# Pressure as a source of gravity

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The active mass density in Einstein's theory of gravitation in the analog of Poisson's equation in a local inertial system is proportional to $\rho + 3p/c^2$. Here $\rho$ is the density of energy and $p$ its pressure for a perfect fluid. By using exact solutions of Einstein's field equations in the static case we study whether the pressure term contributes towards the mass.

## I. INTRODUCTION

Mass is in Newton's theory of gravitation the source of a gravitational field. In a relativistic generalization of Newton's theory we expect energy to take over this rôle. But in Einstein's theory of gravitation it is the full energy-stress tensor that becomes the source of the gravitational field. In the case of a perfect fluid this tensor takes the form

$$T^\mu{}_\nu = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu. \qquad (1.1)$$

Here $p$ denotes the pressure and $\rho$ the density of matter. $u^\mu$ is the four-velocity of matter described by a timelike unit vector with

$$u^\mu u_\mu = 1. \qquad (1.2)$$

The speed of light $c$ has been put equal to one.

To study the rôle of the pressure in contributing to the gravitational field in Einstein's theory we consider a simple case where comparison with Newtonian gravity is straightforward. If we take a gravitating fluid in geodesic flow the spatial gradient of pressure is zero and the pressure becomes a function of time. Such pressure has no influence on the motion of the fluid in Newtonian dynamics. For a local description of the flow we can introduce a comoving local inertial system. In Newton's theory the local acceleration of particles is then given by the negative gradient of a potential $V$ that itself is subject to the Poisson equation

$$\triangle V = 4\pi G\rho, \qquad (1.3)$$

where $G$ is Newton's gravitational constant. The local approximation to the relativistic description should be good since we are dealing with small velocities near the orgin of the comoving inertial system.

A local description of the flow can be obtained in Einstein's theory by using the formulas of geodesic deviation together with the field equations. In this way [1] one obtains a correction of the Poisson equation for the potential $V$

$$\triangle V + \Lambda = 4\pi G(\rho + 3p). \qquad (1.4)$$

The correction term on the left-hand side of the equation is Einstein's cosmological constant that shall not be dis-

cussed further here. On the right-hand side appears now the pressure contributing as source of the gravitational field as first pointed out by Tullio Levi-Civita in the static case [2]. It is easy to estimate the degree of this contribution. The pressure $p$ in an ideal gas of identical particles with number density $n$, momentum $\vec{P}$ and velocity $\vec{v}$ is given by

$$p = \frac{1}{3}n\overline{\vec{P} \cdot \vec{v}}, \qquad (1.5)$$

where the bar indicates averaging and the "·" the scalar product of the two vectors. For highly relativistic particles or photons this gives with $v = c$ and particles of energy $E$

$$3p = n\overline{E} = \rho. \qquad (1.6)$$

This result together with (1.4) indicates that the "active" mass density generating a gravitational field for a photon gas is twice as large as one would have derived from Newton's theory. In fact, if one studies cosmological models in Newtonian theory [3] one obtains for a specific energy of $0$, $\frac{1}{2}$, or $-\frac{1}{2}$ the Friedmann equation for the scale factor $R(t)$ for incoherent matter, meaning vanishing pressure. To obtain the Friedmann equation for a universe filled with radiation, like the early universe, one needs to introduce the $3p$-term into the Poisson equation. This was noticed by William McCrea [4].

It was Richard Tolman [5] who studied universes filled with radiation and began to wonder about the consequences of the $3p$-term for gravitational theory. The following scenario is known as Tolman's paradox: A static spherical box has been filled with a gravitating substance of a given mass. If this substance undergoes an internal transformation (e.g. matter and antimatter turning into radiation) raising the pressure, the active mass in the box would change because of the $3p$-term since the energy is conserved. However, such an internal transformation should not affect the mass measured outside the box, say by an orbiting particle obeying Kepler's third law. In a spherically symmetric field the particle should be oblivious to all spherically symmetric changes inside its orbit, a consequence of the vacuum equations known as Birkhoff's theorem [6].

Charles Misner and Peter Putnam were intrigued by Tolman's paradox [7]. They showed that by increasing the pressure inside the box one has to have stresses in the walls keeping the matter inside confined and the field static [8]. These stresses would make negative contributions to the active mass that would just compensate those arising from the $3p$ term inside. This plausible resolution of the paradox suffered, however, from the restriction that the authors neglected a possible influence of the gravitational field on the cancellation. They solved, essentially, a problem in special relativity. It would, therefore, seem desirable to look at the problem again and study a situation where the gravitational field is fully taken into account.

This issue was discussed by Steven Carlip in the weak field approximation independent of any details of the box by using the virial theorem [9].

To get exact and transparent results we simplify the model as follows: We take a sphere of fluid with constant energy density kept together by its own gravitation and the surface tension of a membrane under a given inside pressure at the surface. This boxed fluid sphere has a mass determined by the Schwarzschild-Droste vacuum field outside. By raising the surface tension of the membrane we squeeze the sphere and increase the pressure inside accordingly to stay in equilibrium. The assumption that the fluid has constant energy density simplifies all calculations because the radius of the sphere does not change, nor does the density of matter inside. Since neither the volume nor the surface area vary, changes in pressure and surface tension do no work and thus do not change the energy.

And now we want to answer the question whether the mass of the sphere measured outside, consisting of the mass of the fluid and the membrane, is affected by the raising of the pressure inside.

To deal with this problem we are fortunate that the gravitational field of a fluid with constant energy density was discovered already by Karl Schwarzschild [10] and is well known. Our next concern is the theory of the membrane.

## II. THE SECOND FUNDAMENTAL FORM AND SURFACE ENERGY-STRESS TENSOR

On a timelike hypersurface the metric components must be continuous across the surface. This holds also for their derivatives along tangential directions to the hypersurface. However, the derivatives of the metric components in the normal direction to the hypersurface in general can have discontinuities. One may hope that an invariant formulation of the problem might be useful because one can then use a system with coordinate singularities for which the calculations become simpler.

Generally speaking we are dealing with a timelike hypersurface in a semi-Riemannian space with metric

$$ds^2 = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu; \qquad \lambda, \mu, \nu = 1, \ldots, n. \quad (2.1)$$

We assume that the hypersurface is defined by

$$F(x^\lambda) = 0, \qquad \text{with} \qquad g^{\mu\nu}F_{,\mu}F_{,\nu} < 0. \quad (2.2)$$

This enables us to define on the hypersurface and its neighborhood a unit spacelike normal vector $n^\nu$ such that

$$n^\nu = \frac{g^{\nu\mu}F_{,\mu}}{\sqrt{|g^{\alpha\beta}F_{,\alpha}F_{,\beta}|}}, \qquad n_\nu n^\nu = -1. \quad (2.3)$$

With the projection tensor $h^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$ the symmetric tensor

$$K_{\nu\lambda} = \frac{1}{2}(n_{\nu;\mu}h^\mu{}_\lambda + n_{\lambda;\mu}h^\mu{}_\nu) \quad (2.4)$$

measures the extrinsic curvature of the hypersurface considered.

If a hypersurface supports an energy-stress tensor $|g^{\alpha\beta}F_{,\alpha}F_{,\beta}|^{1/2}\delta(F)I_{\mu\nu}$, the metric $g_{\mu\nu}$ is continuous, but the extrinsic curvature has different values on the two sides of the hypersurface. Einstein's field equations require then at the hypersurface that

$$\kappa I_{\mu\nu} = [K_{\mu\nu}] - h_{\mu\nu}[K^\lambda_\lambda], \qquad \kappa = 8\pi G, \quad (2.5)$$

where $[K_{\mu\nu}]$ denotes the jump discontinuity of $K_{\mu\nu}$ in passing the hypersurface in the direction of the normal $n^\lambda$, as shown by Lanczos and Israel [11,12].

A translation into English of the original Lanczos paper [11] can be found in Cornelius Lanczos "Collected Published Papers with Commentaries" [13]. The useful commentary by J. David Brown [14] lists the further references [15,16] to which we might add [17–19]. See also Appendix A.

## III. THE EMPTY BUBBLE

For orientation we treat first the equilibrium of a spherical, gravitating empty bubble with a constant surface stress $\tau$ in Newtonian theory. We count $\tau$ positive if it acts as a surface tension that tries to minimize the surface area. As a compressive stress $\tau < 0$ takes negative values. We call the radius of the bubble $r_0$ and assume that matter is distributed on its surface with constant mass density $\sigma$. The total mass $M_S$ of the bubble is given by

$$M_S = 4\pi\sigma r_0^2. \quad (3.1)$$

A surface element of area $dA$ is attracted toward the center of the sphere by a force

$$dF = -G\frac{M_S\sigma dA}{2r_0^2} = -2\pi G\sigma^2 dA. \quad (3.2)$$

It is the mean of the attraction just outside and just inside of the membrane. The surface stress $\tau$ acts on the surface element $dA$ with an outward directed radial force $dF'$

$$\mathrm{d}F' = -\frac{2\tau \mathrm{d}A}{r_0}. \tag{3.3}$$

This formula is easily derived by considering $\mathrm{d}A$ as a circular disc on the sphere with the infinitesimal radius $\epsilon r_0$. The area of this disc is then given by $\mathrm{d}A = \pi(\epsilon r_0)^2$. The radial component of the surface stress $\tau$ is $\epsilon\tau$. Integrating it along the circumference of the circle amounts to multiplying it with $2\pi\epsilon r_0$. We have thus for the circular surface element the radial force given above. The formula is a special case of the formula derived for the theory of capillarity by Thomas Young [20] in 1805 and also a year later by Pierre Laplace [21] to whom it is usually attributed. Instead of the factor $2/r_0$ the general formula has the sum of the principal curvatures that coincide for the sphere in our case.

Adding $F$ and $F'$ to zero for equilibrium gives

$$-\tau = \pi G \sigma^2 r_0 = \frac{GM_S \sigma}{4r_0}, \tag{3.4}$$

from which we have that condition $\sigma > |\tau|$ is equivalent to $GM_S/r_0 < 4$. The gravitational binding energy of the bubble is given by

$$E_{\mathrm{pot}} = -\frac{GM_S^2}{2r_0}. \tag{3.5}$$

As a relativistically corrected mass $M$ of the bubble we might surmise

$$M = M_S - \frac{GM_S^2}{2r_0}. \tag{3.6}$$

We turn now to the relativistic treatment of such a model. It is known that outside of the sphere the space-time has a Schwarzschild geometry and inside it is a flat Minkowki space. Therefore, in the interior $r \leq r_0$ we have the metric

$$\mathrm{d}s^2 = \mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{r}^2 - \tilde{r}^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{3.7}$$

where we have used special notations for time coordinate $\tilde{t}$ and radial coordinate $\tilde{r}$ because they will be further adjusted to match the metric components across the boundary surface. In the exterior $r > r_0$ we take the Schwarzschild-Droste metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{3.8}$$

Matching the metric component $g_{00}$ at $r = r_0$ we find that if $\tilde{t} = \alpha t$

$$\alpha^2 = 1 - \frac{2m}{r_0}. \tag{3.9}$$

Matching $g_{22}$ and $g_{33}$ at $r = r_0$ leads to the conclusion that, if $\tilde{r}$ is a smooth function of $r$, it must satisfies $\tilde{r}(r_0) = r_0$.

Defining $\tilde{r}' = \mathrm{d}\tilde{r}/\mathrm{d}r$ and $\beta \equiv \tilde{r}'(r_0)$ and matching $g_{11}$ gives $\beta = \alpha^{-1}$. Therefore, the interior metric becomes

$$\mathrm{d}s^2 = \alpha^2 \mathrm{d}t^2 - \tilde{r}'^2 \mathrm{d}r^2 - \tilde{r}^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{3.10}$$

where $\tilde{r}$ and $\tilde{r}'$ are understood as functions of $r$.

We want to calculate the tensor $K_{\mu\nu}$ of the hypersurface $r = r_0$ on both sides. To do so we define a unit normal vector

$$n^\nu = \begin{cases} \delta_1^\nu \tilde{r}'^{-1} & 0 < r < r_0 \\ \delta_1^\mu \sqrt{1 - 2m/r} & r \geq r_0. \end{cases} \tag{3.11}$$

On both sides of the surface, the tensor $K_{\mu\nu}$ can be expressed as

$$K_{\mu\nu} = \frac{n^1}{2}\mathrm{diag}(g_{00,1}, 0, g_{22,1}, g_{33,1}). \tag{3.12}$$

Using this expression to calculate the extrinsic curvature we obtain

$$[K_{00}] = \frac{m}{r_0^2}\sqrt{1 - \frac{2m}{r_0}},$$
$$[K_{22}] = [K_{33}]/\sin^2\theta = r_0\left(1 - \sqrt{1 - \frac{2m}{r_0}}\right). \tag{3.13}$$

Using the Lanczos-Israel condition (2.5) and writing $I_\nu^\mu = \mathrm{diag}(\sigma, 0, -\tau, -\tau)$, we find

$$\kappa\sigma = \frac{2}{r_0}(1 - \sqrt{1 - 2m/r_0}),$$
$$\tau = \frac{-\kappa\sigma^2 r_0}{8 - 4\kappa\sigma r_0} = \frac{-\pi G r_0 \sigma^2}{1 - 4\pi G r_0 \sigma}. \tag{3.14}$$

Here $4\pi G r_0 \sigma < 1$. Solving Eq. (3.14) for $m$ leads to

$$m = \frac{\kappa\sigma r_0^2}{2} - \frac{(\kappa\sigma)^2 r_0^3}{8}. \tag{3.15}$$

If we define the surface internal energy as $M_S \equiv 4\pi r_0^2 \sigma$, the active mass measured from outside $M = m/G$ is given by

$$M = M_S - \frac{GM_S^2}{2r_0}, \tag{3.16}$$

which is what we suggested in Eq. (3.6). Here $GM_S/2r_0 < 1$.

Applied to our bubble the general formula given by Edmund Whittaker [22–25] for static mass distributions gives

$$M = \int \sqrt{g_{00}}(2\rho - T)\mathrm{d}^3 V = 4\pi r_0^2 \sqrt{g_{00}(r_0)}(\sigma - 2\tau) = \frac{m}{G}, \tag{3.17}$$

which agrees with what we just found. Finally we point out that with these notations

$$-\tau = \frac{GM_S\sigma/r_0}{4(1 - GM_S/r_0)}. \tag{3.18}$$

The condition of "energy-dominance" $|\tau| < \sigma$ requires that $GM_S/r_0 < 4/5$. In the limit $m/r_0 \ll 1$ we find

$$-\tau = \frac{GM_S\sigma}{4r_0}. \tag{3.19}$$

This gives the correct Newtonian limit of the surface tension as we mentioned above. These results were obtained by Kornel Lanczos [11].

## IV. THE SQUEEZED BALL

We wish to study the following model in Einstein's theory of gravitation. We take a spherical ball of constant energy density. In its interior the ball is kept in equilibrium by its pressure gradient balancing the gravitational pull of the underlying matter. To simplify the following we assume that the energy density is independent of the pressure. Although incompressibility is not an allowed material property since it would lead to an infinite adiabatic speed of sound, constancy of energy density is nevertheless possible in special configurations as discussed by Christian Møller [26]. At the surface of the ball where the pressure is positive or zero we have a membrane. For vanishing surface pressure such a membrane is not necessary, but for a positive pressure at the surface the membrane is there for keeping the ball in equilibrium.

The membrane acts in two ways. Its mass exerts a pressure on the ball and tension squeezes the ball. Since $\rho = $ const. we also have to assume that $\sigma = $ const..

To understand the gravitating rôle of pressure itself—not of pressure gradients—we want to squeeze the ball while keeping it in equilibrium. In this way we raise the overall pressure in the ball and compensate its rise at the surface by increasing the tension in the membrane. To effect such a process one could imagine to lower masses symmetrically and infinitely slowly from all sides to rest on the surface to increase the pressure. The additional weights are then lifted again and replaced by the increased surface tension in the membrane. No work on the ball will be done if its mass before and after is the same.

It is clear that in carrying out these internal transformations in a spherically symmetric fashion we are severely constrained by Birkhoff's theorem. We should imagine that all our machinery needed for lowering and raising weights has to be inside a spherical shell used as a scaffold for these operations that cannot change the total mass (that of the scaffold included) measured at $r \to \infty$.

The ball itself is described by the metric of the interior Schwarzschild solution [10], given for $r \leq r_0$ by

$$\mathrm{d}s^2 = \frac{1}{4}(3a_0 - a)^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{a^2} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{4.1}$$

with

$$a = \sqrt{1 - r^2/R^2} \quad \text{and} \quad a_0 = \sqrt{1 - r_0^2/R^2}. \tag{4.2}$$

The energy density $\rho$ is given by

$$\kappa\rho = \frac{3}{R^2} = \text{const.} \tag{4.3}$$

The constant $R$ is the radius of curvature of 3-space of constant positive curvature described by $t = $ const. The pressure $p$ inside the ball is

$$\kappa p = \frac{3(a - a_0)}{R^2(3a_0 - a)} = \kappa\rho\frac{a - a_0}{3a_0 - a}. \tag{4.4}$$

At the surface of the ball at $r = r_0$ the pressure vanishes. The central pressure $p(0)$ is given by

$$\kappa p(0) = \frac{3(1 - a_0)}{R^2(3a_0 - 1)}. \tag{4.5}$$

This introduces the limitation $a_0 > 1/3$ or $r_0/R < \sqrt{8}/3$.

For $r \geq r_0$ we use the Schwarzschild-Droste vacuum solution (3.8). We fit the two metrics continuously together at $r = r_0$ by putting

$$1 - \frac{2m}{r_0} = a(r_0)^2 = 1 - \frac{r_0^2}{R^2}. \tag{4.6}$$

With (4.3), this equation relates the mass $M = m/G$ to the energy density as

$$M = \frac{4\pi}{3}r_0^3\rho. \tag{4.7}$$

The Euclidean volume appearing on the right-hand side of Eq. (4.7) is not the true volume in the space of constant positive curvature. The volume element of a spherical shell of thickness $\mathrm{d}r$ is given by $\mathrm{d}V = 4\pi r^2\mathrm{d}r/a$. As shown by Edmund Whittaker [22] the mass generating the gravitational field, is obtained for static fields by the integral

$$M = \int_V \sqrt{g_{00}}(\rho + 3p)\mathrm{d}V. \tag{4.8}$$

Besides the addition of $3p$ to the density and the modified volume element, the factor $\sqrt{g_{00}}$ appears that represents the gravitational potential in the weak field approximation. Since we have for the interior Schwarzschild solution that $\sqrt{g_{00}}(\rho + 3p) = \rho a$, the integral (4.8) with $\mathrm{d}V = 4\pi r^2\mathrm{d}r/a$ will give the Euclidean value.

Since we assumed the pressure vanished at $r = r_0$, we should also have continuity of the first derivatives of the two metrics. This is not the case in the coordinates used. However, if the matching problem is formulated in terms of second fundamental form at the hypersurface $r = r_0$ we shall see later that the matching conditions are fulfilled.

We now want to put a deltalike membrane on the boundary $r = r_0$. The membrane, in general, has a nonvanishing energy-stress tensor on it, and therefore the interior of the

bubble might be squeezed and its pressure increased. If the energy density of the fluid remains unchanged, we can take the interior metric to be

$$d s^2 = \frac{1}{4}(3a_1 - a)^2 d\tilde{t}^2 - \frac{d\tilde{r}^2}{a^2} - \tilde{r}^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.9}$$

with

$$a = \sqrt{1 - \tilde{r}^2/R^2} \quad \text{and} \quad a_1 = \sqrt{1 - r_1^2/R^2}, \tag{4.10}$$

where $r_1 > r_0$. Again we have defined new coordinates $\tilde{t}$ and $\tilde{r}$ just so that we can adjust them to match the outside metric. The constant density remains as given in (4.3) while the pressure becomes

$$\kappa p = \frac{3(a - a_1)}{R^2(3a_1 - a)} = \kappa\rho \frac{a - a_1}{3a_1 - a}, \tag{4.11}$$

which no longer vanishes at the boundary $r = r_0$. It follows from (4.11) that $r_1 < \sqrt{8}R/3$ has to hold to avoid that the pressure becomes infinite at the center $\tilde{r} = 0$. In the exterior the metric still takes the form of a Schwarzschild-Droste metric, now with gravitational radius $m'$ to be determined later.

Again matching $g_{22}$ and $g_{33}$ dictates that $\tilde{r}$ as a function of $r$ must obey $\tilde{r}(r_0) = r_0$. The continuity of $g_{11}$ determines that

$$1 - \frac{2m'}{r_0} = \frac{a_0^2}{\beta^2} \quad \text{or} \quad m' = \frac{r_0}{2}\left(1 - \frac{a_0^2}{\beta^2}\right). \tag{4.12}$$

Here $\beta \equiv \frac{d\tilde{r}}{dr}|_{r=r_0}$ is defined in the same way as before and $a_0$ is given by (4.2).

Continuity in $g_{00}$ is achieved by matching the time coordinates $t$ and $\tilde{t}$ at $r = r_0$ by putting $\tilde{t} = \alpha t$. This determines $\alpha$ to be

$$\alpha = \frac{2\sqrt{1 - 2m'/r_0}}{3a_1 - a_0} = \frac{2a_0}{\beta(3a_1 - a_0)}. \tag{4.13}$$

We should from now on take the interior metric as

$$d s^2 = \frac{\alpha^2}{4}(3a_1 - a)^2 dt^2 - \frac{\tilde{r}'^2 dr^2}{a^2} - \tilde{r}^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4.14}$$

where $\tilde{r}$ is understood as a function of $r$. In this way both the coordinates and the metric are continuous across the boundary surface.

The surface pressure $p_s \equiv p(r_0)$ takes the value

$$\kappa p_s = \frac{3(a_0 - a_1)}{R^2(3a_1 - a_0)} = \frac{\beta\alpha - 1}{R^2} = \frac{\beta\alpha - 1}{3}\rho. \tag{4.15}$$

To relate the coefficents $\alpha$ and $\beta$ to the surface energy density and pressure we must again calculate the second fundamental form of the hypersurface $r = r_0$ on either side. We define a unit normal vector

$$n^\mu = \begin{cases} \delta_1^\mu \tilde{r}'^{-1} a & 0 < r < r_0 \\ \delta_1^\mu \sqrt{1 - 2m'/r} & r \geq r_0, \end{cases} \tag{4.16}$$

and they coincide on the boundary $r = r_0$. Clearly, Eq. (3.12) is still valid. We obtain

$$[K_{00}] = \frac{\sqrt{1 - 2m'/r_0}}{2}\left(\frac{1}{r_0} - \frac{\alpha r_0}{R^2} - \frac{a_0^2}{r_0\beta^2}\right), \tag{4.17}$$

$$[K_{22}] = -r_0\sqrt{1 - 2m'/r_0}(1 - \beta),$$
$$[K_{33}] = [K_{22}]\sin^2\theta. \tag{4.18}$$

Using again the Lanczos-Israel condition (2.5) we find with (4.12)

$$\kappa\sigma = \frac{2a_0}{r_0}\left(1 - \frac{1}{\beta}\right) = \frac{2}{r_0}[a_0 - \sqrt{1 - 2m'/r_0}], \tag{4.19}$$

$$\kappa\tau = -\frac{1}{2\sqrt{1 - 2m'/r_0}}\left(\frac{1}{r_0} - \frac{\alpha r_0}{R^2} - \frac{a_0^2}{r_0\beta^2}\right) + \frac{\kappa\sigma}{2}, \tag{4.20}$$

and

$$\frac{1}{\beta} = 1 - \frac{\kappa\sigma r_0}{2a_0}. \tag{4.21}$$

We are now ready to calculate $m'$ following a procedure similar to the one employed in the calculation for the empty bubble. Solving Eq. (4.19) for $m'$ we find that

$$m' = \frac{r_0^3}{2R^2} + \frac{r_0^2\kappa a_0\sigma}{2} - \frac{r_0^3(\kappa\sigma)^2}{8}. \tag{4.22}$$

Here $m'$ is determined by the mass distributions, namely $\rho$, $r_0$ and $\sigma$ and does not depend in addition on pressure $p$ and surface tension $\tau$. This conclusion contains in essence already our result: Internal changes of the pressure and surface stress distribution at fixed $\rho$, $r_0$, and $\sigma$ do not change the total energy. This means that the active mass $M'$ measured from outside is

$$M' = \frac{m'}{G} = \frac{4\pi r_0^3}{3}\rho + 4\pi r_0^2 a_0\sigma - \pi\kappa r_0^3\sigma^2$$
$$= M + a_0 M_S - \frac{GM_S^2}{2r_0}. \tag{4.23}$$

The first term is clearly the contribution from the bulk energy density and the second term is the surface mass scaled by a factor due to the space-time curvature. The fact that it is smaller than $M_S$ can be understood as the effect of the binding energy between the surface mass and the bulk mass. It is more easily seen in the limit $r_0/R \ll 1$ where to the first order of $r_0/R$ we have

$$M' \approx M + M_S - \frac{GM_S M}{r_0} - \frac{GM_S^2}{2r_0}. \tag{4.24}$$

A term of Newtonian potential energy between the surface mass and bulk mass shows up explicitly. The last term of the active mass that is quadratic in $M_S$ comes from the self gravitational binding energy of the surface as we explained before. In the limit $R \to +\infty$ and therefore $\rho \to 0$ the active massive simplifies to

$$M = M_S - \frac{GM_S^2}{2r_0}, \qquad (4.25)$$

which is exactly what we have found already in the previous section.

These results also enable us to write down the surface stress in term of physical quantities. Using Eq. (4.19) we find

$$\sqrt{1 - 2m'/r_0} = a_0 - \frac{\kappa \sigma r_0}{2}. \qquad (4.26)$$

This shows that $\kappa \sigma < 2a_0/r_0$ is necessary to prevent collapsing. Equation (4.15) and (4.21) lead to

$$\alpha = \left(\frac{3p_s}{\rho} + 1\right)\left(1 - \frac{\kappa \sigma r_0}{2a_0}\right). \qquad (4.27)$$

Inserting them into (4.20), after some algebra, eventually leads to

$$-\kappa \tau = -\frac{\kappa r_0 p_s}{2a_0} + \frac{\kappa G \sigma}{r_0(a_0 - \kappa \sigma r_0/2)}\left(\frac{M}{2a_0} + \frac{M_S}{4}\right). \qquad (4.28)$$

As a check on this equation we put the surface pressure $p_s = 0$, and remove the mass from the interior of the bubble, i.e., putting also $M = 0$. Then we obtain for the surface stress $\tau$ of the empty bubble our previous result from Eq. (3.18)

$$\tau = \frac{GM_S \sigma / r_0}{4(1 - GM_S/r_0)}. \qquad (4.29)$$

For the unsqueezed ball with vanishing surface pressure and vanishing surface energy density, $p_s = 0$ and $\sigma = 0$, we find, as expected, that the surface stress $\tau$ vanishes.

We have reached here the main objective of this paper. We wish to demonstrate explicitly the cancellation between the contributions of pressure $p(r)$ and surface tension $\tau$ as a source of the gravitational field. This can only be achieved by calculating $\tau$ in terms of other physical quantities first as we have done above. We can now calculate the active mass $M'$ by the following integration. First we notice that $\sqrt{g_{00}}(\rho + 3p) = \rho \alpha a$, and therefore the active mass can be evaluated as

$$M = \int_V \sqrt{g_{00}}(\rho + 3p)dV + \sqrt{g_{00}(r_0)}r_0^2 \int_S (2\tau + \sigma)d\Omega$$

$$= \frac{4\pi r_0^3}{3}\alpha \rho + 4\pi r_0^2\left[\frac{r_0}{3}\rho(1 - \alpha) + \sigma a_0 - \frac{\kappa \sigma^2 r_0}{4}\right]$$

$$= M + a_0 M_S - \frac{GM_V M_S}{r_0}, \qquad (4.30)$$

where $d\Omega = \sin\theta d\theta d\varphi$. This coincides with (4.19) as it must and we can see the intriguing cancellations in the second step above.

## V. DISCUSSION

We first wish to consider the case of a ball of constant density $\rho$ and radius $r_0$ under its own gravitation. Density and radius of this spherical ball can be chosen freely. The pressure $p_s = p(r_0)$ at the surface vanishes. and $p(r)$ is given by (4.4) and (4.2)

$$p(r) = \rho \frac{\sqrt{1 - r^2/R^2} - \sqrt{1 - r_0^2/R^2}}{3\sqrt{1 - r_0^2/R^2} - \sqrt{1 - r^2/R^2}}. \qquad (5.1)$$

This gives for small values of $r_0 \ll R$

$$p(r) = \rho(r_0^2 - r^2)/4R^2. \qquad (5.2)$$

The Schwarzschild mass $M$ from (4.7) is given by

$$M = \frac{4\pi}{3}\rho r_0^3. \qquad (5.3)$$

If we squeeze this ball by surrounding it by a massless ($\sigma = 0$) membrane with surface tension $\tau$ it will develop a pressure inside given by (4.11)

$$p_1(r) = \rho \frac{\sqrt{1 - r^2/R^2} - \sqrt{1 - r_1^2/R^2}}{3\sqrt{1 - r_1^2/R^2} - \sqrt{1 - r^2/R^2}}, \qquad r_1 \geq r_0. \qquad (5.4)$$

$M$ remains the same according to (4.23) since $M_S = 0$. On the surface this pressure is given by

$$p_s = p_1(r_0) = \rho \frac{\sqrt{1 - r_0^2/R^2} - \sqrt{1 - r_1^2/R^2}}{3\sqrt{1 - r_1^2/R^2} - \sqrt{1 - r_0^2/R^2}}. \qquad (5.5)$$

The pressure at the surface is matched by the surface tension $\tau$ according to (4.28)

$$\tau = r_0 p_s / 2\sqrt{1 - 2GM/r_0}. \qquad (5.6)$$

For small values of $r_0 \ll R$ the pressure $p_1(r)$ is according to (5.4)

$$p_1(r) = \rho(r_1^2 - r^2)/4R^2. \qquad (5.7)$$

On the surface for $r = r_0$ this is

$$p_s = \rho(r_1^2 - r_0^2)/4R^2. \qquad (5.8)$$

Comparing (5.2) with (5.8) we see that

$$p_1(r) - p(r) = p_s \qquad (5.9)$$

for small values of $r_0/R$. Under these conditions the raising of a surface tension in the membrane results in a constant increase of pressure by $p_s$ inside the whole sphere. If $r_0/R$ becomes comparable to 1 the pressure towards the center increases faster than $p_s$ as, is easy to derive from (5.4).

We can now consider $\tau$ or $p_s$ as a new independent parameter for our model. The crucial result of our investigation is that the Schwarzschild mass $M$ of (5.3) is independent of the surface pressure $p_s$. While the density of the active mass inside the sphere depends on the surface pressure $p_s$ it does not influence the mass measured from outside for a system in a static equilibrium.

The case where we remove the unphysical assumption that the membrane be massless ($\sigma = 0$) really shows nothing essentially new. Equation (4.28) relating surface tension $\tau$ with surface pressure $p_s$ becomes more complicated as does the active mass in (4.23). But the conclusion remains the same that $M$ is independent of the surface pressure $p_s$ in equilibrium.

It would seem, therefore, that the $3p$ addition to the active mass density that arose in Tolman's paradox would be compensated by the forces that are responsible for equilibrium. We do not expect that calculations with a more realistic equation of state (we used incompressibility) and a different scenario in raising pressure would change the conclusions. See Appendix B. If one wants to find the effect of the $3p$-term one has to look at nonequilibrium situations as they present themselves in the early universe or in the last stages of a type II supernova core.

## ACKNOWLEDGMENTS

## APPENDIX A

The total energy-momentum tensor $T_\nu^\mu$ for a bulk fluid and a membrane can be written as

$$T^\mu{}_\nu = \rho u^\mu u_\nu - p(\delta_\nu^\mu - u^\mu u_\nu)$$
$$+ \{\sigma u^\mu u_\nu - \tilde{p}(h^\mu{}_\nu - u^\mu u_\nu)\}\delta(n), \quad (A1)$$

where $n$ is such that $n_{,\nu} = n_\nu$. The bulk part of the tensor was given in Eq. (1.1). The shell part $I^\mu{}_\nu$ is constructed in analogy to the bulk part with surface mass density $\sigma$ and surface pressure $\tilde{p} = -\tau$. The metric $h^\mu{}_\nu$ on the timelike hypersurface $n = 0$ created by the history of the membrane is given by $h^\mu{}_\nu = \delta_\nu^\mu + n^\mu n_\nu$. The scalar $n$ measures the geodesic distance from this surface $n = 0$. The divergence for the shell tensor $I_{\mu\nu}$ gives the two "field equations"

$$u^\mu u^\nu[K_{\mu\nu}] + [K_\lambda^\lambda] = -2\kappa\tau,$$
$$u^\mu u^\nu[K_{\mu\nu}] = \frac{1}{2}\kappa(\sigma - 2\tau). \qquad (A2)$$

They correspond in Newtonian theory to Eqs. (3.4) and (3.2).

## APPENDIX B

We sketch here another model that can be used to demonstrate Tolman's paradox while taking the gravitational field fully into account. For this purpose we consider a solution of the Tolman-Oppenheimer-Volkoff equation for a spherically symmetric star consisting of radiation that is regular at the origin at $r = 0$. The equation of state is then given by $p = \rho/3$. Alan Rendall and Bernd Schmidt showed [27] that such solutions exist with the property that their pressure $p$ goes to zero when the radius $r$ goes to infinity. While finite light stars do not exist we can consider a finite piece of radius $R$ and stabilize it with a membrane of suitable energy density and surface tension using the methods of the preceding paper.

Provided that $GM/R < 4/9$ for mass $M$ and radius $R$ of the solution it has the same mass and radius as a Schwarzschild ball with those parameters. One can then imagine that such a Schwarzschild ball with constant energy density and vanishing pressure at its surface can be transformed quasistatically into the previously constructed radiation ball kept together by a membrane.

[1] A. Harvey and E. L. Schucking, Am. J. Phys. **68**, 723 (2000).
[2] T. Levi-Civita, *Opere Matematiche* (Nicola Zanichelli, Bologna, 1960), Vol. 4, 72.
[3] E. A. Milne, Quart. Journ. Math., Oxford Ser. **5**, 64 (1934); and W. H. McCrea and E. A. Milne, Quart. Journ. Math., Oxford Ser. 5, 73–80 (1934).
[4] W. H. McCrea, Proc. R. Soc. A **206**, 562 (1951).
[5] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934).
[6] G. D. Birkhoff, *Relativity and Modern Physics* (Havard University, Cambridge, MA, 1923).
[7] C. W. Misner and P. Putnam, Phys. Rev. **116**, 1045 (1959).
[8] M. Laue, *Das Relativitätsprinzip* (Vieweg, Braunschweig, 1911).
[9] S. Carlip, Am. J. Phys. **66**, 409 (1998).

[10] K. Schwarzschild, Königlich Preussische Akademie der Wissenschaften (Berlin) Sitzungsberichte **424** (1916).

[11] K. Lanczos, Ann. Phys. (Berlin) **74**, 518 (1924).

[12] W. Israel, Nuovo Cimento B **44**, 1 (1966); corrections in **48**, 463 (1967).

[13] *Cornelius Lanczos Collected Published Papers with Commentaries*, edited by William R. Davis *et al.* (North Carolina State University, Raleigh, NC, 1998), Vol. 1, p. 2–219.

[14] J. D. Brown, *Cornelius Lanczos, Collected papers published with commentaries* (North Carolina State University, Raleigh, NC, 1997) Vol. 1, 2–265.

[15] J. Ipser and P. Sikivie, Phys. Rev. D **30**, 712 (1984).

[16] P. Laguna-Castillo and R. A. Matzner, Phys. Rev. D **34**, 2913 (1986).

[17] C. Barrabes, Classical Quantum Gravity **6** 581 (1989).

[18] J. Crisostomo and R. Olea, Phys. Rev. D **69**, 104023 (2004).

[19] R. Mansouri and K. Khorrami, J. Math. Phys. (N.Y.) **37**, 5672 (1996).

[20] T. Young, Phil. Trans. R. Soc. London **95**, 74 (1805).

[21] P. S. Laplace, *Traité de Mechanique Celeste* Vol. 4, Supplement to Book X on Capillary Attraction (1806), translated by N. Bowditch, (Hillard, Gray, Little, and Wilkins, Boston, 1839) 1829–39.

[22] E. Whittaker, Proc. R. Soc. A **149**, 384 (1935).

[23] A. Komar, Phys. Rev. **113**, 934 (1959).

[24] J. Ehlers and W. Kundt, in *Gravitation* edited by L. Witten (Wiley, New York, 1962), p. 68.

[25] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1989), p. 288/289.

[26] C. Møller, Nordita Publications No. 525 (1973), Istituto Nazionale die Alta Matematica, Symposia Matematica Vol. XII.

[27] A. D. Rendall and B. G. Schmidt, Classical Quantum Gravity **8**, 985 (1991).