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Gravity and the quantum

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Abstract. The goal of this review is to present a broad perspective on quantum gravity for non-experts. After a historical introduction, key physical problems of quantum gravity are illustrated. While there are a number of interesting and insightful approaches to address these issues, over the past two decades sustained progress has primarily occurred in two programs: string theory and loop quantum gravity. The first program is described in Horowitz's contribution to this Focus Issue while my article will focus on the second. The emphasis is on underlying ideas, conceptual issues and overall status of the program rather than mathematical details and associated technical subtleties.

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1. Introduction

This section is divided into two parts. The first provides a broad historical perspective and the second illustrates key physical and conceptual problems of quantum gravity.

1.1. A historical perspective

General relativity and quantum theory are among the greatest intellectual achievements of the 20th century. Each of them has profoundly altered the conceptual fabric that underlies our understanding of the physical world. Furthermore, each has been successful in describing the physical phenomena in its own domain to an astonishing degree of accuracy. And yet, they offer us *strikingly* different pictures of physical reality. Indeed, at first one is surprised that physics could keep progressing blissfully in the face of so deep a conflict. The reason of course is the 'accidental' fact that the values of fundamental constants in our universe conspire to make the Planck length so small and Planck energy so high compared to laboratory scales. It is because of this that we can happily maintain a schizophrenic attitude and use the precise, geometric picture of reality offered by general relativity while dealing with cosmological and astrophysical phenomena, and the quantum-mechanical world of chance and intrinsic uncertainties while dealing with atomic and subatomic particles. This strategy is of course quite appropriate as a practical stand. But it is highly unsatisfactory from a conceptual viewpoint. Everything in our past experience in physics tells us that the two pictures we currently use must be approximations, special cases that arise as appropriate limits of a single, universal theory. That theory must therefore represent a synthesis of general relativity and quantum mechanics. This would be the quantum theory of gravity. Not only should it correctly describe all the known physical phenomena, but it should also adequately handle the Planck regime. This is the theory that we invoke when faced with phenomena, such as the big bang and the final state of black holes, where the worlds of general relativity and quantum mechanics must unavoidably meet.

The necessity of a quantum theory of gravity was pointed out by Einstein in a 1916 paper in the Preussische Akademie Sitzungsberichte. He wrote:

'Nevertheless, due to the inner-atomic movement of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics but also the new theory of gravitation'.

Papers on the subject began to appear in the 1930s most notably by Bronstein, Rosenfeld and Pauli. However, detailed work began only in the 1960s. The general developments since then loosely represent four stages, each spanning roughly a decade. In this section, I will present a sketch of these developments.

Firstly, there was the beginning: exploration. The goal was to do unto gravity as one would do unto any other physical field [1].[3] The electromagnetic field had been successfully quantized using two approaches: canonical and covariant. In the canonical approach, electric and magnetic fields obeying Heisenberg's uncertainty principle are at the forefront, and quantum states naturally arise as gauge-invariant functionals of the vector potential on a spatial three-slice. In the covariant approach on the on the other hand, one first isolates and then quantizes the two radiative modes of the Maxwell field in space-time, without carrying out a $(3+1)$-decomposition, and the quantum states naturally arise as elements of the Fock space of photons. Attempts were made to extend these techniques to general relativity. In the electromagnetic case the two methods are completely equivalent. Only the emphasis changes in going from one to another. In the gravitational case, however, the difference is *profound*. This is not accidental. The reason is deeply rooted in one of the essential features of general relativity, namely the dual role of the space-time metric.

To appreciate this point, let us begin with field theories in Minkowski space-time, say Maxwell's theory to be specific. Here, the basic dynamical field is represented by a tensor field $F_{\mu\nu}$ on Minkowski space. The space-time geometry provides the kinematical arena on which the field propagates. The background, Minkowskian metric provides light-cones and the notion of causality. We can foliate this space-time by a one-parameter family of space-like three-planes, and analyse how the values of electric and magnetic fields on one of these surfaces determine those on any other surface. The isometries of the Minkowski metric allow us to construct physical quantities such as fluxes of energy, momentum, and angular momentum carried by electromagnetic waves.

In general relativity, by contrast, there is no background geometry. The space-time metric itself is the fundamental dynamical variable. On the one hand it is analogous to the Minkowski metric in Maxwell's theory; it determines space-time geometry, provides light cones, defines causality, and dictates the propagation of all physical fields (including itself). On the other hand it is the analogue of the Newtonian gravitational potential and therefore the basic dynamical entity of the theory, similar in this respect to the $F_{\mu\nu}$ of the Maxwell theory. This dual role of the metric is in effect a precise statement of the equivalence principle that is at the heart

[3] Since this paper is addressed to non-experts, except in the discussion of very recent developments, I will generally refer to books and review papers which summarize the state of the art at various stages of development of quantum gravity. References to original papers can be found in these reviews.

of general relativity. It is this feature that is largely responsible for the powerful conceptual economy of general relativity, its elegance and its aesthetic beauty, its strangeness in proportion. However, this feature also brings with it a host of problems. We see already in the classical theory several manifestations of these difficulties. It is because there is no background geometry, for example, that it is so difficult to analyse singularities of the theory and to define the energy and momentum carried by gravitational waves. Since there is no *a priori* space-time, to introduce notions as basic as causality, time, and evolution, one must first solve the dynamical equations and *construct* a space-time. As an extreme example, consider black holes, whose definition requires the knowledge of the causal structure of the entire space-time. To find if the given initial conditions lead to the formation of a black hole, one must first obtain their maximal evolution and, using the causal structure determined by that solution, ask if its future infinity has a past boundary. If it does, space-time contains a black hole and the boundary is its event horizon. Thus, because there is no longer a clean separation between the kinematical arena and dynamics, in the classical theory substantial care and effort is needed even in the formulation of basic physical questions.

In quantum theory the problems become significantly more serious. To see this, recall first that, because of the uncertainty principle, already in non-relativistic quantum mechanics, particles do not have well-defined trajectories; time-evolution only produces a probability amplitude, $\Psi(x, t)$, rather than a specific trajectory, $x(t)$. Similarly, in quantum gravity, even after evolving an initial state, one would not be left with a specific space-time. In the absence of a space-time geometry, how is one to introduce even habitual physical notions such as causality, time, scattering states, and black holes?

The canonical and the covariant approaches have adopted dramatically different attitudes to face these problems. In the canonical approach, one notices that, in spite of the conceptual difficulties mentioned above, the Hamiltonian formulation of general relativity is well defined and attempts to use it as a stepping stone to quantization. The fundamental canonical commutation relations are to lead us to the basic uncertainty principle. The motion generated by the Hamiltonian is to be thought of as time evolution. The fact that certain operators on the fixed ('spatial') three-manifold commute is supposed to capture the appropriate notion of causality. The emphasis is on preserving the geometrical character of general relativity, on retaining the compelling fusion of gravity and geometry that Einstein created. In the first stage of the program, completed in the early 1960s, the Hamiltonian formulation of the classical theory was worked out in detail by Dirac, Bergmann, Arnowitt, Deser and Misner and others [2]–[6]. The basic canonical variable was the three-metric on a spatial slice. The 10 Einstein's equations naturally decompose into two sets: four constraints on the metric and its conjugate momentum (analogous to the equation Div $\vec{E} = 0$ of electrodynamics) and six evolution equations. Thus, in the Hamiltonian formulation, general relativity could be interpreted as the dynamical theory of three-geometries. Wheeler therefore baptized it *geometrodynamics* [7, 8].

In the second stage, this framework was used as a point of departure for quantum theory. The basic equations of the quantum theory were written down and several important questions were addressed [6, 8]. Wheeler also launched an ambitious program in which the internal quantum numbers of elementary particles were to arise from non-trivial, microscopic topological configurations and particle physics was to be recast as 'chemistry of geometry'. However, most of the work in quantum geometrodynamics continued to remain formal; indeed, even today the field-theoretic difficulties associated with the presence of an *infinite number of degrees of freedom* remain unresolved. Furthermore, even at the formal level, it has been difficult to solve the quantum Einstein's equations. Therefore, after an initial burst of activity, the quantum geometrodynamics program became stagnant. Interesting results have been obtained in the limited context of quantum cosmology where one freezes all but a finite number of degrees of freedom. However, even in this special case, the initial singularity could not be resolved without additional 'external' inputs into the theory. Sociologically, the program faced another limitation: concepts and techniques which had been so successful in quantum electrodynamics appeared to play no role here. In particular, in quantum geometrodynamics, it is hard to see how gravitons are to emerge, how scattering matrices are to be computed, how Feynman diagrams are to dictate dynamics and virtual processes are to give radiative corrections. To use a well-known phrase [9], the emphasis on geometry in the canonical program 'drove a wedge between general relativity and the theory of elementary particles.'

In the covariant[4] approach [5, 10, 11] the emphasis is just the opposite. Field-theoretic techniques are put at the forefront. The first step in this program is to split the space-time metric $g_{\mu\nu}$ into two parts, $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}$, where $\eta_{\mu\nu}$ is to be a background, kinematical metric, often chosen to be flat, G is Newton's constant, and $h_{\mu\nu}$, the deviation of the physical metric from the chosen background, the dynamical field. The two roles of the metric tensor are now split. The overall attitude is that this sacrifice of the fusion of gravity and geometry is a moderate price to pay for ushering-in the powerful machinery of perturbative quantum field theory. Indeed, with this splitting most of the conceptual problems discussed above seem to melt away. Thus, in the transition to the quantum theory it is only $h_{\mu\nu}$ that is quantized. Quanta of this field propagate on the classical background space-time with metric $\eta_{\mu\nu}$. If the background is in fact chosen to be flat, one can use the Casimir operators of the Poincaré group and show that the quanta have spin two and rest mass zero. These are the gravitons. The Einstein–Hilbert Lagrangian tells us how they interact with one another. Thus, in this program, quantum general relativity was first reduced to a quantum field theory in Minkowski space. One could apply to it all the machinery of perturbation theory that had been so successful in particle physics. One now had a definite program to compute amplitudes for various scattering processes. Unruly gravity appeared to be tamed and forced to fit into the mold created to describe other forces of Nature. Thus, the covariant quantization program was more in tune with the mainstream developments in physics at the time. In 1963 Feynman extended perturbative methods from quantum electrodynamics to gravity. A few years later DeWitt carried this analysis to completion by systematically formulating the Feynman rules for calculating scattering amplitudes among gravitons and between gravitons and matter quanta. He showed that the theory is unitary order by order in the perturbative expansion. By the early 1970s, the covariant approach had led to several concrete results [10].

Consequently, the second stage of the covariant program began with great enthusiasm and hope. The motto was: go forth, perturb, and expand. The enthusiasm was first generated by the discovery that Yang–Mills theory coupled to fermions is renormalizable (if the masses of gauge particles are generated by a spontaneous symmetry-breaking mechanism).[5] This led to a successful theory of electroweak interactions. Particle physics witnessed a renaissance of quantum field theory. The enthusiasm spilled over to gravity. Courageous calculations were performed to estimate radiative corrections. Unfortunately, however, this research soon ran into

[4] In the context of quantum gravity, the term 'covariant' is somewhat misleading because the introduction of a background metric violates diffeomorphism covariance. It is used mainly to emphasize that this approach does not involve a $3+1$ decomposition of space-time.

[5] In fact DeWitt's quantum gravity work [10] played a seminal role in the initial stages of the extension of perturbative techniques from Abelian to non-Abelian gauge theories.

its first road block. The theory was shown to be non-renormalizable when two loop effects are taken into account for pure gravity and already at one loop for gravity coupled with matter [12]. To appreciate the significance of this result, let us return to the quantum theory of photons and electrons. This theory is perturbatively renormalizable. This means that, although individual terms in the perturbation expansion of a physical amplitude may diverge due to radiative corrections involving closed loops of virtual particles, these infinities are of a specific type; they can be systematically absorbed in the values of free parameters of the theory, the fine structure constant and the electron mass. Thus, by renormalizing these parameters, individual terms in the perturbation series can be systematically rendered finite. In quantum general relativity, such a systematic procedure is not available; infinities that arise due to radiative corrections are genuinely troublesome. Put differently, quantum theory acquires an infinite number of undetermined parameters. Although one can still use it as an effective theory in the low-energy regime, regarded as a fundamental theory, it has no predictive power at all!

Buoyed, however, by the success of perturbative methods in electroweak interactions, the community was reluctant to give them up in the gravitational case. In the case of weak interactions, it was known for some time that the observed low energy phenomena could be explained using Fermi's simple four-point interaction. The problem was that this Fermi model led to a non-renormalizable theory. The correct, renormalizable model of Glashow, Weinberg and Salam agrees with Fermi's at low energies but marshals new processes at high energies which improve the ultraviolet behaviour of the theory. It was therefore natural to hope that the situation would be similar in quantum gravity. General relativity, in this analogy, would be similar to Fermi's model. The fact that it is not renormalizable was taken to mean that it ignores important processes at high energies which are, however, unimportant at low energies, i.e., at large distances. Thus, the idea was that the correct theory of gravity would differ from general relativity but only at high energies, i.e., near the Planck regime. With this aim, higher derivative terms were added to the Einstein–Hilbert Lagrangian. If the relative coupling constants are chosen judiciously, the resulting theory does in fact have a better ultraviolet behaviour. Stelle, Tomboulis and others showed that the theory is not only renormalizable but asymptotically free; it resembles the free theory in the high-energy limit. Thus, the initial hope of 'curing' quantum general relativity was in fact realized. However, it turned out that the Hamiltonian of this theory is unbounded from below, and consequently the theory is drastically unstable! In particular, it violates unitarity; probability fails to be conserved. The success of the electroweak theory suggested a second line of attack. In the approaches discussed above, gravity was considered in isolation. The successful unification of electromagnetic and weak interactions suggested the possibility that a consistent theory would result only when gravity is coupled with suitably chosen matter. The most striking implementation of this viewpoint occurred in supergravity. Here, the hope was that the bosonic infinities of the gravitational field would be cancelled by those of suitably chosen fermionic sources, giving us a renormalizable quantum theory of gravity. Much effort went into the analysis of the possibility that the most sophisticated of these theories, $N = 8$ supergravity, can be employed as a genuine grand unified theory.[6] It turned out that some cancellation of infinities does occur and that

[6] For a number of years, there was a great deal of confidence, especially among particle physicists, that supergravity was on the threshold of providing the complete quantum gravity theory. For instance, in the centennial celebration of Einstein's birthday at the Institute of Advanced Study, Princeton [13]—the proceedings of which were videotaped and archived for future historians and physicists—there were two talks on quantum gravity, both devoted to supergravity. A year later, in his Lucasian Chair inaugural address Hawking [14] suggested that the end of theoretical physics was in sight because $N = 8$ supergravity was likely to be the final theory.

supergravity is indeed renormalizable to two loops even though it contains matter fields coupled to gravity. Furthermore, its Hamiltonian is manifestly positive and the theory is unitary. However, it is believed that at fifth and higher loops it is again non-renormalizable.

By and large, the canonical approach was pursued by relativists and the covariant approach by particle physicists. In the mid-1980s, both approaches received unexpected boosts. These launched the third phase in the development of quantum gravity.

A group of particle physicists had been studying string theory to analyse strong interactions from a novel angle. The idea was to replace point particles by one-dimensional extended objects, 'strings', and associate particle-like states with various modes of excitations of the string. Initially there was an embarrassment: in addition to the spin-1 modes characteristic of gauge theories, string theory included also a spin-2, massless excitation. But it was soon realized that this was a blessing in disguise: the theory automatically incorporated a graviton. In this sense, gravity was already built into the theory! However, it was known that the theory had a potential quantum anomaly which threatened to make it inconsistent. In the mid-1980s, Greene and Schwarz showed that there is an anomaly cancellation and perturbative string theory could be consistent in certain space-time dimensions—26 for a purely bosonic string and 10 for a superstring [15, 16]. Since strings were assumed to live in a flat background space-time, one could apply perturbative techniques. However, in this reincarnation, the covariant approach underwent a dramatic revision. Since it is a theory of extended objects rather than point particles, the quantum theory has brand new elements; it is no longer a local quantum field theory. The field-theoretic Feynman diagrams are replaced by world-sheet diagrams. This replacement dramatically improves the ultraviolet behaviour and, although explicit calculations have been carried out only at 2 or 3 loop order, it is widely believed that the perturbation theory is *finite* to all orders; it does not even have to be renormalized. The theory is also unitary. It has a single, new fundamental constant—the string tension—and, since various excited modes of the string represent different particles, there is a built-in principle for unification of all interactions![7] From the viewpoint of local quantum field theories that particle physicists have used in studying electroweak and strong interactions, this mathematical structure seems almost magical. Therefore, there is a hope in the string community that this theory would encompass all of fundamental physics; it would be the 'theory of everything'.

Unfortunately, it soon became clear that string perturbation theory also faces some serious limitations. Perturbative finiteness would imply that each term in the perturbation series is ultra-violet finite.[8] However, Gross and Periwal have shown that in the case of bosonic strings, when summed, the series diverges and does so uncontrollably. (Technically, it is not even Borel-summable.) They also gave arguments that the conclusion would not be changed if one uses superstrings instead. Independent support for these arguments has come from work on random surfaces due to Ambjorn and others. One might wonder why the divergence of the sum should be regarded as a serious failure of the theory. After all, in quantum electrodynamics, the series is also believed to diverge. Recall however that quantum electrodynamics is an inherently incomplete theory. It ignores many processes that come into play at high energies or short distances. In particular, it completely ignores the microstructure of space-time and simply assumes that

[7] To date, none of the low-energy reductions appears to correspond to the world we actually observe. Nonetheless, string theory has provided us with a glimpse of an entirely new vista: the concrete possibility that unification could be brought about by a tightly woven, non-local theory.

[8] But it does appear that there are infrared divergences. As in QED, these are regarded as 'harmless' for calculation of physical effects. I thank Ashoke Sen for discussions on this issue.

space-time can be approximated by a smooth continuum even below the Planck scale. Therefore, it can plead incompleteness and shift the burden of this infinity to a more complete theory. A 'theory of everything' on the other hand, has nowhere to hide. It cannot plead incompleteness and shift its burden. It must face the Planck regime squarely. If the theory is to be consistent, it must have key non-perturbative structures. The current and the fourth stage of the particle physics motivated approaches to quantum gravity are largely devoted to unravelling such structures and using them to solve the outstanding physical problems. Examples of such initiatives are: applications of the AdS/CFT conjecture, use of D-branes and analysis of dualities between various string theories.

On the relativity side, the third stage began with the following observation: the geometrodynamics program laid out by Dirac, Bergmann, Wheeler and others simplifies significantly if we regard a spatial connection—rather than the 3-metric—as the basic object. In fact, we now know that,among others, Einstein and Schrödinger had recast general relativity as a theory of connections already in the 1950s [17]. However, they used the 'Levi-Civita connections' that features in the parallel transport of vectors and found that the theory becomes rather complicated. This episode had been forgotten and connections were re-introduced in the mid-1980s. However, now these were 'spin-connections', required to parallel propagate spinors, and they turn out to *simplify* Einstein's equations considerably. For example, the dynamical evolution dictated by Einstein's equations can now be visualized simply as a *geodesic motion* on the space of spin-connections (with respect to a natural metric extracted from the constraint equations). Since general relativity is now regarded as a dynamical theory of connections, this reincarnation of the canonical approach is called 'connection-dynamics'.

Perhaps the most important advantage of the passage from metrics to connections is that the phase-space of general relativity is now the same as that of gauge theories [18, 19]. The 'wedge between general relativity and the theory of elementary particles' that Weinberg referred to is largely removed without sacrificing the geometrical essence of general relativity. One could now import into general relativity techniques that have been highly successful in the quantization of gauge theories. At the kinematic level, then, there is a unified framework to describe all four fundamental interactions. The dynamics, of course, depends on the interaction. In particular, while there is a background space-time geometry in electroweak and strong interactions, there is none in general relativity. Therefore, qualitatively new features arise. These were exploited in the late 1980s and early 1990s to solve simpler models—general relativity in $2+1$ dimensions [18, 20, 21]; linearized gravity clothed as a gauge theory [18]; and certain cosmological models. To explore the physical, $(3+1)$-dimensional theory, a 'loop representation' was introduced by Rovelli and Smolin. Here, quantum states are taken to be suitable functions of loops on the 3-manifold.[9] This led to a number of interesting and intriguing results, particularly by Gambini, Pullin and their collaborators, relating knot theory and quantum gravity. Thus, there was rapid and unanticipated progress in a number of directions which rejuvenated the canonical quantization program. Since the canonical approach does not require the introduction of a background geometry or use of perturbation theory, and because one now has access to fresh, non-perturbative techniques from gauge theories, in relativity circles there is a hope that this approach may lead to well defined, *non-perturbative* quantum general relativity (or its supersymmetric version, supergravity).

[9] This is the origin of the name 'loop quantum gravity'. The loop representation played an important role in the initial stages. Although this is no longer the case in the current, fourth phase, the name is still used to distinguish this approach from others.

However, a number of these considerations remained rather formal until mid-1990s. Passage to the loop representation required an integration over the infinite-dimensional space of connections and the formal methods were insensitive to possible infinities lurking in the procedure. Indeed, such integrals are notoriously difficult to perform in interacting field theories. To pay due respect to the general covariance of Einstein's theory, one needed diffeomorphism invariant measures and there were folk-theorems to the effect that such measures did not exist!

Fortunately, the folk-theorems turned out to be incorrect. To construct a well-defined theory capable of handling field-theoretic issues, a quantum theory of Riemannian geometry was systematically constructed in the mid-1990s [22]. This launched the fourth (and the current) stage in the canonical approach. Just as differential geometry provides the basic mathematical framework to formulate modern gravitational theories in the classical domain, quantum geometry provides the necessary concepts and techniques in the quantum domain. It is a rigorous mathematical theory which enables one to perform integration on the space of connections for constructing Hilbert spaces of states and to define geometric operators corresponding, e.g. to areas of surfaces and volumes of regions, even though the classical expressions of these quantities involve non-polynomial functions of the Riemannian metric. There are no infinities. One finds that, at the Planck scale, geometry has a definite discrete structure. Its fundamental excitations are one-dimensional, rather like polymers, and the space-time continuum arises only as a coarse-grained approximation. The fact that the structure of space-time at Planck scale is qualitatively different from Minkowski background used in perturbative treatments reinforced the idea that quantum general relativity (or supergravity) may well be non-perturbatively finite. As we will see in section 3 quantum geometry effects have already been shown to resolve the big bang singularity and solve some of the long-standing problems associated with black holes.

The first three stages of developments in quantum gravity taught us many valuable lessons. Perhaps, the most important among them is the realization that perturbative, field-theoretic methods which have been so successful in other branches of physics are simply inadequate in quantum gravity. The assumption that space-time can be replaced by a smooth continuum at arbitrarily small scales leads to inconsistencies. We can neither ignore the microstructure of space-time nor presuppose its nature. We must let quantum gravity itself reveal this structure to us. Irrespective of whether one works with strings or supergravity or general relativity, one has to face the problem of quantization non-perturbatively. In the current, fourth stage both approaches have undergone a metamorphosis. The covariant approach has led to string theory and the canonical approach developed into loop quantum gravity. The mood seems to be markedly different. In both approaches, non-perturbative aspects are at the forefront and conceptual issues are again near centre-stage. However, there are also key differences. Most work in string theory involves background fields and uses higher dimensions and supersymmetry as *essential* ingredients. The emphasis is on unification of gravity with other forces of Nature. Loop quantum gravity, on the other hand, is manifestly background independent. Supersymmetry and higher dimensions do not appear to be essential. However, it has not provided any principle for unifying interactions. In this sense, the two approaches are complementary rather than in competition. Both provide fresh ideas to address some of the key problems but neither is complete.

For brevity and to preserve the flow of discussion, I have restricted myself to the 'main-stream' programs whose development can be continuously tracked over several decades. However, I would like to emphasize that there are a number of other fascinating and highly original approaches—particularly causal dynamical triangulations [23, 24], Euclidean quantum gravity [25, 26], discrete approaches [27], twistor theory [28, 29] and the theory of H-spaces [30], asymptotic quantization [31], non-commutative geometry [32] and causal sets [33].

1.2. Physical questions of quantum gravity

Approaches to quantum gravity face two types of issues: problems that are 'internal' to individual programs and physical and conceptual questions that underlie the whole subject. Examples of the former are: incorporation of physical—rather than half flat—gravitational fields in the twistor program, mechanisms for breaking of supersymmetry and dimensional reduction in string theory, and issues of space-time covariance in the canonical approach. In this subsection, I will focus on the second type of issues by recalling some of the long standing issues that *any* satisfactory quantum theory of gravity should address.

- Big bang and other singularities: it is widely believed that the prediction of a singularity, such as the big bang of classical general relativity, is primarily a signal that the physical theory has been pushed beyond the domain of its validity. A key question to any quantum gravity theory, then, is: what replaces the big bang? Are the classical geometry and the continuum picture only approximations, analogous to the 'mean (magnetization) field' of ferro-magnets? If so, what are the microscopic constituents? What is the space-time analogue of a Heisenberg quantum model of a ferromagnet? When formulated in terms of these fundamental constituents, is the evolution of the *quantum* state of the universe free of singularities? General relativity predicts that the space-time curvature must grow unboundedly as we approach the big bang or the big crunch but we expect the quantum effects, ignored by general relativity, to intervene, making quantum gravity indispensable before infinite curvatures are reached. If so, what is the upper bound on curvature? How close to the singularity can we 'trust' classical general relativity? What can we say about the 'initial conditions', i.e., the quantum state of geometry and matter that correctly describe the big bang? If they have to be imposed externally, is there a *physical* guiding principle?
- Black holes: in the early 1970s, using imaginative thought experiments, Bekenstein argued that black holes must carry an entropy proportional to their area [25, 34, 35]. About the same time, Bardeen, Carter and Hawking (BCH) showed that black holes in equilibrium obey two basic laws, which have the same form as the zeroth and the first laws of thermodynamics, provided one equates the black hole surface gravity κ to some multiple of the temperature T in thermodynamics and the horizon area $a_{\rm hor}$ to a corresponding multiple of the entropy S [25, 34, 35]. However, at first this similarity was thought to be only a formal analogy because the BCH analysis was based on *classical* general relativity and simple dimensional considerations show that the proportionality factors must involve Planck's constant $\hbar$. Two years later, using quantum field theory on a black hole background space-time, Hawking showed that black holes in fact radiate quantum mechanically as though they are black bodies at temperature $T = \hbar\kappa/2\pi$ [25, 36]. Using the analogy with the first law, one can then conclude that the black hole entropy should be given by $S_{\rm BH} = a_{\rm hor}/4G\,\hbar$. This conclusion is striking and deep because it brings together the three pillars of fundamental physics—general relativity, quantum theory and statistical mechanics. However, the argument itself is a rather hodge-podge mixture of classical and semi-classical ideas, reminiscent of the Bohr theory of the atom. A natural question then is: what is the analogue of the more fundamental, Pauli–Schrödinger theory of the hydrogen atom? More precisely, what is the statistical mechanical origin of black hole entropy? What is the nature of a quantum black hole and what is the interplay between the quantum degrees of freedom responsible for entropy

and the exterior curved geometry? Can one derive the Hawking effect from first principles of quantum gravity? Is there an imprint of the classical singularity on the final quantum description, e.g., through 'information loss'?

• Planck scale physics and the low energy world: in general relativity, there is no background metric, no inert stage on which dynamics unfolds. Geometry itself is dynamical. Therefore, as indicated above, one expects that a fully satisfactory quantum gravity theory would also be free of a background space-time geometry. However, of necessity, a background-independent description must use physical concepts and mathematical tools that are quite different from those of the familiar, low energy physics. A major challenge then is to show that this low energy description does arise from the pristine, Planckian world in an appropriate sense, bridging the vast gap of some 16 orders of magnitude in the energy scale. In this 'top-down' approach, does the fundamental theory admit a 'sufficient number' of semi-classical states? Do these semi-classical sectors provide enough of a background geometry to anchor low energy physics? Can one recover the familiar description? If the answers to these questions are in the affirmative, can one pinpoint why the standard 'bottom-up' perturbative approach fails? That is, what is the essential feature which makes the fundamental description mathematically coherent but is absent in the standard perturbative quantum gravity?

There are of course many more challenges: adequacy of standard quantum mechanics, the issues of time and measurement theory and the associated questions of interpretation of the quantum framework, the issue of diffeomorphism invariant observables and convenient ways of computing their properties, practical methods of computing time evolution and S-matrices, exploration of the role of topology and topology change, etc. In loop quantum gravity described in the rest of this review, one adopts the view that the three issues discussed in detail are more basic from a physical viewpoint because they are rooted in general conceptual questions that are largely independent of the specific approach being pursued. Indeed they have been with us longer than any of the current leading approaches.

The rest of this review focuses on Riemannian quantum geometry and loop quantum gravity. It is organized as follows. Section 2 summarizes the underlying ideas, key results from quantum geometry and the status of quantum dynamics in loop quantum gravity. The framework has led to a rich set of results for the first two sets of physical issues discussed above. Section 3 reviews these applications.[10] Section 4 is devoted to outlook. The apparent conflict between the canonical quantization method and space-time covariance is discussed in appendix A.

2. A bird's eye view of loop quantum gravity

In this section, I will briefly summarize the salient features and current status of loop quantum gravity. The emphasis is on structural and conceptual issues; detailed treatments can be found in more complete and more technical recent accounts [22, 37, 39] and references therein. (The development of the subject can be seen by following older monographs [18, 19, 40].)

[10] A summary of the status of semi-classical issues can be found in [22, 37]. Also, I will discuss spin-foams—the path integral partner of the canonical approach discussed here—only in passing. This program has led to fascinating insights on a number of mathematical physics issues—especially the relation between quantum gravity and state sum models—and is better suited to the analysis of global issues such as topology change [38, 39]. However, it is yet to shed new light on conceptual and physical issues discussed in this subsection.

2.1. Viewpoint

In this approach, one takes the central lesson of general relativity seriously: gravity *is* geometry whence, in a fundamental theory, there should be no background metric. In quantum gravity, geometry and matter should *both* be 'born quantum mechanically'. Thus, in contrast to approaches developed by particle physicists, one does not begin with quantum matter on a background geometry and use perturbation theory to incorporate quantum effects of gravity. There *is* a manifold but no metric, or indeed any other physical fields, in the background.[11]

In classical gravity, Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations. This role is now taken by *quantum* Riemannian geometry, discussed below. In the classical domain, general relativity stands out as the best available theory of gravity, some of whose predictions have been tested to an amazing degree of accuracy, surpassing even the legendary tests of quantum electrodynamics. Therefore, it is natural to ask: does quantum general relativity, coupled to suitable matter (or supergravity, its supersymmetric generalization), exist as consistent theories non-perturbatively? There is no implication that such a theory would be the final, complete description of Nature. Nonetheless, this is a fascinating open question, at least at the level of mathematical physics.

As explained in section 1.1, in particle physics circles the answer is often assumed to be in the negative, not because there is concrete evidence against non-perturbative quantum gravity, but because of the analogy to the theory of weak interactions. There, one first had a 4-point interaction model due to Fermi which works quite well at low energies but which fails to be renormalizable. Progress occurred not by looking for non-perturbative formulations of the Fermi model but by replacing the model by the Glashow–Salam–Weinberg renormalizable theory of electro-weak interactions, in which the 4-point interaction is replaced by $W^{\pm}$ and Z propagators. Therefore, it is often assumed that perturbative non-renormalizability of quantum general relativity points in a similar direction. However, this argument overlooks the crucial fact that, in the case of general relativity, there is a qualitatively new element. Perturbative treatments presuppose that the space-time can be assumed to be a continuum *at all scales* of interest to physics under consideration. This assumption is safe for weak interactions. In the gravitational case, on the other hand, the scale of interest is *the Planck length* $\ell_{\rm Pl}$ and there is no physical basis to presuppose that the continuum picture should be valid down to that scale. The failure of the standard perturbative treatments may largely be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies.

Are there any situations, outside loop quantum gravity, where such physical expectations are borne out in detail mathematically? The answer is in the affirmative. There exist quantum field theories (such as the Gross–Gross–Neveau model in three dimensions) in which the standard perturbation expansion is not renormalizable although the theory is *exactly soluble*! Failure of the standard perturbation expansion can occur because one insists on perturbing around the trivial, Gaussian point rather than the more physical, non-trivial fixed point of the renormalization group flow. Interestingly, thanks to recent work by Lauscher, Reuter, Percacci, Perini and others, there is now non-trivial and growing evidence that the situation may be similar in Euclidean quantum

[11] In $2+1$ dimensions, although one begins in a completely analogous fashion, in the final picture one can get rid of the background manifold as well. Thus, the fundamental theory can be formulated combinatorially [18, 20]. To achieve this in $3+1$ dimensions, one needs a more complete theory of (intersecting) knots in three-dimensions.

gravity. Impressive calculations have shown that pure Einstein theory may also admit a non-trivial fixed point. Furthermore, the requirement that the fixed point should continue to exist in the presence of matter constrains the couplings in non-trivial and interesting ways [26].

However, as indicated in section 1, even if quantum general relativity did exist as a mathematically consistent theory, there is no *a priori* reason to assume that it would be the 'final' theory of all known physics. In particular, as is the case with classical general relativity, while requirements of background independence and general covariance do restrict the form of interactions between gravity and matter fields and among matter fields themselves, the theory would not have a built-in principle which *determines* these interactions. Put differently, such a theory would not be a satisfactory candidate for unification of all known forces. However, just as general relativity has had powerful implications in spite of this limitation in the classical domain, quantum general relativity should have qualitatively new predictions, pushing further the existing frontiers of physics. Indeed, unification does not appear to be an essential criterion for usefulness of a theory even in other interactions. QCD, for example, is a powerful theory even though it does not unify strong interactions with electro-weak ones. Furthermore, the fact that we do not yet have a viable candidate for the grand unified theory does not make QCD any less useful.

2.2. *Quantum geometry*

Although loop quantum gravity does not provide a natural unification of dynamics of all interactions, as indicated in section 1.1 this program does provide a kinematical unification. More precisely, in this approach, one begins by formulating general relativity in the mathematical language of connections, the basic variables of gauge theories of electro-weak and strong interactions. Thus, now the configuration variables are not metrics as in Wheeler's geometrodynamics, but certain *spin-connections*; the emphasis is shifted from distances and geodesics to holonomies and Wilson loops [18, 40]. Consequently, the basic kinematical structures are the same as those used in gauge theories. A key difference, however, is that while a background space-time metric is available and crucially used in gauge theories, there are no background fields whatsoever now. Their absence is forced upon us by the requirement of diffeomorphism invariance (or 'general covariance').

Now, as emphasized in section 1.1, most of the techniques used in the familiar, Minkowskian quantum theories are deeply rooted in the availability of a flat background metric. In particular, it is this structure that enables one to single out the vacuum state, perform Fourier transforms to decompose fields canonically into creation and annihilation parts, define masses and spins of particles and carry out regularizations of products of operators. Already, when one passes to quantum field theory in *curved* space-times, extra work is needed to construct mathematical structures that can adequately capture the underlying physics [36]. In our case, the situation is much more drastic [20]: there is no background metric whatsoever! Therefore new physical ideas and mathematical tools are now necessary. Fortunately, they were constructed by a number of researchers in the mid-1990s and have given rise to a detailed quantum theory of geometry [22, 37, 39].

Because the situation is conceptually so novel and because there are no direct experiments to guide us, reliable results require a high degree of mathematical precision to ensure that there are no hidden infinities. Achieving this precision has been a priority in the programme. Thus, while one is inevitably motivated by heuristic, physical ideas and formal manipulations, the final results are mathematically rigorous. In particular, due care is taken in constructing function spaces, defining measures and functional integrals, regularizing products of field operators and calculating eigenvectors and eigenvalues of geometric operators. Consequently, the final results are all free of divergences, well defined and respect the background independence and diffeomorphism invariance.

Let us now turn to specifics. For simplicity, I will focus on the gravitational field; matter couplings are discussed in [18, 22, 37, 39, 40]. The basic gravitational configuration variable is an SU(2)-connection, A_a^i on a 3-manifold M representing 'space'. As in gauge theories, the momenta are the 'electric fields' E_i^a.[12] However, in the present gravitational context, they acquire an additional meaning: they can be naturally interpreted as orthonormal triads (with density weight 1) and determine the dynamical, Riemannian geometry of M. Thus, in contrast to Wheeler's geometrodynamics, the Riemannian structures, including the positive-definite metric on M, is now built from *momentum* variables.

The basic kinematic objects are: (i) holonomies $h_e(A)$ of A_a^i, which dictate how spinors are parallel transported along curves or edges e, and (ii) fluxes $E_{S,t} = \int_S t_i E_i^a \, \mathrm{d}^2 S_a$ of electric fields E_i^a, smeared with test fields t_i on a 2-surface S. The holonomies—the raison d'être of connections—serve as the 'elementary' configuration variables which are to have unambiguous quantum analogues. They form an Abelian $C^\star$ algebra, denoted by Cyl. Similarly, the fluxes serve as 'elementary momentum variables'. Their Poisson brackets with holonomies define a derivation on Cyl. In this sense—as in Hamiltonian mechanics on manifolds—momenta are associated with 'vector fields' on the configuration space.

The first step in quantization is to use the Poisson algebra between these configuration and momentum functions to construct an abstract $\star$-algebra $\mathcal{A}$ of elementary quantum operators. This step is straightforward. The second step is to introduce a representation of this algebra by 'concrete' operators on a Hilbert space (which is to serve as the kinematic setup for the Dirac quantization program) [1, 6, 18]. For systems with an infinite number of degrees of freedom, this step is highly non-trivial. In Minkowskian field theories, for example, the analogous kinematic $\star$-algebra of canonical commutation relations admits infinitely many *inequivalent* representations even after asking for Poicaré invariance! The standard Fock representation is uniquely selected *only* when a restriction to non-interacting theories is made. The general viewpoint is that the choice of representation is dictated by (symmetries and more importantly) the dynamics of the theory under consideration. *A priori*, this task seems daunting for general relativity. However, it turns out that the diffeomorphism invariance—dictated by 'background independence'—is enormously more powerful than Poincaré invariance. Recent results by Fleishhack and Lewandowski *et al* show that *the algebra $\mathcal{A}$ admits a unique diffeomorphism invariant state* [22, 41]! Using it, through a standard procedure due to Gel'fand, Naimark and Segal, one can construct a unique representation of $\mathcal{A}$. Thus, remarkably, there is a unique kinematic framework for *any* diffeomorphism invariant quantum theory for which the appropriate point of departure is provided by $\mathcal{A}$, *irrespective of the details of dynamics*! Chronologically, this concrete representation was in fact introduced in the early 1990s by Ashtekar, Baez, Isham and Lewandowski. It led to the detailed theory of quantum geometry that underlies loop quantum gravity. Once a rich set of results had accumulated, researchers began to analyse the issue of uniqueness of this representation and systematic improvements over several years culminated in the simple statement given above.

[12] Throughout, indices $a, b, \ldots$ will refer to the tangent space of M while the 'internal' indices $i, j, \ldots$ will refer to the Lie algebra of SU(2).

Let me describe the salient features of this representation [22, 37]. Quantum states span a specific Hilbert space $\mathcal{H}$ consisting of wave functions of connections which are square integrable with respect to a natural, diffeomorphism invariant measure. This space is very large. However, it can be conveniently decomposed into a family of orthogonal, *finite* dimensional subspaces $\mathcal{H} = \oplus_{\gamma,\vec{j}}\, \mathcal{H}_{\gamma,\vec{j}}$, labelled by graphs γ, each edge of which itself is labelled by a spin (i.e., half-integer) j [22, 39]. (The vector $\vec{j}$ stands for the collection of half-integers associated with all edges of γ.) One can think of γ as a 'floating lattice' in M—'floating' because its edges are arbitrary, rather than 'rectangular'. (Indeed, since there is no background metric on M, a rectangular lattice has no invariant meaning.) Mathematically, $\mathcal{H}_{\gamma,\vec{j}}$ can be regarded as the Hilbert space of a spin-system. These spaces are extremely simple to work with; this is why very explicit calculations are feasible. Elements of $\mathcal{H}_{\gamma,\vec{j}}$ are referred to as *spin-network states* [22, 39, 42].

In the quantum theory, the fundamental excitations of geometry are most conveniently expressed in terms of holonomies [22]. They are thus *one-dimensional, polymer-like* and, in analogy with gauge theories, can be thought of as 'flux lines' of electric fields/triads. More precisely, they turn out to be *flux lines of area*, the simplest gauge invariant quantities constructed from the momenta E^a_i: an elementary flux line deposits a quantum of area on any 2-surface S it intersects. Thus, if quantum geometry were to be excited along just a few flux lines, most surfaces would have zero area and the quantum state would not at all resemble a classical geometry. This state would be analogous, in Maxwell theory, to a 'genuinely quantum mechanical state' with just a few photons. In the Maxwell case, one must superpose photons coherently to obtain a semi-classical state that can be approximated by a classical electromagnetic field. Similarly, here, semi-classical geometries can result only if a huge number of these elementary excitations are superposed in suitable dense configurations [22, 37]. The state of quantum geometry around you, for example, must have so many elementary excitations that approximately $\sim 10^{68}$ of them intersect the sheet of paper you are reading. Even in such states, the geometry is still distributional, concentrated on the underlying elementary flux lines. But if suitably coarse-grained, it can be approximated by a smooth metric. Thus, the continuum picture is only an approximation that arises from coarse graining of semi-classical states.

The basic quantum operators are the holonomies $\hat{h}_e$ along curves or edges e in M and the fluxes $\hat{E}_{S,t}$ of triads $\hat{E}^a_i$. Both are densely defined and self-adjoint on $\mathcal{H}$. Furthermore, detailed work by Ashtekar, Lewandowski, Rovelli, Smolin, Thiemann and others shows that *all eigenvalues of geometric operators constructed from the fluxes of triad are discrete* [22, 37, 39, 42]. This key property is, in essence, the origin of the fundamental discreteness of quantum geometry. For, just as the classical Riemannian geometry of M is determined by the triads E^a_i, all Riemannian geometry operators—such as the area operator $\hat{A}_S$ associated with a 2-surface S or the volume operator $\hat{V}_R$ associated with a region R—are constructed from $\hat{E}_{S,t}$. However, since even the classical quantities A_S and V_R are non-polynomial functionals of triads, construction of the corresponding $\hat{A}_S$ and $\hat{V}_R$ is quite subtle and requires a great deal of care. But their final expressions are rather simple [22, 37, 39].

In this regularization, the underlying background independence turns out to be a blessing. For, diffeomorphism invariance constrains *severely* the possible forms of the final expressions severely, and the detailed calculations then serve essentially to fix numerical coefficients and other details. Let me illustrate this point with the example of the area operators $\hat{A}_S$. Since they are associated with 2-surfaces S while the states are one-dimensional excitations, the diffeomorphism covariance requires that the action of $\hat{A}_S$ on a state $\psi_{\gamma,\vec{j}}$ must be concentrated at the intersections of S with γ. The detailed expression bears out this expectation: the action of $\hat{A}_S$ on $\Psi_{\gamma,\vec{j}}$ is dictated simply by the spin labels j_I attached to those edges of γ which intersect S. For all the surfaces S and three-dimensional regions R in M, $\hat{A}_S$ and $\hat{V}_R$ are densely defined, self-adjoint operators. *All their eigenvalues are discrete.* Naively, one might expect that the eigenvalues would be uniformly spaced given by, e.g., integral multiples of the Planck area or volume. Indeed, for area, such assumptions were routinely made in the initial investigations of the origin of black hole entropy and, for volume, they are made in quantum gravity approaches based on causal sets where discreteness is postulated at the outset. In quantum Riemannian geometry, this expectation is *not* borne out; the distribution of eigenvalues is quite subtle. In particular, the eigenvalues crowd rapidly as areas and volumes increase. In the case of area operators, the complete spectrum is known in a *closed form*, and the first several hundred eigenvalues have been explicitly computed numerically. For a large eigenvalue a_n, the separation $\Delta a_n = a_{n+1} - a_n$ between consecutive eigenvalues decreases exponentially: $\Delta a_n \leqslant \exp -(\sqrt{a_n}/\ell_{\rm Pl})\ell_{\rm Pl}^2$! Because of such strong crowding, the continuum approximation becomes excellent quite rapidly just a few orders of magnitude above the Planck scale. At the Planck scale, however, there is a precise and very specific replacement. This is the arena of quantum geometry. The premise is that the standard perturbation theory fails because it ignores this fundamental discreteness.

There is however a further subtlety. This non-perturbative quantization has a one parameter family of ambiguities labelled by $\gamma > 0$. This γ is called the Barbero–Immirzi parameter and is rather similar to the well-known θ-parameter of QCD [22, 37, 39]. In QCD, a single classical theory gives rise to inequivalent sectors of quantum theory, labelled by θ. Similarly, γ is classically irrelevant but different values of γ correspond to unitarily inequivalent representations of the algebra of geometric operators. The overall mathematical structure of all these sectors is very similar; the only difference is that the eigenvalues of all geometric operators scale with γ. For example, the simplest eigenvalues of the area operator $\hat{A}_S$ in the γ quantum sector is given by[13]

$$a_{\{j\}} = 8\pi\gamma\ell_{\rm Pl}^2 \sum_I \sqrt{j_I(j_I+1)}, \qquad (2.1)$$

where $\{j\}$ is a collection of $\frac{1}{2}$-integers j_I, with $I = 1, \ldots, N$ for some N. Since the representations are unitarily inequivalent, as usual, one must rely on Nature to resolve this ambiguity: just as Nature must select a specific value of θ in QCD, it must select a specific value of γ in loop quantum gravity. With a single judicious experiment—e.g., measurement of the lowest eigenvalue of the area operator $\hat{A}_S$ for a 2-surface S of any given topology—we could determine the value of γ and fix the theory. Unfortunately, such experiments are hard to perform! However, we will see in section 3.2 that the Bekenstein–Hawking formula of black hole entropy provides an indirect measurement of this lowest eigenvalue of area for the 2-sphere topology and can therefore be used to fix the value of γ.

[13] In particular, the lowest non-zero eigenvalue of area operators is proportional to γ. This fact has led to a misunderstanding in certain particle physics circles where γ is thought of as a regulator responsible for the discreteness of quantum geometry. As explained above, this is *not* the case; γ is analogous to the QCD θ and quantum geometry is discrete in *every* permissible γ-sector. Note also that, at the classical level, the theory is equivalent to general relativity only if γ is *positive*; if one sets $\gamma = 0$ by hand, one cannot recover even the kinematics of general relativity. Similarly, at the quantum level, setting $\gamma = 0$ would lead to a meaningless theory in which *all* eigenvalues of geometric operators vanish identically.

2.3. Quantum dynamics

Quantum geometry provides a mathematical arena to formulate non-perturbative dynamics of candidate quantum theories of gravity, without any reference to a background classical geometry. In the case of general relativity, it provides tools to write down quantum Einstein's equations in the Hamiltonian approach and calculate transition amplitudes in the path integral approach. Until recently, effort was focused primarily on Hamiltonian methods. However, during the last four years or so, path integrals—called *spin foams*—have drawn a great deal of attention. This work has led to fascinating results suggesting that, thanks to the fundamental discreteness of quantum geometry, path integrals defining quantum general relativity may be finite. A summary of these developments can be found in [38, 39]. In this section, I will summarize the status of the Hamiltonian approach. For brevity, I will focus on source-free general relativity, although there has been considerable work also on matter couplings [22, 37, 39].

For simplicity, let me suppose that the 'spatial' 3-manifold M is compact. Then, in any theory without background fields, Hamiltonian dynamics is governed by constraints. Roughly, this is because in these theories diffeomorphisms correspond to gauge in the sense of Dirac. Recall that, on the Maxwell phase space, gauge transformations are generated by the functional $\mathcal{D}_a E^a$, which is constrained to vanish on physical states due to Gauss law. Similarly, on phase spaces of background-independent theories, diffeomorphisms are generated by Hamiltonians which are constrained to vanish on physical states.

In the case of general relativity, there are three sets of constraints. The first set consists of the three Gauss equations

$$\mathcal{G}_i := \mathcal{D}_a E^a_i = 0, \tag{2.2}$$

which, as in Yang–Mills theories, generates internal SU(2) rotations on the connection and the triad fields. The second set consists of a covector (or diffeomorphism) constraint

$$\mathcal{C}_b := E^a_i F^i_{ab} = 0, \tag{2.3}$$

which generates spatial diffeomorphism on M (modulo internal rotations generated by $\mathcal{G}_i$). Finally, there is the key scalar (or Hamiltonian) constraint

$$\mathcal{S} := \epsilon^{ijk} E^a_i E^b_j F_{abk} + \cdots = 0, \tag{2.4}$$

which generates time-evolutions. (In equations (2.4), '. . .' represents extrinsic curvature terms, expressible as Poisson brackets of the connection, the total volume constructed from triads and the first term in the expression of $\mathcal{S}$ given above. We will not need their explicit forms.) Our task in quantum theory is threefold: (i) elevate these constraints (or their 'exponentiated versions') to well-defined operators on the kinematic Hilbert space $\mathcal{H}$, (ii) select physical states by asking that they be annihilated by these constraints and (iii) introduce an inner-product and interesting observables and develop approximation schemes, truncations, etc to explore physical consequences. I would like to emphasize that, even if one begins with Einstein's equations at the classical level, non-perturbative dynamics gives rise to interesting quantum corrections. Consequently, *the effective classical equations derived from the quantum theory exhibit significant departures from classical Einstein's equations*. This fact has had important implications in quantum cosmology.

Let us return to the three tasks. Since the canonical transformations generated by the Gauss and diffeomorphism constraints have a simple geometrical meaning, completion of (i) in these cases is fairly straightforward. For the Hamiltonian constraint, on the other hand, there are no such guiding principles whence the procedure is subtle. In particular, specific regularization choices have to be made. Consequently, the final expression of the Hamiltonian constraint is not unique. A systematic discussion of ambiguities can be found in [22]. At the present stage of the program, such ambiguities are inevitable; one has to consider all viable candidates and analyse if they lead to sensible theories. Interestingly, observational input from cosmology are now being used to constrain the simplest of these ambiguities. In any case, it should be emphasized that the availability of well-defined Hamiltonian constraint operators is by itself a notable technical success. For example, the analogous problem in quantum geometrodynamics—a satisfactory regularization of the Wheeler–DeWitt equation—is still open, although the formal equation was written down some 35 years ago. To be specific, I will first focus on the procedure developed by Lewandowski, Rovelli, Smolin and others which culminated in a specific construction due to Thiemann.

Steps (ii) and (iii) have been completed for the Gauss and the diffeomorphism constraints [22, 37, 39]. Mathematical implementation required a very substantial extension [18] of the algebraic quantization program initiated by Dirac and required the use of the spin-network machinery [22, 37, 39] of quantum geometry. Again, the detailed implementation is a non-trivial technical success and the analogous task has not been completed in geometrodynamics because of difficulties associated with infinite-dimensional spaces. Thiemann's quantum Hamiltonian constraint is first defined *on the space of solutions to the Gauss constraint* [37]. The regularization procedure requires several external inputs. However, a number of these ambiguities disappear when one restricts the action of the constraint operator to the space of solutions of the diffeomorphism constraint. On this space, the problem of finding a general solution to the Hamiltonian constraint can be systematically reduced to that of finding *elementary* solutions, a task that requires only analysis of linear operators on certain finite-dimensional spaces [22]. In this sense, step (ii) has been completed for all constraints.

This is a non-trivial result. However, it is still unclear whether this theory is physically satisfactory; at this stage, it is in principle possible that it captures only an 'exotic' sector of quantum gravity. A *key open problem* in loop quantum gravity is to show that the Hamiltonian constraint—either Thiemann's or an alternative such as the one of Gambini and Pullin—admits a 'sufficient number' of semi-classical states. Progress on this problem has been slow because the general issue of semi-classical limits is itself difficult in *any* background-independent approach.[14] However, a systematic understanding has now begun to emerge and is providing the 'infrastructure' needed to analyse the key problem mentioned above [22, 37]. Finally, while there are promising ideas to complete step (iii), substantial further work is necessary to solve this problem. Recent advances in quantum cosmology, described in section 3.1, is an example of progress in this direction and it provides a significant support for the Thiemann scheme, but of course only within the limited context of mini-superspaces.

To summarize, from a mathematical physics perspective, in the Hamiltonian approach the crux of the dynamics lies in quantum constraints. The quantum Gauss and diffeomorphism constraints have been solved satisfactorily and it is significant that detailed regularization schemes

[14] In the dynamical triangulation [23, 24] and causal set [33] approaches, for example, a great deal of care is required to ensure that even the dimension of a typical space-time is 4.

have been proposed for the Hamiltonian constraint. But it is not clear if any of the proposed strategies to solve this constraint incorporates the familiar low energy physics in full theory, i.e., beyond symmetry reduced models. Novel ideas are being pursued to address this issue. I will summarize them in section 4.

2.4. Remarks

1. There has been another concern about the Thiemann-type regularizations of the Hamiltonian constraint which, however, is less specific. It stems from the structure of the constraint algebra. On the space of solutions to the Gauss constraints, the Hamiltonian constraint operators do not commute. This is compatible with the fact that the Poisson brackets between these constraints do not vanish in the classical theory. However, it is not obvious that the commutator algebra correctly reflects the classical Poisson bracket algebra. To shed light on this issue, Gambini, Lewandowski, Marolf and Pullin introduced a certain domain of definition of the Hamiltonian constraint which is smaller than the space of all solutions to the Gauss constraints but larger than the space of solutions to the Gauss *and* diffeomorphism constraints. It was then found that the commutator between any two Hamiltonian constraints vanishes identically. However, it was also shown that the operator representing the right side of the classical Poisson bracket *also vanishes* on all the quantum states in the new domain. Therefore, while the vanishing of the commutator of the Hamiltonian constraint was initially unexpected, this analysis does not reveal a clear-cut problem with these regularizations.

2. One can follow this scheme step by step in $2+1$ gravity where one knows what the result should be. One can obtain the 'elementary solutions' mentioned above and show that all the standard quantum states—including the semi-classical ones—can be recovered as linear combinations of these elementary ones. As is almost always the case with constrained systems, there are *many more solutions* and the 'spurious ones' have to be eliminated by the requirement that the physical norm be finite. In $2+1$ gravity, the connection formulation used here naturally leads to a complete set of Dirac observables and the inner-product can be essentially fixed by the requirement that they be self-adjoint. In $3+1$ gravity, by contrast, we do not have this luxury and the problem of constructing the physical inner-product is therefore much more difficult. However, the concern here is that of weeding out unwanted solutions rather than having a 'sufficient number' of semi-classical ones, a significantly less serious issue at the present stage of the program.

3. Applications of quantum geometry

In this section, I will summarize two developments that answer several of the questions raised under first two bullets in section 2.1.

3.1. Big bang

Over the last five years, quantum geometry has led to some striking results of direct physical interest. The first of these concerns the fate of the big-bang singularity.

Traditionally, in quantum cosmology one has proceeded by first imposing spatial symmetries—such as homogeneity and isotropy—to freeze out all but a finite number of degrees of freedom *already at the classical level* and then quantizing the reduced system. In the simplest case, the basic variables of the reduced classical system are the scale factor a and matter fields ϕ. The symmetries imply that space-time curvature goes as $\sim 1/a^n$, where $n > 0$ depends on the matter field under consideration. Einstein's equations then predict a big bang, where the scale factor goes to zero and the curvature blows up. As indicated in section 2.1, this is reminiscent of what happens to ferromagnets at the Curie temperature: magnetization goes to zero and the susceptibility diverges. By analogy, the key question is: do these 'pathologies' disappear if we re-examine the situation in the context of an appropriate quantum theory? In traditional quantum cosmologies, without additional input, they do not. That is, typically, to resolve the singularity, one has to either introduce matter with unphysical properties or introduce additional boundary conditions, e.g., by invoking new principles.

In a series of seminal papers, Bojowald has shown that the situation in loop quantum cosmology is quite different: the underlying quantum geometry makes a *qualitative* difference very near the big bang [22, 43, 44]. At first, this seems puzzling because, after symmetry reduction, the system has only a *finite* number of degrees of freedom. Thus, quantum cosmology is analogous to quantum mechanics rather than quantum field theory. How then can one obtain qualitatively new predictions? Ashtekar, Bojowald and Lewandowski clarified the situation: if one follows the program laid out in the full theory, then, even for the symmetry reduced model, one is led to an inequivalent quantum theory—a new quantum mechanics!

Let me make a small detour to explain how this comes about. Consider the simplest case—spatially homogeneous, isotropic models. In the standard geometrodynamic treatment, the operator $\hat{a}$ corresponding to the scale factor is self-adjoint and has zero as part of its *continuous* spectrum. Now, there is a general mathematical result which says that any measurable function of a self-adjoint operator is again self-adjoint. Since the function a^{-1} on the spectrum of $\hat{a}$ is well defined except at $a = 0$ and since this is a subset of zero measure of the continuous spectrum, $(\hat{a})^{-1}$ is self-adjoint and is the natural candidate for the operator analogue of a^{-1}. This operator is unbounded above whence the curvature operator is also unbounded in the quantum theory. In connection-dynamics, the information of geometry is encoded in the triad, which now has a single independent component p related to the scale factor via $|p| = a^2$. To pass to quantum theory, one follows the procedure used in the full theory. Therefore, only the holonomies are well-defined operators; *connections are not*! Since connections in this model have the same information as the (only independent component of the) extrinsic curvature, the resulting theory is *inequivalent* to the one used in geometrodynamics.[15] Specifically, eigenvectors of $\hat{p}$ are now normalizable. Thus $\hat{p} = \sum_p |p > p < p|$; one has a direct sum rather than a direct integral. Hence, the spectrum of $\hat{p}$ is equipped with a *discrete* topology and $p = 0$ is *no longer a subset of zero measure*. Therefore, the naive inverse of $\hat{p}$ is not even densely defined, let alone self-adjoint. The operator corresponding to $a^{-1} = 1/\sqrt{|p|}$ (or any inverse power of a) has to be defined differently. Fortunately, one can again use a procedure introduced by Thiemann in the full theory and show that this can be done. The operator $\widehat{1/a}$ so constructed has the physically expected properties. For instance, it commutes with $\hat{a}$. The product of the eigenvalues of $\hat{a}$

[15] Following the notation of quantum mechanics, let us represent the only independent component of the connection by x. Then, one is led to a quantum theory which is inequivalent from the standard Schrödinger mechanics because, while there is a well-defined operator corresponding to $\exp i\alpha x$ for each real number α, it fails to be weakly continuous. Hence there is no operator corresponding to (the connection) x itself. The operator $\hat{p}$ on the other hand is well defined. The Hilbert space is not $L^2(\mathbb{R}, \mathrm{d}x)$ but $L^2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, \mathrm{d}\mu_o)$ where $\bar{\mathbb{R}}_{\mathrm{Bohr}}$ is the Bohr compactification of the real line and $d\mu_o$ the natural Haar measure thereon. Not surprisingly, the structure of $L^2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, \mathrm{d}\mu_o)$ is the quantum cosmology analogue of that of $\mathcal{H}$ of the full theory [22].

and $\widehat{1/a}$ ranges from 1 to 10^8 already when $a \sim 100\ell_{\rm Pl}$, and becomes even closer to 1 as the universe expands. However, in the deep Planck regime very near the big bang, the operator $\widehat{1/a}$ of loop quantum cosmology is *qualitatively* different from its analogue in geometrodynamics: $\widehat{1/a}$ is *bounded above* in the full Hilbert space! Consequently, the curvature is also bounded above. If classically it goes as $1/a^2$, then the loop quantum cosmology upper bound is about 10^{55} times the curvature at the horizon of a solar mass black hole. This is a huge number. *But it is finite.* The mechanism is qualitatively similar to the one which makes the ground state energy of a hydrogen atom in quantum theory, $E_0 = -(me^4/\hbar^2)$, finite even though it is infinite classically: in the expression of the upper bound of curvature, $\hbar$ again intervenes in the denominator.

This completes the detour. Let us now consider dynamics. Since the curvature is bounded above in the entire Hilbert space, one might hope that the quantum evolution may be well defined right through the big bang singularity. Is this in fact the case? The second surprise is that the answer is in the affirmative. More precisely, the situation can be summarized as follows. As one might expect, the 'evolution' is dictated by the Hamiltonian constraint operator. Let us expand the quantum state as $\mid\Psi>=\sum_p \psi_p(\phi)\mid p\rangle$, where $\mid p\rangle$ are the eigenstates of $\hat{p}$, and ϕ denotes matter fields. Then, the Hamiltonian constraint takes the form:

$$C^+\psi_{p+4p_o}(\phi) + C^o\psi_p(\phi) + C^-\psi_{p-4p_o}(\phi) = \gamma\ell_{\rm Pl}^2\hat{H}_\phi\psi_p(\phi), \tag{3.1}$$

where $C^\pm$, C^o are fixed functions of p; γ is the Barbero–Immirzi parameter; p_o is a constant, determined by the lowest eigenvalue of the area operator, and $\hat{H}_\phi$ is the matter Hamiltonian. Again, using the analogue of the Thiemann regularization from the full theory, one can show that the matter Hamiltonian is a well-defined operator. Primarily, being a constraint equation, (3.1) restricts the physically permissible $\psi_p(\phi)$. However, *if* we choose to interpret the eigenvalues p of $\hat{p}$ (i.e., the square of the scale factor multiplied by the sign of the determinant of the triad) as a time variable, (3.1) can be interpreted as an 'evolution equation' which evolves the state through discrete time steps. The highly non-trivial result is that the coefficients $C^\pm$, C^o are such that *one can evolve right through the classical singularity*, i.e., to the past, right through $p = 0$. Thus, the infinities predicted by the classical theory at the big bang are artifacts of assuming that the classical, continuum space-time approximation is valid right up to the big bang. In the quantum theory, the state can be evolved through the big bang without any difficulty. However, the classical space-time description completely fails near the big bang; figuratively, the classical space-time 'dissolves'. This resolution of the singularity without any 'external' input (such as matter violating energy conditions) is significantly different from what happens with the standard Wheeler–DeWitt equation of quantum geometrodynamics. However, for large values of the scale factor, the two evolutions are close; as one would have hoped, quantum geometry effects intervene only in the 'deep Planck regime' and resolve the singularity. From this perspective, then, one is led to say that the most striking of the consequences of loop quantum gravity are not seen in standard quantum cosmology because it 'washes out' the fundamental discreteness of quantum geometry.

The detailed calculations have revealed another surprising feature. The fact that the quantum effects become prominent near the big bang, completely invalidating the classical predictions, is pleasing but not unexpected. However, prior to these calculations, it was not clear how, soon after the big bang, one can start trusting semi-classical notions and calculations. It would not be surprising if we had to wait till the radius of the universe became, say, a few billion times the Planck length. These calculations strongly suggest that a few tens of Planck lengths should suffice. This is fortunate because it is now feasible to develop quantum numerical relativity; with computational resources commonly available, grids with $(10^9)^3$ points are hopelessly large but one with $(100)^3$ points could be manageable.

Finally, quantum geometry effects also modify the *kinematic* matter Hamiltonian in interesting ways. In particular, they introduce an *anti-damping* term in the evolution of the scalar field during the initial inflationary phase, which can drive the scalar field to values needed in the chaotic inflation scenario [45], without having to appeal to the occurrence of large quantum fluctuations. Such results have encouraged some phenomenologists to seek for signatures of loop quantum gravity effects in the observations of the very early universe. However, these applications lie at the forefront of today's research and are therefore not as definitive.

3.2. Black holes

Loop quantum cosmology illuminates dynamical ramifications of quantum geometry but within the context of mini-superspaces where all but a finite number of degrees of freedom are frozen. In this subsection, I will discuss a complementary application where one considers the full theory but probes the consequences of quantum geometry which are not sensitive to full quantum dynamics—the application of the framework to the problem of black hole entropy. This discussion is based on the work of Ashtekar, Baez, Bojowald, Corichi, Domagala, Krasnov, Lewandowski and Meissner, much of which was motivated by earlier work of Krasnov, Rovelli and others [22, 46, 47].

As explained in section 1, from the mid-1970s, a key question in the subject has been: what is the statistical mechanical origin of the entropy $S_{\mathrm{BH}} = (a_{\mathrm{hor}}/4\ell_{\mathrm{Pl}}^2)$ of large black holes? What are the microscopic degrees of freedom that account for this entropy? This relation implies that a solar mass black hole must have $(\exp 10^{77})$ quantum states, a number that is *huge* even by the standards of statistical mechanics. Where do all these states reside? To answer these questions, in the early 1990s Wheeler had suggested the following heuristic picture, which he christened 'It from Bit'. Divide the black hole horizon into elementary cells, each with one Planck unit of area, ℓ_{Pl}^2, and assign to each cell two microstates or one 'bit'. Then, the total number of states $\mathcal{N}$ is given by $\mathcal{N} = 2^n$ where $n = (a_{\mathrm{hor}}/\ell_{\mathrm{Pl}}^2)$ is the number of elementary cells, whence entropy is given by $S = \ln \mathcal{N} \sim a_{\mathrm{hor}}$. Thus, apart from a numerical coefficient, the entropy ('It') is accounted for by assigning two states ('Bit') to each elementary cell. This qualitative picture is simple and attractive. But can these heuristic ideas be supported by a systematic analysis from first principles? Quantum geometry has supplied such an analysis. As one would expect, while some qualitative features of this picture are borne out, the actual situation is far more subtle.

A systematic approach requires that we first specify the class of black holes of interest. Since the entropy formula is expected to hold unambiguously for black holes in equilibrium, most analyses were confined to *stationary*, eternal black holes (i.e., in four-dimensional general relativity, to the Kerr–Newman family). From a physical viewpoint however, this assumption seems overly restrictive. After all, in statistical mechanical calculations of entropy of ordinary systems, one only has to assume that the given system is in equilibrium, not the whole world. Therefore, it should suffice for us to assume that the black hole itself is in equilibrium; the exterior geometry should not be forced to be time-independent. Furthermore, the analysis should also account for entropy of black holes which may be distorted or carry (Yang–Mills and other) hair. Finally, it has been known since the mid-1970s that the thermodynamical considerations apply not only to black holes but also to cosmological horizons. A natural question is: can these diverse situations be treated in a single stroke? Within the quantum geometry approach,

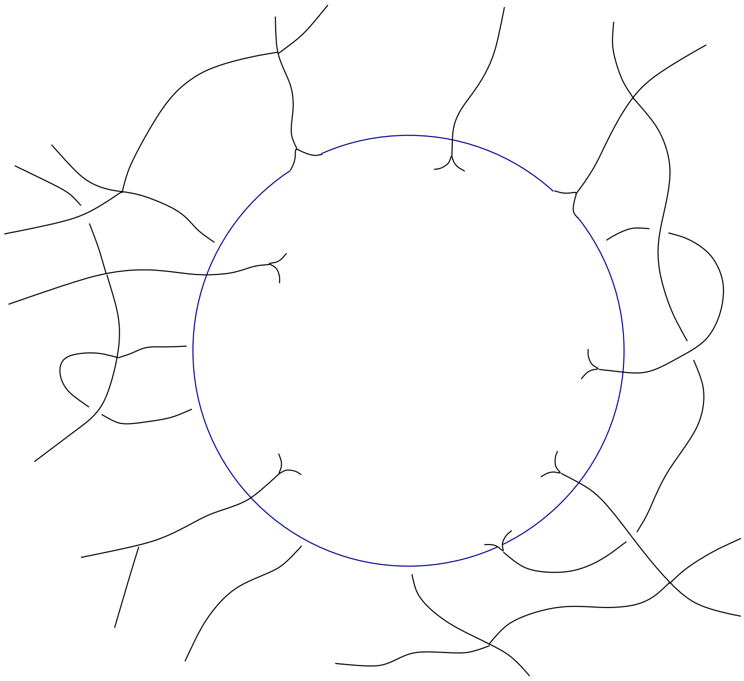

Figure 1. The quantum horizon. Polymer excitations in the bulk puncture the horizon, endowing it with a quantized area. Intrinsically, the horizon is flat, except at punctures where it acquires a quantized deficit angle. These angles add up to endow the horizon with a 2-sphere topology.

the answer is in the affirmative. The entropy calculations have been carried out in the 'isolated horizons' framework which encompasses all these situations. Isolated horizons serve as 'internal boundaries' whose intrinsic geometries (and matter fields) are time-independent, although space-time geometry as well as matter fields in the external space-time region can be fully dynamical. The zeroth and first laws of black hole mechanics have been extended to isolated horizons [35]. Entropy associated with an isolated horizon refers to the family of observers in the exterior, for whom the isolated horizon is a physical boundary that separates the region which is accessible to them from the one which is not. This point is especially important for cosmological horizons where, without reference to observers, one cannot even define horizons. States which contribute to this entropy are the ones which can interact with the states in the exterior; in this sense, they 'reside' on the horizon.

In the detailed analysis, one considers space-times admitting an isolated horizon as inner boundary and carries out a systematic quantization. The quantum geometry framework can be naturally extended to this case. The isolated horizon boundary conditions imply that the intrinsic geometry of the quantum horizon (figure 1) is described by the so-called U(1) Chern–Simons theory on the horizon. This is a well-developed, topological field theory. A deeply satisfying feature of the analysis is that there is a seamless matching of three otherwise independent structures: the isolated horizon boundary conditions, the quantum geometry in the bulk and the Chern–Simons theory on the horizon. In particular, one can calculate eigenvalues of certain physically interesting operators either using purely bulk quantum geometry without any knowledge of the Chern–Simons theory or using the Chern–Simons theory without any knowledge of the bulk quantum geometry. The two theories have never heard of each other. But the isolated horizon boundary conditions require that the two infinite sets of numbers match exactly. This is a highly non-trivial requirement. But the numbers do match, thereby providing a coherent description of the quantum horizon [22].

In this description, the polymer excitations of the bulk geometry, each labelled by a spin j_I, pierce the horizon, endowing it an elementary area a_{j_I} given by (2.1). The sum $\sum_I a_{j_I}$ adds up to the total horizon area $a_{\rm hor}$. The intrinsic geometry of the horizon is flat, except at these punctures, but at each puncture there is a *quantized* deficit angle. These add up to endow the horizon with a 2-sphere topology. For a solar mass black hole, a typical horizon state would have 10^{77} punctures, each contributing a tiny deficit angle. So, although quantum geometry *is* distributional, it can be well approximated by a smooth metric.

The counting of states can be carried out as follows. Firstly, one constructs a micro-canonical ensemble by restricting oneself only to those states for which the total area, mass and angular momentum multipole moments and charges lie in small intervals around fixed values $a_{\rm hor}$, $M^{(n)}_{\rm hor}$, $J^{(n)}_{\rm hor}$, $Q^i_{\rm hor}$. (As is usual in statistical mechanics, the leading contribution to the entropy is independent of the precise choice of these small intervals.) For each set of punctures, one can compute the dimension of the surface Hilbert space, consisting of Chern–Simons states compatible with that set. One allows all possible sets of punctures (by varying both the spin labels and the number of punctures) and adds up the dimensions of the corresponding *surface* Hilbert spaces to obtain the number $\mathcal{N}$ of permissible surface states. One finds that the horizon entropy $S_{\rm hor}$ is given by

$$S_{\rm hor} := \ln \mathcal{N} = \frac{\gamma_o}{\gamma}\frac{a_{\rm hor}}{\ell_{\rm Pl}^2} - \frac{1}{2}\ln\left(\frac{a_{\rm hor}}{\ell_{\rm Pl}^2}\right) + o\ln\left(\frac{a_{\rm hor}}{\ell_{\rm Pl}^2}\right), \tag{3.2}$$

where $\gamma_o \approx 0.2735$ is a root of an algebraic equation[16] and $o(x)$ denote quantities for which $o(x)/x$ tends to zero as x tends to infinity. Thus, for large black holes, the leading term is indeed proportional to the horizon area. This is a non-trivial result; for example, early calculations often led to proportionality to the square root of the area. However, even for large black holes, one obtains agreement with the Hawking–Bekenstein formula only in the sector of quantum geometry in which the Barbero–Immirzi parameter γ takes the value $\gamma = \gamma_o$. Thus, while all γ sectors are equivalent classically, the standard quantum field theory in curved space-times is recovered in the semi-classical theory only in the γ_o sector of quantum geometry. It is quite remarkable that thermodynamic considerations involving *large* black holes can be used to fix the quantization ambiguity which dictates such Planck scale properties as eigenvalues of geometric operators. Note, however, that the value of γ can be fixed by demanding agreement with the semi-classical result just in one case—e.g., a spherical horizon with zero charge, or a cosmological horizon in the de Sitter space-time, or, ... Once the value of γ is fixed, the theory is completely fixed and we can ask: does this theory yield the Hawking–Bekenstein value of entropy for *all* isolated horizons, irrespective of the values of charge, angular momentum and cosmological constant, the amount of distortion or hair? The answer is in the affirmative. Thus, the agreement with quantum field theory in curved space-times holds in *all* these diverse cases.

[16] This value is different from the one originally reported in the detailed analysis of Ashtekar–Baez–Krasnov because their counting contained a subtle error. It was corrected by Domagala and Lewandowski [46] and by Meissner [47]. Their analysis brings out further limitations of Wheeler's 'It from Bit' scenario and of arguments relating entropy to quasi-normal frequencies of black holes.

Why does γ_o not depend on other quantities such as charge? This important property can be traced back to a key consequence of the isolated horizon boundary conditions: detailed calculations show that only the gravitational part of the symplectic structure has a surface term at the horizon; the matter symplectic structures have only volume terms. (Furthermore, the gravitational surface term is insensitive to the value of the cosmological constant.) Consequently, there are no independent surface quantum states associated with matter. This provides a natural explanation of the fact that the Hawking–Bekenstein entropy depends only on the horizon area and is independent of electromagnetic (or other) charges.

So far, all matter fields were assumed to be minimally coupled to gravity (there was no restriction on their couplings to each other). If one allows non-minimal gravitational couplings, the isolated horizon framework (as well as other methods) show that entropy should depend not just on the area but also on the values of non-minimally coupled matter fields at the horizon. At first, this non-geometrical nature of entropy seems to be a major challenge to approaches based on quantum geometry. However, it turns out that, in the presence of non-minimal couplings, the geometrical orthonormal triads E_i^a are no longer functions just of the momenta conjugate to the gravitational connection A_a^i, but depend also on matter fields. Thus quantum Riemannian geometry—including area operators—can no longer be analysed just in the gravitational sector of the quantum theory. The dependence of the triads and area operators on matter fields is such that the counting of surface states leads precisely to the correct expression of entropy, again for the same value of the Barbero–Immirzi parameter γ. This is a subtle and highly non-trivial check on the robustness of the quantum geometry approach to the statistical mechanical calculation of black hole entropy.

Finally, let us return to Wheeler's 'It from Bit'. The horizon can indeed be divided into elementary cells. But they need not have the same area; the area of a cell can be $8\pi\gamma\ell_{\rm Pl}^2\sqrt{j(j+1)}$, where j is an *arbitrary* half-integer subject only to the requirement that $8\pi\gamma\ell_{\rm Pl}^2\sqrt{j(j+1)}$ does not exceed the total horizon area $a_{\rm hor}$. Wheeler assigned to each elementary cell two bits. In the quantum geometry calculation, this corresponds to focusing just on $j = 1/2$ punctures. While the corresponding surface states are already sufficiently numerous to give entropy proportional to area, other states with higher j values also contribute to the leading term in the expression of entropy.[17]

To summarize, quantum geometry naturally provides the micro-states responsible for the huge entropy associated with horizons. In this analysis, all black holes and cosmological horizons are treated in an unified fashion; there is no restriction, e.g., to near-extremal black holes. The sub-leading term has also been calculated and shown to be $-\frac{1}{2}\ln(a_{\rm hor}/\ell_{\rm Pl}^2)$. Finally, in this analysis, quantum Einstein's equations *are* used. In particular, had we not imposed the quantum Gauss and diffeomorphism constraints on surface states, the spurious gauge degrees of freedom would have given an infinite entropy. However, detailed considerations show that, because of the isolated horizon boundary conditions, the Hamiltonian constraint has to be imposed just in the bulk. Since, in the entropy calculation, one traces over bulk states, the final result is insensitive to details on how this (or any other bulk) equation is imposed. Thus, as in other approaches to black hole entropy, the calculation does not require a complete knowledge of quantum dynamics.

[17] These contributions are also conceptually important for certain physical considerations—e.g., to 'explain' why the black hole radiance does not have a purely line spectrum.

4. Summary and outlook

From the historical and conceptual perspectives of section 1, loop quantum gravity has had several successes. Thanks to the systematic development of quantum geometry, several of the roadblocks encountered by quantum geometrodynamics were removed. Functional analytic issues related to the presence of an infinite number of degrees of freedom are now faced squarely. Integrals on infinite-dimensional spaces are rigorously defined and the required operators have been systematically constructed. Thanks to this high level of mathematical precision, the canonical quantization program has leapt past the 'formal' stage of development. More importantly, although some key issues related to quantum dynamics still remain, it has been possible to use the parts of the program that are already well established to extract useful and highly non-trivial physical predictions. In particular, some of the long-standing issues about the nature of the big bang and properties of quantum black holes have been resolved. In this section, I will further clarify some conceptual issues, discuss current research and outline some directions for future.

• *Quantum geometry.* From conceptual considerations, an important issue is the *physical* significance of discreteness of eigenvalues of geometric operators. Recall first that, in the classical theory, differential geometry simply provides us with formulas to compute areas of surfaces and volumes of regions in a Riemannian manifold. To turn these quantities into physical observables of general relativity, one has to define the surfaces and regions *operationally*, e.g. using matter fields. Once this is done, one can simply use the formulas supplied by differential geometry to calculate the values of these observable. The situation is similar in quantum theory. For instance, the area of the isolated horizon is a Dirac observable in the classical theory and the application of the quantum geometry area formula to *this* surface leads to physical results. In $2+1$ dimensions, Freidel, Noui and Perez have recently introduced point particles coupled to gravity [48]. The physical distance between these particles is again a Dirac observable. When used in this context, the spectrum of the length operator has direct physical meaning. In all these situations, the operators and their eigenvalues correspond to the 'proper' lengths, areas and volumes of physical objects, measured in the rest frames. Finally, sometimes questions are raised about compatibility between discreteness of these eigenvalues and Lorentz invariance. As was recently emphasized by Rovelli, there is no tension whatsoever: it suffices to recall that discreteness of eigenvalues of the angular momentum operator $\hat{J}_z$ of non-relativistic quantum mechanics is perfectly compatible with the rotational invariance of that theory.

• *Quantum Einstein's equations.* The challenge of quantum dynamics in the full theory is to find solutions to the quantum constraint equations and endow these physical states with the structure of an appropriate Hilbert space. The general consensus in the loop quantum gravity community is that, while the situation is well understood for the Gauss and diffeomorphism constraints, it is far from being definitive for the Hamiltonian constraint. It *is* non-trivial that well-defined candidate operators representing the Hamiltonian constraint exist on the space of solutions to the Gauss and diffeomorphism constraints. However there are many ambiguities [22] and none of the candidate operators has been shown to lead to a 'sufficient number of' semi-classical states in $3+1$ dimensions. A second important open issue is to find restrictions on matter fields and their couplings to gravity for which this non-perturbative quantization can be carried out to a satisfactory conclusion. As mentioned in section 1.1, the renormalization group approach has provided interesting hints. Specifically, Luscher and Reuter have presented significant evidence for a non-trivial fixed point for pure gravity in four-dimensions. When matter sources are included, it continues to exist only when the matter content and couplings are suitably restricted. For scalar fields, in particular, Percacci and Perini have found that polynomial couplings (beyond the quadratic term in the action) are ruled out, an intriguing result that may 'explain' the triviality of such theories in Minkowski space-times [26]. Are there similar constraints coming from loop quantum gravity?

To address these core issues, at least four different avenues are being pursued. The first, and the closest to ideas discussed in section 2.3, is the 'Master constraint program' recently introduced by Thiemann. The idea here is to avoid using an infinite number of Hamiltonian constraints $\mathcal{S}(N) = \int N(x)\mathcal{S}(x)\, \mathrm{d}^3x$, each smeared by a so-called 'lapse function' N. Instead, one squares the integrand $\mathcal{S}(x)$ itself in an appropriate sense and then integrates it on the 3-manifold M. In simple examples, this procedure leads to physically viable quantum theories. In the gravitational case, however, the procedure does not seem to remove any of the ambiguities. Rather, its principal strength lies in its potential to complete the last step, (iii), in quantum dynamics: finding the physically appropriate scalar product on physical states. The general philosophy is similar to that advocated by John Klauder over the years in his approach to quantum gravity based on coherent states [49]. A second strategy to solve the quantum Hamiltonian constraint is due to Gambini, Pullin and their collaborators. It builds on their extensive work on the interplay between quantum gravity and knot theory [40]. More recent developments use the relatively new invariants of *intersecting* knots discovered by Vassiliev. This is a novel approach which furthermore has a potential of enhancing the relation between topological field theories and quantum gravity. As our knowledge of invariants of intersecting knots deepens, this approach is likely to provide increasingly significant insights. In particular, it has the potential of leading to a formulation of quantum gravity which does not refer even to a background manifold (see footnote 9). The third approach comes from spin-foam models [38, 39], mentioned in section 2.3, which provide a path integral approach to quantum gravity. Transition amplitudes from path integrals can be used to restrict the choice of the Hamiltonian constraint operator in the canonical theory. This is a promising direction and Freidel, Noui, Perez, Rovelli and others are already carrying out detailed analysis of restrictions, especially in $2+1$ dimensions. In the fourth approach, also due to Gambini and Pullin, one first constructs consistent discrete theories at the classical level and then quantizes them [27]. In this programme, there are no constraints: they are solved classically to find the values of the 'lapse and shift fields' which define 'time-evolution'. This strategy has already been applied successfully to gauge theories and certain cosmological models. An added bonus here is that one can revive a certain proposal made by Page and Wootters to address the difficult issues of interpretation of quantum mechanics, which become especially acute in quantum cosmology and, more generally, in the absence of a background physical geometry.

• *Quantum cosmology.* As we saw in section 3, loop quantum gravity has resolved some of the long-standing physical problems about the nature of the big bang. In quantum cosmology, there is ongoing work by Ashtekar, Bojowald, Willis and others on obtaining 'effective field equations' which incorporate quantum corrections. Quantum geometry effects significantly modify the effective field equations and the modifications in turn lead to new physics in the early universe. In particular, Bojowald and Date have shown that not only is the initial singularity resolved but the (Belinski–Khalatnikov–Lifschitz type) chaotic behaviour predicted by classical general relativity and supergravity also disappears! This is perhaps not surprising because the underlying geometry exhibits quantum discreteness: even in the classical theory, chaos disappears if the theory is truncated at any smallest, non-zero volume. There are also less drastic but interesting modifications of the inflationary scenario with potentially observable consequences.

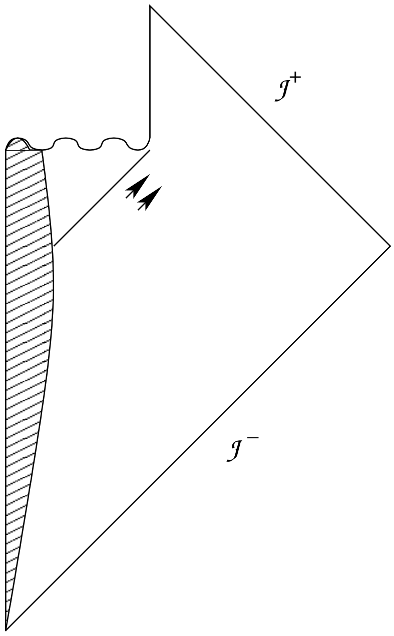

Figure 2. The standard space-time diagram depicting black hole evaporation.

This is a forefront area and it is encouraging that loop quantum cosmology is already yielding some phenomenological results.

• *Quantum black holes.* As in other approaches to black hole entropy, concrete progress could be made because the analysis does not require detailed knowledge of how quantum dynamics is implemented in *full* quantum theory. Also, restriction to large black holes implies that the Hawking radiation is negligible, whence the black hole surface can be modelled by an isolated horizon. To incorporate back-reaction, one would have to extend the present analysis to *dynamical horizons* [35]. It is now known that, in the classical theory, the first law can be extended also to these time-dependent situations and the leading term in the expression of the entropy is again given by $a_{\rm hor}/4\ell_{\rm Pl}^2$. Hawking radiation will cause the horizon of a large black hole to shrink *very* slowly, whence it is reasonable to expect that the Chern–Simons-type description of the quantum horizon geometry can be extended also to this case. The natural question then is: can one describe in detail the black hole evaporation process and shed light on the issue of information loss.

The standard space-time diagram of the evaporating black hole is shown in figure 2. It is based on two ingredients: (i) Hawking's original calculation of black hole radiance, in the framework of quantum field theory on a *fixed* background space-time; and (ii) heuristics of back-reaction effects which suggest that the radius of the event horizon must shrink to zero. It is generally argued that the semi-classical process depicted in this figure should be reliable until the very late stages of evaporation when the black hole has shrunk to Planck size and quantum gravity effects become important. Since it takes a very long time for a large black hole to shrink to this size, one then argues that the quantum gravity effects during the last stages of evaporation will not be sufficient to restore the correlations that have been lost due to thermal radiation over such a long period. Thus there is loss of information. Intuitively, the lost information is 'absorbed' by the final singularity which serves as a new boundary to space-time.

However, loop quantum gravity considerations suggest that this argument is incorrect in two respects. Firstly, the semi-classical picture breaks down not just at the end point of evaporation but in fact *all along what is depicted as the final singularity*. Recently, using ideas from quantum cosmology, the interior of the Schwarzschild horizon was analysed in the context of loop quantum gravity. Again, it was found that the singularity is resolved due to quantum geometry effects [50]. Thus, the space-time does *not* have a singularity as its final boundary. The second limitation of the semi-classical picture of figure 2 is its depiction of the event horizon. The notion of an event horizon is teleological and refers to the *global* structure of space-time. Resolution of the singularity introduces a domain in which there is no classical space-time, whence the notion ceases to be meaningful; it is simply 'transcended' in quantum theory. This leads to a new, possible paradigm for black hole evaporation in loop quantum gravity in which the dynamical horizons evaporate with emission of Hawking radiation, the initial pure state evolves to a final pure state and there is no information loss [51]. Furthermore, the semi-classical considerations are not simply dismissed; they turn out to be valid in certain space-time regions and under certain approximations. But for fundamental conceptual issues, they are simply inadequate. I should emphasize however that, although elements that go into the construction of this paradigm seem to be on firm footing, many details will have to be worked out before it can acquire the status of a model.

• *Semi-classical issues*. A frontier area of research is contact with low energy physics. Here, a number of fascinating challenges appear to be within reach. Fock states have been isolated in the polymer framework [22] and elements of quantum field theory on quantum geometry have been introduced [37]. These developments lead to concrete questions. For example, in quantum field theory in flat space-times, the Hamiltonian and other operators are regularized through normal ordering. For quantum field theory on quantum geometry, on the other hand, the Hamiltonians are expected to be manifestly finite [22, 37]. Can one then show that, in a suitable approximation, normal ordered operators in the Minkowski continuum arise naturally from these finite operators? Can one 'explain' why the so-called Hadamard states of quantum field theory in curved space-times are special? These issues also provide valuable hints for the construction of viable semi-classical states of quantum geometry. The final and much more difficult challenge is to 'explain' why perturbative quantum general relativity fails if the theory exists non-perturbatively. As mentioned in section 1, heuristically the failure can be traced back to the insistence that the continuum space-time geometry is a good approximation even below the Planck scale. But a more detailed answer is needed. Is it because, as recent developments in Euclidean quantum gravity indicate [26], the renormalization group has a non-trivial fixed point?

• *Unification*. Finally, there is the issue of unification. At a kinematical level, there is already a unification because the quantum configuration space of general relativity is the same as in gauge theories which govern the strong and electro-weak interactions. But the non-trivial issue is that of dynamics. I will conclude with a speculation. One possibility is to use the 'emergent phenomena' scenario where new degrees of freedom or particles, which were not present in the initial Lagrangian, emerge when one considers excitations of a non-trivial vacuum. For example, one can begin with solids and arrive at phonons; start with superfluids and find rotons; consider superconductors and discover cooper pairs. In loop quantum gravity, the micro-state representing Minkowski space-time will have a highly non-trivial Planck-scale structure. The basic entities will be one-dimensional and polymer-like. Even in absence of a detailed theory, one can tell that the fluctuations of these one-dimensional entities will correspond not only to gravitons but also to other particles, including a spin-1 particle, a scalar and an anti-symmetric tensor. These 'emergent states' are likely to play an important role in Minkowskian physics derived from loop quantum gravity. A detailed study of these excitations may well lead to interesting dynamics that includes not only gravity but also a select family of non-gravitational fields. It may also serve as a bridge between loop quantum gravity and string theory. For, string theory has two *a priori* elements: unexcited strings which carry no quantum numbers and a background space-time. Loop quantum gravity suggests that both could arise from the quantum state of geometry, peaked at Minkowski (or de Sitter) space. The polymer-like quantum threads which must be woven to create the classical ground state geometries could be interpreted as unexcited strings. Excitations of these strings, in turn, may provide interesting matter couplings for loop quantum gravity.

Acknowledgments

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Appendix A. Canonical approach and covariance

A common criticism of the canonical quantization program pioneered by Dirac and Bergmann is that in the very first step it requires a splitting of space-time into space and time, thereby doing grave injustice to space-time covariance that underlies general relativity. This is a valid concern and it is certainly true that the insistence on using the standard Hamiltonian methods makes the analysis of certain conceptual issues quite awkward. The loop quantum gravity program accepts this price because of two reasons. Firstly, the use of Hamiltonian methods makes it possible to have sufficient mathematical precision in the passage to the quantum theory to resolve the difficult field theoretic problems, ensuring that there are no hidden infinities.[18] The second and more important reason is that the mathematically coherent theory that results has led to novel predictions of direct physical interest.

Note however that the use of Hamiltonian methods by itself does not require a $3+1$ splitting. Following Lagrange, one can construct a 'covariant phase space' from *solutions* to Einstein's equations. This construction has turned out to be extremely convenient in a number of

[18] The only other background-independent approach to quantum general relativity which has faced some of these problems successfully is the causal dynamical triangulation programme [23, 24], which again requires a $3+1$ splitting. The spin-foam approach provides a path integral alternative to loop quantum gravity and does not require a $(3+1)$-decomposition of space-time. If it can be completed and shown to lead to interesting physical predictions, it would provide a more pleasing formulation of ideas underlying loop quantum gravity.

applications: quantum theory of linear fields in curved space-times [36], perturbation theory of stationary stars and black holes, and derivation of expressions of conserved quantities in general relativity, including the 'dynamical' ones such as the Bondi 4-momentum at null infinity [52]. Therefore, it is tempting to use the covariant Hamiltonian formulation as a starting point for quantization. In fact, Irving Segal proposed this strategy for interacting quantum field theories in Minkowski space-time already in the 1970s. However, it was soon shown that his specific strategy is not viable beyond linear systems and no one has been able to obtain a satisfactory substitute. A similar strategy was tried for general relativity as well, using techniques from geometric quantization. Recall that quantum states are square-integrable functions of only 'half' the number of phase space variables—usually the configuration variables. To single out their analogues, in geometric quantization one has to introduce an additional structure on the covariant phase space, called a 'polarization'. Quantization is easiest if this polarization is suitably 'compatible' with the Hamiltonian flow of the theory. Unfortunately, no such polarization has been found on the phase space of general relativity. More importantly, even if this technical problem were to be overcome, the resulting quantum theory would be rather uninteresting for the following reason. In order to have a globally well-defined Hamiltonian vector field, one would have to restrict oneself only to 'weak', four-dimensional gravitational fields. Quantization of such a covariant phase space, then, would not reveal answers to the most important challenges of quantum gravity which occur in the strong field regimes near singularities.

Let us therefore return to the standard canonical phase space and use it as the point of departure for quantization. In the classical regime, the Hamiltonian theory is, of course, completely equivalent to the space-time description. It does have space-time covariance, but it is not 'manifest'. Is this a deep limitation for quantization? Recall that a classical space-time is analogous to a full dynamical trajectory of a particle in non-relativistic quantum mechanics and particle trajectories have no physical role in the full quantum theory. Indeed, even in a semi-classical approximation, the trajectories are fuzzy and smeared. For the same reason, the notion of classical space-times and of space-time covariance is not likely to have a fundamental role in the full quantum theory. These notions have to be recovered only in an appropriate semi-classical regime.

This point is best illustrated in three-dimensional general relativity which shares all the conceptual problems with its four-dimensional analogue, but which is technically much simpler and can be solved exactly. There, one can begin with a $2+1$ splitting and carry out canonical quantization [28]. One can identify, in the canonical phase space, a complete set of functions which commute with all the constraints. These are therefore 'Dirac observables', associated with entire space-times. In quantum theory, they become self-adjoint operators, enabling one to interpret states and extract physical information from quantum calculations, e.g., of transition amplitudes. It turns out that quantum theory—states, inner-products, observables—can be expressed purely combinatorially. In this description, in the full quantum theory there is no space, no time, no covariance to speak of. These notions emerge only when we restrict ourselves to suitable semi-classical states. What is awkward in the canonical approach is the *classical limit* procedure. In the intermediate steps of this procedure, one uses the canonical phase space based on a $2+1$ splitting. But because this phase space description is equivalent to the covariant classical theory, in the final step one again has space-time covariance. To summarize, space-time covariance does not appear to have a fundamental role in the full quantum theory because there is neither space nor time in the full theory and it *is* recovered in the classical limit. The awkwardness arises only in the intermediate steps.

This overall situation has an analogue in ordinary quantum mechanics. Let us take the Hamiltonian framework of a non-relativistic system as the classical theory. Then, we have a 'covariance group'—that of the canonical transformations. To a classical physicist, this is geometrically natural and physically fundamental. Yet, in full quantum theory, it has no special role. The theory of canonical transformations is replaced by the Dirac's transformation theory which enables one to pass from one viable quantum representation (e.g., the q-representation) to another (e.g. the p-representation). The canonical group re-emerges only in the classical limit. However, in the standard q-representation, this recovery takes place in an awkward fashion. In the first step, one recovers just the configuration space. But one can quickly reconstruct the phase space as the cotangent bundle over this configuration space, introduce the symplectic structure and recover the full canonical group as the symmetry group of the classical theory. We routinely accept this procedure and the role of 'phase-space covariance' in quantization in spite of an awkwardness in an intermediate step of taking the classical limit. The canonical approach adopts a similar viewpoint towards space-time covariance.

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