GLOBAL EXISTENCE OF SOLUTIONS OF THE NORDSTRÖM-VLASOV SYSTEM IN TWO SPACE DIMENSIONS

HAYOUNG LEE

Abstract. The dynamics of a self-gravitating ensemble of collisionless particles is modeled by the Nordström-Vlasov system in the framework of the Nordström scalar theory of gravitation. For this system in two space dimensions, integral representations of the first order derivatives of the field are derived. Using these representations we show global existence of smooth solutions for large data.

1. Introduction

The Vlasov equation in general describes a collection of collisionless particles. Each particle is driven by self-induced fields which are generated by all particles together. When the relativistic effects are negligible, the dynamics is described by the Vlasov-Poisson system. Otherwise the relativistic Vlasov-Maxwell system in plasma physics and the Einstein-Vlasov system in stellar dynamics are considered. The Vlasov-Poisson models are well understood by now in the question of global existence of classical solutions [15, 17, 18, 21]. The relativistic models have very different structure and so far they have been considered separately. In the gravitational case, global existence of (asymptotically flat) solutions for the Einstein-Vlasov system is known only for small data with spherical symmetry [20]. For the relativistic Vlasov-Maxwell system the theory is more developed, cf. [2, 6], [8]–[13], [19]. However global existence and uniqueness of classical solutions for large data in three dimensions is still open.

A different relativistic generalization to the Vlasov-Poisson system in the stellar dynamics case has been considered in [1], where the Vlasov dynamics is coupled to a relativistic scalar theory of gravity which goes back, essentially, to Nordström [16]. More precisely, the gravitational theory considered in [1] corresponds to a reformulation of Nordström's theory due to Einstein and Fokker [7]. Therefore the resulting system has been called Nordström-Vlasov system.

Let $f(t,x,p) \geq 0$ denote the density of the particles in phase space, where $t \in \mathbb{R}$ denotes time, $x \in \mathbb{R}^2$ position and $p \in \mathbb{R}^2$ momentum. The gravitational effects are mediated by a scalar field $\phi(t,x)$. The Nordström-Vlasov system in two dimensions is given by

$$\partial_t^2 \phi - \Delta_x \phi = -4\pi \int f \frac{dp}{\sqrt{1+|p|^2}} \tag{1.1}$$

$$\partial_t f + \hat{p} \cdot \nabla_x f - \Big[S(\phi) p + \frac{\nabla_x \phi}{\sqrt{1+|p|^2}} \Big] \cdot \nabla_p f = 3S(\phi) f \tag{1.2}$$

where $\hat{p} = p(1+|p|^2)^{-1/2}$ and $S = \partial_t + \hat{p}\cdot\nabla_x$. Initial data are given by

$$f(0,x,p) = f^{\text{in}}(x,p),$$
$$\phi(0,x) = \phi_0^{\text{in}}(x),$$
$$\partial_t\phi(0,x) = \phi_1^{\text{in}}(x).$$

The spacetime is a Lorentzian manifold with a conformally flat metric which, in the coordinates (t,x), takes the form

$$g_{\mu\nu} = e^{2\phi}\text{diag}(-1,1,1)$$

where the Greek indices run from 0 to 2. The particle distribution f_{physical} defined on the mass shell in this metric is given by

$$f_{\text{physical}}(t,x,p) = e^{-3\phi}f(t,x,e^{\phi}p).$$

Details on the derivation of this system in three dimensions can be founded in [1, 3] and also in general N dimensions in [4].

In [4] a condition is established such that a global classical solution is achieved in three dimensions and existence of global weak solutions of the Nordström-Vlasov system has been shown in [5]. Also the Nordström-Vlasov system has been justified as a genuine relativistic generalization of the (gravitational) Vlasov-Poisson system, by indicating the relation between the solutions of the two systems. Precisely it has been proved in [3] that in the non-relativistic limit $c \to \infty$ the solutions of the Nordström-Vlasov system in three dimensional space converge to solutions of Vlosov-Poisson system in a pointwise sense. One can prove a similar result in the case of two space dimensions, using the analogous argument in [14].

This paper proceeds as follows. In Section 2 we provide representations of the derivatives of the scalar field and state our main results in detail. The first of such results is a global existence theorem of solutions of the Nordström-Vlasov system under the condition that momenta of particles are controlled, which will be proved in Section 3. This control of particle momenta exists in the two space dimensions, which is the second result and the demonstration of this will be shown in Section 4.

2. Preliminaries and the main results

Here are a few notational conventions. C denotes a positive constant which changes from line to line and may depend only on the initial data. Similarly $C(t)$ denotes a positive nondecreasing function of time. Also we use the norms

$$\|f(t)\| = \sup\{f(t,x,p) : (x,p)\in\mathbb{R}^2\times\mathbb{R}^2\},$$
$$\|\phi(t)\| = \sup\{|\phi(t,x)| : x\in\mathbb{R}^2\},$$
$$|||\phi(t)||| = \sup\{\|\phi(\tau)\| : 0\le\tau\le t\}.$$

We also denote

$$\|D\phi(t)\| = \sup\{|\partial_t\phi(t,x)|,\ |\partial_{x_i}\phi(t,x)| : x \in \mathbb{R}^2, i = 1,2\}$$
$$\|D^2\phi(t)\| = \sup\{|\partial_t^2\phi(t,x)|,\ |\partial_t\partial_{x_i}\phi(t,x)|,\ |\partial_{x_i}\partial_{x_j}\phi(t,x)| : x \in \mathbb{R}^2,\ i,j = 1,2\}.$$

Let us recall the representations for the electric and magnetic fields, E_k (with $k = 1,2$) and B, in the case of the relativistic Vlasov-Maxwell system in two space dimensions (Theorem 1, in [10]).

Lemma 1.

$$\begin{aligned}
E_k(t,x) = \tilde{E}_k^0 &- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{(et_k)f}{\sqrt{(t-\tau)^2 - |y-x|^2}}\,dp\,dy\,d\tau \\
&- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{(E_1 + \hat{p}_2 B, E_2 - \hat{p}_1 B)f}{\sqrt{(t-\tau)^2 - |y-x|^2}} \cdot \nabla_p(es_k)\,dp\,dy\,d\tau \\
B(t,x) = \tilde{B}^0 &+ 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{(bt)f}{\sqrt{(t-\tau)^2 - |y-x|^2}}\,dp\,dy\,d\tau \\
&+ 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{(E_1 + \hat{p}_2 B, E_2 - \hat{p}_1 B)f}{\sqrt{(t-\tau)^2 - |y-x|^2}} \cdot \nabla_p(bs)\,dp\,dy\,d\tau
\end{aligned}$$

where $\tilde{E}_k^0$ and $\tilde{B}^0$ are Cauchy data terms and kernels are given by

$$et_k = \frac{\xi_k + \hat{p}_k}{(1+|p|^2)(1+\xi\cdot\hat{p})^2}, \qquad es_k = \frac{\xi_k + \hat{p}_k}{1+\xi\cdot\hat{p}},$$
$$bt = \frac{\xi_1\hat{p}_2 - \xi_2\hat{p}_1}{(1+|p|^2)(1+\xi\cdot\hat{p})^2}, \qquad bs = \frac{\xi_1\hat{p}_2 - \xi_2\hat{p}_1}{1+\xi\cdot\hat{p}}$$

Here $\xi = \frac{y-x}{t-\tau}$.

The next two propositions show that the derivatives of ϕ satisfy similar representations.

Proposition 1.

$$\begin{aligned}
\partial_t\phi(t,x) = \partial_t\phi_{\mathrm{hom}} &- 2\int_{|y-x|<t} \int \frac{f^{\mathrm{in}}(y,p)}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})\sqrt{t^2 - |y-x|^2}}\,dp\,dy \\
&+ 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{a^{\phi_t}(\xi,p)f(\tau,y,p)}{(t-\tau)\sqrt{(t-\tau)^2 - |y-x|^2}}\,dp\,dy\,d\tau \\
&- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{b^{\phi_t}(\xi,p)S(\phi)f(\tau,y,p)}{\sqrt{(t-\tau)^2 - |y-x|^2}}\,dp\,dy\,d\tau \\
&- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{c^{\phi_t}(\xi,p)\cdot(\nabla_x\phi)f(\tau,y,p)}{\sqrt{(t-\tau)^2 - |y-x|^2}}\,dp\,dy\,d\tau
\end{aligned}$$

$$a^{\phi_t}(\xi,p)=\frac{\hat{p}\cdot(\xi+\hat{p})}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})^2}=p\cdot(et_1,et_2)$$
$$b^{\phi_t}(\xi,p)=\frac{1}{\sqrt{1+|p|^2}}$$
$$c^{\phi_t}(\xi,p)=\frac{\xi+\hat{p}}{(1+|p|^2)^{3/2}(1+\xi\cdot\hat{p})^2}$$

Proposition 2.

$$\begin{aligned}\partial_{x_1}\phi(t,x)=\partial_{x_1}\phi_{\text{hom}}&-2\int_{|y-x|<t}\int\frac{\xi_1 f^{\text{in}}(y,p)}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})\sqrt{t^2-|y-x|^2}}\,dp\,dy\\&-2\int_0^t\int_{|y-x|<t-\tau}\int\frac{a^{\phi_{x_1}}(\xi,p)f(\tau,y,p)}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\&-2\int_0^t\int_{|y-x|<t-\tau}\int\frac{b^{\phi_{x_1}}(\xi,p)S(\phi)f(\tau,y,p)}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\&-2\int_0^t\int_{|y-x|<t-\tau}\int\frac{c^{\phi_{x_1}}(\xi,p)\cdot(\nabla_x\phi)f(\tau,y,p)}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\end{aligned}$$

$$a^{\phi_{x_1}}(\xi,p)=\frac{(\xi_1+\hat{p}_1)-\hat{p}_2(\xi_1\hat{p}_2-\xi_2\hat{p}_1)}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})^2}=\sqrt{1+|p|^2}[(et_1)-\hat{p}_2(bt)]$$
$$b^{\phi_{x_1}}(\xi,p)=\frac{\xi_1}{\sqrt{1+|p|^2}}=\xi_1 b^{\phi_t}(\xi,p)$$
$$c^{\phi_{x_1}}(\xi,p)=\frac{\xi_1(\xi+\hat{p})}{(1+|p|^2)^{3/2}(1+\xi\cdot\hat{p})^2}=\xi_1 c^{\phi_t}(\xi,p)$$

The representation for $\partial_{x_2}\phi$ is almost identical to the one for $\partial_{x_1}\phi$ and so we omit it. The proof of Proposition 2 is provided in the appendix.

One basic property of the Vlasov equation is that the distribution f is constant along the characteristic. However, this is no longer true in Nordström-Vlasov system. Nevertheless one can have a similar property. The following lemma is from [3]. It is true also for the two space dimensions and the proof is shown in the appendix.

Lemma 2. *Let $f^{\text{in}}\in C_b^1(\mathbb{R}^4)$, $\phi_0^{\text{in}}\in C_b^3(\mathbb{R}^2)$ and $\phi_1^{\text{in}}\in C_b^2(\mathbb{R}^2)$. Then*

$$\|f(t)\|\le Ce^{ct}$$

for all $t\in\mathbb{R}$.

Here is the main result of this paper :

Theorem 1. *Let $f^{\text{in}}\in C_b^1(\mathbb{R}^4)$ with compact support in p, $\phi_0^{\text{in}}\in C_b^3(\mathbb{R}^2)$ and $\phi_1^{\text{in}}\in C_b^2(\mathbb{R}^2)$. Then there exists a unique classical solution $(f,\phi)\in C^1([0,\infty)\times\mathbb{R}^4)\times C^2([0,\infty)\times\mathbb{R}^2)$ of the Nordström-Vlasov system (1.1)-(1.2).*

We will prove this main result by showing the following two theorems in the rest of the paper.

Theorem 2. *Let $f^{\rm in} \in C_b^1(\mathbb{R}^4)$, $\phi_0^{\rm in} \in C_b^3(\mathbb{R}^2)$ and $\phi_1^{\rm in} \in C_b^2(\mathbb{R}^2)$. Assume that there exists a nondecreasing function $C(t)$ for which*

$$f(t,x,p) = 0 \text{ if } |p| \ge C(t).$$

Then there exists a unique classical solution $(f,\phi) \in C^1([0,\infty)\times\mathbb{R}^4)\times C^2([0,\infty)\times\mathbb{R}^2)$ of the Nordström-Vlasov system (1.1)-(1.2).

Theorem 3. *Assume the initial data from Theorem 2. Also we assume that $f^{\rm in}$ has compact support in p. Then there exists a unique classical solution $(f,\phi) \in C^1([0,\infty)\times\mathbb{R}^4)\times C^2([0,\infty)\times\mathbb{R}^2)$ of the Nordström-Vlasov system (1.1)-(1.2) satisfying*

$$f(t,x,p) = 0 \text{ if } |p| \ge C(t)$$

for some continuous function $C(t)$ and

$$\|f(t)\| + \|\nabla_{(t,x,p)} f(t)\| + \|D\phi(t)\| + \|D^2\phi(t)\| \le C(t)$$

for all $t \ge 0$.

In the notation of the spaces of functions used above, the subscript b means that all the derivatives up to the indicated order are bounded.

3. Proof of Theorem 2

3.1. Estimates on $D\phi$.

Theorem 4. *Assume that $f^{\rm in} \in C_b(\mathbb{R}^4)$, $\phi_0^{\rm in} \in C_b^2(\mathbb{R}^2)$ and $\phi_1^{\rm in} \in C_b^1(\mathbb{R}^2)$. Assume that there exists a nondecreasing function $C(t)$ for which*

$$f(t,x,p) = 0 \text{ if } |p| \ge C(t).$$

Then

$$|||\phi(t)||| + |||D\phi(t)||| \le C(t).$$

PROOF : The classical solution of (1.1) is

$$\phi(t,x) = \phi_{\rm hom}(t,x) - 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{f(\tau,y,p)}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \tag{3.1}$$

where

$$\phi_{\rm hom} = \frac{1}{2\pi}\left[\int_{|y-x|<t} \frac{\phi_1^{\rm in}(y)\,dy}{\sqrt{t^2-|y-x|^2}} + \frac{\partial}{\partial t}\Big(\int_{|y-x|<t} \frac{\phi_0^{\rm in}(y)\,dy}{\sqrt{t^2-|y-x|^2}}\Big)\right]$$

is the solution of the homogeneous wave equation with data $\phi_0^{\rm in}$ and $\phi_1^{\rm in}$ and the second term in (3.1) is the solution of (1.1) with trivial data. Then with the assumption of data in the theorem, one can see that

$$\|\phi_{\rm hom}(t)\| \le C(1+t)\big[\|\phi_0^{\rm in}\| + \|D\phi_0^{\rm in}\| + \|\phi_1^{\rm in}\|\big] \le C(1+t). \tag{3.2}$$

With Lemma 2, the second term in (3.1) becomes

$$\int_0^t \int_{|y-x|<t-\tau} \int_{|p|<C(t)} \frac{f(\tau,y,p)\,dp\,dy\,d\tau}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}} \le C(t)\int_0^t \|f(\tau)\|(t-\tau)\,d\tau \le C(t).$$

Therefore we have

$$\|\|\phi(t)\|\| \le C(t).$$

Using the fact that $D\phi_{\mathrm{hom}}$ satisfies the homogeneous wave equation, we get $\|D\phi_{\mathrm{hom}}(t)\| \le C(t)$. Note that given $|p| < C(t)$, we have $(1+\xi\cdot\hat{p})^{-1} \le C(t)$. Then one can also see that the second terms of the representations $\partial_t\phi$ and $\partial_{x_1}\phi$ are bounded by $C(t)$. Also using a similar argument to the kernels in Propositions 1 and 2, we obtain

$$|f(\tau,y,p)|\big(|a^{\phi_t}|+|b^{\phi_t}|+|c^{\phi_t}|+|a^{\phi_{x_1}}|+|b^{\phi_{x_1}}|+|c^{\phi_{x_1}}|\big) \le C(\tau).$$

Therefore

$$\begin{aligned}\|D\phi(t)\| &\le C(t)+C(t)\int_0^t \int_{|y-x|<t-\tau} \int_{|p|<C(t)} \frac{\big[(t-\tau)^{-1}+\|D\phi(t)\|\big]}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \\ &\le C(t)+C(t)\int_0^t \big[1+\|D\phi(\tau)\|(t-\tau)\big]\,d\tau.\end{aligned}$$

By Gronwall's inequality, $\|D\phi(t)\| \le C(t)$. □

3.2. Estimates on $D^2\phi$.

Theorem 5. *Let (f,ϕ) be as in Theorem 4 and assume that $f^{\mathrm{in}} \in C_b^1(\mathbb{R}^4)$, $\phi_0^{\mathrm{in}} \in C_b^3(\mathbb{R}^2)$ and $\phi_1^{\mathrm{in}} \in C_b^2(\mathbb{R}^2)$. Then*

$$\|\|D^2\phi(t)\|\| \le C(t)\big[1+\ln^*(t\,\|\|\,\nabla_{(x,p)}f(t)\|\|)\big]$$

where

$$\ln^*(t) = \begin{cases} 0 & \text{if } 0\le t\le 1, \\ \ln(t) & \text{if } 1<t. \end{cases}$$

PROOF : Here we will prove the estimate for $\partial_{x_1}^2\phi$. The other derivatives can be obtained with the same argument presented in the following. First, in the representation of $\partial_{x_1}\phi$, define

$$A^{\phi_{x_1}} := \int_0^t \int_{|y-x|<t-\tau} \int \frac{a^{\phi_{x_1}}(\xi,p)f(\tau,y,p)}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.$$

Then using (A.1) we obtain

$$\begin{aligned}
\partial_{x_1} A^{\phi_{x_1}} &= \int_0^t \int_{|y-x|<t-\tau} \int \frac{a^{\phi_{x_1}}(\xi,p)\partial_{x_1} f(\tau,y,p)}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\, dp\, dy\, d\tau \\
&= \int_0^t \int_{|y-x|<t-\tau} \int \frac{a^{\phi_{x_1}}(\xi,p)\xi_1 S f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\, dp\, dy\, d\tau \\
&\quad + \int_0^t \int_{|y-x|<t-\tau} \int \frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1 - \xi_1\hat{p}_2 T_2\big] f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)^2}\, dp\, dy\, d\tau \\
&:= \partial_{x_1} A^{\phi_{x_1}} S + \partial_{x_1} A^{\phi_{x_1}} T.
\end{aligned}$$

Also $\delta \in (0,t)$ and $\epsilon \in (0,1)$. Using (A.3), we get from $\partial_{x_1} A^{\phi_{x_1}} T$

$$\begin{aligned}
&\int_0^{t-\delta} \int_{|y-x|<(1-\epsilon)(t-\tau)} \int \frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1 - \xi_1\hat{p}_2 T_2\big] f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)^2}\, dp\, dy\, d\tau \qquad (3.3)\\
&= -\int_0^{t-\delta} \int_{|y-x|<(1-\epsilon)(t-\tau)} \int \frac{f(\tau,y,p)}{\sqrt{1-|\xi|^2}} \Big[(-\xi_1,1,0)\cdot\nabla_{(\tau,y)} a_1^{\phi_{x_1}}(\xi,p) \\
&\qquad -(-\xi_2,0,1)\cdot\nabla_{(\tau,y)} a_2^{\phi_{x_1}}(\xi,p)\Big]\, dp\, dy\, d\tau \\
&\quad + \int_0^{t-\delta} \int_{|y-x|=(1-\epsilon)(t-\tau)} \int fA\cdot\left(1-\epsilon, \frac{y_1-x_1}{|y-x|}, \frac{y_2-x_2}{|y-x|}\right) dp\, dS_y\, d\tau \\
&\quad + \int_{|y-x|<(1-\epsilon)\delta} \int fA\Big|_{\tau=t-\delta}\cdot(1,0,0)\, dp\, dy + \int_{|y-x|<(1-\epsilon)t} \int fA\Big|_{\tau=0}\cdot(-1,0,0)\, dp\, dy,
\end{aligned}$$

where

$$a_1^{\phi_{x_1}}(\xi,p) := \frac{(1+\xi_2\hat{p}_2)a^{\phi_{x_1}}}{(1+\xi\cdot\hat{p})(t-\tau)^2}, \qquad a_2^{\phi_{x_1}}(\xi,p) := \frac{\xi_1\hat{p}_2 a^{\phi_{x_1}}}{(1+\xi\cdot\hat{p})(t-\tau)^2}$$

and

$$A := [a_1^{\phi_{x_1}}(\xi,p)(-\xi_1,1,0) - a_2^{\phi_{x_1}}(\xi,p)(-\xi_2,0,1)](1-|\xi|^2)^{-1/2}.$$

Applying to the second term in (3.3) the similar argument in (A.5) and then by letting $\epsilon\to 0^+$, we obtain

$$\begin{aligned}
&\int_0^{t-\delta} \int_{|y-x|<t-\tau} \int \frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1 - \xi_1\hat{p}_2 T_2\big] f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)^2}\, dp\, dy\, d\tau \qquad (3.4)\\
&= -\int_0^{t-\delta} \int_{|y-x|<t-\tau} \int \frac{f(\tau,y,p)}{\sqrt{1-|\xi|^2}} \Big[(-\xi_1,1,0)\cdot\nabla_{(\tau,y)} a_1^{\phi_{x_1}}(\xi,p) \\
&\qquad -(-\xi_2,0,1)\cdot\nabla_{(\tau,y)} a_2^{\phi_{x_1}}(\xi,p)\Big]\, dp\, dy\, d\tau \\
&\quad + \int_{|y-x|<\delta} \int fA\Big|_{\tau=t-\delta}\cdot(1,0,0)\, dp\, dy + \int_{|y-x|<t} \int fA\Big|_{\tau=0}\cdot(-1,0,0)\, dp\, dy.
\end{aligned}$$

Note that

$$|A\cdot(1,0,0)| \le C(t)(t-\tau)^{-2}(1-|\xi|^2)^{-1/2}.$$

So we obtain

$$\left|\int_{|y-x|<\delta} \int fA\Big|_{\tau=t-\delta}\cdot(1,0,0)\, dp\, dy\right| \le C(t)\int_{|y-x|<\delta} \delta^{-1}(\delta^2-|y-x|^2)^{-1/2}\, dy \le C(t)$$

and the same estimate holds for the last term in (3.4). Now we compute

$$\Big|(-\xi_1,1,0)\cdot\nabla_{(\tau,y)}a_1^{\phi_{x_1}}(\xi,p)-(-\xi_2,0,1)\cdot\nabla_{(\tau,y)}a_2^{\phi_{x_1}}(\xi,p)\Big|$$
$$=\Big|-\xi_1\Big(\frac{2a_1^{\phi_{x_1}}(\xi,p)}{(t-\tau)^3}+\nabla_\xi a_1^{\phi_{x_1}}(\xi,p)\cdot\frac{\partial\xi}{\partial\tau}\Big)+\nabla_\xi a_1^{\phi_{x_1}}(\xi,p)\cdot\frac{\partial\xi}{\partial y_1}$$
$$+\xi_2\Big(\frac{2a_2^{\phi_{x_1}}(\xi,p)}{(t-\tau)^3}+\nabla_\xi a_2^{\phi_{x_1}}(\xi,p)\cdot\frac{\partial\xi}{\partial\tau}\Big)-\nabla_\xi a_2^{\phi_{x_1}}(\xi,p)\cdot\frac{\partial\xi}{\partial y_2}\Big|\le\frac{C(t)}{(t-\tau)^3}.$$

So we get

$$\Big|\int_0^{t-\delta}\int_{|y-x|<t-\tau}\int\frac{f(\tau,y,p)}{\sqrt{1-|\xi|^2}}\Big[(-\xi_1,1,0)\cdot\nabla_{(\tau,y)}a_1^{\phi_{x_1}}-(-\xi_2,0,1)\cdot\nabla_{(\tau,y)}a_2^{\phi_{x_1}}\Big]\,dp\,dy\,d\tau\Big|$$
$$\le\int_0^{t-\delta}\int_{|y-x|<t-\tau}\frac{C(t)}{(t-\tau)^3\sqrt{1-|\xi|^2}}\,dy\,d\tau=C(t)\ln\frac{t}{\delta}.$$

Therefore (3.4) becomes

$$\int_0^{t-\delta}\int_{|y-x|<t-\tau}\int\frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1-\xi_1\hat{p}_2T_2\big]f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)^2}\,dp\,dy\,d\tau\le C(t)\Big(1+\ln\frac{t}{\delta}\Big).\tag{3.5}$$

For the tip of the cone, with (1.2) and Theorem 4 note that

$$\left|\frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1-\xi_1\hat{p}_2T_2\big]f(\tau,y,p)}{(1+\xi\cdot\hat{p})}\right|\le\frac{C(t)(1+|||\nabla_{(x,p)}f(t)|||)}{\sqrt{1-|\xi|^2}}.$$

So we have

$$\left|\int_{t-\delta}^{t}\int_{|y-x|<t-\tau}\int\frac{a^{\phi_{x_1}}(\xi,p)\big[(1+\xi_2\hat{p}_2)T_1-\xi_1\hat{p}_2T_2\big]f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)^2}\,dp\,dy\,d\tau\right|\tag{3.6}$$
$$\le C(t)(1+|||\nabla_{(x,p)}f(t)|||)\int_{t-\delta}^{t}\int_{|y-x|<t-\tau}\int_{|p|<C(t)}\frac{dp\,dy\,d\tau}{(t-\tau)^2\sqrt{1-|\xi|^2}}=C(t)(1+\delta\,|||\,\nabla_{(x,p)}f(t)|||).$$

Therefore collecting (3.5) and (3.6) we obtain

$$\partial_{x_1}A^{\phi_{x_1}}T\le C(t)\big[1+\ln\frac{t}{\delta}+\delta\,|||\,\nabla_{(x,p)}f(t)\,|||\,\big]$$

and taking $\delta=\min\{t,|||\nabla_{(x,p)}f(t)|||^{-1}\}$ we get

$$\partial_{x_1}A^{\phi_{x_1}}T\le C(t)\big[1+\ln^*(t\,|||\,\nabla_{(x,p)}f(t)|||)\big].\tag{3.7}$$

Recall (A.2) :

$$Sf=F(t,x,p)\cdot\nabla_pf+3(S\phi)f\tag{3.8}$$

where

$$F(t,x,p):=(S\phi)p+\frac{\nabla_x\phi}{\sqrt{1+|p|^2}}.$$

Then

$$\begin{aligned}\partial_{x_1}A^{\phi_{x_1}}S &= \int_0^t\int_{|y-x|<t-\tau}\int \frac{a^{\phi_{x_1}}(\xi,p)\xi_1 Sf(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\ &= \int_0^t\int_{|y-x|<t-\tau}\int \frac{a^{\phi_{x_1}}(\xi,p)\xi_1 F(\tau,y,p)\cdot\nabla_p f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\ &\quad+\int_0^t\int_{|y-x|<t-\tau}\int \frac{3a^{\phi_{x_1}}(\xi,p)S(\phi)\xi_1 f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.\end{aligned}$$

Note that $\nabla_p F = 2S(\phi)$.

$$\begin{aligned}\partial_{x_1}A^{\phi_{x_1}}S &= -\int_0^t\int_{|y-x|<t-\tau}\int \nabla_p\left(\frac{a^{\phi_{x_1}}(\xi,p)}{1+\xi\cdot\hat{p}}\right)\cdot\frac{\xi_1 F(\tau,y,p)f(\tau,y,p)}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\ &\quad+\int_0^t\int_{|y-x|<t-\tau}\int \frac{a^{\phi_{x_1}}(\xi,p)S(\phi)\xi_1 f(\tau,y,p)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.\end{aligned}$$

By Theorem 4, one can see that

$$|\partial_{x_1}A^{\phi_{x_1}}S| \le C(t). \tag{3.9}$$

Now collecting (3.7) and (3.9) we get

$$|\partial_{x_1}A^{\phi_{x_1}}| \le C(t)\big[1+\ln^*(t\,\|\,\nabla_{(x,p)}f(t)\|\|)\big]. \tag{3.10}$$

Define

$$B^{\phi_{x_1}} := \int_0^t\int_{|y-x|<t-\tau}\int \frac{b^{\phi_{x_1}}(\xi,p)S(\phi)f(\tau,y,p)}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.$$

Then

$$\partial_{x_1}B^{\phi_{x_1}} = \int_0^t\int_{|y-x|<t-\tau}\int \frac{b^{\phi_{x_1}}(\xi,p)\big[\partial_{x_1}[S(\phi)]f+S(\phi)\partial_{x_1}f\big]}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.$$

The first term becomes :

$$\left|\int_0^t\int_{|y-x|<t-\tau}\int \frac{b^{\phi_{x_1}}(\xi,p)\partial_{x_1}[S(\phi)]f}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\right| \le C(t)\int_0^t \|D^2\phi(\tau)\|\,d\tau. \tag{3.11}$$

Using (A.1) the second term becomes :

$$\begin{aligned}&\int_0^t\int_{|y-x|<t-\tau}\int \frac{b^{\phi_{x_1}}(\xi,p)S(\phi)\partial_{x_1}f}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\ &= \int_0^t\int_{|y-x|<t-\tau}\int \frac{\xi_1 b^{\phi_{x_1}}(\xi,p)S(\phi)Sf}{(1+\xi\cdot\hat{p})\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau\\ &\quad+\int_0^t\int_{|y-x|<t-\tau}\int \frac{b^{\phi_{x_1}}(\xi,p)S(\phi)\big[(1+\xi_2\hat{p}_2)T_1-\xi_1\hat{p}_2T_2\big]f}{(1+\xi\cdot\hat{p})(t-\tau)}\,dp\,dy\,d\tau\\ &:= \partial_{x_1}B^{\phi_{x_1}}S+\partial_{x_1}B^{\phi_{x_1}}T.\end{aligned}$$

Again use (3.8) for the term $\partial_{x_1} B^{\phi_{x_1}} S$:

$$\partial_{x_1} B^{\phi_{x_1}} S = -\int_0^t \int_{|y-x|<t-\tau} \int \nabla_p \left(\frac{b^{\phi_{x_1}}(\xi,p)}{1+\xi\cdot\hat{p}} \right) \cdot \frac{\xi_1 F(\tau,y,p) S(\phi) f}{\sqrt{(t-\tau)^2-|y-x|^2}}\, dp\, dy\, d\tau$$
$$+\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1 b^{\phi_{x_1}}(\xi,p)\big[S(\phi)\big]^2 f}{(1+\xi\cdot\hat{p})\sqrt{(t-\tau)^2-|y-x|^2}}\, dp\, dy\, d\tau.$$

So one can see that

$$|\partial_{x_1} B^{\phi_{x_1}} S| \le C(t). \tag{3.12}$$

For the term $\partial_{x_1} B^{\phi_{x_1}} T$, let us carry out the computation with the term involved T_1. The argument is same for the term with T_2 and so will be omitted. Let

$$B := \frac{b^{\phi_{x_1}}(\xi,p)(1+\xi_2\hat{p}_2)(-\xi_1,1,0)}{(1+\xi\cdot\hat{p})(t-\tau)\sqrt{1-|\xi|^2}}.$$

Note that for $\epsilon\in(0,1)$ we have

$$\int_0^t \int_{|y-x|<(1-\epsilon)(t-\tau)} \int \frac{b^{\phi_{x_1}}(\xi,p)S(\phi)(1+\xi_2\hat{p}_2)T_1 f}{(1+\xi\cdot\hat{p})(t-\tau)}\, dp\, dy\, d\tau$$
$$= -\int_0^t \int_{|y-x|<(1-\epsilon)(t-\tau)} \int \nabla_{(\tau,y)} \left[\frac{b^{\phi_{x_1}}(\xi,p)S(\phi)(1+\xi_2\hat{p}_2)}{(1+\xi\cdot\hat{p})(t-\tau)} \right] \cdot \frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\, dp\, dy\, d\tau$$
$$+\int_0^t \int_{|y-x|=(1-\epsilon)(t-\tau)} \int S(\phi) f B \cdot \left(1-\epsilon, \frac{y_1-x_1}{|y-x|}, \frac{y_2-x_2}{|y-x|} \right) dp\, dS_y\, d\tau$$
$$+\int_0^t \int_{|y-x|<(1-\epsilon)t} \int S(\phi) f B|_{\tau=0} \cdot (-1,0,0)\, dp\, dy.$$

Applying the similar argument in (A.5) to the second term and then by letting $\epsilon\to 0^+$, we obtain

$$\int_0^t \int_{|y-x|<t-\tau} \int \frac{b^{\phi_{x_1}}(\xi,p)S(\phi)(1+\xi_2\hat{p}_2)T_1 f}{(1+\xi\cdot\hat{p})(t-\tau)}\, dp\, dy\, d\tau$$
$$= -\int_0^t \int_{|y-x|<t-\tau} \int \nabla_{(\tau,y)} \left[\frac{b^{\phi_{x_1}}(\xi,p)S(\phi)(1+\xi_2\hat{p}_2)}{(1+\xi\cdot\hat{p})(t-\tau)} \right] \cdot \frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\, dp\, dy\, d\tau$$
$$+\int_0^t \int_{|y-x|<t} \int S(\phi) f B|_{\tau=0} \cdot (-1,0,0)\, dp\, dy.$$

Then using Theorem 4, one can see that the last term is bounded by $C(t)$. Let

$$\tilde{b}^{\phi_{x_1}(\xi,p)} := b^{\phi_{x_1}}(\xi,p)(1+\xi_2\hat{p}_2)(1+\xi\cdot\hat{p})^{-1}.$$

Compute the following :

$$
\begin{aligned}
&\left|\nabla_{(\tau,y)}\left[\frac{\tilde{b}^{\phi_{x_1}}(\xi,p)S(\phi)}{(t-\tau)}\right]\cdot(-\xi_1,1,0)\right| \\
&\leq \left|-\xi_1\left(\frac{\tilde{b}^{\phi_{x_1}}(\xi,p)}{(t-\tau)^2}+\frac{\nabla_\xi\tilde{b}^{\phi_{x_1}}(\xi,p)S(\phi)}{(t-\tau)}\cdot\frac{\partial\xi}{\partial\tau}+\frac{\tilde{b}^{\phi_{x_1}}(\xi,p)(|\partial_t^2\phi|+|\partial_t\partial_{x_i}\phi|)}{(t-\tau)}\right)\right. \\
&\qquad\left.+\frac{\nabla_\xi\tilde{b}^{\phi_{x_1}}(\xi,p)S(\phi)}{(t-\tau)}\cdot\frac{\partial\xi}{\partial y_1}+\frac{\tilde{b}^{\phi_{x_1}}(\xi,p)}{(t-\tau)}\big(|\partial_{x_1}\partial_t\phi|+|\partial_{x_1}\partial_{x_i}\phi|\big)\right| \\
&\leq C(t)\big[(t-\tau)^{-2}+(t-\tau)^{-1}\|D^2\phi(\tau)\|\big].
\end{aligned}
$$

So we get

$$
\begin{aligned}
&\left|\int_0^t\int_{|y-x|<t-\tau}\int\frac{b^{\phi_{x_1}}(\xi,p)S(\phi)(1+\xi_2\hat{p}_2)T_1f}{(1+\xi\cdot\hat{p})(t-\tau)}\,dp\,dy\,d\tau\right| \qquad (3.13) \\
&\leq C(t)+C(t)\int_0^t\int_{|y-x|<t-\tau}\frac{(t-\tau)^{-1}+\|D^2\phi(\tau)\|}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dy\,d\tau \\
&= C(t)+C(t)\int_0^t\big(1+(t-\tau)\|D^2\phi(\tau)\|\big)\,d\tau\leq C(t)\left[1+\int_0^t\|D^2\phi(\tau)\|\,d\tau\right].
\end{aligned}
$$

Therefore collecting (3.11), (3.12) and (3.13) we obtain

$$
|\partial_{x_1}B^{\phi_{x_1}}|\leq C(t)\left[1+\int_0^t\|D^2\phi(\tau)\|\,d\tau\right].
$$

A similar argument is applied to the x_1 derivative of the term involved with the kernel $c^{\phi_{x_1}}$ in the representation of ϕ_{x_1}. Note that $\partial^2_{x_1}\phi_{\rm hom}$ satisfies a homogeneous wave equation with the given initial data and so it is bounded by $C(t)$. Since $\partial_{x_1}f^{\rm in}$ is bounded, the x_1 derivative of the second term in the representation of ϕ_{x_1} is bounded by $C(t)$ as well. Therefore we obtain

$$
\|D^2\phi(t)\|\leq C(t)\left[1+\ln^*(t\,\|\,\nabla_{(x,p)}f(t)\|\|)+\int_0^t\|D^2\phi(\tau)\|\,d\tau\right].
$$

So by Gronwall's inequality the theorem follows. □

3.3. **Estimates on Df and Proof of Theorem 2.**

Theorem 6. *Let (f,ϕ) and initial data be as in Theorem 5. Then*

$$
\|\nabla_{(t,x,p)}f(t)\|+\|D^2\phi(t)\|\leq C(t).
$$

PROOF : First we assume more smoothness on the initial data, i.e., $f^{\rm in}\in C^2$, $\phi_0^{\rm in}\in C^4$ and $\phi_1^{\rm in}\in C^3$. Then applying ∂_{x_1} to the Vlasov equation (1.2) and integrating it along the characteristics, by Lemma 2 and Theorem 4 we get

$$
|\partial_{x_1}f(t,x,p)|\leq\|\partial_{x_1}f^{\rm in}\|+C(t)\int_0^t\big(1+\|D^2\phi(\tau)\|\big)\big(1+\|\nabla_{(x,p)}f(\tau)\|\big)\,d\tau.
$$

Similarly for $\partial_{p_1} f$ we have

$$|\partial_{p_1} f(t,x,p)| \leq \|\partial_{p_1} f^{\mathrm{in}}\| + C(t)\int_0^t (1+\|\nabla_{(x,p)} f(\tau)\|)\,d\tau.$$

Therefore using Theorem 5, we get

$$\|\nabla_{(x,p)} f(t)\| \leq \|\nabla_{(x,p)} f^{\mathrm{in}}\| + C(t)\int_0^t \big[1+\ln^*(\tau \,\|\!|\, \nabla_{(x,p)} f(\tau)\|\!|)\big]\big(1+\|\!|\nabla_{(x,p)} f(\tau)\,\|\!|\,\big)\,d\tau.$$

This equation is still satisfied with our original data in the theorem by a limiting argument. For fixed t and $s \in [0,t]$, consider

$$3 + \|\!|\nabla_{(x,p)} f(s)\|\!| \leq Q(s) \tag{3.14}$$

where Q is defined by

$$Q(s) := (3+\|\nabla_{(x,p)} f^{\mathrm{in}}\|) + C(t)\int_0^s Q(\tau)\ln Q(\tau)\,d\tau.$$

Then one can see that

$$Q(s) = \exp\big(e^{C(t)s}\ln(3+\|\nabla_{(x,p)} f^{\mathrm{in}}\|)\big).$$

So taking $s=t$ with (3.14) we obtain

$$\|\!|\nabla_{(x,p)} f(t)\|\!| \leq C(t). \tag{3.15}$$

The bound on $\|\partial_t f(t)\|$ follows by (1.2), (3.15), Lemma 2 and Theorem 4 :

$$\|\partial_t f(t)\| \leq \|\nabla_x f(t)\| + \|D\phi(t)\|\,\|\nabla_p f(t)\| + \|D\phi(t)\|\,\|f(t)\| \leq C(t).$$

With Theorem 5, the proof of the theorem completes. □

PROOF OF THEOREM 2 : In [4], the three dimensional version of the iteration scheme is presented and the convergence is shown. Even though the representations of the derivatives of ϕ are different in the two dimensional case, the iteration scheme and the proof of the convergence in [4] can be applied directly with Lemma 2, Theorems 4 and 6, to end the argument of the existence of solution. We also refer to the same reference for the uniqueness of the solution. □

4. PROOF OF THEOREM 3

In the previous section, it was shown that the solution is continued as long as the p support of f remains bounded for bounded time. By the assumption that f^{in} has compact support in p, there is a smooth solution with

$$f(t,x,p) = 0 \text{ if } |p| \geq C(t),$$

on some time interval $[0,T)$. Without loss of generality we take T to be maximal and consider $t \in [0,T)$ for the rest of the paper. Now define

$$P(t) = \sup\{|p| : f(s,x,p) \neq 0 \text{ for some } (s,x) \in [0,t]\times\mathbb{R}^2\} + 3.$$

Note that $P(t) \le C(t)$ implies that $T = \infty$ and so the bounds stated in Theorems 4 and 6 hold for all t. Therefore once we achieve that $P(t) \le C(t)$ then the theorem follows.

4.1. **Preliminaries and Lemmas.** We present some lemmas and notations to use frequently to prove Theorem 3. To keep notations not to heavy we write $v \wedge w = v_1 w_2 - v_2 w_1$, for any two vectors (v_1, v_2) and (w_1, w_2). and also define $\omega = (y-x)/|y-x|$.

Lemma 3.

$$(1+|p|^2)^{-1} \le 2(1+\xi\cdot\hat{p}).$$

Proof : The lemma follows by the fact that

$$\sqrt{1+|p|^2}(1+\xi\cdot\hat{p}) = \sqrt{1+|p|^2} + \xi\cdot p = \frac{1+|p|^2-(\xi\cdot p)^2}{\sqrt{1+|p|^2}-\xi\cdot p} \ge 1/(2\sqrt{1+|p|^2}).$$

□

The following lemma is from [11]. We state it without the proof.

Lemma 4.

$$(\hat{p}\wedge\omega)^2 \le 2(1+\xi\cdot\hat{p}).$$

Now for the next lemma, we define the energy density e by

$$e(t,x) := 4\pi\int\sqrt{1+|p|^2}f(t,x,p)\,dp + \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}|\nabla_x\phi|^2.$$

Lemma 5. *Let the assumptions of Theorem 3 hold. Then for each* $R \ge 0$*,*

$$\sup_{x\in\mathbb{R}^2}\int_{|y-x|<R} e(t,y)\,dy \le C(R+t)^2, \tag{4.1}$$

$$\sup_{x\in\mathbb{R}^2}\int_0^t\int_{|y-x|=t-\tau+R}\left(\frac{1}{2}(\omega\wedge\nabla_x\phi)^2 + \frac{1}{2}(\partial_t\phi-\nabla_x\phi\cdot\omega)^2 + 4\pi\int\sqrt{1+|p|^2}(1+\hat{p}\cdot\omega)\,dp\right)dS_y\,d\tau \le C(R+t)^2, \tag{4.2}$$

$$\sup_{x\in\mathbb{R}^2}\int_{|y-x|<R}\left(\int\frac{f(t,x,p)}{\sqrt{1+|p|^2}}\,dp\right)^3 dy \le C(t)(R+t)^2. \tag{4.3}$$

Proof : First note that

$$\int_{|y-x|\le R+t} e(0,y)\,dy \le C(R+t)^2. \tag{4.4}$$

We have the energy identity :

$$\partial_t e(t,x) + \nabla_x\cdot\left(-\partial_t\phi\nabla_x\phi + 4\pi\int pf(t,x,p)\,dp\right) = 0.$$

So we have

$$\begin{aligned}0 &= \int_0^t \int_{|y-x|<t-\tau+R} \Big[\partial_\tau e + \nabla_y \cdot \Big(-\partial_t\phi\nabla_x\phi + 4\pi \int pf(\tau,y,p)\,dp\Big)\Big]\,dy\,d\tau \\ &= \int_0^t \int_{|y-x|=t-\tau+R} \Big[e + \omega \cdot \Big(-\partial_t\phi\nabla_x\phi + 4\pi \int pf(\tau,y,p)\,dp\Big)\Big]\,dS_y\,d\tau \\ &\quad - \int_{|y-x|<t+R} e(0,y)\,dy + \int_{|y-x|<R} e(t,y)\,dy.\end{aligned}$$

Also one can see that

$$\begin{aligned}&e + \omega \cdot \Big(-\partial_t\phi\nabla_x\phi + 4\pi \int pf(t,x,p)\,dp\Big) \\ &\quad = \frac{1}{2}(\omega\wedge\nabla_x\phi)^2 + \frac{1}{2}(\partial_t\phi - \nabla_x\phi\cdot\omega)^2 + 4\pi\int \sqrt{1+|p|^2}(1+\hat{p}\cdot\omega)f\,dp \ge 0.\end{aligned}$$

Therefore with (4.4), we obtain (4.1) and (4.2). Note that for each $r>0$,

$$\int \frac{f(t,x,p)}{\sqrt{1+|p|^2}}\,dp \le C(t)\int_{|p|<r} |p|^{-1}\,dp + r^{-2}\int_{|p|>r}\sqrt{1+|p|^2}f\,dp \le C(t)(r+r^{-2}e).$$

and so taking $r = e^{1/3}$ and with (4.1) we obtain (4.3). □

The following lemma is almost identical to Lemma 3 in [11], except that we have Lemma 2. So we state it without the proof.

Lemma 6. *For* $|\xi|<1$, *define*

$$\sigma_{BC}(t,x,\xi) := \int \frac{f(t,x,p)}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})}\,dp.$$

Then

$$0 \le \sigma_{BC} \le C(t)P(t)\min\{P(t),(1-|\xi|^2)^{-1/2}\}.$$

4.2. **Fields estimates.** Let A^{ϕ_l}, B^{ϕ_l} and C^{ϕ_l} be terms with kernels a^{ϕ_l}, b^{ϕ_l} and c^{ϕ_l} respectively in the representations of the derivatives of ϕ, where $l = t, x_1$ and x_2.

Lemma 7.

$$|B^{\phi_t}| + |C^{\phi_t}| + |B^{\phi_{x_i}}| + |C^{\phi_{x_i}}| \le C\int_0^t\int_{|y-x|<t-\tau}\int \frac{f\big(|\partial_t\phi| + |\nabla_x\phi| + (1+\xi\cdot\hat{p})^{-1}|\omega\wedge\nabla_x\phi|\big)\,dp\,dy\,d\tau}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}$$

where $i = 1$ *and* 2.

PROOF : Define $\omega^\perp = (-\omega_2,\omega_1)$ and then for every $z\in\mathbb{R}^2$ we have

$$z = (\omega\cdot z)\omega + (\omega\wedge z)\omega^\perp, \quad z\cdot\omega^\perp = \omega\wedge z, \quad z\wedge\omega^\perp = \omega\cdot z.$$

For fixed ξ and $\hat{p}$ define $\mathcal{F}:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ by

$$\mathcal{F}(g,\vec{h}) := \xi_i(1+\xi\cdot\hat{p})^2(g+\hat{p}\cdot\vec{h}) + \xi_i(\xi+\hat{p})\cdot\vec{h}(1+|p|^2)^{-1}.$$

Then from $b^{\phi_{x_i}}$ and $c^{\phi_{x_i}}$, we have

$$b^{\phi_{x_i}} S(\phi) + c^{\phi_{x_i}} \cdot \nabla_x \phi = \frac{\mathcal{F}(\partial_t \phi, \nabla_x \phi)}{\sqrt{1+|p|^2}(1+\xi\cdot\hat{p})^2}.$$

Using $\{(0,\omega^\perp),(1,\omega),(-1,\omega)\}$ as an orthogonal basis of $\mathbb{R}^3$ one can write

$$\mathcal{F}(\partial_t\phi,\nabla_x\phi) = \big(A_1(0,\omega^\perp) + A_2(1,\omega) + A_3(-1,\omega)\big)\cdot(\partial_t\phi,\nabla_x\phi)$$

for all $\partial_t\phi\in\mathbb{R}$ and $\nabla_x\phi\in\mathbb{R}^2$ where

$$A_1 = \mathcal{F}(0,\omega^\perp), \quad 2A_2 = \mathcal{F}(1,\omega), \quad 2A_3 = \mathcal{F}(-1,\omega).$$

Now we estimate A_1, A_2 and A_3. First note that applying Lemmas 4 and 3 to the first and the second terms respectively we have

$$|A_1| = |\mathcal{F}(0,\omega^\perp)| = |\xi_i|(1+\xi\cdot\hat{p})^2|\omega\wedge\hat{p}| + |\xi_i||\omega\wedge\hat{p}|(1+|p|^2)^{-1} \le C\big[(1+\xi\cdot\hat{p})^2 + (1+\xi\cdot\hat{p})\big]. \tag{4.5}$$

Also we get

$$\begin{aligned}|2A_2| = |\mathcal{F}(1,\omega)| &= |\xi_i|(1+\xi\cdot\hat{p})^2|1+\hat{p}\cdot\omega| + |\xi_i||(\xi+\hat{p})\cdot\omega|(1+|p|^2)^{-1}\\ &\le C(1+\xi\cdot\hat{p})^2 + C(1+\xi\cdot\hat{p})\big||\xi|+\hat{p}\cdot\omega\big|\\ &\le C(1+\xi\cdot\hat{p})^2 + C(1+\xi\cdot\hat{p})\big||\xi|-1+(1+\xi\cdot\hat{p})+(\omega-\xi)\cdot\hat{p}\big|\\ &\le C(1+\xi\cdot\hat{p})^2 + C(1+\xi\cdot\hat{p})\big(1-|\xi|+(1+\xi\cdot\hat{p})+|\omega-\xi|\big) \le C(1+\xi\cdot\hat{p})^2 \end{aligned} \tag{4.6}$$

by the fact that $|\omega-\xi| = 1-|\xi| \le 1+\xi\cdot\hat{p}$. Similarly

$$|2A_3| = |\mathcal{F}(-1,\omega)| = |\xi_i|(1+\xi\cdot\hat{p})^2|-1+\hat{p}\cdot\omega| + |\xi_i||(\xi+\hat{p})\cdot\omega|(1+|p|^2)^{-1} \le C(1+\xi\cdot\hat{p})^2. \tag{4.7}$$

Collecting these bounds (4.5) - (4.7), we have

$$\begin{aligned}|\mathcal{F}(\partial_t\phi,\nabla_x\phi)| &\le C\big[(1+\xi\cdot\hat{p})^2+(1+\xi\cdot\hat{p})\big]|(0,\omega^\perp)\cdot(\partial_t\phi,\nabla_x\phi)|\\ &\qquad + C(1+\xi\cdot\hat{p})^2|(1,\omega)\cdot(\partial_t\phi,\nabla_x\phi)| + C(1+\xi\cdot\hat{p})^2|(-1,\omega)\cdot(\partial_t\phi,\nabla_x\phi)|\\ &\le C(1+\xi\cdot\hat{p})^2\big(|\partial_t\phi|+|\nabla_x\phi|\big) + C(1+\xi\cdot\hat{p})|\omega\wedge\nabla_x\phi|.\end{aligned}$$

Therefore we obtain

$$|B^{\phi_{x_i}}| + |C^{\phi_{x_i}}| \le C\int_0^t\int_{|y-x|<t-\tau}\int \frac{f\big(|\partial_t\phi|+|\nabla_x\phi|+(1+\xi\cdot\hat{p})^{-1}|\omega\wedge\nabla_x\phi|\big)\,dp\,dy\,d\tau}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}.$$

A same argument works for B^{ϕ_t} and C^{ϕ_t} and so the lemma follows. □

Lemma 8.

$$\int_0^t\int_{|y-x|<t-\tau}\frac{\sigma_{BC}(\tau,y,\xi)|\omega\wedge\nabla_x\phi(\tau,y)|\,dy\,d\tau}{\sqrt{(t-\tau)^2-|y-x|^2}} \le C(t)P(t)\ln P(t).$$

PROOF : Let $r = |y-x|$ and $s = (t-\tau-r)/2$ in the r integration. We invert the order of the τ and s and then change back to r in the τ integration :

$$\begin{aligned}
&\int_0^t \int_{|y-x|<t-\tau} \frac{\sigma_{BC}(\tau,y,\xi)|\omega\wedge\nabla_x\phi(\tau,y)|\,dy\,d\tau}{\sqrt{(t-\tau)^2-|y-x|^2}} \\
&\quad = \int_0^t \int_0^{t-\tau} \int_{|y-x|=r} \frac{\sigma_{BC}(\tau,y,\xi)|\omega\wedge\nabla_x\phi(\tau,y)|\,dS_y\,dr\,d\tau}{\sqrt{(t-\tau)^2-r^2}} \\
&\quad = \int_0^t \int_0^{\frac{1}{2}(t-\tau)} \int_{|y-x|=t-\tau-2s} \frac{\sigma_{BC}(\tau,y,\xi)|\omega\wedge\nabla_x\phi(\tau,y)|\,dS_y\,ds\,d\tau}{\sqrt{s}\sqrt{t-\tau-s}} \\
&\quad = \int_0^{t/2} \int_0^{t-2s} \int_{|y-x|=t-\tau-2s} \frac{\sigma_{BC}(\tau,y,\xi)|\omega\wedge\nabla_x\phi(\tau,y)|\,dS_y\,d\tau\,ds}{\sqrt{s}\sqrt{t-\tau-s}} \\
&\quad = \int_0^{t/2} \int_0^{t-2s} \int_{|y-x|=r} \frac{\sigma_{BC}(t-r-2s,y,(y-x)(r+2s)^{-1})|\omega\wedge\nabla_x\phi(t-r-2s,y)|\,dS_y\,dr\,ds}{\sqrt{s}\sqrt{r+s}}.
\end{aligned}$$

Let $\epsilon \in (0, t/2]$ and consider $\tau \in (\epsilon, t/2)$. From Lemma 6 and with $|y-x| = r = t-\tau-2s$, we have

$$\sigma_{BC}(t-r-2s,y,(y-x)(r+2s)^{-1}) \le \frac{C(t)P(t)(r+2s)}{\sqrt{s}\sqrt{r+s}} \le C(t)P(t)\sqrt{r+s}/\sqrt{s}.$$

Hence we get

$$\begin{aligned}
&\int_\epsilon^{t/2} \int_0^{t-2s} \int_{|y-x|=r} \frac{\sigma_{BC}(t-r-2s,y,(y-x)(r+2s)^{-1})|\omega\wedge\nabla_x\phi(t-r-2s,y)|\,dS_y\,dr\,ds}{\sqrt{s}\sqrt{r+s}} \\
&\quad \le C(t)P(t) \int_\epsilon^{t/2} \int_0^{t-2s} \int_{|y-x|=r} s^{-1}|\omega\wedge\nabla_x\phi(t-r-2s,y)|\,dS_y\,dr\,ds.
\end{aligned} \tag{4.8}$$

By (4.2) and letting $r = t-\tau-2s$ we have

$$\begin{aligned}
&\int_0^{t-2s} \int_{|y-x|=r} |\omega\wedge\nabla_x\phi(t-r-2s,y)|^2\,dS_y\,dr \\
&\quad = \int_0^{t-2s} \int_{|y-x|=t-\tau-2s} |\omega\wedge\nabla_x\phi(\tau,y)|^2\,dS_y\,d\tau \le C(t-2s)^2 \le Ct^2.
\end{aligned} \tag{4.9}$$

So by Schwarz's inequality (4.8) becomes

$$\begin{aligned}
&\int_\epsilon^{t/2} \int_0^{t-2s} \int_{|y-x|=r} \frac{\sigma_{BC}(t-r-2s,y,(y-x)(r+2s)^{-1})|\omega\wedge\nabla_x\phi(t-r-2s,y)|\,dS_y\,dr\,ds}{\sqrt{s}\sqrt{r+s}} \\
&\quad \le C(t)P(t) \int_\epsilon^{t/2} \left(\int_0^{t-2s} \int_{|y-x|=r} s^{-2}\,dS_y\,dr \right)^{1/2} ds = C(t)P(t)\ln\frac{t}{2\epsilon}.
\end{aligned} \tag{4.10}$$

Consider $\tau \in (0,\epsilon)$. From Lemma 6, $\sigma_{BC} \le C(t)P^2(t)$ and by (4.9) we get

$$\int_0^\epsilon \int_0^{t-2s} \int_{|y-x|=r} \frac{\sigma_{BC}(t-r-2s, y, (y-x)(r+2s)^{-1})|\omega \wedge \nabla_x \phi(t-r-2s,y)|\, dS_y\, dr\, ds}{\sqrt{s}\sqrt{r+s}} \tag{4.11}$$

$$\le C(t)P^2(t) \int_0^\epsilon \int_0^{t-2s} \int_{|y-x|=r} |\omega \wedge \nabla_x \phi(t-r-2s,y)| [s(r+s)]^{-1/2}\, dS_y\, dr\, ds$$

$$\le C(t)P^2(t) \int_0^\epsilon \left(\int_0^{t-2s} \int_{|y-x|=r} [s(r+s)]^{-1}\, dS_y\, dr \right)^{1/2} ds \le C(t)P^2(t) \int_0^\epsilon s^{-1/2}\, ds = C(t)P^2(t)\sqrt{\epsilon}.$$

Collecting (4.10) and (4.11) we obtain that

$$\int_0^t \int_{|y-x|<t-\tau} \frac{\sigma_{BC}(\tau,y,\xi)|\omega \wedge \nabla_x \phi(\tau,y)|\, dy\, d\tau}{\sqrt{(t-\tau)^2 - |y-x|^2}} \le C(t)P(t) \left(\ln \frac{t}{2\epsilon} + P(t)\sqrt{\epsilon} \right).$$

Taking $\epsilon = \min\{t/2, P^{-2}(t)\}$ completes the proof. □

Proposition 3.

$$|B^{\phi_t}| + |C^{\phi_t}| + |B^{\phi_{x_i}}| + |C^{\phi_{x_i}}| \le C(t)P(t)\ln P(t) + C(t)\int_0^t (\|\partial_t \phi(\tau)\| + \|\nabla_x \phi(\tau)\|)\, d\tau.$$

Proof : By Lemmas 7 and 8 we obtain

$$|B^{\phi_t}| + |C^{\phi_t}| + |B^{\phi_{x_i}}| + |C^{\phi_{x_i}}|$$

$$\le C\int_0^t \int_{|y-x|<t-\tau} \int \frac{(|\partial_t \phi| + |\nabla_x \phi|) f\, dp\, dy\, d\tau}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2 - |y-x|^2}} + C\int_0^t \int_{|y-x|<t-\tau} \frac{\sigma_{BC}|\omega \wedge \nabla_x \phi|\, dy\, d\tau}{\sqrt{(t-\tau)^2 - |y-x|^2}}$$

$$\le C\int_0^t (\|\partial_t \phi(\tau)\| + \|\nabla_x \phi(\tau)\|) \int_{|y-x|<t-\tau} \int \frac{f\, dp\, dy\, d\tau}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2 - |y-x|^2}} + C(t)P(t)\ln P(t).$$

By (4.3) and Hölder's inequality , note that

$$\int_{|y-x|<t-\tau} \int \frac{f(\tau,y,p)\, dp\, dy}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2 - |y-x|^2}}$$

$$\le \left(\int_{|y-x|<t-\tau} \left(\int \frac{f(\tau,y,p)}{\sqrt{1+|p|^2}}\, dp \right)^3 dy \right)^{1/3} \left(\int_{|y-x|<t-\tau} \left((t-\tau)^2 - |y-x|^2\right)^{-3/4} dy \right)^{2/3} \le C(t).$$

Therefore the proposition follows. □

Lemma 9.

$$\int (|a^{\phi_t}| + |a^{\phi_{x_i}}|) f\, dp \le C(t)\min\{P^3(t), P^{3/2}(t)e^{1/2}(t,x)(1-|\xi|^2)^{-1/4}\}.$$

The proof of this lemma is almost identical to Lemma 5 in [11], except the fact that the kernels a^{ϕ_t} and $a^{\phi_{x_i}}$ have one higher order of p comparing with those in [11] and we have Lemma 2. For the precise relation, recall that Propositions 1 and 2 in the present paper. For this reason, we leave the sketch of the proof in the appendix.

Proposition 4.

$$|A^{\phi_t}| + |A^{\phi_{x_i}}| \le C(t)P^2(t)\ln^{2/3} P(t).$$

PROOF : First note that

$$|A^{\phi_t}| + |A^{\phi_{x_i}}| \le C\int_0^t \int_{|y-x|<t-\tau}\int \frac{f(|a^{\phi_t}| + |a^{\phi_{x_i}}|)\,dp\,dy\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}.$$

Let $\delta \in (0,t]$ and $\epsilon \in (0,1)$. By Lemma 9, we get

$$\begin{aligned}&\int_0^\delta \int_{1-\epsilon<|\xi|<1}\int \frac{f(|a^{\phi_t}| + |a^{\phi_{x_i}}|)\,dp\,dy\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\\ &\quad\le C(t)P^3(t)\int_0^t\int_{(1-\epsilon)(t-\tau)}^{t-\tau}\frac{r\,dr\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-r^2}} \le C(t)P^3(t)\sqrt{\epsilon}.\end{aligned}$$

Again by Lemma 9 and (4.1) we get

$$\begin{aligned}&\int_0^{t-\delta}\int_{|y-x|<(1-\epsilon)(t-\tau)}\int \frac{f(|a^{\phi_t}| + |a^{\phi_{x_i}}|)\,dp\,dy\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\\ &\quad\le C(t)P^{3/2}(t)\int_0^{t-\delta}\int_{|y-x|<(1-\epsilon)(t-\tau)}\frac{e^{1/2}(\tau,y)(1-|\xi|)^{-1/4}\,dy\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\\ &\quad\le C(t)P^{3/2}(t)\int_0^{t-\delta}(t-\tau)^{-1}\left(\int_{|y-x|<(1-\epsilon)(t-\tau)}\frac{(1-|\xi|^2)^{-1/2}\,dy}{(t-\tau)^2-|y-x|^2}\right)^{1/2} d\tau \le C(t)\ln\frac{t}{\delta}P^{3/2}(t)\epsilon^{-1/4}.\end{aligned}$$

For the tip of the cone we have by Lemma 9 that

$$\begin{aligned}&\int_{t-\delta}^t\int_{|y-x|<t-\tau}\int \frac{f(|a^{\phi_t}| + |a^{\phi_{x_i}}|)\,dp\,dy\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-|y-x|^2}}\\ &\quad\le C(t)P^3(t)\int_{t-\delta}^t\int_0^{t-\tau}\frac{r\,dr\,d\tau}{(t-\tau)\sqrt{(t-\tau)^2-r^2}} = C(t)P^3(t)\delta.\end{aligned}$$

Collecting all above three estimates, we have

$$|A^{\phi_t}| + |A^{\phi_{x_i}}| \le C(t)\big[P^3(t)\sqrt{\epsilon} + \ln\frac{t}{\delta}P^{3/2}(t)\epsilon^{-1/4} + P^3(t)\delta\big].$$

We take $\delta = \min\{t, P^{-1}(t)\}$ then

$$|A^{\phi_t}| + |A^{\phi_{x_i}}| \le C(t)\big[P^3(t)\sqrt{\epsilon} + \ln P(t)P^{3/2}(t)\epsilon^{-1/4} + P^2(t)\big].$$

Taking $\epsilon = P^{-2}(t)\ln^{4/3} P(t)$ the proposition follows. □

4.3. **Proof of Theorem 3.** We note that from Subsection 3.1 we have proved that $\|D\phi(t)\| \le C(t)$. Since $|p| < P(0)$ in the second terms of the representations $\partial_t\phi$ and $\partial_{x_i}\phi$, one can see that these terms are also bounded by $C(t)$. Now with Propositions 3 and 4 we obtain

$$|\partial_t\phi(t,x)| + |\nabla_x\phi(t,x)| \le C(t)P^2(t)\ln^{3/2} P(t) + C(t)\int_0^t \|\partial_t\phi(\tau)\| + \|\nabla_x\phi(\tau)\|\,d\tau.$$

So for fixed t and $s \in [0,t]$

$$\|\partial_t \phi(s)\| + \|\nabla_x \phi(s)\| \le C(t)P^2(t)\ln P(t) + C(t)\int_0^s \|\partial_t\phi(\tau)\| + \|\nabla_x\phi(\tau)\|\, d\tau.$$

By Gronwall's inequality and taking $s = t$ we achieve

$$\|\partial_t \phi(t)\| + \|\nabla_x \phi(t)\| \le C(t)P^2(t)\ln P(t). \tag{4.12}$$

We define the characteristics $(\mathcal{X},\mathcal{P})(s,t,x,p)$ for (1.2) by

$$d/ds\mathcal{X} = \hat{\mathcal{P}} \tag{4.13}$$

$$d/ds\mathcal{P} = -(\partial_t\phi(s,\mathcal{X}) + \hat{\mathcal{P}}\cdot\nabla_x\phi(s,\mathcal{X}))\mathcal{P} - \frac{\nabla_x\phi(s,\mathcal{X})}{\sqrt{1+|\mathcal{P}|^2}}, \tag{4.14}$$

with $\mathcal{X}(t,t,x,p) = x$, $\mathcal{P}(t,t,x,p) = p$. Also define

$$\bar{P}(t) := \sup\{e^{\phi(s,x)}|p| : f(s,x,p) \neq 0 \text{ for some } (s,x) \in [0,t]\times\mathbb{R}^2\} + 3.$$

Consider $e^{2\phi}|p|^2$ along the characteristics :

$$d/ds(e^{2\phi(s,\mathcal{X})}|\mathcal{P}|^2) = -2e^{2\phi(s,\mathcal{X})}\hat{\mathcal{P}}\cdot\nabla_x\phi(s,\mathcal{X}).$$

Then with (4.12) one can see that when $f(0,x,p)\neq 0$,

$$\begin{aligned} e^{2\phi(0,\mathcal{X}(0,t,x,p))}|\mathcal{P}(0,t,x,p)|^2 &\le e^{2\phi(t,x)}|p|^2 + C\int_0^t e^{2\phi(\tau,\mathcal{X})}\|\nabla_x\phi(\tau)\|\, d\tau \\ &\le e^{2\phi(t,x)}|p|^2 + C(t)\int_0^t e^{2\phi(\tau,\mathcal{X})}P^2(\tau)\ln P(\tau)\, d\tau. \end{aligned}$$

Note that in (3.1) we have $e^{\phi} \le e^{\phi_{\text{hom}}}$. Also we have seen that in Subsection 3.1 $\|\phi_{\text{hom}}(t)\| \le C(1+t)$. Therefore with the definition $\bar{P}$, for fixed t and $s\in[0,t]$ we get

$$\bar{P}^2(s) \le C(t) + C(t)\int_0^s \bar{P}^2(\tau)\ln\bar{P}^2(\tau)\, d\tau.$$

Therefore we have

$$\bar{P}^2(s) \le \exp(e^{C(t)s}\ln C(t))$$

and taking $s = t$ we get

$$\bar{P}(t) \le C(t). \tag{4.15}$$

Note that by (4.15) we have

$$\int \frac{f(t,x,p)}{\sqrt{1+|p|^2}}\, dp \le C(t)\int_{e^{\phi}|p|\le C(t)} \left(e^{\phi}|p|\right)^{-1} d(e^{\phi}p) \le C(t).$$

So in (3.1) we get $-\phi \le C(t)$ and so $e^{-\phi}\le C(t)$. Therefore with the definition $\bar{P}$ and (4.15) we achieve $P(t)\le C(t)$. □

Appendix A. Proof of Proposition 2

Using the following operations

$$S := \partial_t + \hat{p} \cdot \nabla_x$$
$$T_k := \frac{1}{\sqrt{1-|\xi|^2}}(\partial_{x_k} - \xi_k \partial_t) \quad k = 1, 2.$$

we obtain

$$\begin{aligned}\partial_t &= \frac{S - \sqrt{1-|\xi|^2}(\hat{p}_1 T_1 + \hat{p}_2 T_2)}{1+\xi\cdot\hat{p}} \\ \partial_{x_1} &= \frac{\xi_1 S + \sqrt{1-|\xi|^2}[(1+\xi_2\hat{p}_2)T_1 - \xi_1\hat{p}_2 T_2]}{1+\xi\cdot\hat{p}} \\ \partial_{x_2} &= \frac{\xi_2 S + \sqrt{1-|\xi|^2}[-\xi_2\hat{p}_1 T_1 + (1+\xi_1\hat{p}_1)T_2]}{1+\xi\cdot\hat{p}}.\end{aligned} \tag{A.1}$$

By these definitions it follows from (3.1) that

$$\begin{aligned}\partial_{x_1}\phi(t,x) = \partial_{x_1}\phi_0 & \\ &- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1 Sf(\tau,y,p)}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \\ &- 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{(1+\xi_2\hat{p}_2)T_1 f(\tau,y,p)}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}(t-\tau)}\,dp\,dy\,d\tau \\ &+ 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1\hat{p}_2 T_2 f(\tau,y,p)}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}(t-\tau)}\,dp\,dy\,d\tau \\ &:= \partial_{x_1}\phi_0 + S\mathrm{term}_{x_1} + T_1\mathrm{term}_{x_1} + T_2\mathrm{term}_{x_1}.\end{aligned}$$

Note that from (1.2)

$$Sf = F(t,x,p)\cdot\nabla_p f + 3S(\phi)f \tag{A.2}$$

where

$$F(t,x,p) := S(\phi)p + \frac{\nabla_x\phi}{\sqrt{1+|p|^2}}.$$

So $S\mathrm{term}_{x_1}$ becomes

$$\begin{aligned}S\mathrm{term}_{x_1} &= -2\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1 F(\tau,y,p)\cdot\nabla_p f(\tau,y,p)}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \\ &\quad - 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{3\xi_1 S(\phi)f(\tau,y,p)}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \\ &= -2\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1 S(\phi)f(\tau,y,p)}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau \\ &\quad - 2\int_0^t \int_{|y-x|<t-\tau} \int \frac{\xi_1(\xi+\hat{p})\cdot(\nabla_x\phi)f(\tau,y,p)}{(1+\xi\cdot\hat{p})^2(1+|p|^2)^{3/2}\sqrt{(t-\tau)^2-|y-x|^2}}\,dp\,dy\,d\tau.\end{aligned}$$

Also note that

$$\frac{\partial}{\partial y_k}\left(\frac{g(\tau,y)}{\sqrt{1-|\xi|^2}}\right)+\frac{\partial}{\partial \tau}\left(\frac{-\xi_k g(\tau,y)}{\sqrt{1-|\xi|^2}}\right)=T_k g(\tau,y). \tag{A.3}$$

So we get

$$T_1\mathrm{term}_{x_1}=-2\int_0^t\int_{|y-x|<t-\tau}\int\frac{(1+\xi_2\hat{p}_2)\nabla_{(\tau,y)}}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}(t-\tau)}\cdot\left(\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\right)dp\,dy\,d\tau.$$

We integrate by parts :

$$\begin{aligned}&\int_0^t\int_{|y-x|<(1-\epsilon)(t-\tau)}\int K_{T_1}\nabla_{(\tau,y)}\cdot\left(\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\right)dp\,dy\,d\tau \qquad\qquad (\mathrm{A.4})\\ &=-\int_0^t\int_{|y-x|<(1-\epsilon)(t-\tau)}\int\nabla_{(\tau,y)}K_{T_1}\cdot\left(\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\right)dp\,dy\,d\tau\\ &\quad+\int_0^t\int_{|y-x|=(1-\epsilon)(t-\tau)}\int K_{T_1}\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\cdot\left(1-\epsilon,\frac{y_1-x_1}{|y-x|},\frac{y_2-x_2}{|y-x|}\right)dp\,dS_y\,d\tau\\ &\quad+\int_{|y-x|<(1-\epsilon)t}\int K_{T_1}\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\bigg|_{\tau=0}\cdot(-1,0,0)\,dp\,dy\end{aligned}$$

where $\epsilon\in(0,1)$ and

$$K_{T_1}:=\frac{1+\xi_2\hat{p}_2}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}(t-\tau)}.$$

In the second term in (A.4) note that

$$\left|\frac{(-\xi_1,1,0)}{\sqrt{1-|\xi|^2}}\cdot\left(1-\epsilon,\frac{y_1-x_1}{|y-x|},\frac{y_2-x_2}{|y-x|}\right)\right|=\sqrt{2\epsilon-\epsilon^2}\frac{|y_1-x_1|}{(1-\epsilon)(t-\tau)}\leq\sqrt{2\epsilon}. \tag{A.5}$$

So let $\epsilon\to 0^+$. We get

$$\begin{aligned}T_1\mathrm{term}_{x_1}&=-2\int_{|y-x|<t}\int\frac{\xi_1 K_{T_1}f^{\mathrm{in}}(y,p)}{\sqrt{1-|\xi|^2}}\,dp\,dy\\ &\quad+2\int_0^t\int_{|y-x|<t-\tau}\int\nabla_{(\tau,y)}K_{T_1}\cdot\left(\frac{(-\xi_1,1,0)f}{\sqrt{1-|\xi|^2}}\right)dp\,dy\,d\tau.\end{aligned}$$

Note that

$$\nabla_{(\tau,y)}K_{T_1}\cdot(-\xi_1,1,0)=-\frac{\xi_1\xi_2\hat{p}_2}{(1+\xi\cdot\hat{p})\sqrt{1+|p|^2}(t-\tau)^2}-\frac{(\xi_1+\hat{p}_1)(1+\xi_2\hat{p}_2)}{(1+\xi\cdot\hat{p})^2\sqrt{1+|p|^2}(t-\tau)^2}.$$

One can identify $T_2\mathrm{term}_{x_1}$ by the same argument. Combining these terms, we obtain the representation of $\partial_{x_1}\phi$.

APPENDIX B. PROOF OF LEMMA 2

In (3.1) note that $e^{\phi} \leq e^{\phi_{\text{hom}}}$. Let $(\mathcal{X}, \mathcal{P})(s,t,x,p)$ denote the characteristics as in (4.13) and (4.14). In short, we use $\mathcal{X}(s) := \mathcal{X}(s,t,x,p)$ and $\mathcal{P}(s) := \mathcal{P}(s,t,x,p)$. Note that the function $e^{-3\phi} f$ is constant along these curves. Hence the solution of (1.2) is given by

$$\begin{aligned} f(t,x,p) &= f^{\text{in}}(\mathcal{X}(0), \mathcal{P}(0)) \exp\left[3\phi(t,x)\right] \exp\left[-3\phi_0^{\text{in}}(\mathcal{X}(0))\right] \\ &\leq f^{\text{in}}(\mathcal{X}(0), \mathcal{P}(0)) \exp\left[3\phi_{\text{hom}}(t,x)\right] \exp\left[-3\phi_0^{\text{in}}(\mathcal{X}(0))\right]. \end{aligned}$$

So with (3.2) the lemma follows.

APPENDIX C. PROOF OF LEMMA 9

Note that

$$\begin{aligned} |a^{\phi_t}| + |a^{\phi_{x_i}}| &\leq C(1+|p|)(1-|\hat{p}|^2)(1+\xi\cdot\hat{p})^{-2}[|\xi+\hat{p}| + |\xi\wedge\hat{p}|] \\ &\leq C(1+|p|)(1-|\hat{p}|^2)(1+\xi\cdot\hat{p})^{-3/2}. \end{aligned}$$

Then we have

$$\int (|a^{\phi_t}| + |a^{\phi_{x_i}}|) f \, dp \leq C P(t) \int_{|p|<P(t)} f (1-|\hat{p}|^2)(1+\xi\cdot\hat{p})^{-3/2} \, dp. \tag{C.1}$$

Let $\hat{u} = (1+u^2)^{-1/2} u$. One can see that

$$\int_0^{\pi} \frac{d\theta}{(1-\hat{u}\cos\theta)^{3/2}} \leq \frac{1}{\sqrt{1-\hat{u}|\xi|}} \int_0^{\pi} \frac{d\theta}{1-\hat{u}|\xi|\cos\theta} = \frac{1}{\sqrt{1-\hat{u}|\xi|}} \frac{\pi}{\sqrt{1-\hat{u}^2|\xi|^2}} \leq C(1-\hat{u}|\xi|)^{-1}. \tag{C.2}$$

So for any $R \in (0, P(t)]$ using (C.2) we get

$$\int_{|v|<R} (1-|\hat{p}|^2)(1+\xi\cdot\hat{p})^{-3/2} \, dp \leq C \int_0^R (1+\hat{u}) u \, du \leq C \int_0^R u \, du \leq C R^2. \tag{C.3}$$

If $R = P(t)$, then with (C.1) we get

$$\int (|a^{\phi_t}| + |a^{\phi_{x_i}}|) f \, dp \leq C(t) P^3(t). \tag{C.4}$$

For $R < P(t)$ we use Hölder's inequality and (C.2) to obtain

$$\begin{aligned} &\int_{R<|p|<P(t)} f(1-|\hat{p}|^2)(1+\xi\cdot\hat{p})^{-3/2} \, dp \\ &\leq \int_{R<|p|<P(t)} 2f(1+\xi\cdot\hat{p})^{-1/2} \, dp \leq 2\left(\int_{R<|p|} f^{3/2} \, dp\right)^{2/3} \left(\int_{|p|<P(t)} (1+\xi\cdot\hat{p})^{-3/2} \, dp\right)^{1/3} \\ &\leq C\left((1+R^2)^{-1/2} \int_{R<|p|} f\sqrt{1+|p|^2} \, dp\right)^{2/3} \left(\int_0^{P(t)} (1-u|\xi|)^{-1} u \, du\right)^{1/3} \\ &\leq C R^{-2/3} e^{2/3} \left(\int_0^{P(t)} (1-|\xi|^2)^{-1} u \, du\right)^{1/3} \leq C R^{-2/3} e^{2/3} P^{2/3}(t) (1-|\xi|^2)^{-1/3}. \end{aligned}$$

Now combining this and (C.3) we get

$$\int(|a^{\phi_t}|+|a^{\phi_{x_i}}|)f\,dp \leq C(t)P(t)(R^2+R^{-2/3}e^{2/3}P^{2/3}(t)(1-|\xi|^2)^{-1/3})$$

for all $R\in(0,P(t))$. Taking $R=e^{1/4}P^{1/4}(t)(1-|\xi|^2)^{-1/8}$ the lemma follows with (C.4).

References

[1] S. Calogero: *Spherically symmetric steady states of galactic dynamics in scalar gravity.* Class. Quant. Gravity **20**, 1729–1741 (2003)

[2] S. Calogero: *Global Small Solutions of the Vlasov-Maxwell System in the Absence of Incoming Radiation.* Indiana Univ. Math. Journal (to appear), Preprint: math-ph/0211013

[3] S. Calogero, H. Lee: *The non-relativistic limit of the Nordström-Vlasov system.* Preprint: math-ph/0309030

[4] S. Calogero, G. Rein: *On classical solutions of the Nordström-Vlasov system.* Comm. Partial Diff. Eqns. (to appear), Preprint: math-ph/0304021

[5] S. Calogero, G. Rein: *Global weak solutions to the Nordström-Vlasov system.* Preprint: math-ph/0309046.

[6] R. DiPerna, R. J. Lions: *Global weak solutions of Vlasov-Maxwell systems.* Comm. Pure Appl. Math. **42**, no. 6, 729–757 (1989)

[7] A. Einstein, A. D. Fokker: *Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls.* Annalen der Physik **44**, 321–328 (1914)

[8] R. Glassey, J. Schaeffer: *Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data.* Comm. Math. Phys. **119**, 353–384 (1988)

[9] R. Glassey, J. Schaeffer: *The "Two and One-Half Dimensional" Relativistic Vlasov-Maxwell System.* Comm. Math. Phys. **185**, 257–284 (1997)

[10] R. Glassey, J. Schaeffer: *The relativistic Vlasov-Maxwell system in two space dimensions, Part I.* Arch. Rational Mech. Anal. **141**, 331–354 (1998)

[11] R. Glassey, J. Schaeffer: *The relativistic Vlasov-Maxwell system in two space dimensions, Part II.* Arch. Rational Mech. Anal. **141**, 335–374 (1998)

[12] R. Glassey, W. Strauss: *Singularity formation in a collisionless plasma could occur only at high velocities.* Arch. Rational Mech. Anal. **92**, 59–90 (1986)

[13] R. Glassey, W. Strauss: *Absence of shocks in an initially dilute collisionless plasma.* Comm. Math. Phys. **113**, 191–208 (1987)

[14] H. Lee: *The classical limit of the relativistic Vlasov-Maxwell system in two space dimensions.* Math. Methods Appl. Sci. (to appear)

[15] P.-L. Lions, B. Perthame: *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system.* Invent. Math. **105**, 415–430 (1991)

[16] G. Nordström: *Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips.* Ann. Phys. Lpz. **42**, 533 (1913)

[17] K. Pfaffelmoser: *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data.* J. Diff. Eqns. **95**, 281–303 (1992)

[18] G. Rein: *Selfgravitating systems in Newtonian theory—the Vlasov-Poisson system.* Banach Center Publications **41**, Part I, 179–194 (1997)

[19] G. Rein: *Generic global solutions of the relativistic Vlasov-Maxwell system of plasma physics.* Comm. Math. Phys. **135**, 41–78 (1990)

[20] G. Rein, A. D. Rendall: *Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data.* Commun. Math. Phys. **150**, 561–583 (1992)

[21] J. Schaeffer: *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions.* Comm. Partial Diff. Eqns. **16**, 1313–1335 (1991)

MAX-PLANCK-INSTITUT FÜR GRAVITATIONSPHYSIK, AM MÜHLENBERG 1, GOLM, 14476, GERMANY

E-mail address: hayoung@aei.mpg.de