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Linking Bäcklund and monodromy charges for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Gleb Arutyunov

Institute for Theoretical Physics, Utrecht University
Leuvenlaan 4, 3508 TD Utrecht, The Netherlands
E-mail: `g.arutyunov@phys.uu.nl`

Marija Zamaklar

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut
Am Mühlenberg 1, D-14476 Golm, Germany
E-mail: `marija.zamaklar@aei.mpg.de`

Abstract: We find an explicit relation between the two known ways of generating an infinite set of local conserved charges for the string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$: the Bäcklund and monodromy approaches. We start by constructing the two-parameter family of Bäcklund transformations for the string with an arbitrary world-sheet metric. We then show that only for a special value of one of the parameters the solutions generated by this transformation are compatible with the Virasoro constraints. By solving the Bäcklund equations in a non-perturbative fashion, we finally show that the generating functional of the Bäcklund conservation laws is equal to a certain sum of the quasi-momenta. The positions of the quasi-momenta in the complex spectral plane are uniquely determined by the real parameter of the Bäcklund transform.

Keywords: AdS-CFT and dS-CFT Correspondence, Integrable Field Theories, Bethe Ansatz.

Contents

1. Introduction and summary

New important insights into the conjectured duality between gauge and string theories [1] have been gained in the last two years. Although this duality relates a weakly coupled gauge theory to a strongly coupled string theory in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, it has been realized that certain (plane-wave) string states do admit a direct comparison with composite operators of the $\mathcal{N} = 4$ SYM [2]. Furthermore, extending the semi-classical approach of [3], a large sector of rotating multi-spin string solutions has been found [4]. These string solitons are naturally described by a simple finite-dimensional integrable system [5, 6] and they probe the structure of the space-time beyond the plane-wave limit. From mathematical point of view these solitons are the simplest examples of more general finite-gap solutions of the classical string sigma model [7]. The related progress on the gauge theory side has been based upon understanding the integrable properties of the dilatation operator at leading [8] and higher orders of perturbation theory [9]. This nicely generalizes and extends the integrable structures found in QCD [10].

The observed integrability of the *classical* string sigma model and the integrability of various spin chain hamiltonians emerging from the dilatation operator in the gauge theory open up a new avenue to address the issue of string quantization in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background.[1] The necessity of understanding the string spectrum is especially sharpened in

[1]While in the non-planar sector integrability is generically broken, it seems that for certain processes, integrable structures might remain preserved [11].

the light of recently observed discrepancies between the gauge and string theory calculations [12 − 14]. Although these mismatches can presumably be attributed to the order in which limits are taken in the string and gauge theories, the only way to fully resolve this issue is to find the exact string spectrum in $AdS_5 \times S^5$ and compare it to that of gauge theory.

At present it is unknown how to promote the observed *classical* integrability of the sigma model to the quantum level. Inspired by the findings in the gauge theory [12, 15], one can make an educated guess [16] for the *quantum* version of the classical Bethe equations of the string sigma model [7]. However, the quantum string Bethe equations describe the so-called $\mathfrak{su}(2)$ subsector of the theory and they are asymptotic, i.e. they require the R-charges of the string states to be large. It is not yet clear to which extent these equations can reproduce the full string spectrum. Recently these equations were extended in a beautiful way to the other subsectors, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1,1)$, and further intriguing relations to the gauge theory quantities were found [17].

Success of string quantization crucially depends on the choice of dynamical variables. Since it is not clear what kind of gauge is most suitable for the quantization, it is important to extract information in a covariant manner as much as possible. Even classically, restricting oneself to a particular gauge may simplify some computations, but can make other computations extremely difficult. For instance, the uniform gauge of [18] is convenient for the construction of the perturbative expansion (in the inverse powers of curvature) of the string hamiltonian around the plane-wave limit. However, this gauge is not suitable to reach the flat space limit; here the AdS light-cone gauge [19] is appropriate. Thus, the best option would be to explore as far as possible the classical/quantum integrability of the string sigma model in a covariant manner.

The integrability of the classical sigma model is manifested through an infinite set of (commuting) conserved charges.[2] Two ways to construct these charges[3] are known. One of them is through the so-called Bäcklund transformations [22, 23]. The other is based on the fundamental linear problem and the associated monodromy matrix [24] (see also [25] for a comprehensive review). Recently, both of these methods have been used successfully to reveal a close relation between the integrable structures of gauge and string theories [26, 7]. More precisely, in [26] the generating function for the Bäcklund conservation laws associated with the classical bosonic string sigma-model in the *conformal gauge* has been constructed. It has been evaluated on the so-called rigid string solutions and furthermore shown to match (at two leading orders of perturbation theory) with the generating function of the commuting charges obtained on the gauge theory side (see also [27]). Further interesting developments in this direction include linking the Bäcklund transform with the geometric U(1) symmetry of the string phase space [28].

On the other hand, in [7] the monodromy approach has been used to classify the finite-gap solutions of the classical bosonic string sigma-model. Here the spectrum of the model appears to be encoded in integral equations of the Bethe type which also exhibit an agreement with the Bethe equations describing the spectrum of long operators in the

[2]There is also the issue of the non-abelian symmetry which has been investigated in [20].

[3]The principal sigma model and its reductions also admit local *higher spin* conserved currents [21]. It would be interesting to clarify their meaning in the context of the AdS/CFT duality conjecture.

Yang-Mills theory [7]. Recently, this approach has been extended to the supersymmetric string sigma-model [29] (see also [30] on related issues) and furthermore used to show an agreement with the spectrum of the one-loop gauge theory.

The features discussed above all point out that the two approaches of studying strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$— the Bäcklund and the monodromy approaches — are undoubtedly related. Surprisingly, we have not found a simple explanation of this fact in the existing literature. The main aim of the present paper is therefore to understand the precise relation between these two, apparently different approaches. Let us now describe the content of the paper and the results obtained.

In view of the importance of the covariant approach, we start by constructing the covariant form of the Bäcklund transformations. The Bäcklund transformations transform a solution of the second-order evolution equations X_0 (the *reference* solution) into a new solution $X(\lambda, x)$ (the *dressed* solution). We identify a family of such transformations which depend on two continuous (spectral) parameters, λ and x. These transformations are defined in an arbitrary but *fixed* world-sheet metric γ. In string theory, in addition to the dynamical equations for the string embedding coordinates, one also has the Virasoro constraints. These are just the equations of motion for the world-sheet metric and once the field configuration X is known, the metric is fully fixed. Since in our construction we *assume* that the reference and dressed solutions have *the same* world-sheet metric, one also has to check that the new solution is compatible with this fixed metric (i.e. one has to check that the new solution satisfies the Virasoro constraints).[4] We show that this is not always the case, and that this requirement introduces a restriction on the value of the spectral parameter. By computing the two-dimensional stress tensor for the dressed solution, assuming the invariance of the world-sheet metric under the Bäcklund transform, we find that it vanishes only for special values of the spectral parameter λ, namely for $\lambda = \pm 1$. Thus, not every solution of the Bäcklund transform is compatible with the conformal constraints. It seems that this issue plays only a minor role in the general construction of integrable models, but it is crucial for applications in string theory.

Given a periodic solution to the Bäcklund equations one can construct an infinite set of conserved currents characterizing the reference solution. Building on the approach of Hanrad et al. [31], we solve the Bäcklund equations in a non-perturbative fashion. In this process the Bäcklund equations are reduced to the fundamental linear problem [24] and, therefore, their solutions are expressed in terms of the wave function which solves the linear problem. Using this explicit solution, we rewrite the generating function of the Bäcklund conservation laws via the eigenvalues of the monodromy matrix associated to the linear problem. It turns out that the Bäcklund generating function is equal to a certain sum of the logarithms of eigenvalues (quasi-momenta) of the monodromy matrix; the positions of the individual quasi-momenta in the complex plane are uniquely determined by the real spectral parameter x of the Bäcklund transform. This establishes a direct relation between the Bäcklund and monodromy approaches for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[4]It would also be interesting to understand whether the whole construction of the Bäcklund transform can be carried out without assuming the invariance of the world-sheet metric but requiring the reference and dressed solutions to obey the Virasoro constraints.

JHEP07(2005)026

In summary, we have discovered a very simple relation between two apparently different approaches to the construction of an infinite set of conserved charges for integrable models. In the context of string theory, our result could be used to shed some light on the formidable problem of quantizing strings on $AdS_5 \times S^5$. In particular, it seems to suggest that the "quantization" of the classical Bethe equation [16], originating from the monodromy approach, could equally well be described in terms of the yet unknown *quantum Bäcklund transform*; the latter should be understood as quantization of the classical Bäcklund equations. To see whether this is really the case for the full string theory on $AdS_5 \times S^5$, it would be important to first understand whether and how the relations which we have discovered for the bosonic string prevail once the fermions are taken into account, i.e. for the classical Green-Schwarz superstring on $AdS_5 \times S^5$. This question will be discussed elsewhere.

For the convenience of the reader, let us summarize the organization of the paper. In section 2 we introduce and describe the general properties of the Bäcklund transform. In section 3 we determine the general requirements for the Bäcklund transform to be compatible with the Virasoro constraints. In the next section we discuss a general solution of the Bäcklund equations in terms of the fundamental linear problem. Furthermore, in section 5 we find a relation between the Bäcklund and the monodromy conservation laws. In section 6 we present an independent perturbative check of our basic formula relating the Bäcklund and the monodromy charges. Finally, in two appendices attached, we discuss the perturbative construction of Bäcklund charges and also give some details on the gamma-matrix algebra.

2. Preliminaries

In this section we set up the notation and review some background material which will be necessary for the derivation of the covariant form of the Bäcklund equations. The starting point is the (bosonic) part of the sigma model action for strings in the $AdS_5 \times S^5$ background,

$$S = \frac{1}{2}\int d\tau d\sigma\, \gamma^{\alpha\beta}\, \mathrm{Tr}\left(\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\right), \tag{2.1}$$

where the indices $\alpha, \beta = (\tau, \sigma)$ refer to the world-sheet time and space directions. Here the matrix g describes an embedding of the $AdS_5 \times S^5$ space into the group $SU(2,2) \times SU(4)$ and $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ is the Weyl invariant tensor constructed from the lorentzian world-sheet metric $h_{\alpha\beta}$, and it has $\det\gamma = -1$. The string tension in (2.1) is set to unity. The equations of motion for the dynamical fields $g, \gamma^{\alpha\beta}$ derived from the action (2.1) are

$$\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta g g^{-1}) = \partial_\alpha(\gamma^{\alpha\beta} g^{-1}\partial_\beta g) = 0\,, \tag{2.2}$$

$$\mathrm{Tr}(\partial_\alpha g g^{-1}\partial_\beta g g^{-1}) - \frac{1}{2}\gamma_{\alpha\beta}\,\mathrm{Tr}(\partial_\rho g g^{-1}\partial_\delta g g^{-1})\gamma^{\rho\delta} = 0\,. \tag{2.3}$$

Equations (2.2) are the conservation laws for the left, A_L, and the right, A_R, currents

$$A_L^\alpha = \gamma^{\alpha\beta}\partial_\beta g g^{-1}\,, \qquad A_R^\alpha = \gamma^{\alpha\beta} g^{-1}\partial_\beta g\,. \tag{2.4}$$

In what follows we will mainly use A_L and therefore to save notation we will drop the subscript L.

We will assume that the group element g has the block-diagonal structure

$$g = \begin{pmatrix} g_{\rm a} & 0 \\ 0 & g_{\rm s} \end{pmatrix}, \tag{2.5}$$

where the matrices $g_{\rm a}$ and $g_{\rm s}$ belong to SU(2,2) and SU(4) respectively. To describe an embedding of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space into $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ it is convenient to choose the following parametrization for the group elements [6]

$$g_{\rm a} = p^i \Gamma^i_{\rm a}\,, \qquad g_{\rm s} = q^i \Gamma^i_{\rm s}\,. \tag{2.6}$$

Here $i = 1, \ldots, 6$. The 4×4 gamma-matrices $\Gamma^i_{\rm a}$ and $\Gamma^i_{\rm s}$ realize the *chiral* representations of SO(4,2) and SO(6) respectively. We summarize some of their properties in appendix B. The variables p^i and q^i parametrize the AdS space and the five-sphere and they obey the constraints $q^i q^i = 1$ and $\eta_{ij} p^i p^j = -1$, where η_{ij} has AdS signature.

Before solving the Virasoro constraints (2.3), the block-diagonal structure of g implies that the Bäcklund transformations and the conservation laws associated to the AdS and sphere sectors of the model are completely independent. Thus, it is sufficient to discuss the corresponding theory for the sphere part of the model; extension to the AdS sector goes without any difficulty. Therefore in what follows we set

$$g \equiv g_{\rm s} \tag{2.7}$$

confining our explicit treatment of the Bäcklund theory to the sphere case.

Finally we have to take into account the Virasoro constraints (2.3) which express the condition of vanishing of the two-dimensional stress tensor. Given a solution g of equations (2.2), equation (2.3) can be solved for the world-sheet metric $\gamma^{\alpha\beta}$:

$$\gamma_{\alpha\beta} = \frac{\theta_{\alpha\beta}}{\sqrt{-\det \theta_{\alpha\beta}}}\,, \qquad \theta_{\alpha\beta} = \mathrm{Tr}(A_\alpha A_\beta)\,. \tag{2.8}$$

Thus, the AdS and sphere sectors of the model are related through the Virasoro constraints. Generically the world-sheet metric appears to be a function of both the AdS and sphere coordinates. On the other hand, our definition of the Bäcklund transform (see the next section) implies that the reference and dressed solutions have the same wold-sheet metric (2.8). A priori such a definition is not necessarily compatible with the Virasoro constraints and below we will find further restrictions on the Bäcklund transform which guarantee that this is indeed the case.

3. The Bäcklund transformations and conservation laws

Given any two solutions of the equations of motion, g and $\tilde{g}$, we can construct two conserved currents $\tilde{A}_\alpha$ and A_α. Let us now require the difference of these quantities to be a topological current which is therefore trivially conserved:[5]

$$\gamma^{\alpha\beta} \tilde{A}_\beta - \gamma^{\alpha\beta} A_\beta = \epsilon^{\alpha\beta} \partial_\beta \chi\,. \tag{3.1}$$

[5] The currents $\tilde{A}_\alpha$ and A_α are periodic functions of σ. This implies for χ that $\chi(\sigma + 2\pi) = \chi(\sigma) + \mathrm{const.}$ However, if we require coincidence of the corresponding Noether charges, χ must be periodic.

JHEP07(2005)026

Here we assume that the indices of both A and $\tilde{A}$ are raised and lowered with a one and the same world-sheet metric γ. The matrix χ depends on $\tilde{g}$ and g, and it becomes constant when $\tilde{g} = g$. Subtracting from equation (3.1) its hermitian conjugate and using the fact that A and $\tilde{A}$ are anti-hermitian we obtain

$$\chi + \chi^\dagger = \mathrm{C}\,, \tag{3.2}$$

where C is a constant matrix. The matrix χ must be also invariant under the global transformation $g \to gh$ and should transform as $\chi \to h\chi h^{-1}$ under the global rotations $g \to hg$. In particular, equation (3.2) will remain invariant under these symmetry transformations provided we choose C to be proportional to the identity matrix. Obviously, any χ with such properties can be constructed in terms of a unitary matrix

$$U = \tilde{g}g^{-1} \tag{3.3}$$

or its inverse. To restrict possible choices for χ we therefore have to impose certain conditions on U which would allow one to express all higher powers of U or U^{-1} in terms of U. The simplest possibility is to take $\chi = \lambda U$, where λ is a complex (spectral) parameter. Since a phase of λ can always be absorbed by redefining U we may assume that λ is real. In appendix B we will verify that reality of λ is compatible with our definition of the coset model. With this choice we have

$$U + U^\dagger = 2\frac{x}{\lambda}\mathbb{I}\,, \tag{3.4}$$

where x and λ are real numbers. Hence, in what follows we assume that the difference of the Noether currents is of the form

$$\tilde{A}_\alpha - A_\alpha = \lambda\,\epsilon_\alpha{}^\beta\,U_\beta \tag{3.5}$$

where we defined $\epsilon_\alpha{}^\beta \equiv \gamma_{\alpha\delta}\,\epsilon^{\delta\beta}$ and also $U_\beta \equiv \partial_\beta U$. Obviously equation (3.5) represents a non-trivial condition on $\tilde{g}$. We will refer to $\tilde{g}$ as the Bäcklund transform of g.

Since U is unitary it can be diagonalized with a proper unitary matrix. Then equation (3.4) allows one to determine the eigenvalues of U. The eigenvalues appear to be degenerate — two of them are equal to ℓ and the other two to its complex conjugate, $\bar{\ell}$, where

$$\ell = \frac{x}{\lambda} - i\sqrt{1 - \left(\frac{x}{\lambda}\right)^2}\,. \tag{3.6}$$

Moreover, the eigenvalue problem imposes a restriction on the spectral parameter x:

$$-\lambda \le x \le \lambda\,. \tag{3.7}$$

The current $\tilde{A}_\alpha$ can be written via U as follows

$$\tilde{A}_\alpha = \partial_\alpha U U^{-1} + U A_\alpha U^{-1}\,. \tag{3.8}$$

Thus, we see that the Bäcklund transformation is in fact a certain *gauge transformation*.[6] The basic relation (3.5) can be written as the differential equation for the matrix U:

$$U_\beta(\delta_\alpha{}^\beta - \lambda\epsilon_\alpha{}^\beta U) = [A_\alpha, U]\,. \tag{3.9}$$

Using equation (3.4) the last equation can be brought to the form

$$\begin{aligned}\kappa U_\alpha &= -2x\lambda A_\alpha + (1+\lambda^2)A_\alpha U - (1+\lambda^2-4x^2)UA_\alpha - 2x\lambda UA_\alpha U + \\ &\quad + \epsilon_\alpha{}^\beta\Big(-\lambda(1+\lambda^2)A_\beta + 2xA_\beta U + 2x\lambda^2 UA_\beta - \lambda(1+\lambda^2)UA_\beta U\Big)\,,\end{aligned} \tag{3.10}$$

where $\kappa = (1+\lambda^2)^2 - 4x^2$. This is a matrix differential equation of the *Riccati type*; its solutions depend on two spectral parameters x and λ.

Equation (3.10) implies an infinite number of conservation laws. Indeed, define the following current

$$J^\alpha = \frac{1+\lambda^2}{\kappa}\Big[(1+\lambda^2)\gamma^{\alpha\beta}\,\mathrm{Tr}(A_\beta U) + 2x\epsilon^{\alpha\beta}\,\mathrm{Tr}(A_\beta U)\Big]\,. \tag{3.11}$$

Using the Riccati equation (3.10), the equations of motion for A_α and the zero-curvature condition $\partial_{[\alpha}A_{\beta]} = [A_\alpha, A_\beta]$ one can easily prove that $\partial_\alpha J^\alpha = 0$. The normalisation of the current is chosen for later convenience. Assuming the solution U to be a periodic function of σ, $U(\sigma+2\pi) = U(\sigma)$, the generating function of conserved charges is obtained by integrating the τ-component of the current:

$$\mathbf{Q}(x,\lambda) = \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}J^\tau(\sigma)\,. \tag{3.12}$$

Upon expansion over the spectral parameters the function $\mathbf{Q}(x,\lambda)$ generates an infinite set of integrals of motion. It is worth stressing that both the Bäcklund transformations and the conservation laws are determined for an arbitrary world-sheet metric.

We further notice that if we define $\mathscr{L}$ as

$$\mathscr{L}_\alpha = \frac{1+\lambda^2}{\kappa}\Big((1+\lambda^2)A_\alpha + 2x\epsilon_\alpha{}^\beta A_\beta\Big) \tag{3.13}$$

then $\mathscr{L}$ satisfies the zero curvature condition. The conserved current (3.11) takes a very simple form

$$J^\alpha = \gamma^{\alpha\beta}\,\mathrm{Tr}(\mathscr{L}_\beta U)\,. \tag{3.14}$$

More generally, introducing a connection $\mathscr{L}(a,b)$ parametrized by the coefficients a and b

$$\mathscr{L}_\alpha(a,b) = aA_\alpha + b\epsilon_\alpha{}^\beta A_\beta \tag{3.15}$$

one can check that it has zero curvature provided

$$a^2 - b^2 - a = 0\,. \tag{3.16}$$

We will refer to such a connection as the $\mathscr{L}$-operator.

Finally we note that the Riccati equation (3.10) can be expressed in terms of the $\mathscr{L}$-operator (3.13) only

$$(1+\lambda^2)U_\alpha = [\mathscr{L}_\alpha, U] - \epsilon_\alpha{}^\beta\left(\lambda\mathscr{L}_\beta + \lambda U\mathscr{L}_\beta U - 2xU\mathscr{L}_\beta\right)\,. \tag{3.17}$$

[6] See [32] on the relation of Bäcklund transformations with the theory of Poisson-Lie groups and dressing symmetries.

4. The stress tensor

The definition of a conserved current requires a world-sheet metric. In our construction of the Bäcklund transform we assumed that the conserved currents A^α and $\tilde{A}^\alpha$ are defined with one and the same world-sheet metric $\gamma^{\alpha\beta}$. As we have already discussed in the introduction, in string theory the Weyl-invariant metric $\gamma^{\alpha\beta}$ is fully determined by the Virasoro constraints $T_{\alpha\beta} = 0$, where $T_{\alpha\beta}$ is the two-dimensional stress tensor. Suppose we are given a pair g and $\gamma^{\alpha\beta}$ which solves both the dynamical equations for g and the Virasoro constraints. An important question we want to address here is what are the general conditions on the Bäcklund solution $\tilde{g}$ so that $\tilde{g}$ still solves the Virasoro constraints with the same metric $\gamma^{\alpha\beta}$. In other words, we require vanishing of the stress-energy tensor $T_{\alpha\beta}(\tilde{g},\gamma)$ for the Bäcklund solution.

To elaborate on this issue, we first compute $\delta\theta_{\alpha\beta} = \mathrm{Tr}(\tilde{A}_\alpha\tilde{A}_\beta - A_\alpha A_\beta)$. Using equation (3.4) it is easy to see that U_α obeys the following equation

$$U_\alpha = U U_\alpha U\,. \tag{4.1}$$

This equation together with equation (3.8) leads to

$$\delta\theta_{\alpha\beta} = \mathrm{Tr}\left(U_\alpha U A_\beta + U_\beta U A_\alpha + U_\alpha U_\beta\right). \tag{4.2}$$

Furthermore, we use the Bäcklund equations (3.10) to exclude the derivatives of U. After rather tedious computation we arrive at

$$\begin{aligned}\delta\theta_{\alpha\beta} \;=\; & -\frac{\lambda^2}{\kappa}({\delta_\alpha}^\mu{\delta_\beta}^\nu - {\epsilon_\alpha}^\mu{\epsilon_\beta}^\nu)\,\mathrm{Tr}\left[A_\mu(A_\nu - UA_\nu U^{-1}) + A_\mu(A_\nu - UA_\nu U^{-1})\right] + \\ & +\frac{\lambda(\lambda^2-1)}{\kappa}\,\mathrm{Tr}\left({\epsilon_\alpha}^\mu[A_\mu, A_\beta]U + {\epsilon_\beta}^\mu[A_\mu, A_\alpha]U\right). \end{aligned} \tag{4.3}$$

By using this formula we can now find how the stress tensor varies under the Bäcklund transform

$$\begin{aligned}\delta T_{\alpha\beta} \;=\; & \frac{\lambda^2}{\kappa}(\gamma_{\alpha\beta}\gamma^{\mu\nu} - {\delta_\alpha}^\mu{\delta_\beta}^\nu + {\epsilon_\alpha}^\mu{\epsilon_\beta}^\nu)\,\mathrm{Tr}\left[A_\mu(A_\nu - UA_\nu U^{-1}) + A_\mu(A_\nu - UA_\nu U^{-1})\right] + \\ & +\frac{\lambda(\lambda^2-1)}{\kappa}\,\mathrm{Tr}\,U\left({\epsilon_\alpha}^\mu[A_\mu, A_\beta] + {\epsilon_\beta}^\mu[A_\mu, A_\alpha] + \gamma_{\alpha\beta}\epsilon^{\mu\nu}[A_\mu, A_\nu]\right). \end{aligned} \tag{4.4}$$

Let us consider the first term in the expression above. It involves three tensor structures which are however not independent. Indeed, there is the *epsilon identity* which reads

$$\epsilon^{\mu\nu}\epsilon_{\alpha\beta} = {\delta_\alpha}^\nu{\delta_\beta}^\mu - {\delta_\alpha}^\mu{\delta_\beta}^\nu\,. \tag{4.5}$$

This identity implies

$${\epsilon_\alpha}^\mu{\epsilon_\beta}^\nu = {\delta_\alpha}^\nu{\delta_\beta}^\mu - \gamma_{\alpha\beta}\gamma^{\mu\nu}\,. \tag{4.6}$$

Therefore,

$$\gamma_{\alpha\beta}\gamma^{\mu\nu} - {\delta_\alpha}^\mu{\delta_\beta}^\nu + {\epsilon_\alpha}^\mu{\epsilon_\beta}^\nu = {\delta_\alpha}^\nu{\delta_\beta}^\mu - {\delta_\alpha}^\mu{\delta_\beta}^\nu\,. \tag{4.7}$$

JHEP07(2005)026

Since this expression is multiplied by a tensor which is symmetric under the permutation of the μ and ν indices, the contribution of the term under consideration vanishes. Thus,

$$\delta T_{\alpha\beta} = \frac{\lambda(\lambda^2-1)}{\kappa}\,\mathrm{Tr}\,U\Big(\epsilon_\alpha{}^\mu[A_\mu, A_\beta] + \epsilon_\beta{}^\mu[A_\mu, A_\alpha] + \gamma_{\alpha\beta}\epsilon^{\mu\nu}[A_\mu, A_\nu]\Big)\,.$$

Thus, we see that the compatibility of the Virasoro constraints with the general solution of the Bäcklund transform requires $\lambda = \pm 1$. Note that $\delta T_{\alpha\beta}$ also vanishes for $\lambda = 0$ which must be the case since for this value of λ we trivially have $\tilde{A} = A$.

5. General solution of the Bäcklund equations

In the previous section, we have found that the new solution, generated via the Bäcklund transform, satisfies the Virasoro constraints if and only if the spectral paramter λ is restricted to be $\lambda = \pm 1$. Therefore, in the following we only consider this case. The equation (3.6) then implies that the spectral parameter x has to be in the range $-1 \le x \le 1$.

One way of solving the Riccati equation (3.10) is using the perturbation method, i.e. by expanding the variable U in a power series around the points $x = \pm 1$. We present this computation in appendix A. Only these perturbative solutions have been so far been used in the literature, to determine the *local* conservation laws [23].

In this section we show how to express the solutions of the Riccati equation via solutions of the Riemann-Hilbert problem in a *non-perturbative* manner. The main result is given in formula (5.12). We will furthermore use this result to establish a simple relation between the Bäcklund charges and the local integrals of motion generated by the monodromy matrix of the fundamental linear problem.

To obtain solutions of the Riccati equation we employ the linearization method of [31]. According to this method, a solution U of the Riccati equation can be factorized as

$$U(\sigma,\tau) = XY^{-1}\,. \tag{5.1}$$

Here the matrices X and Y are obtained by applying to the initial data, X_0 and Y_0, an element $\mathbf{G}$ of the group SU(4,4)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{G}\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, \qquad \Omega = X_0 Y_0^{-1}\,, \tag{5.2}$$

so that the initial value of U is $U(0,0) = \Omega$.[7] of U is $U(0,0) = \Omega$.

The matrix $\mathbf{G}$ is subject to the following two conditions

$$\mathbf{G}^\dagger h_1 \mathbf{G} = h_1\,, \qquad \mathbf{G}^\dagger h_2 \mathbf{G} = h_2\,, \tag{5.3}$$

where h_1 and h_2 are the following block matrices

$$h_1 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & -2x\mathbb{I} \end{pmatrix}.$$

[7] Since the Riccati equation is a differential equation its solutions depend on an integration constant; this constant is Ω.

The first condition in (5.3) means that $\mathbf{G}$ belongs to SU(4,4) and is necessary for U to be unitary. The second condition is equivalent to the requirement (3.4). The constant matrix Ω obeys the same constraints as the matrix U, namely,

$$\Omega^\dagger\Omega = \mathbb{I}\,, \qquad \Omega + \Omega^\dagger = 2x\mathbb{I}\,.$$

Solving equations (5.3) we find

$$\mathbf{G} = \frac{1}{\ell - \bar{\ell}} \begin{pmatrix} \ell\Psi - \bar{\ell}(\Psi^\dagger)^{-1} & (\Psi^\dagger)^{-1} - \Psi \\ \Psi - (\Psi^\dagger)^{-1} & \ell(\Psi^\dagger)^{-1} - \bar{\ell}\Psi \end{pmatrix}, \tag{5.4}$$

where ℓ is the same complex parameter as in equation (3.6), i.e. it is related to the spectral parameter x as $(\lambda = 1)$

$$\ell = x - i\sqrt{1 - x^2}\,. \tag{5.5}$$

Note also that ℓ is on the unit circle because $\ell\bar{\ell} = 1$. Finally, if the complex matrix Ψ in equation (5.4) satisfies the differential equation

$$\partial_\alpha \Psi = \mathscr{L}_\alpha(\ell)\Psi\,, \tag{5.6}$$

where by definition $\mathscr{L}_\alpha(\ell)$ is the $\mathscr{L}$-operator (3.15) with the following coefficients a and b

$$a = \frac{1}{1 - \ell^2}\,, \qquad b = \frac{\ell}{1 - \ell^2}\,, \tag{5.7}$$

then the Riccati equation for U is satisfied. We will refer to equation (5.6) as *the fundamental linear problem.*

To verify the last statement we first find the matrices[8] X and Y

$$\begin{aligned} X &= \big[\ell\Psi(\ell) - \bar{\ell}\Psi(\bar{\ell})\big]\Omega + \Psi(\bar{\ell}) - \Psi(\ell)\,, \\ Y &= \big[\Psi(\ell) - \Psi(\bar{\ell})\big]\Omega + \ell\Psi(\bar{\ell}) - \bar{\ell}\Psi(\ell)\,, \end{aligned} \tag{5.8}$$

where the matrix Ψ is normalized as

$$\Psi(0,0) = \mathbb{I}\,. \tag{5.9}$$

Next we note the following two identities valid for the spectral parameter ℓ on a circle

$$\begin{aligned} \mathscr{L}_\alpha(\ell) &= \frac{\epsilon_\alpha{}^\beta - \bar{\ell}\delta_\alpha{}^\beta}{\ell - \bar{\ell}}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big)\,, \\ \mathscr{L}_\alpha(\bar{\ell}) &= \frac{\epsilon_\alpha{}^\beta - \ell\delta_\alpha{}^\beta}{\ell - \bar{\ell}}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big)\,. \end{aligned}$$

These identities together with equation (5.6) for Ψ are used to obtain the system of evolution equations for X and Y:

$$\begin{aligned} \partial_\alpha X &= \frac{\epsilon_\alpha{}^\beta}{\ell - \bar{\ell}}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big)X - \frac{1}{\ell - \bar{\ell}}\Big(\mathscr{L}_\alpha(\ell) - \mathscr{L}_\alpha(\bar{\ell})\Big)Y\,, \\ \partial_\alpha Y &= \frac{\epsilon_\alpha{}^\beta - 2x\delta_\alpha{}^\beta}{\ell - \bar{\ell}}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big)Y + \frac{1}{\ell - \bar{\ell}}\Big(\mathscr{L}_\alpha(\ell) - \mathscr{L}_\alpha(\bar{\ell})\Big)X\,. \end{aligned} \tag{5.10}$$

[8]Note the conjugation rule: $\Psi^\dagger(\ell)\Psi(\bar{\ell}) = \mathbb{I}$.

In writing these formulae the following relation has been used

$$(\bar{\ell}\Psi(\ell) - \ell\Psi(\bar{\ell}))\Omega + \ell^2\Psi(\bar{\ell}) - \bar{\ell}^2\Psi(\ell) = -X + 2xY\,. \tag{5.11}$$

Now the Riccati equation for U easily follows from the system (5.10).

Thus, the solution for U reads as

$$U = \Big[\ell\Psi(\ell)\big(\Omega - \bar{\ell}\big) + \bar{\ell}\Psi(\bar{\ell})\big(\Omega - \ell\big)\Big]\Big[\Psi(\ell)\big(\Omega - \bar{\ell}\big) - \Psi(\bar{\ell})\big(\Omega - \ell\big)\Big]^{-1}. \tag{5.12}$$

Note that the matrices $\Omega - \ell$ and $\Omega - \bar{\ell}$ are not invertible. As was already mentioned, since

$$\ell\bar{\ell} = 1\,, \qquad \ell + \bar{\ell} = 2x\,,$$

the variables ℓ and $\bar{\ell}$ are the eigenvalues of Ω and

$$(\Omega - \bar{\ell})(\Omega^\dagger - \bar{\ell}) = 0\,, \qquad \Omega - \ell = -(\Omega^\dagger - \bar{\ell}), \qquad (\Omega - \bar{\ell})(\Omega - \ell) = 0\,.$$

These properties allow us to define two hermitian (and orthogonal) projectors

$$\Omega^+ = \frac{\Omega - \bar{\ell}}{\ell - \bar{\ell}}\,, \qquad \Omega^- = -\frac{\Omega - \ell}{\ell - \bar{\ell}} \tag{5.13}$$

which provide an orthogonal decomposition of the identity: $\Omega^+ + \Omega^- = \mathbb{I}$, $\Omega^\pm\Omega^\mp = 0$.

6. Matching the Bäcklund and monodromy charges

We would now like to establish a connection between the conservation laws generated by the Bäcklund transform and the conservation laws arising in the standard monodromy approach. Our starting point is the nonperturbative solution (3.10) of the Bäcklund equation, and the expression for the Bäcklund current (3.14). Let us start by noting the following important relation

$$\gamma^{\alpha\beta}\mathscr{L}_\beta\left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) = \frac{2}{\ell - \bar{\ell}}\epsilon^{\alpha\beta}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big)\,. \tag{6.1}$$

Here on the left hand side, the Lax operator $\mathscr{L}$ is the same as in equation (3.15), with the coefficients a and b parametrized by x. This is this Lax operator which determines the conserved Bäcklund current (3.14). The Lax operator which appears on the right hand side in (6.1) is the same as $\mathscr{L}(\ell)$ which defines the fundamental linear problem (5.6). The coefficients a and b of the operator $\mathscr{L}(\bar{\ell})$ are given by equations (5.7) with the obvious substitution $\ell \to \bar{\ell}$.

Using the equation (5.10) for Y, we thus obtain

$$\partial_\alpha YY^{-1} = \frac{\epsilon_\alpha{}^\beta - 2x\delta_\alpha{}^\beta}{\ell - \bar{\ell}}\Big(\mathscr{L}_\beta(\ell) - \mathscr{L}_\beta(\bar{\ell})\Big) + \frac{1}{\ell - \bar{\ell}}\Big(\mathscr{L}_\alpha(\ell) - \mathscr{L}_\alpha(\bar{\ell})\Big)U\,. \tag{6.2}$$

Taking the trace of this equation, we arrive at

$$\mathrm{Tr}\left(\partial_\alpha YY^{-1}\right) = \mathrm{Tr}\Big(\frac{\mathscr{L}_\alpha(\ell) - \mathscr{L}_\alpha(\bar{\ell})}{\ell - \bar{\ell}}\Big)U\,. \tag{6.3}$$

Therefore, by using equation (6.1) for the current (3.14), we find the simple expression

$$J^{\alpha} = \gamma^{\alpha\beta} \operatorname{Tr} \mathscr{L}_{\beta}(x) U = 2\epsilon^{\alpha\beta} \operatorname{Tr} \Big(\frac{\mathscr{L}_{\beta}(\ell) - \mathscr{L}_{\beta}(\bar{\ell})}{\ell - \bar{\ell}} \Big) U = 2\epsilon^{\alpha\beta} \operatorname{Tr} \big(\partial_{\beta} Y Y^{-1} \big) . \tag{6.4}$$

In this form, the conservation of the current is obvious. The current is topological and thus is conserved without using the equations of motion for the fundamental fields. However, the corresponding charge is conserved if and only if the current is a periodic function of σ. Periodicity of J^{α} then imposes certain restrictions on the initial value Ω. To better understand this issue we compute the charge (3.12),

$$\pi \mathbf{Q}(x) = \operatorname{Tr} \log \big(Y(2\pi, \tau) Y(0, \tau)^{-1} \big) . \tag{6.5}$$

Differentiating this expression with respect to τ and using equations (5.10) we find that it is indeed time-independent provided

$$X(2\pi, \tau) Y(2\pi, \tau)^{-1} = X(0, \tau) Y(0, \tau)^{-1}$$

or, in other words, that the Bäcklund solution is periodic: $U(2\pi) = U(0)$.

Let us now study the periodicity property of the Bäcklund solution in more detail. Clearly, periodicity of U is equivalent to the following requirement

$$\begin{aligned} X(2\pi, \tau) &= X(0, \tau) \mathrm{M} \\ Y(2\pi, \tau) &= Y(0, \tau) \mathrm{M} , \end{aligned} \tag{6.6}$$

i.e. the matrices X and Y have to have the same monodromy M. By using equations (5.8), it is easy to see that this requirement is equivalent to

$$\begin{aligned} \Psi(0, \tau)^{-1} \Psi(2\pi, \tau) \Omega^{+} &= \Omega^{+} \mathrm{M} , \\ \bar{\Psi}(0, \tau)^{-1} \bar{\Psi}(2\pi, \tau) \Omega^{-} &= \Omega^{-} \mathrm{M} , \end{aligned} \tag{6.7}$$

where we have introduced the concise notation $\Psi \equiv \Psi(\ell)$ and $\bar{\Psi} \equiv \Psi(\bar{\ell})$. There are several important facts following from these equations.

First, let us note that these equations make sense because both $\Psi(0, \tau)^{-1} \Psi(2\pi, \tau)$ and $\bar{\Psi}(0, \tau)^{-1} \bar{\Psi}(2\pi, \tau)$ are time-independent as straightforwardly follows from the evolution equation for Ψ. Thus, M is also time-independent. Introducing the monodromy $\mathrm{T}(\tau)$ for the solution Ψ

$$\Psi(2\pi, \tau) = \mathrm{T}(\tau) \Psi(0, \tau) \tag{6.8}$$

we can express our basic τ-independent quantities via the value of the corresponding monodromy matrix[9] at $\tau = 0$ as $\Psi(0, \tau)^{-1} \Psi(2\pi, \tau) = \mathrm{T}(0)$ and $\bar{\Psi}(0, \tau)^{-1} \bar{\Psi}(2\pi, \tau) = \bar{\mathrm{T}}(0)$. Second, by adding the equations (6.7) we obtain

$$\mathrm{M} = \mathrm{T}(0) \Omega^{+} + \bar{\mathrm{T}}(0) \Omega^{-} . \tag{6.9}$$

[9]Note that the monodromy T do depend on τ, only its spectral invariants are conserved.

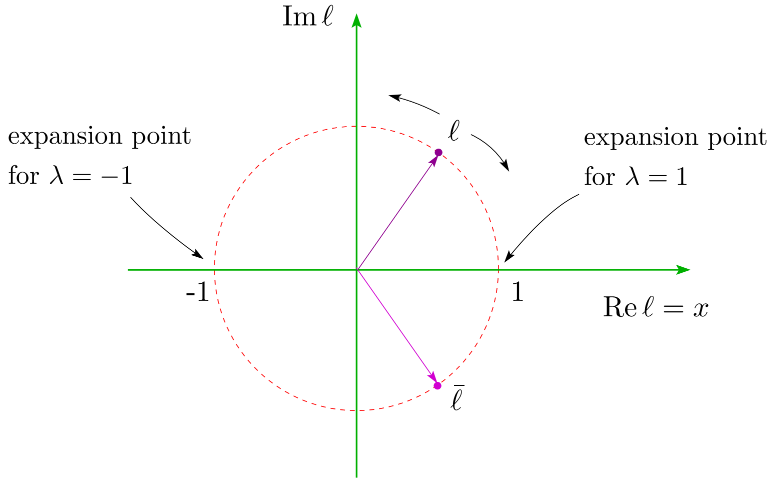

Figure 1: The Bäcklund generating function $\mathbf{Q}(x)$ for $x \in [-1, 1]$ is given by the sum of the quasi-momenta in the upper and lower half-planes (semi-circles).

Furthermore, we derive from equations (6.7) the equations which determine $\Omega^{\pm}$:

$$\Omega^{-}\mathrm{T}(0)\Omega^{+} = 0\,, \qquad \Omega^{+}\bar{\mathrm{T}}(0)\Omega^{-} = 0\,. \tag{6.10}$$

As a side remark, note that the determinants of both Ψ and $\bar{\Psi}$ are τ- and σ-independent and therefore that $\det\Psi = \det\bar{\Psi} = 1$ (since this is the case at the initial point $\tau = \sigma = 0$). As a consequence, $\det\mathrm{T} = \det\bar{\mathrm{T}} = 1$.

Let us now diagonalize Ω with some unitary matrix h so that $\Omega^{\pm}$ take the block-form

$$h\Omega^{+}h^{-1} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \qquad h\Omega^{-}h^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{6.11}$$

Then according to equations (6.10) we see that the matrices $h^{-1}\mathrm{T}(0)h$ and $h^{-1}\bar{\mathrm{T}}(0)h$ must have the (block) lower and upper triangular structure respectively

$$\mathbf{T} = h^{-1}\mathrm{T}(0)h = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ 0 & \mathbf{T}_4 \end{pmatrix}, \qquad \bar{\mathbf{T}} = h^{-1}\bar{\mathrm{T}}(0)h = \begin{pmatrix} \bar{\mathbf{T}}_1 & 0 \\ \bar{\mathbf{T}}_3 & \bar{\mathbf{T}}_4 \end{pmatrix}. \tag{6.12}$$

This allows one to write the conserved charge in the following factorized form

$$\pi\mathbf{Q}(x) = \mathrm{Tr}\log\mathrm{M} = \mathrm{Tr}\log\mathbf{T}_1 + \mathrm{Tr}\log\bar{\mathbf{T}}_4\,. \tag{6.13}$$

The reality property of the Bäcklund charge implies the conjugation rule according to which $\mathbf{T}_1^{\dagger}$ is related to $\bar{\mathbf{T}}_4$ by a similarity transformation.

Let us denote by $\exp(ip_k(\ell))$, where $k = 1, \ldots, 4$, the eigenvalues of the monodromy matrix T. The function $p_k(\ell)$ is known as the quasi-momentum or the Floquet function. An important property of the quasi-momentum is that it generates *local* integrals of motion upon expansion around the poles of the Lax connections [25], which in our case are at $\ell = \pm 1$.

Finally, it remains to note that the triangle monodromies $\mathbf{T}$ and $\bar{\mathbf{T}}$ in equations (6.12) can be brought to diagonal form by corresponding similarity transformations and their spectra coincide with that of T and $\bar{\mathrm{T}}$ respectively. In this way we have established the following remarkably simple relation between the local charges generated by the Bäcklund transform [26] and their cousins arising in the conventional monodromy approach [7],

$$\mathbf{Q}(x) = \frac{i}{\pi}\big(p_1(\ell) + p_2(\ell) + p_3(\bar{\ell}) + p_4(\bar{\ell})\big)\,. \tag{6.14}$$

Here $\sum_{k=1}^{4} p_k(\ell) = 0$ and the spectral parameter ℓ of the linear problem (5.6) is related to the spectral parameter x of the Bäcklund transform as

$$\ell = x - i\sqrt{1-x^2}\,. \tag{6.15}$$

We stress that our derivation does not require any gauge fixing and the result is valid for an arbitrary world-sheet metric γ. Also taking the log's in equation (6.13) we assumed that all p_i's are on the principle branch of the log.

Equation (6.14) should be understood in a perturbative sense when the left and the right hand side admit a well-defined asymptotic expansion around $x = \ell = \bar{\ell} = 1$. Of course, the same relation is true for the second series of the conservation laws upon expanding around $x = \ell = \bar{\ell} = -1$.

So far our discussion was quite general and applied to the principal sigma-model. To carry over this construction to the coset model describing strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ one has to find an embedding of the coset into the group $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ that is compatible with additional constraints like equation (3.4). This is also needed to guarantee that the form of the coset element is preserved under the Bäcklund transformations. In appendix B we show that the embedding of $\mathrm{AdS}_5 \times \mathrm{S}^5$ into $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ described in section 2 obeys these compatibility requirements. This allows us to conclude that the same formula (6.14) remains valid for the sphere part of the coset model. For the AdS sector the matching formula (6.14) looks the same provided $p_k(x)$ are quasi-momenta related to the AdS monodromy.

Finally we note that the quasi-momenta are defined up to permutations. On the other hand the formula (6.14) does not seem to be permutation invariant. For the coset model in question it is known [18] that *at leading order* in the perturbative expansion around a pole the quasi-momenta exhibit a degenerate behavior: they all coincide up to a sign, two of them are positive and the other two are negative. Fixing up $p_1(\ell)$ we then define $p_2(\ell)$ to be such a quasi-momentum for which $-i \log\det \mathrm{T}_1 = p_1(\ell) + p_2(\ell)$ is non-zero at leading order in $1/(1-x)$ expansion. In the next section we will check equation (6.14) for the first two orders in the perturbative expansion.

7. Monodromy vs. Bäcklund for rigid strings

Here we would like to check the basic formula (6.14) by explicitly comparing the few leading charges arising in the expansion of the generating function for Bäcklund charges $\mathbf{Q}(x)$, with that of the quasi-momentum $p(\ell)$. In general, finding the higher charges from the mondoromy is rather involved; however we make progress by computing their values on certain string configurations. In particular, the rigid string solutions [5, 6] provide an excellent tool for probing the higher hidden charges [26].

We choose to work with a solution which describes a rigid string with a circular profile, and carrying two non-vanishing spins in the five-sphere. This solution can be conveniently written in terms of the standard Jacobi elliptic functions as follows [5]

$$\begin{aligned} q^1 + iq^2 &= \mathrm{sn}(a\sigma, \mathrm{t}) \exp(iw_1\tau)\,, \\ q^3 + iq^4 &= \mathrm{cn}(a\sigma, \mathrm{t}) \exp(iw_2\tau)\,, \\ q^5 &= q^6 = 0\,. \end{aligned} \tag{7.1}$$

Here $w_{12}^2 = w_1^2 - w_2^2$ is related to the elliptic modulus t through the closed string periodicity condition:

$$a \equiv \sqrt{\frac{w_{12}^2}{\mathrm{t}}} = \frac{2}{\pi}\mathrm{K}(\mathrm{t})\,, \tag{7.2}$$

where K is the complete elliptic integral of the first kind. The modulus t can be further expressed via the S^5 spins but we do not need this here.

The generating function for the Bäcklund charges $\mathcal{E}(\gamma)$ on rigid string solutions was obtained in [26]. In this work another spectral parameter denoted by γ has been used (not to be confused with our definition of the world-sheet metric). It is related to x we use here by

$$\gamma^2 = \frac{1-x}{1+x}\,. \tag{7.3}$$

The generating function $\mathbf{Q}(x)$ is obtained from the generating function $\mathcal{E}(\gamma)$ of [26] by multiplying it with a certain factor, namely,

$$\mathbf{Q}(x) = -4(1-x)^{-\frac{3}{2}}\sqrt{1+x}\;\mathcal{E}(\gamma)\,, \tag{7.4}$$

where the spectral parameter γ is related to x through (7.3). Using these relations we then extract from the results of [26] the following asymptotics

$$\mathbf{Q}(x) \overset{x\to 1}{=} \frac{2\sqrt{2}}{\sqrt{1-x}}Q_{-1} + \frac{1}{\sqrt{2}}Q_1\sqrt{1-x} + \cdots\,, \tag{7.5}$$

where all the coefficients Q_k are functions of K and E — the elliptic integrals of the first and second kind respectively. In particular,

$$Q_{-1} = \mathcal{E}\,, \tag{7.6}$$

$$Q_1 = \mathcal{E} - \frac{32}{\pi^2\mathcal{E}}\mathrm{K}(t)\mathrm{E}(t) - \frac{64(t-1)}{\pi^4\mathcal{E}^3}\mathrm{K}(t)^4\,. \tag{7.7}$$

Here $\mathcal{E} = \sqrt{\frac{1}{t}(w_1^2 + (t-1)w_2^2)}$ is the space-time energy of the string.

JHEP07(2005)026

Now let us consider the monodromy (6.8) of the fundamental linear problem related to the solution (7.1). Due to $q_5 = q_6 = 0$ the $\mathfrak{su}(4)$ Lax connection $\mathscr{L}$ can be split by an appropriate (constant) similarity transformation into two independent $\mathfrak{su}(2)$ connections $\mathscr{L}^{\pm}$. Moreover, the time dependence of the latter is trivially factored out as

$$\mathscr{L}_\alpha \to \begin{pmatrix} \mathscr{R}_+ \mathscr{L}_\alpha^+ \mathscr{R}_+^\dagger & 0 \\ 0 & \mathscr{R}_- \mathscr{L}_\alpha^- \mathscr{R}_-^\dagger \end{pmatrix} . \tag{7.8}$$

Here

$$\mathscr{R}_\pm = \begin{pmatrix} e^{i\frac{w_1 \pm w_2}{2}\tau} & 0 \\ 0 & e^{-i\frac{w_1 \pm w_2}{2}\tau} \end{pmatrix} \tag{7.9}$$

and the $\mathfrak{su}(2)$ matrices $\mathscr{L}_\alpha^\pm$ are time-independent. In particular, the σ-components of $\mathscr{L}^\pm$ read

$$\mathscr{L}_\sigma^+(\ell) = \frac{1}{1-\ell^2}\begin{pmatrix} i\ell(w_1 \operatorname{sn}^2 a\sigma - w_2 \operatorname{cn}^2 a\sigma) & -a \operatorname{dn} a\sigma - i\ell(w_1{+}w_2) \operatorname{sn} a\sigma \operatorname{cn} a\sigma \\ a \operatorname{dn} a\sigma - i\ell(w_1{+}w_2) \operatorname{sn} a\sigma \operatorname{cn} a\sigma & -i\ell(w_1 \operatorname{sn}^2 a\sigma - w_2 \operatorname{cn}^2 a\sigma) \end{pmatrix},$$
$$\mathscr{L}_\sigma^-(\ell) = \frac{1}{1-\ell^2}\begin{pmatrix} i\ell(w_1 \operatorname{sn}^2 a\sigma + w_2 \operatorname{cn}^2 a\sigma) & -a \operatorname{dn} a\sigma - i\ell(w_1{-}w_2) \operatorname{sn} a\sigma \operatorname{cn} a\sigma \\ a \operatorname{dn} a\sigma - i\ell(w_1{-}w_2) \operatorname{sn} a\sigma \operatorname{cn} a\sigma & -i\ell(w_1 \operatorname{sn}^2 a\sigma + w_2 \operatorname{cn}^2 a\sigma) \end{pmatrix}.$$

Clearly they just differ by the substitution $w_2 \to -w_2$. These connections are used to construct the corresponding monodromies

$$\mathrm{T}_\pm(\ell) = \overleftarrow{\exp} \int_0^{2\pi} \mathrm{d}\sigma \mathscr{L}_\sigma^\pm(\ell) . \tag{7.10}$$

Furthermore we define the following σ-dependent matrices

$$\mathrm{T}_\pm(\sigma) = \overleftarrow{\exp} \int_0^{\sigma} \mathrm{d}\sigma \mathscr{L}_\sigma^\pm(\ell) \tag{7.11}$$

which are solutions to the differential equations

$$\partial_\sigma \mathrm{T}_\pm = \mathscr{L}_\sigma^\pm \mathrm{T}_\pm . \tag{7.12}$$

In what follows we will discuss $\mathrm{T} \equiv \mathrm{T}_-$, the results for T_+ are obtained by the substitution $w_2 \to -w_2$.

Let us represent $\mathrm{T}(\sigma)$ as

$$\mathrm{T}(\sigma) = g(\sigma)\mathscr{D}(\sigma)g^{-1}(0) , \qquad \mathscr{D}(\sigma) = \exp(id(\sigma)\sigma_3) , \tag{7.13}$$

where $g(\sigma)$ is a *periodic unitary* gauge transformation.[10] Thus, the trace of the monodromy satisfies $\operatorname{Tr} \mathrm{T}(2\pi) = 2\cos d(2\pi)$ which implies that the quasi-momentum $p(\ell) = d(2\pi)$ is

$$p(\ell) = \frac{1}{2} \arccos \operatorname{Tr} \mathrm{T}(2\pi) = \int_0^{2\pi} \mathrm{d}\sigma \partial_\sigma d(\sigma) . \tag{7.14}$$

[10] The procedure we use here is equivalent to diagonalizing the Lax connection around one of its poles by an appropriate regular unitary gauge transformation.

Here the last formula is a consequence of $d(0) = 0$. Introducing the parametrization

$$g = \frac{1}{\sqrt{1+\rho\bar{\rho}}} \begin{pmatrix} 1 & \rho \\ -\bar{\rho} & 1 \end{pmatrix}, \qquad \mathscr{L}_\sigma = \begin{pmatrix} iu & v \\ -\bar{v} & -iu \end{pmatrix} \tag{7.15}$$

one finds that the differential equation for $\mathrm{T}(\sigma)$ boils down to the following system

$$\partial_\sigma \rho = v + 2iu\rho + \bar{v}\rho^2 \,, \tag{7.16}$$

$$\partial_\sigma d = u + \frac{1}{2i}(\rho\bar{v} - v\bar{\rho}) \,. \tag{7.17}$$

In particular, the first equation is of the (scalar!) Riccati type. Then the generating function for the string integrals of motion is given by

$$p(\ell) = \int_0^{2\pi} d\sigma \Big[u + \frac{1}{2i}(\rho\bar{v} - v\bar{\rho}) \Big] \,. \tag{7.18}$$

To solve equation (7.16) we assume the expansion

$$\rho = \rho_0 + (1-\ell)\rho_1 + \cdots \tag{7.19}$$

around the pole of $\mathscr{L}$ at $\ell = 1$. In particular, the solution for ρ_0 is

$$\rho_0 = -i \frac{w_2 + \mathcal{E} + (w_1 - w_2)\,\mathrm{sn}^2(a\sigma, t)}{a\,\mathrm{dn}^2(a\sigma, t) - i(w_1 - w_2)\,\mathrm{sn}(a\sigma, t)\,\mathrm{cn}(a\sigma, t)} \,. \tag{7.20}$$

Around $\ell = 1$ the quasi-momentum is expanded as

$$\frac{p(\ell)}{\pi} = \frac{1}{\ell - 1} p_{-1} + p_0 + (\ell - 1)p_1 + \cdots \,. \tag{7.21}$$

Performing rather involved integrations we have found a few leading charges. They are

$$\begin{aligned} p_{-1} &= \mathcal{E} \,, \\ p_0 &= \frac{1}{2}\mathcal{E} + \frac{1}{\mathcal{E}}(\mathcal{E} - w_1)(\mathcal{E} - w_2) - \frac{2(\mathcal{E} - w_1)}{\mathrm{K}(t)} \Pi\left(\frac{w_2 - w_1}{w_2 - \mathcal{E}}, t \right) , \\ p_1 &= -\frac{1}{4}\mathcal{E} + \frac{4}{\pi^2\mathcal{E}}\mathrm{K}(t)\mathrm{E}(t) + \frac{8(t-1)}{\pi^4\mathcal{E}^3}\mathrm{K}(t)^4 \,, \\ p_2 &= \frac{1}{8}\mathcal{E} - \frac{2}{\pi^2\mathcal{E}}\mathrm{K}(t)\mathrm{E}(t) - \frac{4(t-1)}{\pi^4\mathcal{E}^3}\mathrm{K}(t)^4 - \\ &\quad - \frac{w_1 w_2}{\mathcal{E}^2}\Big[\frac{2}{\pi^2\mathcal{E}}\mathrm{K}(t)\mathrm{E}(t) + \frac{8(t-1)}{\pi^4\mathcal{E}^3}\mathrm{K}(t)^4 \Big] \,. \end{aligned} \tag{7.22}$$

Here the frequencies $w_{1,2}$ are expressed via the space-time energy of the string

$$w_1 = \frac{1}{\pi}\sqrt{\pi^2\mathcal{E}^2 + 4(t-1)\mathrm{K}(t)^2} \,, \qquad w_2 = \frac{1}{\pi}\sqrt{\pi^2\mathcal{E}^2 - 4\mathrm{K}(t)^2} \tag{7.23}$$

and Π stands for the standard elliptic integral of the third kind.

Some comments are in order. The charge p_{-1} is independent of w_2 and therefore it is the same for both monodromies $\mathrm{T}_\pm$. This means that the quasi-momenta exhibit a degenerate behavior at leading order in the $1/(\ell-1)$ expansion [18]. Furthermore we note that the charge p_0 is rather distinguished from the rest as it is the only one which contains the elliptic integral of the third kind. Most importantly, this charge is not invariant under $w_2 \to -w_2$. Therefore, the degeneracy of the quasi-momenta observed at leading order gets removed. This is an important fact because it allows one to treat the quasi-momenta $p_k(\ell)$ with the corresponding pole part subtracted as analytic functions associated to different sheets of a unique Riemann surface [7]. To observe the splitting of the eigenvalues of the monodromy, unitarity of the gauge transformation diagonalizing the $\mathscr{L}$-operator around its pole is essential.

Finally, we compute

$$i\left(\frac{p(\ell)}{\pi}-\frac{p(\bar{\ell})}{\pi}\right)=\frac{\sqrt{2}}{\sqrt{1-x}}p_{-1}+\frac{1}{2\sqrt{2}}(-p_{-1}-8p_1)\sqrt{1-x}+\cdots .$$

Note that the charge p_0 does not appear in this expansion. Since p_{-1} and p_1 are independent of w_2 the contribution of the quasi-momenta associated to T_+ will be the same at this order. Therefore,

$$\frac{i}{\pi}\left(p_1(\ell)+p_2(\ell)-p_1(\bar{\ell})-p_2(\bar{\ell})\right)=\frac{2\sqrt{2}}{\sqrt{1-x}}p_{-1}+\frac{1}{\sqrt{2}}(-p_{-1}-8p_1)\sqrt{1-x}+\cdots . \tag{7.24}$$

Now substituting here the expressions (7.22) for the $p's$ we observe that equation (7.24) perfectly reproduces the first two terms in the expansion (7.5)! This provides another non-trivial check of our basic formula (6.14).

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A. Perturbative solution of the Bäcklund equations

Here we discuss the perturbative solutions of the Bäcklund equations which allow one to determine the *local* conservation laws of the model. Our treatment can be viewed as the matrix generalization of the vector approach of [23]. We also assume $\lambda = \pm 1$ so that the Bäcklund solutions satisfy the Virasoro constraints.

To start, we write the matrix U in the form

$$U=\frac{x}{\lambda}\mathbb{I}+\mathbf{P}\,, \tag{A.1}$$

where $\mathbf{P}$ is anti-hermitian, $\mathbf{P}^\dagger+\mathbf{P}=0$, and obeys the condition

$$\mathbf{P}^\dagger\mathbf{P}=\mathbb{I}-\left(\frac{x}{\lambda}\right)^2=-\mathbf{P}^2\,. \tag{A.2}$$

Thus, the matrix $\mathbf{P}$ has degenerate eigenvalues which are $\pm i\sqrt{1-\frac{x^2}{\lambda^2}}$. In terms of $\mathbf{P}$ the Riccati equation acquires the form

$$\begin{aligned}\kappa\mathbf{P}_\alpha &= \frac{2x}{\lambda}(x^2-\lambda^2)A_\alpha+(1+\lambda^2-2x^2)[A_\alpha,\mathbf{P}]-2x\lambda\mathbf{P}A_\alpha\mathbf{P}+\\ &+\epsilon_\alpha{}^\beta\left(\frac{1}{\lambda}(x^2-\lambda^2)(1+\lambda^2)A_\beta+x(1-\lambda^2)[A_\beta,\mathbf{P}]-\lambda(1+\lambda^2)\mathbf{P}A_\beta\mathbf{P}\right).\end{aligned}$$

Equivalently this can be cast into the form

$$\mathbf{P}_\alpha=\frac{1}{1+\lambda^2}\left[\mathscr{L}_\alpha-x\epsilon_\alpha{}^\beta\mathscr{L}_\beta-\frac{1}{2}\lambda\epsilon_\alpha{}^\beta[\mathbf{P},\mathscr{L}_\beta],\mathbf{P}\right], \tag{A.3}$$

where we made use of equation (A.2). One can easily check that this equation is compatible with the symmetry properties of $\mathbf{P}$. In fact, it can be viewed as the matrix differential equation for an element $\mathbf{P}=\tilde{q}^iq^j\Gamma_{ij}$ of $\mathfrak{su}(4)$.

From equation (A.1) we see that for $\lambda=1$ the matrix $U\to\mathbb{I}$ when $x\to1$ while for $\lambda=-1$ we will have the same asymptotic behavior provided $x\to-1$. This shows that for $\lambda=1$ the Riccati equation will have a well-defined perturbative expansion around $x=1$ and for $\lambda=-1$ the expansion must be concentrated around $x=-1$.

Let us define $\zeta=\pm\sqrt{1-x^2}$ where " $+$ " is for $\lambda=1$ and " $-$ " is for $\lambda=-1$. It is convenient to introduce a rescaled matrix $\mathscr{P}$:

$$\mathscr{P}=\frac{\mathbf{P}}{\zeta}=\sum_{n=0}^{\infty}\zeta^n\mathscr{P}_n \tag{A.4}$$

obeying the condition $\mathscr{P}^\dagger\mathscr{P}=\mathbb{I}$. For $\lambda=\pm1$ the Riccati equation boils down to

$$\zeta(-2\mathscr{P}_\alpha+[A_\alpha,\mathscr{P}])=\pm\sqrt{1-\zeta^2}A_\alpha+\epsilon_\alpha{}^\beta A_\beta+\mathscr{P}\left(\pm\sqrt{1-\zeta^2}A_\alpha+\epsilon_\alpha{}^\beta A_\beta\right)\mathscr{P}. \tag{A.5}$$

Now one can realize that upon substituting here the expansion (A.4) we will get recurrent relations which would allow us to solve for $\mathscr{P}_n$ in terms of lower coefficients $\mathscr{P}_k$, $k<n$. Most importantly, we see that in the perturbative treatment the original differential problem was replaced by an algebraic one. Thus, finding solutions does not involve integration and, as a consequence, the solution appears to be a local function of the fields and their derivatives.

As an example, let us find explicitly the leading term $\mathscr{P}_0$. We get

$$[A_\alpha\pm\epsilon_\alpha{}^\beta A_\beta,\mathscr{P}_0]=0\,, \tag{A.6}$$

together with the condition

$$\mathscr{P}_0^2=-\mathbb{I} \tag{A.7}$$

that follows from the linearized expression (A.2). Recall that the reference gauge connection A_α equals

$$A_\alpha=g_\alpha g^{-1}=q^i_\alpha q^j\Gamma_{ij}\,. \tag{A.8}$$

One can show that equations (A.6), (A.7) have the following solutions

$$\mathscr{P}_0 = \frac{2}{\|q^{\pm}_{\alpha}\|}(q^i_\alpha \pm \epsilon_\alpha{}^\beta q^i_\beta) q^j \Gamma_{ij} \qquad \text{or} \qquad \mathscr{P}_0 = -\frac{2}{\|q^{\pm}_{\alpha}\|}(q^i_\alpha \pm \epsilon_\alpha{}^\beta q^i_\beta) q^j \Gamma_{ij}\,, \tag{A.9}$$

where there is no summation over the index α. Here we use the notation

$$q^{\pm}_{\alpha} = \left(\delta_\alpha{}^\beta \pm \epsilon_\alpha{}^\beta\right) q_\beta\,, \qquad \|q^{\pm}_{\alpha}\|^2 \equiv q^{\pm}_{\alpha} \cdot q^{\pm}_{\alpha}\,. \tag{A.10}$$

Since

$$q_{\pm\sigma} = \pm\frac{1 \pm \gamma_{\tau\sigma}}{\gamma_{\tau\tau}} q_{\pm\tau} \tag{A.11}$$

one finds that

$$\frac{q^{\pm}_{\tau}}{\|q^{\pm}_{\tau}\|} = \frac{q^{\pm}_{\sigma}}{\|q^{\pm}_{\sigma}\|} = \frac{q^{\pm\tau}}{\|q^{\pm\tau}\|} = \frac{q^{\pm\sigma}}{\|q^{\pm\sigma}\|}\,. \tag{A.12}$$

Now using the properties (A.12) we can see that (A.9) manifestly satisfies the relations (A.6) and (A.7).

B. Coset model

Introduce the following six unitary 4×4 matrices

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_6 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

These matrices satisfy the algebra

$$\Gamma_i \Gamma_j^\dagger + \Gamma_j \Gamma_i^\dagger = 2\delta_{ij}\,.$$

An embedding g of a coset element describing the five-sphere into SU(4) is conveniently described as

$$g = q_i \Gamma_i\,, \tag{B.1}$$

where the coordinates q_i satisfy the constraint $q_i q_i = 1$. Note that the six Γ-matrices above are antisymmetric. Therefore, the coset element is obtained by intersecting the unitarity condition $g^\dagger g = 1$ with the requirement $g^t = -g$.

Let us define $\Gamma_{ij} = \frac{1}{2}(\Gamma_i \Gamma_j^\dagger - \Gamma_j \Gamma_i^\dagger)$. The matrices Γ_{ij} obey the following algebra

$$\{\Gamma_{ij}, \Gamma_{kl}\} = 2\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right), \tag{B.2}$$
$$[\Gamma_{ij}, \Gamma_{kl}] = 2\left(\delta_{jk}\Gamma_{il} + \delta_{il}\Gamma_{jk} - \delta_{ik}\Gamma_{jl} - \delta_{jl}\Gamma_{ik}\right), \tag{B.3}$$

JHEP07(2005)026

i.e. Γ_{ij} generate the $\mathfrak{su}(4)$ algebra. Another way to represent the $\mathfrak{su}(4)$ generators is to use $\bar{\Gamma}_{ij} = \frac{1}{2}(\Gamma_i^\dagger \Gamma_j - \Gamma_j^\dagger \Gamma_i)$. One can easily see that $\bar{\Gamma}_{ij} = (\Gamma_{ij})^*$ corresponds to the anti-chiral representation of $\mathfrak{su}(4)$.

The element U is then

$$U = \frac{1}{2}\tilde{q}_i q_j \big(\Gamma_i \Gamma_j^\dagger + \Gamma_j \Gamma_i^\dagger\big) + \frac{1}{2}\tilde{q}_i q_j \big(\Gamma_i \Gamma_j^\dagger - \Gamma_j \Gamma_i^\dagger\big) = (\tilde{q}q)\mathbb{I} + \tilde{q}_i q_j \Gamma_{ij}\,, \tag{B.4}$$

where we have introduced $\Gamma_{ij} = \frac{1}{2}(\Gamma_i \Gamma_j^\dagger - \Gamma_j \Gamma_i^\dagger)$. If we define $\chi = \lambda U$, then equation (3.2) acquires the form

$$\lambda U + \bar{\lambda} U^\dagger = 2x\mathbb{I}\,. \tag{B.5}$$

Upon substitution of the coset element U equation (B.5) reduces to

$$\begin{aligned} (\lambda + \bar{\lambda})(\tilde{q}q) &= 2x \\ \lambda \Gamma_{ij} + \bar{\lambda}\Gamma_{ij}^\dagger &= 0\,, \qquad i \neq j\,. \end{aligned}$$

Since the matrices Γ_{ij} are anti-hermitian we have to require that $\lambda = \bar{\lambda}$ and, therefore, for the scalar product $(\tilde{q}q)$ we obtain $(\tilde{q}q) = \frac{x}{\lambda}$.

For completeness we also provide a similar representation for $\mathfrak{su}(2,2)$ and describe the AdS sector of the model. Consider the following metric

$$\eta_{ij} = \mathrm{diag}(1,1,1,1,-1,-1)$$

and the matrix E:

$$\mathrm{E} = \mathrm{diag}(1,1,-1,-1)\,. \tag{B.6}$$

Once again we introduce the six 4×4 matrices

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \Gamma_6 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

These matrices obey the following algebra

$$\Gamma^i \mathrm{E} \Gamma^{j\dagger} + \Gamma^j \mathrm{E} \Gamma^{i\dagger} = -2\eta^{ij}\mathrm{E}\,. \tag{B.7}$$

If we introduce $g = p^i \Gamma^i$ we then see that this element satisfies

$$g^\dagger \mathrm{E} g = \mathrm{E}\,, \tag{B.8}$$

provided p^i obey $\eta_{ij}p^ip^j = -1$. Equation (B.8) defines the group SU(2,2). We further introduce

$$\Gamma_{ij} = \frac{1}{2}(\Gamma_i \mathrm{E}\Gamma_j^\dagger \mathrm{E} - \Gamma_j \mathrm{E}\Gamma_i^\dagger \mathrm{E}), \qquad \Gamma_{ij}^\dagger = -\mathrm{E}\Gamma_{ij}\mathrm{E}. \tag{B.9}$$

These matrices obey the commutation relations of the $\mathfrak{su}(2,2)$ algebra

$$[\Gamma_{ij}, \Gamma_{kl}] = 2(\eta_{ik}\Gamma_{jl} - \eta_{jk}\Gamma_{jl} + \eta_{jl}\Gamma_{ik} - \eta_{il}\Gamma_{jk}) \tag{B.10}$$

and also

$$\{\Gamma_{ij}, \Gamma_{kl}\} = 2(\eta_{il}\eta_{jk} - \eta_{ik}\eta_{jl}). \tag{B.11}$$

The AdS element $U = \tilde{g}g^{-1}$ is then represented as

$$U = -(\tilde{p}p) + \tilde{p}^i p^j \Gamma_{ij}, \tag{B.12}$$

where $(\tilde{p}p) = \eta_{ij}p^ip^j$. It is easy to see that the AdS analogue of equation (B.5) is

$$\lambda \mathrm{E} U + \bar{\lambda} U^\dagger \mathrm{E} = 2x\mathrm{E}. \tag{B.13}$$

This equation is compatible with the coset element (B.12) provided λ is real and $(\tilde{p}p) = -\frac{x}{\lambda}$. Further we note that equation (B.13) can be rewritten as

$$U + U^{-1} = 2\frac{x}{\lambda}\mathbb{I}. \tag{B.14}$$

Thus, being written in terms of U and U^{-1} equations (3.4) and (B.14) look the same and, therefore, lead to the same form of the Riccati equation (3.10).

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