a contribution to the matrix of anomalous dimensions.

4. Fermion zero modes

In order to evaluate instanton induced correlation functions we need to integrate the classical profiles of the relevant composite operators over the instanton moduli space. We are interested in the dependence on the collective coordinates and of particular relevance will be the way the fermionic modes enter into the expressions for the various fields. The zero-mode dependence in the elementary fields of the $\mathcal{N}=4$ SYM multiplet was reviewed in detail in [27]. Here we briefly summarise the features which will be relevant for the analysis of two-point functions of BMN operators.

The field equations of the $\mathcal{N}=4$ SYM theory admit a solution in which the gauge potential corresponds to a standard instanton of SU(N) pure Yang-Mills theory and all the other fields vanish,

$$A_\mu = A^I_\mu\,, \qquad \varphi^{AB} = \lambda^A_\alpha = \bar{\lambda}^{\dot{\alpha}}_A = 0\,. \tag{4.1}$$

However, the Dirac operator has zero modes in the background of this solution, i.e. the equation $\bar{\not{D}}_{\dot{\alpha}\alpha}\lambda^{\alpha A} = 0$ has non-trivial solutions when the covariant derivative is evaluated in the background of an instanton. The general solution to the Dirac equation is linear in the instanton fermion zero modes. This non-trivial solution gives rise to a non-zero solution for the scalar fields when plugged into the corresponding equation, $D^2\varphi^{AB} = \sqrt{2}[\lambda^A, \lambda^B]$. The latter admits a solution for the scalar which is bilinear in the fermion modes. Proceeding with this iterative solution of the field equations one generates a complete supermultiplet and further iterations give rise to additional terms with more fermion modes in each field. The general solution obtained through this procedure is schematically of the form

$$\begin{aligned} A_\mu &= \sum_{\substack{n=0\\4n\le 8N}} A_\mu^{(4n)}\,, & \varphi^{AB} &= \sum_{\substack{n=0\\4n+2\le 8N}} \varphi^{(2+4n)AB} \\ \lambda^A_\alpha &= \sum_{\substack{n=0\\4n+1\le 8N}} \lambda_\alpha^{(1+4n)A}\,, & \bar{\lambda}_{\dot{\alpha}A} &= \sum_{\substack{n=0\\4n+3\le 8N}} \bar{\lambda}^{(3+4n)}_{\dot{\alpha}A}\,, \end{aligned} \tag{4.2}$$

where the notation $\Phi^{(n)}$ is used to denote a term in the solution for the field Φ containing n fermion zero modes. It is also understood that in (4.2) the number of superconformal modes in each field does not exceed 16 and the remaining modes are of ν^A_u and $\bar{\nu}^{Au}$ type.

In computing the expressions for gauge invariant composite operators we shall make use of the ADHM description in which the elementary fields are written as $[N+2]\times[N+2]$ matrices. In particular, the leading order term in the solution for the scalars φ^{AB} is given explicitly in appendix A. The solution of the iterative equations becomes very involved after a few steps. However the flavour structure of the combination of fermion zero modes in each term can be determined without actually solving the equations and is sufficient to identify which operators can get an instanton correction to their scaling dimension.

All the fermion zero modes, both the superconformal ones, η^A_α and $\bar{\xi}^{\dot{\alpha}A}$, and the modes of type ν^A_u and $\bar{\nu}^{Au}$, transform in the **4** of SU(4). We shall denote a generic fermion mode by ${}_{m}\mathrm{f}^{A}$. The starting point for the construction of the instanton supermultiplet is the classical instanton, $A^{(0)}_\mu$, which has no fermions. The first term in λ^A_α is linear in the fermion modes

$$\lambda^{(1)A}_\alpha \sim {}_{m}\mathrm{f}^{A}\,. \tag{4.3}$$

For the term $\varphi^{(2)AB}$ in the scalar solution one finds

$$\varphi^{(2)AB} \sim {}_{m}\mathrm{f}^{[A}\,{}_{m}\mathrm{f}^{B]}\,, \tag{4.4}$$

i.e. the two fermion modes are antisymmetrised in order to form a combination in the **6**. The $\overline{\mathbf{4}}$ spinor $\bar{\lambda}^{(3)\dot{\alpha}}_A$ contains three fermion modes in the combination

$$\bar{\lambda}^{(3)\dot{\alpha}}_A \sim \varepsilon_{ABCD}\,{}_{m}\mathrm{f}^{B}\,{}_{m}\mathrm{f}^{C}\,{}_{m}\mathrm{f}^{D}\,, \tag{4.5}$$

so that the component $\lambda^{(3)}_A$ has one mode of each flavour apart from A. Proceeding in the multiplet we find the quartic term in the solution for the vector, $A^{(4)}_\mu$, which contains one fermion mode of each flavour in a singlet combination

$$A^{(4)}_\mu \sim \varepsilon_{ABCD}\,{}_{m}\mathrm{f}^{A}\,{}_{m}\mathrm{f}^{B}\,{}_{m}\mathrm{f}^{C}\,{}_{m}\mathrm{f}^{D}\,. \tag{4.6}$$

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The following term is $\lambda^{(5)A}_{\alpha}$, which has flavour structure

$$\lambda^{(5)A}_{\alpha} \sim \varepsilon_{BCDE}\, {}_{m}\mathrm{f}^{A}\, {}_{m}\mathrm{f}^{B}\, {}_{m}\mathrm{f}^{C}\, {}_{m}\mathrm{f}^{D}\, {}_{m}\mathrm{f}^{E}\,, \tag{4.7}$$

i.e. it involves a mode of flavour A plus one of each flavour. Then we find $\varphi^{(6)AB}$ that contains an antisymmetric combination of a mode of flavour A and one of flavour B plus one mode of each flavour antisymmetrised in a singlet,

$$\varphi^{(6)AB} \sim \varepsilon_{CDEF}\, {}_{m}\mathrm{f}^{[A}\, {}_{m}\mathrm{f}^{B]}\, {}_{m}\mathrm{f}^{C}\, {}_{m}\mathrm{f}^{D}\, {}_{m}\mathrm{f}^{E}\, {}_{m}\mathrm{f}^{F}\,. \tag{4.8}$$

The previous expressions are symbolic and the products of ${}_{m}\mathrm{f}$'s in (4.4)-(4.8) correspond to different combinations of the modes η^{A}_{α}, $\bar{\xi}^{\dot{\alpha}A}$, ν^{A}_{u} and $\bar{\nu}^{Au}$ in the various entries of the ADHM matrix for each field. The structure of the terms with more fermion modes in the multiplet can be determined analogously. The iterative solution of the field equation to construct the first few terms in the multiplet was carried out explicitly in [31].

From the above equations we can deduce the form of the component fields in the decomposition relevant for the BMN limit. For the scalars in (2.4) we have

$$\begin{aligned} &Z^{(2)} \sim {}_{m}\mathrm{f}^{[1}\, {}_{m}\mathrm{f}^{4]}\,, && \bar{Z}^{(2)} \sim {}_{m}\mathrm{f}^{[2}\, {}_{m}\mathrm{f}^{3]} \\ &\varphi^{(2)\,1,3} \sim {}_{m}\mathrm{f}^{[1}\, {}_{m}\mathrm{f}^{3]} + {}_{m}\mathrm{f}^{[2}\, {}_{m}\mathrm{f}^{4]}\,, && \varphi^{(2)\,2,4} \sim {}_{m}\mathrm{f}^{[1}\, {}_{m}\mathrm{f}^{2]} + {}_{m}\mathrm{f}^{[3}\, {}_{m}\mathrm{f}^{4]}\,. \end{aligned} \tag{4.9}$$

For the fermions in (2.7) and (2.9) we have respectively

$$\psi^{-(1)\,a} \sim {}_{m}\mathrm{f}^{a}\,, \qquad \bar{\psi}^{+(1)}_{a} \sim \left(M^{+}\, {}_{m}\mathrm{f}\right)_{a}\,, \qquad a = 1,4 \;\; \dot{a} = 2,3 \tag{4.10}$$

and

$$\psi^{+(3)}_{\dot{a}} \sim \varepsilon_{\dot{r}BCD}\, {}_{m}\mathrm{f}^{B}\, {}_{m}\mathrm{f}^{C}\, {}_{m}\mathrm{f}^{D}\,, \qquad \bar{\psi}^{-(3)\,\dot{a}} \sim \left(M^{-}\varepsilon\right)^{\dot{a}}{}_{BCD}\, {}_{m}\mathrm{f}^{B}\, {}_{m}\mathrm{f}^{C}\, {}_{m}\mathrm{f}^{D}\,, \qquad a = 1,4\,,\; \dot{a} = 2,3\,, \tag{4.11}$$

so that

$$\psi^{-(1)\,1} \sim {}_{m}\mathrm{f}^{1}\,, \qquad \psi^{-(1)\,4} \sim {}_{m}\mathrm{f}^{4}\,, \qquad \bar{\psi}^{+(1)\,1} \sim {}_{m}\mathrm{f}^{2}\,, \qquad \bar{\psi}^{+(1)\,4} \sim {}_{m}\mathrm{f}^{3}\,, \tag{4.12}$$

$$\psi^{+(3)\,2} \sim {}_{m}\mathrm{f}^{1}\, {}_{m}\mathrm{f}^{3}\, {}_{m}\mathrm{f}^{4}\,, \quad \psi^{+(3)\,3} \sim {}_{m}\mathrm{f}^{1}\, {}_{m}\mathrm{f}^{2}\, {}_{m}\mathrm{f}^{4}\,, \quad \bar{\psi}^{-(3)\,2} \sim {}_{m}\mathrm{f}^{2}\, {}_{m}\mathrm{f}^{3}\, {}_{m}\mathrm{f}^{4}\,, \quad \bar{\psi}^{-(3)\,3} \sim {}_{m}\mathrm{f}^{1}\, {}_{m}\mathrm{f}^{2}\, {}_{m}\mathrm{f}^{3}\,. \tag{4.13}$$

The terms of higher order are easily deduced from the previous general discussion. Notice that we can assign U(1) charge $+\frac{1}{2}$ to the fermion modes ${}_{m}\mathrm{f}^{1}$ and ${}_{m}\mathrm{f}^{4}$ and charge $-\frac{1}{2}$ to the modes ${}_{m}\mathrm{f}^{2}$ and ${}_{m}\mathrm{f}^{3}$.

The dependence on the superconformal modes, η^{A}_{α} and $\bar{\xi}^{\dot{\alpha}A}$, can be obtained using supersymmetry without solving the field equations. These modes are associated with superconformal symmetries broken in the instanton background. The corresponding terms in the $\mathscr{N}$=4 supermultiplet can thus be generated acting with the broken Poincaré and special supersymmetries, $Q_{\alpha A}$ and $\bar{S}_{\dot{\alpha}A}$, on the classical instanton solution for the gauge potential. In the case of SU(2) gauge group there are no additional fermion modes and the complete solution can be obtained in this way. In general, however, the dependence on the ν^{A}_{u} and $\bar{\nu}^{Au}$ modes can be determined only by solving the equations of motion.

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It is useful to discuss the derivation of the dependence on the superconformal modes using supersymmetry since it allows us to clarify how different combinations of fermion modes appear in various operators.

Substituting $A_\mu^{(0)} \equiv A_\mu^I$ in the supersymmetry transformation of λ_α^A gives $\lambda_\alpha^{(1)A}$, which is linear in η_α^A and $\bar\xi^{\dot\alpha A}$ and solves the corresponding field equation. Replacing λ_α^A by $\lambda^{(1)A}$ in the variation of φ^{AB} generates the solution $\varphi^{(2)AB}$ for the scalar. The iteration of this procedure gives rise to $\bar\lambda_{\dot\alpha A}^{(3)}$, then to the correction $A_\mu^{(4)}$ to the gauge field and so on.

In the examples studied in [15 − 17, 27] the superconformal modes always appear in the expressions of gauge-invariant composite operators in the combination

$$\zeta_\alpha^A(x) = \frac{1}{\sqrt{\rho}} \left[\rho\, \eta_\alpha^A - (x-x_0)_\mu \sigma^\mu_{\alpha\dot\alpha}\, \bar\xi^{\dot\alpha A}\right] . \tag{4.14}$$

In general, however, the dependence on the fermion superconformal modes, even in gauge-invariant operators, is not only through this combination and instead the moduli η_α^A and $\bar\xi^{\dot\alpha A}$ appear explicitly. This can be understood analysing the form of the Poincaré and special supersymmetry variations of the fields. Under a combination of the broken supersymmetries, $\eta^{\alpha A} Q_{\alpha A} + \bar\xi_{\dot\alpha}^A \bar S_A^{\dot\alpha}$, we have

$$\delta A_\mu = (\eta^A + \sigma\!\cdot\! x\, \bar\xi^A)\sigma_\mu \bar\lambda_A \tag{4.15}$$

$$\delta\lambda^A = F_{\mu\nu}\sigma^{\mu\nu}(\eta^A + \sigma\!\cdot\! x\, \bar\xi^A) + [\varphi^{AB}, \bar\varphi_{BC}](\eta^C + \sigma\!\cdot\! x\, \bar\xi^C) \tag{4.16}$$

$$\delta\varphi^{AB} = \lambda^A(\eta^B + \sigma\!\cdot\! x\, \bar\xi^B) - (A \leftrightarrow B) \tag{4.17}$$

$$\delta\bar\lambda_A = \not{D}\bar\varphi_{AB}(\eta^B + \sigma\!\cdot\! x\, \bar\xi^B) + \bar\varphi_{AB}\bar\xi^B\,, \tag{4.18}$$

which shows that, whereas the variations of A_μ, λ_α^A and φ^{AB} involve the combination ζ_α^A, the superconformal variation of $\bar\lambda_A^{\dot\alpha}$ contains an extra term. Therefore the profiles of operators involving $\bar\lambda_A^{\dot\alpha}$ in general depend separately on η_α^A and $\bar\xi_{\dot\alpha}^A$. Since further application of the broken supersymmetries generates new terms in the solution for the elementary fields, it follows that not only operators containing $\bar\lambda_{\dot\alpha A}$, but also those in which any elementary field contain a non-minimal number of fermion modes (e.g. $A_\mu^{(4)}$, $\lambda_\alpha^{(5)A}$, $\varphi^{(6)AB}$) will depend on η_α^A and $\bar\xi_{\dot\alpha}^A$ not only via ζ_α^A. This observation will play an important rôle in the case of two impurity BMN operators. As will be shown in the next section, a naive counting of zero-modes including only terms with the minimal number of fermion modes in each field would lead to conclude that these operators have vanishing two-point functions in the instanton background. We will, however, argue that the inclusion of the term $\varphi^{(6)AB}$ in the solution is needed in order to compute the leading non-zero instanton contributions to these two-point functions.

5. Two-point functions of BMN operators

In this section we analyse instanton contributions to two-point functions of the BMN operators described in section 2.2. Using the results of the previous sections we shall determine which operators have non-zero two-point functions in the instanton background and the dependence of the instanton induced anomalous dimensions on the parameters,

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$g_{\rm YM}$, N and J as well as the integers corresponding to the mode numbers in the dual string states. Zero and one impurity operators are protected, therefore their two-point functions are not renormalised and receive no instanton contribution. We shall therefore discuss two and four impurity operators. Operators with an odd number of impurities are expected not to receive instanton contributions. The analysis of two impurity operators in the next subsection will be rather qualitative because the leading non-zero contribution to their two-point functions involves the six-fermion term in the scalar solution which is not known explicitly. The four impurity case which is fully under control will be discussed in the following subsection.

5.1 Two impurity operators

At the two impurity level we focus on the bosonic $\mathrm{SO}(4)_C$ singlet operator (2.17) in the $\mathbf{9}$ of $\mathrm{SO}(4)_R$. Since this sector contains only one operator there is no problem of mixing and the anomalous dimension of the operator $\mathscr{O}^{\{ij\}}_{J,\mathbf{9};n}$ can be read directly from the coefficient of the two-point function $\langle \mathscr{O}^{\{ij\}}_{J,\mathbf{9};n}(x_1)\bar{\mathscr{O}}^{\{kl\}}_{-J,\mathbf{9};m}(x_2)\rangle$.

As usual it is convenient to compute this two-point function for a particular choice of components, rather than work in a manifestly $\mathrm{SO}(4)_R$ covariant way. Therefore we consider[4]

$$G_{\mathbf{9}}(x_1,x_2) = \langle \mathscr{O}^{\{13\}}_n(x_1)\bar{\mathscr{O}}^{\{13\}}_m(x_2)\rangle_{\rm inst}\,, \tag{5.1}$$

so that there is no trace to subtract.

The component $\mathscr{O}^{\{13\}}_n$ is

$$\mathscr{O}^{\{13\}}_n = \frac{i}{\sqrt{J\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+2}}}\sum_{p=0}^{J} {\rm e}^{2\pi i p n/J}\left[{\rm Tr}\left(Z^{J-p}\varphi^{13}Z^p\varphi^{13}\right) - {\rm Tr}\left(Z^{J-p}\varphi^{24}Z^p\varphi^{24}\right)\right] \tag{5.2}$$

and the conjugate operator is

$$\bar{\mathscr{O}}^{\{13\}}_n = \frac{-i}{\sqrt{J\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+2}}}\sum_{p=0}^{J} {\rm e}^{-2\pi i p n/J}\left[{\rm Tr}\left(\bar{Z}^{J-p}\varphi^{13}\bar{Z}^p\varphi^{13}\right) - {\rm Tr}\left(\bar{Z}^{J-p}\varphi^{24}\bar{Z}^p\varphi^{24}\right)\right]. \tag{5.3}$$

The semi-classical approximation in the one-instanton sector for (5.1) gives

$$\begin{aligned} G_{\mathbf{9}}(x_1,x_2) = {}& \frac{\pi^{-4N}g^{4N}_{\rm YM}{\rm e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int {\rm d}\rho\,{\rm d}^4x_0\prod_{A=1}^{4}{\rm d}^2\eta^A{\rm d}^2\bar{\xi}^A\,{\rm d}^{N-2}\nu^A{\rm d}^{N-2}\bar{\nu}^A\,\rho^{4N-13}\times \\ &\times{\rm e}^{\frac{\pi^2}{16g^2_{\rm YM}\rho^2}\varepsilon_{ABCD}(\bar{\nu}^{[A}\nu^{B]})(\bar{\nu}^{[C}\nu^{D]})}\,\hat{\mathscr{O}}^{\{13\}}_{J;n}(x_1;x_0,\rho,\eta,\bar{\xi},\nu,\bar{\nu})\,\hat{\bar{\mathscr{O}}}^{\{13\}}_{-J;m}(x_2;x_0,\rho,\eta,\bar{\xi},\nu,\bar{\nu})\,. \end{aligned} \tag{5.4}$$

In order to have a non-zero contribution to this two-point function the classical profiles of the two operators must contain, when combined, the sixteen fermion modes corresponding to the broken supersymmetries.

[4]The subscripts indicating the $\mathrm{SO}(4)_R$ representation and the U(1) charge will be omitted except in situations where this may cause confusion.

It is easy to verify that substituting for each scalar field in (5.1) the leading order solution, $\varphi^{(2)}$, does not allow to soak up all the superconformal modes. Using for each scalar field the bilinear term (4.9) in the solution we find that $\mathscr{O}^{\{13\}}$ contains the following combinations of fermion modes

$$\left(m^{[1}_{\rm f} m^{4]}_{\rm f} \right)^J \left(m^{[1}_{\rm f} m^{3]}_{\rm f} m^{[1}_{\rm f} m^{3]}_{\rm f} + m^{[2}_{\rm f} m^{4]}_{\rm f} m^{[2}_{\rm f} m^{4]}_{\rm f} \right) . \tag{5.5}$$

Similarly, the conjugate operator $\hat{\bar{\mathscr{O}}}^{\{13\}}$ contains

$$\left(m^{[2}_{\rm f} m^{3]}_{\rm f} \right)^J \left(m^{[1}_{\rm f} m^{3]}_{\rm f} m^{[1}_{\rm f} m^{3]}_{\rm f} + m^{[2}_{\rm f} m^{4]}_{\rm f} m^{[2}_{\rm f} m^{4]}_{\rm f} \right) . \tag{5.6}$$

The argument given at the end of section 4 shows that in these traces, involving only $Z^{(2)}$ and $\varphi^{(2)AB}$, the superconformal modes always appear in the combination ζ^A of (4.14). This can be easily verified using the explicit expression for the scalar ADHM matrices given in (A.7). Then, because of the condition $\left(\zeta^A(x)\right)^3 = 0$ satisfied by the Weyl spinors ζ^A, in order to soak up the sixteen superconformal modes in the two-point function (5.4) each of the operators should contain two factors of ζ^A for each flavour. In other words the sixteen superconformal modes should appear in the two-point function in the form

$$\prod_{A=1}^{4} \left[\zeta^A(x_1)\right]^2 \left[\zeta^A(x_2)\right]^2 . \tag{5.7}$$

Examining the combinations (5.5) and (5.6) it is clear that this is not possible. $\hat{\mathscr{O}}^{\{13\}}$ cannot soak up the required superconformal modes of flavour 2 and 3, since it does not contain two factors of both ζ^2 and ζ^3, while $\hat{\bar{\mathscr{O}}}^{\{13\}}$ cannot soak up all the superconformal modes of flavour 1 and 4, since it does not contain two factors of both ζ^1 and ζ^4. This simple analysis of the flavour structure of the superconformal modes in the classical profiles of the operators shows that the two-point function (5.1) vanishes at leading order in the instanton background. This argument does not rely on the way the remaining $J-2$ $(\bar{\nu}^A\nu^B)$ bilinears are distributed in the two operators. According to the discussion in section 3 the leading contribution in the large-N limit would come from terms in which all the $(\bar{\nu}^A\nu^B)$ bilinears are antisymmetrised. However, since the above argument is based only on the analysis of the superconformal modes the conclusion that the leading $g_{\rm YM}$ contribution to the two-point function (5.1) vanishes is valid at all orders in $1/N$.

In order to saturate the integrations over the superconformal modes in (5.4) we need to use for some of the scalar fields the solution containing six fermionic modes, $\varphi^{(6)AB}$. Inspecting the combinations (5.5) and (5.6) found at leading order and recalling (4.8) it is easy to verify that it is sufficient to consider one $\varphi^{(6)AB}$ (or $Z^{(6)}$ and $\bar{Z}^{(6)}$ respectively) insertion in each operator. These are the leading order contributions, the insertion of more six-fermion scalars leads to contributions of higher order in $g_{\rm YM}$ since in this case more $\bar{\nu}^A\nu^B$ bilinears appear.

Recalling the form of $\varphi^{(6)AB}$ given in (4.8) we find that the combinations of fermionic modes in ${\rm Tr}\left(Z^{J-p}\varphi^{AB}Z^p\varphi^{CD}\right)$ and ${\rm Tr}\left(\bar{Z}^{J-p}\varphi^{AB}\bar{Z}^p\varphi^{CD}\right)$ are respectively

$$\varepsilon_{A'B'C'D'}\, m^{A'}_{\rm f} m^{B'}_{\rm f} m^{C'}_{\rm f} m^{D'}_{\rm f} \left(m^{[1}_{\rm f} m^{4]}_{\rm f} \right)^J \left(m^{[A}_{\rm f} m^{B]}_{\rm f} m^{[C}_{\rm f} m^{D]}_{\rm f} \right) \tag{5.8}$$

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and

$$\varepsilon_{A'B'C'D'}\, m_{\rm f}^{A'}\, m_{\rm f}^{B'}\, m_{\rm f}^{C'}\, m_{\rm f}^{D'} \left(m_{\rm f}^{[2}\, m_{\rm f}^{3]} \right)^J \left(m_{\rm f}^{[A}\, m_{\rm f}^{B]}\, m_{\rm f}^{[C}\, m_{\rm f}^{D]} \right) . \tag{5.9}$$

Notice that here the superconformal modes do not necessarily enter via (4.14), since we are using the term with six fermions in the solution for one of the fields in each operator. More precisely the structure of the supersymmetry transformations (4.15)-(4.18) shows that both traces contain one single $\bar{\xi}$ mode which is not part of a ζ. This is crucial in order to get a non-zero result from the moduli space integration, because it allows, for two flavours, to distribute three fermionic superconformal modes in one operator and one in the other.

The resulting non-zero contribution to the two-point function is

$$\begin{aligned} G_{\mathbf{9}}(x_1,x_2) \sim \frac{g_{\rm YM}^4 \mathrm{e}^{2\pi i\tau}}{JN^{3/2}} \int \mathrm{d}^4 x_0\, \mathrm{d}\rho \rho^{2J-5} f(x_1,x_2;x_0,\rho) \int \mathrm{d}^5\Omega \left(\Omega^{14}\right)^{J-1} \left(\Omega^{23}\right)^{J-1} \Omega^{13}\Omega^{24} \times \\ \times \int \prod_{A=1}^{4} \mathrm{d}^2\eta^A\, \mathrm{d}^2\bar{\xi}^A \left\{ \left[\left(\zeta^1\right)^2 \zeta^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2 \bar{\xi}^1 \right](x_1) \left[\zeta^1 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2 \bar{\xi}^2 \right](x_2) + \right. \\ \left. + \left[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \zeta^3 \left(\zeta^4\right)^2 \bar{\xi}^4 \right](x_1) \left[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \zeta^4 \bar{\xi}^3 \right](x_2) \right\} , \end{aligned} \tag{5.10}$$

where the $(\bar{\nu}^A \nu^B)$ bilinears have been rewritten in terms of Ω^{AB}'s as described in section 3. The overall powers of $g_{\rm YM}$ and N come from the normalisation of the operators, the moduli space integration measure and the $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears, see equations (3.16) and (3.17). The fermion superconformal modes are saturated and the corresponding integration is non-zero. In (5.10) the dependence on the bosonic moduli has been collected in the function $f(x_1,x_2;x_0,\rho)$, which can only be computed knowing the explicit form of the solution $\varphi^{(6)AB}$ which we have not determined. The exact form of the solution is also needed in order to compute the overall coefficient and, in particular, the dependence on J. More details of the derivation of (5.10) as well as of the evaluation of the moduli space integrals are given in appendix B.

The final result for the two-point function is of the form

$$G_{\mathbf{9}}(x_1,x_2) \sim \frac{g_{\rm YM}^4 J^3 \mathrm{e}^{2\pi i\tau}}{N^{3/2}} \frac{1}{(x_1-x_2)^{2(J+2)}}\, I\,, \tag{5.11}$$

where I is a logarithmically divergent integral, to be regulated e.g. by dimensional regularisation of the x_0 integral. The logarithmic divergence is due to the bosonic integrations over x_0 and ρ, as can be verified by dimensional analysis. The presence of this divergence signals an instanton contribution to the anomalous dimension of the operator $\mathscr{O}_{\mathbf{9}}^{\{ij\}}$.

As already observed there is only one operator in the representation $\mathbf{9}$ of $\mathrm{SO}(4)_R$ and thus there is no mixing to resolve and the present analysis directly determines the instanton correction to the scaling dimension. We thus find that the instanton induced anomalous dimension of $\mathscr{O}_{\mathbf{9}}^{\{13\}}$ behaves as

$$\gamma_{\mathbf{9}}^{\rm inst} \sim \frac{g_{\rm YM}^4 J^3}{N^{3/2}}\, \mathrm{e}^{-\frac{8\pi^2}{g_{\rm YM}^2}+i\theta} \sim \left(g_2\right)^{7/2} \left(\lambda'\right)^2\, \mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta} . \tag{5.12}$$

This is in agreement with the non-perturbative correction to the mass of the dual string state computed in [14]. In particular, the anomalous dimension (5.12) is independent of the parameter n corresponding to the mode number of the plane-wave string state. Apart from the exponential factor characteristic of instanton effects, (5.12) contains an additional factor of $(\lambda')^2$. This is due to the inclusion of six-fermion scalars which give rise to additional $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears, each of which brings one more power of $g_{_{\rm YM}}$. As will be shown in the next subsection in the case of four impurity $\mathrm{SO}(4)_R$ singlets it is sufficient to consider the bilinear solution for all the scalars and as a consequence we shall find a leading contribution of order $(g_2)^{7/2}\mathrm{e}^{-8\pi^2/g_2\lambda'}$.

Contributions in which some of the $\bar{\nu}\nu$ bilinears are in the $\mathbf{10}$ of SU(4) give rise to subleading corrections which are suppressed by powers of $1/N$.

Another class of contributions to (5.1) which are suppressed in the large-N limit are those in which pairs of scalars are contracted. In these terms the analysis of the superconformal modes is unaltered and in order to soak them up it is again necessary to use the solution $\varphi^{(6)AB}$ for two of the scalars. Two of the scalars which were previously replaced by $\bar{\nu}\nu$ bilinears are now contracted and do not contain any fermion modes. Hence the integration over the moduli space produces one less power of $g_{_{\rm YM}}^2 N$. However, with the normalisation we are using the propagator is proportional to $g_{_{\rm YM}}^2$, so that in conclusion the contribution of these terms is down by $1/N$ with respect to (5.11) because there is no power of N associated with the contraction.

A careful analysis of both types of $1/N$ corrections shows that they give a contribution to the anomalous dimension of the operator $\mathscr{O}_{\mathbf{9}}$ of order $(g_2)^{9/2}(\lambda')^2$. These are the leading terms in a power series in g_2. In general the corrections to the semi-classical approximation in the BMN limit can be reorganised into a double series in g_2 and λ'.

Operators in different sectors can be studied along the same lines. However, superconformal invariance implies that all the two impurity operators have the same anomalous dimension [28] and thus the above result can be extended to two impurity operators in all the other sectors with no further calculations required.

Arguments similar to those discussed here, showing the vanishing of the leading one-instanton contribution to the two-point function $G_{\mathbf{9}}(x_1, x_2)$, have been used to prove various non-renormalisation properties in [32, 33, 27]. In view of the results we found for $\mathscr{O}_{\mathbf{9}}$, one can expect that some of the non-renormalisation results of these papers may not be extended to higher orders in the coupling.

5.2 Four impurity operators

The calculation of two-point functions of four impurity operators is more involved than the corresponding calculation in the two impurity case from the point of view of the combinatorial analysis. However, at the four impurity level, in the case of $\mathrm{SO}(4)_R$ singlets, the leading instanton contributions do not involve the six fermion solution for the scalar fields. A non-zero result is obtained using only the bilinear solution, which is known explicitly and given in (A.7), in computing the classical profiles of the operators. Therefore we can analyse in a quantitative way the semi-classical contributions to the two-point functions. The fact that non-zero correlation functions of singlet operators are obtained using the minimal number of fermion modes for each field also implies that in this case a contribution to the matrix of anomalous dimensions arises at leading order in the instanton background. As we shall see these operators have instanton induced anomalous dimension of order $(g_2)^{7/2}\mathrm{e}^{2\pi i\tau}$. Another difference with respect to the two impurity case studied in the previous section is that two-point functions of four impurity operators depend explicitly on the integers dual to the string mode numbers. We shall discuss in detail a $\mathrm{SO}(4)_R \times \mathrm{SO}(4)_C$ singlet with four scalar impurities and show that the behaviour of its two-point functions is in remarkable agreement with the corresponding string calculation of [14]. Other singlet operators can be analysed in a similar fashion. Operators in other sectors will be shown to receive contribution only at higher order in λ'. This result follows simply from the analysis of fermion zero modes and is also in agreement with the string theory prediction.

5.2.1 ε-singlet operator

In this subsection we present the calculation of the one-instanton contribution to the two-point function of one particular $\mathrm{SO}(4)_R$ singlet. More details are provided in appendix B. We focus on the four scalar impurity operator in which the $\mathrm{SO}(4)_R$ indices are contracted via an ε-tensor,

$$\begin{aligned}\mathcal{O}_{\mathbf{1};J;n_1,n_2,n_3} = \frac{\varepsilon_{ijkl}}{\sqrt{J^3\left(\frac{g^2_{\mathrm{YM}}N}{8\pi^2}\right)^{J+4}}} \sum_{\substack{q,r,s=0\\ q+r+s\leq J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3s]/J} \times \\ \times \mathrm{Tr}\left(Z^{J-(q+r+s)}\varphi^i Z^q \varphi^j Z^r \varphi^k Z^s \varphi^l\right) . \end{aligned} \tag{5.13}$$

The string state in the plane-wave background which is naturally identified as being dual to this operator is of the form

$$\varepsilon_{ijkl}\, \alpha^i_{-n_1}\alpha^j_{-n_2}\widetilde{\alpha}^k_{-n_3}\widetilde{\alpha}^l_{-(n_1+n_2-n_3)}|0\rangle_h , \tag{5.14}$$

where $|0\rangle_h$ is the BMN ground state and the contraction runs over values of the indices in one of the two SO(4) factors. D-instanton contributions to the renormalisation of the mass of this state were computed in [14]. We shall return to the comparison with the string results at the end of this section. Notice, however, that the state (5.14) is antisymmetric under the exchange of the two left-moving or right-moving modes. The operator (5.13) on the other hand has no definite symmetry under permutations of the parameters n_1, n_2 and n_3. Therefore in order to construct a gauge theory operator that can be identified with (5.14) it will be necessary to explicitly antisymmetrise (5.13). This point will prove crucial when comparing instanton corrections to the scaling dimension of $\mathcal{O}_{\mathbf{1}}$ to D-instanton induced corrections to the mass of the string state.

We are interested in the two-point function

$$\begin{aligned} G_{\mathbf{1}}(x_1,x_2;n_1,n_2,n_3;m_1,m_2,m_3) &= \langle \mathcal{O}_{\mathbf{1};n_1,n_2,n_3}(x_1)\,\bar{\mathcal{O}}_{\mathbf{1};m_1,m_2,m_3}(x_2)\rangle_{\mathrm{inst}} \\ &= \frac{\pi^{-4N}g^{4N}_{\mathrm{YM}}\mathrm{e}^{2\pi i\tau}}{(N-1)!(N-2)!}\int \mathrm{d}\rho\,\mathrm{d}^4x_0 \prod_{A=1}^{4}\mathrm{d}^2\eta^A\mathrm{d}^2\bar{\xi}^A\,\mathrm{d}^{N-2}\nu^A\mathrm{d}^{N-2}\bar{\nu}^A\,\rho^{4N-13}\times \end{aligned} \tag{5.15}$$

$$
\begin{aligned}
&\times \mathrm{e}^{\frac{\pi^2}{16 g_{\rm YM}^2 \rho^2}\varepsilon_{ABCD}(\bar\nu^{[A}\nu^{B]})(\bar\nu^{[C}\nu^{D]})} \times \\
&\times \hat{\mathscr{O}}_{\mathbf{1};n_1,n_2,n_3}\big(x_1;x_0,\rho,\eta,\bar\xi,\nu,\bar\nu\big)\hat{\bar{\mathscr{O}}}_{\mathbf{1};m_1,m_2,m_3}\big(x_2;x_0,\rho,\eta,\bar\xi,\nu,\bar\nu\big)\,.
\end{aligned}
$$

As usual the semi-classical approximation requires the calculation of the classical profiles of $\mathscr{O}_{\mathbf{1}}$ and $\bar{\mathscr{O}}_{\mathbf{1}}$ in the instanton background.

Summing over the SO(4) indices in (5.13) and using the relations (2.4) we find that the operator $\mathscr{O}_{\mathbf{1}}$ contains the independent traces

$$
\begin{aligned}
&+\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{13}Z^r\varphi^{24}Z^s\varphi^{34}\right)-\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{34}Z^r\varphi^{24}Z^s\varphi^{13}\right)+\\
&+\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{24}Z^r\varphi^{34}Z^s\varphi^{13}\right)+\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{34}Z^r\varphi^{13}Z^s\varphi^{24}\right)-\\
&-\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{13}Z^r\varphi^{34}Z^s\varphi^{24}\right)-\mathrm{Tr}\left(Z^p\varphi^{12}Z^q\varphi^{24}Z^r\varphi^{13}Z^s\varphi^{34}\right)\,,
\end{aligned}
\tag{5.16}
$$

where $p=J-(q+r+s)$, plus three other groups of six traces obtained by cyclic permutations of the indices on the impurities in (5.16). The conjugate operator, $\bar{\mathscr{O}}_{\mathbf{1}}$, contains the same terms, but with the Z's replaced by $\bar Z$'s.

It is straightforward to verify that these traces, when evaluated in the instanton background, contain the correct combination of fermions required to soak up the superconformal modes in a two-point function and that this can be achieved using only the bilinear solution for all the scalars. In this case all the η^A and $\bar\xi^A$ modes in the gauge invariant traces are combined into ζ^A's. In order to give rise to a non-zero two-point function in the one instanton sector both operators should then contain the combination $\prod_{A=1}^4\left(\zeta^A\right)^2$. To achieve this in each trace in (5.16) the four impurities must provide two superconformal modes of flavours 2 and 3, whereas the modes of flavour 1 and 4 can be taken from the impurities or from the Z's. Similarly in the case of $\bar{\mathscr{O}}_{\mathbf{1}}$ the superconformal modes of flavour 1 and 4 come necessarily from the impurities and those of flavour 2 and 3 can be provided by the impurities or by the $\bar Z$'s. As in the two impurity case studied in the previous section the leading contribution is obtained taking all the remaining modes in $(\bar\nu\nu)_{\mathbf{6}}$ bilinears. In all the traces appearing in both $\mathscr{O}_{\mathbf{1}}$ and $\bar{\mathscr{O}}_{\mathbf{1}}$ the impurities contain two fermion modes of each flavour. The combination of fermion modes entering into all the terms in $\mathscr{O}_{\mathbf{1}}$ is

$$
\left(m\,{}^{1}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{2}_{\mathrm{f}}\right)^{2}\left(m\,{}^{3}_{\mathrm{f}}\right)^{2}\left(m\,{}^{4}_{\mathrm{f}}\right)^{J+2}\,,
\tag{5.17}
$$

whereas all the terms in the expansion of $\bar{\mathscr{O}}_{\mathbf{1}}$ contain

$$
\left(m\,{}^{1}_{\mathrm{f}}\right)^{2}\left(m\,{}^{2}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{3}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{4}_{\mathrm{f}}\right)^{2}\,.
\tag{5.18}
$$

The leading contribution to the two-point function $G_{\mathbf{1}}(x_1,x_2)$ in the semi-classical approximation arises from terms in the profiles of the operators containing the following combinations of fermion modes

$$
\begin{aligned}
\mathscr{O}_{\mathbf{1}} &\to \left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\left(\bar\nu^{[1}\nu^{4]}\right)^J\\
\bar{\mathscr{O}}_{\mathbf{1}} &\to \left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\left(\bar\nu^{[2}\nu^{3]}\right)^J\,.
\end{aligned}
\tag{5.19}
$$

As previously observed these combinations can be obtained in many different ways corresponding to the choice of which field, Z or φ, provides each of the ζ's of flavour 1 and

4 in $\mathscr{O}$. An equal number of different contributions arises from the ways of distributing the ζ's of flavour 2 and 3 among the impurities or the $\bar{Z}$'s in $\bar{\mathscr{O}}$. In order to simplify the discussion of the associated combinatorics it is convenient to introduce the following notation. We denote by $\check{\varphi}^{AB}$ a scalar solution in which only the ν and $\bar{\nu}$ modes are kept and all the superconformal modes are set to zero; scalars containing only bilinears in the superconformal modes are indicated by $\widetilde{\varphi}^{AB}$; the symbol $\widehat{\varphi}^{AB}$ is used for scalar profiles in which only mixed terms, $\zeta\nu$ or $\zeta\bar{\nu}$, are included,

$$\check{\varphi}^{AB} \equiv \varphi^{AB}(x, x_0, \rho; \nu, \bar{\nu}; \eta = \bar{\xi} = 0) \tag{5.20}$$

$$\widetilde{\varphi}^{AB} \equiv \varphi^{AB}(x, x_0, \rho; \eta, \bar{\xi}; \nu = \bar{\nu} = 0) \tag{5.21}$$

$$\widehat{\varphi}^{AB} \equiv \varphi^{AB}(x, x_0, \rho; \eta\nu, \eta\bar{\nu}, \bar{\xi}\nu, \bar{\xi}\bar{\nu}; \zeta\zeta = \bar{\nu}\nu = 0)\,. \tag{5.22}$$

The same notation is also used for $Z \sim \varphi^{14}$ and $\bar{Z} \sim \varphi^{23}$.

We are only interested in contributions to the two-point function $G_{\mathbf{1}}(x_1, x_2)$ which survive in the BMN limit, $N \to \infty$, $J \to \infty$, with J^2/N fixed. The leading large-N contributions are those in which the combinations (5.19) are selected, i.e. the superconformal modes are soaked up and all the remaining fields are replaced by $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears. Within this class of terms the dominant ones in the large J limit are those in which as many superconformal modes as possible are extracted from the Z's and $\bar{Z}$'s, because there is roughly a multiplicity factor of J associated with the choice of each Z or $\bar{Z}$ providing one such mode. We first discuss these leading terms and we will then show that these are the only non-vanishing contributions in the BMN limit.

As shown by the previous preliminary analysis, in the operator $\mathscr{O}_{\mathbf{1}}$ the modes ζ^2 and ζ^3 necessarily come from the impurities and thus the leading large J terms arise from traces in which we take the two ζ^1 and the two ζ^4 modes from four distinct Z's. Similar considerations apply to the $\bar{\mathscr{O}}_{\mathbf{1}}$ operator with the rôle of the flavours $(1,4)$ and $(2,3)$ exchanged. Using the notation introduced in (5.20)-(5.22) this means that we consider traces in which all four impurities are $\widehat{\varphi}^{AB}$ matrices and we choose four Z's to be $\widehat{Z}$ matrices, with all the others being $\check{Z}$'s. There is a total of 35 different traces of this type for each of the 6×4 terms in the operator (5.13) and a similar counting applies to its conjugate. The 35 traces correspond to the inequivalent ways of choosing the four $\widehat{Z}$'s from the four groups of Z's in (5.13). For the generic term in the operator, $\mathrm{Tr}\left(Z^p\varphi^{A_1B_1}Z^q\varphi^{A_2B_2}Z^r\varphi^{A_3B_3}Z^s\varphi^{A_4B_4}\right)$, with $p = J-(q+r+s)$, we need to consider

$$\begin{aligned}
&\mathrm{Tr}\left(\check{Z}^{p_1}\widehat{Z}\check{Z}^{p_2}\widehat{Z}\check{Z}^{p_3}\widehat{Z}\check{Z}^{p_4}\widehat{Z}\check{Z}^{p_5}\widehat{\varphi}^{A_1B_1}\check{Z}^{q}\widehat{\varphi}^{A_2B_2}\check{Z}^{r}\widehat{\varphi}^{A_3B_3}\check{Z}^{s}\widehat{\varphi}^{A_4B_4}\right)\\
&\mathrm{Tr}\left(\check{Z}^{p_1}\widehat{Z}\check{Z}^{p_2}\widehat{Z}\check{Z}^{p_3}\widehat{Z}\check{Z}^{p_4}\widehat{\varphi}^{A_1B_1}\check{Z}^{q_1}\widehat{Z}\check{Z}^{q_2}q\widehat{\varphi}^{A_2B_2}\check{Z}^{r}\widehat{\varphi}^{A_3B_3}\check{Z}^{s}\widehat{\varphi}^{A_4B_4}\right)\\
&\dots\\
&\mathrm{Tr}\left(\check{Z}^{p}\widehat{\varphi}^{A_1B_1}\check{Z}^{q}\widehat{\varphi}^{A_2B_2}\check{Z}^{r}\widehat{\varphi}^{A_3B_3}\check{Z}^{s_1}\widehat{Z}\check{Z}^{s_2}\widehat{Z}\check{Z}^{s_3}\widehat{Z}\check{Z}^{s_4}\widehat{Z}\check{Z}^{s_5}\widehat{\varphi}^{A_4B_4}\right)\,,
\end{aligned} \tag{5.23}$$

where in the first trace $\sum_i p_i = p-4 = J-(q+r+s+4)$, in the second $\sum_i p_i = p-3$ and $\sum_i q_i = q-1$ and so on until the last sum where $\sum_i s_i = s-4$. The ellipsis in (5.23) refers to other combinations in which the four $\widehat{Z}$'s are gradually moved to the right. All these traces can be evaluated using the ADHM solution for the scalars given in (A.7) and selecting for each factor the matrix elements containing the appropriate fermion bilinears. The calculation is rather involved. As explained in appendix B it can be carried out most efficiently defining a more general trace from which the 35 distinct traces (5.23) can be obtained for different choices of indices.

In order to compute the relevant part of the profile of $\mathscr{O}_{\mathbf{1}}$ we need to sum the contributions of the traces (5.23) corresponding to the 6×4 choices of indices, (A_i, B_i), $i = 1, \ldots, 4$, on the impurities. A key feature of all these traces is that they do not depend on the way the Z's are grouped, i.e. they do not depend on the exponents, $(p_1, \ldots, p_5, q, r, s)$, $(p_1, \ldots, p_4, q_1, q_2, r, s)$ etc. in (5.23), but only on the ordering of the four $\widehat{Z}$'s with respect to the four impurities, $\widehat{\varphi}^{A_i B_i}$, $i = 1, \ldots, 4$. This is a consequence of the structure of the ADHM matrices and the restrictions imposed by the ADHM constraints. Keeping only the terms with two ζ's of each flavour all the traces in $\mathscr{O}_{\mathbf{1}}$ produce expressions which after simple Fierz rearrangements can be brought to the form

$$\frac{\rho^8}{[(x_1 - x_0)^2 + \rho^2]^{J+8}} \left(\bar{\nu}^{[1}\nu^{4]}\right)^J \left[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2\right](x_1) . \tag{5.24}$$

Similarly all the contributions from the traces in $\bar{\mathscr{O}}_{\mathbf{1}}$ containing the required eight superconformal modes can be put in the form

$$\frac{\rho^8}{[(x_2 - x_0)^2 + \rho^2]^{J+8}} \left(\bar{\nu}^{[1}\nu^{4]}\right)^J \left[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2\right](x_2) . \tag{5.25}$$

Each set of indices (A_i, B_i), $i = 1, \ldots, 4$ on the impurities in each of the 35 traces leads to a contribution of the form (5.24)-(5.25) with a different numerical coefficient.

The fact that the result of all the traces can be reduced to the above expressions implies that when substituting into the definition of the operator (5.13) and its conjugate a common factor (5.24) or, respectively, (5.25) can be taken out of the traces. Associated with each of the 35 types of traces there are, however, multiplicity factors which make the sums in the definition of the operator non-trivial. For instance in the last trace in (5.23) there are s choices for the first $\widehat{Z}$ among the Z's, $(s-1)$ choices for the second $\widehat{Z}$, $(s-2)$ for the third and $(s-3)$ for the fourth. After substituting into the definition (5.13) and factoring out the moduli dependence in the form (5.24), the contribution of the last trace in (5.23) involves the sums

$$\sum_{\substack{q,r,s=0 \\ q+r+s\leq J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} s(s-1)(s-2)(s-3) , \tag{5.26}$$

with a numerical coefficient resulting from the contributions of the 6×4 permutations of indices of the impurities. Repeating the same analysis for all the traces means combining a huge number of terms which makes the calculation extremely laborious. Completely analogous steps go into the calculation of the profile of the conjugate operator.

After lengthy algebraic manipulations and the use of the formalism described in section 3, the semi-classical result for the two-point function (5.16) takes the form

$$G_{\mathbf{1}}(x_1, x_2) = \frac{\mathrm{e}^{2\pi i\tau}}{J^3 N^{7/2}} \int \frac{\mathrm{d}^4 x_0 \, \mathrm{d}\rho}{\rho^5} \frac{\rho^{J+8}}{[(x_1 - x_0)^2 + \rho^2]^{J+8}} \frac{\rho^{J+8}}{[(x_2 - x_0)^2 + \rho^2]^{J+8}} \times$$

$$\times \int \prod_{A=1}^{4} \mathrm{d}^2\eta^A \mathrm{d}^2\bar{\xi}^A \prod_{B=1}^{4} \left[\left(\zeta^B\right)^2(x_1)\right] \left[\left(\zeta^B\right)^2(x_2)\right] \times$$

$$\times \int \mathrm{d}^5\Omega \left(\Omega^{14}\right)^J \left(\Omega^{23}\right)^J K(n_1,n_2,n_3;J)K(m_1,m_2,m_3;J)\,, \tag{5.27}$$

where following the discussion in section 3 the $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears have been expressed in terms of the angular variables Ω^{AB}. In (5.27) overall numerical coefficients have been omitted. The J and N dependence in the prefactor in (5.27) is obtained combining the normalisation of the operators, the contribution of the measure on the instanton moduli space and the factors of $g_{\rm YM}\sqrt{N}$ associated with $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears. The origin of the various factors which determine the dependence on the parameters $g_{\rm YM}$, N and J will be summarised shortly. The expression (5.27) contains integrations over the bosonic moduli, x_0 and ρ, the sixteen superconformal fermion modes and the five-sphere coordinates Ω^{AB}. The dependence on the integers n_i, m_i, $i=1,2,3$, dual to the mode numbers of the corresponding string state is contained in the functions $K(n_1,n_2,n_3;J)$ and $K(m_1,m_2,m_3;J)$. These are given by the sum of 35 terms,

$$K(n_1,n_2,n_3;J) = \sum_{a=1}^{35} c_a\, \mathcal{S}_a(n_1,n_2,n_3;J)\,, \tag{5.28}$$

where the symbols $\mathcal{S}_a$ indicate 35 different sums over the indices q,r,s in which the summands are given by the phase factor $\exp\{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J\}$ times the multiplicity factors associated with the different distributions of $\widehat{Z}$'s in each case. The numerical coefficients c_a are obtained combining the contributions of the different permutations of indices on the impurities for each of the 35 terms. See appendix B.2 for more details.

In the large J limit the leading order contribution to the sums $\mathcal{S}_a$ can be obtained using a continuum approximation by setting $x=q/J$, $y=r/J$, $z=s/J$, so that $x,y,z\in[0,1]$. For instance the sum (5.26) is approximated as

$$\sum_{\substack{q,r,s=0\\ q+r+s\le J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} s(s-1)(s-2)(s-3) \to$$

$$\to J^7 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} \mathrm{d}z\, z^4 \mathrm{e}^{2\pi i[(n_1+n_2+n_3)x+(n_2+n_3)y+n_4 z]}\,, \tag{5.29}$$

which shows that it behaves as J^7 for large J. We shall denote by $\kappa(n_1,n_2,n_3)$ the function of the mode numbers arising from these sums/integrals after extracting a factor of J^7,

$$K(n_1,n_2,n_3;J) = J^7\kappa(n_1,n_2,n_3)\,. \tag{5.30}$$

The two-point function is then

$$G_{\mathbf{1}}(x_1,x_2) = \frac{J^{11}\,\mathrm{e}^{2\pi i\tau}}{N^{7/2}} \int \frac{\mathrm{d}^4x_0\,\mathrm{d}\rho}{\rho^5} \frac{\rho^{J+8}}{[(x_1-x_0)^2+\rho^2]^{J+8}} \frac{\rho^{J+8}}{[(x_2-x_0)^2+\rho^2]^{J+8}} \times$$

$$\times \int \prod_{A=1}^{4} \mathrm{d}^2\eta^A \mathrm{d}^2\bar{\xi}^A \prod_{B=1}^{4} \left[\left(\zeta^B\right)^2(x_1)\right]\left[\left(\zeta^B\right)^2(x_2)\right] \times$$

$$\times \int \mathrm{d}^5\Omega \, \left(\Omega^{14}\right)^J \left(\Omega^{23}\right)^J \, \kappa(n_1,n_2,n_3;J)\kappa(m_1,m_2,m_3;J) \,. \qquad (5.31)$$

Unlike the case of two impurity operators discussed in the previous subsection, here the dependence on the instanton moduli is known explicitly and we can compute the associated integrations. More details are given in appendix B.2. The integration over the five-sphere in (5.31) is a special case of the general integral (3.19) and gives

$$I_{S^5} = \int \mathrm{d}^5\Omega \, \left(\Omega^{14}\right)^J \left(\Omega^{23}\right)^J = \frac{\pi^3}{(J+1)(J+2)} \,. \qquad (5.32)$$

The integration over the superconformal modes is also straightforward. It does not depend on N or J. For each flavour the result is

$$I_\zeta = \int \mathrm{d}^2\eta \mathrm{d}^2\bar{\xi} \, \left[(\zeta)^2(x_1)\right]\left[(\zeta)^2(x_2)\right] = -(x_1-x_2)^2 \,, \qquad (5.33)$$

so that the fermionic integrals contribute a factor of $(x_1-x_2)^8$. The integration over the bosonic part of the moduli space must be treated carefully since it is logarithmically divergent as expected in the presence of a contribution to the matrix of anomalous dimensions. The integrals need to be regulated for instance by dimensional regularisation of the x_0 integral and can then be computed using standard techniques, e.g. introducing Feynman parameters. The result is

$$\begin{aligned} I_{\mathrm{b}} &= \int \frac{\mathrm{d}^4x_0 \, \mathrm{d}\rho}{\rho^5} \frac{\rho^{J+8}}{[(x_1-x_0)^2+\rho^2]^{J+8}} \frac{\rho^{J+8}}{[(x_2-x_0)^2+\rho^2]^{J+8}} \\ &= \frac{1}{\epsilon} \frac{\Gamma(J+6)\Gamma(J+8+\epsilon)}{[\Gamma(J+8)]^2} \, \pi^{2-\epsilon} \frac{1}{(x_{12}^2)^{J+8+\epsilon}} \,, \qquad \epsilon \to 0 \,. \end{aligned} \qquad (5.34)$$

The $1/\epsilon$ pole is the manifestation of a logarithmic divergence in dimensional regularisation. The contribution (5.34) behaves as $1/J^2$ in the large J limit.

Putting together all the contributions the dependence on the parameters, g_{YM}, N and J, in the correlation function can be summarised as follows

$$\underbrace{\left(\frac{1}{\sqrt{J^3(g_{\mathrm{YM}}^2 N)^{J+4}}}\right)^2}_{\text{normalised op. profile}} \underbrace{\left(g_{\mathrm{YM}}\sqrt{N}\right)^{2J}}_{\nu,\,\bar{\nu}\text{ bilinears}} \underbrace{\mathrm{e}^{2\pi i\tau} g_{\mathrm{YM}}^8 \sqrt{N}}_{\text{measure}} \underbrace{\frac{1}{J^2}}_{S^5\text{ integral}} \underbrace{\frac{1}{J^2}}_{x_0,\,\rho\text{ integrals}} \underbrace{(J^7)^2}_{\mathcal{S}_a\text{ sums}} \sim$$

$$\sim \frac{J^7}{N^{7/2}} \, \mathrm{e}^{2\pi i\tau} = (g_2)^{7/2} \, \mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta} \,. \qquad (5.35)$$

The final result for the two-point function is thus, up to a numerical coefficient,

$$G_{\mathbf{1}}(x_1,x_2) = (g_2)^{7/2}\mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}\kappa(n_1,n_2,n_3)\kappa(m_1,m_2,m_3)\,\frac{1}{(x_{12}^2)^{J+4}}\log\left(\Lambda^2 x_{12}^2\right) \,. \qquad (5.36)$$

where the scale Λ appears as a consequence of the $1/\epsilon$ divergence. It has no observable effect. The physical information contained in the two-point function is in the contribution to the matrix of anomalous dimensions which is read from the coefficient in (5.36) and does not depend on Λ.

The result is expressed in terms of the double scaling parameters λ' and g_2. Note that, unlike the two-point functions of two impurity operators (5.36) is independent of λ' apart from the dependence in the exponential instanton weight.

The non-perturbative mass correction computed in [14] for the state (5.14), in terms of the same gauge theory parameters, is of the form

$$\delta m \sim (g_2)^{7/2} \mathrm{e}^{-\frac{8\pi^2}{g_2 \lambda'} + i\theta} \frac{1}{(n_1 n_2)^2} . \tag{5.37}$$

so that the λ' and g_2 dependence is in agreement with the gauge theory calculation. The mode number dependence in (5.37) is remarkably simple and a special feature of the string result, which is a direct consequence of the structure of the D-instanton boundary state, is that it is non-vanishing only if the mode numbers in both the incoming and the outgoing states are pairwise equal. The only states which couple to the D-instanton are of the form

$$\varepsilon_{ijkl}\, \alpha^i_{-n_1} \alpha^j_{-n_2} \widetilde{\alpha}^k_{-n_1} \widetilde{\alpha}^l_{-n_2} |0\rangle_h . \tag{5.38}$$

On the other hand in the gauge theory result (5.36) obtained for the operator (5.13) the mode number dependence is contained in $\kappa(n_1, n_2, n_3)$ and $\kappa(m_1, m_2, m_3)$, which are extremely complicated rational functions of their arguments. In particular the condition that the integers n_i be equal in pairs does not seem to be required.

However, as observed after (5.14) in order to correctly match the properties of the dual string state, the operator (5.13) must be explicitly antisymmetrised under the exchange of pairs of mode numbers. This antisymmetrisation induces dramatic simplifications. Working with the correctly antisymmetrised operators the result for the two-point function is

$$G_1(x_1, x_2) = \frac{3^2 (g_2)^{7/2} \mathrm{e}^{-\frac{8\pi^2}{g_2 \lambda'} + i\theta}}{2^{41} \pi^{9/2}} \frac{1}{(n_1 n_2)(m_1 m_2)} \frac{1}{(x_{12}^2)^{J+4}} \log\left(\Lambda^2 x_{12}^2\right) \tag{5.39}$$

if the mode numbers in each of the two operators are equal in pairs and vanishes otherwise. In (5.39) we have reinstated all the numerical coefficients coming from the profiles of the operators, the combinatorics previous described and the moduli space measure. This result is in perfect agreement with the string result (5.37) of [14]. It is worth stressing that the simplification found after the antisymmetrisation is extraordinary given the complexity of the function $\kappa(n_1, n_2, n_3)$. Moreover the condition of pairwise equal mode numbers which is also imposed in this way is far from obvious and highly non-trivial from the point of view of the gauge theory calculation.

As we have seen, in the two-point function computed in the semi-classical approximation the mode number dependence factorises. A consequence of this is that in $G_{\mathbf{1}}(x_1, x_2)$ the two independent mode numbers in $\mathscr{O}_{\mathbf{1}}$, n_1 and n_2, do not have to equal those in $\bar{\mathscr{O}}_{\mathbf{1}}$, m_1 and m_2. This appears to contradict energy conservation in the dual string amplitude, which requires the mode numbers of the outgoing state to match one to one those of the incoming state. However, the fact that the condition $m_i = n_i$, $i = 1, 2$ does not arise is an effect of the semi-classical approximation. This is valid in the $\lambda' \to 0$ limit which corresponds to the $m \to \infty$ limit in the plane-wave string theory (where m is the mass parameter entering the string action). In this strict limit energy conservation in a two-point string amplitude only requires that the number of oscillators in the incoming and outgoing states be equal, with no constraint on the associated mode numbers. Therefore (5.39) is indeed in agreement with the string theory result. On the other hand the instanton corrections discussed here should be considered as subleading corrections on top of the perturbative effects. The condition $m_i = n_i$ on the operators in a two-point function is already imposed at the perturbative level and should therefore be assumed when computing instanton contributions in the semi-classical approximation.

The calculation presented here is not sufficient to determine the actual instanton induced anomalous dimension of the operator $\mathscr{O}_{\mathbf{1}}$. This requires the diagonalisation of the matrix of anomalous dimensions of which we have not computed all the entries. Other entries are determined by the corresponding two-point functions whose calculation follows the same steps described here and results in expressions similar to (5.39). From this we can conclude that the behaviour of the leading instanton contribution to the anomalous dimensions of singlet operators is

$$\gamma_{\mathbf{1}}^{\rm inst} \sim (g_2)^{7/2} \mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta} \frac{1}{(n_1 n_2)^2} \,. \tag{5.40}$$

As a further test of the result we find that the two point function vanishes in the limit of zero mode numbers. The function $\kappa(n_1, n_2, n_3)$ is identically zero when $n_1 = n_2 = n_3 = 0$. This is the expected behaviour because in this limit the operator is expected to become protected and no corrections to its free theory two-point functions should arise. The string theory counterpart of this result is the decoupling of the supergravity modes (dual to the protected operators with $\{n_i = 0\}$), which was also verified in [14].

In the previous analysis we have considered only a class of contributions in which in each operator as many superconformal modes as possible were taken from the Z's. It is easy to verify that these are the only relevant terms at leading order in the BMN limit. All the other types of traces are suppressed and vanish in the $J \to \infty$ limit. As an example consider a contribution to the profile of $\mathscr{O}_{\mathbf{1}}$ in which the ζ^2 and ζ^3 modes as well as one of either the ζ^1 or ζ^4 modes are taken from the impurities. Instead of the last trace in (5.23) we would then consider traces of the type

$$\mathrm{Tr}\left(\check{Z}^p \widetilde{\varphi}^{A_1B_1} \check{Z}^q \widehat{\varphi}^{A_2B_2} \check{Z}^r \widehat{\varphi}^{A_3B_3} \check{Z}^{s_1} \widehat{Z} \check{Z}^{s_2} \widehat{Z} \check{Z}^{s_3} \widehat{Z} \check{Z}^{s_4} \widehat{\varphi}^{A_4B_4} \right), \tag{5.41}$$

where the first impurity contains two superconformal modes and thus only three $\widehat{Z}$'s are needed. An analysis similar to that carried out for the traces (5.23) can be repeated in this case and one finds that associated with such a trace there is sum of the form

$$\sum_{\substack{q,r,s=0\\ q+r+s\leq J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} \frac{1}{3!} s(s-1)(s-2) \,, \tag{5.42}$$

which behaves as J^6 in the large J limit. The remaining J, N and $g_{\rm YM}$ dependence in the correlation function is unmodified and thus the combined behaviour of this contribution can be read from (5.35) replacing the last factor on the first line by $(J^6)^2$ leading to

$$
{\rm e}^{2\pi i\tau}\frac{J^5}{N^{7/2}} \sim \frac{{\rm e}^{2\pi i\tau}(g_2)^{7/2}}{J^2}\,,
$$

which vanishes in the BMN limit. Similar arguments can be repeated for all the contributions other than those leading to (5.39), which is therefore the complete leading instanton contribution to this singlet two-point function in the BMN limit. We shall briefly comment on corrections to this result of higher order in λ' and g_2 in the discussion section.

5.2.2 Other four impurity singlets

There are two other independent four impurity singlet operators involving four scalar impurities. They correspond to the two inequivalent ways of contracting the $\mathrm{SO}(4)_R$ indices with Kronecker delta's,

$$
\begin{aligned}
\mathscr{O}^{(d_1)}_{\mathbf{1};J;n_1,n_2,n_3} &= \frac{1}{\sqrt{J^3\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+4}}}\sum_{\substack{q,r,s=0\\ q+r+s\le J}}^{J}{\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J}\,\times \\
&\quad\times \mathrm{Tr}\left(Z^{J-(q+r+s)}\varphi^i Z^q\varphi^i Z^r\varphi^j Z^s\varphi^j\right)\,, && (5.43)\\
\mathscr{O}^{(d_2)}_{\mathbf{1};J;n_1,n_2,n_3} &= \frac{1}{\sqrt{J^3\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+4}}}\sum_{\substack{q,r,s=0\\ q+r+s\le J}}^{J}{\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J}\,\times \\
&\quad\times \mathrm{Tr}\left(Z^{J-(q+r+s)}\varphi^i Z^q\varphi^j Z^r\varphi^i Z^s\varphi^j\right)\,. && (5.44)
\end{aligned}
$$

The calculation of instanton contributions to the two-point functions of these operators proceeds in complete analogy with the discussion in the previous subsection. These operators are expected to receive contributions of the same type as the ε-singlet (5.13) and to mix with the latter much in the same way as was found in the string theory analysis of [14].

Other singlet operators can be constructed using all the other combinations of impurities in table 2. In all these cases the analysis of fermion zero modes shows that a non-zero contribution to the corresponding two-point functions can arise at the same leading order as (5.40). Operators containing $D_\mu Z$ insertions correspond to string states involving bosonic oscillators which are vectors of the second SO(4). Operators containing the $\psi^{-\,a}_{a}$ and $\psi^{+\,\dot\alpha}_{\dot a}$ fermions are dual to states created by the $S^{\pm}$ oscillators. The calculation of two-point functions of all these operators is similar to that described in the previous section with the additional technical complication that in the presence of covariant derivatives the solution $A^{(4)}_\mu$ for the gauge potential is needed and for operators containing fermions the solution $\lambda^{(5)A}_\alpha$ is needed.

As observed in the case of the operator (5.13), two-point functions in the semi-classical approximation factorise, with the two operators being related only by the five-sphere integration. Because of this property mixing is expected among all the operators which receive instanton contributions.

In the $\mathcal{N}=4$ theory it is in principle possible to construct a large number of other operators which potentially mix with those considered here, being $\mathrm{SO}(4)_R\times\mathrm{SO}(4)_C$ singlets with $\Delta - J = 4$. These involve $\Delta - J = 2$ impurities and thus do not correspond to new independent states having vanishing two-point functions in free theory. However, it is known that the inclusion of such operators is needed in perturbation theory to properly resolve the mixing beyond the zeroth order approximation in the g_2 expansion. Since instanton effects are exponentially suppressed in g_2 one should in principle expect these operators to be relevant at leading order in the instanton background. This is, however, not the case. The combinatorial analysis involved in computing the classical profiles of the operators shows that those containing $\Delta - J = 2$ impurities are suppressed in the large J limit.

5.2.3 Operators in other sectors

As observed in section 2.2.2 the spectrum of four impurity BMN operators is rather rich. Instanton contributions to the anomalous dimensions of operators in other sectors can be studied with the same methods used for the singlets. D-instanton induced amplitudes for string states in the plane wave background dual to non-singlet operators are suppressed with respect to those in the singlet sector. Hence string theory predicts that the leading instanton contributions to the anomalous dimensions of non-singlet four impurity operators should be suppressed with respect to (5.40). More precisely the string prediction is that the leading non-zero contributions should arise at order $\mathrm{e}^{2\pi i\tau}(g_2)^{7/2}(\lambda')^2$. We shall not discuss in detail the calculation of two-point functions needed to verify this prediction, but we present here an argument indicating that the gauge theory result is indeed in agreement with string theory. We focus on an operator with four scalar impurities which is a singlet of $\mathrm{SO}(4)_C$ and belongs to the $\mathbf{3}^+\oplus\mathbf{3}^-$ of $\mathrm{SO}(4)_R$,

$$\mathcal{O}^{[ij]}_{\mathbf{3}^+\oplus\mathbf{3}^-;J;n_1,n_2,n_3} = \frac{1}{\sqrt{J^3\left(\frac{g^2_{\mathrm{YM}}N}{8\pi^2}\right)^{J+4}}}\sum_{\substack{q,r,s=0\\ q+r+s\leq J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J}\times$$
$$\times\,\mathrm{Tr}\left(Z^{J-(q+r+s)}\varphi^k Z^q \varphi^k Z^r \varphi^{[i} Z^s \varphi^{j]}\right)\,. \quad (5.45)$$

The study of other non-singlet operators is completely analogous. Considering for concreteness the component $i=1$, $j=2$ in (5.45), we find that using for all the scalars the bilinear solution the combinations of fermion modes contained in the classical profiles of the operator and its conjugate are respectively

$$\left({}_{m}{}^{1}_{\mathrm{f}}\right)^{J+3}\left({}_{m}{}^{2}_{\mathrm{f}}\right)^{2}\left({}_{m}{}^{3}_{\mathrm{f}}\right)^{2}\left({}_{m}{}^{4}_{\mathrm{f}}\right)^{J+1}+ \quad (5.46)$$
$$+\left({}_{m}{}^{1}_{\mathrm{f}}\right)^{J+2}\left({}_{m}{}^{2}_{\mathrm{f}}\right)^{3}\left({}_{m}{}^{3}_{\mathrm{f}}\right)\left({}_{m}{}^{4}_{\mathrm{f}}\right)^{J+2}+ \quad (5.47)$$
$$+\left({}_{m}{}^{1}_{\mathrm{f}}\right)^{J+2}\left({}_{m}{}^{2}_{\mathrm{f}}\right)\left({}_{m}{}^{3}_{\mathrm{f}}\right)^{3}\left({}_{m}{}^{4}_{\mathrm{f}}\right)^{J+2}+ \quad (5.48)$$

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$$+\left(m\,{}^{1}_{\mathrm{f}}\right)^{J+1}\left(m\,{}^{2}_{\mathrm{f}}\right)^{2}\left(m\,{}^{3}_{\mathrm{f}}\right)^{2}\left(m\,{}^{4}_{\mathrm{f}}\right)^{J+3} \tag{5.49}$$

and

$$\left(m\,{}^{1}_{\mathrm{f}}\right)^{3}\left(m\,{}^{2}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{3}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{4}_{\mathrm{f}}\right)+ \tag{5.50}$$
$$+\left(m\,{}^{1}_{\mathrm{f}}\right)^{2}\left(m\,{}^{2}_{\mathrm{f}}\right)^{J+3}\left(m\,{}^{3}_{\mathrm{f}}\right)^{J+1}\left(m\,{}^{4}_{\mathrm{f}}\right)^{2}+ \tag{5.51}$$
$$+\left(m\,{}^{1}_{\mathrm{f}}\right)^{2}\left(m\,{}^{2}_{\mathrm{f}}\right)^{J+1}\left(m\,{}^{3}_{\mathrm{f}}\right)^{J+3}\left(m\,{}^{4}_{\mathrm{f}}\right)^{2}+ \tag{5.52}$$
$$+\left(m\,{}^{1}_{\mathrm{f}}\right)\left(m\,{}^{2}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{3}_{\mathrm{f}}\right)^{J+2}\left(m\,{}^{4}_{\mathrm{f}}\right)^{3}. \tag{5.53}$$

This shows that the integrations over the superconformal modes in the two-point function can be saturated, selecting terms containing (5.46) or (5.49) in $\mathscr{O}^{[12]}$ and terms containing (5.51) and (5.52) in $\bar{\mathscr{O}}^{[12]}$. However, with these choices the resulting five-sphere integrals vanish. For instance combining (5.46) and (5.51) leads to the following moduli space integrals

$$\begin{aligned}&\int \mathrm{d}^8\eta\mathrm{d}^8\bar{\xi}\left[\left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\right](x_1)\left[\left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\right](x_2)\times\\ &\qquad\times\int \mathrm{d}^{4(N-2)}\nu\mathrm{d}^{4(N-2)}\bar{\nu}\left(\bar{\nu}^{[1}\nu^{4]}\right)^{J-1}\left(\bar{\nu}^{(1}\nu^{1)}\right)\left(\bar{\nu}^{[2}\nu^{3]}\right)^{J-1}\left(\bar{\nu}^{(2}\nu^{2)}\right).\end{aligned} \tag{5.54}$$

The integration over the five-sphere arising from the second line of (5.54) vanishes because the multiplicity of the flavours 1 and 2 exceeds that of the flavours 3 and 4.

In order to soak up the superconformal modes while avoiding the obstruction from the five sphere integral it is necessary to include a six-fermion term in each operator. In this way the combinations of modes in the two operators are the same as in (5.46)-(5.53) with the addition of one mode of each flavour. The same arguments given in section 5.1 in connection with two impurity operators can be repeated here and for instance combining (5.46) and (5.53) we get moduli space integrations of the type

$$\begin{aligned}&\int \mathrm{d}^8\eta\mathrm{d}^8\bar{\xi}\left[\left(\zeta^1\right)^2\zeta^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\bar{\xi}^1\right](x_1)\left[\zeta^1\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\bar{\xi}^2\right](x_2)\times\\ &\qquad\times\int \mathrm{d}^{4(N-2)}\nu\mathrm{d}^{4(N-2)}\bar{\nu}\left(\bar{\nu}^{[1}\nu^{4]}\right)^{J+1}\left(\bar{\nu}^{[2}\nu^{3]}\right)^{J+1}\left(\bar{\nu}^{[1}\nu^{2]}\right)\left(\bar{\nu}^{[3}\nu^{4]}\right).\end{aligned} \tag{5.55}$$

Just as in the two impurity case the resulting non-vanishing contribution to the two-point function is suppressed by a factor of $(\lambda')^2$ due to the additional $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears in (5.55). In conclusion the analysis of fermion zero modes confirms that the leading non-zero instanton contribution to the anomalous dimensions of four impurity operators in the $\mathbf{3}^+$ and $\mathbf{3}^-$ representations behaves as

$$\gamma^{\mathrm{inst}}_{\mathbf{3}^+\oplus\mathbf{3}^-}\sim \mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}(g_2)^{7/2}(\lambda')^2\,, \tag{5.56}$$

in agreement with the string prediction of [14].

Other sectors can be analysed in a similar fashion and we find that all non-singlet operators receive leading non-zero contributions of the same order as (5.56).

6. Discussion and conclusions

This paper has considered one-instanton contributions to two-point correlation functions of gauge invariant operators in the BMN sector of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory. These determine the leading instanton contributions to the anomalous dimensions of operators dual to physical string states in the maximally supersymmetric plane-wave background obtained as Penrose limit of $\mathrm{AdS}_5 \times S^5$. The basic message is that we find striking agreement between these instanton effects in the gauge theory and those of the plane-wave string theory calculated in [14].

We focused on operators with two and four scalar impurities. The four impurity case, although more involved, is fully under control, whereas the two impurity case presents subtleties due to the fact the leading semi-classical approximation vanishes and the first non-zero contribution arises at higher order. We have explicitly computed a two-point function of four impurity operators which are $\mathrm{SO}(4)_R \times \mathrm{SO}(4)_C$ singlets. Our analysis shows that instanton induced contributions to the anomalous dimensions of operators in this sector behave as $1/(n_1 n_2)^2 \exp\left(-8\pi^2/g_2\lambda' + i\theta\right) g_2^{7/2}$, where λ' and g_2 are the effective coupling constant and genus counting parameter in the BMN limit and n_1 and n_2 correspond to the mode numbers of the dual string state. The result is in perfect agreement with the D-instanton correction to the mass matrix elements of the corresponding states in the plane-wave string theory which was computed in [14]. Even without directly matching the numerical values of the anomalous dimensions and the string mass renormalisation, the agreement with the string calculation appears highly non-trivial. The correct dependence on the parameters λ' and g_2 is obtained by combining contributions arising from the integrations over the instanton moduli space and various combinatorial factors. Even more impressively, the mode number dependence found in [14] is reproduced after spectacular cancellations.

In the case of two impurity operators the leading instanton correction vanishes. The first non-zero contribution is awkward to calculate completely, but with mild assumptions we showed that it has the form $\exp\left(-8\pi^2/g_2\lambda' + i\theta\right) g_2^{7/2}\lambda'^2$ and does not depend on the single mode number characterising the dual string state. This behaviour is also in agreement with the results of [14], although the subtleties presented by the gauge theory calculation did not arise on the string side. Four impurity operators in sectors other than the singlet have the same λ' and g_2 dependence as two impurity operators, again in agreement with the string prediction of [14].

Our results provide a significant new test of the duality proposed in [1]. The fact that non-perturbative contributions obey BMN scaling, i.e. can be re-expressed in terms of the effective parameters λ' and g_2, strongly supports the conjecture that this property should hold at all orders. This can be further tested by analysing subleading effects in the instanton background. A class of higher order contributions can easily be obtained from the calculations presented in this paper, relaxing the requirement that all the fermion modes of type ν and $\bar{\nu}$ be combined in $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears. As already observed, replacing a $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinear with a $(\bar{\nu}\nu)_{\mathbf{10}}$ leads to a suppression by a factor of $1/\sqrt{N}$, see (3.17). The profile of BMN operators of the type that we considered contains J $(\bar{\nu}\nu)$'s after the superconformal modes have been soaked up. Hence each factor of $1/\sqrt{N}$ coming from the replacement of a $(\bar{\nu}\nu)_{\mathbf{6}}$ by a $(\bar{\nu}\nu)_{\mathbf{10}}$ is associated with a factor of J corresponding to the number of choices for the $(\bar{\nu}\nu)_{\mathbf{10}}$ bilinear, resulting in a suppression by $g_2^{1/2}$. Moreover in order to get a non-zero result from the five sphere integration an even number of $(\bar{\nu}\nu)_{\mathbf{10}}$ is required. Therefore contributions of this type with an increasing number of $(\bar{\nu}\nu)_{\mathbf{10}}$ insertions give rise to subleading corrections which form a series in integer powers of g_2. More complicated terms with the same behaviour correspond to contributions in which pairs of fields are contracted between the two-operators. We can also identify a class of subleading corrections suppressed by powers of λ'. These are generated by including in the profile of the operators a number of fermion modes greater than the minimal number required by the moduli space integration. For instance in the case of the four impurity singlets that we studied in section 5.2.1 this is achieved by including one six-fermion scalar in one of the two operators. In this case the calculation is analogous to that of section 5.1 for the two impurity case and we can argue that the resulting contribution should be suppressed by a factor of λ'. Including more six-fermion terms gives rise to higher powers of λ'. Although these arguments are rather qualitative they indicate how the perturbative series of corrections to the semi-classical one-instanton contributions can be reorganised into a double series in λ' and g_2.

Some rather striking general properties of the D-instanton induced corrections to the string mass spectrum of [14] can be immediately deduced from the structure of the D-instanton boundary state in the plane-wave background, whereas the corresponding effects in the gauge theory are far from obvious. In fact, the string theory results suggest a number of extensions and generalisations of the gauge theory results. For example, one generic feature of the string calculation is that only states with an even number of non-zero mode insertions receive D-instanton corrections. Zero mode oscillators can appear in odd numbers with the condition that they be contracted into a $\mathrm{SO}(4) \times \mathrm{SO}(4)$ scalar between the incoming and outgoing states. The simplest example in which these properties can be verified involves five impurity operators and the calculation of the necessary two-point function in the gauge theory is extremely complicated[5]. In general, contributions to operators with a larger (even) number of impurities are expected to be non-zero at leading order in the instanton background. However, the complexity of the combinatorics involved in such calculations grows rapidly with the number of impurities.

Another peculiarity observed in the string theory calculation [14] is that the D-instanton contribution to the masses of certain states with a large number of fermionic non-zero mode excitations involves large powers of the mass parameter m. When expressed in terms of gauge theory parameters this corresponds to large inverse powers of λ'. As observed in [14] the behaviour of these mass corrections is not pathological in the $\lambda' \to 0$ limit, because the inverse powers of λ' are accompanied by the instanton factor $\exp\left(-8\pi^2/g_2\lambda'\right)$. From the point of view of the gauge theory this result is particularly intriguing not only because of the unusual coupling constant dependence that the anomalous

[5] Three impurity operators present technical difficulties similar to those encountered in the two impurity case.

dimensions of the dual operators should display, but also because there are no other known examples of operators in $\mathcal{N}=4$ SYM whose anomalous dimension receives instanton but not perturbative corrections. We will study this particular class of BMN operators in a future publication [25].

In the original formulation of the AdS/CFT duality, relating $\mathcal{N}=4$ SYM to type-IIB string theory in $\mathrm{AdS}_5 \times S^5$, the effects of multi-instantons in the large-N limit of the gauge theory and of multi D-instantons in string theory were shown to be in remarkable agreement [16]. Clearly it would be of interest to generalise the present work from the one-instanton sector to the multi-instanton sector. However, such a generalisation is technically very challenging both on the string and on the gauge side.

Instanton effects have been studied in a number of different supersymmetric field theories [34 − 40] in the context of the AdS/CFT correspondence, and agreement has been found between string and gauge theory. The example of the $\mathcal{N}=2$ $\mathrm{Sp}(N)$ superconformal field theory studied in [34, 37, 38] is particularly interesting in connection with our work because in this case the analogue of the BMN limit has been studied in [41]. In this case the duality involves a theory of open and closed strings in a plane-wave background and the dual gauge theory has a rich spectrum of gauge-invariant operators and possesses a Higgs branch [42]. It would be interesting to study instanton effects in the BMN sector of this theory.

In a conformal field theory, the problem of computing the scaling dimensions of gauge invariant operators can be reformulated as an eigenvalue problem for the dilation operator of the theory. At the perturbative level this observation leads to a very efficient approach to the calculation of anomalous dimensions in $\mathcal{N}=4$ SYM [43]. Some comments about the possibility of extending this approach to non-perturbative sectors were made in [27], but there has been no further progress in this direction. A remarkable consequence of recasting the problem of computing anomalous dimensions as an eigenvalue problem for the dilation operator is the emergence of connections with integrable systems. In the planar limit the dilation operator can be related to the hamiltonian of an integrable spin chain, leading to the possibility of applying techniques such as the Bethe ansatz to the computation of anomalous dimensions [44], see [45] for a review and references. The integrability structure observed in the $\mathcal{N}=4$ Yang-Mills theory appears, however, to be spoiled by the inclusion of non-planar contributions. Therefore instanton effects, which are exponentially suppressed in the large-N limit, are unlikely to be relevant in connection with integrability.

Instantons play a special rôle in the $\mathcal{N}=4$ theory in connection with the $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry, which transforms the complex coupling so that $\tau \to \frac{a\tau+b}{c\tau+d}$ (with $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = 1$) thereby mixing perturbative and non-perturbative effects. In the conformal phase invariance of the theory requires that the full spectrum of scaling dimensions be invariant. The anomalous dimensions are therefore naturally expressed as functions $\gamma(\tau, \bar{\tau})$. Similarly D-instantons are instrumental in the implementation of S-duality in type-IIB string theory. Their rôle is well understood at the level of the effective action for the supergravity states, but little is known at the level of the massive string excitations. As in the SYM case, invariance of the theory requires that the complete spectrum be invariant. In general $\mathrm{SL}(2,\mathbb{Z})$ transformations relate operators of small and

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large dimension, just as in string theory they relate fundamental strings to D-strings, which have large masses of order $1/g_s$, in the limit of weak string coupling, $g_s \ll 1$. It would be interesting to understand how S-duality is realised in type-IIB string theory in the plane-wave background. A corresponding symmetry should exist in the BMN sector of $\mathcal{N}=4$ SYM and the instanton effects which we studied in this paper should be important in its implementation.

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A. Useful formulae

This appendix contains some definitions and formulae used in the paper. The Clebsch-Gordan coefficients Σ^i_{AB} ($\bar{\Sigma}^{AB}_i$) projecting the product of two $\mathbf{4}$'s ($\overline{\mathbf{4}}$'s) onto the $\mathbf{6}$ are defined as

$$\begin{aligned}\Sigma^i_{AB} &= (\Sigma^a_{AB}, \Sigma^{a+3}_{AB}) = (\eta^a_{AB}, i\bar{\eta}^a_{AB}) \\ \bar{\Sigma}^{AB}_i &= (\bar{\Sigma}^a_{AB}, \bar{\Sigma}^{a+3}_{AB}) = (-\eta^{AB}_a, i\bar{\eta}^{AB}_a)\,,\end{aligned} \tag{A.1}$$

where $a = 1, 2, 3$ and the 't Hooft symbols η^a_{AB} and $\bar{\eta}^a_{AB}$ are

$$\begin{aligned}&\eta^a_{AB} = \bar{\eta}^a_{AB} = \varepsilon_{aAB}\,, \qquad A, B = 1, 2, 3\,, \\ &\eta^a_{A4} = \bar{\eta}^a_{4A} = \delta^a_A\,, \\ &\eta^a_{AB} = -\eta^a_{BA}\,, \qquad\qquad \bar{\eta}^a_{AB} = -\bar{\eta}^a_{BA}\,.\end{aligned} \tag{A.2}$$

In some situations the $\mathcal{N}=1$ formulation proves very useful. The $\mathcal{N}=1$ decomposition of the $\mathcal{N}=4$ supermultiplet consists of three chiral multiplets and one vector multiplet and under this decomposition only a $\mathrm{SU}(3) \times \mathrm{U}(1)$ subgroup of the SU(4) R-symmetry group is manifest. The six scalars are combined into three complex fields, ϕ^I, $I = 1, 2, 3$, according to

$$\begin{aligned}\phi^I &= \frac{1}{\sqrt{2}}\left(\hat{\varphi}^I + i\hat{\varphi}^{I+3}\right) \\ \phi^\dagger_I &= \frac{1}{\sqrt{2}}\left(\hat{\varphi}^I - i\hat{\varphi}^{I+3}\right).\end{aligned} \tag{A.3}$$

The complex scalars ϕ^I and $\phi^\dagger_I$ transform respectively in the $\mathbf{3}_1$ and $\overline{\mathbf{3}}_{-1}$ of $\mathrm{SU}(3) \times \mathrm{U}(1)$ (where the subscript indicates the U(1) charge). The fermions in the chiral multiplets are

$$\psi^I_\alpha = \lambda^I_\alpha\,, \qquad \bar{\psi}^{\dot{\alpha}}_I = \bar{\lambda}^{\dot{\alpha}}_I\,, \qquad I = 1, 2, 3\,, \tag{A.4}$$

transforming in the $\mathbf{3}_{3/2}$ and $\overline{\mathbf{3}}_{-3/2}$. The fourth fermion and the vector form the $\mathcal{N}=1$ vector multiplet, $\{\lambda_\alpha = \lambda^4_\alpha\,, A_\mu\}$, and are $\mathrm{SU}(3) \times \mathrm{U}(1)$ singlets.

Using these definitions we find the following relations among the scalars in the different formulations

$$\begin{aligned}
\hat\varphi^1 &= \frac{\sqrt{2}}{2}\left(\varphi^{14}+\varphi^{23}\right), \quad \hat\varphi^2 = \frac{\sqrt{2}}{2}\left(-\varphi^{13}+\varphi^{24}\right), \quad \hat\varphi^3 = \frac{\sqrt{2}}{2}\left(\varphi^{12}+\varphi^{34}\right), \\
\hat\varphi^4 &= \frac{i\sqrt{2}}{2}\left(-\varphi^{14}+\varphi^{23}\right), \quad \hat\varphi^5 = \frac{i\sqrt{2}}{2}\left(-\varphi^{13}-\varphi^{24}\right), \quad \hat\varphi^6 = \frac{i\sqrt{2}}{2}\left(\varphi^{12}-\varphi^{34}\right)
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
\phi^1 &= 2\varphi^{14}, \quad \phi^2 = 2\varphi^{24}, \quad \phi^3 = 2\varphi^{34}, \\
\phi^\dagger_1 &= 2\varphi^{23}, \quad \phi^\dagger_2 = -2\varphi^{13}, \quad \phi^\dagger_3 = 2\varphi^{12}.
\end{aligned} \tag{A.6}$$

In the ADHM formalism the expressions for the $\mathscr{N}=4$ elementary fields in the background of an instanton are conveniently given as $[N+2]\times[N+2]$ matrices. In particular, the two-fermion solution for the scalar field φ^{AB} in the one instanton sector can be written in the block-form

$$\begin{aligned}
\left(\hat\varphi^{(1)AB}\right)^{v}_{u;} &= \frac{1}{4(y^2+\rho^2)^2}\Big\{y^2\Big[-16(\bar\xi^{\dot\alpha B}\bar\xi^{A}_{\dot\beta}-\bar\xi^{\dot\alpha A}\bar\xi^{B}_{\dot\beta})w_{u;\dot\alpha}\bar w^{\dot\beta;v}+ \\
&\qquad +4w_{u;\dot\alpha}(\bar\xi^{\dot\alpha B}\bar\nu^{A\,v}-\bar\xi^{\dot\alpha A}\bar\nu^{B\,v})\Big]+ \\
&\qquad +(y^2+\rho^2)\left[-4(\bar\xi^{B}_{\dot\alpha}\nu^{A}_{u}-\bar\xi^{A}_{\dot\alpha}\nu^{B}_{u})\bar w^{\dot\alpha;v}+(\nu^{B}_{u}\bar\nu^{A\,v}-\nu^{A}_{u}\bar\nu^{B\,v})\right]+ \\
&\qquad +y^{\dot\alpha\delta}\Big[16(\eta^{B}_{\delta}\bar\xi^{A}_{\dot\beta}-\eta^{A}_{\delta}\bar\xi^{B}_{\dot\beta})w_{u;\dot\alpha}\bar w^{\dot\beta;v}-4w_{u;\dot\alpha}(\eta^{B}_{\delta}\bar\nu^{Av}-\eta^{A}_{\delta}\bar\nu^{Bv})\Big]\Big\} \\
\left(\hat\varphi^{(2)AB}\right)^{\gamma}_{u;} &= \frac{1}{4(y^2+\rho^2)^2}\Big\{16y^2w_{u;\dot\alpha}(\bar\xi^{\dot\alpha B}\eta^{\gamma A}-\bar\xi^{\dot\alpha A}\eta^{\gamma B})+4(y^2+\rho^2)(\nu^{B}_{u}\eta^{\gamma A}-\nu^{A}_{u}\eta^{\gamma B})- \\
&\qquad -w_{u;\dot\alpha}\Big[16y^{\dot\alpha\delta}(\eta^{B}_{\delta}\eta^{\gamma A}-\eta^{A}_{\delta}\eta^{\gamma B})+ \\
&\qquad\qquad +\frac{1}{2}\frac{y^2+\rho^2}{\rho^2}y^{\dot\alpha\gamma}(\bar\nu^{Au}\nu^{B}_{u}-\bar\nu^{Br}\nu^{A}_{r})\Big]\Big\} \\
\left(\hat\varphi^{(3)AB}\right)^{v}_{\beta;} &= \frac{1}{4(y^2+\rho^2)^2}\Big\{\rho^2\Big[16y_{\beta\dot\alpha}(\bar\xi^{\dot\alpha B}\bar\xi^{A}_{\dot\beta}-\bar\xi^{\dot\alpha A}\bar\xi^{B}_{\dot\beta})\bar w^{\dot\beta;v}-4y_{\beta\dot\alpha}(\bar\xi^{\dot\alpha B}\bar\nu^{Av}-\bar\xi^{\dot\alpha A}\bar\nu^{Bv})- \\
&\qquad -16(\eta^{B}_{\beta}\bar\xi^{A}_{\dot\alpha}-\eta^{A}_{\beta}\bar\xi^{B}_{\dot\alpha})\bar w^{\dot\alpha;v}+4(\eta^{B}_{\beta}\bar\nu^{Av}-\eta^{A}_{\beta}\bar\nu^{Bv})\Big]\Big\} \\
\left(\hat\varphi^{(4)AB}\right)^{\gamma}_{\beta;} &= \frac{\rho^2}{4(y^2+\rho^2)^2}\Big[-16y_{\beta\dot\alpha}(\bar\xi^{\dot\alpha B}\eta^{\gamma A}-\bar\xi^{\dot\alpha A}\eta^{\gamma B})+16(\eta^{B}_{\beta}\eta^{\gamma A}-\eta^{B}_{\beta}\eta^{\gamma A})+ \\
&\qquad +\frac{1}{2}\frac{y^2+\rho^2}{\rho^2}\delta^{\gamma}_{\beta}(\bar\nu^{Ar}\nu^{B}_{r}-\bar\nu^{Br}\nu^{A}_{r})\Big].
\end{aligned} \tag{A.7}$$

B. Instanton induced two-point functions of BMN operators

In this appendix we present some details of the calculations of one-instanton contributions to the two-point functions of BMN operators discussed in section 5.

B.1 Two-impurity operator in the 9 of $\mathrm{SO}(4)_R$

As shown in section 5.1 the leading semi-classical contribution to the two-point functions of two impurity operators in the **9** of $\mathrm{SO}(4)_R$ vanishes because the superconformal modes cannot be soaked up. A non-zero result is obtained including for one scalar field in each operator the six fermion solution.

In the case of the component considered in section 5.1 the terms in the two-point function which contain the correct combination of fermion modes to give a non vanishing contribution are

$$
\begin{aligned}
G_{\mathbf{9}}(x_1,x_2)=&\frac{1}{J\left(\frac{g_{\rm YM}^2 N}{8\pi^2}\right)^{J+2}}\sum_{p,q=0}^{J}\cos\left(\frac{2\pi i p n}{J}\right)\cos\left(\frac{2\pi i q m}{J}\right)\times\\
&\times\Big\{\langle {\rm Tr}\left[\left(Z^{J-p}\varphi^{13}Z^{p}\varphi^{13}\right)(x_1)\right]{\rm Tr}\left[\left(\bar Z^{J-q}\varphi^{24}\bar Z^{q}\varphi^{24}\right)(x_2)\right]\rangle+\\
&\quad+\langle {\rm Tr}\left[\left(Z^{J-p}\varphi^{24}Z^{p}\varphi^{24}\right)(x_1)\right]{\rm Tr}\left[\left(\bar Z^{J-q}\varphi^{13}\bar Z^{q}\varphi^{13}\right)(x_2)\right]\rangle\Big\}\,. \qquad \text{(B.1)}
\end{aligned}
$$

The other terms vanish in the instanton background either because they do not contain all the required superconformal modes or because of the integration over the five-sphere. For instance if one considers $\langle {\rm Tr}\left[\left(Z^{J-p}\varphi^{13}Z^{p}\varphi^{13}\right)(x_1)\right]{\rm Tr}\left[\left(\bar Z^{J-q}\varphi^{13}\bar Z^{q}\varphi^{13}\right)(x_2)\right]\rangle$ it is easy to verify that the superconformal modes can be soaked up, but the resulting five-sphere integral vanishes because among the remaining fermion modes different flavours appear with different multiplicities.

Let us consider the terms in (B.1) where in each trace one scalar is understood to be replaced with the six-fermion solution. In the first expectation value the two traces contain respectively the following combinations of fermion modes

$$
\begin{aligned}
&\left({}_{m}\,{}^{1}_{\mathrm{f}}\right)^{J+3}\left({}_{m}\,{}^{2}_{\mathrm{f}}\right)^{1}\left({}_{m}\,{}^{3}_{\mathrm{f}}\right)^{3}\left({}_{m}\,{}^{4}_{\mathrm{f}}\right)^{J+1}\\
&\left({}_{m}\,{}^{1}_{\mathrm{f}}\right)^{1}\left({}_{m}\,{}^{2}_{\mathrm{f}}\right)^{J+3}\left({}_{m}\,{}^{3}_{\mathrm{f}}\right)^{J+1}\left({}_{m}\,{}^{4}_{\mathrm{f}}\right)^{3}. \qquad \text{(B.2)}
\end{aligned}
$$

Using the fact that each trace contains one $\bar\xi$ mode not part of a ζ we can soak up the superconformal modes selecting the following combinations of fermion modes in the two traces

$$
\begin{aligned}
&{\rm Tr}\left(Z^{J-p}\varphi^{13}Z^{p}\varphi^{13}\right)\to\left(\zeta^1\right)^2\zeta^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\bar\xi^1\left(\bar\nu^{[1}\nu^{4]}\right)^{J-1}\left(\bar\nu^{[1}\nu^{3]}\right)\\
&{\rm Tr}\left(\bar Z^{J-p}\varphi^{24}\bar Z^{p}\varphi^{24}\right)\to\zeta^1\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\bar\xi^2\left(\bar\nu^{[2}\nu^{3]}\right)^{J-1}\left(\bar\nu^{[2}\nu^{4]}\right). \qquad \text{(B.3)}
\end{aligned}
$$

Similarly the two traces in the second term in (B.1) contain respectively

$$
\left({}_{m}\,{}^{1}_{\mathrm{f}}\right)^{J+1}\left({}_{m}\,{}^{2}_{\mathrm{f}}\right)^{3}\left({}_{m}\,{}^{3}_{\mathrm{f}}\right)^{1}\left({}_{m}\,{}^{4}_{\mathrm{f}}\right)^{J+3} \qquad \text{(B.4)}
$$

and

$$
\left({}_{m}\,{}^{1}_{\mathrm{f}}\right)^{3}\left({}_{m}\,{}^{2}_{\mathrm{f}}\right)^{J+1}\left({}_{m}\,{}^{3}_{\mathrm{f}}\right)^{J+3}\left({}_{m}\,{}^{4}_{\mathrm{f}}\right)^{1} \qquad \text{(B.5)}
$$

and we need to consider

$$
\begin{aligned}
&{\rm Tr}\left(Z^{J-p}\varphi^{24}Z^{p}\varphi^{24}\right)\to\left(\zeta^1\right)^2\left(\zeta^2\right)^2\zeta^3\left(\zeta^4\right)^2\bar\xi^4\left(\bar\nu^{[1}\nu^{4]}\right)^{J-1}\left(\bar\nu^{[2}\nu^{4]}\right)\\
&{\rm Tr}\left(\bar Z^{J-p}\varphi^{13}\bar Z^{p}\varphi^{13}\right)\to\left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\zeta^4\bar\xi^3\left(\bar\nu^{[2}\nu^{3]}\right)^{J-1}\left(\bar\nu^{[1}\nu^{3]}\right). \qquad \text{(B.6)}
\end{aligned}
$$

These expressions contain the correct combinations of fermion superconformal modes such that the corresponding integration is non-zero. The two terms in (B.1) give rise to

$$\begin{aligned}\int \prod_{A=1}^{4} \mathrm{d}^2\eta^A\, \mathrm{d}^2\bar{\xi}^A \Big\{ &\Big[\left(\zeta^1\right)^2 \zeta^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2 \bar{\xi}^1\Big](x_1) \Big[\zeta^1 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \left(\zeta^4\right)^2 \bar{\xi}^2\Big](x_2) + \\ &+ \Big[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \zeta^3 \left(\zeta^4\right)^2 \bar{\xi}^4\Big](x_1) \Big[\left(\zeta^1\right)^2 \left(\zeta^2\right)^2 \left(\zeta^3\right)^2 \zeta^4 \bar{\xi}^3\Big](x_2)\Big\} \sim \\ &\sim (x_1 - x_0)\cdot(x_2 - x_0)\,(x_1 - x_2)^4 .\end{aligned} \tag{B.7}$$

After re-expressing the $(\bar{\nu}^A \nu^B)$ bilinears in terms of Ω^{AB}'s as described in section 3 both (B.3) and (B.6) lead to the same five-sphere integral,

$$I_{S^5} = \int \mathrm{d}^5\Omega\, \left(\Omega^{14}\right)^{J-1} \left(\Omega^{23}\right)^{J-1} \Omega^{13}\Omega^{24} . \tag{B.8}$$

This can be evaluated rewriting it as

$$I_{S^5} = \int_{\sum_{i=1}^{6}\Omega_i^2=1} \mathrm{d}^6\Omega\, \left(\Sigma^{14}_i\Omega^i\right)^{J-1} \left(\Sigma^{23}_j\Omega^j\right)^{J-1} \left(\Sigma^{13}_k\Omega^k\right) \left(\Sigma^{24}_l\Omega^l\right) \tag{B.9}$$

where the symbols Σ^{AB}_i are defined in (A.1). Defining $\Omega \equiv \Sigma^{14}_i\Omega^i = (\Omega^1 + i\Omega^4)$, $\bar{\Omega} \equiv \Sigma^{23}_i\Omega^i = (\Omega^1 - i\Omega^4)$, $\widetilde{\Omega} \equiv \Sigma^{13}_i\Omega^i = (\Omega^2 + i\Omega^5)$ and $\bar{\widetilde{\Omega}} \equiv \Sigma^{24}_i\Omega^i = (\Omega^2 - i\Omega^5)$ the integral reduces to

$$\begin{aligned} I_{S^5} &= \int \mathrm{d}^6\Omega\, \delta\!\left(\sum_{i=1}^{6}\Omega_i^2 - 1\right) \left(\Omega\bar{\Omega}\right)^{J-1} \left(\widetilde{\Omega}\bar{\widetilde{\Omega}}\right) \\ &= \int \mathrm{d}\Omega\, \mathrm{d}\bar{\Omega}\, \mathrm{d}\widetilde{\Omega}\, \mathrm{d}\bar{\widetilde{\Omega}}\, \mathrm{d}^2\Omega^I\, \delta(\Omega^I\Omega^I + \Omega\bar{\Omega} + \widetilde{\Omega}\bar{\widetilde{\Omega}} - 1) \left(\Omega\bar{\Omega}\right)^{J-1} \left(\widetilde{\Omega}\bar{\widetilde{\Omega}}\right), \end{aligned} \tag{B.10}$$

where $\Omega^I = (\Omega^3, \Omega^6)$. Introducing polar coordinates for the Ω^I directions

$$\begin{aligned} I_{S^5} &= 2\pi \int \mathrm{d}r\, r\, \mathrm{d}\Omega\, \mathrm{d}\bar{\Omega}\, \mathrm{d}\widetilde{\Omega}\, \mathrm{d}\bar{\widetilde{\Omega}}\, \delta(r^2 + \Omega\bar{\Omega} + \widetilde{\Omega}\bar{\widetilde{\Omega}} - 1) \left(\Omega\bar{\Omega}\right)^{J-1} \left(\widetilde{\Omega}\bar{\widetilde{\Omega}}\right) \\ &= \pi \int_{\Omega\bar{\Omega} + \widetilde{\Omega}\bar{\widetilde{\Omega}} \leq 1} \mathrm{d}\Omega\, \mathrm{d}\bar{\Omega}\, \mathrm{d}\widetilde{\Omega}\, \mathrm{d}\bar{\widetilde{\Omega}}\, \left(\Omega\bar{\Omega}\right)^{J-1} \left(\widetilde{\Omega}\bar{\widetilde{\Omega}}\right). \end{aligned} \tag{B.11}$$

The remaining integrals are straightforward

$$\begin{aligned} I_{S^5} &= 2\pi^2 \int_{\Omega\bar{\Omega}\leq 1} \mathrm{d}\Omega\, \mathrm{d}\bar{\Omega}\, \left(\Omega\bar{\Omega}\right)^{J-1} \int_0^{\sqrt{1-\Omega\bar{\Omega}}} \mathrm{d}z z^3 \\ &= \frac{\pi^2}{2} \int_{\Omega\bar{\Omega}\leq 1} \mathrm{d}\Omega\, \mathrm{d}\bar{\Omega}\, \left(1 - \Omega\bar{\Omega}\right)^2 \left(\Omega\bar{\Omega}\right)^{J-1} = \frac{\pi^3}{2} \int_0^1 \mathrm{d}z\, z \left(1 - z^2\right)^2 z^{2J} \\ &= \frac{\pi^3}{J(J+1)(J+2)} . \end{aligned} \tag{B.12}$$

As observed in section 5.1 the exact dependence on the bosonic moduli in the two-point function $G_{\mathbf{9}}(x_1, x_2)$ cannot be determined without knowing the six-fermion solution. Dimensional analysis indicates that the final bosonic integrations over position and size of the instanton are logarithmically divergent as expected in the presence of an instanton contribution to the anomalous dimension of the operator $\mathscr{O}^{\{ij\}}$.

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B.2 ε-singlet four impurity operator

In order to compute the profiles of the operator $\mathscr{O}_{\mathbf{1}}$ in (5.13) and its conjugate, which are needed in the calculation of the two-point function (5.16), we have to evaluate the traces (5.23). In the instanton background these are rewritten as traces of $[N+2]$-dimensional ADHM matrices. To compute these traces more efficiently it is convenient to define the $[N+2]\times[N+2]$ matrix

$$\left[U^{C_1,D_1;C_2,D_2}_{k_1,k_2}(\zeta,\nu,\bar\nu)\right]_{u,\alpha}^{v,\beta} = \left[\left(\check\varphi^{14}\right)^{k_1}\widehat\varphi^{C_1D_1}\left(\check\varphi^{14}\right)^{k_2}\widehat\varphi^{C_2D_2}\right]_{u,\alpha}^{v,\beta}. \tag{B.13}$$

where the notation used is that introduced in (5.20)-(5.22). This has the standard block-form of ADHM matrices and the range of the indices here is the same as in (A.7) for the elementary scalar fields.

In terms of the matrix $U^{C_i,D_i;C_j,D_j}_{k_i,k_j}(\zeta,\nu,\bar\nu)$ all the 35 traces we are interested in can be written as

$$\mathrm{Tr}\left[U^{C_1,D_1;C_2,D_2}_{k_1,k_2}(\zeta,\nu,\bar\nu)U^{C_3,D_3;C_4,D_4}_{k_3,k_4}(\zeta,\nu,\bar\nu)U^{C_5,D_5;C_6,D_6}_{k_5,k_6}(\zeta,\nu,\bar\nu)U^{C_7,D_7;C_8,D_8}_{k_7,k_8}(\zeta,\nu,\bar\nu)\right], \tag{B.14}$$

for appropriate choices of the indices C_i, D_i and the exponents k_i, $i=1,\ldots,8$. For example the three traces written explicitly in (5.23) become

$$\mathrm{Tr}\left[U^{1,4;1,4}_{p_1,p_2}(\zeta,\nu,\bar\nu)U^{1,4;1,4}_{p_3,p_4}(\zeta,\nu,\bar\nu)U^{A_1,B_1;A_2,B_2}_{p_5,q}(\zeta,\nu,\bar\nu)U^{A_3,B_3;A_4,B_4}_{r,s}(\zeta,\nu,\bar\nu)\right],$$
$$\mathrm{Tr}\left[U^{1,4;1,4}_{p_1,p_2}(\zeta,\nu,\bar\nu)U^{1,4;A_1,B_1}_{p_3,p_4}(\zeta,\nu,\bar\nu)U^{1,4;A_2,B_2}_{q_1,q_2}(\zeta,\nu,\bar\nu)U^{A_3,B_3;A_4,B_4}_{r,s}(\zeta,\nu,\bar\nu)\right]$$

and

$$\mathrm{Tr}\left[U^{A_1,B_1;A_2,B_2}_{p,q}(\zeta,\nu,\bar\nu)U^{A_3,B_3;1,4}_{r,s_1}(\zeta,\nu,\bar\nu)U^{1,4;1,4}_{s_2,s_3}(\zeta,\nu,\bar\nu)U^{1,4;A_4,B_4}_{s_4,s_5}(\zeta,\nu,\bar\nu)\right].$$

The generic trace (B.14) is thus the only one that needs to be evaluated. It can be computed using the building blocks (A.7) and the result is

$$\begin{aligned}
&\mathrm{Tr}\Big[U^{C_1,D_1;C_2,D_2}_{k_1,k_2}(\zeta,\nu,\bar\nu)U^{C_3,D_3;C_4,D_4}_{k_3,k_4}(\zeta,\nu,\bar\nu)\times\\
&\quad\times U^{C_5,D_5;C_6,D_6}_{k_5,k_6}(\zeta,\nu,\bar\nu)U^{C_7,D_7;C_8,D_8}_{k_7,k_8}(\zeta,\nu,\bar\nu)\Big]=\frac{1}{2^{3J-8}}\frac{\rho^8}{[(x-x_0)^2+\rho^2]^{J+8}}\left(\bar\nu^{[1}\nu^{4]}\right)^{J-8}\times\\
&\quad\times\Big\{\left[\left(\zeta^{D_1}\zeta^{D_2}\right)\left(\zeta^{D_3}\zeta^{D_4}\right)\left(\zeta^{D_5}\zeta^{D_6}\right)\left(\zeta^{D_7}\zeta^{D_8}\right)\right]\times\\
&\qquad\times\left[\left(\bar\nu^{[C_8}\nu^{4]}\right)\left(\bar\nu^{[1}\nu^{C_1]}\right)+\left(\bar\nu^{[C_8}\nu^{1]}\right)\left(\bar\nu^{[4}\nu^{C_1]}\right)\right]\times\\
&\qquad\times\left[\left(\bar\nu^{[C_2}\nu^{4]}\right)\left(\bar\nu^{[1}\nu^{C_3]}\right)+\left(\bar\nu^{[C_2}\nu^{1]}\right)\left(\bar\nu^{[4}\nu^{C_3]}\right)\right]\times\\
&\qquad\times\left[\left(\bar\nu^{[C_4}\nu^{4]}\right)\left(\bar\nu^{[1}\nu^{C_5]}\right)+\left(\bar\nu^{[C_4}\nu^{1]}\right)\left(\bar\nu^{[4}\nu^{C_5]}\right)\right]\times\\
&\qquad\times\left[\left(\bar\nu^{[C_6}\nu^{4]}\right)\left(\bar\nu^{[1}\nu^{C_7]}\right)+\left(\bar\nu^{[C_6}\nu^{1]}\right)\left(\bar\nu^{[4}\nu^{C_7]}\right)\right]+\\
&\qquad+\left[\left(\zeta^{D_2}\zeta^{D_3}\right)\left(\zeta^{D_4}\zeta^{D_5}\right)\left(\zeta^{D_6}\zeta^{D_7}\right)\left(\zeta^{D_8}\zeta^{D_1}\right)\right]\times\\
&\qquad\times\left[\left(\bar\nu^{[C_1}\nu^{4]}\right)\left(\bar\nu^{[1}\nu^{C_2]}\right)+\left(\bar\nu^{[C_1}\nu^{1]}\right)\left(\bar\nu^{[4}\nu^{C_2]}\right)\right]\times
\end{aligned}$$

$$\times \left[\left(\bar{\nu}^{[C_3}\nu^{4]}\right) \left(\bar{\nu}^{[1}\nu^{C_4]}\right) + \left(\bar{\nu}^{[C_3}\nu^{1]}\right) \left(\bar{\nu}^{[4}\nu^{C_4]}\right) \right] \times$$
$$\times \left[\left(\bar{\nu}^{[C_5}\nu^{4]}\right) \left(\bar{\nu}^{[1}\nu^{C_6]}\right) + \left(\bar{\nu}^{[C_5}\nu^{1]}\right) \left(\bar{\nu}^{[4}\nu^{C_6]}\right) \right] \times$$
$$\times \left[\left(\bar{\nu}^{[C_7}\nu^{4]}\right) \left(\bar{\nu}^{[1}\nu^{C_8]}\right) + \left(\bar{\nu}^{[C_7}\nu^{1]}\right) \left(\bar{\nu}^{[4}\nu^{C_8]}\right) \right] + \text{permutations}\Big\}, \tag{B.15}$$

where the permutations not indicated explicitly correspond to antisymmetrisation in all the (C_i, D_i) pairs.

The key feature of (B.15) is that it depends on the set of indices (C_i, D_i), but not on the exponents k_i. The traces (5.23) require four pairs of (C_i, D_i) indices to be $(1,4)$ while the remaining four pairs correspond to the (A_k, B_k) indices carried by the impurities. The fact that (B.15) does not depend on the k_i's means that the traces (5.23) do not depend on the exponents on the Z's, but only on the relative positions of the $\widehat{Z}$'s with respect to the impurities. Therefore when substituting into the definition (5.13) of the operator the traces (B.15) can be taken out of the sums over the indices q, r, s. After substituting the values of the indices corresponding to the various terms in the expansion (5.16) and some simple Fierz rearrangements the profile of the operator $\mathscr{O}_{\mathbf{1}}$ takes the form of a common factor containing the dependence on the bosonic and fermionic moduli, multiplying the combination $K(n_1, n_2, n_3; J)$ of 35 sums which contain the dependence on the mode numbers n_1, n_2 and n_3, see (5.28). To illustrate more concretely how this works let us describe explicitly one particular term. We consider the first trace in (5.16) and compute the contribution of the last type in (5.23) for this trace. We have to evaluate

$$\frac{1}{\sqrt{J^3\left(\frac{g_{\rm YM}^2 N}{8\pi^2}\right)^{J+4}}} \sum_{\substack{q,r,s_1,\ldots,s_5=0 \\ q+r+s_1+\cdots+s_5\le J-4 \\ s_1+\cdots+s_5=s-4}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} \times$$
$$\times \,{\rm Tr}\left(\check{Z}^{J-(q+r+s)}\widehat{\varphi}^{12}\check{Z}^{q}\widehat{\varphi}^{13}\check{Z}^{r}\widehat{\varphi}^{24}\check{Z}^{s_1}\widehat{Z}\check{Z}^{s_2}\widehat{Z}\check{Z}^{s_3}\widehat{Z}\check{Z}^{s_4}\widehat{Z}\check{Z}^{s_5}\widehat{\varphi}^{34}\right) =$$
$$= \frac{1}{\sqrt{J^3\left(\frac{g_{\rm YM}^2 N}{8\pi^2}\right)^{J+4}}} \sum_{\substack{q,r,s_1,\ldots,s_5=0 \\ q+r+s_1+\cdots+s_5\le J-4 \\ s_1+\cdots+s_5=s-4}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} \times$$
$$\times \,{\rm Tr}\left(U^{1,2;1,3}_{p,q}U^{2,4;1,4}_{r,s_1}U^{1,4;1,4}_{s_2,s_3}U^{1,4;3,4}_{s_4,s_5}\right). \tag{B.16}$$

Using (B.15) with the particular choice of indices in the trace in (B.16) we get (up to a numerical constant)

$${\rm Tr}\left(U^{1,2;1,3}_{p,q}U^{2,4;1,4}_{r,s_1}U^{1,4;1,4}_{s_2,s_3}U^{1,4;3,4}_{s_4,s_5}\right) =$$
$$= \frac{1}{2^{3J+8}}\frac{\rho^8}{[(x_1-x_0)^2+\rho^2]^{J+8}}\left(\bar{\nu}^{[1}\nu^{4]}\right)^J \times$$
$$\times \left[(\zeta^1\zeta^1)(\zeta^2\zeta^3)(\zeta^2\zeta^3)(\zeta^4\zeta^4) - (\zeta^1\zeta^4)(\zeta^4\zeta^4)(\zeta^2\zeta^3)(\zeta^2\zeta^3)\right]$$
$$= -\frac{3}{2}\frac{1}{2^{3J+8}}\frac{\rho^8}{[(x_1-x_0)^2+\rho^2]^{J+8}}\left(\bar{\nu}^{[1}\nu^{4]}\right)^J\left[\left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\right], \tag{B.17}$$

where the last line has been obtained using simple Fierz rearrangements on the ζ's. As anticipated the trace is independent of the exponents, q, r, s_i. Equation (B.16) then becomes

$$-\frac{3}{2}\,\frac{1}{2^{3J+8}}\,\frac{1}{\sqrt{J^3\left(\frac{g_{\rm YM}^2 N}{8\pi^2}\right)^{J+4}}}\,\frac{\rho^8}{[(x_1-x_0)^2+\rho^2]^{J+8}}\,\left(\bar\nu^{[1}\nu^{4]}\right)^J\left[\left(\zeta^1\right)^2\left(\zeta^2\right)^2\left(\zeta^3\right)^2\left(\zeta^4\right)^2\right]\times$$
$$\times\sum_{\substack{q,r,s=0\\ q+r+s\leq J}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J}\,\frac{1}{4!}s(s-1)(s-2)(s-3)\,, \qquad \text{(B.18)}$$

where we have used the fact that there is no dependence on the single exponents, $s_1,\ldots,s_5$, so that the result is independent of the way the four $\widehat{Z}$ are distributed among the last s Z's. This leads to the factor $\frac{1}{4!}s(s-1)(s-2)(s-3)$ which is a multiplicity coefficient associated with the number of ways of picking four identical $\widehat{Z}$'s out of s Z's. Equation (B.18) illustrates the factorisation of the result into two terms, the first line containing the dependence on the instanton moduli and the second line containing the dependence on the mode numbers.

The function $K(n_1,n_2,n_3;J)$ takes the form

$$K(n_1,n_2,n_3;J)=\sum_{a=1}^{35} c_a\,\mathcal{S}_a(n_1,n_2,n_3;J)\,, \qquad \text{(B.19)}$$

where each of the $\mathcal{S}_a(n_1,n_2,n_3;J)$ is a sum similar to the second line of (B.18) with different summand corresponding to the different multiplicity factors associated with the distributions of $\widehat{Z}$'s in the traces (5.23). Table 3 summarises the contributions to (B.19).

Using the coefficients given in table 3, and noting that the phase factor factorises, (B.19) can be written as

$$\begin{aligned} K(n_1,n_2,n_3) &= -\frac{3}{4}\sum_{\substack{p,q,r,s=0\\ q+r+s+p=J}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J}(p-q+r-s)(p+q+r+s)^3 \\ &= -\frac{3J^3}{4}\sum_{\substack{p,q,r=0\\ p+q+r\leq J}}^{J} {\rm e}^{2\pi i[(n_1+n_2+n_3)p+(n_2+n_3)q+n_3 r]/J}(2p+2r-J)\,. \end{aligned} \qquad \text{(B.20)}$$

The sums in (B.20) can be approximated with integrals in the $J\to\infty$ limit, which can then be evaluated differentiating a generating function. The relevant generating function is given by

$$\begin{aligned} g(a_1,a_2,a_3,a_4) &= \int_0^1 {\rm d}x\int_0^{1-x}{\rm d}y\int_0^{1-x-y}{\rm d}z\,{\rm e}^{2\pi i[a_1x+a_2y+a_3z+a_4(1-x-y-z)]} \\ &= \frac{i}{8\pi^3}\sum_{i=1}^{4}\frac{{\rm e}^{2\pi i a_i}}{\prod_{j=1,j\neq i}^4(a_i-a_j)}\,. \end{aligned} \qquad \text{(B.21)}$$

Summand	Coefficient (c_a)	Summand	Coefficient (c_a)
$\frac{1}{4!}p(p-1)(p-2)(p-3)$	-18	$\frac{1}{2!}prs(s-1)$	0
$\frac{1}{3!}p(p-1)(p-2)q$	-9	$\frac{1}{3!}ps(s-1)(s-2)$	9
$\frac{1}{3!}p(p-1)(p-2)r$	-18	$\frac{1}{4!}q(q-1)(q-2)(q-3)$	18
$\frac{1}{3!}p(p-1)(p-2)s$	-9	$\frac{1}{3!}q(q-1)(q-2)r$	9
$\frac{1}{(2!)^2}p(p-1)q(q-1)$	0	$\frac{1}{3!}q(q-1)(q-2)s$	18
$\frac{1}{2!}p(p-1)qr$	-9	$\frac{1}{(2!)^2}q(q-1)r(r-1)$	0
$\frac{1}{2!}p(p-1)qs$	0	$\frac{1}{2!}q(q-1)rs$	9
$\frac{1}{(2!)^2}p(p-1)r(r-1)$	-18	$\frac{1}{(2!)^2}q(q-1)s(s-1)$	18
$\frac{1}{2!}p(p-1)rs$	-9	$\frac{1}{3!}qr(r-1)(r-2)$	-9
$\frac{1}{(2!)^2}p(p-1)s(s-1)$	0	$\frac{1}{2!}qr(r-1)s$	0
$\frac{1}{3!}pq(q-1)(q-2)$	9	$\frac{1}{2!}qrs(s-1)$	9
$\frac{1}{2!}pq(q-1)r$	0	$\frac{1}{3!}qs(s-1)(s-2)$	18
$\frac{1}{2!}pq(q-1)s$	9	$\frac{1}{4!}r(r-1)(r-2)(r-3)$	-18
$\frac{1}{2!}pqr(r-1)$	-9	$\frac{1}{3!}r(r-1)(r-2)s$	-9
$pqrs$	0	$\frac{1}{(2!)^2}r(r-1)s(s-1)$	0
$\frac{1}{2!}pqs(s-1)$	9	$\frac{1}{3!}rs(s-1)(s-2)$	9
$\frac{1}{3!}pr(r-1)(r-2)$	-18	$\frac{1}{4!}s(s-1)(s-2)(s-3)$	18
$\frac{1}{2!}pr(r-1)s$	-9		

Table 3: Contributions to the function $K(n_1, n_2, n_3; J)$. $\mathcal{S}_a$ are sums over the indices $q, r, s \in [0, J]$ (with the constraint $q+r+s \leq J$) in which the summands are those indicated in the table multiplied by the phase factor $\exp\left(2\pi i[(n_1+n_2+n_3)q + (n_2+n_3)r + n_3 s]/J\right)$. Here $p = J-(q+r+s)$. The c_a's are the combined coefficients taking into account all the terms in the operator.

Thus the above sum requires evaluating

$$f(a_1, a_2, a_3, a_4) = \frac{3i}{8\pi}\left(2\frac{\partial}{\partial a_1} + 2\frac{\partial}{\partial a_3} - 2\pi i\right) g(a_1, a_2, a_3, a_4)\,. \tag{B.22}$$

As discussed in section 5.2.1 in order to get the correct mode number dependence we need to antisymmetrise (B.22) with respect to the exchange of pairs of mode numbers. Therefore we need to compute

$$\begin{aligned}\lim_{n_3 \to -n_1} & \big[f(n_1+n_2+n_3, n_2+n_3, n_3, 0) - f(n_1+n_2+n_3, n_1+n_3, n_3, 0) - \\ & - f(-n_3, -n_1-n_3, -n_1-n_2-n_3, 0) + \\ & + f(-n_3, -n_2-n_3, -n_1-n_2-n_3, 0)\big] = \frac{3}{8\pi^2}\frac{1}{(n_1 n_2)}\,,\end{aligned} \tag{B.23}$$

where only in the case where we impose pairwise equality do we get a non-zero result. In conclusion the mode number dependence in the profile of the operator $\mathscr{O}_{\mathbf{1}}$ is

$$K(n_1,n_2,n_3;J)=\frac{3}{8\pi^2}\frac{J^7}{(n_1n_2)}\,, \tag{B.24}$$

where the factor of J^7 is the combination of the J^3 in (B.20) and a J^4 arising from the conversion of the sums into integrals in the continuum limit.

The two-point function $G_{\mathbf{1}}(x_1,x_2)$ thus becomes

$$\begin{aligned} G_{\mathbf{1}}(x_1,x_2) = {} & \frac{J^{11}\,\mathrm{e}^{2\pi i\tau}}{N^{7/2}}\frac{1}{(n_1n_2)(m_1m_2)}\times \\ & \times\int\frac{\mathrm{d}^4x_0\,\mathrm{d}\rho}{\rho^5}\frac{\rho^{J+8}}{[(x_1-x_0)^2+\rho^2]^{J+8}}\frac{\rho^{J+8}}{[(x_2-x_0)^2+\rho^2]^{J+8}} \\ & \times\int\prod_{A=1}^{4}\mathrm{d}^2\eta^A\mathrm{d}^2\bar\xi^A\prod_{B=1}^{4}\left[\left(\zeta^B\right)^2(x_1)\right]\left[\left(\zeta^B\right)^2(x_2)\right]\int\mathrm{d}^5\Omega\,\left(\Omega^{14}\right)^J\left(\Omega^{23}\right)^J. \end{aligned} \tag{B.25}$$

The integrations in the second line of (B.26) are straightforward. The integrals over the fermion superconformal modes give $(x_1-x_2)^8$, see (5.33). The five sphere integral is similar to that encountered in the two impurity case and can be calculated in a similar fashion. Proceeding as in (B.8)-(B.12) we get

$$I_{S^5}=\int\mathrm{d}^5\Omega\,\left(\Omega^{14}\right)^J\left(\Omega^{23}\right)^J=\int\mathrm{d}\Omega\,\mathrm{d}\bar\Omega\,\mathrm{d}^4\Omega^I\,\delta(\Omega^I\Omega^I+\Omega\bar\Omega-1)\left(\Omega\bar\Omega\right)^J\,, \tag{B.26}$$

where $\Omega=(\Omega^1+i\Omega^4)$, $\bar\Omega=(\Omega^1-i\Omega^4)$ and $\Omega^I=(\Omega^2,\Omega^3,\Omega^5,\Omega^6)$, so that introducing spherical coordinates

$$\begin{aligned} I_{S^5} &= 2\pi^2\int\mathrm{d}r\,r\,\mathrm{d}\Omega\,\mathrm{d}\bar\Omega\,\,\delta(r^2+\Omega\bar\Omega-1)\left(\Omega\bar\Omega\right)^J \\ &= 2\pi^2\int_{\Omega\bar\Omega\le1}\mathrm{d}\Omega\,\mathrm{d}\bar\Omega\,\left(1-\Omega\bar\Omega\right)\left(\Omega\bar\Omega\right)^J=\frac{\pi^3}{(J+1)(J+2)}\,. \end{aligned} \tag{B.27}$$

The integration over the bosonic part of the moduli space must be treated carefully since it is logarithmically divergent as expected in the presence of a contribution to the matrix of anomalous dimensions. The integrals need to be regulated for instance by dimensional regularisation of the x_0 integral. Introducing Feynman parameters we get

$$\begin{aligned} I_{\mathrm{b}} &= \int\frac{\mathrm{d}^4x_0\,\mathrm{d}\rho}{\rho^5}\frac{\rho^{J+8}}{[(x_1-x_0)^2+\rho^2]^{J+8}}\frac{\rho^{J+8}}{[(x_2-x_0)^2+\rho^2]^{J+8}} \\ &= \frac{\Gamma(2J+16)}{[\Gamma(J+8)]^2}\int_0^1\mathrm{d}\alpha_1\mathrm{d}\alpha_2\,\delta(1-\alpha_1-\alpha_2)\alpha_1^{J+7}\alpha_2^{J+7}\times \\ &\quad\times\int\mathrm{d}^4x_0\,\mathrm{d}\rho\,\frac{\rho^{2J+11}}{[(x_0-\alpha_1x_1-\alpha_2x_2)^2+\rho^2+\alpha_1\alpha_2x_{12}^2]^{2J+16}}\,. \end{aligned} \tag{B.28}$$

After dimensional regularisation,

$$I_{\mathrm{b}}\to I_{\mathrm{b}}^{(\epsilon)}=\frac{\Gamma(J+6)\Gamma(J+8+\epsilon)}{[\Gamma(J+8)]^2}\,\pi^{2-\epsilon}\frac{1}{(x_{12}^2)^{J+8+\epsilon}}\int_0^1\mathrm{d}\alpha\,\frac{1}{[\alpha(1-\alpha)]^{1+\epsilon}}$$

$$= \frac{1}{\epsilon}\frac{\Gamma(J+6)\Gamma(J+8+\epsilon)}{[\Gamma(J+8)]^2}\,\pi^{2-\epsilon}\frac{1}{(x_{12}^2)^{J+8+\epsilon}}\,. \tag{B.29}$$

The $1/\epsilon$ pole corresponds to a logarithmic divergence in dimensional regularisation.

Substituting into (B.26) we finally get

$$G_1(x_1,x_2) \sim (g_2)^{7/2}\mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}\frac{1}{(n_1n_2)(m_1m_2)}\frac{1}{(x_{12}^2)^{J+4}}\,\log\left(\Lambda^2 x_{12}^2\right)\,, \tag{B.30}$$

where the exact numerical coefficient was given in (5.39).

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