Lie algebroid morphisms, Poisson Sigma Models, and off-shell closed gauge symmetries

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May 14, 2004

Abstract

Chern-Simons gauge theories in 3 dimensions and the Poisson Sigma Model (PSM) in 2 dimensions are examples of the same theory, if their field equations are interpreted as morphisms of Lie algebroids and their symmetries (on-shell) as homotopies of such morphisms. We point out that the (off-shell) gauge symmetries of the PSM in the literature are not globally well-defined for non-parallelizable Poisson manifolds and propose a covariant definition of the off-shell gauge symmetries as left action of some finite-dimensional Lie algebroid.

Our approach allows to avoid complications arising in the infinite dimensional super-geometry of the BV- and AKSZ-formalism. This preprint is a starting point in a series of papers meant to introduce Yang-Mills type gauge theories of Lie algebroids, which include and generalize the standard YM theory, gerbes, and the PSM.

1 Introduction

Yang-Mills (YM) gauge theories are an important ingredient in our present-day understanding of fundamental forces. On the mathematical side they are governed by a principal fiber bundle $\pi\colon P \to \Sigma$, where Σ is our spacetime manifold. Here any fiber $\pi^{-1}(x)$, $x \in \Sigma$, is a G-torsor, where G is the Lie structure group of P. In the simplest case when P is a trivial bundle, $P \cong \Sigma\times G$, π is the projection to the first factor, and the G-action on P is defined by right multiplication in the second factor. For the standard model of elementary particle physics $G = SU(3)\times SU(2)\times U(1)$, but also other, "larger" Lie groups come into mind in the context of a further unification of fundamental interactions.

Gauge bosons correspond to connections in P, matter fields are sections in associated fiber bundles (usually vector bundles), and local gauge symmetries are the vertical automorphisms $\mathrm{Aut}_v(P)$ of P. In the case of a trivial bundle the gauge bosons are just $\mathfrak{g}$-valued 1-forms $A = A^I b_I$ on Σ, where $\mathfrak{g}$ is the Lie algebra of the gauge or structure group G, b_I is some basis in $\mathfrak{g}$, and $I = 1, \ldots \dim\mathfrak{g}$. Sections of vector bundles then correspond to vector-valued functions (or spinors) on Σ and the infinite dimensional group of gauge transformations $\mathrm{Aut}_v(P)$ just becomes isomorphic to $\mathrm{Map}(\Sigma, G)$. Thus infinitesimally local gauge symmetries are parametrized by $\epsilon = \epsilon^I b_I \in \mathrm{Map}(\Sigma, \mathfrak{g})$ and one has $\delta_\epsilon A^I = \mathrm{d}\epsilon^I + C^I_{JK} A^J \epsilon^K$, where C^I_{JK} denote the structure constants of the Lie algebra $\mathfrak{g}$.

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All fundamental interactions fit into this framework except for gravity. Even though it is possible to cast general relativity in the language of a gauge theory of connections [2], the local gauge symmetries contain the diffeomorphisms of Σ. The Lie algebra of $\mathrm{Diff}(\Sigma)$ consists of vector fields on Σ. This has to be contrasted to elements of $\mathrm{aut}_v(P) \equiv \mathrm{Lie}(\mathrm{Aut}_v(P))$, which always have a trivial projection to $T\Sigma$. On the level of a Hamiltonian formulation of the theory this usually leads to structure functions in the algebra of constraints, whereas for YM gauge theories the algebra of constraints is governed by the structure constants C^I_{JK} (cf., e.g., [10] for details). Structure functions of first class constraints are a typical feature of a formulation of a theory with an open algebra of local symmetries, where the commutator of infinitesimal local or gauge symmetries closes only on-shell, i.e. upon use of the field equations. In YM theories, on the other hand, gauge symmetries always form a closed algebra. In a way, within YM theories many considerations of local symmetries can be reduced to a finite dimensional group, the structure group G, whereas for gravitational theories all of the infinite-dimensional group of local symmetries seems unavoidable. This may be regarded as maybe one of the main obstacles in a successful quantization of gravity along the lines of YM-gauge theories.

It may be an important step to broaden the framework of YM-gauge theories in such a way that also some gauge theories with an open algebra of gauge transformations fit into it, while still many considerations can be reduced to a purely finite-dimensional setting. In [24] (cf. also [25]) a particular program in this direction has been proposed. Essentially, the structural Lie group G of a YM theory is replaced by (or generalized to) a so-called Lie groupoid; correspondingly, the Lie algebra $\mathfrak{g}$ generalizes to a so-called Lie algebroid E.[4] The present paper is the first one in a series of papers devoted to this subject and aims at providing part of the mathematical basis for the others.

From some other perspective our goal is to provide a better understanding and definition of "non-linear gauge theories", as they have been suggested already quite some time ago by van Nieuwenhuizen and collaborators, cf., e.g., [23]. Heuristically, in such a theory one wants to replace the structure constants C^I_{JK} of a standard YM-theory by some field-dependent quantities, which then generically will lead to a theory with an open algebra of local symmetries, due to the transformation of Cs. In our approach, C^I_{JK} will be the structure functions of a Lie algebroid $E \to M$. M then serves as a target space for a Sigma Model so that the map $\Sigma \to M$ locally corresponds to a set of scalar fields $X^i(x)$ and the coefficients C^I_{JK} depend on these fields in general; from a physical perspective these fields can be some kind of Higgs fields or, as shown in [25], they can turn out to be just some auxiliary fields that do not carry any propagating degrees of freedom, but serve as moduli parameteres. In addition to them locally one still has a set of 1-form gauge fields A^I.

In two spacetime dimensions, $\dim \Sigma = 2$, a prototype of such a non-linear gauge theory is provided by the Poisson Sigma Model (PSM) [22, 12]. It is worth mentioning here that in this particular spacetime dimension, essentially all possible YM gauge theories and 2d gravity theories find a unifying formulation as particular PSMs (cf., e.g., [21, 13]). In all of our work we want to use the PSM as a kind of main guiding example for developing a more general theory. In particular, in the present first paper we show how the field equations and the gauge symmetries of this model are related to Lie algebroids. Focusing on the case corresponding to a trivial principal bundle, the field content of the PSM, locally described by a set of couples $(X^i, A_i)_{i=1}^n$ of scalar and

[4] Among others a Lie algebroid is a vector bundle $E \to M$ carrying a Lie bracket for its sections; for M a point one obtains a Lie algebra, while $E = TM$ with the Lie-Jacobi bracket for vector fields on M is another prominent example. We will recall the notion of a Lie algebroid in the subsequent section; for further background material on Lie algebroids and Lie groupoids we refer to the monograph [8] and references therein.

1-form fields, respectively, corresponds to vector bundle morphisms $\phi\colon T\Sigma \to T^*M$. Since M is a Poisson manifold, both the source and target vector bundle carry Lie algebroid structures. The content of the field equations will then be shown to be equivalent to requiring ϕ to respect the Lie algebroid structures, i.e. to be Lie algebroid morphisms.

Whereas for Lie algebras it is very straightforward to define the notion of a morphism, for Lie algebroids the situation is somewhat more intricate. After setting the notation and collecting some background material in section 2, in section 3 several formulations of such a morphism will be mentioned and related to one another. Essentially one needs to dualize the map ϕ, requiring it to be an appropriate chain map. However, in our final formulation, using the graph of ϕ, this can also be circumvented.

An important observation in this context is that also YM-type gauge theories such as the Chern-Simons theory fit into that framework: Flatness of a connection $A = A^I b_I$ in a trivial principal fiber bundle is tantamount to the condition that the corresponding map from $T\Sigma \to \mathfrak{g}$, $\xi \mapsto A^I(\xi) b_I$, is a Lie algebroid morphism. Correspondingly, in our investigations we will replace T^*M of the PSM by an arbitrary Lie algebroid E_2. In fact, for means of generality we will also generalize $T\Sigma$ to an arbitrary Lie algebroid E_1, although the main example of physical interest may still be provided by the tangent bundle of spacetime.

For the formulation of $\phi\colon E_1 \to E_2$ in terms of the graph map one uses the fact that the set $E_1 \times E_2$ can be given the structure of a Lie algebroid $E = E_1 \boxplus E_2$ itself (details of this will be provided in section 2 below). It will then be shown that ϕ is a morphism of Lie algebroids, iff $\phi^{\text{gra}}\colon E_1 \to E$ is a morphism. By construction, the base map of ϕ^{gra} is an embedding, permitting to work with ϕ^{gra}-related sections instead of with the dual map.

In section 4 finally we turn to the issue of local gauge symmetries. We first point out that the local infinitesimal symmetries usually used in the PSM are in general not well-defined globally. They make sense only if the target Poisson manifold M can be covered by a single chart, or if it carries some flat connection, implicit but not transparent in the usual formulas (Eqs. (15) and (16) below). This is somewhat remarkable in view of the already relatively large, and in part also mathematical literature on the PSM; partially this may be related to the fact that in many physical examples of the PSM such as 2d YM- and/or 2d gravity models a flat target $M \cong \mathbb{R}^n$ is used (cf., e.g., [13, 26, 9]), which moreover also underlies the Kontsevich formula [14], resulting from the perturbative quantization of the PSM [4].

In section 4 we present one possible way of curing this deficiency, simultaneously generalizing the local symmetries also to the context of arbitrary Lie algebroids. This is done in such a way that for the particular case $E_2 := \mathfrak{g}$ and $E_1 := T\Sigma$ one indeed reobtains the usual YM gauge transformations. Moreover, also in the general case, we will be able to trace back everything to purely finite dimensional terms. Employing the picture with the graph, $\phi^{\text{gra}}\colon E_1 \to E$, the infinitesimal gauge symmetries (and also what corresponds to infinitesimal diffeomorphisms of Σ) result from particular, structure preserving infinitesimal automorphisms of E, acting from the left on ϕ^{gra} (or from the right in the dualized picture $\Phi^{\text{gra}}\colon \Gamma(\Lambda^\cdot E^*) \to \Gamma(\Lambda^\cdot E_1^*)$), and generated by particular sections of E via a Lie algebroid generalization of the Lie derivative. As a byproduct we find that the gauge symmetries formulated in this way close even off-shell. But also if one needs to calculate e.g. the commutator of the original symmetries of the PSM for $M \cong \mathbb{R}^n$ the present approach provides a significant technical advance.

Although this approach may be related also to an infinite-dimensional Lie algebroid $\mathcal{E}$ of infinitesimal gauge transformations [19], the base manifold $\mathcal{M}$ of which are maps $\Sigma \to M$ (or, more generally, maps from the base of E_1 to the base of E_2), one can consistently—and with conceptual profit—truncate $\Gamma(\mathcal{E})$ to the space of sections in the finite dimensional algebroid E. For the PSM a likewise statement applies to its AKSZ-formulation [1, 6], which yields in a most transparent way the BV-form of the PSM.

As an alternative, one may also employ a connection in the target Lie algebroid E_2 for providing another possible global definition of the local gauge symmetries. While some elementary formulas in this direction will be displayed at the end of section 4, a more abstract analysis along the lines of the present paper can be found in another, accompanying paper [17].

Both definitions of gauge symmetries can be made to agree for the PSM on $M \cong \mathbb{R}^n$, as well as certainly in the YM-case. Also they always agree globally upon use of the field equations, i.e. on-shell. Already the standard gauge symmetries of the PSM have a good global on-shell meaning, as an infinitesimal homotopy of Lie algebroids. Correspondingly, a homotopy of Lie algebroids defines an integrated version of the on-shell gauge symmetries (section 4). Globally and off-shell, however, the gauge symmetries defined via an E-Lie derivative and those defined by means of a connection Γ on E_2 are different; in particular also the latter do not close off-shell, their commutator containing contributions of Γ.

The formulation in the present paper as well as in [17] is put in such a form that a generalization to non-trivial fibrations is rather straightforward. Essentially E, as a manifold, is then not just a direct product $E_1 \times E_2$, but a particular fiber bundle over the base of E_1. In order to not overload the presentation, we found it useful to present this generalization in another separate work [18]. All three papers together then are meant to provide, among others, a basic mathematical framework for the definition of Lie algebroid Yang-Mills type gauge theories.

Some examples for action functionals of this kind of gauge theories are presented in [25]. They generalize e.g. usual YM gauge theories in arbitrary dimensions of Σ. It turns out that one can use such theories to effectively glue together YM theories with different structure groups, which can even differ in their dimension, this theory being then governed by one Lagrangian; due to the gauge symmetries, the map from $\Sigma \to M$, corresponding to the scalar fields $X^i(x)$, carries only global degrees of freedom, and the representative maps have the role of moduli for changing from one YM theory to another one. But also topological action functionals can be constructed, generalizing e.g. the Chern-Simons gauge theory in three and the PSM in two spacetime dimensions. Also one can extend the relation of the PSM to 2d gravity theories, to the definition of topological gravity theories in arbitrary spacetime dimensions; for this cf. [27]. Still further work is necessary to see how far one can push this approach and what kind of different theories can be constructed. The present paper is meant to set part of the basis for it. One may expect, however, that the resulting theories have a wide range of implications, in physics as well as in mathematics.

2 Preliminaries

In this section we mainly set the notation and recall some background material needed later on. We start with the Poisson Sigma Model (PSM) [22, 12], presenting a slightly more abstract definition of its action functional S. S is a functional of the vector bundle morphisms $\phi\colon T\Sigma \to T^*M$, where Σ is a two-dimensional manifold, called the world sheet, and M some Poisson manifold. We denote the Poisson bivector by $\mathcal{P} \in \Gamma(\Lambda^2 TM)$, $\{f, g\} = \langle \mathcal{P}, \mathrm{d}f \wedge \mathrm{d}g \rangle$; in local coordinates X^i on M, $\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(X)\partial_i \wedge \partial_j \Rightarrow \{X^i, X^j\} = \mathcal{P}^{ij}$, and

$$[\mathcal{P}, \mathcal{P}]_{\text{Schouten}} \equiv \mathcal{P}^{ij}{}_{,s} \mathcal{P}^{ks}\, \partial_i \wedge \partial_j \wedge \partial_k = 0, \tag{1}$$

as a manifestation of the Jacobi identity for the Poisson bracket.

Any morphism $\phi\colon E_1 \to E_2$ between two vector bundles $\pi_i\colon E_i \to M_i$, $i = 1, 2$, may be expressed in different equivalent ways. One of them is by specifying the induced base map $\phi_0\colon M_1 \to M_2$ and, in addition, by providing a section A of the bundle $E_1^* \otimes \phi_0^* E_2$. If b_I, $I = 1, \ldots \mathrm{rank}(E_2)$, denotes a local basis of E_2 and $\mathfrak{b}_I$ the corresponding induced basis in the pullback bundle $\phi_0^* E_2$, and if $E_1 = T\Sigma$, then $A = A^I \otimes \mathfrak{b}_I$, where $A^I \in \Omega^1(\Sigma) \equiv \Gamma(T^*\Sigma)$ (possibly also defined locally on Σ only, however).

Later on we will also need the graph ϕ^{gra} of the above map ϕ as well as its trivial extension ${}^E\phi$,

$$\phi^{\mathrm{gra}}\colon E_1 \to E := E_1 \boxplus E_2\,, \quad e_1 \mapsto e_1 \boxplus \phi(e_1)\,, \tag{2}$$

$${}^E\phi\colon E \to E\,, \quad {}^E\phi = \phi^{\mathrm{gra}} \circ p_1\,. \tag{3}$$

Here $\pi\colon E \to M$, the exterior sum of E_1 and E_2, is a vector bundle over $M := M_1 \times M_2$ defined as $\mathrm{pr}_1^* E_1 \oplus \mathrm{pr}_2^* E_2$, where $\mathrm{pr}_i : M \to M_i$ is the projection to the i–th factor of the Cartesian product, and p_1 is the canonical projection bundle morphism $E \to E_1$ covering $\mathrm{pr}_1 : M \to M_1$:

$$\begin{array}{ccc} E & \xrightarrow{p_1} & E_1 \\ \pi\downarrow & & \downarrow\pi_1 \\ M & \xrightarrow{\mathrm{pr}_1} & M_1 \end{array} \tag{4}$$

Alternatively, the vector bundle morphism ϕ induces a map $\Phi\colon \Gamma(\overset{p}{\otimes} E_2^*) \to \Gamma(\overset{p}{\otimes} E_1^*)$. For $p = 0$ it is given by the pullback of functions, $C^\infty(M_2) \ni f \mapsto \phi_0^* f \in C^\infty(M_1)$, while for $p = 1$, $\Phi(u_2)$ for $u_2 \in \Gamma(E_2^*)$ is defined by $\langle \Phi(u_2), s_1\rangle|_x = \langle u_2|_{\phi_0(x)}, \phi(s_1|_x)\rangle$ $\forall x \in M_1$ and $\forall s_1 \in \Gamma(E_1)$. In the particular case of $E_1 = T\Sigma$ mentioned previously, with b^I denoting the local basis in E_2^* dual to b_I, one has $\Phi(b^I) = A^I$. The extension to arbitrary p is canonical now. Mostly we will use only the restriction of the above map Φ to the antisymmetric subspace $\Gamma(\Lambda^p E_2^*) =: \Omega^p_{E_2}(M_2)$ (the space of E_2-forms) only, which we denote by the same letter.

The above map Φ can be extended also to all E_2-tensors, and we will denote this extension by

$$\Phi^!\colon \Gamma(\overset{p}{\otimes} E_2^* \overset{q}{\otimes} E_2) \to \Gamma(\overset{p}{\otimes} E_1^* \overset{q}{\otimes} (\phi_0)^* E_2)\,, \tag{5}$$

where on the first factor $\Phi^!$ acts as Φ above and on E_2 it is defined as $\Gamma(E_2) \ni s_2 \mapsto s_2 \circ \phi_0$, viewed as a section of the pullback bundle $(\phi_0)^* E_2$. With this map the above section $A \in E_1^* \otimes \phi_0^* E_2$ is nothing but the image of the canonical identity section $\delta \in E_2^* \otimes E_2$, $A = \Phi^!(\delta)$ (in local terms $\delta = b^I \otimes b_I$ and $\Phi^!(b_I) \equiv \mathfrak{b}_I$).

In the particular case $E_2 = T^*M$ and $E_1 = T\Sigma$ (and only in this case!) the map Φ can be extended to all E_2-tensors also in another way, which we denote by

$$\Phi^*\colon \Gamma(\overset{p}{\otimes}\, TM \overset{q}{\otimes}\, T^*M) \to \Gamma(\overset{p+q}{\otimes}\, T^*\Sigma)\,. \tag{6}$$

Here 1-forms on M, corresponding to $p = 0$, $q = 1$, are mapped by the pullback ϕ_0^* to 1-forms on Σ—and, as before, this map is extended canonically to all possible choices for p and q.

Such as ϕ permits the dual formulation in terms of $\Phi\colon \Omega^p_{E_2}(M_2) \to \Omega^p_{E_1}(M_1)$, also the maps (2) and (3) induce reverse maps:

$$\Phi^{\mathrm{gra}}\colon \Omega^p_E(M) \to \Omega^p_{E_1}(M_1)\,, \quad {}^E\Phi\colon \Omega^p_E(M) \to \Omega^p_E(M)\,. \tag{7}$$

Note that due to the isomorphism $\bigoplus_{p+q=k} \Omega^p_{E_1}(M_1) \otimes \Omega^q_{E_2}(M_2) \cong \Omega^k_E(M)$, where multiplication is defined according to $(\omega_1 \otimes \omega_2) \wedge (\omega_1' \otimes \omega_2') = (-1)^{qp'} (\omega_1 \wedge \omega_1') \otimes (\omega_2 \wedge \omega_2')$, there is a natural bigrading for E-forms; if we want to stress the bigrading, we write $\Omega_E^{p,q}(M)$, while k in $\Omega_E^k(M)$ denotes the total degree, which is the sum of the two individual degrees on E_1 and E_2. The above maps (7) are related to Φ in the following way:

$$\Phi^{\mathrm{gra}} \colon \Omega_E^{p,q}(M) \cong \Omega_{E_1}^p(M_1) \otimes \Omega_{E_2}^q(M_2) \xrightarrow{\mathrm{id}\otimes\Phi} \Omega_{E_1}^p(M_1) \otimes \Omega_{E_1}^q(M_1) \xrightarrow{\wedge} \Omega_{E_1}^{p+q}(M_1)\,, \tag{8}$$

$$^E\Phi \colon \Omega_E^{p,q}(M) \xrightarrow{\Phi^{\mathrm{gra}}} \Omega_{E_1}^{p+q}(M_1) \xrightarrow{P_1} \Omega_{E_1}^{p+q}(M_1) \otimes \Omega_{E_2}^0(M_2) \cong \Omega_E^{p+q,0}(M)\,, \tag{9}$$

where $P_1 \colon \Omega_{E_1}^p(M_1) \ni \omega_1 \mapsto \omega_1 \otimes 1 \in \Omega_{E_1}^p(M_1) \otimes \Omega_{E_2}^0(M_2) \subset \Omega_E^p(M)$ is the map induced by the bundle morphism $p_1 \colon E \to E_1$. So, $^E\Phi$ preserves only the total degree, but not the bigrading.

By definition, $p_1 \circ \phi^{\mathrm{gra}} = \mathrm{id}_{E_1}$, which translates into the dual relation $\Phi^{\mathrm{gra}} \circ P_1 = \mathrm{id}$. For $^E\Phi = P_1 \circ \Phi^{\mathrm{gra}}$ we then obtain $^E\Phi \circ P_1 = P_1$, implying that $^E\Phi$ is a projector to the image of P_1 (i.e. on $\Omega_E^{\cdot}(M)$ one has ${^E\Phi}^2 = {^E\Phi}$ and $\mathrm{im}{^E\Phi} = \mathrm{im}P_1$).

Using the map Φ^*, we may now give a concise global definition of the action functional of the PSM:[5]

$$S[\phi] = \int_\Sigma \mathrm{Alt}\ \Phi^*(\delta + \mathcal{P})\,, \tag{10}$$

where δ is the canonical identity section in $TM \otimes T^*M$ and Alt denotes the antisymmetrization. In local coordinates X^i on M and with the induced local basis $\partial_i \sim b_I$ and $\mathrm{d}X^i \sim b^I$ in TM and $T^*M = E_2$, respectively, one has $\delta = \partial_i \otimes \mathrm{d}X^i$ and $\mathrm{Alt}\ \Phi(\delta) = \mathrm{Alt}\,(A_i \otimes \mathrm{d}X^i) = A_i \wedge \mathrm{d}X^i$ (where $A_i \sim A^I$, as introduced above, and $X^i = X^i(x)$ denotes the scalar field corresponding to the map $\phi_0 \colon \Sigma \to M$, just expressed in local coordinates). $\mathcal{P}$, on the other hand, is the Poisson tensor on M, and for the second term simply $\mathrm{Alt}\ \Phi^*(\mathcal{P}) = \Phi(\mathcal{P}) = \frac{1}{2}\mathcal{P}^{ij}A_i \wedge A_j$. Thus in the more familiar and for practical purposes most useful local description, S takes the form

$$S = S[\phi_0, A] = \int\limits_\Sigma A_i \wedge \mathrm{d}X^i + \frac{1}{2}\mathcal{P}^{ij}A_i \wedge A_j\,. \tag{11}$$

For completeness we also mention another possible covariant presentation of the action functional: For this purpose we first rewrite $\mathcal{P}$ as $\frac{1}{2}\langle\mathcal{P}, \delta \wedge \delta\rangle$, then the second term in (10), which may be also written as $\Phi^!(\mathcal{P})$, becomes $\frac{1}{2}\langle\Phi^!(\mathcal{P}), A\wedge A\rangle$ with $A\wedge A \in \Omega^2(\Sigma, \Lambda^2\phi_0^*T^*M)$. Moreover, $(\phi_0)_* \colon T\Sigma \to TM$ is a vector bundle morphism covering ϕ_0. Thus, according to the above discussion, it induces a section of $T^*\Sigma \otimes \phi_0^*TM$, which we denote suggestively by $\mathrm{d}\phi_0$. It clearly can be contracted with $A \in \Gamma(T^*\Sigma \otimes \phi_0^*T^*M)$. In this way we obtain

$$S = \int_\Sigma \langle A \overset{\wedge}{,} \mathrm{d}\phi_0\rangle + \frac{1}{2}\langle\mathcal{P} \circ \phi_0, A \wedge A\rangle\,. \tag{12}$$

Concerning the field equations and the symmetries of the PSM action functional, we let it suffice here to just recall the local basis expressions—anyway, much to follow will be devoted to a more abstract and covariant formulation of precisely these two issues.

The field equations of the action functional (11) are

$$\frac{\delta S}{\delta A_i} \equiv \mathrm{d}X^i + \mathcal{P}^{ij}(X)A_j = 0 \tag{13}$$

$$\frac{\delta S}{\delta X^i} \equiv \mathrm{d}A_i + \frac{1}{2}\mathcal{P}^{kl}{}_{,i}(X)\,A_k \wedge A_l = 0 \tag{14}$$

[5] In section 4 this formula is rewritten in two further quite similar fashions, cf. Eq. (64) below, which will be explained only there to not overload the presentation here.

The gauge symmetries are generated by

$$\delta_\epsilon X^i = \mathcal{P}^{ji}\epsilon_j \tag{15}$$

$$\delta_\epsilon A_i = \mathrm{d}\epsilon_i + \mathcal{P}^{jk}{}_{,i} A_j \epsilon_k \tag{16}$$

where $\epsilon \equiv \epsilon_i \mathrm{d}X^i \in \Gamma(\phi_0^* T^* M)$ may be chosen arbitrarily. The obvious Diff(Σ) invariance of the action functional S, e.g., can be generated by means of (15) and (16) with the choice $\epsilon_i = \langle v, A_i \rangle$ with $v \in \Gamma(T\Sigma)$ being the infinitesimal generator of a diffeomorphism in the above group. For further remarks in the context of symmetries, somewhat complementary to what will follow in the present paper, we also refer to the introductory section 2.1 of [3].

We now recall the definition of a Lie algebroid. First of all, $E = T^*M$, M Poisson, is a particular example, and many things become more transparent when they are formulated in this somewhat more general context and language. Moreover, although the action functional S, as introduced above, is quite particular to morphisms from only $T\Sigma \to T^*M$, where Σ is two-dimensional and M Poisson, the field equations and symmetries generalize easily to arbitrary Lie algebroid morphisms $\phi\colon E_1 \to E_2$. Moreover, we believe that the corresponding considerations are of interest in this more general context as well. Finally, we remark that it is even possible to construct action functionals for this more general setting, too, but this is not subject of the present paper.

A Lie algebroid over a base manifold M is a vector bundle E with a Lie algebra structure $[\cdot,\cdot]$ on the space of sections $\Gamma(E)$ together with a bundle map $\rho\colon E \to TM$, called the *anchor*, which by definition governs the following Leibniz rule: for any $s, s' \in \Gamma(E)$, $f \in C^\infty(M)$,

$$[s, fs'] = f[s,s'] + \rho_s(f)s' \,, \tag{17}$$

where $\rho_\cdot$ denotes the induced map of sections from $\Gamma(E)$ to $\Gamma(TM)$. It is not difficult to see that $\rho_\cdot$ provides a representation of $(\Gamma(E), [\,,])$ in the Lie algebra of vector fields, i.e. that $[\rho_s, \rho_{s'}] = \rho_{[s,s']}$. We briefly recall the list of standard examples of Lie algebroids: Lie algebras, M being a point, or bundles of Lie algebras, for $\rho \equiv 0$. The tangent bundle, $E = TM$, $\rho = \mathrm{i}d$. And, finally, $E = T^*M$, M Poisson, where $\rho = \mathcal{P}^\sharp$, $\rho(\alpha_i \mathrm{d}X^i) = \alpha_i \mathcal{P}^{ij}\partial_j$, and the bracket $[\mathrm{d}f, \mathrm{d}g] := \mathrm{d}\{f,g\}$ between exact 1-forms is extended to all 1-forms by means of (17).

There is also an equivalent definition of a Lie algebroid $(E, M, \rho, [\cdot,\cdot])$ as the differential graded algebra $(\Gamma(\Lambda^\cdot E^*), \wedge, {}^E\mathrm{d})$, where ${}^E\mathrm{d}$ is defined by ($\omega \in \Gamma(\Lambda^p E^*)$, $s_i \in \Gamma(E)$)

$${}^E\mathrm{d}\omega(s_1, ..., s_{p+1}) = \sum_i (-1)^{i+1} \rho(s_i)\omega(..., \hat{s}_i, ...) + \sum_{i<j} (-1)^{i+j} \omega([s_i, s_j], ..., \hat{s}_i, ..., \hat{s}_j, ...), \tag{18}$$

which is a generalization of the Cartan formula for the exterior derivative in the standard Lie algebroid TM.

An anchor map of a Lie algebroid E provides a representation of $\Gamma(E)$ in $C^\infty(M)$. One can lift this action to a representation in $\Gamma(\Lambda^\cdot E^*)$: Taking any section s of E, we associate a Lie derivative (*E-Lie derivative*) ${}^E L_s$ along s by generalization of Cartan's magic formula

$${}^E L_s = [{}^E\mathrm{d}, \imath_s] = {}^E\mathrm{d}\imath_s + \imath_s {}^E\mathrm{d}, \tag{19}$$

where $\imath_s$ denotes contraction with s and ${}^E\mathrm{d}$ is defined in (18) above. It is now straightforward to prove that indeed one has a representation, i.e. that $[{}^E L_s, {}^E L_{s'}] = {}^E L_{[s,s']}$ holds true.[6] (In general, for

[6] This is done most easily by noting that the operator $[\imath_s, [\imath_{s'}, {}^E\mathrm{d}]]$ on $\Omega_E^\cdot(M)$ is $C^\infty(M)$-linear and agrees with $\imath_{[s,s']}$, cf. also [16, 15].

operators $\mathcal{V}_1, \mathcal{V}_2$ of some fixed degree in a graded vector space, we define the graded commutator bracket according to $[\mathcal{V}_1, \mathcal{V}_2] := \mathcal{V}_1 \circ \mathcal{V}_2 - (-1)^{\deg \mathcal{V}_1 \deg \mathcal{V}_2} \mathcal{V}_2 \circ \mathcal{V}_1$. In the above, ${}^E\mathrm{d}$, i_s, and ${}^E L_s$ are of degree $+1$, -1, and 0, respectively.)

For later use we will need some of the above formulas in more explicit form: Let $(U, \{X^i\})$ be a local coordinate chart, b_I be a frame of E_U over U, and b^I its dual frame in E_U^*.Then with $\rho(b_I) \equiv \rho_I =: \rho_I^i \partial_i$ and $[b_I, b_J] =: C_{IJ}^K b_K$ one finds

$$ {}^E\mathrm{d}X^i = b^I \rho_I^i(X), \quad {}^E\mathrm{d}b^I = -\frac{1}{2} C^I_{JK}(X) b^J \wedge b^K, \tag{20} $$

as well as

$$ {}^E L_s X^i = s^I \rho_I^i, \quad {}^E L_s b^I = \rho_J(s^I) b^J + C^I_{JK}(X) b^J s^K. \tag{21} $$

In the Poisson case, $b_I \sim \mathrm{d}X^i$, $b^I \sim \partial_i$, $\rho_I^j \sim \mathcal{P}^{ij}$, and $C^I_{JK} \sim \mathcal{P}^{jk}{}_{,i}$.

Some words about conventions may be in place: If there are two Lie algebroids involved, $E_i \to M_i$, $i = 1, 2$, such as already above in the context of a bundle map $\phi \colon E_1 \to E_2$, we will mostly mark objects of the respective algebroid with the corresponding index. E.g. $s_2, s_2' \in \Gamma(E_2)$ for sections of the target bundle. Similarly, for the respective Lie algebroid exterior derivatives, we will use the abbreviations ${}^{E_i}\mathrm{d} =: \mathrm{d}_i$. However, to simplify notation we will make exceptions from the above rule for what concerns e.g. local coordinates and frames: x^μ, b_α denote coordinates and frame in the source M_1 and E_1, respectively, while X^i and b_I do so for the target. Correspondingly, then C^K_{IJ} ($C^\gamma_{\alpha\beta}$) denote structure functions in E_2 (E_1), and likewise for connection coefficients etc. Depending on the context, furthermore, X^i may just denote coordinates on M_2 or, as e.g. already in (13) above, the collection of functions on (parts of) M_1 corresponding to the base map $\phi_0 \colon M_1 \to M_2$; otherwise we would have to write $\Phi(X^i) \equiv (\phi_0)^* X^i$, in the previously introduced notation, where, moreover, Φ and $(\phi_0)^*$ are the canonical restrictions of the respective maps to functions defined on the neighborhood $U \subset M_2$ on which the coordinates X^i are defined. Likewise $\mathrm{d}X^i$ may denote a basis of local 1-forms in T^*M or its pullback, which more carefully we would have to write as $(\phi_0)^*\mathrm{d}X^i \equiv \Phi^*\mathrm{d}X^i$. On the other hand, for the induced basis in $(\phi_0)^*T^*M$ for clarity we use $\mathrm{đ}X^i := \Phi^!(\mathrm{d}X^i) \equiv \mathrm{d}X^i \circ \phi_0$. In generalization of the 1-form fields A_i of the PSM, we have the (locally defined) set of E_1-1-forms $A^I \equiv A^I_\alpha \otimes b^\alpha = \Phi(b^I)$; they combine into (the globally defined) $A = \Phi^!(b^I \otimes b_I) = A^I \otimes \mathfrak{b}_I \in \Gamma(E_1^* \otimes \phi_0^* E_2)$, which in the PSM case becomes $A = A_i \otimes \mathrm{đ}X^i$.

Finally we mention that if $E_i \to M_i$, $i = 1, 2$, are two Lie algebroids, then also $E \to M$, where $E \equiv E_1 \boxplus E_2$ and $M \equiv M_1 \times M_2$ as introduced above, can be endowed canonically with a Lie algebroid structure (generalizing the direct sum of two Lie algebras). For this purpose we use the isomorphism $\Omega^\cdot_E(M) \cong \Omega^\cdot_{E_1}(M_1) \otimes \Omega^\cdot_{E_2}(M_2)$, and define ${}^E\mathrm{d} := {}^E\mathrm{d}_1 + {}^E\mathrm{d}_2$, where ${}^E\mathrm{d}_1 = \mathrm{d}_1 \otimes \mathrm{id}$ and, similarly, ${}^E\mathrm{d}_2 = (-1)^{\varepsilon_1} \mathrm{id} \otimes \mathrm{d}_2$, with ε_1 being the grading operator acting as multiplication by p on $\Omega^p_{E_1}(M_1)$. By construction, $({}^E\mathrm{d}_i)^2 = 0$, and, due to the grading operator ε_1, also ${}^E\mathrm{d}_1$ and ${}^E\mathrm{d}_2$ anticommute, so that indeed $({}^E\mathrm{d})^2 = 0$.

3 Morphisms and field equations

Assume that $E_1 \to M_1$ and $E_2 \to M_2$ are Lie algebroids with the anchors ρ_1 and ρ_2, respectively and that $\phi \colon E_1 \to E_2$ is a vector bundle morphism. For the particular case $E_1 = T\Sigma$ and $E_2 = T^*M$, M Poisson, ϕ reproduces the content of the fields in the PSM; it is worthwhile, however, to discuss the more general situation $\phi\colon E_1 \to E_2$ (cf. also [27, 25] for further motivation for this perspective). In the beginning of the present section we address the question, under what conditions we may call ϕ a morphism of Lie algebroids, as well as how, in the particular case of the PSM, this is related to its field equations. On our way we will prove also some helpful reformulations of the notion of Lie algebroid morphisms in terms of the maps introduced in the previous section.

For $M_1 = M_2 = \{pt\}$ the above Lie algebroids simply become Lie algebras. By definition, $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of Lie algebras iff $[\phi(s_1), \phi(s_1')] - \phi([s_1, s_1']) = 0 \;\; \forall s_1, s_1' \in \mathfrak{g}_1$. But, in general a vector bundle morphism $\phi\colon E_1 \to E_2$ does not induce a map of sections of those bundles (except if, say, the induced base map $\phi_0\colon M_1 \to M_2$ is a diffeomorphism). Instead, as with vector fields and the tangent map φ_* of a map $\varphi\colon M_1 \to M_2$ (corresponding to the example of standard Lie algebroids $E_i = TM_i$ with $\phi = \varphi_*$), one may speak of relation of sections only: Sections $s_i \in \Gamma(E_i)$ are called *ϕ-related*, $s_1 \overset{\phi}{\sim} s_2$, iff $\phi \circ s_1 = s_2 \circ \phi_0$. Following [11] we also say that $s_1 \in \Gamma(E_1)$ is ϕ−projectable if it is ϕ−related to some $s_2 \in \Gamma(E_2)$. The most straightforward attempt to generalize the morphism of Lie algebras would then be

Definition 1 *Let ϕ be a vector bundle morphism $\phi\colon E_1 \to E_2$ between two Lie algebroids $(E_i, M_i, \rho_i, [\cdot,\cdot]_i)$, $i = 1, 2$. We say that E_1* and *E_2* are *ϕ*−related, *$E_1 \overset{\phi}{\sim} E_2$, iff*

$$\rho_2 \circ \phi \;=\; (\phi_0)_* \circ \rho_1 \tag{22}$$

$$s_1 \overset{\phi}{\sim} s_2, s_1' \overset{\phi}{\sim} s_2' \;\Rightarrow\; [s_1, s_1']_1 \overset{\phi}{\sim} [s_2, s_2']_2 \qquad \forall s_i, s_i' \in \Gamma(E_i) \tag{23}$$

where $(\phi_0)_*\colon TM_1 \to TM_2$ denotes the push forward of tangent vectors induced by ϕ_0.

In general, however, ϕ-relation of Lie algebroids is too weak a notion to deserve being called also a morphism of Lie algebroids. We thus take recourse to a dual perspective, using the map Φ introduced in the previous section (in the example of standard Lie algebroids $E_i = TM_i$ and $\phi = \varphi_*$, the map Φ is just the pull back of differential forms):

Definition 2 *A vector bundle morphism $\phi\colon E_1 \to E_2$ between two Lie algebroids $(E_i, M_i, \rho_i, [\cdot,\cdot]_i) \simeq (\Gamma(\Lambda E_i^*), \wedge, \mathrm{d}_i)$, $i = 1, 2$, is a* morphism of Lie algebroids, *iff the induced map $\Phi\colon \Gamma(\Lambda E_2^*) \to \Gamma(\Lambda E_1^*)$ is a chain map:*

$$\mathrm{d}_1\Phi - \Phi\mathrm{d}_2 = 0\,. \tag{24}$$

In other words, ϕ is a morphism iff

$$F_\phi\colon \Omega^{\cdot}_{E_2}(M_2) \to \Omega^{\cdot+1}_{E_1}(M_1)\,, \quad F_\phi := \mathrm{d}_1\Phi - \Phi\mathrm{d}_2 \tag{25}$$

vanishes. Before continuation, we show that Definition 2 indeed serves the purpose of giving a mathematical meaning to the field equations of the PSM:

Proposition 1 *A bundle map ϕ between $T\Sigma$ and T^*M is a solution of the PSM equations (13, 14), iff Φ is a morphism of Lie algebroids.*

Proof. Let us choose a local chart $U \subset M$ supplied with coordinate functions $\{X^i\}$, inducing the local frame ∂_i of TU. Applying $\mathrm{d}\Phi - \Phi\partial$ to X^i and ∂_i, we immediately obtain the first and the second field equations, (13) and (14), respectively. Here d is the usual de Rham operator on Σ and ∂ is the Lichnerowicz-Poisson differential acting on $\Gamma(\Lambda^{\cdot} TM)$, which is a particular case of the canonical Lie algebroid differential on T^*M determined by the Poisson structure $\mathcal{P}$. Since both the conditions (13), (14) and (24) are local, this completes the proof. ■

In [11], instead of the above, one finds the following definition:

Definition 3 *Let E_1, E_2 be Lie algebroids on bases M_1, M_2 with anchors ρ_1, ρ_2. Then* a morphism of Lie algebroids $E_1 \to E_2$ *is a vector bundle morphism $\phi\colon E_1 \to E_2$, $\phi_0\colon M_1 \to M_2$ such that equation (22) holds and such that for arbitrary $s_1, s_1' \in \Gamma(E_1)$ with $\phi-$decomposition*

$$\phi \circ s_1 = \sum a_i(\eta_i \circ \phi_0), \quad \phi \circ s_1' = \sum a_i'(\eta_i' \circ \phi_0) \tag{26}$$

we have

$$\phi \circ [s_1, s_1'] = \sum a_i a_j'([\eta_i, \eta_j'] \circ \phi_0) + \sum \rho_1(s_1)(a_j')(\eta_j' \circ \phi_0) - \sum \rho_1(s_1')(a_j)(\eta_j \circ \phi_0)\,. \tag{27}$$

Here $\{\eta_i\}, \{\eta_i'\}$ are sections of E_2 and a_i, a_i' functions over M_1. Let us mention that *any* section $s \in \Gamma(E_1)$ has *some* $\phi-$decomposition (e.g. choose for $\{\eta_i\}$ a (possibly overcomplete) basis of sections in E_2—the definition then may be shown to be also independent of this choice of basis).

Proposition 2 *Definitions 2 and 3 are equivalent.*

Proof. As seen by a simple straightforward calculation, application of (24) to functions yields a dual formulation of (22) (just contract the former equation with sections of E_1).

It remains to show equivalence of the second defining property in Definition 3 to the application of (24) to sections of E_2^*. In other words we need to prove that for any $u \in \Gamma(E_2^*)$ and $s_1, s_1' \in \Gamma(E_1)$ with decompositions (26) one has

$$\begin{aligned}\langle(\mathrm{d}_1\Phi - \Phi\mathrm{d}_2)u, s_1 \wedge s_1'\rangle = \langle u \circ \phi_0, \sum \rho_1(s_1)(a_j')(\eta_j' \circ \phi_0) - \\ - \sum \rho_1(s_1')(a_j)(\eta_j \circ \phi_0) - \phi \circ [s_1, s_1'] + \sum a_i a_j'([\eta_i, \eta_j'] \circ \phi_0)\rangle\,.\end{aligned} \tag{28}$$

In fact, using (18), we obtain

$$\begin{aligned}\langle \mathrm{d}_1\Phi(u), s_1 \wedge s_1'\rangle = \rho_1(s_1)\langle\Phi u, s_1'\rangle - \rho_1(s_1')\langle\Phi u, s_1\rangle - \langle\Phi u, [s_1, s_1']\rangle = \\ = \rho_1(s_1)(\sum a_j'\phi_0^*\langle u, \eta_j'\rangle) - \rho_1(s_1')(\sum a_j\phi_0^*\langle u, \eta_j\rangle) - \langle u \circ \phi_0, \phi \circ [s_1, s_1']\rangle\,.\end{aligned} \tag{29}$$

The Leibniz rule for the anchor map action of s_1, s_1' gives

$$\begin{aligned}\langle \mathrm{d}_1\Phi(u), s_1 \wedge s_1'\rangle = \langle u \circ \phi_0, \sum \rho_1(s_1)(a_j')(\eta_j' \circ \phi_0) - \sum \rho_1(s_1')(a_j)(\eta_j \circ \phi_0) - \phi \circ [s_1, s_1']\rangle + \\ + \sum a_j'\rho_1(s_1)\phi_0^*\langle u, \eta_j'\rangle - \sum a_j'\rho_1(s_1')\phi_0^*\langle u, \eta_j\rangle\,.\end{aligned} \tag{30}$$

On the other hand,

$$\begin{aligned}\langle\Phi\mathrm{d}_2 u, s_1 \wedge s_1'\rangle = \sum a_i a_j'\phi_0^*\langle\mathrm{d}_2 u, \eta_i \wedge \eta_j'\rangle = \sum a_j a_j'\phi_0^*\left(\rho_2(\eta_i)\langle u, \eta_j'\rangle - \rho_2(\eta_j')\langle u, \eta_i\rangle\right) - \\ -\langle u \circ \phi_0, \sum a_i a_j'([\eta_i, \eta_j'] \circ \phi_0)\rangle\end{aligned} \tag{31}$$

Eq. (22) implies that $\forall h \in C^\infty(M_2), s \in \Gamma(E_1), x \in M_1$, one has $\rho_1(s)_{|x}\phi_0^* h = \rho_2(\phi \circ s_x)h$. Hence, taking into account the $\phi-$decompositions of s_i, s_j', we get

$$\rho_1(s_1)\phi_0^*\langle u, \eta_j'\rangle = \sum a_i\phi_0^*\rho_2(\eta_i)\langle u, \eta_j'\rangle, \tag{32}$$

and a likewise formula with primed and unprimed quantities exchanged. Thus all additional contributions in the difference $\langle \mathrm{d}_1\Phi u, s_1 \wedge s_2\rangle - \langle\Phi\mathrm{d}_2 u, s_1 \wedge s_1'\rangle$ vanish, i.e. the last two terms in (30) cancel against the first two terms in (31). ■

From Definition 3 it is also obvious that for Lie algebras, corresponding to $M_1 = M_2 = \{pt\}$, the chain property (24) is equivalent to ϕ being a morphism in the usual sense. Also, from this version we see that if $\phi\colon E_1 \to E_2$ is a morphism of Lie algebroids, then E_1 and E_2 are $\phi-$related: Indeed, the condition on the left-hand side of (23) implies a $\phi-$decomposition (26) with only one term, $a = 1$ and $\eta = s_2$ (and likewise for the primed quantity), in which case Eq. (27) just reduces to the right-hand side of (23).

However, in general the converse conclusion is not true as illustrated e.g. by the following example in the context of the PSM (cf. our discussion above and in particular Proposition 1):

Example 1 *Let x^1, x^2 be coordinates on the world-sheet $\Sigma := \mathbb{R}^2$ and let $M := \mathbb{R}^4$ be a target manifold supplied with a zero Poisson tensor. Assume that ϕ is specified by the following choice of fields: $A := A_i \otimes \mathrm{d}X^i$ with $A_1 := \mathrm{d}x^1$, $A_2 := x^2\mathrm{d}x^1$, $A_3 := \mathrm{d}x^2$, $A_4 = x^2\mathrm{d}x^2$ and ϕ_0, in accordance with the first morphism property (22), which is equivalent to the first set of field equations (13), is chosen to map to a single point in $\mathbb{R}^4$. This provides a $\phi-$relation of $T\mathbb{R}^2$ and $T^*\mathbb{R}^4$, because there is not even a single vector field ξ on $\mathbb{R}^2$ that—for this choice of ϕ—is ϕ-related to* any *section of $\Gamma(T^*\mathbb{R}^4)$, and thus the condition (ii) in Definition 1 becomes empty. But this does not satisfy the morphism property (24) since A_i clearly does not satisfy also the second set of field equations (14) (which would imply that all A_is are closed).*

Under suitable further conditions it is nevertheless possible to reverse the above mentioned implication: In the above example the main problem was that the given vector bundle morphism excludes the existence of *any* projectable section.

Proposition 3 *[11] If the sections of E_1 which are projectable with respect to a given vector bundle morphism $\phi : E_1 \to E_2$ generate all of $\Gamma(E_1)$, then $\phi-$relation implies the morphism property. The assumption holds true in particular, if ϕ is fiberwise surjective.*

Proof. According to the assumption any $s_1, s_1' \in \Gamma(E_1)$ decompose as $s_1 = \sum a_i\xi_i, \quad s_1' = \sum a_i'\xi_i'$ such that ξ_i, ξ_i' are $\phi-$related to some $\eta_i, \eta_i' \in \Gamma(E_2)$, respectively. Since obviously (26) holds true, we should prove (27): By $\phi-$relation $\phi\circ[\xi_i, \xi_j'] = [\eta_i, \eta_j']\circ\phi_0$. Using this relation in the application of ϕ to

$$[s_1, s_1'] = \sum a_i a_j'([\xi_i, \xi_j'] + \sum \rho_1(s_1)(a_j')\xi_j' - \sum \rho_1(s_1')(a_j)\xi_j \tag{33}$$

we indeed find (27).

Finally, if ϕ is fiberwise surjective, there exists an isomorphism between E_1 and $\ker\phi \oplus \phi_0^*E_2$. Evidently, any section of $\ker\phi$ is $\phi-$related to the zero section of E_2 and all sections of $\phi_0^*E_2$ are generated by $\phi^*(\Gamma(E_2))$; thus all sections of $\ker\phi \oplus \phi_0^*E_2 \cong E_1$ are generated by projectable sections. ■

Let us notice that since the morphism equation (24) and the proof above are local, the statement of Proposition 3 remains unchanged if we replace M_1 with an open neighborhood of any point $x_0 \in M_1$. This argument is used in the next

Proposition 4 *Any $\phi-$relation with a base map that is a local immersion is a morphism of Lie algebroids.*

Proof. If $\phi_0\colon M_1 \to M_2$ is a local immersion then for any point $x_0 \in M_1$ there exists a coordinate chart (U, X^i) around $\phi_0(x_0)$ and an open neighborhood $V \subset M_1$ of x_0 such that $\phi_0(V) \subset U$ is given by the set of equations $X^{1+\dim M_1} = \ldots = X^{\dim M_2} = 0$ and $\phi_0\colon V \to \phi_0(V)$ is a diffeomorphism.

Now one can show that any section of $(E_1)_{|V}$ is projectable with respect to the restriction of ϕ on $(E_1)_{|V}$, i.e. $\phi-$related to some section of $(E_2)_{|U}$ as a consequence of the following simple facts:

- The restriction of ϕ defines a map of sections $\Gamma((E_1)_{|V}) \to \Gamma((E_2)_{|\phi_0(V)})$
- Any section of $(E_2)_{|\phi_0(V)}$ can be extended as a section of $(E_2)_{|U}$.

■

The last statement is of particular interest due to

Proposition 5 *$\phi\colon E_1 \to E_2$ is a morphism of Lie algebroids, iff its graph $\phi^{\mathrm{gra}}\colon E_1 \to E \equiv E_1 \boxplus E_2$ is a morphism of Lie algebroids.*

Proof. With $\Omega^{\cdot}_E(M) \cong \Omega^{\cdot}_{E_1}(M_1) \otimes \Omega^{\cdot}_{E_2}(M_2)$, ${}^E\mathrm{d} = {}^E\mathrm{d}_1 + {}^E\mathrm{d}_2$, and $\Phi^{\mathrm{gra}}(\omega_1 \otimes \omega_2) = \omega_1 \wedge \Phi(\omega_2)$ for all $\omega_i \in \Omega^{q_i}_{E_i}(M_i)$ (so that $\varepsilon_1(\omega_1) = \deg \omega_1 = q_1$) one has

$$(\mathrm{d}_1 \Phi^{\mathrm{gra}} - \Phi^{\mathrm{gra}}\, {}^E\mathrm{d})(\omega_1 \otimes \omega_2) = (-1)^{q_1} \omega_1 \wedge (\mathrm{d}_1 \Phi - \Phi \mathrm{d}_2)\omega_2$$

which vanishes identically if and only if Φ is a chain map. ■

Since the base map of ϕ^{gra} is even an embedding, the general notion of Lie algebroid morphism can be reduced to the simplified notion of ϕ-relation of Lie algebroids, $E_1 \overset{\phi^{\mathrm{gra}}}{\sim} E$.

Finally, the chain property (24) may be reformulated also nicely in terms of operators living in one and the same bundle. Recall that ${}^E\Phi$ and ${}^E\mathrm{d}$ both act inside $\Omega^{\cdot}_E(M)$ (cf. Eq. (7) and end of the previous section); while ${}^E\Phi$ is of (total) degree zero, ${}^E\mathrm{d}$ is of degree one. We have:

Proposition 6 *$\phi\colon E_1 \to E_2$ is a morphism of Lie algebroids, iff the induced operator ${}^E\Phi$ commutes with ${}^E\mathrm{d}$ on $\Omega^{\cdot}_E(M)$, i.e. iff the operator*

$$ {}^E F_\phi := [{}^E\mathrm{d}, {}^E\Phi] \tag{34}$$

vanishes.

Proof. By definition, ${}^E\Phi = P_1 \circ \Phi^{\mathrm{gra}}$. Since evidently ${}^E\mathrm{d} \circ P_1 = P_1 \circ \mathrm{d}_1$ holds true, we obtain

$${}^E\mathrm{d}\, {}^E\Phi - {}^E\Phi\, {}^E\mathrm{d} = P_1 \circ (\mathrm{d}_1 \Phi^{\mathrm{gra}} - \Phi^{\mathrm{gra}}\, {}^E\mathrm{d}),$$

which concludes the proof due to Proposition 5 and the fact that P_1 is an injection. ■

Maybe some warning is in place: The above notion of a morphism, in any of its formulations, applied to the cotangent bundle of two Poisson manifolds, does *not* coincide with a Poisson morphism. In contrast, a Poisson map, i.e. a map $\hat{\phi}_0\colon (M_1, \mathcal{P}_1) \to (M_2, \mathcal{P}_2)$ with $(\hat{\phi}_0)_* \mathcal{P}_1|_x = \mathcal{P}_2|_{\phi_0(x)}\ \forall x \in M_1$, gives rise only to a bundle morphism $\hat{\phi}\colon TM_1 \to TM_2$ by means of the tangent map $\hat{\phi} := (\hat{\phi}_0)_*$. This generalizes in the following way

Definition 4 *Let $(E_i, [\cdot,\cdot], \rho_i)$ be Lie algebroids over base manifolds M_i, $i = 1, 2$. We say that a bundle map $\hat{\phi}\colon E_1^* \to E_2^*$ is* a comorphism *if the induced operator $\hat{\Phi}\colon \Gamma(E_2) \to \Gamma(E_1)$ satisfies the following properties:*

$$\begin{aligned} (\hat{\phi}_0)^*(\rho_2(s_2)(f)) &= \rho_1(\hat{\Phi}(s_2))((\hat{\phi}_0)^*(f)) && \forall f \in C^\infty(M_2), s_2 \in \Gamma(E_2) \\ \hat{\Phi}([s_2, s_2']) &= [\hat{\Phi}(s_2), \hat{\Phi}(s_2')] && \forall s_2, s_2' \in \Gamma(E_2) \end{aligned}$$

In this terminology a Poisson map thus corresponds to a comorphism of the respective Poisson Lie algebroids, $\hat{\Phi}$ then being nothing but the pullback of differential 1-forms.

An algebraic generalization of these notions may be found in [20], such that a morphism (comorphism) of Lie algebroids corresponds to a comorphism (morphism) of the related pseudoalgebras, respectively.

We conclude this section with a short remark about covariance of the field equations (13), (14). Obviously the total set of field equations must be covariant—they are the Euler Lagrange equations of a completely covariant action functional, cf., e.g., (10) or (12), or, likewise, they can be reformulated frame independently as in (24). On the other hand, the field equations (14) are not only not written in an explicitly covariant form, by themselves they even are not frame independent. The reason for this is the (kind of) Leibniz rule satisfied by the operator (25),

$$F_\phi(\omega_2 \wedge \omega_2') \quad = \quad F_\phi(\omega_2) \wedge \Phi(\omega_2') + (-1)^p \Phi(\omega_2) \wedge F_\phi(\omega_2')\,, \tag{35}$$

which holds for arbitrary $\omega_2 \in \Gamma(\Lambda^p E_2^*)\,,\ \omega_2' \in \Gamma(\Lambda^q E_2^*)$. Indeed, with the abbreviations[7]

$$F^i \quad := \quad F_\phi(X^i) = \mathrm{d}_1 X^i - \rho^i_I A^I\,, \tag{36}$$

$$F^I \quad := \quad F_\phi(b^I) = \mathrm{d}_1 A^I + \frac{1}{2} C^I_{JK} A^J \wedge A^K\,, \tag{37}$$

the first and second set of field equations are $F^i = 0$ and $F^I = 0$, respectively. Suppose now we change frame from b^I to a new one, $\tilde{b}^I$, by means of $b^I = B^I_J \tilde{b}^J$. Then, by means of (35), we find $F^I = \Phi(B^I_J)\, F^{\widetilde{J}} + \Phi(B^I_{J,i}) F^i \wedge A^J$. This obviously implies that only upon usage of $F^i = 0$, which itself clearly *is* covariant (also with respect to change of coordinates $X^i \to \widetilde{X^i}$), we may conclude $F^{\widetilde{I}} = 0$ from $F^I = 0$.

This may be cured by means of an auxiliary connection Γ on E_2, introducing

$$F^I_{(\Gamma)} := F^I + \Gamma^I_{iJ} F^i \wedge A^J\,. \tag{38}$$

This option shall be investigated into further depth in a separate paper [17]. In the present paper we are interested particularly in morphisms, $F^i = 0 = F^I$, in which case covariance of (37) is of subordinate importance. The issue of covariance will become more important in the context of the following section, however.

4 Generalized gauge symmetries

We now turn to interpreting and generalizing the gauge symmetries of the PSM. In view of the generalization (36) and (37) of the field equations (13) and (14), it is suggestive to replace the gauge symmetries (15) and (16) by

$$\delta^0_\epsilon X^i \quad = \quad \rho^i_I \epsilon^I\,, \tag{39}$$

$$\delta^0_\epsilon A^I \quad = \quad \mathrm{d}_1 \epsilon^I + C^I_{JK} A^J \epsilon^K\,; \tag{40}$$

without further mention it is assumed furthermore that δ^0_ϵ obeys an (ungraded) Leibniz rule (which is used e.g. when establishing gauge invariance of (11) up to boundary terms).

Such as we were able to cast (36) and (37) into a more elegant and covariant form, cf., e.g., (34), and prove the equivalence of their vanishing with the morphism property of Lie algebroids,

[7] For notation and conventions recall end of section 2.

we may now strive for similar issues in the context of (39) and (40). This indeed is part of the intention of the present section. However, first we need to notice that in the context of symmetries the non-covariance of the formulas (39), (40) or (15), (16) is much more severe than in the case of the field equations, which are only not written in explicitly covariant form in (13), (14), while, as a total set, they certainly are covariant. As written, the symmetries either have only on-shell meaning (when there is an action functional like in the PSM this is tantamount to having meaning only as quotient of all symmetries modulo so-called trivial ones, cf. also [3]) or they are defined only for trivial or flat bundles E_2 (respectively for topologically rather trivial Poisson manifolds)!

Let us be more explicit about this: An infinitesimal gauge symmetry such as (39) and (40) is supposed to be a vector field on the (infinite dimensional) space $\mathcal{M} = \{\phi\colon E_1 \to E_2\} \cong \{\Phi\}$ of fields and thus, for a fixed element ϕ in $\mathcal{M}$, a vector $\mathcal{V} \in T_\phi\mathcal{M}$. Note that $\mathcal{M}$ is a bundle over $\mathcal{M}_0 = \{\phi_0\colon M_1 \to M_2\}$, the space of base maps. The projection of $\mathcal{V}$ to $\mathcal{M}_0$ then gives a vector $\mathcal{V}_0 \in T_{\phi_0}\mathcal{M}_0$. Eq. (39) indeed corresponds to a vector on $\mathcal{M}_0$, as may be seen by changing coordinates on M_2 (or likewise also local frames in E_2). However, (39) and (40) together do *not* give a well-defined vector on the total space $\mathcal{M}$. Indeed, if we change frame in E_2^*, $b^I = B^I_J\tilde{b}^J$, such that $\epsilon^I = B^I_J(X(x))\,\tilde{\epsilon}^J$ etc, a straightforward calculation yields

$$\mathrm{d}_1\epsilon^I + C^I_{JK}A^J\epsilon^K = B^I_J\left(\mathrm{d}_1\tilde{\epsilon}^J + \tilde{C}^J_{KL}\tilde{A}^K\tilde{\epsilon}^L\right) + B^I_{J,i}\mathrm{d}_1X^i\tilde{\epsilon}^J + B^I_{J,i}\tilde{\rho}^i_K\tilde{A}^J\tilde{\epsilon}^K - B^I_{J,i}\tilde{\rho}^i_K\tilde{A}^K\tilde{\epsilon}^J\,; \qquad (41)$$

on the other hand, by the Leibniz rule we obtain,

$$\delta^0_\epsilon(B^I_J\,\tilde{A}^J) = B^I_J\,\delta^0_\epsilon\tilde{A}^J + B^I_{J,i}\tilde{A}^J\delta^0_\epsilon X^i\,. \qquad (42)$$

The difference of the right hand sides of (41) and (42) is

$$B^I_{J,i}F^i\tilde{\epsilon}^J\,. \qquad (43)$$

Therefore, in general (39) and (40) do not provide a vector in $T_{\phi_0}\mathcal{M}_0$; it is globally well-defined only on fields satisfying $F^i = 0$ or when B^I_J can be chosen consistently to be X-independent. The first option is (part of) the on-shell condition, the second one corresponds to the existence of a flat connection in E_2. In this case (40) depends implicitly on the frame and on the flat connection chosen, which is zero in the particular frame chosen, but becomes non-zero if we change the frame.

At this point let us emphasize that δ_ϵ is *not* a tangent vector field to $\mathcal{M}$ if it satisfies $\delta_\epsilon A^I = B^I_J\delta_\epsilon\tilde{A}^J$ (which would correspond to the absence of all three terms in (41)) with respect to a change of frame $b^I = B^I_J\tilde{b}^J$; it is an element of $T_\phi\mathcal{M}$ only when it satisfies an ungraded Leibniz rule, i.e. in particular

$$\delta_\epsilon A^I = B^I_J\delta_\epsilon\tilde{A}^J + B^I_{J,i}\tilde{A}^J\delta_\epsilon X^i \qquad (44)$$

(which would correspond to the absence of the last and the third to last term in (41), which together combined into (43)). As a consequence, even if one uses a connection on E_2 to provide a global and frame independent definition of the tangent vectors δ_ϵ, the explicit formula for $\delta_\epsilon A^I$ will *not* be covariant (in the usual sense) with respect to capital indices (containing only covariant derivatives and E_2-tensors).[8] In contrast, $\delta_\epsilon X^i$ is covariant with respect to i, since multiplication by (the pullback of) the Jacobian of a coordinate change on M_2 is in agreement with the Leibniz property of δ_ϵ.

[8]There is one trivial exception to this statement, namely the case for which the second term in (44) vanishes identically (for all choices of $B^I_J(X)$): This happens iff $\delta_\epsilon X^i \equiv 0$ for all ϵ, which, in view of the covariance and off-shell validity of (39), in turn is tantamount to $\rho \equiv 0$, i.e. this happens iff E_2 is a bundle of Lie algebras.

For the rest of the section, we will proceed as follows: In view of the above observation, δ^0_ϵ as defined in (39) and (40) should have a good, more abstract *on-shell* interpretation. Indeed, we will see that it corresponds to an infinititesimal homotopy of Lie algebroid morphisms. Simultaneously this picture provides an on-shell integration of the infinitesimal symmetries δ^0_ϵ. Next we want to lift the on-shell symmetry to a well-defined off-shell symmetry. This is not unique certainly. One option is to do this in such a way that the (infinitesimal) inner automorphisms of E_1 and E_2 are contained as Lie subalgebras. This will turn out to be done most efficiently in terms of E-Lie derivatives of the exterior sum Lie algebroid $E = E_1 \boxplus E_2$. The second option is to employ a connection on E_2, such that for flat connections Γ, and in a frame for which $\Gamma = 0$, one reobtains the original formulas for δ^0_ϵ. This second option shall be mentioned at the end of this section peripherically only; for more details we refer to [17].

Definition 5 *Let E_1 and E_2 be Lie algebroids over smooth manifolds M_1 and M_2, respectively. We say that the two morphisms $\phi, \phi' : E_1 \to E_2$ are* homotopic, *iff there is a morphism $\overline{\phi}$ from the Lie algebroid $\overline{E} := E_1 \boxplus TI$ over the manifold $N = M_1 \times I$, $I \equiv [0,1]$, such that the restriction of $\overline{\phi}$ to the boundary components $M_1 \times \{0\}$ and $M_1 \times \{1\}$ coincides with ϕ and ϕ', respectively.*

Proposition 7 *Two Lie algebroid morphisms ϕ and ϕ' are homotopic, iff they can be connected by a flow of $\delta^0_\cdot$ as defined in (39), (40).*

Note that, as outlined above, δ^0_ϵ is well-defined on-shell, i.e. as a vector field on the subset of $\mathcal{M}$ satisfying the field equations $F^i = 0 = F^I$; in the above proposition δ^0_ϵ is understood in this on-shell sense.

Proof. Given a local frame $\{b_I\}$ in E_2 over a coordinate chart $\{X^i\}$, we immediately obtain the following system of equalities from the chain property of $\overline{\phi}$:

$$\overline{F}^i = {}^{\overline{E}}\mathrm{d}X^i - \rho^i_I \overline{A}^I \equiv 0, \quad \overline{F}^I = {}^{\overline{E}}\mathrm{d}\overline{A}^I + \frac{1}{2} C^I_{JK} \overline{A}^J \wedge \overline{A}^K \equiv 0\,, \tag{45}$$

where the structure functions C^I_{JK} and ρ^i_I depend on $X(x)$, $x \in M_1$, but not on t. On the other hand, by definition, $\overline{E} = E_1 \boxplus TI$ and ${}^{\overline{E}}\mathrm{d}_U = \mathrm{d}_1 + \mathrm{d}t \wedge \partial_t$; correspondingly, $\overline{A}^I = A^I + \overline{A}^I_t \mathrm{d}t$, with $A^I \equiv A^I(t) \equiv A^I_\alpha b^\alpha$ being (local) t-dependent E_1-1-forms. Adapting (45) to this splitting, and renaming $\overline{A}^I_t$ to ϵ^I, we obtain

$$\overline{F}^i = F^i(t) + \mathrm{d}t \left(\partial_t X^i - \rho^i_I \epsilon^I_t \right) \tag{46}$$

$$\overline{F}^I = F^I(t) + \mathrm{d}t \wedge \left(\partial_t A^I - \mathrm{d}_1 \epsilon^I + C^I_{JK} \epsilon^J \overline{A}^K \right) \tag{47}$$

where F^i and F^I are of the form (36) and (37) and $\partial_t A^I \equiv (\partial_t \overline{A}^I_\alpha) b^\alpha$. This proves that $\overline{F}^i = 0 = \overline{F}^I$, iff for any t one has $F^i = 0 = F^I$ and $\partial_t X^k = \delta^0_\epsilon X^k$, $\partial_t A^K = \delta^0_\epsilon A^K$. ■

If M_i are manifolds with boundary one has to take care about boundary conditions. In particular, the space of morphisms from TI to an arbitrary Lie algebroid E over a manifold M modulo homotopies (with fixed boundary contribution) gives the *fundamental* or *Weinstein's groupoid* of E, cf. [5]. Thus, the on-shell part of gauge symmetries (39), (40) is well-motivated now. It corresponds to the infinitesimal flow of a homotopy of Lie algebroid morphisms.

We now turn to a possible off-shell definition of the gauge symmetries without the introduction of any further structures such as a connection on E_2, employed in an alternative approach in [17].

Concretely this means that we want to extend (39), (40) to a differential δ_ϵ, satisfying (44), where for $F^i = 0 = F^I$ the gauge transformation δ_ϵ reduces to δ^0_ϵ—and we want to relate this differential on field space to a differential operator on or between finite dimensional bundles, in analogy of what we did with the field equations.

Definition 6 *We call an operator* $\mathcal{V}\colon \Omega^{\cdot}_{E_2}(M_2) \to \Omega^{\cdot+\deg\mathcal{V}}_{E_1}(M_1)$ *a* Φ-Leibniz *operator, if it satisfies* $\forall \omega, \omega' \in \Omega^{\cdot}_{E_1}(M_1)$ *(ω homogeneous)*

$$\mathcal{V}(\omega \wedge \omega') = \mathcal{V}(\omega) \wedge \Phi(\omega') + (-1)^{\deg\mathcal{V}\deg\omega}\, \Phi(\omega) \wedge \mathcal{V}(\omega'), \tag{48}$$

and likewise an operator ${}^E\mathcal{V}$ *in* $\Omega^{\cdot}_E(M)$ *(of fixed degree)* ${}^E\Phi$-Leibniz *if it satisfies the above equation with* $\mathcal{V}$ *and* Φ *replaced by* ${}^E\mathcal{V}$ *and* ${}^E\Phi$*, respectively.*

An example for a degree one Φ-Leibniz operator is provided by F_ϕ, cf. Eq. (35); likewise ${}^EF_\phi$, defined in (34), is ${}^E\Phi$-Leibniz. More generally, obviously any consecutive application (in both possible orders) of a (standard) Leibniz operator with Φ (${}^E\Phi$) gives a Φ-Leibniz (${}^E\Phi$-Leibniz) operator.

Definition 7 *We call* $\delta\Phi\colon \Omega^{\cdot}_{E_2}(M_2) \to \Omega^{\cdot}_{E_1}(M_1)$ *an* infinitesimal gauge symmetry*, if it is a degree zero* Φ*-Leibniz operator satisfying* $\mathrm{d}_1\delta\Phi \approx \delta\Phi\mathrm{d}_2$*, where* $\approx$ *denotes an on-shell equality (i.e. it has to be an equality for all* Φ *with* $F_\phi = 0$*). Likewise a degree zero* ${}^E\Phi$*-Leibniz operator* ${}^E\delta\Phi$ *is an infinitesimal gauge symmetry if it satisfies*

$$[{}^E\delta\Phi, {}^E\mathrm{d}] \approx 0 \quad \Leftrightarrow \quad [{}^E\delta\Phi, {}^E\mathrm{d}]|_{\phi:[{}^E\Phi,{}^E\mathrm{d}]=0} = 0 \tag{49}$$

and $\mathrm{im}{}^E\delta\Phi \subset \mathrm{im}P_1$, ${}^E\delta\Phi \circ P_1 = 0$.

This is motivated as follows: $\delta\Phi \sim \mathrm{d}\Phi_t/\mathrm{d}t|_{t=0}$ for some family of Φs parametrized by t. Correspondingly, since Φ is of degree zero, also $\delta\Phi$ is, and functoriality of Φ, $\Phi(\omega\wedge\omega') = \Phi(\omega)\wedge\Phi(\omega')$, results in the Φ-Leibniz property. Finally, Φ_t satisfying the field equations implies that $\delta\Phi$ does so on use of the field equation for $\Phi \sim \Phi_{t=0}$. All this applies analogously to ${}^E\delta\Phi$, where, however, in addition we need to take care of the fact that ${}^E\Phi$ is not an arbitrary operator in $\Omega^{\cdot}_E(M)$, but restricted as specified in (9) and the text thereafter.

One of the main features of a gauge symmetry is that it maps solutions of field equations into solutions. Here, the solutions have the meaning of a morphism (of Lie algebroids) $\phi\colon E_1 \to E_2$. To construct gauge symmetries we may thus proceed as follows: Let the gauge transformed morphism $\tilde\phi$ be given by $\tilde\phi := (a_1)^{-1} \circ \phi \circ a_2$ where $a_i \in \mathrm{Aut}(E_i)$, $i = 1, 2$, the respective group of automorphisms of E_i. This defines a right action of $\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)$ on $\mathcal{M} = \{\phi\}$, which on the level of Lie algebras provides a *homomorphism* $\mathrm{aut}(E_1) \oplus \mathrm{aut}(E_2) \to \Gamma(T\mathcal{M})$.

A subgroup of the automorphism group of a Lie algebra $E_i \cong \mathfrak{g}_i$ is the group of inner automorphisms, given by the adjoint action of the Lie group G_i which integrates $\mathfrak{g}_i$; infinitesimally, this is just the regular representation of the Lie algebra $\mathfrak{g}_i$, i.e. the action of $\mathfrak{g}_i$ onto itself given by multiplication in the Lie algebra, $v_i \mapsto [v_i, \cdot]$ (a homomorphism of $\mathfrak{g}_i \to \mathrm{aut}(\mathfrak{g}_i)$). Although not every Lie algebroid has a (sufficiently smooth) Lie groupoid integrating it (cf. [7] for the necessary and sufficient conditions), we still may generalize the infinitesimal picture to the setting of Lie algebroids: Given a section $s_i \in \Gamma(E_i)$, we may regard ${}^{E_i}L_{s_i}$ as a vector field on E_i, which, due to ${}^{E_i}L_{s_i}(s'_i) = [s_i, s'_i]$ and the Jacobi property of the Lie algebroid bracket, is an infinitesimal automorphism of E_i.

That ${}^{E_i}L_{s_i}$ indeed can be regarded as a vector field on E_i may be seen as follows: $C^\infty(M_i)$ and $\Omega^1_{E_i}(M_i)$ are fiberwise constant and linear functions on E_i, respectively. Together they generate all of $C^\infty(E_i)$. Local coordinates X on M_i and a local coframe b^I provide a local coordinate system on E_i. Applying a vector field to local coordinates gives its components in this coordinate system; these components may be easily extracted from Eq. (21), showing that they are linear in the fiber coordinates. The E_i-Lie derivative ${}^{E_i}L_{s_i}$ provides a uniquely defined lift of $\rho(s_i) \in \Gamma(TM_i)$ to $\Gamma(T(E_i))$; in contrast to the lift given by a contravariant connection this lift is not C^∞-linear in s_i, certainly.

Proposition 8 *For arbitrary sections $s_i \in \Gamma(E_i)$, $i = 1, 2$,*

$$\delta\Phi := \Phi \circ {}^{E_2}L_{s_2} - {}^{E_1}L_{s_1} \circ \Phi \tag{50}$$

is an infinitesimal gauge symmetry. For any $\Phi \in \mathcal{M}$, its action on a local coordinate system X^i, b^I on E_2 defines a Leibniz operator δ_ϵ (an element in $\Gamma(T_\phi\mathcal{M})$), which agrees with δ^0_ϵ given in (39), (40) on-shell where $\epsilon = \Phi^!(s_2) - \iota_{s_1}A$ (and $A \equiv \Phi^!(\delta) = A^I \otimes \mathfrak{b}_I$). Moreover, the commutator of two such infinitesimal gauge transformations is again of the same form, $[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{\epsilon''}$, where ϵ'' results from $s''_1 = [s_1, s'_1]$ and $s''_2 = [s_2, s'_2]$.

The statement in this proposition may be simplified by saying that there exists a *homomorphism* $\Gamma(E_1) \oplus \Gamma(E_2) \to \Gamma(T_\phi\mathcal{M})$, $\delta_{(s_1,s_2)}\Phi \mapsto \delta_\epsilon$; however, we refrained from doing so, since, at least at this point, we did not want to go into the details of defining properly the infinite dimensional tangent vector bundle $T\mathcal{M}$ (while still we will come back to this perspective in more detail below). Let us remark already at this point, moreover, that the set of ϵ's that one may obtain in this fashion is too restrictive, yet. Assume e.g. that ϕ corresponds to $A^I = 0$ and $X^i(x) = const$. Then any ϵ of the above form is necessarily constant, while it need not be so in (39), (40), where $\epsilon \in \Gamma(M_1, \phi_0^*E_2)$ arbitrary.

Proof. First it is easy to see that (50) provides an infinitesimal gauge symmetry in agreement with definition 7. As a composition of Leibniz operators with Φ it is Φ-Leibniz, and since ${}^{E_i}L$-Lie derivatives commute with the respective differential d_i, $\mathrm{d}_1\,\Phi \approx \Phi\,\mathrm{d}_2$ is seen to result in $\mathrm{d}_1\,\delta\Phi \approx \delta\Phi\,\mathrm{d}_2$.

To determine the desired map $(s_1, s_2) \in \Gamma(E_1) \oplus \Gamma(E_2)$ to $\epsilon \in \Gamma(M_1, \phi_0^*E_2)$, we may use Cartan's magic formula (19) to rewrite $\delta\Phi = \delta_{(s_1,s_2)}\Phi$ according to

$$\delta_{(s_1,s_2)}\Phi = \Phi\,{}^{E_2}L_{s_2} - {}^{E_1}L_{s_1}\,\Phi = \delta^0_{(s_1,s_2)}\Phi - \left(F_\phi\,\iota_{s_2} + \iota_{s_1}\,F_\phi\right)\,, \tag{51}$$

where

$$\delta^0_{(s_1,s_2)}\Phi \equiv \mathrm{d}_1\,(\Phi\,\iota_{s_2} - \iota_{s_1}\,\Phi) + (\Phi\,\iota_{s_2} - \iota_{s_1}\,\Phi)\,\mathrm{d}_2\,. \tag{52}$$

While the last two terms in (51) vanish on-shell obviously, it is easy to verify that $\delta^0_{(s_1,s_2)}$ acting on X^i and b^I agrees with δ^0_ϵ in (39), (40) with the parameter ϵ as given above. Finally, since actions coming from the right and from the left commute, it is obvious that $[\delta_\epsilon, \delta_{\epsilon'}]$ (with ϵ and ϵ' of the given form) when applied to A^I and $\phi_0^*X^i$ is tantamount to the application of $\Phi \circ [{}^{E_2}L_{s_2}, {}^{E_2}L_{s'_2}] - [{}^{E_1}L_{s_1}, {}^{E_1}L_{s'_1}] \circ \Phi$ to b^I and X^i, respectively. The statement now follows since E_i-Lie derivatives are a representation of $\Gamma(E_i)$. ■

Note that in contrast to δ^0_ϵ, the operator $\delta^0_{(s_1,s_2)}\Phi$ in Eq. (52) is defined frame-independently. However, it now is not a Φ-Leibniz operator (only on-shell it is). We remark in parenthesis that one may also generalize the operator in Eq. (52) to one defined for arbitrary sections $\epsilon \in \Gamma(M_1, \phi_0^* E_2)$: $\delta^0_\epsilon \Phi := \mathrm{d}_1\, i_\epsilon + i_\epsilon\, \mathrm{d}_2$ with the operator i_ϵ being defined by means of

$$i_\epsilon(f b^{I_1} \wedge \ldots \wedge b^{I_k}) := \sum_{j=1}^{k} (-1)^{j+1} \epsilon^{I_j}\, \Phi\left(f b^{I_1} \wedge \ldots \wedge \widehat{b^{I_j}} \wedge \ldots \wedge b^{I_k} \right) . \tag{53}$$

But such somewhat artificial constructions do not seem very promising. Instead, the right step is to take recourse to the exterior sum bundle $E = E_1 \boxplus E_2$. This has the effect that in the end the section $\Phi^! s_2|_x = s_2(X(x))$, $x \in M_1$, of the previous proposition is replaced by a likewise section that depends on both variables, $X(x)$ and x, independently.

Theorem 1 *Any section $\epsilon \in \Gamma(E)$ which is projectable to a section of E_1 (p_1-projectable) defines an infinitesimal gauge symmetry by means of*

$${}^E\delta_\epsilon \Phi := [{}^E\Phi, {}^E L_\epsilon] , \tag{54}$$

and the commutator of two such gauge transformations for ϵ, ϵ' is the gauge transformation associated to $[\epsilon, \epsilon'] \in \Gamma(E)$. In particular, for "vertical" sections $\epsilon \in \Gamma(\mathrm{pr}_2^* E_2) \subset \Gamma(E)$ its action on local fields X^i, A^I is given by:

$$\delta_\epsilon X^i = \delta^0_\epsilon X^i \quad , \qquad \delta_\epsilon A^I = \delta^0_\epsilon A^I - \epsilon,_i F^i , \tag{55}$$

where δ^0 was defined in Eqs. (39), (40) and $F^i \approx 0$ in Eq. (36).

Proof. Obviously ${}^E\delta_\epsilon \Phi$ is ${}^E\Phi$-Leibniz, and it obeys Eq. (49) since ${}^E\mathrm{d}$ commutes with any E-Lie derivative and on-shell (by definition) also with ${}^E\Phi$. Thus it remains to check the final two restrictions on an infinitesimal gauge transformation specified in definition 7. It is these conditions that make the restriction to p_1-projectability (as defined in the beginning of Sec. 3, where the bundle map ϕ is replaced by $p_1 \colon E \to E_1$, cf. diagram 1) of $\epsilon \in \Gamma(E)$ necessary. To see this we first split ϵ according to $E = \mathrm{pr}_1^* E_1 \oplus \mathrm{pr}_2^* E_2$ into $\epsilon = \epsilon_1 + \epsilon_2$ and use linearity in ϵ. Due to $[{}^E\Phi, {}^E L_{e_2}] = {}^E\Phi {}^E L_{e_2}$, the image of ${}^E\delta_{\epsilon_2}\Phi$ lies trivially in $\mathrm{im} P_1 = \mathrm{im}{}^E\Phi$, and also obviously it acts trivially on $P_1(\omega_1) = \omega_1 \otimes 1$ for all $\omega_1 \in \Omega^{\cdot}_{E1}(M_1)$. To ensure that also ${}^E\delta_{\epsilon_1}\Phi$ kills all $\omega_1 \otimes 1$, we introduced the commutator of ${}^E L_{\epsilon_1}$ with ${}^E\Phi$, the latter operator acting as the identity on the image of P_1. However, in this case both conditions are satisfied if and only if ϵ_1 depends on $x \in M_1$ only, but not also on $X \in M_2$ (consider e.g. ${}^E L_{\epsilon_1} {}^E\Phi = \imath_{\epsilon_1} \mathrm{d}^E\Phi + \ldots$); more abstractly this means that ϵ is p_1-projectable, the corresponding E-Lie derivative generating only automorphisms of E that are preserving fibers over M_1.

Two successive gauge transformations with parameter ϵ and ϵ' are characterized by the operator $[[{}^E\Phi, {}^E L_\epsilon], {}^E L_{\epsilon'}]$.[9] Subtracting from this the corresponding operator with ϵ and ϵ' exchanged and using the Jacobi condition for the (graded) commutator bracket, we obtain $[{}^E\Phi, [{}^E L_\epsilon, {}^E L_{\epsilon'}]] = [{}^E\Phi, {}^E L_{[\epsilon,\epsilon']}]$, a gauge transformation with parameter $[\epsilon, \epsilon']$.

To relate the gauge transformations above to explicit transformations acting on the fields, we proceed similarly to before (cf. Eqs. (51) and (52)), where now the splitting becomes a bit more elegant:

$${}^E\delta_\epsilon \Phi = [{}^E\Phi, [{}^E\mathrm{d}, \imath_\epsilon]] = {}^E\delta^0_\epsilon \Phi - [{}^E F_\phi, \imath_\epsilon] , \tag{56}$$

$${}^E\delta^0_\epsilon \Phi \equiv [{}^E\mathrm{d}, [{}^E\Phi, \imath_\epsilon]] , \tag{57}$$

[9] That the successive application of a vector field in field space $\mathcal{M}$ has again such a simple operator-description (being a second order differential operator on $\mathcal{M}$, it now is no more ${}^E\Phi$-Leibniz, certainly, but satisfies a similar higher analog of this property), is also a benefit of the present approach using operators on $\Omega^{\cdot}_E(M)$, $E = E_1 \boxplus E_2$.

where we made use of the (graded) Jacobi property and the definition (34) for ${}^E F_\phi$. Upon action on X^i, b^I (or, more generally, the image of $P_2 \colon \Omega^\cdot_{E2}(M_2) \to \Omega^\cdot_E(M)$)—and for $\epsilon = s_1 + s_2$— the operator ${}^E\delta^0_\epsilon \Phi$ is identified easily with the one in (52); for general p_1-projectable ϵ it just provides formulas (39) and (40). The on-shell vanishing contributions, necessary to render the gauge transformation globally defined and Leibniz, are now easily calculated to be

$$[{}^E F_\phi, \iota_\epsilon] X^i = \iota_{\epsilon_1} F^i , \tag{58}$$

$$[{}^E F_\phi, \iota_\epsilon] b^I = -\epsilon^I_{2},_i F^i + \iota_{\epsilon_1} F^I . \tag{59}$$

Note that here we used that ${}^E F_\phi(\epsilon_2^I)$ contributes only by its derivative with respect to X, but not also with respect to x; the latter terms cancel in the commutator (34). Eq. (55) now follows by specialization to $\epsilon = \epsilon_2$. ■

Given sections $s_i \in \Gamma(E_i)$, $i = 1, 2$, there is a natural inclusion as sections of the exterior sum E of E_1 and E_2. With $\epsilon := s_1 + s_2$ it is easy to see that the action of ${}^E\delta_\epsilon \Phi$ on X^i and b^I precisely reduces to $\delta\Phi$ as given in (50). The extension of the present approach is that now s_2 may effectively depend also on x (and that due to using the graph both, the action from the left and the action from the right in Prop. 8 now come from the right); due to this x-dependence of ϵ_2 (while ϵ_1 is still not permitted to depend on X), the total action is no more a direct sum of $\Gamma(E_1)$ with $\Gamma(E_2)$ as in Prop. 8, but a semidirect sum, spanned by the two Lie subalgebras generated by ϵ_1 and ϵ_2, respectively.

It is needless to say that an explicit verification of the closure of the symmetries (55) (or even as the one with $\epsilon = \epsilon_1 + \epsilon_2$, cf. Eqs. (58), (59)) would be a formidable task. This now was reduced to a simple line only. We may even use the above approach to simplify the likewise calculation of the commutator of the initial symmetries (say in a flat bundle or used in one particular coordinate patch):

Corollary: The commutator of two symmetries (39), (40) corresponding to $\epsilon, \epsilon' \in \Gamma(\phi_0^* E_2)$ is

$$[\delta^0_\epsilon, \delta^0_{\epsilon'}] X^i = \delta^0_{[\epsilon,\epsilon']} X^i \ , \quad [\delta^0_\epsilon, \delta^0_{\epsilon'}] A^I = \delta^0_{[\epsilon,\epsilon']} A^I - C^I_{JK,i} F^i \epsilon^J \epsilon'^K, \tag{60}$$

where $[\epsilon, \epsilon']^I := \phi_0^* \left(C^I_{JK} \right) \epsilon^J \epsilon'^K$.

Proof. Any section of $\Gamma(\phi_0^* E_2)$ can be regarded as the restriction of some section in $\Gamma(M, \mathrm{pr}_2^* E_2)$ to the graph of $\phi_0 \colon M_1 \to M_2$ inside M. Notice that this choice is not unique, certainly; given a flat connection on E_2, or in a particular local frame b^I, (which underlies the definition of δ^0!), we can choose this extension to be constant along M_1-fibers or independent of X. We denote these extensions again by the same letters. Note that the bracket in E induces the bracket as specified above when restricted to the $\phi_0^{\mathrm{gra}}(M_1) \subset M$; however, the bracket $[\epsilon, \epsilon'] \subset \Gamma(M, \mathrm{pr}_2^* E_2)$ is in general *not* constant along M_1-fibers; in general it depends on X due to the X-dependence of the structure functions C^I_{JK}. By use of Eq. (55) we thus obtain immediately

$$[\delta^0_\epsilon, \delta^0_{\epsilon'}] X^i = [\delta_\epsilon, \delta_{\epsilon'}] X^i = \delta_{[\epsilon,\epsilon']} X^i = \delta^0_{[\epsilon,\epsilon']} X^i \ , \tag{61}$$

$$[\delta^0_\epsilon, \delta^0_{\epsilon'}] A^I = [\delta_\epsilon, \delta_{\epsilon'}] A^I = \delta_{[\epsilon,\epsilon']} A^I = \delta^0_{[\epsilon,\epsilon']} A^I - C^I_{JK,i} F^i \epsilon^J \epsilon^K \ . \tag{62}$$

■

In the particular case of the PSM this reproduces the well-known contribution rendering the algebra to be an "open" algebra. We now see that this may be avoided by the additional contribution in (55) at the cost of keeping track of the $X(x)$ dependence of ϵ, which, however, anyway cannot be avoided in the case of a general, non-flat bundle E_2.

Summing up, we see that the gauge symmetries (55) are well-defined off-shell and globally. They are one possible off-shell extension of the always defined on-shell version, recognized above as a homotopy. Another extension is provided by a connection on E_2. In rather explicit terms this takes the form (besides the obvious $\delta_\epsilon^{(\Gamma)} X^i = \delta_\epsilon X^i$):

$$\delta_\epsilon^{(\Gamma)} A^I = \delta_\epsilon^0 A^I + \Gamma^I_{iJ} F^i \epsilon^J \,. \tag{63}$$

Let us remark that similarly to our considerations about homotopy—but without requiring F to vanish—it is possible to view these transformations as the components of the covariant curvatures F^i and $F^I_{(\Gamma)}$ in a $(1 + \dim(M_1))$–dimensional spacetime, cf. Eq. (38). For a more detailed and coordinate independent explanation of this alternative we refer to [17].

For both off-shell extensions it is clear by construction that they map solutions to the field equations into other solutions. However, it is not clear that, when specialized to the PSM, they would leave invariant the action functional (since then the invariance needs to hold off-shell). In fact, if e.g. one wants to check invariance of the PSM action (11) with respect to (63), specialized to the Poisson case, one finds invariance for all $\epsilon_i = \epsilon_i(x, X(x))$ if and only if the connection Γ is torsion-free.

We now want to discuss the same issue for the case of (55), also in a more coordinate independent way. For this purpose we return to (10), rewriting it, however, in a way more suitable to the graph map ϕ^{gra} (we prefer to use ϕ^{gra} here instead of ${}^E\phi$, since for an action functional we need a volume form on M_1, not a form on all of $M = M_1 \times M_2$). We first remark that the joint map $\mathrm{Alt} \circ \Phi^*$ can be obtained also as the dual map to $\widetilde{\phi} := \phi \oplus (\phi_0)_* \colon TM_1 \to T^*M_2 \oplus TM_2$. Indeed, the induced map $\widetilde{\Phi}$ then just maps $\Gamma(\Lambda^\cdot(TM_2)) \otimes \Omega^\cdot(M_2)$ to forms over M_1 and $\widetilde{\Phi} = \mathrm{Alt}\ \Phi^*$. Next, we may repeat the steps above for the map ϕ^{gra} instead of ϕ by replacing the target Lie algebroid in the map $\phi \colon E_1 \to E_2$ by $E = E_1 \boxplus E_2$. So, $\widetilde{\phi^{\mathrm{gra}}} = \phi^{\mathrm{gra}} \oplus (\phi_0^{\mathrm{gra}})_* \colon TM_1 \to E \oplus TM$ and $\widetilde{\Phi^{\mathrm{gra}}}$ acts from $\Gamma(\Lambda^\cdot(E \oplus TM)^*) \cong \Omega^\cdot_E(M) \otimes \Omega^\cdot(M)$ to $\Omega^\cdot(M_1)$. In this way we obtain

$$S[\phi] = \int_\Sigma \widetilde{\Phi}(\delta + \mathcal{P}) = \int_\Sigma \widetilde{\Phi^{\mathrm{gra}}}(\delta + \mathcal{P}) \,. \tag{64}$$

To determine the variation of $\widetilde{\Phi^{\mathrm{gra}}}$ with respect to a gauge transformation, we first need to extend the E-Lie derivative ${}^E L$ defined on E to $E \oplus TM$ (which is *not* a Lie algebroid itself in general), i.e. to define $\widetilde{{}^E L_\epsilon}$ on elements of $\Omega^\cdot_E(M) \otimes \Omega^\cdot(M)$ for any $\epsilon \in \Gamma(E)$: let $\widetilde{{}^E L_\epsilon}$ restrict to ${}^E L_\epsilon$ on $\Omega^\cdot_E(M)$ and act as $L_{\rho(\epsilon)}$ on $\Omega^\cdot(M)$; this gives a well-defined action on the tensor product since the two actions agree on functions. Then for any projectable section $\epsilon \in \Gamma(E)$ one has $\delta_\epsilon(\widetilde{\Phi^{\mathrm{gra}}}) = \widetilde{\Phi^{\mathrm{gra}}}\, {}^E L_\epsilon - L_{(p_1)_*(\epsilon)}\, \widetilde{\Phi^{\mathrm{gra}}}$, where $(p_1)_*(\epsilon) \in \Gamma(TM_1)$ is the projection of ϵ to $E_1 = TM_1$, and L denotes the ordinary Lie derivative. The second contribution in $\delta_\epsilon(\widetilde{\Phi^{\mathrm{gra}}})$ takes care of the fact that one respects the graph property. Now we are ready to state

Proposition 9 *The PSM action (10) or (64) is invariant with respect to the gauge transformations (55), if the projectable section $\epsilon \in \Gamma(E) \cong \Omega^1(M_2)$ satisfies*

$$\left(\mathcal{P}^\sharp \otimes \mathrm{id}\right)\, (\mathrm{d}_2 \epsilon) = 0 \,, \tag{65}$$

where d_2 is the de Rham operator over M_2 (extended trivially to $M = M_1 \times M_2$).

Proof. In this situation we now have the identifications: $E_1 = TM_1$, $E_2 = T^*M_2$, $E = TM_1 \boxplus T^*M_2$, and $\Omega^m_E(M) = \oplus_{p+q=m} \Omega^p(M_1) \boxtimes \Gamma(\Lambda^q TM_2)$. Thus, d_1 coincides with the de Rham operator on M_1.

Here $\mathcal{P} \in \Gamma(\Lambda^2 TM_2)$ and $\delta \in \Gamma(TM_2) \otimes \Omega^1(M_2)$ are sections of $\Omega^{\cdot}_E(M) \otimes \Omega^{\cdot}(M)$ living only over M_2. Since $\int_{M_1} L_\xi$ equals zero for any vector field $\xi \in \Gamma(TM_1)$ (taking into account that $L_\xi(\cdot)$ is always exact when acting on a form of highest degree), it is sufficient to check the statement for an arbitrary "vertical" section $\epsilon \in \Gamma(\mathrm{pr}_2^* E_2)$ (whose projection to TM_1 vanishes). One can easily calculate that

$$ {}^E L_\epsilon(\delta) = \mathrm{d}_1\epsilon_i \otimes \mathrm{d}X^i + \mathcal{P}^{ji}\partial_i \otimes \mathrm{d}_1\epsilon_j + (\epsilon_{j,i} - \epsilon_{i,j})\mathcal{P}^{jk}\partial_k \otimes \mathrm{d}X^i \in \Omega^1_E(M) \otimes \Omega^1(M) \tag{66} $$

$$ {}^E L_\epsilon(\mathcal{P}) = \mathcal{P}^{ji}\mathrm{d}_1\epsilon_j \otimes \partial_i + \frac{1}{2}(\epsilon_{j,i} - \epsilon_{i,j})\mathcal{P}^{ki}\mathcal{P}^{lj}\partial_k \wedge \partial_l \in \Omega^2_E(M) \tag{67} $$

which implies that the corresponding variation of the PSM action in the form (10) equals

$$ {}^E L_\epsilon S_{PSM} = \int\limits_{M_1} \mathrm{d}_1\epsilon_i \wedge \mathrm{d}X^i + (\epsilon_{j,i} - \epsilon_{i,j})\mathcal{P}^{jk} A_k \otimes \mathrm{d}X^i + \frac{1}{2}(\epsilon_{j,i} - \epsilon_{i,j})\mathcal{P}^{ki}\mathcal{P}^{lj} A_k \wedge A_l \,. \tag{68} $$

Clearly, the expression (68) vanishes if $\partial M_1 = \emptyset$ and the required condition (65) holds, which implies that $(\epsilon_{j,i} - \epsilon_{i,j})\mathcal{P}^{jk}\partial_k \equiv 0$.

■

In the remainder we briefly compare with another point of view on gauge transformations, viewed as an action of a certain infinite dimensional Lie algebroid living on the space of base maps, c.f. [19]. Let E_i be Lie algebroids over M_i, $i = 1, 2$. Then there is a vector bundle $\mathcal{E}$ over the space $\mathcal{M}_0$ of smooth maps ϕ_0 acting from M_1 to M_2, defined such that the infinite dimensional fiber $\mathcal{E}_{\phi_0}$ at any point ϕ_0 is $\Gamma(M_1, \phi_0^* E_2)$.

One has a natural map Ind^E acting from sections of $\mathrm{pr}_2^* E_2$ over M, as used before, to sections of $\mathcal{E}$ over $\mathcal{M}$: any section $s \in \Gamma(M, \mathrm{pr}_2^* E_2)$ gives a section of $\mathcal{E}$ by the map $s \mapsto \mathrm{Ind}^E_s$, such that $\mathrm{Ind}^E_s(\phi_0) := (\phi_0^{\mathrm{gra}})^* s \in \Gamma(M_1, \phi_0^* E_2)$. The map Ind^E is an embedding; moreover, the space of all sections $\Gamma(\mathcal{M}_0, \mathcal{E})$ is generated by Ind^E_s, $s \in \Gamma(M, \mathrm{pr}_2^* E_2)$ over an appropriate algebra of "smooth" functions on $\mathcal{M}_0$. For example, if $E_2 = TM_2$ then the corresponding bundle over $\mathcal{M}_0$ can be thought of as $T\mathcal{M}_0$. Let us notice that $T\mathcal{M}_0$ is also a Lie algebroid, such that the map $\mathrm{Ind}^T \colon \Gamma(M, \mathrm{pr}_2^* TM_2) \to \Gamma(\mathcal{M}_0, T\mathcal{M}_0)$ respects the Lie brackets. We can easily extend this fact for a general $\mathcal{E} \to \mathcal{M}_0$ obtained as above. For this purpose we introduce an anchor map ${}^{\mathcal{E}}\rho \colon \mathcal{E} \to T\mathcal{M}_0$ such that the following diagram is commutative:

$$ \begin{array}{ccc} \Gamma(M, \mathrm{pr}_2^* E_2) & \overset{\mathrm{Ind}^E}{\longrightarrow} & \Gamma(\mathcal{M}_0, \mathcal{E}) \\ \rho \downarrow & & {}^{\mathcal{E}}\rho \downarrow \\ \Gamma(M, \mathrm{pr}_2^* TM_2) & \overset{\mathrm{Ind}^T}{\longrightarrow} & \Gamma(\mathcal{M}_0, T\mathcal{M}_0) \end{array} \tag{69} $$

Now the Lie bracket on the image of $\Gamma(M, \mathrm{pr}_2^* E_2)$ can be extended to the space of all sections of $\Gamma(\mathcal{M}_0, \mathcal{E})$ by which it becomes a Lie algebroid bracket.

As an example, consider $M_2 = pt$, E_2 a Lie algebra $\mathfrak{g}$ with trivial anchor map $\rho \equiv 0$. Then $\mathcal{M}_0$ consists of only one element, and $\mathcal{E} = C^\infty(M_1, \mathfrak{g})$ is an infinite-dimensional Lie algebra of "multiloops".

In the language of Poisson Sigma Models, or more generally in the setting of Theorem 1, δ_ϵ defines a gauge transformation for any section $\epsilon \in \Gamma(\mathcal{M}_0, \mathcal{E})$. The previous discussion, however, only led to an action of $\Gamma(\mathcal{E})$ on base maps $\phi_0 \colon M_1 \to M_2$ via the vector field ${}^{\mathcal{E}}\rho(\epsilon)$ on $\mathcal{M}_0$. More generally, all vector fields $v_1 \in \Gamma(TM_1)$ and $v_2 \in \Gamma(TM_2)$ define sections $\overline{v}_1$ and $\overline{v}_2$ of $\Gamma(\mathcal{M}_0)$ which in a point $\phi_0 \in \mathcal{M}_0$ take the value

$$\overline{v}_1(\phi_0)(x) := \mathrm{d}\phi_0 \circ v_1(x) \quad \text{and} \tag{70}$$

$$\overline{v}_2(\phi_0)(x) := v_2 \circ \phi_0(x)\,, \tag{71}$$

respectively. Here, we use $T_{\phi_0}\mathcal{M}_0 \cong \Gamma(M_1, \phi_0^* TM_2)$ such that a vector field on $\mathcal{M}_0$ is defined by giving its value $v(\phi_0)(x) \in \phi_0^* TM_2$ in a map ϕ_0 and a point $x \in M_1$. Both vector fields can be seen to generate left and right compositions of diffeomorphisms on M_1 and M_2, respectively, with maps in $\mathcal{M}_0$. As such, those vector fields always commute with each other. Sections of E_1 and E_2 then define vector fields on $\mathcal{M}_0$ through $\rho_1(\epsilon_1) \in \Gamma(TM_1)$ and $\rho_2(\epsilon_2) \in \Gamma(TM_2)$.

This construction is clearly not general enough for our purposes. For gauge transformations we need vector fields which act on the set of bundle maps $E_1 \to E_2$ (i.e. "classical fields") denoted as $\mathcal{M}$. This space $\mathcal{M}$ is a bundle over $\mathcal{M}_0$ with fiber over a point $\phi_0 \in \mathcal{M}_0$ equal to $\Omega^1_{E_1}(M_1, \phi_0^* E_2)$.

Vector fields on $\mathcal{M}$ suitable for gauge transformations can advantageously be defined in the framework of infinite-dimensional supergeometry (however, an advantage of our independent construction is that we avoid infinite-dimensional supercomplications). A vector bundle $E \to M$ can be thought of as a $\mathbb{Z}-$graded manifold, denoted as $E[1]$, with the parity of the fibers defined to be odd. The algebra of smooth functions $C^\infty(E[1])$ on $E[1]$ is naturally isomorphic to $\Gamma(M, \Lambda^\cdot E^*)$, and any bundle map $E_1 \to E_2$ between two vector bundles becomes a degree preserving map $E_1[1] \to E_2[1]$. For any Lie algebroid $E \to M$ the canonical differential ${}^E\mathrm{d}$ defines a (super-) vector field of degree one tangent to $E[1]$, endowing $E[1]$ with a Q-structure. (A $\mathbb{Z}$-graded manifold endowed with an odd nilpotent vector field is called a Q-manifold [1].) Using this formalism, we can reformulate the chain property (24): a Lie algebroid morphism is a map $\phi\colon E_1[1] \to E_2[1]$ of degree zero, such that $\phi_*(\mathrm{d}_1) = \mathrm{d}_2$.

Denote the space of all graded maps $E_1[1] \to E_2[1]$ as $\mathcal{M}_\mathbb{Z}$ (containing $\mathcal{M}$ as the zero degree part). Analogously to the previous construction (70), the vector fields d_1 and d_2 on $E_1[1]$ and $E_2[1]$ naturally generate commuting vector fields δ_1 and δ_2 on $\mathcal{M}_Z$, respectively (corresponding to left and right compositions of morphisms). Since d_1 and d_2 are odd and nilpotent, so are δ_1 and δ_2. The difference $\delta := \delta_1 - \delta_2$ is again a nilpotent vector field of degree 1. Moreover, δ vanishes on the set of maps which preserve the Q-structures (in particular, on the set of Lie algebroid morphisms).

A Lie algebroid E can be identified with the tangent bundle $TE[1]$ where the action of a vector field on functions $C^\infty(E[1]) \cong \Gamma(M, \Lambda^\cdot E^*)$ is obtained by contraction between E and E^*. If we have a section $\epsilon \in \Gamma(\mathcal{M}_0, \mathcal{E})$ taking values in E_2, we obtain a vector field $\overline{\epsilon}$ on $\mathcal{M}$. Using $T_\phi\mathcal{M} \cong \Gamma(E_1, \phi^* TE_2)$, the vector field $\overline{\epsilon} \in \Gamma(T\mathcal{M})$ is defined by

$$\overline{\epsilon}(\phi)(x) := \epsilon_{\phi_0}(\pi_1(x)) \circ \phi(x)$$

for $x \in E_1$ and with $\pi_1\colon E_1 \to M_1$. Using the super structure of $\mathcal{M}_\mathbb{Z}$, $\overline{\epsilon}$ is a vector field of degree -1. A straightforward computation shows that the supercommutator between δ and the contraction with ϵ is a degree preserving vector field (therefore it is tangent to the subspace $\mathcal{M}$). This formula for a generalized gauge flow expressed as a supercommutator is an analog of Cartan's magic formula (19), which now holds in the context of an infinite-dimensional geometry of graded maps. One can use this infinitesimal transformation to generalize the gauge transformation (55) to sections ϵ which not only depend on $X \in M_2$, but also depend on the map ϕ_0 nontrivially. In particular, ϵ might be a local functional determined by higher jets of a base map $M_1 \to M_2$. In a similar way, we can express sections of $\epsilon_1 \in \Gamma(E_1)$ as vector fields on $\mathcal{M}$:

$$\overline{\epsilon_1}(\phi)(x) := \phi \circ \epsilon_1(\pi_1(x))\,.$$

Note that, unlike the vector fields defined in (70), vector fields obtained in this way from $\epsilon_1 \in \Gamma(E_1)$ and $\epsilon \in \Gamma(\mathcal{M}_0, \mathcal{E})$ do not commute in general since ϵ also depends on M_1.

Acknowledgement

T.S. gratefully acknowledges helpful discussions with A. Alekseev and D. Roytenberg in an early stage of this work.

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