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Non-Perturbative Contributions in the Plane-Wave/BMN Limit

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Abstract: This talk surveys recent work on the contribution of instantons to the anomalous dimensions of BMN operators in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and the corresponding non-perturbative contributions to the mass-matrix of excited string states in maximally supersymmetric plane-wave string theory. The dependence on the coupling constants and the impurity mode numbers in the gauge theory and string theory are in striking agreement.
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Contents

1. Introduction

The conjectured correspondence between the BMN sector of $\mathcal{N} = 4, d = 4$ supersymmetric Yang-Mills and type IIB string theory in the maximally supersymmetric plane-wave background has been examined in some detail at the perturbative level. However, the understanding of non-perturbative aspects of the correspondence has been very limited. Such non-perturbative effects are well-studied in the context of the AdS/CFT correspondence where Yang–Mills instanton effects in $\mathcal{N} = 4$ supersymmetric Yang-Mills correspond closely to D-instanton effects in type IIB superstring theory in $AdS_5 \times S^5$. A natural question to ask is whether there is a similar relationship between non-perturbative effects in plane-wave string theory and the BMN limit of the gauge theory.

The correspondence relates the plane-wave string mass spectrum to the spectrum of scaling dimensions of gauge theory operators in the so called BMN sector of $\mathcal{N}$=4 SYM. This consists of gauge invariant operators of large conformal dimension, Δ, and large charge, J, with respect to a U(1) subgroup of the SU(4) R-symmetry group. The duality involves the double limit $\Delta \to \infty$, $J \to \infty$, while $\Delta - J$ is kept finite and related to the string theory hamiltonian by

$$\Delta - J = \frac{1}{\mu} H^{(2)} , \tag{1.1}$$

The background value of the Ramond–Ramond ($R - R$) five-form flux, μ, is related to the mass parameter, m, which appears in the light cone string action by $m = \mu p_- \alpha'$, where p_- is a component of light cone momentum. The two-particle hamiltonian is the sum of two pieces

$$H^{(2)} = H^{(2)}_{\rm pert} + H^{(2)}_{\rm nonpert} . \tag{1.2}$$

The perturbative part, $H^{(2)}_{\rm pert}$, is a power series in the string coupling, g_s, while $H^{(2)}_{\rm nonpert}$ is the non-perturbative part, which is suppressed by powers of e^{-1/g_s}.

The correspondence between the spectra of the two theories is the statement that the eigenvalues of the operators on the two sides of the equality (1.1) coincide. A quantitative comparison is possible if one considers the large N limit in the gauge theory focusing on operators in the BMN sector. As a result of combining the large N limit with the limit of large Δ and J, new effective parameters arise, which are related to the ordinary 't Hooft parameters, λ and $1/N$, by a rescaling,

$$\lambda' = \frac{g_{\rm YM}^2 N}{J^2}\,, \qquad g_2 = \frac{J^2}{N}\,. \tag{1.3}$$

The correspondence relates these effective gauge theory couplings to string theory parameters in the plane-wave background,

$$m^2 = (\mu p_- \alpha')^2 = \frac{1}{\lambda'}\,, \qquad 4\pi g_{\rm s} m^2 = g_2\,. \tag{1.4}$$

The double scaling limit, $N \to \infty$, $J \to \infty$, with J^2/N fixed, connects the weak coupling regime of the gauge theory to string theory at small g_s and large m.

Perturbative contributions to the mass spectrum have been analysed in some detail on both the string side and compared with corresponding contributions to the anomalous dimensions of BMN operators in the gauge theory. However, there have been no calculations of non-perturbative corrections due to D-instanton effects, which contribute to $H^{(2)}_{\rm nonpert}$ or of the corresponding Yang–Mills instanton contributions in the BMN limit. Indeed, it is not at all obvious at first sight that Yang–Mills instantons survive the BMN limit, but the correspondence with string theory D-instantons implies that they must. This talk, which is necessarily brief, reviews the contents of [1, 2] that study the BMN/plane-wave correspondence at the non-perturbative level and details can be found in these papers[1]. The next section summarizes the results of [1] on plane-wave string theory while the gauge theory results of [2] are summarized in section 3. The agreement between the dependence of the instanton contributions on the two sides of the correspondence is impressive. Further details concerning states with fermionic impurities are in a forthcoming publication [3].

2. Mass-matrix elements in plane-wave string theory

In the maximally supersymmetric plane-wave background the five-form $R-R$ potential has a non-zero value that sets the scale for the masses of the supergravity fields and reduces the isometry of the background to SO(4)× SO(4). Light-cone gauge string theory in this background is a free world-sheet theory with eight massive world-sheet bosons, X^I, and eight massive world-sheet fermions, θ^A, and may be described by the world-sheet action

$$\mathcal{L} = \frac{1}{2}\left(\partial_+ X\, \partial_- X - m^2 X_I^2\right) + i\left(\theta^1 \partial_+ \theta^1 + \theta^2 \partial_- \theta^2 - 2m\, \theta^1\, \Pi\, \theta^2\right)\,, \tag{2.1}$$

where θ^1 and θ^2 are SO(8) Grassmann spinors and $\Pi = \gamma^1\gamma^2\gamma^3\gamma^4$ (where γ^I are SO(8) gamma matrices). In the quantum theory the zero modes in the eight transverse directions, X^I, define harmonic oscillators with strength proportional to m.

The classical supergravity states are obtained by applying the zero mode bosonic and fermionic creation operators to the ground state (the BMN vacuum). Excited string states

[1]No further bibliographic references will be given here but full bibliographies are contained in these publications.

are constructed as usual by applying higher mode creation operators to the zero mode states. A state constructed by applying p excited bosonic or fermionic creation operators is said to have p 'impurities', a terminology that makes contact with the corresponding operators in the gauge theory. Each oscillator can be in any excited state subject to the usual 'level-matching' restriction which means that there are, in general, $p-1$ independent mode numbers that enter into the definition of the p-impurity state. Some effort has been expended in constructing a three-string vertex, from which a certain amount of perturbative information concerning string two-point functions – or, equivalently, the mass matrix elements – beyond free string theory can be extracted. We are concerned with non-perturbative contributions to the hamiltonian due to D-instantons. The single D-instanton sector has a measure that is proportional to $e^{2i\pi\tau}$ where $\tau = \tau_1 + i\tau_2 \equiv C^{(0)} + ie^{-\phi}$ ($C^{(0)}$ is the $R-R$ pseudoscalar, ϕ is the dilaton and $g_s = e^{\phi}$). Although this is exponentially small, it is the leading contribution with the phase factor $e^{2\pi i C^{(0)}}$. It is therefore of interest to understand how the mass matrix is modified by these contributions. In the following we will outline the calculation of such D-instanton contributions to mass matrix elements, or two-point functions, to leading order in the string coupling.

A D-instanton with position x_0 is described, to lowest order in the string coupling, by world-sheet disks with Dirichlet boundaries fixed at x_0. The light-cone boundary state description of the D-instanton in plane-wave string theory, generalizes that of the Minkowski space theory. The D-instanton boundary state couples to single closed-string states and preserves eight kinematical and eight dynamical supersymmetries and is given (at a specific value of x_0^+) by

$$|B, \mathbf{x_0}\rangle = \mathcal{N}_{(0,0)} \exp\left(\sum_{k=1}^{\infty} \frac{1}{\omega_k} \alpha^I_{-k}\tilde{\alpha}^I_{-k} - i\eta S_{-k} M_k \tilde{S}_{-k}\right) |\mathbf{x_0}\,\rangle_0\,, \tag{2.2}$$

where α, $\tilde{\alpha}$, S and $\tilde{S}$ are the left and right-moving non-zero modes of the bosonic and fermionic coordinates, X^I, θ^1 and θ^2, and $|\mathbf{x_0}\,\rangle_0$ is the ground state of all the oscillators of non-zero mode number. The coordinate $\mathbf{x_0}^I$ is the eigenvalue of the position operator constructed from the zero-mode oscillators, $a^{\dagger\, I}$ and a^I. The quantity M_k is a matrix in spinor space and is a function of m that reduces to the unit matrix in the flat-space limit, and $\omega_k = \sqrt{m^2+k^2}$.

The leading contribution to the two-point function of string states in a D-instanton background comes from a disconnected world-sheet that is the product of two disks, with one closed-string state attached to each, and with Dirichlet boundary conditions. The two-boundary state is simply the product of two single-boundary states acting in distinct Fock spaces, $||B\,\mathbf{x_0}\rangle\rangle_2 = |B, \mathbf{x_0}\rangle \otimes |B, \mathbf{x_0}\rangle'$. The position coordinates x_0 are interpreted as bosonic moduli associated with broken translation invariance. Similarly, the eight broken kinematical supersymmetries, q^a ($a = 1, \ldots, 8$), and the eight broken dynamical supersymmetries, $Q^{\dot{a}}$ ($\dot{a} = 1, \ldots, 8$), lead to the presence of sixteen fermionic moduli, ϵ^a, $\eta^{\dot{a}}$. The dependence of the state on the fermionic moduli is included by summing over all possible ways of attaching sixteen fermionic open-string states to the boundaries of the two disks.

A dressed boundary state, including these supermoduli can be defined by

$$||V, x_0\rangle\rangle_2 = g_s^{7/2}\, e^{2\pi i\tau} \prod_{a=1}^{8} (\epsilon^a q^a)\, (\eta^a Q^a)\, ||B\, \mathbf{x_0}\rangle\rangle_2\, e^{ix_0^+(p_{1+}+p_{2+})}\, e^{ix_0^-(p_{1-}+p_{2-})}\,. \tag{2.3}$$

The factor of $g_s^{7/2}$ in the D-instanton measure can be extracted from previous work on D-instanton contributions in $AdS_5 \times S^5$ (and we are not keeping overall multiplicative constants). The on-shell two-point function between string states $|\chi_1\rangle$ and $\langle\chi_2|$ is given by the integrated matrix element

$$\int d^8\epsilon\, d^8\eta\, d^{10}x_0\, \langle\chi_2| \otimes \langle\chi_1||V, x_0\rangle\rangle_2\,. \tag{2.4}$$

Integration over the light-cone moduli, $x_0^\pm$, ensures the conservation of $p_\pm$ in any process while integration over the other supermoduli generate correlations between the two disks.

For a state with occupation numbers n_r (where r labels the oscillator levels) the light-cone energy is given by the nonlinear formula $p_+ = \sum_r \omega_r/2p_-$. It follows that conservation of p_+ implies that the number of impurities is preserved by this process, so that $|\chi_1\rangle$ and $\langle\chi_2|$ have the same number of impurities. Generally, conservation of p_+ imposes the even stronger condition that the non-zero mode numbers of oscillators in the incoming state coincide with those of the outgoing state. The nonlinear energy relation is seen on the gauge side after summing perturbative planar contributions to all orders in λ'. However, to leading order in the $1/m^2 \sim \lambda'$ expansion $\omega_{n_r} = m$ and conservation of p_+ imposes no relation between the mode numbers of the incoming and outgoing states. Therefore, since we are interested in comparing with perturbative gauge theory we need not impose the equality of incoming and outgoing oscillator mode numbers in the following.

Certain other general features of D-instanton dominated matrix elements follow from general properties of the boundary state (2.3). For example, the boundary state couples to arbitrary numbers of pairs of modes, where each pair consists of one left-moving mode with a mode-number n and a right-moving mode with the *same* mode number. This means that it only has non-zero coupling to states that are level-matched in this pairwise fashion – a feature that must therefore also be true on the gauge theory side although it will prove much harder to see this from a conventional Yang–Mills instanton calculation.

Examples of D-instanton contributions to matrix elements between states with various numbers of bosonic and fermionic impurities were considered in [1]. Those results that are particularly relevant for comparison with the gauge theory results of [2] are the following.

Two bosonic impurities

A level-matched state with two bosonic impurities is associated with a single mode number. The two-state bra vector in (2.4) is given by

$${}_1\langle\chi_1| \otimes {}_2\langle\chi_2| = \frac{1}{\omega_m\omega_n}\, t^{(1)}_{IJ}\, t^{(2)}_{KL}\, {}_h\langle 0|\alpha_m^{(1)I}\, \tilde\alpha_m^{(1)J} \otimes {}_h\langle 0|\alpha_n^{(2)K}\, \tilde\alpha_n^{(2)L}\,, \tag{2.5}$$

where the wave functions $t^{(1)}_{IJ}$ and $t^{(2)}_{IJ}$ are tensors of SO(4)× SO(4) (with indices that take the values $I, J, K, L = 1, \ldots, 8$). The ground state $|0\rangle_h$ denotes the BMN ground state, which is the state of lowest p_+. The leading semi-classical one D-instanton contribution to the two-string mass-matrix element is independent of the mode number and has the form (ignoring a constant overall factor)

$$\begin{aligned}\frac{1}{\mu} H^{(2)}_{\rm nonpert} &= e^{2\pi i\tau}\, g_s^{7/2}\, m^3\, t^{(1)}_{ij}\, t^{(2)}_{pq}\, (\delta^{pi}\delta^{qj} + \delta^{pj}\delta^{iq}) \\ &= e^{i\theta - \frac{8\pi^2}{g_{YM}}}\, \lambda'^2\, g_2^{7/2}\, t^{(1)}_{ij}\, t^{(2)}_{pq}\, (\delta^{pi}\delta^{qj} + \delta^{pj}\delta^{iq})\,, \end{aligned} \tag{2.6}$$

in the large-m limit (where $m = \mu\, p_-\, \alpha'$). We have indicated the expression in terms of the gauge theory parameters in the second line for future reference. In this expression we have also specialized to vector indices i, j, p, q lying in one of the SO(4) factors of the SO(4)× SO(4) isometry group since this is the case that is easiest to calculate in the gauge theory. Although the exact string theory expression includes all non-leading terms, it is only the large-m limit that can be compared with the gauge theory calculations. Note, in particular, that this leading contribution is of order λ'^2 and is suppressed relative to potential $O(\lambda'^0)$ effects. This fact makes the Yang–Mills instanton contribution to the two impurity case more difficult to evaluate in precise detail than cases with higher numbers of impurities.

Four bosonic impurities

With four bosonic impurities there are three independent non-zero mode numbers for each external state after taking level matching into account. However, as we remarked above, the only non-zero matrix elements are those in which each α_n mode is accompanied by a $\tilde\alpha_n$ with the same mode number, n. In this case the bra state in (2.4) is given by

$$\begin{aligned} {}_1\langle\chi_1| \otimes {}_2\langle\chi_2| &= \frac{1}{\omega_{m_1}\omega_{m_2}\omega_{n_1}\omega_{n_2}}\, t^{(1)}_{j_1 j_2 j_3 j_4}\, t^{(2)}_{p_1 p_2 p_3 p_4} \\ {}_h\langle 0|\alpha^{(1)j_1}_{m_1}\, \tilde\alpha^{(1)j_2}_{m_1}\, \alpha^{(1)j_3}_{m_2}\, \tilde\alpha^{(1)j_4}_{m_2} &\otimes {}_h\langle 0|\alpha^{(2)p_1}_{n_1}\, \tilde\alpha^{(2)p_2}_{n_1}\, \alpha^{(2)p_3}_{n_2}\, \tilde\alpha^{(2)p_4}_{n_2}\,. \end{aligned} \tag{2.7}$$

The tensor wave functions $t_{p_1p_2p_3p_4}$ of the in and out states have again been restricted to have indices in a single SO(4) factor of the isometry group simply because that is the easiest case to consider in the dual gauge theory. In this case the mass-matrix is given, at leading order in powers of $1/m$, by

$$\begin{aligned}\frac{1}{\mu} H^{(2)}_{\rm nonpert} &= e^{2\pi i\tau}\, t^+\, t^-\, g_s^{7/2}\, m^7\, \frac{1}{m_1 m_2 n_1 n_2} \\ &= e^{i\theta - \frac{8\pi^2}{g_{YM}}}\, t^+\, t^-\, g_2^{7/2}\, \frac{1}{m_1 m_2 n_1 n_2}\,, \end{aligned} \tag{2.8}$$

where

$$t^{\pm} = t_{j_1 \tilde j_2 j_3 \tilde j_4}\, (\delta_{j_1 j_3}\delta_{j_2 j_4} - \delta_{j_1 j_4}\delta_{j_2 j_3} \pm \epsilon_{j_1 j_2 j_3 j_4})\,. \tag{2.9}$$

In this case the result is zeroth order in λ' perturbation theory. The expression (2.8) implies that to leading order in m only scalar states have an induced D-instanton coupling. The rest of the possible bosonic four-impurity states have couplings that are suppressed by powers of m compared to this leading result. Further details of these four-impurity matrix elements are given in [1].

3. Anomalous dimensions of BMN states in $\mathcal{N}=4$ Yang–Mills theory

We will now discuss semi-classical instanton contributions to the anomalous dimensions of BMN operators in $\mathcal{N}=4$ SU(N) Yang–Mills theory, which are extracted from two-point correlation functions.

Conformal invariance determines the form of two-point functions of primary operators, $\mathscr{O}$ and $\bar{\mathscr{O}}$, to be

$$\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = \frac{c}{(x_1-x_2)^{2\Delta}}\,, \tag{3.1}$$

where Δ is the scaling dimension. In general in the quantum theory Δ acquires an anomalous term, $\Delta(g_{\rm YM}) = \Delta_0 + \gamma(g_{\rm YM})$. At weak coupling the anomalous dimension $\gamma(g_{\rm YM})$ is small and substituting in (3.1) gives

$$\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = \frac{c\,\Lambda^{2\gamma(g_{\rm YM})}}{(x_1-x_2)^{2\Delta_0}}\left(1-\gamma(g_{\rm YM})\log\left[\Lambda^2(x_1-x_2)^2\right]+\cdots\right)\,, \tag{3.2}$$

where Λ is an arbitrary renormalisation scale. As a function of the coupling constant the anomalous dimension admits an expansion consisting of a perturbative series plus non-perturbative corrections. The generic two-point function at weak coupling takes the form

$$\begin{aligned}\langle \mathscr{O}(x_1)\bar{\mathscr{O}}(x_2)\rangle = {} & \frac{c(g_{\rm YM})}{(x_1-x_2)^{2\Delta_0}}\Big(1-g_{\rm YM}^2\gamma^{(1)}\log\left[\Lambda^2(x_1-x_2)^2\right] \\ & +\cdots - {\rm e}^{2\pi i\tau}\gamma^{({\rm inst})}\log\left[\Lambda^2(x_1-x_2)^2\right]+\cdots\Big)\,. \end{aligned}\tag{3.3}$$

Therefore perturbative and instanton contributions to the anomalous dimension are extracted from the coefficients of the logarithmically divergent terms in a two-point function. The general structure of these anomalous dimensions is an expansion of the form

$$\gamma(g_{\rm YM},\theta,N) = \sum_{n=1}^{\infty}\gamma_n^{\rm pert}(N)\,g_{\rm YM}^{2n} + \sum_{K>0}\sum_{m=0}^{\infty}\left[\gamma_m^{(K)}(N)\,g_{\rm YM}^{2m}\,{\rm e}^{2\pi i\tau K} + \text{ c.c.}\right]\,, \tag{3.4}$$

where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{\rm YM}^2}$. The double series in the second term in (3.4) contains the contributions of multi-instanton sectors as well as the perturbative fluctuations in each such sector. If the BMN sector of the gauge theory scales appropriately (3.4) becomes a series in the scaled couplings λ' and g_2, as we will see in the examples to be reviewed in this section.

The gauge invariant composite BMN operators are defined by normalized traces of products of scalar fields and are labelled by the total value of $\Delta - J$, the impurity number. The six scalar fields in the Yang–Mills multiplet are decomposed according to the value of J they carry: Z has $J=1$ or $\Delta-J=0$, $\bar{Z}$ has $J=-1$ or $\Delta+J=0$ and φ^i $(i=1,2,3,4)$ have $J=0$ or $\Delta\pm J=1$. Since we want to take the limit of large J for a fixed value of $\Delta-J$ a BMN operator has a large number of Z's and a finite number of other scalar impurities. Operators with $\Delta-J=0$ or 1 are protected and so their two-point functions do not receive instanton contributions. With two or more impurities the situation is more interesting although the analysis is quite complicated. We are considering BMN operators with k bosonic impurities that are linear combinations of colour traces of products of scalar fields of the form

$$\mathcal{O}^{i_1\ldots i_k}_{J;n_1\ldots n_k} = \frac{1}{\sqrt{J^{k-1}\left(\frac{g^2_{\rm YM}N}{8\pi^2}\right)^{J+k}}} \sum_{\substack{p_1,\ldots,p_{k-1}=0 \\ p_1+\cdots+p_{k-1}\leq J}}^{J} {\rm e}^{2\pi i[(n_1+\cdots+n_{k-1})p_1+(n_2+\cdots+n_{k-1})q_3+\cdots+n_{k-1}p_{k-1}]/J}$$

$$\times {\rm Tr}\left(Z^{J-(p_1+\cdots+p_{k-1})}\varphi^{i_1}Z^{p_1}\varphi^{i_2}\cdots Z^{p_{k-1}}\varphi^{i_k}\right) . \qquad (3.5)$$

The conjugate operator $\bar{\mathcal{O}}$ has a large number of $\bar{Z}$'s instead of Z's. For the two-point function (3.3) to be non-vanishing the operators must have equal and opposite values of J.

Instanton contributions to two-point correlation functions

In semi-classical approximation, correlation functions of composite operators are computed by replacing each field by the solution of its corresponding field equation in the presence of an instanton, expressed in terms of the fermionic and bosonic moduli. These moduli encode the broken superconformal symmetries together with the broken (super)symmetries associated with the orientation of a SU(2) instanton within SU(N). The symmetries are restored by integration over the supermoduli. For large N, the integration is carried out by a saddle point procedure. In the case of a two-point function of a generic local operator, $\mathcal{O}(x)$, and its conjugate the supermoduli integral takes the form

$$\langle\bar{\mathcal{O}}(x_1)\mathcal{O}(x_2)\rangle_{\rm inst} = \int {\rm d}\mu_{\rm inst}(\mathscr{m}_{\rm b},\mathscr{m}_{\rm f})\,{\rm e}^{-S_{\rm inst}}\,\hat{\bar{\mathcal{O}}}(x_1;\mathscr{m}_{\rm b},\mathscr{m}_{\rm f})\hat{\mathcal{O}}(x_2;\mathscr{m}_{\rm b},\mathscr{m}_{\rm f})\,, \qquad (3.6)$$

where we have denoted the bosonic and fermionic collective coordinates by $\mathscr{m}_{\rm b}$ and $\mathscr{m}_{\rm f}$ respectively. In (3.6) ${\rm d}\mu_{\rm inst}(\mathscr{m}_{\rm b},\mathscr{m}_{\rm f})$ is the integration measure on the instanton moduli space, $S_{\rm inst}$ is the classical action evaluated on the instanton solution and $\hat{\mathcal{O}}$ and $\hat{\bar{\mathcal{O}}}$ denote the classical expressions for the operators $\mathcal{O}$ and $\bar{\mathcal{O}}$ computed in the instanton background.

A one-instanton configuration in SU(N) Yang–Mills theory is characterised by $4N$ bosonic moduli that can be identified with the size, ρ, and position, x_0, of the instanton as well as its global gauge orientation. The latter can be described by three angles identifying the iso-orientation of a SU(2) instanton and $4N$ additional constrained variables, $w_{u\dot{\alpha}}$ and $\bar{w}^{\dot{\alpha}u}$ (where $u=1,\ldots,N$ is a colour index), in the coset SU(N)/(SU($N-2$)×U(1)) describing the embedding of the SU(2) configuration into SU(N). In the one-instanton sector in the $\mathcal{N}$=4 theory there are additionally $8N$ fermionic collective coordinates corresponding to zero modes of the Dirac operator in the background of an instanton. They comprise the 16 moduli associated with Poincaré and special supersymmetries broken by the instanton and denoted respectively by η^A_α and $\bar{\xi}^{\dot{\alpha}A}$ (where A is an index in the fundamental of the SU(4) R-symmetry group) and $8N$ additional parameters, ν^A_u and $\bar{\nu}^{Au}$, which can be considered as the fermionic superpartners of the gauge orientation parameters. The sixteen superconformal moduli are exact, *i.e.* they enter the expectation values (3.6) only through the classical profiles of the operators. The other fermion modes, ν^A_u and $\bar{\nu}^{Au}$, appear explicitly in the integration measure via the classical action, $S_{\rm inst}$, and are therefore 'non-exact' moduli. This distinction plays a crucial rôle in the calculation of correlation functions. The

ν_u^A and $\bar{\nu}^{Au}$ modes satisfy the fermionic ADHM constraints

$$\bar{w}^{\dot{\alpha}u}\nu_u^A = 0\,, \quad \bar{\nu}^{Au}w_{u\dot{\alpha}} = 0\,, \tag{3.7}$$

which effectively reduce their number to $8(N-2)$. The manner in which these moduli enter into the expressions for the fields is determined by the solution of the field equations for $\mathcal{N}$=4 SYM theory in an instanton background. The solution for each field in the Yang–Mills suprermultiplet can be written as a sum of terms containing different numbers of fermionic zero modes. For the purpose of this talk let us note that a scalar field has the form

$$\Phi^{AB} = \sum_{\substack{n=0 \\ 4n+2\leq 8N}} \Phi^{(2+4n)AB}\,, \tag{3.8}$$

where the notation $\Phi^{(4n+2)AB}$ denotes a term in the solution for the field Φ containing a product of $4n+2$ fermion zero modes. The minimum number of fermionic moduli in a scalar field is therefore two, while the next term contains a product of six fermionic moduli and so on. It is understood that the number of superconformal modes in each field cannot exceed 16 and the remaining modes are of ν_u^A and $\bar{\nu}^{Au}$ type. Furthermore, terms with higher numbers of moduli are suppressed by powers of the coupling, so the leading contribution to the two-point function is that with the minimal number of moduli in each scalar field.

In order to evaluate the two-point function (3.6) the expressions for the fields in terms of moduli must be substituted into each composite operator and the resulting traces must then be evaluated. The actual integration over the large number of supermoduli is reasonably straightforward, but there are complicated combinatorics involved in distributing the moduli among the fields in the two operators, that we will now outline (and are discussed in detail in [2]).

The $J+k$ scalar fields in the operator $\mathscr{O}$ defined in (3.6) each contain at least two fermionic moduli, which may be chosen from the superconformal moduli, η and $\bar{\xi}$, or from the non-exact moduli, ν and $\bar{\nu}$. The sixteen fermionic superconformal moduli naturally arise in the combination

$$\zeta_\alpha^A(x) = \frac{1}{\sqrt{\rho}}\left[\rho\,\eta_\alpha^A - (x-x_0)_\mu \sigma^\mu_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha}A}\right]\,, \tag{3.9}$$

where $\zeta_\alpha^A(x)$ are eight position-dependent Grassmann variables. This means that there has to be a factor of $\prod_{A=1}^4(\zeta^A(x_1))^2$ in each operator in the two-point correlation function. In other words each of the two operators in the correlation function has to contain eight of the superconformal moduli. Taking their SU(4) quantum numbers into account only four of these can be soaked up by the Z fields and the rest have to be contained in the impurity fields, φ^{AB}. Once the sixteen superconformal moduli are distributed among some of the scalar fields the non-exact moduli are soaked up by the remaining (large number) of fields, which are mostly Z's.

The bosonic integrations over the position and size of the instanton are left as a last step. These integrals are logarithmically divergent, the coefficient of the logarithm corresponding to the contribution to the matrix of anomalous dimensions.

In [2] we considered the two impurity and four-impurity cases in detail. The results were as follows.

Two bosonic impurities

For the two impurity case there is a technical problem in carrying out a complete analysis. The point is that in order to soak up all sixteen of the fermionic supermoduli, at least one of the scalars in each operator has to soak up six fermionic moduli, rather than the minimum number of two. This means that the contribution is of higher order in λ' than a leading contribution would be, which is in line with the two-impurity result in plane-wave string theory described earlier. It is technically very complicated to derive the precise from of this six-fermion contribution, but this is needed to determine the J-dependence of the two-point function. Nevertheless, if we *assume* BMN scaling the analysis can be carried through sufficiently to argue that the result is in agreement with the string calculation. This follows since the dependence on $g_{\rm YM}$ and N can be determined without knowledge of the details of the six-fermion term, and this uniquely fixes the power of J needed for BMN scaling. This requirement, in turn, constrains the way in which the fermion zero modes can appear in the profile of the operator. Specifically, the two-point function can obey BMN scaling only if the distribution of the zero modes is such that the final result is independent of the single mode number entering the definition of the two impurity operators.

Since in this case the analysis is incomplete we will only state the final result here, but will give a more detailed description of our method in the four-impurity case. It is simplest to choose the two states to be in the representation **9** of $\mathrm{SO}(4)_R$, since this sector contains only one operator which cannot mix with any other. The result yfor the two-point function of this operator, assuming BMN scaling, has the form

$$G_{\mathbf{9}}(x_1,x_2) \sim \frac{g_{\rm YM}^4 J^3 \mathrm{e}^{2\pi i\tau}}{N^{3/2}} \frac{1}{(x_1-x_2)^{2(J+2)}}\, I\,, \tag{3.10}$$

where I is a logarithmically divergent integral over the bosonic moduli, which can be regulated by dimensional regularisation of the x_0 integral. The coefficient of this divergence gives the instanton induced anomalous dimension of $\mathscr{O}_{\mathbf{9}}^{\{13\}}$,

$$\gamma_{\mathbf{9}}^{\rm inst} \sim \frac{g_{\rm YM}^4 J^3}{N^{3/2}}\, \mathrm{e}^{-\frac{8\pi^2}{g_{\rm YM}^2}+i\theta} \sim (g_2)^{7/2} \left(\lambda'\right)^2\; \mathrm{e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}\,. \tag{3.11}$$

This is in agreement with the non-perturbative correction to the mass of the dual string state computed in [1]. In particular, the anomalous dimension (3.11) is independent of the parameter n corresponding to the mode number of the plane-wave string state. Apart from the exponential factor characteristic of instanton effects, (3.11) contains an additional factor of $(\lambda')^2$. This is due to the inclusion of six-fermion scalars which give rise to additional $(\bar{\nu}\nu)_{\mathbf{6}}$ bilinears, each of which brings one more power of $g_{\rm YM}$. As will be seen in the next subsection, in the case of four impurity $\mathrm{SO}(4)_R$ singlets it is sufficient to consider the solution for all the scalars that is bilinear in the fermions and as a consequence we shall find a leading contribution of order $(g_2)^{7/2}\mathrm{e}^{-8\pi^2/g_2\lambda'}$.

Four bosonic impurities

The calculation of two-point functions of four impurity operators is more involved than the corresponding calculation in the two impurity case from the point of view of the combinatorial analysis. However, at the four impurity level, in the case of $\mathrm{SO}(4)_R\times\mathrm{SO}(4)_C$ singlets, the calculation of the leading instanton contributions requires only the inclusion of the quadratic fermionic terms in the classical profiles of the scalar fields, which are known explicitly.

However, at the four impurity level, in the case of $\mathrm{SO}(4)_R\times\mathrm{SO}(4)_C$ singlets, the leading instanton contributions the classical profiles of the scalar fields involve only the quadratic fermionic terms and are known explicitly. Therefore, in this case the semi-classical contributions to the two-point functions can be analyzed more completely. The fact that non-zero correlation functions are obtained using the minimal number of fermion modes for each field also implies that in this case a contribution to the matrix of anomalous dimensions arises at leading order in λ'. The case in which the external state is an $\mathrm{SO}(4)\times\mathrm{SO}(4)$ singlet with four scalar impurities is the simplest to analyze and also corresponds to the states we discussed in the context of the plane-wave string theory. More explicitly, the operators to be considered are of the form

$$\mathscr{O}_{\mathbf{1};J;n_1,n_2,n_3} = \frac{\varepsilon_{ijkl}}{\sqrt{J^3\left(\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\right)^{J+4}}} \sum_{\substack{q,r,s=0\\ q+r+s\le J}}^{J} \mathrm{e}^{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3 s]/J} \times \mathrm{Tr}\left(Z^{J-(q+r+s)}\varphi^i Z^q \varphi^j Z^r \varphi^k Z^s \varphi^l\right) , \qquad (3.12)$$

which is dual to the scalar plane-wave string state $\varepsilon_{ijkl}\,\alpha^i_{-n_1}\alpha^j_{-n_2}\widetilde{\alpha}^k_{-n_3}\widetilde{\alpha}^l_{-(n_1+n_2-n_3)}|0\rangle_h$. The conjugate operator involves $\bar{Z}$'s instead of Z's.

As before, in considering the distribution of the fermionic moduli among the $J+4$ fields within a trace, half of the superconformal modes (i.e., eight) must be soaked up by each of the two operators in the two-point correlation function. Furthermore, at least four of these have to be soaked up by the impurity scalar fields since the quantum numbers of the Z's are such that they can soak up at most four of the superconformal modes. The number of possible ways of distributing each kind of fermionic modulus among the $J+4$ scalar fields is very large and we will not describe the combinatorics here. After summing this very large number of terms the resulting expression for the correlator is (omitting overall coefficients)

$$G_{\mathbf{1}}(x_1,x_2) = \frac{\mathrm{e}^{2\pi i\tau}}{J^3 2^J N^{7/2}} \int \frac{\mathrm{d}^4x_0\,\mathrm{d}\rho}{\rho^5} \frac{\rho^{J+8}}{[(x_1-x_0)^2+\rho^2]^{J+8}} \frac{\rho^{J+8}}{[(x_2-x_0)^2+\rho^2]^{J+8}} \times \int \prod_{A=1}^{4} \mathrm{d}^2\eta^A \mathrm{d}^2\bar{\xi}^A \prod_{B=1}^{4} \left[\left(\zeta^B\right)^2(x_1)\right]\left[\left(\zeta^B\right)^2(x_2)\right] \times \int \mathrm{d}^5\Omega\ \left(\Omega^{14}\right)^J \left(\Omega^{23}\right)^J\ K(n_1,n_2,n_3;J)K(m_1,m_2,m_3;J)\,, \qquad (3.13)$$

where Ω^{AB} are angular variables on the five-sphere that emerge from the integration over the ν and $\bar{\nu}$ moduli. The J and N dependence in the prefactor is obtained combining the normalisation of the operators, the contribution of the measure on the instanton moduli space and the factors of $g_{\rm YM}\sqrt{N}$ associated with the ν and $\bar{\nu}$ variables. The expression (3.13) contains integrations over the bosonic moduli, x_0 and ρ, the sixteen superconformal fermion modes and the five-sphere coordinates Ω^{AB}.

The dependence on the integers n_i, m_i, $i = 1,2,3$, dual to the mode numbers of the corresponding string states is contained in the functions $K(n_1,n_2,n_3;J)$ and $K(m_1,m_2,m_3;J)$. These are given by the sum of 35 terms, which are sums over integers q,r,s of phases $\exp\{2\pi i[(n_1+n_2+n_3)q+(n_2+n_3)r+n_3s]/J\}$ multiplying the multiplicity factors associated with the different distributions of $\widehat{Z}$'s in each case. There are very many contributions to each of these 35 terms and the sums over this very large number of phase factors lead to some very impressive cancellations of what would otherwise be large and unlikely looking expressions.

The final result is obtained after performing the bosonic integrals. At each step various powers of g_{YM}, N and J enter, and it appears rather miraculous that in the end they all combine into a function that depends only on g_2 and λ', in accord with the BMN scaling. We can indicate where these different powers of the couplings come from as follows,

$$\underbrace{\left(\frac{1}{\sqrt{J^3(g_{\rm YM}^2 N)^{J+4}}}\right)^2}_{\text{normalised op. profile}} \underbrace{\left(g_{\rm YM}\sqrt{N}\right)^{2J}}_{\nu,\,\bar{\nu}\text{ bilinears}} \underbrace{\frac{{\rm e}^{2\pi i\tau} g_{\rm YM}^8\sqrt{N}}{2^J}}_{\text{measure}} \underbrace{\frac{2^J}{J^2}}_{S^5\text{ integral}} \underbrace{\frac{1}{J^2}}_{x_0,\,\rho\text{ integrals}} \underbrace{(J^7)^2}_{\text{sums}}$$
$$\sim \frac{J^7}{N^{7/2}}\,{\rm e}^{2\pi i\tau} = (g_2)^{7/2}\,{\rm e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}\,. \tag{3.14}$$

The final result for the two-point function turns out to vanish unless the mode numbers of the operators are equal in pairs – just as in the string theory D-instanton calculation. The result is

$$G_1(x_1,x_2) = \frac{3^2(g_2)^{7/2}{\rm e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta}}{2^{41}\pi^{13/2}}\,\frac{1}{m_1m_2n_1n_2}\,\frac{1}{(x_{12}^2)^{J+4}}\,\log\left(\Lambda^2 x_{12}^2\right)\,, \tag{3.15}$$

where the scale Λ appears as a consequence of the $1/\epsilon$ divergence in the ρ, x_0 integration. The physical information contained in the two-point function is in the contribution to the matrix of anomalous dimensions which is read from the coefficient in (3.15) and does not depend on Λ. Unlike the two-point functions of two impurity operators (3.15) is independent of λ', apart from the dependence in the exponential instanton weight. The mode-number dependence in (3.15) is extremely simple, given the very large number of terms that had to be summed.

The calculation presented here is not sufficient to determine the actual instanton induced anomalous dimension of the operator $\mathscr{O}_{\mathbf{1}}$. This requires the diagonalisation of the matrix of anomalous dimensions of which we have not computed all the entries. Other entries are determined by the corresponding two-point functions whose calculation follows the same steps described here and results in expressions similar to (3.15). From this we can conclude that the behaviour of the leading instanton contribution to the anomalous dimensions of singlet operators is

$$\gamma_{\mathbf{1}}^{\rm inst} \sim (g_2)^{7/2} {\rm e}^{-\frac{8\pi^2}{g_2\lambda'}+i\theta} \frac{1}{m_1 m_2 n_1 n_2} \tag{3.16}$$

which is in agreement with the string result (2.8). It is worth stressing that the the condition of pairwise equality of mode numbers appears in a highly non-obvious manner in the gauge theory calculation, while it followed rather trivially from the form of the boundary state in the plane-wave string theory.

The Yang–Mills instanton contributions to other (non-singlet) four-impurity operators are suppressed by powers of λ', as in the two-impurity case. This is also in qualitative agreement with the string side of the correspondence. However, in order to evaluate the semi-classical profiles of the BMN operators we would again have to use the contribution to some of the scalar fields that contains a product of six fermionic moduli, which presents the same technical obstacle as in the two impurity case.

4. Other issues

The basic message is that we find striking agreement between instanton effects in the gauge theory and those calculated in the plane-wave string theory. We focused on operators with two and four scalar impurities since these are the easiest to calculate on the gauge side. The four impurity case, although more involved, is fully under control, whereas the two impurity case presents subtleties due to the fact the leading semi-classical approximation vanishes and the first non-zero contribution arises at higher order. Clearly it would be interesting but very challenging to generalize the present work from the one-instanton sector to multi-instanton sectors.

The structure of the string theory side of the calculation was much simpler than the gauge side. In fact, many properties of the Yang–Mills side would be very difficult to calculate without the insights provided by the string calculation. For example, one generic feature of the string calculation is that only states with an even number of non-zero mode insertions receive D-instanton corrections. Zero mode oscillators can appear in odd numbers with the condition that they be contracted into a SO(4)×SO(4) scalar between the incoming and outgoing states. Another peculiarity observed in the string theory calculation is that the D-instanton contribution to the masses of certain states with a large number of fermionic non-zero mode excitations involves large powers of the mass parameter m. These mass-matrix elements are ones that receive no perturbative contributions. When expressed in terms of gauge theory parameters this behaviour corresponds to large *inverse* powers of λ'. This is not pathological in the $\lambda' \to 0$ limit, because the inverse powers of λ' are accompanied by the instanton factor, $\exp\left(-8\pi^2/g_2\lambda'\right)$. From the point of view of the gauge theory this result is intriguing, not only because of the unusual coupling constant dependence that the anomalous dimensions of the dual operators display, but also because there are no other known examples of operators in $\mathcal{N}$=4 SYM whose anomalous dimension receives instanton but not perturbative corrections. This particular class of BMN operators will be discussed in [3].

Finally, we should note that the issue of non-perturbative corrections to anomalous dimensions is very far removed from the interesting issues surrounding the integrability of string theory in $AdS_5 \times S^5$. Integrability is expected to be a property of tree-level string theory and the corresponding planar approximation to $\mathcal{N} = 4$ Yang–Mills, which can be successfully modelled by local spin chains. In contrast, an instanton affects all the fields in the BMN operator equally, and is therefore highly non-local along the chain. However, instantons are crucial in describing the SL(2,Z) S-duality transformations of the theory and, in particular, for understanding how SL(2,Z) acts on the anomalous dimensions. In general SL(2,Z) transformations relate operators of small and large dimension, just as in string theory they relate fundamental strings to D-strings, which have large masses of order $1/g_s$, in the limit of weak string coupling, $g_s \ll 1$. It would be interesting to understand how S-duality is realised in type IIB string theory in the plane-wave background. A corresponding symmetry should exist in the BMN sector of $\mathcal{N}$=4 SYM and the instanton effects which we have described should be relevant to its implementation.

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