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Irreversibility of world-sheet renormalization group flow

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Abstract

We demonstrate the irreversibility of a wide class of world-sheet renormalization group (RG) flows to first order in α' in string theory. Our techniques draw on the mathematics of Ricci flows, adapted to asymptotically flat target manifolds. In the case of somewhere-negative scalar curvature (of the target space), we give a proof by constructing an entropy that increases monotonically along the flow, based on Perelman's Ricci flow entropy. One consequence is the absence of periodic solutions, and we are able to give a second, direct proof of this. If the scalar curvature is everywhere positive, we instead construct a regularized volume to provide an entropy for the flow. Our results are, in a sense, the analogue of Zamolodchikov's c-theorem for world-sheet RG flows on noncompact spacetimes (though our entropy is not the Zamolodchikov C-function).

0. Introduction

For a wide class of 2-dimensional quantum field theories, Zamolodchikov's c-theorem [1] demonstrates the irreversibility of renormalization group (RG) flow. In particular, there exists a function defined on the space of 2D renormalizable field theories, the C-function, which decreases along RG trajectories and is stationary only at RG fixed points. The fixed points of the RG flow are conformal field theories. The C-function equals the central charge of the conformal field theory at the fixed points. However, as shown by Polchinski [2], the c-theorem is not valid for a very important class of 2-dimensional quantum field theories of relevance to string theory: the world-sheet nonlinear sigma model on *noncompact* target spaces. There exists no general proof that RG flows of world-sheet sigma models are irreversible. Indeed, violations of irreversibility are known for other kinds of field theory RG flows (cf. [3] and references therein). A recent focus of study in this area has been RG flows of sigma models with target spaces that are 2-dimensional and noncompact, and the question of whether the *mass* of

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the target spacetime changes monotonically along the flow has been investigated [4–7]. However, the mass at infinity does not change *along* the flow; it changes only at the final fixed point.

In this Letter, we demonstrate the irreversibility of world-sheet RG flow to first order in α' on complete, asymptotically flat Riemannian manifolds (or static slices of spacetimes) when all fields other than the metric and the dilaton are set to zero. In addition to proving that this class of flows does not contain periodic solutions, we construct an *entropy* (by which we mean a Liapunov function) that increases monotonically along the flow. Thus, in a sense, our result generalises Zamolodchikov's c-theorem to a wide class of string theory world-sheet RG flows on noncompact spacetimes. Our entropy is not, however, the C-function. The C-function for the world-sheet sigma model was computed by Tseytlin [8], who showed that it could be obtained as a generalized transform of the low-energy string effective action divided by manifold volume. Thus for the class of RG flows on noncompact spacetimes that we are interested in, the C-function is zero and cannot serve as an entropy along the flow.

We caution that we work only to first order in α', the square of the string scale. As is well known for compact target spaces with $R > 0$, for example, higher order stringy corrections cannot be neglected. For flows where even just the second order corrections in α' to the beta functions become significant, we are presently unable to rule out periodic solutions using our methods. This is because our techniques draw on the theory of quasi-linear differential equations, which does not apply (at least not straightforwardly) to higher-order RG flow.

To first order in α', the RG equations for the (string frame) metric g_{ab} and the dilaton $\Phi_D = \Psi/2$ are

$$\frac{\partial}{\partial\tau} g_{ij} = -\alpha'(R_{ij} + \nabla_i\nabla_j\Psi), \tag{1}$$

$$\frac{\partial}{\partial\tau}\Psi = \frac{\alpha'}{2}\bigl(\Delta\Psi - |\nabla\Psi|^2\bigr). \tag{2}$$

Here τ is the logarithm of the world-sheet RG scale. The RG equation for the metric first appeared in the 1970s [9,10]. Subsequent papers [11] generalised this analysis to include the effects of other background fields like the dilaton and the antisymmetric tensor field.

The form of these equations is not diffeomorphism invariant. A τ-dependent diffeomorphism generated by $\xi_i = \frac{\alpha'}{2}\nabla_i\Psi$ decouples the metric flow from the dilaton field, so (1) becomes

$$\frac{\partial}{\partial\tau} g_{ij} = -\alpha' R_{ij}. \tag{3}$$

This equation, which we call the *Hamilton gauge* RG equation for the metric, arises in mathematics as a tool in Hamilton's programme [12] to address Thurston's geometrization conjecture for closed 3-manifolds, where it is called the *Ricci flow*.

Herein we consider instead asymptotically flat manifolds with asymptotic structure fixed along the flow. We take such metrics to satisfy

$$\begin{aligned}
&\bigl(g_{ij}(\tau) - \delta_{ij}\bigr) \in \mathcal{O}\bigl(1/r^{D-2-\epsilon}\bigr),\\
&\partial_k g_{ij}(\tau) \in \mathcal{O}\bigl(1/r^{D-1-\epsilon}\bigr),\\
&\partial_k\partial_l g_{ij}(\tau) \in \mathcal{O}\bigl(1/r^{D-\epsilon}\bigr),
\end{aligned} \tag{4}$$

and so on up to at least 4 derivatives, uniformly in τ for any $\epsilon > 0$. (Local existence of such solutions will be presented elsewhere.)

In the sequel, we will study the irreversibility of (3). This implies the irreversibility of the system (1), (2), but is more general. To see this, observe that the system (1), (2) is the pullback along ξ_i of the system comprised of (3) and the Hamilton gauge dilaton equation

$$\frac{\partial}{\partial\tau}\Psi = \frac{\alpha'}{2}\Delta\Psi. \tag{5}$$

By the maximum principle for parabolic equations [13], (5) has only monotonic solutions (if, say, $\Psi \to$ const at spatial infinity). In consequence, the flows (3), (5) and, therefore, (1), (2) are irreversible unless the dilaton is constant: if it is, then the question reduces to whether (3) has reversible flows. But this question is interesting even when the dilaton is not constant, and this more general case entails no addition burden since (3) is independent of the dilaton.

Flows that are periodic up to diffeomorphism are called *breathers*. More precisely, a solution of (3) is a *steady breather* if, for some $\tau_1 < \tau_2$ and a diffeomorphism ϕ,

$$g(\tau_1) = \phi^* g(\tau_2). \tag{6}$$

Expanding and *shrinking* breathers are defined by including as well an overall rescaling in (6), but cannot occur for flows through asymptotically flat spaces with Euclidean structure at infinity fixed in τ, so we do not consider them further (though they can produce interesting physics; cf. [5]). We will demonstrate that asymptotically flat breathers occur only in the trivial case in which the metrics *all* along the flow are related by diffeomorphisms and the breather is a *steady Ricci soliton*.

Let R be the scalar curvature of the target manifold. From (3) we can derive its flow, which is given by

$$\frac{\partial R}{\partial \tau} = \frac{\alpha'}{2}\left(\Delta R + 2R_{ij}R^{ij}\right). \tag{7}$$

Our problem separates into cases according to whether R can be somewhere negative along the flow, never negative but somewhere zero, or always positive. In Sections 1 and 2, we will adapt Perelman's recently discovered Ricci flow entropy for compact manifolds [14]) to the asymptotically flat case and use this entropy to rule out breathers if $R < 0$ somewhere. This entropy is not useful if $R \geqslant 0$ everywhere along the flow, so in Section 3 we give an entirely different entropy-type argument based on regularized volume. In Appendix A, we give alternative, direct (entirely non-entropic) geometric arguments based on work of Ivey [15] to rule out nontrivial breathers if $R \leqslant 0$ somewhere.

1. A Perelman-type entropy

Following Perelman, we will establish an entropy by examining the spectrum of a certain Schrödinger operator. Since we work with noncompact manifolds, a certain amount of mathematical care is important. Thus certain function spaces must make an appearance. The first is $H^1(M)$, the space of functions that are square-integrable with respect to the metric volume element on M and have square-integrable (distributional) derivative. For $u \in H^1(M)$, R falling off (or merely bounded), and $\kappa \in \mathbb{R}$, we can define the functional

$$F^{(\kappa)}[g,u] := \int_M \left(4|\nabla u|^2 + \kappa R u^2\right) dV(g). \tag{8}$$

Next we consider the subset $\mathcal{C} \subseteq H^1(M)$ of normalized ($\int u^2\, dV(g) = 1$) non-negative functions in $H^1(M)$. We use these to define the entropy, which is

$$\lambda^{(\kappa)}(\tau) := \inf_{u\in\mathcal{C}} F^{(\kappa)}\big(g(\tau), u\big). \tag{9}$$

We take $\kappa \geqslant 1$: this, we will see, will ensure that the entropy is monotonic. Now integrate (8) by parts and impose fall-off conditions on u to neglect boundary terms. This shows that $\lambda^{(\kappa)}(\tau)$ is the left endpoint of the spectrum of the Schrödinger operator $-4\Delta + \kappa R$. Our arguments will require a discrete spectrum. Perelman worked with compact manifolds where this is always the case, so he set $\kappa = 1$. In our noncompact case, if $R < 0$ somewhere, there will be a discrete spectrum of eigenfunctions if we choose κ large enough. Then $\lambda^{(\kappa)}(\tau)$ will belong to the minimum eigenfunction $\bar{u} \in \mathcal{C}$. (For $R \geqslant 0$ everywhere, the spectrum is continuous and starts from zero, so $\lambda^{(\kappa)}(\tau) = 0\ \forall\tau$: this is not a useful entropy.)

We will sometimes "approximate" the elements of $\mathcal{C}$ by functions that are exactly $\propto 1/r^m$ near infinity, $m > D/2$. These functions belong to $\mathcal{D} = \{u \in \mathcal{C} \mid u = w + k/r^m,\ w \in C_0^\infty,\ k = \text{const}\}$. $\mathcal{D}$ is dense in $\mathcal{C}$, so continuity of the map $H^1(M) \ni u \mapsto F^{(\kappa)}(g(t), u) \in \mathbb{R}$ gives

$$\lambda^{(\kappa)}(\tau) = \inf_{u\in\mathcal{D}} F^{(\kappa)}\big(g(\tau), u\big), \tag{10}$$

which is more useful than (9) for calculations.

Now say that $g(\tau)$ solves (3) on $\tau \in [0, \tau_*]$. To find $u(\tau)$, use $g(\tau)$ to write the backwards evolution equation

$$\frac{\partial v}{\partial \tau} = \frac{\alpha'}{2}(-\Delta v + R v). \tag{11}$$

Solve this for some given "initial" data $v(\tau_*)$, where $\sqrt{v(\tau_*)} \in \mathcal{D}$. Such a solution always exists for $\tau \leqslant \tau_*$; moreover, $v(\tau) > 0$, so we can define $u(\tau) := \sqrt{v(\tau)} > 0$ and $P(\tau) := -\ln(v(\tau))$. It can be shown that

$$u(\tau) \equiv \sqrt{v(\tau)} \equiv e^{-P/2} \in H^1(M), \quad \tau \leqslant \tau_*, \tag{12}$$

so $F^{(\kappa)}[g(\tau), u(\tau)]$ is defined, and $e^{-P(\tau)}$ and the derivatives of P fall off uniformly, fast enough to allow us to discard boundary terms when integrating by parts.[2] We note that $\frac{d}{d\tau}(u^2\,dV(g)) \equiv \frac{d}{d\tau}(v\,dV(g)) = 0$ (since from (3) we have $\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\tau} = -\frac{\alpha'}{2}R$) and $u(\tau_*) \in \mathcal{D}$ implies that

$$\|u(\tau)\| = 1, \quad \tau \leqslant \tau_*, \tag{13}$$

so in fact $u(\tau) \in \mathcal{C}$, $\tau \leqslant \tau_*$. From (11), $P(\tau)$ satisfies

$$\frac{\partial P}{\partial\tau} = \frac{\alpha'}{2}\bigl(-\Delta P + |\nabla P|^2 - R\bigr). \tag{14}$$

Now we are in a position to differentiate

$$F^{(\kappa)}(\tau) := F^{(\kappa)}\bigl(g(\tau), e^{-P(\tau)/2}\bigr) = \int \bigl(|\nabla P|^2 + \kappa R\bigr)e^{-P}\,dV(g). \tag{15}$$

Integrating by parts to simplify the result, we get

$$\frac{dF^{(\kappa)}}{d\tau} = \int_M \biggl\{(\nabla_i P\nabla_j P)\frac{\partial g^{ij}}{\partial\tau} + 2\bigl(|\nabla P|^2 - \Delta P\bigr)\frac{\partial P}{\partial\tau} + \kappa\frac{\partial R}{\partial\tau} + \bigl(|\nabla P|^2 + \kappa R\bigr)\frac{\partial}{\partial\tau}\log\bigl(e^{-P}\sqrt{g}\bigr)\biggr\} \times e^{-P}\,dV(g). \tag{16}$$

We could now insert the flow Eqs. (3), (7), (14) into (16), but the result would not appear manifestly non-negative. To make it manifest, recall that the flow equations are not form-invariant with respect to τ-dependent diffeomorphisms. Under the diffeomorphism generated by $-\frac{\alpha'}{2}\nabla P$, Eqs. (3), (7), and (14) become

$$\frac{\partial}{\partial\tau}g_{ij} = -\alpha'(R_{ij} + \nabla_i\nabla_j P), \tag{17}$$

$$\frac{\partial R}{\partial\tau} = \frac{\alpha'}{2}\bigl(\Delta R + 2R_{ij}R^{ij} - \nabla_i R\nabla^i P\bigr), \tag{18}$$

$$\frac{\partial P}{\partial\tau} = -\frac{\alpha'}{2}(\Delta P + R). \tag{19}$$

We call these the *Perelman gauge* flow equations. But integrals over M, such as $F^{(\kappa)}$, are invariant under diffeomorphisms. If the diffeomorphism preserves the asymptotic structure, we remain justified in discarding boundary terms, so (16) holds in such a gauge. Inserting the Perelman gauge flow equations into (16), integrating by parts, and using the boundary conditions, the Ricci identity, and the contracted second Bianchi identity, we get

$$\frac{dF^{(\kappa)}}{d\tau} = \alpha'\int_M \bigl(|R_{ij} + \nabla_i\nabla_j P|^2 + (\kappa - 1)|R_{ij}|^2\bigr) \times e^{-P}\,dV(g), \tag{20}$$

which is manifestly non-negative if $\kappa \geqslant 1$ (which we take from here on). Hence $\gamma(\tau) \leqslant \gamma(\tau_*)$ for $\tau \leqslant \tau_*$. In other words we have

$$F^{(\kappa)}\bigl(g(\tau), u(\tau)\bigr) \leqslant F^{(\kappa)}\bigl(g(\tau_*), u(\tau_*)\bigr) \tag{21}$$

for $\tau \leqslant \tau_*$. Using $u(\tau) > 0$, (12) and (13) we see that $u(\tau) \in \mathcal{C}$ and hence by the definition of the entropy and (21) it follows that $\lambda^{(\kappa)}(\tau) \leqslant F^{(\kappa)}(g(\tau_*), u(\tau_*))$ for $\tau \leqslant \tau_*$. But recall that $u(\tau_*)$ was taken to be an arbitrary element of $\mathcal{D}$ and hence we get by (10) that

$$\lambda^{(\kappa)}(\tau) \leqslant \lambda^{(\kappa)}(\tau_*) \quad \text{for } \tau \leqslant \tau_* \tag{22}$$

which proves that the *entropy is increasing*.

2. No breathers I: $R < 0$ somewhere

If g is a breather, then $\lambda^{(\kappa)}(\tau_1) = \lambda^{(\kappa)}(\tau_2) =: \Lambda$. Because entropy is monotonic, then $\lambda^{(\kappa)}(\tau) = \Lambda$ for all $\tau \in [\tau_1, \tau_2]$. We must now show that this statement has geometrical consequences. The trick is to construct a function $u = \bar{u}(\tau)$ that realizes the entropy Λ at all τ. Now if $R < 0$ somewhere, we can choose κ large enough so that there is a minimizer $\bar{u}(\tau_2) \in \mathcal{C}$ for the entropy realizing $\lambda^{(\kappa)}(\tau_2)$:

$$\lambda^{(\kappa)}(\tau_2) = F^{(\kappa)}\bigl(g(\tau_2), \bar{u}(\tau_2)\bigr). \tag{23}$$

We choose a sequence $u_n(\tau_2) \in \mathcal{D}$ that converges to $\bar{u}(\tau_2)$ in $H^1(M)$ and, as above, use the squares $v_n(\tau_2) := u_n^2(\tau_2)$ as "initial" data at τ_2 for a sequence of solutions $v_n(\tau) := v(\tau)$ of (11) on $\tau \in [\tau_1, \tau_2]$ (where now $\tau_* = \tau_2$). Because we start with $u_n(\tau_2) \in \mathcal{D}$, we are assured that $u_n(\tau) := \sqrt{v_n(\tau)}$ is defined (and in $\mathcal{C}$), and so

$$F_n^{(\kappa)}(\tau) := F^{(\kappa)}\bigl(g(\tau), u_n(\tau)\bigr), \quad \tau \in [\tau_1, \tau_2] \tag{24}$$

[2] The fall-off rates are $e^{-P(\tau)} \in \mathcal{O}(1/r^{2m-\epsilon})$, $\partial^I P(\tau) \in \mathcal{O}(1/r^{|I|-\epsilon})$, $\partial^I \frac{\partial}{\partial\tau}P(\tau) \in \mathcal{O}(1/r^{|I|+2-\epsilon})$, uniformly in τ for any $\epsilon > 0$, and I a multi-index with $1 \leqslant |I| \leqslant 2$.

is also defined. Repeating the calculations leading to (20) (using $P_n(\tau) := -\ln(v_n(\tau))$), we obtain

$$\frac{dF_n^{(\kappa)}}{d\tau} \geqslant 0, \tag{25}$$

and therefore $F_n^{(\kappa)}(\tau) \leqslant F_n^{(\kappa)}(\tau_2)$. Putting everything together, we have for $\tau \in [\tau_1, \tau_2]$ that the breather obeys

$$\begin{aligned} \Lambda = \lambda^{(\kappa)}(\tau_2) = \lambda^{(\kappa)}(\tau_1) \leqslant \lambda^{(\kappa)}(\tau) \\ \leqslant F_n^{(\kappa)}(\tau) \leqslant F_n^{(\kappa)}(\tau_2) \to \lambda^{(\kappa)}(\tau_2) \end{aligned} \tag{26}$$

as $n \to \infty$. In particular, this shows that $F_n^{(\kappa)}(\tau) \to \lambda^{(\kappa)}(\tau) = \Lambda$ for each $\tau \in [\tau_1, \tau_2]$, so $u_n(\tau)$ is a *minimizing sequence*. Thus, passing to a subsequence if necessary, we have $u_n(\tau) \to \bar{u}(\tau)$ weakly in $H^1(M)$ with $\bar{u}(\tau) \in \mathcal{C} \subseteq H^1(M)$ and $\lambda^{(\kappa)}(\tau) = F^{(\kappa)}(g(\tau), \bar{u}(\tau))$, as desired.

Next, integrating Eq. (25) yields $\int_{\tau_1}^{\tau_2} \frac{dF_n^{(\kappa)}}{d\tau} d\tau = F_n^{(\kappa)}(\tau_2) - F_n^{(\kappa)}(\tau_1) \to 0$ as $n \to \infty$. Since $\frac{dF_n^{(\kappa)}}{d\tau} \geqslant 0$, then it follows that, once again passing to a subsequence if necessary, $\lim_{n\to\infty} \frac{dF_n^{(\kappa)}}{d\tau} = 0$ pointwise on (τ_1, τ_2) except perhaps on a set S of measure zero. Then (20) (with κ large and $e^{-P} = u_n^2$) implies that $\int_M |R_{ij}|^2 u_n^2 \, dV \to 0$ as $n \to \infty$ for each $\tau \notin S$. It follows that $|R_{ij}| u_n \to 0$ in $H^1(M)$, $\tau \notin S$. But we also know that $|R_{ij}| u_n \to |R_{ij}| \bar{u}$ weakly in $H^1(M)$. By the uniqueness of weak limits and the fact that $\bar{u} > 0$ we get that $R_{ij}(\tau) = 0$ for each $\tau \in \Sigma$. By continuity in τ, $R_{ij}(\tau) = 0$ for all $\tau \in [\tau_1, \tau_2]$. Since we assume $R < 0$ somewhere, this is a contradiction.

3. No breathers II: $R \geqslant 0$

We now assume that $0 \leqslant R(\tau)$ and that $\int_M R(\tau) < \infty$ for all $\tau \in [\tau_1, \tau_2]$. A wide class of local solutions satisfy this condition: in fact, this is what is needed to ensure that the mass of the manifold is well-defined and unique. However, for this case the entropy defined above is always zero and does not rule out breathers. We therefore need a new entropy.

Since the scalar curvature satisfies (7), for which the minimum principle applies, it follows that $R(\tau_1) \geqslant 0$ implies $R(\tau) \geqslant 0$ all $\tau \geqslant \tau_1$. Thus if the scalar curvature is initially non-negative then it stays non-negative along the flow. The volume element obeys $\frac{\partial}{\partial \tau} dV(g) = -\frac{\alpha'}{2} R \, dV(g)$. Integrating this equation yields

$$\sqrt{g(\tau)} - \sqrt{g(\tau_1)} = -\frac{\alpha'}{2} \int_{\tau_1}^{\tau} R(s) \sqrt{g(s)} \, ds. \tag{27}$$

Since $\int_M R(\tau) \sqrt{g(\tau)} \, d^D x$ is bounded on $\tau \in [\tau_1, \tau_2]$, it follows that $\int_{\tau_1}^{\tau} R(s) \sqrt{g(s)} \, ds$ is integrable over M and hence the new entropy

$$\begin{aligned} \mu(\tau) &:= -\int_M \left(\sqrt{g(\tau)} - \sqrt{g(\tau_1)}\right) d^D x \\ &= \frac{\alpha'}{2} \int_M \int_{\tau_1}^{\tau} R(s) \sqrt{g(s)} \, ds \, d^D x \\ &= \frac{\alpha'}{2} \int_{\tau_1}^{\tau} \int_M R(s) \, dV\big(g(s)\big) \, ds \end{aligned} \tag{28}$$

is finite where, for simplicity, we have taken $M = \mathbb{R}^D$. Now, $R(\tau) \geqslant 0$ implies that $\mu(\tau)$ is nondecreasing. Moreover we have that $\mu(\tau_2) > \mu(\tau_1) = 0$ unless $R(\tau) = 0$ for all $\tau \in [\tau_1, \tau_2]$. If $R(\tau) = 0$ for all $\tau \in [\tau_1, \tau_2]$, then from (7) we see that $R_{ij}(\tau) = 0$ for all τ which shows that $g(\tau_1)$ must be a fixed point. So now let us assume that there exists a $\tau \in [\tau_1, \tau_2]$ for which $R(\tau) \neq 0$. Then

$$\mu(\tau_2) > \mu(\tau_1) = 0. \tag{29}$$

Let G denote $\det(g_{ij}(\tau_2))$. Then, applying the breather condition (6) to (28), we get

$$\mu(\tau_2) = -\frac{\alpha'}{2} \int_M \left[\sqrt{G} - \det\big(J(\phi)\big)\sqrt{G} \circ \phi\right] d^D x, \tag{30}$$

where $J(\phi)$ is the Jacobian matrix, i.e., $J(\phi)^i{}_j := \partial_j \phi^i$. To proceed, we assume that the diffeomorphism ϕ lies in the connected component of the identity so that there exists a 1-parameter family of diffeomorphisms ψ_t $(0 \leqslant t \leqslant 1)$ such that $\psi_0 = \mathrm{id}$, $\psi_1 = \phi$, and ψ_t preserves the asymptotic structure for each $t \in [0, 1]$. Now consider the Lagrangian density

$$\mathcal{L}\big(\psi_t^i, \partial_j \psi^i\big) := \det\big(J(\psi_t)\big)\sqrt{G} \circ \psi_t - \sqrt{G}. \tag{31}$$

A straightforward calculation shows that the Euler–Lagrange equations are automatically satisfied; that is, we have the identity

$$-\partial_j \frac{\partial \mathcal{L}}{\partial \partial_j \psi_t^i} + \frac{\partial \mathcal{L}}{\partial \psi_t^i} = 0. \tag{32}$$

Thus if we define

$$I(t) := \frac{\alpha'}{2} \int_M \mathcal{L}(\psi_t^i, \partial_j \psi_t^i)\, d^D x, \tag{33}$$

then differentiating $I(t)$, integrating by parts, and using (32) yields

$$\frac{dI}{dt} = \frac{\alpha'}{2} \int_{S_\infty} \det(J(\psi_t)) J(\psi_t)^{-1j}{}_i \frac{d\psi_t^i}{dt}\, dS_j. \tag{34}$$

If ψ_t approaches identity as $r \to \infty$ and $\int_{S_\infty} \frac{d\psi_t^i}{dt} dS_i = 0$ then we get that $I(t) = \text{const}$. But $I(1) = \mu(\tau_2)$, and $I(0) = 0$, so $\mu(\tau_2) = 0$. This shows that if the diffeomorphism ϕ has the form $\phi^i(x) = x^i + \bar{\phi}^i(x)$ where $\int_{S_\infty} \bar{\phi}^i\, dS_i = 0$ then we must have $\mu(\tau_2) = 0$ which contradicts (29). Therefore if $g(\tau_1) = \phi^* g(\tau_2)$, then the diffeomorphism will, at least, violate

$$\int_{S_\infty} (\phi^i(x) - x^i)\, dS_i = 0. \tag{35}$$

Whether there exist solutions periodic modulo diffeomorphisms that violate this condition is an interesting question which deserves attention.

In closing, we ask, can we pass now to RG flows wherein the second-order corrections are important? Unfortunately, in such circumstances, we do not know whether the flow will always preserve asymptotic flatness, even for short times. If it does, then it still may not have other necessary properties, such as the preservation of positive scalar curvature upon which the volume entropy argument depends. This is illustrative of the potential difficulties in generalizing our arguments to the second-order case.

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Appendix A

Here we give distinct geometric arguments for the case where $R < 0$ somewhere. Our arguments closely parallel those of Ivey [15] for compact manifolds. By asymptotic flatness, $R \to 0$ uniformly in τ at spatial infinity. Since the flow is smooth, R has an infimum, say $-k$, by assumption negative. Outside a big enough compact set $C := [0, T] \times K$ asymptotic flatness guarantees that $R > -k/2$. Thus the infimum must be approached inside C, which is compact, so R achieves the minimum value $-k$ (in C and thus in $[0, T] \times M$). Moreover, if the solution is a breather with period $< T$, the minimum must be achieved at least once in the open region $(0, T) \times M$. But if R has a minimum in $(0, T) \times M$, then $\frac{\partial R}{\partial t} = 0$ and $\Delta R \geqslant 0$ there, so we conclude from (7) (or (18)—the argument is gauge invariant) that R_{ij} vanishes there. Taking the trace, $R = 0$ at the minimum. This is a contradiction, so there cannot be a breather with period $< T$ and somewhere negative scalar curvature. But T is arbitrary.

To conclude, consider the case of $R \geqslant 0$ with $R = 0$ at an isolated point. The argument given by Ivey [15] for this case is essentially local and carries over to the case of asymptotic flatness (though we constructed our own version to verify details). The basic idea is that (7) obeys a Hopf lemma, which gives that $dR \neq 0$ if $R = 0$ at an isolated point with $\tau > 0$. Since dR must be zero at an interior minimum, either we never have $R = 0$ for $\tau > 0$, or $R = 0$ everywhere and throughout the flow. In the latter case, we see from (7) that $R_{ij} = 0$ everywhere and the breather is a fixed point of the flow.

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