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Quasi-local rotating black holes in higher dimension: geometry

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Abstract

With the help of a generalized Raychaudhuri equation non-expanding null surfaces are studied in an arbitrary dimensional case. The definition and basic properties of non-expanding and isolated horizons known in the literature in the four- and three-dimensional cases are generalized. A local description of the horizon's geometry is provided. The zeroth law of black-hole thermodynamics is derived. The constraints have a similar structure to that of the four-dimensional spacetime case. The geometry of a vacuum isolated horizon is determined by the induced metric and the rotation 1-form potential, local generalizations of the area and the angular momentum typically used in the stationary black-hole solutions case.

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1. Introduction

The theory of non-expanding and isolated horizons in four-dimensional spacetime [1–10] is a quasi-local approach to a black hole in equilibrium. A horizon is a compact, spacelike 2-surface expanding at the speed of light, however, not changing its area element. No symmetry assumptions are made about a spacetime neighbourhood surrounding the horizon. In fact, generically there is no Killing vector [11]. The parameters characterizing stationary black-hole solutions, such as the area and the angular momentum, are replaced by appropriate local fields [4, 12]. Despite this enormous change in the number of the degrees of freedom, the zeroth and the first law of black-hole thermodynamics still hold (see also [13, 14]). On the other hand, an interest in higher-dimensional black-hole solutions is growing [15–19]. The goal of the current and a forthcoming paper [20] is a generalization of those results to

the higher-dimensional case. Whether or not the generalization would be straightforward was *a priori* not known. In the calculations concerning the four- and three-dimensional [21] cases the Newman–Penrose formalism (and its adaptation to three dimensions) was used many times, for example in the proof of the zeroth law.

We consider an n-dimensional spacetime of the signature $(-,+,\ldots,+)$ and arbitrary $n > 2$. First, we derive a higher-dimensional Raychaudhuri equation for a null, geodesic flow. This is an easy generalization of the derivation one can find in [22].

Next, we study non-expanding null surfaces. Our considerations are local, therefore the results may be applied to the surfaces of arbitrary topology. Assuming the usual energy inequalities (classical), we find that the vanishing of the expansion of a null surface implies the vanishing of the shear. Consequently, the spacetime covariant derivative preserves the tangent bundle of each non-expanding null surface, and induces a covariant derivative therein. The induced degenerate metric tensor and the induced covariant derivative (partially independent of each other) constitute the *geometry* of a non-expanding null surface. The geometry is the subject of our study. The induced degenerate metric tensor can be locally identified with a metric tensor defined on the $(n-2)$-dimensional space of the tangent null curves. We do not find any restrictions on that $n-2$ metric tensor. The rotation of a given non-expanding null surface is described by a differential 2-form invariant derived from the covariant derivative, the *rotation 2-form*. Its properties imply the zeroth law upon quite weak energy conditions. The remaining components of the surface covariant derivative—briefly speaking, the shear and expansion of a transversal null vector field—are subject to constraint equations which dictate a null evolution along the surface.

The constraint equations become particularly important in the case of a surface admitting a null symmetry, called an isolated null surface. Due to them, in the vacuum case, the whole geometry of a given non-extremal[4] null isolated surface is locally characterized by the induced degenerate metric tensor, the rotation 2-form (or even by their pullbacks to a spacelike $(n-2)$-dimensional subsurface) and the value of the cosmological constant. We also derive the equations constraining the induced metric and the rotation 2-form in the vacuum extremal isolated null surface case.

In the last section, we apply our local results to the non-expanding and isolated horizons, which are defined by assuming the existence of a global, compact spacelike cross-section and the product structure.

Our characterization of the geometry is used in a coming paper [20] to introduce a canonical framework for the isolated horizons and to derive the first law in a way analogous to the four- and three-dimensional cases.

1.1. Assumptions and notation convention

We consider a manifold $\mathcal{M}$ of the dimension $n > 2$ (our primary interest is in the case $n > 4$). $\mathcal{M}$ is equipped with a (pseudo) metric tensor field $g_{\alpha\beta}$ of the signature $(-,+,\ldots,+)$ (one minus and $n-1$ pluses) and the corresponding Levi-Civita connection ∇_α. The corresponding Riemann[5], Weyl, Ricci and Einstein tensors are denoted, respectively, by ${}^{(n)}R^{\alpha}{}_{\beta\gamma\delta}$, ${}^{(n)}C^{\alpha}{}_{\beta\gamma\delta}$, ${}^{(n)}\mathcal{R}_{\alpha\beta}$ and ${}^{(n)}G_{\alpha\beta}$. We often refer to the Einstein equations, which read

$$ {}^{(n)}G_{\alpha\beta} = -\Lambda g_{\alpha\beta} + T_{\alpha\beta}, \tag{1.1} $$

where Λ is a constant called cosmological and $T_{\alpha\beta}$ is the matter energy–momentum tensor.

[4] An isolated null surface $(\triangle, [\ell])$ (see definition 5.1) is referred to as extremal (non-extremal) whenever the surface gravity $\kappa^{(\ell)}$ corresponding to null field ℓ vanishes (does not vanish) respectively.

[5] We use the following convention: $[\nabla_\alpha, \nabla_\beta]X^\gamma = R^\gamma{}_{\delta\alpha\beta}X^\delta$.

The following (abstract) index notation will be used in this paper.

(i) Indices of the spacetime tensors will be (and have already been) denoted by lower Greek letters: $\alpha, \beta, \gamma, \delta \ldots$.
(ii) Tensors defined in $(n-1)$-dimensional null subspaces (tangent to a null surface except section 2) will carry indices denoted by lower Latin letters: $a, b, c, d \ldots$.
(iii) Capital Latin letters $A, B, C, D, \ldots$ will be used as the indices of tensors considered in $(n-2)$-dimensional spaces (the quotient of a null space by the null direction, the space tangent to a spacelike section of a null surface, the space tangent to the manifold of null curves in a null surface).

2. Null geodesic flows, null surfaces

2.1. Null geodesic flows, generalized Raychaudhuri equation

Consider a null geodesic vector field ℓ, that is a null vector field such that

$$\nabla_\ell \ell = \kappa^{(\ell)} \ell, \tag{2.1}$$

where $\kappa^{(\ell)}$ is an arbitrary function. We are assuming that ℓ is a section of a sub-bundle L of the tangent bundle $T\mathcal{M}$ whose fibres are one dimensional (the assumption is satisfied by every nowhere-vanishing ℓ). It follows from (2.1) that the sub-bundle $L^\perp \subset T\mathcal{M}$ consisting of all the vectors tangent to $\mathcal{M}$ and orthogonal to the fibres of L is preserved by the flow of the vector field ℓ. Therefore, the null flow determines an evolution of tensors defined in the fibres of $L^\perp$. Particularly important will be for us the tensor $q_{ab}(x)$ induced in each fibre $L^\perp_x$ of $L^\perp$ by the restriction of the spacetime metric tensor $g_{\alpha\beta}(x)$. The induced tensor is often referred to as the degenerate metric tensor. Indeed, for every point $x \in \mathcal{M}$, $q_{ab}(x)$ is symmetric and being defined in the $(n-1)$-dimensional fibre $L^\perp_x$, it has the signature $(0, n)$. The evolution of the field q_{ab} defined by the null flow is just

$$\mathcal{L}_\ell q_{ab} = 2B^{(\ell)}{}_{(ab)}, \tag{2.2}$$

where $B^{(\ell)}{}_{(ab)}(x)$ is the restriction of the derivative tensor

$$B^{(\ell)}{}_{\alpha\beta} = \nabla_\beta \ell_\alpha, \tag{2.3}$$

to the vector space $L^\perp_x$, at each $x \in \mathcal{M}$. Note that

$$\ell^\alpha B^{(\ell)}{}_{\alpha\beta} = 0, \qquad \ell^\beta B^{(\ell)}{}_{\alpha\beta} = \kappa^{(\ell)} \ell_\alpha, \tag{2.4}$$

therefore the restriction $B^{(\ell)}{}_{ab}$ is annihilated by ℓ,

$$\ell^a B^{(\ell)}{}_{ab} = 0, \qquad \ell^b B^{(\ell)}{}_{ab} = 0. \tag{2.5}$$

The null flow evolution of $B^{(\ell)}{}_{\alpha\beta}$ involves the spacetime Riemann tensor $R_{\alpha\beta\gamma\delta}$,

$$\mathcal{L}_\ell B^{(\ell)}{}_{\alpha\beta} = \kappa^{(\ell)} B^{(\ell)}{}_{\alpha\beta} + \ell_\alpha \kappa^{(\ell)}{}_{,\beta} + B^{(\ell)}{}_{\gamma\alpha} B^{(\ell)\gamma}{}_{\beta} - {}^{(n)}R_{\mu\alpha\nu\beta} \ell^\mu \ell^\nu. \tag{2.6}$$

To read from this equation the equation of the null flow evolution of q_{ab} it is convenient to consider the quotient bundle $L^\perp/L$, whose fibre at every $x \in \mathcal{M}$ is the quotient vector space $L^\perp_x \big/ L_x$ of the fibres. Given a covariant tensor $\bar{C}_{A\ldots B}$ in $L^\perp_x \big/ L_x$, we denote its pullback to $L^\perp_x$ by $C_{a\ldots b}$; given a vector X^a at x orthogonal to L_x we denote by $\bar{X}^A$ its projection onto $L^\perp_x \big/ L_x$. Examples are the very tensors $q_{ab}(x)$ and $B^{(\ell)}{}_{ab}(x)$, which are in fact pullbacks of tensors defined in $L^\perp_x \big/ L_x$, consequently denoted by $\bar{q}_{AB}(x)$ and $\bar{B}^{(\ell)}{}_{AB}(x)$, respectively. The first of them, $\bar{q}_{AB}$, is a non-degenerate, positive definite metric tensor in each fibre of $L^\perp/L$. The quotient bundle is also preserved by the null flow. Using (2.2) we can see that

$$\mathcal{L}_\ell \bar{q}_{AB} = 2\bar{B}^{(\ell)}{}_{(AB)}. \tag{2.7}$$

The tensor $\bar{B}^{(\ell)}{}_{AB}$ can be decomposed into three parts:

- the trace with respect to $\bar{q}_{AB}$ (with $\bar{q}^{AB}$ being an inverse of $\bar{q}_{AB}$)
$$\theta^{(\ell)} := \bar{q}^{AB}\bar{B}^{(\ell)}{}_{AB}, \tag{2.8}$$
which is called the expansion scalar;
- the traceless symmetric part:
$$\sigma^{(\ell)}{}_{AB} := \bar{B}^{(\ell)}{}_{(AB)} - \frac{1}{n-2}\theta^{(\ell)}\bar{q}_{AB}, \tag{2.9}$$
called the shear tensor; and
- the antisymmetric part $\bar{B}^{(\ell)}{}_{[AB]}$.

Since the transformation law for the tensor $B^{(\ell)}{}_{\alpha\beta}$ upon rescalings $\ell \mapsto \ell' = f\ell$ by a function f is quite simple, namely
$$B^{(\ell')}{}_{\alpha\beta} = fB^{(\ell)}{}_{\alpha\beta} + \ell_\alpha \nabla_\beta f, \tag{2.10}$$
and in particular
$$B^{(\ell')}{}_{ab} = fB^{(\ell)}{}_{ab}, \qquad \bar{B}^{(\ell')}{}_{AB} = f\bar{B}^{(\ell)}{}_{AB}, \tag{2.11}$$
it is often convenient to choose a section ℓ_o of the bundle L such that
$$\kappa^{(\ell_o)} = 0 \tag{2.12}$$
in (2.1). The evolution of the corresponding $\bar{B}^{(\ell_o)}{}_{AB}$ defined by the flow of ℓ_o is
$$\mathcal{L}_{\ell_o}\bar{B}^{(\ell_o)}{}_{AB} = \bar{q}^{CD}\bar{B}^{(\ell_o)}{}_{CA}\bar{B}^{(\ell_o)}{}_{DB} - {}^{(n)}\bar{R}_{\alpha A\beta B}\ell_o^\alpha \ell_o^\beta. \tag{2.13}$$
In particular,
$$\mathcal{L}_{\ell_o}\theta^{(\ell_o)} = -\frac{1}{n-2}(\theta^{(\ell_o)})^2 - \sigma^{(\ell_o)}{}_{AB}\sigma^{(\ell_o)AB} + \bar{B}^{(\ell_o)}{}_{[AB]}\bar{B}^{(\ell_o)[AB]} - \ell_o{}^\mu \ell_o{}^\nu\, {}^{(n)}\mathcal{R}_{\mu\nu}, \tag{2.14}$$
where ${}^{(n)}\mathcal{R}_{\mu\nu}$ is the spacetime Ricci tensor and the capital indices are raised with the inverse metric tensor $\bar{q}^{AB}$. Note that this geometric identity defines the dynamics of the geometry $\bar{q}_{AB}$ if we use the Einstein equations and replace the Ricci tensor by the matter energy–momentum. This is a straightforward generalization of the famous Raychaudhuri equation in four-dimensional spacetime. The essential feature of this equation is still present in this n-dimensional case: all the terms on the right-hand side, except $\bar{B}^{(\ell_o)}{}_{[AB]}\bar{B}^{(\ell_o)[AB]}$, are non-positive, provided the Einstein equations hold and the energy condition
$$T_{\alpha\beta}\ell^\alpha \ell^\beta \geqslant 0 \tag{2.15}$$
is satisfied by the matter. In particular, the non-negativity of $\sigma^{(\ell)}{}_{AB}\sigma^{(\ell)AB}$ follows from the positive definiteness of the metric tensor field $\bar{q}_{AB}$.

We have not exhausted all the information contained in the tensor $B^{(\ell)}{}_{\alpha\beta}$ and in equation (2.6). We will go back to them in the context of the zeroth law of the non-expanding null surface thermodynamics.

2.2. Null surfaces

An $(n-1)$-dimensional submanifold $\triangle$ in $\mathcal{M}$ is called a null surface if at every point $x \in \triangle$ the pullback $q_{ab}(x)$ of the metric tensor $g_{\alpha\beta}(x)$ onto $\triangle$ is degenerate. Denote by L_x the degeneracy subspace $L_x \subset T_x\triangle$. It follows from the algebra of a metric tensor of the signature $(1, n-1)$ that L_x is one dimensional at each point x, provided the $g_{\alpha\beta}(x)$ is non-degenerate. It consists of the null vectors tangent to the surface $\triangle$ at x. The spaces L_x form a sub-bundle $L \subset T\triangle$

referred to throughout this paper as *the null direction bundle*. Consider an arbitrary null vector field ℓ^a defined (locally) on $\triangle$, a (local) section of the bundle L. It is geodesic, that is it satisfies (2.1). The function $\kappa^{(\ell)}$ is referred to as the surface gravity corresponding to ℓ. To apply the definitions and results of section 2.1 (in particular, the Raychaudhuri equation) to the vector field ℓ^a defined on $\triangle$, it is enough to consider at each point $x \in \triangle$ an appropriate local extension of the bundle L and of the vector field ℓ^a to a neighbourhood of x in $\mathcal{M}$. Such extension always exists. Obviously, the tensor $B^{(\ell)}{}_{\beta\alpha} = \nabla_\alpha \ell_\beta$ depends on the extension; however, at the surface $\triangle$, the part $B^{(\ell)}{}_{\beta a}$ defined by the restriction of the derivative to the tangent space to $\triangle$ is extension independent. Moreover, due to (2.4) the tensor $B^{(\ell)\beta}{}_a$ considered as a vector-valued 1-form defined on $\triangle$ takes values in the tangent bundle $T\triangle$, therefore it is defined intrinsically on $\triangle$ and can be denoted by $B^{(\ell)b}{}_a$. Another object defined intrinsically on $\triangle$ is $\mathcal{L}_\ell B^{(\ell)\beta}{}_a$. In this way equations (2.6), (2.13), (2.14) can be applied to every null vector field ℓ^a defined on and tangent to $\triangle$. The equations describe the evolution of the tensors q_{ab} and $B^b{}_a$ along $\triangle$, defined by ℓ^a. The existence of the surface implies that the antisymmetric part of the pullback $B^{(\ell)}{}_{ab}$ vanishes,

$$B^{(\ell)}{}_{[ab]} = 0. \tag{2.16}$$

To see this, owing to (2.11), it suffices to show (2.16) for an arbitrary non-trivial example of ℓ^a. Consider a function r defined in a neighbourhood of a point of $\triangle$ in $\mathcal{M}$ such that

$$r|_\triangle = \text{const}, \qquad \mathrm{d}r|_\triangle \neq 0. \tag{2.17}$$

Then $\ell_{o\mu} = \nabla_\mu r$ defines a vector field ℓ_o^a tangent to the surface $\triangle$ and null thereon. $B^{(\ell_o)}{}_{ab}$ is the pullback to $\triangle$ of the symmetric spacetime tensor $\nabla_\beta \nabla_\alpha r$, so it is symmetric itself. As a consequence of (2.16), the Raychaudhuri equation reads

$$\mathcal{L}_{\ell_o}\theta^{(\ell_o)} = -\frac{1}{n-2}(\theta^{(\ell_o)})^2 - \sigma^{(\ell_o)}{}_{AB}\sigma^{(\ell_o)AB} - \ell_o{}^\mu \ell_o{}^\nu {}^{(n)}\mathcal{R}_{\mu\nu}, \tag{2.18}$$

in the case of $\ell^a = \ell_o^a$ such that the corresponding surface gravity $\kappa^{(\ell_o)}$ vanishes.

3. Non-expanding null surfaces

3.1. Definition

Suppose that given a null surface $\triangle$, for every point $x \in \triangle$ the expansion $\theta^{(\ell)}$ of some non-trivial null vector field ℓ^a tangent to $\triangle$ at x vanishes,

$$\theta^{(\ell)} = 0. \tag{3.1}$$

Then, we say that $\triangle$ is non-expanding. This is a property of the surface $\triangle$ only, independent of a choice of ℓ. Indeed, it follows from (2.11) that if at a given point x the expansion $\theta^{(\ell)}$ vanishes, then the same is true for every other section ℓ' of the bundle L.

3.2. The vanishing of the shear and ${}^{(n)}\mathcal{R}_{\ell\ell}$

To learn more about the non-expanding null surface case, consider again a vector field ℓ_o^a, a section of the bundle L, such that

$$\kappa^{(\ell_o)} = 0. \tag{3.2}$$

The vanishing of the left-hand side (the expansion $\theta^{(\ell_o)}$) of (2.18) and the vanishing of the antisymmetric part of $\bar{B}^{(\ell_o)}{}_{AB}$ lead to

$$0 = \sigma^{(\ell_o)}{}_{AB}\sigma^{(\ell_o)AB} + \ell_o{}^\mu \ell_o{}^\nu {}^{(n)}\mathcal{R}_{\mu\nu}. \tag{3.3}$$

The first term on the right-hand side is non-negative. We assume the energy inequality (2.15) which makes the second term also non-negative. Hence, all of them necessarily vanish on $\triangle$. Moreover, it follows from the positivity of the metric tensor $\bar{q}_{AB}$ and the symmetry of ${\sigma^{(\ell)}}_{AB}$ that

$$ {\sigma^{(\ell_o)}}_{AB}\sigma^{(\ell_o)AB} = 0 \quad \Rightarrow \quad {\sigma^{(\ell_o)}}_{AB} = 0. \tag{3.4} $$

Consequently,

$$ {B^{(\ell_o)}}_{ab} = 0 = \ell_o{}^{\mu}\ell_o{}^{\nu}\,{}^{(n)}\mathcal{R}_{\mu\nu}, \tag{3.5} $$

on $\triangle$.

Since the tensor ${\bar{B}^{(\ell)}}_{ab}$ transforms as presented in (2.11) the final conclusion is true for arbitrary choice of a section ℓ of the bundle L.

Theorem 3.1. *Suppose $\triangle$ is a non-expanding, null, $(n-1)$-dimensional surface contained in a spacetime of signature $(1, n-1)$; suppose the Einstein field equation holds on $\triangle$ with a cosmological constant and with the matter fields which satisfy the energy condition (2.15). Then*

(i) the surface is shear-free, that is for every null vector field ℓ^{α} defined on and tangent to $\triangle$

$$ \nabla_a\ell_b = 0, \tag{3.6} $$

where $\nabla_a\ell_b$ is the pullback of the spacetime $\nabla_{\alpha}\ell_{\beta}$ to $\triangle$;

(ii) the induced degenerate metric q_{ab} in $\triangle$ is invariant with respect to the flow of every null vector field ℓ^a tangent to $\triangle$,

$$ \mathcal{L}_{\ell}q_{ab} = 0; \tag{3.7} $$

(iii) the spacetime Ricci tensor satisfies on $\triangle$ the following condition:

$$ {}^{(n)}\mathcal{R}_{\alpha\beta}\ell^{\alpha}\ell^{\beta} = 0. \tag{3.8} $$

Property (i) above combined with $\ell^a q_{ab} = 0$ means that locally q_{ab} is the pullback of certain metric tensor field $\hat{q}_{AB}$ defined on an $(n-2)$-dimensional manifold $\hat{\triangle}'$. The manifold $\hat{\triangle}'$ is the set of the null curves tangent to the null direction bundle L in an appropriate neighbourhood $\triangle' \subset \triangle$ open in $\triangle$, and the map is the natural projection,

$$ \pi : \triangle' \to \hat{\triangle}', \qquad q_{ab} = \pi^{*}\hat{q}_{AB}. \tag{3.9} $$

3.3. The induced covariant derivative

If the assumptions of theorem 3.1 are satisfied, then for any vector fields X, Y, sections of the tangent bundle $T\triangle$, the covariant derivative $\nabla_X Y$ is again a vector field tangent to $\triangle$. Indeed, it is easy to see that $\nabla_X Y$ is necessarily orthogonal to ℓ,

$$ \ell_{\mu}X^{\nu}\nabla_{\nu}Y^{\mu} = X^{\nu}\nabla_{\nu}(\ell_{\mu}Y^{\mu}) - X^{\nu}Y^{\mu}B_{\nu\mu} = 0, \tag{3.10} $$

where the first term vanishes by the definition of ℓ and the second due to theorem 3.1. The induced covariant derivative will be denoted by D_a. For the vector fields X^a, Y^a, sections of the bundle $T\triangle$, the derivative is just

$$ D_X Y^a := \nabla_X Y^a, \tag{3.11} $$

whereas for a covector W_a, a section of the dual bundle $T^{*}\triangle$, the derivative $D_X W_a$ is determined by the Leibnitz rule,

$$ Y^a D_X W_a = D_X(Y^a W_a) - (D_X Y^a)W_a. \tag{3.12} $$

Obviously, the derivative D_a is torsion free and annihilates the degenerate metric tensor q_{ab},

$$ D_a D_b f = D_b D_a f, \qquad D_a q_{bc} = 0, \tag{3.13} $$

for every function f.

3.4. Further conditions on the Riemann tensor necessary at $\triangle$

The conclusions of theorem 3.1 lead to stronger restrictions on the Riemann tensor at $\triangle$, namely

$$ {}^{(n)}R_{abc\alpha}\ell^{\alpha} = 0, \tag{3.14} $$

where ℓ^a is a null vector tangent to $\triangle$. Indeed, the contraction of the Riemann tensor with ℓ and any vector fields X^a, Y^a, Z^a, sections of $T\triangle$, can be expressed as a functional homogeneous in the derivative tensor ${B^{(\ell)}}_{ab}$:

$$ \begin{aligned} X^{\mu}\ell^{\nu}Y^{\alpha}Z^{\beta\,(n)}R_{\mu\nu\alpha\beta} &= X^{\mu}Y^{\alpha}Z^{\beta}\nabla_{[\alpha}\nabla_{\beta]}\ell_{\mu} \\ &= Y^a D_a X^m Z^b {B^{(\ell)}}_{bm} - {B^{(\ell)}}_{bm}(X^m Y^a D_a Z^b + Z^b Y^a D_a X^m) \\ &\quad - Z^b D_b X^m Y^a {B^{(\ell)}}_{am} + {B^{(\ell)}}_{am}(X^m Z^b D_b Y^a + Y^a Z^b D_b X^m). \end{aligned} \tag{3.15} $$

Note that in the calculation we have used the fact that the spacetime covariant derivative applied in any direction tangent to $\triangle$ preserves the tangent bundle $T\triangle$.

Thus far only the inequality $T_{\alpha\beta}\ell^{\alpha}\ell^{\beta} \geqslant 0$ was used apart from the zero expansion assumption and the Einstein equations with possibly non-zero cosmological constant. A somewhat stronger but still quite mild assumption about the energy–momentum tensor $T_{\alpha\beta}$ is[6]

Condition 3.2 (stronger energy condition)**.** *At every point of the surface* $\triangle$*, the vector field*

$$ -{T^{\mu}}_{\nu}\ell^{\nu} \tag{3.16} $$

is causal, that is

$$ g^{\mu\nu}T_{\mu\alpha}\ell^{\alpha}T_{\nu\beta}\ell^{\beta} \leqslant 0, \tag{3.17} $$

and future oriented, for every future oriented null vector field ℓ *defined on and tangent to* $\triangle$*.*

This condition implies, in particular, the previous

$$ T_{\ell\ell} \geqslant 0. \tag{3.18} $$

Now, the vanishing of the Ricci tensor component $\ell^{\mu}\ell^{\nu}\,{}^{(n)}\mathcal{R}_{\mu\nu}$ on $\triangle$ combined with stronger energy condition (3.2) leads to further restrictions on the Ricci tensor. Consider the 1-form

$$ {\mathcal{R}^{(\ell)}}_{a} := {}^{(n)}\mathcal{R}_{a\beta}\ell^{\beta}, \tag{3.19} $$

a section of the cotangent bundle $T^*\triangle$. Due to the vanishing of ${}^{(n)}\mathcal{R}_{\ell\ell}$, at each $x \in \triangle$, ${\mathcal{R}^{(\ell)}}_{a} \in T^*_x\triangle$ is the pullback of some $\bar{\mathcal{R}}^{(\ell)}{}_A \in (T_x\triangle/L)^*$. The Einstein field equations allow us to express the non-positive spacetime norm of the field ${T^{\mu}}_{\nu}\ell^{\nu}$ by the non-negative norm of $\bar{\mathcal{R}}^{(\ell)}{}_A$ with respect to $\bar{q}^{AB}$,

$$ 0 \geqslant g^{\mu\nu}T_{\mu\alpha}\ell^{\alpha}T_{\nu\beta}\ell^{\beta} = \bar{q}^{AB}\bar{\mathcal{R}}^{(\ell)}{}_A\bar{\mathcal{R}}^{(\ell)}{}_B \geqslant 0. \tag{3.20} $$

Hence, the pullback onto $\triangle$ of the Ricci tensor contracted with ℓ is identically zero at $\triangle$,

$$ {}^{(n)}\mathcal{R}_{a\beta}\ell^{\beta} = 0. \tag{3.21} $$

Combining this result with condition (3.14) on the Riemann tensor one can obtain the following condition on the spacetime Weyl tensor at $\triangle$:

$$ {}^{(n)}C_{abc\delta}\ell^{\delta}|_{\triangle} = 0. \tag{3.22} $$

In the $n = 4$ case the condition means that the null direction tangent to the surface $\triangle$ is a double principal null direction of the Weyl tensor. In [23] the Petrov classification of the Weyl tensor was generalized to an arbitrary dimension. The Weyl tensor was expressed in a frame built of real vectors $(n, \ell, \theta_{(A)})$ such that[7]

[6] This inequality automatically holds when the dominant energy condition is assumed, but is much weaker. The consequences of dropping the condition will be briefly discussed in the appendix.

[7] The proposed frame is an analogue of the Newman–Penrose complex null tetrad [24]. The pair of complex null vectors $(m, \bar{m})$ is replaced by a set of real spacelike unit vectors which allows us to generalize the frame to arbitrary dimension.

- on a given n-dimensional manifold $\mathcal{M}$ the vectors (n, ℓ) are null and normalized by the condition $\ell^\mu n_\mu = -1$;
- the spacelike vectors $\theta_{(A)}$ constitute the orthonormal basis of the subspace of $T\mathcal{M}$ orthogonal to (n, ℓ).

The proposed classification is based on the behaviour of the Weyl tensor under the boost transformations $(\ell \mapsto f\ell, n \mapsto f^{-1}n)$. For a given (fixed) ℓ the Weyl tensor can be decomposed onto the sum of the terms $C^{(b)}$, such that each of them transforms under the boost in the following way: $C^{(b)} \mapsto f^b C^{(b)}$. The integer power b is called the boost weight. The weight of the leading term (denoted as the boost order $\mathcal{B}(\ell)$) depends on the ℓ only. Therefore for a given Weyl tensor one can distinguish the set of *aligned* vectors of the boost order $\mathcal{B}(\ell) \leqslant 1$. The Weyl tensor is classified as being of type I (II, III, N) if there exists a null vector ℓ of the boost order 1 $(0, -1, -2)$, and there does not exist a null vector of a lower order. Condition (3.22) implies that the boost order of the null direction tangent to $\triangle$ is at most 0, so the Weyl tensor is at least of type II with respect to the principal classification introduced in [23] and sketched above.

3.5. Rotation

The covariant derivative D_a induced on $\triangle$ preserves the null direction bundle L. Indeed, for every section ℓ^a of L, and every vector field X^a, a section of $T\triangle$, the vector field $D_X\ell^a$ is orthogonal to every vector Y^a tangent to $\triangle$,

$$q_{ab}Y^aX^cD_c\ell^b = -q_{ab}\ell^bX^cD_cY^a = 0. \tag{3.23}$$

This implies that the derivative $D_a\ell^b$ is proportional to ℓ^b itself,

$$D_a\ell^b = B^{(\ell)b}{}_a = \omega^{(\ell)}{}_a\ell^b, \tag{3.24}$$

where $\omega^{(\ell)}{}_a$ is a 1-form defined uniquely on this subset of $\triangle$ on which $\ell \neq 0$ is defined. We call $\omega^{(\ell)}{}_a$ the rotation 1-form potential, as a generalization of the $(n = 4)$-dimensional case [4].

In four dimensions, the evolution of $\omega^{(\ell)}{}_a$ along the surface $\triangle$ upon the null flow is responsible for the zeroth law of the non-expanding horizon thermodynamics. Therefore, we study this equation in the current case. It is convenient to investigate the behaviour of the following object:

$$\ell^b\mathcal{L}_\ell\omega^{(\ell)}{}_a = \mathcal{L}_\ell B^{(\ell)b}{}_a. \tag{3.25}$$

The right-hand side is given by (2.6), and after a short calculation it reads

$$\ell^b\mathcal{L}_\ell\omega^{(\ell)}{}_a = \ell^bD_a\kappa^{(\ell)} - {}^{(n)}R_c{}^b{}_{da}\ell^c\ell^d, \tag{3.26}$$

where we used the fact that

$$\kappa^{(\ell)} = \omega^{(\ell)}{}_a\ell^a. \tag{3.27}$$

The vanishing of the components ${}^{(n)}R_{abcd}\ell^d$ (see (3.15)) allows us to express the Riemann tensor component appearing in (3.26) by the Ricci tensor

$${}^{(n)}R_c{}^b{}_{da}\ell^c\ell^d = -{}^{(n)}\mathcal{R}_{ca}\ell^b\ell^c, \tag{3.28}$$

hence the vector field ℓ^a can be completely factored out,

$$\mathcal{L}_\ell\omega^{(\ell)}{}_a = D_a\kappa^{(\ell)} + {}^{(n)}\mathcal{R}_{ab}\ell^b. \tag{3.29}$$

If stronger energy condition (3.2) holds, then the last term above also vanishes (see section 3.4).

In conclusion, the evolution of the rotation potential is described by the following theorem:

Theorem 3.3 (the zeroth law)**.** *Suppose $\triangle$ is an $(n-1)$-dimensional, non-expanding, null surface; suppose that the assumptions of theorem 3.1 and stronger energy condition (3.2) are satisfied. Then, for every null vector field ℓ^a defined on and tangent to $\triangle$, the corresponding rotation 1-form potential $\omega^{(\ell)}$ and the surface gravity $\kappa^{(\ell)}$ satisfy the following constraint:*

$$\mathcal{L}_\ell \omega^{(\ell)}{}_a = D_a \kappa^{(\ell)}. \tag{3.30}$$

Theorem 3.3 tells us that there is always a choice of the section ℓ of the null direction bundle L such that $\omega^{(\ell)}$ is Lie dragged by ℓ. For we can always find a non-trivial section ℓ of L such that $\kappa^{(\ell)}$ is constant. The relation of black-hole thermodynamic to the original zeroth law goes the other way around. Indeed, if the vector field ℓ^a admits an extension to a Killing vector defined in a neighbourhood of $\triangle$, then $\omega^{(\ell)}$ is Lie dragged by the flow; therefore the left hand side is zero, and hence $\kappa^{(\ell)}$ is necessarily (locally) constant.

The dependence of the rotation 1-form potential $\omega^{(\ell)}{}_a$ on a choice of the section ℓ^a of L follows from (2.10): If $\ell'^a = f\ell^a$, then

$$\omega^{(\ell')}{}_a = \omega^{(\ell)}{}_a + D_a \ln f. \tag{3.31}$$

As one can see, its exterior derivative (in the sense of the manifold $\triangle$) is the surface $\triangle$ invariant,

$$\Omega_{ab} := D_a \omega^{(\ell)}{}_b - D_b \omega^{(\ell)}{}_a = D_a \omega^{(\ell')}{}_b - D_b \omega^{(\ell')}{}_a. \tag{3.32}$$

We call it the rotation 2-form. Note that whereas the sections of the null direction bundle L were considered on $\triangle$ locally, and so were the corresponding rotation 1-form potentials $\omega^{(\ell')}{}_a$, the rotation 2-form is defined globally on $\triangle$. An immediate consequence of theorem (3.3) is that whenever the assumptions are satisfied, the rotation 2-form is orthogonal to the bundle L, and Lie dragged by any (local) null flow defined by a section ℓ of L,

$$\ell^a \Omega_{ab} = \mathcal{L}_\ell \omega^{(\ell)}{}_b - D_b\big(\ell^a \omega^{(\ell)}{}_a\big) = 0, \tag{3.33a}$$

$$\mathcal{L}_\ell \Omega_{ab} = 0. \tag{3.33b}$$

Therefore, the rotation 2-form Ω_{ab} is at every point $x \in \triangle$ the pullback with respect to $T_x\triangle \to T_x\triangle/L_x$ of a tensor $\bar{\Omega}_{AB}$ defined in $T_x\triangle/L_x$, such that

$$\mathcal{L}_\ell \bar{\Omega}_{AB} = 0. \tag{3.34}$$

3.6. Geometry and the constraints

Given a non-expanding null surface $\triangle$, the pair (q_{ab}, D_a), that is the induced degenerate metric and the induced covariant derivative, respectively, is referred to as the geometry of $\triangle$. By a 'constraint' on the non-expanding surface geometry we mean here every geometric identity $F(q_{ab}, D_a, {}^{(n)}\mathcal{R}_{\alpha\beta}) = 0$ involving the geometry (q_{ab}, D_a) and the spacetime Ricci tensor at $\triangle$ only. Part of the constraints is already solved by theorem 3.1(ii), that is by the conclusion that q_{ab} be Lie dragged by every null flow generated by a null vector field ℓ tangent to $\triangle$. Another example of a constraint is the zeroth law (3.29), (3.30). A complete[8] set of the functionally independent constraints is formed by $\mathcal{L}_\ell q_{ab} = 0$ and by an identity satisfied by the commutator $[\mathcal{L}_\ell, D_a]$, where ℓ is a fixed, non-vanishing section of the null direction bundle L. We turn now to the second identity mentioned above.

[8] Among all the components of the Einstein tensor only its pullback to $\triangle$ can be expressed by (q_{ab}, D_a) only. It will be shown further that its value is determined by the commutator $[\mathcal{L}_\ell, D_a]$. The remaining components involve transversal derivatives of the components of ∇_μ (where the number of determined transversal derivatives is equal to the number of the remaining components of the Einstein tensor).

Using formula (3.24), and using (3.14), after simple calculations one can express the value of the commutator $[\mathcal{L}_\ell, D_a]$ by the rotation potential, its derivative and the spacetime Riemann tensor,

$$[\mathcal{L}_\ell, D_a]X^b = \left[\ell^b\left(D_{(a}\omega^{(\ell)}{}_{c)} + \omega^{(\ell)}{}_a\omega^{(\ell)}{}_c\right) - \ell^{\delta\,(n)}R^b{}_{(ca)\delta}\right]X^c \tag{3.35}$$

(where $D_a\omega^{(\ell)}{}_c$ stands for the tensor, not for a derivative operator acting on X^b). It follows from condition (3.14) that $\ell^{\delta\,(n)}R^b{}_{(ca)\delta}$ is also proportional to ℓ^b, and we can write

$$[\mathcal{L}_\ell, D_a]X^b = \ell^b N_{ac}X^c. \tag{3.36}$$

To spell out what the proportionality factor N_{ac} is we need to recall that the degenerate metric tensor field q_{ab} can be locally defined as the pullback (3.9) of the $(n-2)$-dimensional metric tensor $\hat{q}_{AB}$ defined on the manifold of the curves generating $\triangle$. The proportionality factor can be expressed by the pullback of the spacetime Ricci tensor and the pullback[9] of the Ricci tensor ${}^{(n-2)}\mathcal{R}_{AB}$ of the metric $\hat{q}_{AB}$, namely

$$N_{ac} = D_{(a}\omega^{(\ell)}{}_{c)} + \omega^{(\ell)}{}_a\omega^{(\ell)}{}_c + \tfrac{1}{2}\left({}^{(n)}\mathcal{R}_{ac} - \pi^{*\,(n-2)}\mathcal{R}_{ac}\right). \tag{3.37}$$

Identities (3.36), (3.37) are the constraints in the sense explained at the beginning of this subsection. They become the gravitational part of the genuine Einstein constraints when the spacetime Ricci tensor is replaced by the cosmological constant part and by the energy–momentum tensor of the matter field. As an example, later we will consider the vacuum case. We did not assume stronger energy condition (3.2) to derive (3.36), (3.37).

The contraction of (3.36), (3.37) with ℓ^a is equivalent to (3.29). Hence, it defines the evolution of the rotation 1-form potential $\omega^{(\ell)}{}_a$ already used in the proof of the zeroth law. Recall that locally there is a nowhere vanishing tangent, null vector field ℓ_o on $\triangle$ such that

$$\kappa^{(\ell_o)} = 0. \tag{3.38}$$

The corresponding $\omega^{(\ell_o)}$ is Lie dragged by the vector field ℓ_o, provided the assumptions of the zeroth law are satisfied,

$$\mathcal{L}_\ell\omega^{(\ell_o)} = 0. \tag{3.39}$$

The meaning of the remaining part of the constraint (3.36), (3.37) is explained in the next subsection after we itemize the derivative D_a into components and provide a more explicit form of equations (3.36), (3.37).

4. Elements of the non-expanding null surface geometry

4.1. Compatible coordinates, foliations

To understand better the elements of the covariant derivative D_a induced on a null, non-expanding surface $\triangle$, and to investigate further its relation with the spacetime Ricci tensor, we need to introduce an extra local structure on $\triangle$.

Let ℓ^a be a nowhere vanishing local section of the null direction bundle L. Given ℓ^a, let v be a real function defined in the domain of ℓ^a, compatible with ℓ^a, that is such that

$$\ell^a D_a v = 1. \tag{4.1}$$

The function v exists provided we sufficiently reduce the domain of ℓ^a. We will refer to v as a coordinate compatible with ℓ. The function v is used to define a covector field on $\triangle$,

$$n_a := -D_a v. \tag{4.2}$$

[9] The first pullback is defined by the embedding $\triangle \to \mathcal{M}$, whereas the second one corresponds to the locally defined projection $\pi : \triangle' \to \hat{\triangle}'$ of a neighbourhood $\triangle' \subset \triangle$ onto the space of the null curves in $\triangle'$.

The covector field has the following properties:

(i) It is normalized in the sense that

$$\ell^a n_a = -1, \tag{4.3}$$

(ii) it is orthogonal to the constancy surfaces $\tilde{\triangle}_v$ of the function v.

The surfaces $\tilde{\triangle}_v$ will be referred to as slices. The family of the slices is preserved by the null flow of ℓ, and so is n_a,

$$\mathcal{L}_\ell n_a = 0. \tag{4.4}$$

At every point $x \in \triangle$, the tensor

$$\tilde{q}^a{}_b := \delta^a{}_b + \ell^a n_b \tag{4.5}$$

defines the orthogonal to ℓ^a projection

$$T_x\triangle \ni X^a \mapsto \tilde{X}^a = \tilde{q}^a{}_b X^b \in T_x\tilde{\triangle}_v \tag{4.6}$$

onto the tangent space $T_x\tilde{\triangle}_v$, where $\tilde{\triangle}_v$ is the slice passing through x. Hence, instead of $\tilde{X}^a$ we will write $\tilde{X}^A$, according to the index notation explained in the introduction. Applied to the covectors, elements of $T^*_x M$, on the other hand, $\tilde{q}^a{}_b$ maps each of them into the pullback onto $\tilde{\triangle}_v$,

$$T^*_x\triangle \ni Y_a \mapsto \tilde{Y}_a := \tilde{q}^b{}_a Y_b \in T^*_x\tilde{\triangle}_v. \tag{4.7}$$

Hence the result will also be denoted by using a capital Latin index, such as for example $\tilde{Y}_A$.

The field n_a could be extended to a section of the pullback $T^*_\triangle\mathcal{M}$ to $\triangle$ of the cotangent bundle T^*M, by the requirement that

$$g^{\mu\nu} n_\mu n_\nu = 0. \tag{4.8}$$

Hence n_a can be thought of as a transversal to $\triangle$ null vector field from the spacetime point of view.

4.2. The elements of D_a

Each slice $\tilde{\triangle}_v$ of the foliation introduced above is equipped with the induced metric tensor $\tilde{q}_{AB}$ defined by the pullback of q_{ab} (and of $g_{\alpha\beta}$) to $\tilde{\triangle}_v$. Denote by $\tilde{D}_A$ the torsion-free and metric covariant derivative determined on $\tilde{\triangle}_v$ by the metric tensor $\tilde{q}_{AB}$. All the slices are naturally isometric.

The covector field n_a gives rise to the following symmetric tensor defined on $\triangle$:

$$S_{ab} := D_a n_b. \tag{4.9}$$

Given the structure introduced in the previous subsection on $\triangle$ locally (the null vector field ℓ^a, the foliation by slices $\tilde{\triangle}_v$ and the covector field n_a), the derivative D_a defined on $\triangle$ is determined by the following information:

- the torsion-free covariant derivative $\tilde{D}_A$ corresponding to the Levi-Civita connection of the induced metric tensor $\tilde{q}_{AB}$,
- the rotation 1-form potential $\omega^{(\ell)}{}_a$ and
- a symmetric tensor $\tilde{S}_{AB}$ defined in each slice $\tilde{\triangle}_v$ by the pullback of $D_a n_b$,

$$\tilde{S}_{AB} = \tilde{q}^a{}_A \tilde{q}^b{}_B S_{ab}, \tag{4.10}$$

and referred to as the transversal expansion-shear tensor.

Indeed, for every vector field X^a and every covector field Y_a, the sections of $T\triangle$ and $T^*\triangle$, respectively, their derivative can be composed from the following parts:

$$\tilde{q}^a{}_A\tilde{q}^B{}_b D_a X^b = \tilde{D}_A\tilde{X}^B \tag{4.11a}$$

$$\tilde{q}^a{}_A n_b D_a X^b = \tilde{D}_A(X^b n_b) - \tilde{q}^a{}_A S_{ab} \tag{4.11b}$$

$$\ell^a D_a X^b = \mathcal{L}_\ell X^b + X^a \omega^{(\ell)}{}_a \ell^b \tag{4.11c}$$

$$\tilde{q}^a{}_A\tilde{q}^b{}_B D_a Y_b = \tilde{D}_A\tilde{Y}_B - (Y_b\ell^b)\tilde{q}^a{}_A\tilde{q}^b{}_B S_{ab} \tag{4.11d}$$

$$\tilde{q}^a{}_A\ell^b D_a Y_b = \tilde{D}_A(\ell^b Y_b) - \tilde{\omega}^{(\ell)}{}_A\ell^b Y_b \tag{4.11e}$$

$$\ell^a D_a Y_b = \mathcal{L}_\ell Y_b - \omega^{(\ell)}{}_b Y_a \ell^a, \tag{4.11f}$$

where we have used the notation introduced in the previous section: $\tilde{X}^A = \tilde{q}^A{}_a X^a$, $\tilde{Y}_A = \tilde{q}^a{}_A Y_a$, $\tilde{\omega}^{(\ell)}{}_A = \tilde{q}^a{}_A\omega^{(\ell)}{}_a$.

The careful reader will have noticed that all the components of the tensor S_{ab} were used above, not only the $\tilde{S}_{AB}$ part. However, due to the normalization (4.3) the contraction of the tensor with the null normal to $\triangle$ is equal to

$$\ell^a S_{ab} = \omega^{(\ell)}{}_b. \tag{4.12}$$

4.3. The constraints on the elements of D_a

The constraints satisfied by D_a are expressed in the previous section by the commutator $[\mathcal{L}_\ell, D_a]$ (3.36), (3.37). Since the foliation we used to decompose D_a into the elements $\tilde{D}_A$, $\omega^{(\ell)}{}_a$ and $\tilde{S}_{AB}$ is invariant with respect to the flow of ℓ, the evolution of D_a comes down to an evolution of $\tilde{D}_A$, $\omega^{(\ell)}{}_a$ and $\tilde{S}_{AB}$. The slice connection $\tilde{D}_A$ is invariant with respect to the flow because of (3.7). The evolution of $\omega^{(\ell)}{}_a$ is already given by the zeroth law (3.30). To describe the evolution of the remaining element $\tilde{S}_{AB}$ we calculate the action of the commutator on the covector n_a and find that the tensor N_{ab} defined in (3.36) can be expressed by $\mathcal{L}_\ell S_{ab}$, namely

$$N_{ab} = \mathcal{L}_\ell S_{ab}. \tag{4.13}$$

Therefore, by (3.37),

$$\mathcal{L}_\ell S_{ab} = D_{(a}\omega^{(\ell)}{}_{b)} + \omega^{(\ell)}{}_a\omega^{(\ell)}{}_b - \tfrac{1}{2}{}^{(n)}R_{c(ab)}{}^d\ell^c n_d. \tag{4.14}$$

The contraction of the expression with ℓ^a reproduces the zeroth law. The remaining component, namely the pullback of $\mathcal{L}_\ell S_{ab}$ onto a slice $\tilde{\triangle}_v$, determines the evolution of the transversal expansion-shear tensor $\tilde{S}_{AB}$,

$$\mathcal{L}_\ell\tilde{S}_{AB} = -\kappa^{(\ell)}\tilde{S}_{AB} + \tilde{D}_{(A}\tilde{\omega}^{(\ell)}{}_{B)} + \tilde{\omega}^{(\ell)}{}_A\tilde{\omega}^{(\ell)}{}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} + \tfrac{1}{2}{}^{(n)}\tilde{\mathcal{R}}_{AB}, \tag{4.15}$$

where the tilde consequently means the projection (4.7), and ${}^{(n-2)}\mathcal{R}_{AB}$ is the Ricci tensor of the metric tensor induced in slice $\tilde{\triangle}_v$ (since locally, every slice $\triangle_v$ is naturally isometric with the space of the null curves $\hat{\triangle}'$ equipped with the metric tensor $\hat{q}_{AB}$ we denote the corresponding Ricci tensors in the same way).

5. Isolated null surfaces

5.1. Definition, assumptions, constraints

In this section we continue with the study of the non-expanding, null surfaces. We assume that the Einstein equations with a (possibly zero) cosmological constant hold on the surface,

with the energy–momentum tensor $T_{\alpha\beta}$, which satisfies stronger energy conditions (3.2). As was shown, these assumptions imply that the spacetime Ricci tensor satisfies (3.21).

Let $\triangle$ be a non-expanding null surface. Whereas the induced metric tensor is Lie dragged by every null vector field tangent to $\triangle$ we could see that the remaining ingredient of the geometry, the covariant derivative, is subject to the null evolution equation (3.36), (3.37) implied by the constraints. The equation depends on a choice of the null vector field ℓ; however, in general (and generically in the four-spacetime-dimensional case) a geometry (q_{ab}, D_a) does not admit any choice of ℓ such that $[\mathcal{L}_\ell, D_a] = 0$.

Definition 5.1. *An isolated null surface is a non-expanding null surface $\triangle$ equipped with a class $[\boldsymbol{\ell}]$ of tangent, null, non-vanishing vector fields $\boldsymbol{\ell}$ such that*

$$\mathcal{L}_{\boldsymbol{\ell}} q_{ab} = 0, \qquad [\mathcal{L}_{\boldsymbol{\ell}}, D_a] = 0 \tag{5.1}$$

where q_{ab} is the induced degenerate metric tensor and D_a is the induced covariant derivative, and $\boldsymbol{\ell}, \boldsymbol{\ell}' \in [\boldsymbol{\ell}]$ provided $\boldsymbol{\ell}' = c\boldsymbol{\ell}$, where c is a constant.

In this section we consider an isolated null surface $(\triangle, [\boldsymbol{\ell}])$. We assume $\triangle$ is connected. Note that given the flow $[\boldsymbol{\ell}]$, the rotation 1-form $\omega^{(\boldsymbol{\ell})}{}_a$ is defined uniquely owing to (3.31). Obviously, it is Lie dragged by $[\boldsymbol{\ell}]$,

$$\boldsymbol{\ell}^b \mathcal{L}_{\boldsymbol{\ell}} \omega^{(\boldsymbol{\ell})}{}_a = \mathcal{L}_{\boldsymbol{\ell}}(D_a \boldsymbol{\ell}^b) = D_a \mathcal{L}_{\boldsymbol{\ell}} \boldsymbol{\ell}^b = 0. \tag{5.2}$$

As a consequence, theorem 3.3 takes the familiar form of the zeroth law of black-hole thermodynamics,

$$\kappa^{(\boldsymbol{\ell})} = \text{const}, \tag{5.3}$$

where the value of the surface gravity depends on the choice of $\ell \in [\ell]$ unless $\kappa^{(\ell)} = 0$.

The constraint (3.36), (3.37) takes the following form:

$$D_{(a}\omega^{(\boldsymbol{\ell})}{}_{c)} + \omega^{(\boldsymbol{\ell})}{}_a \omega^{(\boldsymbol{\ell})}{}_c + \tfrac{1}{2}({}^{(n)}\mathcal{R}_{ac} - \pi^{*\,(n-2)}\mathcal{R}_{ac}) = 0. \tag{5.4}$$

Not surprisingly, a necessary condition is that the pullback of the spacetime Ricci tensor on $\triangle$ is Lie dragged by $[\boldsymbol{\ell}]$,

$$\mathcal{L}_{\boldsymbol{\ell}}\, {}^{(n)}\mathcal{R}_{ab} = 0. \tag{5.5}$$

To understand better the meaning of equation (5.4) let us apply the (local) decomposition of D_a introduced in section 4.2. Introduce a foliation of $\triangle$ preserved by $[\boldsymbol{\ell}]$ and use the corresponding covector n_a, orthogonal to the slices and normalized to an arbitrarily fixed null vector field $\boldsymbol{\ell}^a$ generating the flow $[\boldsymbol{\ell}]$. If the derivative D_a satisfies the definition of the isolated null surface, then the corresponding transversal expansion-shear tensor $\tilde{S}_{AB}$ defined on the slices is invariant with respect to the null flow

$$\mathcal{L}_{\boldsymbol{\ell}} \tilde{S}_{AB} = \mathcal{L}_{\boldsymbol{\ell}}\big(\tilde{q}^a{}_A \tilde{q}^b{}_B D_a n_b\big) = 0, \tag{5.6}$$

because all the factors in the parenthesis are invariant. Conversely, given a non-expanding null surface $\triangle$, a null flow $[\ell]$ generated by the nowhere vanishing vector field ℓ^a, and one of the foliations defined in section 4.1, the invariance of $\omega^{(\ell)}{}_a$ and $\tilde{S}_{AB}$ with respect to the null flow implies that $(\triangle, [\ell])$ is an isolated null surface.

Now, the constraint (4.15) implies

$$\kappa^{(\boldsymbol{\ell})} \tilde{S}_{AB} = \tilde{D}_{(A}\tilde{\omega}^{(\boldsymbol{\ell})}{}_{B)} + \tilde{\omega}^{(\boldsymbol{\ell})}{}_A \tilde{\omega}^{(\boldsymbol{\ell})}{}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} + {}^{(n)}\tilde{\mathcal{R}}_{AB}. \tag{5.7}$$

A characterization of the isolated null surface depends crucially on whether $\kappa^{(\ell)}$ vanishes or not; therefore we define two types of the isolated null surfaces:

(i) *extremal*, if $\kappa^{(\ell)} = 0$, or
(ii) *non-extremal*, whenever $\kappa^{(\ell)} \neq 0$.

The meaning of the constraint (5.7) depends on the type. In the non-extremal case (5.7) determines $\tilde{S}_{AB}$ given $\tilde{q}_{AB}, \tilde{\omega}^{(\ell)}{}_A$ and the pullback ${}^{(n)}\tilde{\mathcal{R}}_{AB}$ of the spacetime Ricci tensor expressed by the cosmological constant and the matter energy–momentum tensor.

Theorem 5.2 (non-extremal, vacuum isolated null surface)**.** *Let* $(\triangle, [\ell])$ *be a non-extremal isolated null surface; suppose the vacuum Einstein equations with a cosmological constant* Λ *are satisfied. Then, the geometry of* $\triangle$ *is determined by the induced metric tensor* q_{ab}*, the rotation 1-form potential* $\omega^{(\ell)}{}_a$ *and the value* Λ *of the cosmological constant.*

If matter fields are present, then typically the geometry is determined just by adding to $\left(q_{ab}, \omega^{(\ell)}{}_a, \Lambda\right)$ appropriate information on the field on $\triangle$.

In the extremal case, on the other hand, equation (5.4) becomes a condition on $\tilde{\omega}^{(\ell)}{}_A$, q_{AB} and ${}^{(n)}\tilde{\mathcal{R}}_{AB}$.

Theorem 5.3 (extremal, isolated null surface)**.** *Suppose* $(\triangle, [\ell])$ *is an extremal isolated null surface contained in n-dimensional spacetime; then, for every* $(n-2)$*-dimensional spacelike submanifold* $\tilde{\triangle}$ *transversal to the orbits of the null flow, the following constraint is satisfied,*

$$\tilde{D}_{(A}\tilde{\omega}^{(\ell)}{}_{B)} + \tilde{\omega}^{(\ell)}{}_A\tilde{\omega}^{(\ell)}{}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} + \tfrac{1}{2}{}^{(n)}\tilde{\mathcal{R}}_{AB} = 0, \tag{5.8}$$

where $\tilde{D}_A$ *and* ${}^{(n-2)}\mathcal{R}_{AB}$ *are, respectively, the torsion-free connection and the corresponding Ricci tensor of the metric tensor* $\tilde{q}_{AB}$ *induced on* $\tilde{\triangle}$.

In the vacuum case, the geometry of extremal isolated surfaces gives rise to an equation which can be formulated in a self-contained way. Given an $(n-2)$-dimensional manifold $\tilde{\triangle}$, consider a pair $\left(\tilde{q}_{AB}, \tilde{\omega}^{(\ell)}{}_A\right)$, which consists of, respectively, a metric tensor field (of the Riemannian signature) and a differential 1-form. The equation reads

$$\tilde{D}_{(A}\tilde{\omega}^{(\ell)}{}_{B)} + \tilde{\omega}^{(\ell)}{}_A\tilde{\omega}^{(\ell)}{}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} = \tfrac{1}{2}\Lambda\tilde{q}_{AB}, \tag{5.9}$$

where Λ is the cosmological constant and $\tilde{q}_{AB}$ is still defined by (4.7). In the case when $\tilde{\triangle}$ is compact and $\Lambda = 0$, the equation has quite interesting properties. They were discussed in [25] in the $n = 4$ case. In particular, it was shown there that if $\tilde{\triangle}$ is topologically a 2-sphere, then the only axially symmetric solutions are those defined by the extremal Kerr solutions at their event horizons. The general solution to equation (5.9) is not known.

5.2. Non-expanding null surfaces admitting a two-dimensional null symmetry group

Given an isolated null surface $(\triangle, [\ell])$, *a priori* there may exist another null flow $[\ell']$ defining a symmetry of the geometry (q_{ab}, D_a) and being another isolated null surface structure. In the 4-spacetime dimensions this non-generic case of two-dimensional null symmetry group was studied in detail (see [4, 25]). In particular, an unexpected relation with the extremal isolated null surface constraints was discovered and used in the construction of examples [25]. It turns out that those results can be easily generalized to the surfaces embedded in a higher-dimensional spacetime. We are concerned with this issue in this subsection.

Suppose then that a non-expanding null surface $\triangle$ admits two distinct isolated null surface structures $[\ell]$ and $[\ell']$. Let vector fields ℓ and ℓ' be generators of the flows. There exists a real nowhere vanishing function f defined on $\triangle$ such that

$$\ell' = f\ell. \tag{5.10}$$

Each of the commutators $[\mathcal{L}_\ell, D_a]$ and $[\mathcal{L}_{\ell'}, D_a]$ is represented, respectively, by the tensors N_{ab} and N'_{ab} (5.4). According to the very assumption made in this subsection, they both identically vanish. On the other hand, generally one is related to another by the following transformation law:

$$fN'_{bc} = fN_{bc} + \omega^{(\ell)}{}_c D_b f + \omega^{(\ell)}{}_b D_c f + D_b D_c f. \tag{5.11}$$

If both vector fields ℓ and $f\ell$ are the symmetries of (q_{ab}, D_a), then both Lie derivatives vanish. The equation above becomes then a differential condition on the function f, namely

$$D_a D_b f + 2\omega^{(\ell)}{}_{(a} D_{b)} f = 0. \tag{5.12}$$

Contraction of the condition with ℓ^b gives the equation

$$D_a(\mathcal{L}_\ell f + \kappa^{(\ell)} f) = 0, \tag{5.13}$$

which allows us to determine the null evolution of f

$$\mathcal{L}_\ell f + \kappa^{(\ell)} f = \kappa^{(\ell')} = \text{const}. \tag{5.14}$$

By integrating this equation we obtain a solution whose form depends on the surface gravity $\kappa^{(\ell)}$

$$f = \begin{cases} B\,\mathrm{e}^{-\kappa^{(\ell)} v} + \frac{\kappa^{(\ell')}}{\kappa^{(\ell)}} & \kappa^{(\ell)} \neq 0 \\ \kappa^{(\ell')} v - B & \kappa^{(\ell)} = 0, \end{cases} \tag{5.15}$$

where v is a coordinate compatible with ℓ (defined via (4.1)) and B is an arbitrary real function constant along the null generators.

Note that we used the zeroth law according to which the surface gravity is constant at the surface. The zeroth law relies on stronger energy condition (3.2).

To determine the function B we need to use the remaining part of (5.12), namely its projection onto the slice $\tilde{\triangle}_v$ (see (4.9), (4.10))

$$\begin{aligned} \tilde{q}^a{}_A \tilde{q}^b_B (D_a D_b f + 2\omega^{(\ell)}{}_{(a} D_{b)} f) &= \tilde{D}_A \tilde{D}_B f + 2\tilde{\omega}^{(\ell)}{}_{(A} \tilde{D}_{B)} f - \tilde{q}^a{}_A \tilde{q}^b{}_B D_a(\ell^{b'} n_b + n^{b'} \ell_b) D_{b'} f \\ &= \tilde{D}_A \tilde{D}_B f + 2\tilde{\omega}^{(\ell)}{}_{(A} \tilde{D}_{B)} f - \tilde{S}_{AB} \mathcal{L}_\ell f. \end{aligned} \tag{5.16}$$

Without loss of generality we can restrict ourselves to the following cases:

(i) $\kappa^{(\ell)} \neq 0$
(ii) $\kappa^{(\ell)} = \kappa^{(\ell')} = 0$.

In both of them equation (5.16) is equivalent to the following differential constraint for B:

$$[\tilde{D}_A \tilde{D}_B + 2\tilde{\omega}^{(\ell)}{}_{(A} \tilde{D}_{B)} + \kappa^{(\ell)} \tilde{S}_{AB}] B = 0. \tag{5.17}$$

By the comparison with the constraint (5.7) on the isolated null surface geometry we can see that the term $\kappa^{(\ell)} \tilde{S}_{AB}$ can be replaced by the appropriate functional of $(\tilde{q}_{AB}, \tilde{\omega}^{(\ell)}{}_A, {}^{(n)}\tilde{\mathcal{R}}_{AB})$. The resulting equation leads to an interesting conclusion. Since B vanishes nowhere (the flows are both non-trivial and distinct) the set of data $(\tilde{q}_{AB}, \tilde{\omega}^{(\ell)}{}_A, {}^{(n)}\tilde{\mathcal{R}}_{AB}, B)$ satisfies the constraint (5.17) if and only if the set $(\tilde{q}_{AB}, \tilde{\omega}^{(\ell_o)}{}_A = \tilde{\omega}^{(\ell)}{}_A + \tilde{D}_A \ln B, {}^{(n)}\tilde{\mathcal{R}}_{AB})$ satisfies the constraint (5.8) for the geometry of the extremal isolated null surface. We will go back to the consequence of this result at the end of the following section.

6. Non-expanding horizons and isolated horizons

Thus far our considerations were purely local. No global assumptions concerning the null surfaces topology were made. The specific property of a quasi-locally defined black-hole is its compact character in spacelike dimensions. This notion has not been defined on the most general level. We consider in our paper the topologically simplest and, at the same time, the typical case of the Cartesian product structure:

6.1. Non-expanding horizons

Definition 6.1. *A non-expanding null surface $\triangle$ in an n-dimensional spacetime $\mathcal{M}$ is called a non-expanding horizon (NEH) if there is an embedding*

$$\hat{\triangle}'' \times \mathbb{R} \to \mathcal{M} \tag{6.1}$$

such that

- $\triangle$ *is the image,*
- $\hat{\triangle}''$ *is an* $(n-2)$*-dimensional compact manifold,*
- $\mathbb{R}$ *is the real line,*
- *for every maximal null curve in* $\triangle$ *there is* $\hat{x} \in \hat{\triangle}''$ *such that the curve is the image of* $\{\hat{x}\} \times \mathbb{R}$.

The *base space* $\hat{\triangle}$ defined as the space of all the maximal null curves in $\triangle$ can be identified with the manifold $\hat{\triangle}''$ given an embedding used in definition 6.1. Whereas the embedding is not unique, the manifold structure defined in this way on $\hat{\triangle}$ is unique. There is also a uniquely defined projection

$$\pi : \triangle \to \hat{\triangle}. \tag{6.2}$$

In this section we consider a NEH $\triangle$. Of course, it inherits all the properties of the non-expanding null surfaces. The following theorems are applications of the results of section 3 to the non-expanding horizons.

The first theorem summarizes the properties following from the weaker energy assumption (2.15):

Theorem 6.2. *Suppose $\triangle$ is a non-expanding horizon in a spacetime $\mathcal{M}$. Suppose at $\triangle$ the spacetime Einstein field equations hold and the matter fields satisfy condition (2.15). Let ℓ be an arbitrary null vector field tangent to $\triangle$ (in items (iii–vii) below); then*

(i) there is a metric tensor field $\hat{q}_{AB}$ (called a projective metric) defined on the base space $\hat{\triangle}$ such that the degenerate metric tensor q_{ab} induced in $\triangle$ by the spacetime metric tensor is given by the pullback,

$$q_{ab} = \pi^* \hat{q}_{ab}; \tag{6.3}$$

(ii) there is a covariant derivative D_a defined in the tangent bundle $T\triangle$ such that, for every two vector fields X, Y,

$$D_X Y = \nabla_X Y, \tag{6.4}$$

where ∇_α is the spacetime covariant derivative;

(iii) there is a 1-form $\omega^{(\ell)}{}_a$ (called the rotation 1-form potential) defined on $\triangle$ such that

$$D_a \ell^b = \omega^{(\ell)}{}_a \ell^b; \tag{6.5}$$

(iv) the rotation 2-form (invariant)

$$\Omega_{ab} := D_a \omega^{(\ell)}{}_b - D_b \omega^{(\ell)}{}_a \tag{6.6}$$

is uniquely independent of ℓ;

(v) the rotation 1-form potential and the self-acceleration $\kappa^{(\ell)} := \ell^a \omega^{(\ell)}{}_a$ of ℓ satisfy

$$\mathcal{L}_\ell \omega^{(\ell)}{}_a = D_a \kappa^{(\ell)} + {}^{(n)}\mathcal{R}_{ab} \ell^b; \tag{6.7}$$

(vi) the infinitesimal Lie transport of D_a with respect to the null flow of ℓ is the following tensor:

$$[\mathcal{L}_\ell, D_a]^b_c = \ell^b \big(D_{(a}\omega^{(\ell)}{}_{c)} + \omega^{(\ell)}{}_a \omega^{(\ell)}{}_c + \tfrac{1}{2}({}^{(n)}\mathcal{R}_{ac} - \pi^{*(n-2)}\mathcal{R}_{ac}) \big) \tag{6.8}$$

where ${}^{(n-2)}\mathcal{R}_{AB}$ is the Ricci tensor of the metric tensor $\hat{q}_{AB}$;

(vii) the following components of the pullback onto $\triangle$ of the spacetime Ricci and Riemann tensor vanish:

$$\ell^\alpha \ell^\beta \, {}^{(n)}\mathcal{R}_{\alpha\beta} = 0 = \ell^\alpha \, {}^{(n)}R_{\alpha bcd}. \tag{6.9}$$

In the previous sections we also considered stronger energy condition (3.2). The following theorem summarizes its consequences for a non-expanding horizon:

Theorem 6.3. *Suppose all the assumptions of theorem 6.2 are satisfied and additionally the matter fields at $\triangle$ satisfy stronger energy condition (3.2); then*

- *on the base space $\hat{\triangle}$, there is a uniquely defined 2-form $\hat{\Omega}_{AB}$ such that the rotation 2-form invariant is its pullback*

$$\Omega_{ab} = \pi^* \hat{\Omega}_{ab}; \tag{6.10}$$

- *the rotation 1-form potential and the self-acceleration satisfy*

$$\mathcal{L}_\ell \omega^{(\ell)}{}_a = D_a \kappa^{(\ell)}; \tag{6.11}$$

- *the pullback of the spacetime Ricci tensor and, respectively, the spacetime Weyl tensor onto $\triangle$ is transversal to the null direction tangent to $\triangle$,*

$$\ell^\alpha \, {}^{(n)}\mathcal{R}_{\alpha b} = 0 = \ell^\alpha \, {}^{(n)}C_{\alpha bcd}. \tag{6.12}$$

We will consider now the non-expanding horizons where theorem 6.3 applies.

In the case of the non-expanding horizons, there are globally defined, nowhere vanishing null vector fields ℓ^a tangent to $\triangle$ at our disposal. In particular, there is a vector field ℓ^a_o of the identically vanishing self-acceleration, $\kappa^{(\ell_o)} = 0$. There is also a null vector field ℓ^a of $\kappa^{(\ell)}$ being an arbitrary constant[10],

$$\kappa^{(\ell)} = \text{const}. \tag{6.13}$$

The vector field ℓ^a can vanish in a harmless, for our purposes, way on an $(n-2)$-dimensional section of $\triangle$ only. We fix one of the vector fields ℓ^a (including (ℓ^a_o)) throughout this section. We will also use a coordinate v compatible with the vector field ℓ^a ($\ell^a D_a v = 1$), and the covector field n_a ($= -D_a v$), both introduced in section 4.1 defined on $\triangle$ (except the zero slice of ℓ). It follows from the zeroth law (6.11) that the rotation 1-form potential is Lie dragged by ℓ,

$$\mathcal{L}_\ell \omega^{(\ell)}{}_a = 0. \tag{6.14}$$

We discuss below two independent consequences of this fact. The first one is the existence of a new invariant of the geometry of $\triangle$, a certain harmonic 1-form invariantly defined on the base manifold $\hat{\triangle}$. The second one concerns the degrees of freedom of a general vacuum solution (q_{ab}, D_a) of the constraints (3.7), (3.36), (3.37).

[10] The first one, ℓ_o, can be defined by fixing appropriately the affine parameter v at each null curve in $\triangle$. Then, the second vector field is just $\ell = v\ell_o$.

6.1.1. Harmonic invariant. It turns out that $\omega^{(\ell)}{}_a$ defines on the base space $\hat{\triangle}$ a unique 1-form depending only on the geometry (q_{ab}, D_a) of $\triangle$. Indeed, given the function v, there is a differential 1-form field $\hat{\omega}^{(\ell)}{}_A$ defined on $\hat{\triangle}$ and called the projective rotation 1-form potential such that

$$\omega^{(\ell)}{}_a = \pi^* \hat{\omega}^{(\ell)}{}_a + \kappa^{(\ell)} D_a v. \tag{6.15}$$

The 1-form $\hat{\omega}^{(\ell)}{}_A$ is not uniquely defined, though. It depends on the choice of the function v compatible with ℓ^a, and on the choice of ℓ^a itself. Given ℓ^a, the freedom is in the transformations

$$v = v' + B, \qquad \mathcal{L}_\ell B = 0, \tag{6.16a}$$

$$\hat{\omega}^{(\ell)\prime}{}_A = \hat{\omega}^{(\ell)}{}_A + \kappa^{(\ell)} \hat{D}_A B. \tag{6.16b}$$

The transformations $\ell'^a = f\ell^a$ which preserve condition (6.13) are given by (5.15), and it can be shown using (3.31) that the only possible form of the corresponding $\tilde{\omega}^{(\ell')}{}_A$ is again that of (6.16*b*) with possibly different function B. Therefore, if we apply the (unique) Hodge decomposition [26] onto the exact, co-exact and the harmonic part, respectively, to $\hat{\omega}^{(\ell)}{}_A$,

$$\hat{\omega}^{(\ell)}{}_A = \hat{\omega}^{(\ell)\mathrm{ex}}{}_A + \hat{\omega}^{(\ell)\mathrm{co}}{}_A + \hat{\omega}^{(\ell)\mathrm{ha}}{}_A, \tag{6.17}$$

then the parts $\hat{\omega}^{(\ell)\mathrm{co}}{}_A$ and $\hat{\omega}^{(\ell)\mathrm{ha}}{}_A$ are invariant, that is determined by the geometry (q_{ab}, D_a) of $\triangle$ only. The co-exact part is determined by the already defined invariant 2-form (3.32), via

$$\hat{\Omega}_{AB} = \hat{D}_A \hat{\omega}^{(\ell)\mathrm{co}}{}_B - \hat{D}_B \hat{\omega}^{(\ell)\mathrm{co}}{}_A. \tag{6.18}$$

The harmonic part of $\hat{\omega}^{(\ell)}{}_A$ is the new invariant. It did not occur in the case of spherical $\hat{\triangle}$ considered in [4]. In the case of a general topology of $\hat{\triangle}$, the invariant may be relevant. There are known non-trivial examples of black holes with non-spherical base spaces. For instance, in five dimensions there exists asymptotically flat, regular, axi-symmetric solutions (see [16] for details) of the horizon base space topology $S^1 \times S^2$. The space of harmonic 1-forms is finite dimensional, so the degrees of freedom identified with the harmonic component of the rotation 1-form potential are global in character.

6.1.2. Degrees of freedom. Let ℓ^a, v and n_a be still the same, respectively, vector field, a compatible coordinate and a covector field introduced below theorem 6.3. The covariant derivative D_a is characterized by the elements defined in section 4.2, subject to the constraints (3.36), (3.37). Suppose the vacuum Einstein equations with a (possibly zero) cosmological constant are satisfied on $\triangle$.

The geometry (q_{ab}, D_a) can be completely characterized by the following data:

(i) the data defined on the space of the null geodesics $\hat{\triangle}$:
- the projective metric tensor $\hat{q}$ (6.3)
- the projective rotation 1-form potential $\hat{\omega}^{(\ell)}{}_A$ (3.32)
- the projective transversal expansion-shear data $\hat{S}^o_{AB}$ (see (6.19) below)

(ii) the values of the surface gravity $\kappa^{(\ell)}$ and the cosmological constant Λ,

where the projective transversal expansion-shear data $\hat{S}^o_{AB}$ is a tensor defined on $\hat{\triangle}$ by the following form of a general solution of (4.15),

$$\tilde{S}_{AB} = \begin{cases} \big(\hat{D}_{(A}\hat{\omega}^{(\ell)}{}_{B)} + \hat{\omega}^{(\ell)}{}_A\hat{\omega}^{(\ell)}{}_B - \frac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} - \frac{1}{2}\Lambda\hat{q}_{AB}\big)v + \pi^*\hat{S}^o{}_{AB} \\ \quad \text{for} \quad \kappa^{(\ell)} = 0, \\ e^{-\kappa^{(\ell)}v}\pi^*\hat{S}^o{}_{AB} + \frac{1}{\kappa^{(\ell)}}\big(\hat{D}_{(A}\hat{\omega}^{(\ell)}{}_{B)} + \hat{\omega}^{(\ell)}{}_A\hat{\omega}^{(\ell)}{}_B - \frac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} - \frac{1}{2}\Lambda\hat{q}_{AB}\big) \\ \quad \text{otherwise.} \end{cases} \tag{6.19}$$

A part of data depends on the choice of the vector field ℓ^a and the compatible coordinate v. Given ℓ^a such that $\kappa^{(\ell)} \neq 0$, the compatible coordinate v can be fixed up to a constant by requiring that the exact part in the Hodge decomposition of the projective rotation 1-form potential $\hat{\omega}^{(\ell)}{}_A$ vanishes (see section 6.1.1). The vector ℓ^a itself, generically, can be fixed up to a constant factor by requiring that the projective transversal expansion-shear data $\hat{S}^o{}_{AB}$ be traceless. Indeed, the transformations of ℓ^a such that $\kappa^{(\ell)}$ remains a non-zero constant are given by (5.10), (5.15). They are accompanied by the transformations $\mathcal{L}_\ell S_{ab} \mapsto \mathcal{L}_{\ell'} S'_{ab}$ determined by (5.11). Contraction of the mentioned equation with the tensor $\tilde{q}^{ab}$ defined as follows,

$$\pi_* \tilde{q}^{ab} = \hat{q}^{ab}, \qquad \tilde{q}^{ab} n_b = 0, \tag{6.20}$$

and the assumption that $\tilde{q}^{ab} \mathcal{L}_{\ell'} S_{ab} = 0$ (equivalent to $\hat{q}^{AB} \hat{S}^o{}_{AB} = 0$) produces the following gauge condition defined on the slices:

$$[\tilde{D}^2 + 2\tilde{\omega}^{(\ell)A} \tilde{D}_A + \tilde{q}^{AB}(\kappa^{(\ell)} \tilde{S}_{AB} + \mathcal{L}_\ell \tilde{S}_{AB})]B = \frac{\kappa^{(\ell')}}{\kappa^{(\ell)}} \mathrm{e}^{\kappa^{(\ell)} v} \tilde{q}^{AB} \mathcal{L}_\ell \tilde{S}_{AB}. \tag{6.21}$$

According to the zeroth law and (6.19) the above equation defines at each slice the same constraint for a NEH's geometry, and can be rewritten in terms of the objects defined on the base space $\hat{\triangle}$. Hence, the condition that $\hat{S}_{AB}$ be traceless takes the form of the following elliptic equation on the function B,

$$\left[\hat{D}^2 + 2\hat{\omega}^{(\ell)A} \hat{D}_A + \operatorname{div} \hat{\omega}^{(\ell)} + |\hat{\omega}^{(\ell)}|^2 - \frac{1}{2} \hat{q}^{AB\ (n-2)}\mathcal{R}_{AB} - \frac{n-2}{2}\Lambda\right] B = \kappa^{(\ell')} \hat{q}^{AB} \hat{S}^o{}_{AB}, \tag{6.22}$$

where $\operatorname{div} \hat{\omega}^{(\ell)} := \hat{D}^A \hat{\omega}^{(\ell)}{}_A$ and $|\hat{\omega}^{(\ell)}|^2 := \hat{\omega}^{(\ell)A} \hat{\omega}^{(\ell)}{}_A$. The equation generically has a unique solution. Finally, the remaining rescaling freedom by a constant can be removed by fixing the value of the surface gravity $\kappa^{(\ell)}$ arbitrarily (the area of $\triangle$ can be used as a quantity providing the appropriate units).

6.1.3. Abstract non-expanding null surface/horizon geometry. Non-expanding null surface geometry can be defined abstractly. Consider an $(n-1)$-dimensional manifold $\triangle$. Let q_{ab} be a symmetric tensor of the signature $(0, +, \ldots, +)$. Let D_a be a covariant, torsion-free derivative such that

$$D_a q_{bc} = 0. \tag{6.23}$$

A vector ℓ^a tangent to $\triangle$ is called null whenever

$$\ell^a q_{ab} = 0. \tag{6.24}$$

Even though we are not assuming any symmetry, every null vector field ℓ^a is a symmetry of q_{ab}:

Lemma 6.4. *Suppose ℓ^a is a null vector field tangent to $\triangle$; then*

$$\mathcal{L}_\ell q_{ab} = 0. \tag{6.25}$$

Despite the fact that the lemma is quite surprising, the proof is not difficult. We leave it to the interested reader.

Given a null vector field ℓ^a, we can repeat the definitions of section 3, and associate with it the surface gravity $\kappa^{(\ell)}$, and the rotation 1-form potential $\omega^{(\ell)}$. Now, a vacuum Einstein constraint can be defined as an equation on the geometry (q_{ab}, D_a) per analogy with the non-expanding null surface case. To spell it out we need one more definition. Introduce on

$\triangle$ a symmetric tensor ${}^{(n-2)}\mathcal{R}_{ab}$ such that for every $(n-2)$-subsurface contained in $\triangle$ the pullback of ${}^{(n-2)}\mathcal{R}_{ab}$ to the subsurface coincides with the Ricci tensor of the induced metric, provided the pullback $\tilde{q}_{AB}$ of q_{ab} is a non-degenerate metric tensor. The vacuum constraint is defined as

$$[\mathcal{L}_\ell, D_a]^b_c = \left(D_{(a}\omega^{(\ell)}{}_{c)} + \omega^{(\ell)}{}_a\omega^{(\ell)}{}_c - \tfrac{1}{2}\Lambda q_{ac}\right) - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{ac}, \tag{6.26}$$

and it involves an arbitrary cosmological constant Λ.

Suppose now that

$$\triangle = \hat{\triangle} \times \mathbb{R}, \tag{6.27}$$

and the tensor q_{ab} is the product tensor defined naturally by a metric tensor $\hat{q}_{AB}$ defined in $\hat{\triangle}$ and the identically zero tensor defined in $\mathbb{R}$. The analysis of sections 6.1.1 and 6.1.2 can be repeated for solutions of the vacuum Einstein constraint (6.26). Again the base space $\hat{\triangle}$ is equipped with the data of section 6.1.2, that is the projective: metric tensor $\hat{q}_{AB}$, rotation 1-form potential $\hat{\omega}^{(\ell)}{}_A$, transversal expansion-shear data $\hat{S}^o{}_{AB}$. Completed by the values of the surface gravity $\kappa^{(\ell)}$ and the cosmological constant Λ the data are free, in the sense that every data set defines a single solution (q_{ab}, D_a).

6.2. Isolated horizons

Definition 6.5. *An isolated null surface* $(\triangle, [\ell])$ *such that the surface* $\triangle$ *is a non-expanding horizon is called an isolated horizon (IH).*

Consider an arbitrary isolated horizon $(\triangle, [\ell])$. A generator ℓ of the null symmetry is defined globally on $\triangle$, and it is unique modulo the rescaling $\ell' = a_0\ell$ by a constant a_0. Therefore, the rotation 1-form potential ω is defined globally on $\triangle$ and in an independent of the choice of $\ell' \in [\ell]$ manner (hence we will drop in this section the suffix (ℓ) on ω but keep it on the surface gravity). It follows from section 5 that for every isolated horizon $(\triangle, [\ell])$ the rotation 1-form potential is Lie dragged by the vector field ℓ, additionally the energy condition (2.15) is satisfied necessarily and the left-hand side of (6.8) is assumed to be zero. In conclusion,

(i) the rotation 1-form potential is Lie dragged by the horizon symmetry ℓ

$$\mathcal{L}_\ell\omega_a = 0; \tag{6.28}$$

(ii) if the matter fields satisfy stronger energy condition (3.2) on $\triangle$, then the surface gravity $\kappa^{(\ell)}$ is constant,

$$\kappa^{(\ell)} = \text{const}; \tag{6.29}$$

(iii) the pullback of the spacetime Ricci tensor on $\triangle$ is Lie dragged by $[\ell]$,

$$\mathcal{L}_\ell\,{}^{(n)}\mathcal{R}_{ab} = 0; \tag{6.30}$$

(iv) in the case when stronger energy condition (3.2) holds, the tensor S_{ab} has the form

$$S_{ab} = \pi^*\hat{S}_{ab} - 2\omega_{(a}n_{b)} - \kappa^{(\ell)}n_an_b; \tag{6.31}$$

where $\hat{S}_{AB}$ is a symmetric tensor defined on $\hat{\triangle}$;

(v) the constraint (4.15) applied to S_{ab} above reads

$$\kappa^{(\ell)}\hat{S}_{AB} = \hat{D}_{(A}\hat{\omega}_{B)} + \hat{\omega}_A\hat{\omega}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} + \tfrac{1}{2}{}^{(n)}\hat{\mathcal{R}}_{AB}, \tag{6.32}$$

where by ${}^{(n)}\hat{\mathcal{R}}_{AB}$ we denoted the tensor uniquely defined on $\hat{\triangle}$ such that $\pi^{*(n)}\hat{\mathcal{R}}_{ab} = {}^{(n)}\mathcal{R}_{ab}$.

The classification of the isolated null surfaces with respect to whether $\kappa^{(\ell)}$ vanishes or not applies to the isolated horizons, therefore we call an isolated horizon extremal whenever $\kappa^{(\ell)} = 0$, and non-extremal otherwise.

6.2.1. Degrees of freedom: the non-extremal case. Suppose the vacuum Einstein equations (with a possibly non-zero cosmological constant) hold on an isolated horizon $(\triangle, [\ell])$, and

$$\kappa^{(\ell)} \neq 0. \tag{6.33}$$

Since $\triangle$ is a non-expanding horizon, its geometry can be characterized by the data (i) and (ii) discussed in section 6.1.2. Now, however, the data satisfy an extra constraint following from (5.7), namely

$$\hat{S}^o_{AB} = 0 \tag{6.34}$$

in (6.19). Therefore, in the non-extremal isolated horizon case, given a generator ℓ of the flow $[\ell]$, the geometry (q_{ab}, D_a) is completely determined by the projective metric and the projective 1-form potential $(\hat{q}_{AB}, \hat{\omega}_A)$ defined on the base manifold $\hat{\triangle}$, provided the surface gravity $\kappa^{(\ell)}$ and the cosmological constant are given. The discussion of the gauge degrees of freedom in the data of section 6.1.2 applies, except that in this case the null flow $[\ell]$ is given. The data $\hat{q}_{ab}$ and $\hat{\omega}_A$ are free in the sense of section 6.1.3.

6.2.2. Degrees of freedom—the extremal case. In the extremal case, on the other hand, the vacuum isolated horizon constraints (5.7) do not constrain the projective transversal expansion-shear data $\hat{S}^o_{AB}$ at all. On the other hand, the projective metric tensor $\hat{q}_{AB}$ and the projective rotation 1-form potential $\hat{\omega}_A$ necessarily satisfy a constraint

$$\big(\hat{D}_{(A}\hat{\omega}_{B)} + \hat{\omega}_A\hat{\omega}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} - \tfrac{1}{2}\Lambda\hat{q}_{AB}\big) = 0. \tag{6.35}$$

The general solution to this equation is not known even in the case of four-dimensional spacetimes; however, the number $\frac{1}{2}n(n-1)(n-2)$ of the equations is in the space of the solutions to the constraints (5.1) equal to the number of the independent variables: $\frac{1}{2}(n-2)(n-3)$ for the metric [27] plus $n-2$ for the rotation. Therefore, one can expect that extremal isolated horizons should be described by the global degrees of freedom.

In the extremal case, as opposed to the non-extremal case, the projective rotation 1-form potential $\hat{\omega}_A$ is uniquely defined on $\hat{\triangle}$, including the exact part. The projective transversal expansion-shear tensor $\hat{S}_{AB}$, on the other hand, is still the choice of the compatible coordinate v dependent. The transformation of projective tensor $\hat{S}_{AB} = \hat{S}^o{}_{AB}$ is

$$\hat{S}_{AB} \to \hat{S}_{AB} + [\hat{D}_A\hat{D}_B + 2\hat{\omega}_{(A}\hat{D}_{B)}]f. \tag{6.36}$$

The trace of this equation with respect to $\hat{q}_{AB}$ becomes an elliptic PDE for the function f. Therefore generically there is a possibility of distinguishing the coordinate v (and the corresponding family of sections of $\triangle$) by the requirement that the trace of $\hat{S}_{AB}$, as well as the trace of $\tilde{S}_{AB}$, be zero.

Finally, in the sense of section 6.1.3, the degrees of freedom in the space of the extremal isolated horizons, solutions to the constraints (5.1) are given by the solutions $(\hat{q}_{AB}, \hat{\omega}_A)$ of the constraint (6.35), and the traceless part of the transversal expansion shear tensor $\hat{S}_{AB}$.

6.3. Non-expanding horizons with two-dimensional null symmetry group

In the four-dimensional case (see [4]) there exist non-expanding horizons which admit a two-dimensional group of the null symmetries. In section 5.2 we investigated the conditions for the existence of more than one null symmetry of an isolated null surface in arbitrary dimension. In this section we will investigate further the geometries of the isolated horizons admitting more than one null symmetry. We will show the following theorem:

Theorem 6.6. *Suppose $\triangle$ is a non-expanding horizon which admits two distinct isolated horizon structures. Suppose the energy condition (3.2) holds on $\triangle$. Then, $\triangle$ admits an extremal isolated horizon structure $[\ell']$ and a compatible coordinate v' such that the corresponding transversal expansion-shear tensor $\tilde{S}_{AB}$ identically vanishes at $\triangle$.*

Proof. Let $[\ell]$ and $[\ell']$ be two different isolated horizon structures at $\triangle$ generated by ℓ and ℓ', respectively. According to the zeroth law, the surface gravities $\kappa^{(\ell)}$ and $\kappa^{(\ell')}$ are constant on the horizon. Suppose

$$\kappa^{(\ell)} \neq 0. \tag{6.37}$$

Let $v : \triangle \to \mathbb{R}$ be a compatible coordinate. The relation between ℓ and ℓ' is

$$\ell' = f\ell. \tag{6.38}$$

where the function f is of the form

$$f = \begin{cases} B\,\mathrm{e}^{-\kappa^{(\ell)}v} + \frac{\kappa^{(\ell')}}{\kappa^{(\ell)}} & \kappa^{(\ell)} \neq 0 \\ \kappa^{(\ell')}v - B & \kappa^{(\ell)} = 0, \end{cases} \tag{6.39}$$

where the function B is constant along the null curves in $\triangle$, and the necessary and sufficient condition (5.17) for the function B thought of as a function $B : \hat{\triangle} \to \mathbb{R}$ can be rewritten in terms of the data defined on the base manifold $\hat{\triangle}$ of the null curves:

$$\big[\hat{D}_A\hat{D}_B + 2\hat{\omega}^{(\ell)}{}_{(A}\hat{D}_{B)} + \hat{D}_{(A}\hat{\omega}^{(\ell)}{}_{B)} + \hat{\omega}^{(\ell)}{}_A\hat{\omega}^{(\ell)}{}_B - \tfrac{1}{2}{}^{(n-2)}\mathcal{R}_{AB} + \tfrac{1}{2}{}^{(n)}\hat{\mathcal{R}}_{AB}\big]B = 0. \tag{6.40}$$

where $\hat{D}_{(A}\hat{\omega}^{(\ell)}{}_{B)}$ should be considered as a tensor, not as an operator.

Note, however, that the function B is independent of the surface gravity of another vector field we construct with, therefore, either $\kappa^{(\ell'')} = 0$, or, given the functions f'' and B, we can define a new function f',

$$f' := B\,\mathrm{e}^{-\kappa^{(\ell)}v}. \tag{6.41}$$

Then, the vector field $\ell' = f'\ell$ defines an extremal isolated horizon. We will show now that there is a coordinate $v' : \triangle \to \mathbb{R}$ compatible with ℓ' such that the corresponding projective transversal expansion-shear data $\hat{S}'^{o}_{AB}$ are identically zero. According to equation (6.41) the general form of a coordinate v' compatible with ℓ' is

$$v' = (\kappa^{(\ell)}B)^{-1}\big(\mathrm{e}^{\kappa^{(\ell)}v} - 1\big) + v'_0, \tag{6.42}$$

where v'_0 is a function constant along the null curves in $\triangle$. Let us choose v'_0 to be

$$v'_0 := (\kappa^{(\ell)}B)^{-1}. \tag{6.43}$$

Then the correspondence between the vector fields $n'_a = -D_a v'$ and $n_a = -D_a v$ can be described by the following equation:

$$n'_a = v'(\kappa^{(\ell)}n_a - D_a \ln B). \tag{6.44}$$

The covariant derivative of the above equation determines the transformation $S_{ab} \mapsto S'_{ab}$:

$$\frac{1}{v'}S'_{ab} - \frac{1}{v'^2}n'_a n'_b = S_{ab} - D_a D_b \ln B. \tag{6.45}$$

Taking the Lie derivative of this formula with respect to ℓ and taking into account that $\mathcal{L}_{\ell'}S'_{ab} = \mathcal{L}_\ell S_{ab} = 0$ we get the following result:

$$S'_{ab} = \omega^{(\ell')}{}_{(a}n'_{b)}. \tag{6.46}$$

Therefore the pullback $\tilde{S}'_{AB}$ of S'_{ab} identically vanishes, and so does the projective part $\hat{S}'_{AB}$ defined by (6.19). □

In the case when both the symmetries ℓ and ℓ' admit extremal isolated horizon structures equation (6.40) takes the following form:

$$[\hat{D}_A\hat{D}_B + 2\hat{\omega}^{(\ell)}{}_{(A}\hat{D}_{B)}]B = 0. \tag{6.47}$$

Together with the constraint (6.35) the equation above forms an overdefined system involving the data $\hat{q}, \hat{\omega}^{(\ell)}$. The non-existence of the solutions to the system in the case of the horizon embedded in a four-dimensional electrovac spacetime with vanishing cosmological constant was shown in [4, 25]; however, one cannot expect to repeat this result in the general case. It seems that the answer for the question whether the solutions to the system (6.35), (6.47) do exist requires an analysis for each case of the assumed dimension and the topology of a horizon base space separately.

7. Conclusion

It turns out that the basic properties of null, non-expanding surfaces are not sensitive to the spacetime dimension. We have discussed only those properties which were found relevant in the four-dimensional case. The exception is the characterization of the surfaces admitting a two-dimensional group of null symmetries and the relation with the extremal isolated surface constraint.

A new element in the characterization of the non-expanding, null surfaces is the harmonic 1-form invariant defined by the rotation 1-form potential on the space of the null generators of the surface.

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Appendix. Remarks on the exotic matter case

In the development of the objects describing the geometrical structure of the non-expanding and isolated horizons, it was assumed that the energy condition (3.2) holds for the matter fields on the horizon. On the other hand, one may need to deal with the models in which the considered condition has to be dropped. Then the question arises: How many of the structures developed here still apply? The current section is an attempt to answer this question.

A.1. Non-expanding horizons

In this paper the energy condition (3.2) was in fact used to develop identity (3.21) only. As the mentioned identity is equivalent to the condition

$$T_{ab}\ell^b = 0, \tag{A1}$$

involving the pullback of $T_{\mu\nu}$ onto the horizon, even if condition (3.2) is not fulfilled, all the statements still apply as long as equation (A1) holds. As a 'toy'[11] example of the matter

[11] This particular case is already exhibited in this paper as the Einstein equation with cosmological constant was considered. We use it only to illustrate the existence of possible models for which (A1) holds and condition (3.2) does not. The real applications would be the 'varying cosmological constant' models.

field satisfying (A1) we can consider the cosmological constant represented as a matter field $T_{\mu\nu} = -\Lambda g_{\mu\nu}$. For negative Λ the stronger energy condition (3.2) does not hold any longer ($-T^{\nu}{}_{\nu}\ell^{\nu}$ is past-oriented) but equality (A1) is still true.

Consider now the most general situation when nothing about the energy–momentum tensor is assumed. Then the statements of theorem 6.2 are no longer true: the shear of the NEH does not have to vanish because the $|\sigma^{(\ell)}|^2$ term in the Raychaudhuri equation can be balanced by the negative ${}^{(n)}\mathcal{R}_{\ell\ell}$. The presence of shear affects the entire geometrical structure. For example, only the area form of the metric tensor $\bar{q}_{AB}$ is preserved by the null flow

$$\mathcal{L}_\ell \bar{\epsilon} = 0. \tag{A2}$$

Moreover, the spacetime covariant derivative does not preserve the tangent bundle of the horizon. Therefore, the internal connection D of the horizon must be introduced another way than (3.11) and none of the statements in section 6.1 will hold.

One of methods to deal with the problem is to restrict investigated objects to the *non-expanding shear-free horizons* defined as the null surfaces equipped with a metric q preserved by the null flow. The restriction makes sense as the NEHs admitting the isolated horizon structure (which includes also Killing horizons) necessarily have to belong to this class. Note that for that class of the horizons condition (2.15) is satisfied due to the Raychaudhuri equation, so the discussion in the main body of the paper applies here.

The other restriction we can make is to impose the weaker energy condition (2.15) only. This case has been investigated thorough the main part of the paper: the statements of theorem 6.2 are true for the considered NEH, whereas those for theorem 6.3 are not. Note that however the horizon can no longer be of type II in principal classification, the component ${}^{(n)}C_{a\ell b\ell}$ still vanishes, so the horizon is at least of type I (remaining algebraically special).

A.2. Isolated horizons

If we assume that the horizon admits an isolated horizon structure, its shear and Ricci component ${}^{(n)}\mathcal{R}_{\ell\ell}$ vanish due to the existence of a null symmetry without any assumption imposed on the energy–momentum tensor of the matter fields.

Because of the modification of the zeroth law the statement $\kappa^{(\ell)} = \text{const}$ must be replaced by the following one:

$$\partial_a \kappa^{(\ell)} = -{}^{(n)}\mathcal{R}_{ab}\boldsymbol{\ell}^b, \tag{A3}$$

so the 'surface gravity' defined as $\boldsymbol{\ell}^a\omega_a$ becomes a function constant along the null generators

$$\mathcal{L}_{\boldsymbol{\ell}}\kappa^{(\ell)} = 0. \tag{A4}$$

Now without any additional energy assumptions the division into the non-extremal and extremal IH structures is no longer valid as the structures with $\kappa^{(\ell)} = 0$ at some open subset of $\tilde{\Delta}$ and $\kappa^{(\ell)} \neq 0$ elsewhere are possible. Therefore the structure of the constraint (6.32) (so the structure of the local degrees of freedom) can be different at distinct open subsets of the horizon base space.

The problem of classification (and description of the degrees of freedom) can be dealt with by imposing other assumptions which are satisfied by some class of exotic matter fields.

As the (spacetime) energy–momentum tensor of the matter fields is necessarily divergence free, the following constraint is true in particular at the horizon:

$$\boldsymbol{\ell}^\mu T_{\mu\nu}{}^{;\nu} = 0. \tag{A5}$$

Suppose now the field ℓ and a null vector field n orthogonal to surfaces $\tilde{\triangle}_v := \{p \in \triangle : v(p) = \text{const}\}$ (with v being the coordinate compatible with ℓ) are extended to the spacetime neighbourhood of $\triangle$ such that at the horizon

$$n^\mu \nabla_\mu \ell^\nu = n^\mu \nabla_\mu n^\nu = 0. \tag{A6}$$

Using identity (6.30) we can after simple calculations express condition (A5) as an $(n-2)$-dimensional differential equation defined on each surface $\tilde{\triangle}_v$

$$0 = \tilde{\Delta}\kappa^{(\ell)} - \tilde{q}^{AB}\tilde{\omega}_B \tilde{D}_A \kappa^{(\ell)} + \mathcal{L}_n{}^{(n)}\mathcal{R}_{\ell\ell}. \tag{A7}$$

The only part inhomogeneous in $\kappa^{(\ell)}$ is a transversal derivative of the Ricci tensor component $\mathcal{L}_n{}^{(n)}\mathcal{R}_{\ell\ell}|_{\tilde{\triangle}}$. If it vanishes on the horizon

$$\mathcal{L}_n{}^{(n)}\mathcal{R}_{\ell\ell}|_{\triangle} = 0, \tag{A8}$$

then according to the vanishing of $\mathcal{L}_\ell \tilde{q}_{AB}$ and (A4) the equation can be rewritten as a PDE defined on the base space:

$$0 = \hat{\Delta}\kappa^{(\ell)} - \hat{q}^{AB}\hat{\omega}_B \hat{D}_A \kappa^{(\ell)}. \tag{A9}$$

The equation is now a homogeneous elliptic PDE defined on a compact manifold. Therefore if $\kappa^{(\ell)}$ vanishes on some open subset of $\hat{\triangle}$ then it must vanish on the entire horizon (as both $\kappa^{(\ell)}$ and $\hat{D}_A\kappa^{(\ell)}$ vanish at the edge of the considered subset). Hence the following is true:

Corollary A.1. *Given an isolated horizon $\triangle$ equipped with a symmetry ℓ and embedded in a spacetime satisfying the Einstein field equations, assume that*

$$\mathcal{L}_n T_{\ell\ell}|_{\tilde{\triangle}} = 0, \tag{A10}$$

for some null vector field n transversal to the leaves of the horizon foliation preserved by the flow $[\ell]$. Then the horizon must belong to one of the following classes:

(i) extremal isolated horizons: surface gravity vanishes everywhere
(ii) non-extremal ones: $\kappa^{(\ell)} \neq 0$ on a dense subset of $\triangle$.

The assumed condition is equivalent to (A8). When it is satisfied by the matter fields the partition defined in corollary A.1 can be used instead of the partition proposed in section 6.2. The structure of the constraint (6.32) then remains global.

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