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On $K(\mathrm{E}_9)$

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Abstract

We study the maximal compact subgroup $K(\mathrm{E}_9)$ of the affine Lie group $\mathrm{E}_{9(9)}$ and its on-shell realization as an R symmetry of maximal $N = 16$ supergravity in two dimensions. We first give a rigorous definition of the group $K(\mathrm{E}_9)$, which lives on the double cover of the spectral parameter plane, and show that the infinitesimal action of $K(\mathrm{E}_9)$ on the chiral components of the bosons and the fermions is determined in terms of an expansion of the Lie algebra of $K(\mathrm{E}_9)$ about the two branch points of this cover; this implies in particular that the fermions of $N = 16$ supergravity transform in a spinor representation of $K(\mathrm{E}_9)$. The fermionic equations of motion can be fitted into the lowest components of a single $K(\mathrm{E}_9)$ covariant 'Dirac equation', with the linear system of $N = 16$ supergravity as the gauge connection. These results suggest the existence of an 'off-shell' realization of $K(\mathrm{E}_9)$ in terms of an infinite component spinor representation. We conclude with some coments on 'generalized holonomies' of M theory.

1 Introduction

The R symmetries of maximally extended supergravities are the maximal compact subgroups of the global $\mathrm{E}_{n(n)}$ symmetries known or conjectured to arise in the dimensional reduction of $D = 11$ supergravity with n (spacelike) Killing vectors to $11-n$ dimensions [1, 2]. For all n, these are defined as the invariant subgroups w.r.t. the Chevalley involution θ, which is defined by its action on the Chevalley generators (see e.g. [3])

$$\theta(e_i) = -f_i \ , \quad \theta(f_i) = -e_i \ , \quad \theta(h_i) = -h_i \ , \tag{1.1}$$

In three or more space-time dimensions, $n \leq 8$, and these groups, which we will denote here generally by $K(\mathrm{E}_n)$, are finite dimensional and well understood. By contrast, in less than three dimensions, where $n \geq 9$, $\mathrm{E}_{n(n)}$ and $K(\mathrm{E}_n)$ are both infinite dimensional groups. In this paper, we will focus on the case $n = 9$, i.e. the Lie group $K(\mathrm{E}_9) \subset \mathrm{E}_{9(9)}$ and its associated involutory Lie algebra $\mathfrak{ke}_9 := \mathrm{Lie}\,(K(\mathrm{E}_9)) \subset \mathfrak{e}_9 := \mathrm{Lie}\,(\mathrm{E}_{9(9)})$ [1]; more specifically, we will study the realization of this symmetry in the context of maximal $N = 16$ supergravity in two dimensions. This case is still far simpler than $n \geq 10$, for which the Cartan matrices of E_n become indefinite. However, even though the affine Lie group $\mathrm{E}_{9(9)}$ and its realization as a loop group are fairly well understood, very little seems to be known about its compact subgroup $K(\mathrm{E}_9)$ or the irreducible representations of $K(\mathrm{E}_9)$: for instance, the standard textbook on loop groups [4] contains no information on this topic. Involutory subalgebras of Kac Moody algebras and their relation with Slodowy algebras were studied in [5][2], but these require in addition outer automorphisms of the Dynkin diagram, which the E_9 algebra does not possess. Furthermore, by all appearances, the subject of *spinorial (i.e. double valued) representations* of $K(\mathrm{E}_9)$ (or any other infinite dimensional involutory subgroup of an affine Lie group) is mostly *terra incognita*.

Our study is motivated by the structure of the linear systems for gravity and supergravity in two dimensions, and by the desire to understand the as yet unknown infinite dimensional symmetries underlying string and M theory. For the bosonic theories, it has been known for a long time from the study of the Geroch group in general relativity and its generalizations, that the affine symmetries arising in the reduction to two dimensions can be realized on an infinite set of 'dual potentials' which in the non-linear realization are non-linear and non-local functions of a finite number of physical fields (see [6] for a review from the relativist's perspective, and [7] for a more general treatment emphasizing the group theoretical aspects). These dual potentials are known to arise from the solution of the associated linear system via an expansion in the spectral parameter γ about the point $\gamma = 0$. For the locally supersymmetric models and their linear systems, which were investigated in [8, 9, 10], a direct fermionic analog of the bosonic dual potentials does not appear to exist, for the very simple reason that fermions obey first order

[1] The group $K(\mathrm{E}_9)$ is sometimes also designated by $\mathrm{SO}(16)^\infty$.
[2] We are grateful to A. Kleinschmidt for bringing this reference to our attention.

equations of motion, and hence cannot be dualized. In this paper, we will present evidence that an infinite hierarchy of fermionic fields nevertheless does exist.

In more mathematical terms, we propose that the fermionic fields belong to a *spinor representation of* $K(\mathrm{E}_9)$, which arises not by dualization or an expansion about $\gamma = 0$, but by an expansion about the two branch points $\gamma = \pm 1$ in the spectral parameter plane, which are associated to the positive and negative chirality components of the fermionic fields. Our main point here is the observation that the fermionic multiplet considered in [10], which contains the fermion fields of $N = 16, d = 2$ supergravity, already by itself constitutes a spinorial, albeit non-faithful, representation of $K(\mathrm{E}_9)$. In this paper, we will work out these transformations in more detail, reducing the complicated contour integrals of [10] to the much simpler formulas (5.16), (5.17). As we will furthermore show in section 6, the fermionic equations of motion can be combined into a single $K(\mathrm{E}_9)$ covariant 'Dirac equation', with the linear system of $N = 16$ supergravity [8, 9, 10] serving as the $K(\mathrm{E}_9)$ connection.

A crucial distinction that we will make in this paper is between 'on-shell' and 'off shell' realizations of $K(\mathrm{E}_9)$. By the former we mean those (field and coordinate dependent) $K(\mathrm{E}_9)$ transformations leaving the $N = 16$ supergravity equations of motion form invariant. By an 'off shell' realization, on the other hand, we mean a realization in terms of infinitely many fields, that does not require the equations of motion to be satisfied. In section 4 we will explain how such a representation can be arrived at for the bosonic theory. We will describe the peculiar difficulties that one encounters when trying to construct 'off shell' spinor representations of $K(\mathrm{E}_9)$ in section 6. Although we are not able so far to give a complete characterization of the latter, we believe that the present results constitute a first step towards their systematic construction.

2 Preliminaries

For the reader's convenience, we start out with a summary of some pertinent facts about the linear systems and spectral parameters appearing in $2d$ gravity and supergravity, following [7] and [11], before giving a rigorous definition of the groups $\mathrm{E}_{9(9)}$ and $K(\mathrm{E}_9)$ in the next section.

2.1 Spectral parameter

We briefly recall some basic facts about the spectral parameters appearing in the linear systems for $2d$ gravity and supergravity, see [7, 11, 10] for our conventions and more details. For Lorentzian worldsheets, the spectral parameter γ entering the linear system is introduced as the function

$$\gamma(\rho, \tilde{\rho}; w) = \frac{1}{\rho} \left(w + \tilde{\rho} - \sqrt{(w + \tilde{\rho})^2 - \rho^2} \right) \; ; \tag{2.1}$$

it depends on the $2d$ coordinates via the 'dilaton' ρ and the 'axion' $\tilde{\rho}$, which are dual to one another: [3]

$$\partial_\pm \rho = \pm \partial_\pm \tilde{\rho} \, , \tag{2.2}$$

and hence both obey free field equations of motion $\partial_+ \partial_- \rho = \partial_+ \partial_- \tilde{\rho} = 0$.

The third variable w arises as a constant of integration in the first order differential equations satisfied by $\gamma(\rho, \tilde{\rho}; w)$

$$\gamma^{-1} \partial_\pm \gamma = \frac{1 \mp \gamma}{1 \pm \gamma} \, \rho^{-1} \partial_\pm \rho \, , \tag{2.3}$$

which are compatible by virtue of (2.2). We will occasionally refer to γ and w as the 'variable' and the 'constant' spectral parameter, respectively, as both play an important role in understanding the symmetries of $2d$ (super)gravity. From (2.1), we immediately obtain

$$\gamma^{-1} \partial_w \gamma \; = \; - \frac{2\gamma}{\rho(1 - \gamma^2)} \, , \tag{2.4}$$

and the inverse relation

$$w(\gamma) \; = \; \frac{\rho}{2} \left(\gamma + \frac{1}{\gamma} \right) - \tilde{\rho} \, . \tag{2.5}$$

From (2.1) we see that the function $\gamma(w)$ lives on a two-sheeted cover of the complex w-plane, with the branch points

$$\gamma = \pm 1 \iff w = \pm \rho - \tilde{\rho} \, . \tag{2.6}$$

Notice that these are 'moving' branch points, as they depend on the spacetime coordinates via the fields ρ and $\tilde{\rho}$. At the same time they are fixed points of the involution

$$\mathcal{I} : \gamma \longrightarrow \frac{1}{\gamma} \qquad \Longrightarrow \qquad w = w \circ \mathcal{I} \, , \tag{2.7}$$

which exchanges the two sheets. The branch points play a special role because we will show that they are associated with the two chiralities of the fermions and the bosonic currents on the world sheet. For later use, we therefore introduce the new variables

$$u_\pm = u_\mp^{-1} = \frac{1 \pm \gamma}{1 \mp \gamma} = \sqrt{\frac{w + \tilde{\rho} \pm \rho}{w + \tilde{\rho} \mp \rho}} \qquad \Longrightarrow \qquad \gamma = \mp \frac{1 - u_\pm}{1 + u_\pm} \, , \tag{2.8}$$

[3] With $x^\pm := \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ and $\partial_\pm := \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$.

which may be viewed as local coordinates in the γ plane around the two branch points $u_+ = 0$ and $u_- = 0$, respectively (with $u_\pm = \infty$ at the opposite branch point). With this Eq. (2.3) becomes

$$\gamma^{-1}\partial_\pm\gamma = u_\pm^{-1}\rho^{-1}\partial_\pm\rho \, . \tag{2.9}$$

Let us record the following formulas which will be useful later

$$\gamma\,\partial_\gamma = \tfrac{1}{2}(u_\pm^2 - 1)\,\partial_{u_\pm} \, , \qquad u_\pm^{-1}\partial_\pm u_\pm = -u_\mp^{-1}\partial_\pm u_\mp = \tfrac{1}{2}(1-u_\pm^{-2})\rho^{-1}\partial_\pm\rho \, , \tag{2.10}$$

and

$$v - w \;=\; \frac{\rho}{2}\,\frac{(\gamma(v)-\gamma(w))\,(\gamma(v)\gamma(w)-1)}{\gamma(v)\gamma(w)} \, . \tag{2.11}$$

2.2 Linear system

As explained in [7, 8, 9, 10], the linear system of $N = 16$ supergravity in two spacetime dimensions is formulated in terms of an E_8 matrix $\hat{\mathcal{V}}(x,\gamma)$ which depends on the spacetime coordinates $x \equiv (x^0, x^1)$ (or $x \equiv (x^+, x^-)$) and the variable spectral parameter γ, and which for real values of γ is an element of the real form $\mathrm{E}_{8(8)}$. It is subject to the transformations

$$\hat{\mathcal{V}}(x,\gamma) \longrightarrow G(w)\hat{\mathcal{V}}(x,\gamma)H(x,\gamma) \, , \tag{2.12}$$

where $G(w)$ is an element of the loop group $\mathrm{E}_{9(9)}$, and $H(x,\gamma)$ an element of its ‘maximal compact’ subgroup $K(\mathrm{E}_9)$. We will properly define these groups in the following section, but here already note that the latter consists of those $\mathrm{E}_{8(8)}$ valued functions $H(\gamma)$ satisfying

$$H(\gamma)^{-1} = H(1/\gamma)^T \, . \tag{2.13}$$

The matrix $\hat{\mathcal{V}}$ satisfies the linear partial differential equations

$$\partial_\pm\hat{\mathcal{V}}(x,\gamma) = \hat{\mathcal{V}}(x,\gamma)L_\pm(x,\gamma) \, , \tag{2.14}$$

with the Lax connection $L_\pm(x,t,\gamma)$ defined as function of the physical fields. In the absence of fermionic fields, the Lax connection is [7]

$$L_\pm(x,\gamma) = Q_\pm(x) + \frac{1 \mp \gamma}{1 \pm \gamma}\,P_\pm(x) \in \mathfrak{e}_{8(8)} \, . \tag{2.15}$$

The bosonic currents $Q_\pm \in \mathfrak{so}(16)$ and $P_\pm \in \mathfrak{e}_{8(8)}$ are defined by

$$\mathcal{V}^{-1}\partial_\pm\mathcal{V} \;\equiv\; Q_\pm + P_\pm \;\equiv\; \tfrac{1}{2}\,Q_\pm^{IJ}\,X^{IJ} + P_\pm^A\,Y^A \, , \tag{2.16}$$

where $\mathcal{V}$ is an $\mathrm{E}_{8(8)}$ valued matrix in which the bosonic fields of the theory are assembled. The X^{IJ} and Y^A denote the 120 compact and 128 noncompact generators of $\mathfrak{e}_{8(8)}$, respectively (see [10] for more detailed explanation of our notations and conventions). The equations (2.14) are compatible if and only if the equations of motion of the original $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ σ-model are satisfied. In addition to these second order equations of motion, the model exhibits the first order *conformal constraints*

$$T_{\pm\pm} \equiv \tfrac{1}{2}\rho\, P^A_\pm P^A_\pm - \partial_\pm\rho\,\partial_\pm\sigma \;\approx\; 0\;, \tag{2.17}$$

that determine the conformal factor σ of the two-dimensional metric in terms of the matter currents.

The fermionic fields of $N=16$ supergravity are the nonpropagating gravitino fields ψ^I_μ, the dilatino fields ψ^I_2, which both originate from the gravitino in three dimensions, and the spin-$1/2$ fields $\chi^{\dot A}$, transforming in the $\mathbf{16}$ and $\mathbf{128_c}$ of $Spin(16)$, respectively. With the chiral notation of [10] the full Lax connection is given by

$$\begin{aligned} L_\pm(\gamma) &\equiv \tfrac{1}{2}\hat{Q}^{IJ}_\pm(\gamma)X^{IJ} + \hat{P}^A_\pm(\gamma)Y^A\;, \\ \hat{Q}^{IJ}_\pm(\gamma) &= Q^{IJ}_\pm - \frac{2\mathrm{i}\gamma}{(1\pm\gamma)^2}\,\left(8\psi^{[I}_{2\,\pm}\psi^{J]}_\pm \pm \Gamma^{IJ}_{\dot A\dot B}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\right) - \frac{32\mathrm{i}\gamma^2}{(1\pm\gamma)^4}\,\psi^I_{2\,\pm}\psi^J_{2\,\pm}\;, \\ \hat{P}^A_\pm(\gamma) &= \frac{1\mp\gamma}{1\pm\gamma}\,P^A_\pm + \frac{4\mathrm{i}\gamma(1\mp\gamma)}{(1\pm\gamma)^3}\,\Gamma^I_{A\dot B}\psi^I_{2\,\pm}\chi^{\dot B}_\pm\;, \end{aligned} \tag{2.18}$$

with $\mathrm{SO}(16)$ Γ-matrices $\Gamma^I_{A\dot A}$. The compatibility equations of (2.14) with this connection reproduce the full supergravity equations of motion including fermionic terms to all orders [10]. For later use, let us also write out the linear system in terms of the coordinates $u_\pm$:

$$\begin{aligned} \hat{Q}^{IJ}_\pm(u_\pm) &= Q^{IJ}_\pm \pm \frac{\mathrm{i}}{2}\left(u_\pm^{-2}-1\right)\left(8\psi^{[I}_{2\,\pm}\psi^{J]}_\pm \pm \Gamma^{IJ}_{\dot A\dot B}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\right) - 2\mathrm{i}\left(u_\pm^{-2}-1\right)^2\psi^I_{2\,\pm}\psi^J_{2\,\pm}\;, \\ \hat{P}^A_\pm(u_\pm) &= u_\pm^{-1}\,P^A_\pm \mp \mathrm{i}u_\pm^{-1}\left(u_\pm^{-2}-1\right)\Gamma^I_{A\dot B}\psi^I_{2\,\pm}\chi^{\dot B}_\pm\;. \end{aligned} \tag{2.19}$$

The linear system thus is singular at the branch points $u_\pm = 0$, with the bosonic chiral currents appearing as the coefficients of the first order pole, and the fermionic bilinears as the coefficients of the higher order poles. By local supersymmetry we can set the dilatino $\psi^I_2 = 0$, in which case the linear system has only a second order pole multiplying a fermionic bilinear in the fields $\chi^{\dot A}_\pm$ in addition to the first oder pole from the bosons [8]. The conformal constraints (2.17) receive quadratic and quartic corrections in the fermions. In addition there are the *superconformal constraints*

$$\mathcal{S}^I_\pm \;\equiv\; D_\pm(\rho\psi^I_{2\,\pm}) - \rho\partial_\pm\sigma\psi^I_{2\,\pm} \mp \rho\,\Gamma^I_{A\dot A}\,\chi^{\dot A}_\pm\,P^A_\pm \pm \partial_\pm\rho\,\psi^I_\pm \;\approx\; 0\;, \tag{2.20}$$

modulo cubic fermion terms[4], that allow to express the gravition field $\psi^I_\pm$ in terms of the other fields.

For the bosonic theory, it was shown in [7] that the phase space of the two-dimensional theory — in this case, the space of solutions of the equations of motion of $D = 11$ supergravity with nine commuting Killing vectors — can be described in terms of the infinite-dimensional coset space

$$\mathrm{G/H} \;=\; \mathrm{E}_{9(9)}/K(\mathrm{E}_9)\;, \tag{2.21}$$

parametrized by the matrices $\hat{\mathcal{V}}(x,\gamma)$. In accordance with (2.12), the global $\mathrm{E}_{9(9)}$ symmetry acts from the left on this coset space, while the local $K(\mathrm{E}_9)$ symmetry acts on it from the right. The latter may be used to bring $\hat{\mathcal{V}}$ into a generalized triangular gauge, defined by requiring $\hat{\mathcal{V}}$ to be holomorphic in a neighborhood of $\gamma = 0$ [13, 7]. This gauge choice allows to read off the solution of the bosonic field equations by setting $\gamma = 0$:

$$\mathcal{V}(x) = \hat{\mathcal{V}}(x,\gamma)\Big|_{\gamma=0}\;. \tag{2.22}$$

In the triangular gauge, $\hat{\mathcal{V}}$ can be represented in the form

$$\hat{\mathcal{V}}(x,\gamma) \;=\; S(w)H(x,\gamma)\;, \tag{2.23}$$

where $\hat{\mathcal{V}}(\gamma)$ is holomorphic inside the unit disc in the complex γ-plane, and $H(x,\gamma)$ belongs to $K(\mathrm{E}_9)$, i.e. obeys (2.13). This fixes $H(x,\gamma)$ up to constant $\mathrm{SO}(16)$ transformations. On such $\hat{\mathcal{V}}$, the global non-linear and non-local action of $\mathrm{E}_{9(9)}$ takes the (infinitesimal) form

$$\delta_\Lambda\,\hat{\mathcal{V}}(x,\gamma(w)) \;=\; \Lambda(w)\hat{\mathcal{V}}(x,\gamma(w)) - \hat{\mathcal{V}}(x,\gamma(w))\,\Upsilon_\Lambda(x,\gamma(w))\;, \tag{2.24}$$

Here, $\Lambda(w)$ is an $\mathfrak{e}_8$ valued, but coordinate independent, function on the complex w-plane parametrizing the infinitesimal action of $\mathfrak{e}_9 \equiv \mathrm{Lie}(E_9)$. The matrix $\Upsilon_\Lambda(x,\gamma(w))$ is a special field dependent element of the algebra $\mathfrak{ke}_9 \equiv \mathrm{Lie}(K(\mathrm{E}_9))$: it restores the holomorphic gauge which is violated by the pure action of $\Lambda(w)$ in (2.24). On the physical fields, $\mathrm{E}_{9(9)}$ thus acts by non-linear and non-local transformations. The loop algebra is recovered by chosing $\Lambda(w) = \Lambda_n w^n$. It follows from (2.5) that compensating $K(\mathrm{E}_9)$ transformations are only required for $n \geq 0$, while transformations with $n < 0$ do not violate the triangular gauge; the latter only shift the dual potentials by constants (and are thus related to the integration constants arising at each step of the dualization), and have no effect on the physical fields.

[4]The full expression for the superconformal constraints including all higher order fermionic terms has been worked out in [10].

3 $\mathrm{E}_{9(9)}$ and $K(\mathrm{E}_9)$

We now give a rigorous definition of the two groups $\mathrm{E}_{9(9)}$ and $K(\mathrm{E}_9)$, following [7, 12]. In view of possible later applications, we will aim for a 'coordinate free' description, where we can regard w and γ merely as local coordinates on some Riemann surface Σ, and its double cover $\tilde{\Sigma}$, respectively (although this is not strictly necessary for the purposes of the present paper). The involution $\mathcal{I}$ exchanges the two sheets of the cover; on $\tilde{\Sigma}$, it generalizes (2.7) in such a way that its two fixed points map to the branch points of $\tilde{\Sigma}$. For stationary axisymmetric or colliding plane wave solutions, Σ is just the complex plane, but there do exist solutions for which Σ is a hyperelliptic Riemann surface, namely the algebro-geometric solutions of [14, 15]. One can discuss the Ernst equation and its generalizations on even more general Riemann surfaces [16], although the spectral problem is not completely understood in that case. Nevertheless, we will keep the discussion in this section quite general with these possible generalizations in mind.

Viewing w and γ as local coordinates, it was shown in [12] how to reformulate the defining equation (2.1) as a linear system taking values in a certain subalgebra of the Lie algebra of vector fields on $\tilde{\Sigma}$. For this purpose, one generalizes the inverse relation (2.5) by considering coordinate dependent maps

$$Y : \tilde{\Sigma} \longrightarrow \Sigma \qquad \text{with} \quad Y \circ \mathcal{I} = Y \, . \tag{3.1}$$

The variable Y here corresponds to the variable w^{-1}, and the maps Y should be thought of as generalizing the relation (2.5). The w-diffeomorphisms f and the γ-diffeomorphisms k then act from left and right on Y according to

$$Y \longrightarrow f \circ Y \circ k \qquad \text{for} \quad f \in \mathrm{Diff}^+(\Sigma) \quad , \quad k \in \mathrm{Diff}^+(\tilde{\Sigma}) \, . \tag{3.2}$$

In order to preserve the double cover, we demand that the diffeomorphims k acting on γ satisfy

$$k \circ \mathcal{I} = \mathcal{I} \circ k \, , \tag{3.3}$$

ensuring in particular that k preserves the two branch points. The associated Lie algebra is the involutory subalgebra of the Witt algebra generated by [12]

$$\mathcal{K}_n = \left(-\gamma^{n+1} + \gamma^{-n+1}\right) \partial_\gamma \, . \tag{3.4}$$

These generators define a maximal 'anomaly free' subalgebra (i.e. without central extension) of the Witt-Virasoro algebra. In terms of the local coordinates $u^{\pm}$, we have

$$\mathcal{K}_n = \mathcal{K}_n(u_\pm) \frac{\partial}{\partial u_\pm} \, , \tag{3.5}$$

with $\mathcal{K}_n(u_\pm) = -\mathcal{K}_n(-u_\pm)$; therefore these vector fields generate 'parity preserving' holomorphic reparametrizations in a neighborhood of $u_\pm = 0$. In particular, we have

$$\mathcal{K}_1 = \pm 2u_\pm \frac{\partial}{\partial u_\pm} , \tag{3.6}$$

i.e. $\mathcal{K}_1$ acts as a dilatation operator. In terms of local coordinates $u_\pm$ the 'linear system' is (see [12] for details)

$$Y^{-1}\partial_\pm Y = \frac{1}{2}\rho^{-1}\partial_\pm \rho \left(u_\pm - \frac{1}{u_\pm} \right) \frac{\partial}{\partial u_\pm} , \tag{3.7}$$

and takes values in the Lie algebra of vector fields generated by $\{\mathcal{K}_n \,|\, n \in \mathbb{N}\}$. The ensuing compatiblity condition is just the equation of motion for ρ.

We now define the groups $\mathrm{E}_{9(9)}$ and $K(\mathrm{E}_9)$, respectively, in terms of maps from Σ and $\tilde{\Sigma}$ into the complexified group $\mathrm{E}_8(\mathbb{C})$. In order to ensure that the solutions of the equations of motion are real, we have to impose certain reality constraints, which in turn requires that Σ and $\tilde{\Sigma}$ admit a generalized 'complex conjugation', i.e. a reflection r such that the points of Σ and $\tilde{\Sigma}$ invariant under r define 'real sections' $R(\Sigma) \subset \Sigma$ and $R(\tilde{\Sigma}) \subset \tilde{\Sigma}$, respectively. We furthermore demand that this reflection commute with the involution $\mathcal{I}$

$$r \circ \mathcal{I} = \mathcal{I} \circ r . \tag{3.8}$$

Of course, not every Riemann surface may admit such a reflection. Examples of surfaces which do are those surfaces which can be realized as multisheeted coverings of the complex plane, such that the set of branch cuts is invariant w.r.t. reflection on the real axis; the reflection r is then just the one induced by ordinary complex conjugation. In particular, if Σ is the Riemann sphere, we have $R(\Sigma) = \mathbb{R}$. The 'moving' branch cut is also constrained and must be invariant under r: for stationary axisymmetric solutions, it lies on the imaginary axis and is symmetric w.r.t. the real axis, while for Lorentzian solutions, it is a part of the real axis, or of $R(\Sigma)$ for more general Σ.

The group $\mathrm{E}_{9(9)}$ is defined as

$$\mathrm{E}_{9(9)} := \left\{ G \in \mathcal{M}\left(\Sigma, \mathrm{E}_8(\mathbb{C})\right) \middle| G(r(w)) = \overline{G(w)} \; ; \; G(w) \in \mathrm{E}_{8(8)} \text{ for } r(w) = w \right\} , \tag{3.9}$$

where $\mathcal{M}$ denotes the *meromorphic* maps. That is, $\mathrm{E}_{9(9)}$ consists of meromorphic mappings of Σ into the complexified group $\mathrm{E}_8(\mathbb{C})$ with the additional reality constraint that on the real section of Σ the elements belong to the particular real form $\mathrm{E}_{8(8)}$. The above definition generalizes the corresponding one of [7] for stationary axisymmetric solutions (where $\mathrm{G} = \mathrm{SL}(2,\mathbb{C})$), and can be shown to act transitively on the multi-soliton solutions of [17, 7].

Similarly, $K(\mathrm{E}_9)$ is defined as the group of meromorphic mappings

$$K(\mathrm{E}_9) \subset \mathcal{M}\,(\tilde{\Sigma}, \mathrm{E}_8(\mathbb{C}))\;, \tag{3.10}$$

subject to the three requirements

1. $H(r(\gamma)) = \overline{H(\gamma)}$ for all $\gamma \in \tilde{\Sigma}$.
2. For all $H \in K(\mathrm{E}_9)$

$$H^{-1} = \tau \circ H \circ \mathcal{I}\;, \tag{3.11}$$

   where τ is the symmetric space involution defining the real form $\mathrm{E}_{8(8)}$. In particular, we have $H \in \mathrm{SO}(16)$ at the two fixed points of $\mathcal{I}$.

3. All $H \in K(\mathrm{E}_9)$ are *holomorphic at the branch points*, i.e. the fixed points of $\mathcal{I}$.

The first of these conditions is the reality constraint. The second is a restatement of the condition (2.13); it restricts the affine Lie algebra to a maximal subalgebra without central extension. The third (regularity) requirement will be motivated below.

Next we spell out what these conditions imply for the Lie algebra $\mathfrak{ke}_9$. For an element $h \in \mathfrak{ke}_9$, we have

$$\begin{aligned} h(\gamma) &= -h^T\left(\frac{1}{\gamma}\right) \quad \Longrightarrow \\ h(\gamma) &= \tfrac{1}{2}h_0^{IJ}X^{IJ} + \sum_{n=1}^{\infty}\left[\tfrac{1}{2}\left(\gamma^{-n}+\gamma^{n}\right)h_n^{IJ}\,X^{IJ} + \left(\gamma^{-n}-\gamma^{n}\right)h_n^{A}\,Y^{A}\right]. \end{aligned} \tag{3.12}$$

The reality constraint (1) above then implies that the coefficients h_n^{IJ} and h_n^A are *real*. Furthermore, expressing the nine Chevalley generators of $\mathfrak{ke}_9$ in terms of the current algebra representation above, it is an elementary exercise (which we leave to the reader) to check that the parametrization (3.12) is equivalent to the abstract definition (1.1) of $\mathfrak{ke}_9$ given in the introduction. The sum on the r.h.s. can be treated as a formal power series in γ, but it is often more convenient to work directly with the condition on the function $h(\gamma)$. The expansion shows that the involutory subalgebra $\mathfrak{ke}_9$ is itself *not* a Kac Moody algebra; for instance, it does not possess a triangular decomposition. It is also easy to see that the central term of $\mathfrak{e}_9$ does not belong to the subalgebra $\mathfrak{ke}_9$, so the latter is a maximal 'anomaly-free' subalgebra of $\mathfrak{e}_9$.

Because $h(\gamma)$ is holomorphic at the branch points $\gamma = \pm 1$ by assumption, it can be expanded into convergent series about the two branch points in terms of the local coordinates $u_\pm$, viz.

$$h^\pm(u_\pm) = \tfrac{1}{2}h^{\pm IJ}(u_\pm)X^{IJ} + h^{\pm A}(u_\pm)Y^A\;, \tag{3.13}$$

with

$$h^{\pm IJ}(u_\pm) = h^{\pm IJ}(-u_\pm)\ , \quad h^{\pm A}(u_\pm) = -h^{\pm A}(-u_\pm)\ , \tag{3.14}$$

where the superscripts on $h^\pm$ are to indicate that we are dealing with two expansions of the *same* function h about the two different points. Hence,

$$h^\pm(u_\pm) = \tfrac{1}{2}h_0^{\pm IJ}X^{IJ} + \sum_{n=1}^{\infty}\left[\tfrac{1}{2}u_\pm^{2n}h_{2n}^{\pm IJ}\,X^{IJ} + u_\pm^{2n-1}h_{2n-1}^{\pm A}\,Y^A\right]\ . \tag{3.15}$$

The expansions in terms of the coefficients $h_n^\pm$ cannot terminate at finite order for either u_+ or u_-, because otherwise h would blow up at the opposite branch points contrary to assumption. We remark that $\mathfrak{so}(16)\oplus\mathfrak{so}(16)$ is *not* a subalgebra of $\mathfrak{ke}_9$, despite the occurrence of two expansion coefficients $h_0^{\pm IJ}$, since both expansions arise from a single function $h(\gamma)$. When expanded about the two branch points, the algebra $\mathfrak{ke}_9$ looks like 'half' of a twisted version of $\mathfrak{e}_9$, i.e. like a Borel subalgebra thereof. However, we cannot extend it to a full twisted affine Lie algebra because the required holomorphicity of h at $u_\pm = 0$ eliminates 'one half' of the latter.

4 Some remarks on bosonic representations of $K(\mathrm{E}_9)$

It is rather straightforward to construct bosonic (i.e. single valued) representations of $K(\mathrm{E}_9)$ by 'lifting' representations of $\mathrm{E}_{8(8)}$. The more difficult task, however, is to come up with *irreducible* representations [4]. The representation which is perhaps most easily understood is the *adjoint representation*: it is realized in terms of meromorphic functions $\{\phi^{IJ}(\gamma), \phi^A(\gamma)\}$ satisfying the same constraints as (3.14) when expanded in local coordinates about $u_\pm = 0$, i.e.

$$\phi_\pm^{IJ}(u_\pm) = \phi_\pm^{IJ}(-u_\pm)\quad ,\quad \phi_\pm^A(u_\pm) = -\phi_\pm^A(-u_\pm)\ , \tag{4.1}$$

(note that ϕ is allowed to have poles of any given order at $u_\pm = 0$). The transformations

$$\begin{aligned}\delta\phi_\pm^{IJ}(u_\pm) &= 2h^{\pm K[I}(u_\pm)\phi_\pm^{J]K}(u_\pm) + \tfrac{1}{2}h^{\pm A}(u_\pm)\Gamma^{IJ}_{AB}\phi_\pm^B(u_\pm)\ ,\\ \delta\phi_\pm^A(u_\pm) &= \tfrac{1}{4}h^{\pm IJ}(u_\pm)\Gamma^{IJ}_{AB}\phi_\pm^B(u_\pm) + \tfrac{1}{4}h^{\pm B}(u_\pm)\Gamma^{IJ}_{BA}\phi_\pm^{IJ}(u_\pm)\ ,\end{aligned} \tag{4.2}$$

then preserve (4.1). The commutator of two such transformations with parameters h_1 and h_2 is a new transformation with parameter

$$\begin{aligned}\tilde{h}^{\pm IJ}(u_\pm) &= [h_1^\pm(u_\pm), h_2^\pm(u_\pm)]^{IJ} + \tfrac{1}{2}h_1^{\pm A}(u_\pm)\Gamma^{IJ}_{AB}h_2^{\pm B}(u_\pm)\ ,\\ \tilde{h}^{\pm A}(u_\pm) &= \tfrac{1}{2}h_1^{\pm B}(u_\pm)h_2^{\pm KL}(u_\pm)\Gamma^{KL}_{BA}\ .\end{aligned} \tag{4.3}$$

Due to the required regularity and the parity constraints (3.14), the second term on the r.h.s. of the first equation starts only at $\mathcal{O}(u_\pm^2)$. The full symmetry acting on this, and in fact any other, representation of $K(\mathrm{E}_9)$ is the semi-direct product of $K(\mathrm{E}_9)$ and the restricted diffeomorphisms (3.4), which is a maximal 'anomaly free' subalgebra of the semidirect product of the Witt-Virasoro (pseudo)group and $\mathrm{E}_{9(9)}$ preserving the parity conditions (4.1).

In an analogous fashion, any representation of $\mathrm{E}_{8(8)}$ can be 'lifted' to produce a representation of $K(\mathrm{E}_9)$ and its semidirect product with the involutory diffeomorphisms, if suitable 'parity conditions' analogous to (4.1) are imposed. In contradistinction to the Lie algebra $\mathfrak{ke}_9$ itself, its representations may have poles of bounded (but arbitrary) order at $u_\pm = 0$; the holomorphicity requirement ensures that the order of the pole is unchanged under the action of $\mathfrak{ke}_9$, and also preserved under the diffeomorphisms generated by the vector fields $\mathcal{K}_n$ by virtue of the assumed regularity around $u_\pm = 0$. The requirement that the representations be complex analytic functions is essential here. If we were dealing with arbitrary functions instead, these representations would be highly reducible: a smaller representation can always be obtained restricting to the set of functions which vanish on an arbitrary closed set [4]. However, this way of reducing a given representation obviously does not work for analytic functions.

The linear systems (2.15) and (2.18) transform both as $\mathfrak{ke}_9$ *gauge connections*. To see this in more detail, observe that the r.h.s. of the linear systems (2.15) and (2.18) both belong to $\mathfrak{ke}_9$ — in contrast to the Cartan form (2.16), which has components in all of the Lie algebra $\mathfrak{e}_{8(8)}$ (and not just its compact subalgebra $\mathfrak{so}(16)$). For instance, under (2.12), the bosonic linear system

$$\hat{\mathcal{V}}^{-1}\partial_\pm \hat{\mathcal{V}}(u_\pm) \;\; = \;\; Q_\pm + u_\pm^{-1} P_\pm \quad ; \tag{4.4}$$

is inert under rigid $\mathrm{E}_{9(9)}$ (i.e. does not transform under $G(w)$), and transforms with an *inhomogeneous term* under $K(\mathrm{E}_9)$; the same is true for the linear system with fermions (2.18)). However, a general $K(\mathrm{E}_9)$ gauge transformation will not preserve the particular $u_\pm$ dependence on the r.h.s. of (4.4). This is only the case for very special $K(\mathrm{E}_9)$ transformations with parameters depending on the physical fields in a special way — leading to the non-linear and non-local realization of the affine symmetry on the finitely many physical fields that is known from the realization of the Geroch group in general relativity [6, 7]. Because they leave the equations of motion form invariant, we will refer to these restricted $K(\mathrm{E}_9)$ transformations as 'on shell'.

To allow for the most general 'off shell' $K(\mathrm{E}_9)$ gauge transformation, we relax the special $u_\pm$ dependence of (4.4) by writing [5]

$$\hat{\mathcal{V}}^{-1}\partial_\pm \hat{\mathcal{V}}(x;\gamma) \;\; = \;\; \mathcal{Q}_\pm(x;\gamma) + \mathcal{P}_\pm(x;\gamma) \;\equiv\; \mathcal{J}_\pm(x;\gamma) = -\mathcal{J}_\pm^T\left(x;\frac{1}{\gamma}\right) \in \mathfrak{ke}_9 \, . \tag{4.5}$$

The 'dual potentials', which were previously constrained by the special $u_\pm$ dependence on the r.h.s. of (2.15) (or (2.18)) to be non-linear and non-local functions of the physical fields now

[5] We will not always indicate the x-dependence in the remainder.

become independent gauge degrees of freedom. The currents $\mathcal{J}_\pm$ can be expanded about both branch points, such that

$$\mathcal{J}_\pm\Big|_{\gamma\,\sim\,\mp1} \;=\; \mathcal{Q}_\pm(u_\pm) + \mathcal{P}_\pm(u_\pm)\;, \tag{4.6}$$

with

$$\begin{aligned}\mathcal{Q}_\pm(u_\pm) &= \mathcal{Q}_\pm(-u_\pm) = Q_\pm + \mathcal{O}(u_\pm^2)\;,\\ \mathcal{P}_\pm(u_\pm) &= -\mathcal{P}_\pm(-u_\pm) = u_\pm^{-1}\,P_\pm + \mathcal{O}(u_\pm)\;,\end{aligned} \tag{4.7}$$

where we only require that $\mathcal{P}_\pm$ has at most a first order pole, and regular $Q_\pm$. At the opposite branch point we demand

$$\mathcal{J}_\pm\Big|_{\gamma\,\sim\,\pm1} \;=\; \tilde{\mathcal{Q}}_\pm(u_\mp) + \tilde{\mathcal{P}}_\pm(u_\mp)\;, \tag{4.8}$$

with

$$\begin{aligned}\tilde{\mathcal{Q}}_\pm(u_\mp) &= \tilde{\mathcal{Q}}_\pm(-u_\mp) = \tilde{Q}_\pm + \mathcal{O}(u_\mp^2)\;,\\ \tilde{\mathcal{P}}_\pm(u_\mp) &= -\tilde{\mathcal{P}}_\pm(-u_\mp) = u_\mp\,\tilde{P}_\pm + \mathcal{O}(u_\mp^3)\;.\end{aligned} \tag{4.9}$$

While $\mathcal{P}_\pm$ thus has a first order pole at $u_\pm = 0$, it has a zero at the opposite branch point $u_\pm = \infty$, and this property is preserved by the transformations (4.10). Analogous comments apply to the linear system with fermions, except that now $\mathcal{P}_\pm(u_\pm) \sim u_\pm^{-3}$ and $\mathcal{Q}_\pm(u_\pm) \sim u_\pm^{-4}$ for $u_\pm \sim 0$. It is only for the gauge-fixed form of the linear system (4.4) that $Q_\pm = \tilde{Q}_\pm$ and $P_\pm = \tilde{P}_\pm$.

Under $\delta\hat{\mathcal{V}} = \hat{\mathcal{V}}\,h$, the connection $\mathcal{J}_\pm(\gamma)$ transforms according to

$$\delta\mathcal{J}_\pm(\gamma) \;=\; \partial_\pm h(\gamma) + [\mathcal{J}_\pm(\gamma), h(\gamma)]\;. \tag{4.10}$$

In terms of the coordinates $u_\pm$ and making use of (2.3) we have

$$\delta\mathcal{J}_\pm(u_\pm) \;=\; \partial_\pm h^\pm(u_\pm) + \tfrac{1}{2}(u_\pm - u_\pm^{-1})\,\rho^{-1}\partial_\pm\rho\,\partial h^\pm(u_\pm) + \big[\mathcal{J}_\pm(u_\pm), h^\pm(u_\pm)\big]\;, \tag{4.11}$$

where the derivative in the first term does not act on $u_\pm$. (4.10) implies

$$\begin{aligned}\delta\mathcal{Q}_\pm^{IJ}(u_\pm) &= \partial_\pm h^{\pm IJ}(u_\pm) + \tfrac{1}{2}(u_\pm - u_\pm^{-1})\,\rho^{-1}\partial_\pm\rho\,\partial h^{\pm IJ}(u_\pm)\\ &\quad + 2\mathcal{Q}_\pm^{K[I}(u_\pm)h^{\pm J]K}(u_\pm) + \tfrac{1}{2}\mathcal{P}_\pm^{A}(u_\pm)\Gamma^{IJ}_{AB}h^{\pm B}(u_\pm)\;,\\ \delta\mathcal{P}_\pm^{A}(u_\pm) &= \partial_\pm h^{\pm A}(u_\pm) + \tfrac{1}{2}(u_\pm - u_\pm^{-1})\,\rho^{-1}\partial_\pm\rho\,\partial h^{\pm A}(u_\pm)\\ &\quad + \tfrac{1}{4}\mathcal{Q}_\pm^{IJ}(u_\pm)\Gamma^{IJ}_{AB}h^{\pm B}(u_\pm) + \tfrac{1}{4}\mathcal{P}_\pm^{B}(u_\pm)\Gamma^{IJ}_{BA}h^{\pm IJ}(u_\pm)\;.\end{aligned} \tag{4.12}$$

Substituting (3.15), we obtain for instance

$$\delta P_\pm^A = -\tfrac{1}{2}\rho^{-1}\partial_\pm\rho\, h_1^{\pm A} - \tfrac{1}{4}P_\pm^B\Gamma^{IJ}_{AB}h_0^{\pm IJ}\ . \tag{4.13}$$

The key point here is that the information about the physical fields (the currents $P_\pm$ and the fermionic bilinears) *is encoded in the poles of the $K(\mathrm{E}_9)$ connection*, whereas the higher order regular terms in $u_\pm$ in the expansion of $\mathcal{J}_\pm(u_\pm)$ are to be interpreted as $K(\mathrm{E}_9)$ gauge degrees of freedom, which can be removed by the adjoint action of $K(\mathrm{E}_9)$. The residue is only affected by the inhomogeneous term in (4.10), and it is precisely this term which allows to generate non-trivial solutions from the vacuum by the nonlinear and non-local action of $\mathrm{E}_{9(9)}$ via the induced action of $K(\mathrm{E}_9)$.

5 Induced action of $K(\mathrm{E}_9)$ on the supergravity multiplet

In our previous work [10], we have studied the action of the infinite dimensional global $\mathrm{E}_{9(9)}$ symmetry on all fields of $N = 16$ supergravity. On the chiral bosonic 'currents' and on the chiral components of the fermionic fields, the global $\mathrm{E}_{9(9)}$ was shown to act via an induced $K(\mathrm{E}_9)$ transformation, which can be canonically generated via a Lie-Poisson action. We refer readers to [10] for the detailed derivations and explicit expressions, and here only summarize the pertinent formulas for the variations of the various $N = 16$ supergravity fields. If the action of the global $\mathrm{E}_{9(9)}$ is parametrized by a Lie-algebra-valued function $\Lambda(w) \in \mathfrak{e}_{8(8)}$, the solution $\hat{\mathcal{V}}$ of the linear system transforms as

$$\delta_\Lambda\, \hat{\mathcal{V}}(x,\gamma(w)) \;=\; \oint_{\mathcal{C}} \frac{dv}{2\pi\mathrm{i}\,(v-w)}\,\Lambda(v)\hat{\mathcal{V}}(x,\gamma(w)) - \hat{\mathcal{V}}(x,\gamma(w))\,\Upsilon_\Lambda(x,\gamma(w))\ , \tag{5.1}$$

with the matrix

$$\begin{aligned}\Upsilon_\Lambda(x,\gamma(w)) \;\;\equiv\;\; &\oint_{\mathcal{C}} \frac{d\gamma'}{2\pi\mathrm{i}\,\gamma'}\,\frac{\gamma(w)(1-\gamma'^2)}{(\gamma'-\gamma(w))\,(1-\gamma(w)\gamma')}\,\tfrac{1}{2}\hat{\Lambda}^{IJ}(\gamma')\,X^{IJ} \\ &+ \oint_{\mathcal{C}} \frac{d\gamma'}{2\pi\mathrm{i}\,\gamma'}\,\frac{\gamma'(1-\gamma(w)^2)}{(\gamma'-\gamma(w))\,(1-\gamma(w)\gamma')}\,\hat{\Lambda}^{A}(\gamma')\,Y^A\ ,\end{aligned} \tag{5.2}$$

and the dressed parameters

$$\hat{\Lambda}(\gamma') \;\equiv\; \tfrac{1}{2}\hat{\Lambda}^{IJ}(\gamma')\,X^{IJ} + \hat{\Lambda}^A(\gamma')\,Y^A \;\;\equiv\;\; \hat{\mathcal{V}}^{-1}(\gamma')\Lambda(w(\gamma'))\hat{\mathcal{V}}(\gamma')\ . \tag{5.3}$$

Replacing $\gamma \to 1/\gamma$ in (5.2) one directly verifies that $\Upsilon_\Lambda(x,\gamma(w)) \in \mathfrak{ke}_9$ according to (3.12), independently of the choice of integration contout $\mathcal{C}$. In order to recover the affine (loop) algebra, choose $\Lambda(w) = \Lambda_n w^n$ and a path $\mathcal{C}$ encircling the point $\gamma = 0$ in the complex γ-plane.

Expanding $\hat{\Lambda}$ around $\gamma = 0$ into its singular and regular part, we get

$$\hat{\mathcal{V}}^{-1}\Lambda\hat{\mathcal{V}} \equiv \hat{\Lambda}_{\rm sing} + \hat{\Lambda}_{\rm reg} , \qquad \hat{\Lambda}_{\rm sing} \equiv \sum_{k=0}^{n} \gamma^{-k}\hat{\Lambda}_{-k} . \tag{5.4}$$

Formula (5.2) then yields

$$\begin{aligned} \Upsilon_\Lambda(x,\gamma) &= \sum_{k=0}^{n} \tfrac{1}{2}\left(\gamma^{-k}+\gamma^{k}\right)\hat{\Lambda}^{IJ}_{-k}\, X^{IJ} + \left(\gamma^{-k}-\gamma^{k}\right)\hat{\Lambda}^{A}_{-k}\, Y^{A} \\ &= \hat{\Lambda}_{\rm sing}(\gamma) - \hat{\Lambda}^{T}_{\rm sing}(1/\gamma) , \end{aligned}$$

and thus

$$\delta_\Lambda \hat{\mathcal{V}}(\gamma) = \hat{\mathcal{V}}(\gamma)\left(\hat{\Lambda}_{\rm reg}(\gamma) + \hat{\Lambda}^{T}_{\rm sing}(1/\gamma)\right) , \tag{5.5}$$

(note that $\hat{\Lambda}^{T}_{\rm sing}(1/\gamma)$ is regular near $\gamma = 0$).

These considerations show that our definition (5.2) selects $\Upsilon_\Lambda(\gamma)$ precisely such that $\hat{\mathcal{V}}$ remains holomorphic around $\gamma = 0$, in accordance with the prescription of [7]. Below we will spell out in more detail the conditions that the expansion coefficients of these 'on shell $K(\mathrm{E}_9)$ transformations' have to obey, but let us note already here that (5.3) and the linear system together imply the differential identities

$$D_\pm\hat{\Lambda}^{IJ} = \frac{1}{2u_\pm}\Gamma^{IJ}_{AB}\hat{\Lambda}^{A}P^{B}_{\pm} , \qquad D_\pm\hat{\Lambda}^{A} = \frac{1}{4u_\pm}\Gamma^{IJ}_{AB}\hat{\Lambda}^{IJ}P^{B}_{\pm} , \tag{5.6}$$

which express the dependence of the dressed parameters on the physical fields.

Working out the action of δ_Λ on the various fields, we obtain the explicit formulas for the bosonic fields [10]

$$\begin{aligned} \delta_\Lambda \mathcal{V}(x) &= \int_{\mathcal{C}} \frac{dv}{2\pi\mathrm{i}} \left(\frac{2\gamma}{\rho(1-\gamma^2)}\mathcal{V}(x)\hat{\Lambda}^{A}Y^{A}\right) , \\ \delta_\Lambda P_\pm(x) &= \int_{\mathcal{C}} \frac{dv}{2\pi\mathrm{i}} \left[\frac{\gamma}{\rho(1\pm\gamma)^2}\hat{\Lambda}^{IJ}X^{IJ},\, P_\pm(x)\right] \\ &\qquad \mp \int_{\mathcal{C}} \frac{dv}{2\pi\mathrm{i}} \frac{4\gamma^2\partial_\pm\rho}{\rho^2(1\pm\gamma)^2(1-\gamma^2)}\hat{\Lambda}^{A}Y^{A} , \end{aligned} \tag{5.7}$$

and the fermionic fields[6]

$$\delta_\Lambda \psi^{I}_{2\,\pm} = \oint_{\mathcal{C}} \frac{dv}{2\pi\mathrm{i}} \left(\frac{2\gamma}{\rho(1\pm\gamma)^2}\hat{\Lambda}^{IJ}(\gamma)\psi^{J}_{2\,\pm}\right) ,$$

[6]Note the change w.r.t. the formulas in [10] by a relative factor of $-1/2$.

$$\delta_\Lambda \chi^{\dot{A}}_\pm = \oint_{\mathcal{C}} \frac{dv}{2\pi \mathrm{i}} \left(\frac{\gamma}{2\rho(1\pm\gamma)^2} \Gamma^{IJ}_{\dot{A}\dot{B}} \hat{\Lambda}^{IJ}(\gamma)\chi^{\dot{B}}_\pm + \frac{4\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)} \Gamma^I_{A\dot{A}} \hat{\Lambda}^A(\gamma)\psi^I_{2\pm} \right),$$

$$\delta_\Lambda \psi^I_\pm = \oint_{\mathcal{C}} \frac{dv}{2\pi \mathrm{i}} \left(\frac{2\gamma}{\rho(1\pm\gamma)^2} \hat{\Lambda}^{IJ}(\gamma)\psi^J_\pm + \frac{8\gamma^2}{\rho(1\pm\gamma)^4} \hat{\Lambda}^{IJ}(\gamma)\psi^J_{2\pm} \right)$$
$$\mp \oint_{\mathcal{C}} \frac{dv}{2\pi i} \left(\frac{4\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)} \Gamma^I_{A\dot{B}} \hat{\Lambda}^A(\gamma)\chi^{\dot{B}}_\pm \right) . \qquad (5.8)$$

Writing

$$\Upsilon_\Lambda(x,\gamma(w)) \equiv \tfrac{1}{2} \Upsilon^{IJ} X^{IJ} + \Upsilon^A Y^A , \qquad (5.9)$$

making use of the formula (cf. (2.4))

$$\frac{d\gamma}{\gamma} = -\frac{2\gamma}{\rho(1-\gamma^2)} dw , \qquad (5.10)$$

and recalling the definition (5.2), we see that the symmetry action of $K(\mathrm{E}_9)$ on the bosonic physical fields may be written in a much simpler form as

$$\delta_\Lambda \mathcal{V} = -\mathcal{V} \Upsilon_\Lambda|_{\gamma=0} ,$$
$$\delta_\Lambda P^A_\pm = \tfrac{1}{4} \Gamma^{IJ}_{AB} P^B_\pm \Upsilon^{IJ}|_{\gamma=\mp1} \pm \rho^{-1}\partial_\pm\rho\, \partial_\gamma \Upsilon^A|_{\gamma=\mp1} ,$$
$$\delta_\Lambda \sigma = \oint_{\mathcal{C}} \frac{d\gamma}{2\pi \mathrm{i}} \,\mathrm{tr} \left[\Lambda \partial_\gamma \hat{\mathcal{V}} \hat{\mathcal{V}}^{-1}\right] . \qquad (5.11)$$

The action on the fermionic fields of the model takes the form

$$\delta_\Lambda \psi^I_{2\pm} = \psi^J_{2\pm} \Upsilon^{IJ}|_{\gamma=\mp1} ,$$
$$\delta_\Lambda \chi^{\dot{A}}_\pm = \tfrac{1}{4}\Gamma^{IJ}_{\dot{A}\dot{B}} \chi^{\dot{B}}_\pm \Upsilon^{IJ}|_{\gamma=\mp1} - \Gamma^I_{A\dot{A}} \psi^I_{2\pm} \partial_\gamma \Upsilon^A|_{\gamma=\mp1} ,$$
$$\delta_\Lambda \psi^I_\pm = \psi^J_\pm \Upsilon^{IJ}|_{\gamma=\mp1} \pm \Gamma^I_{A\dot{B}} \chi^{\dot{B}}_\pm \partial_\gamma \Upsilon^A|_{\gamma=\mp1} \mp 2\psi^J_{2\pm} (\partial^2_\gamma \Upsilon^{IJ} \mp \partial_\gamma \Upsilon^{IJ})_{\gamma=\mp1} . \qquad (5.12)$$

Switching to local coordinates $u_\pm$, we have the expansion coefficients $\{\Upsilon^{\pm IJ}, \Upsilon^{\pm A}\}$ about the branch points

$$\Upsilon^\pm_\Lambda(x,u_\pm) \equiv \sum_{n=0}^\infty \left(\tfrac{1}{2} u^{2n}_\pm \Upsilon^{\pm IJ}_{2n} X^{IJ} + u^{2n+1}_\pm \Upsilon^{\pm A}_{2n+1} Y^A\right) , \qquad \Xi^A_0 \equiv \Upsilon^A|_{\gamma=0} . \qquad (5.13)$$

They satisfy differential equations which follow from (5.2) and (5.6), namely

$$D_\pm \Upsilon^{\mp IJ}_{2n} = \tfrac{1}{2} \Gamma^{IJ}_{AB} P^B_\pm \Upsilon^{\mp A}_{2n-1} + \left((1-n)\Upsilon^{\mp IJ}_{2n-2} + n\Upsilon^{\mp IJ}_{2n}\right) \rho^{-1}\partial_\pm\rho , \qquad (n \geq 1)$$

$$
\begin{aligned}
D_\pm \Upsilon_0^{\mp IJ} &= -\tfrac{1}{2}\,\Gamma^{IJ}_{AB} P^B_\pm\, \Xi_0^A \;, \\
D_\pm \Upsilon_{2n+1}^{\mp A} &= \tfrac{1}{4}\,\Gamma^{IJ}_{AB} P^B_\pm\, \Upsilon_{2n}^{\mp IJ} + \big((n{+}\tfrac{1}{2})\Upsilon_{2n+1}^{\mp A} + (\tfrac{1}{2}{-}n)\Upsilon_{2n-1}^{\mp A}\big)\,\rho^{-1}\partial_\pm\rho \;, \qquad (n\geq 1) \\
D_\pm \Upsilon_1^{\mp A} &= \tfrac{1}{4}\,\Gamma^{IJ}_{AB} P^B_\pm \big(\Upsilon_0^{\mp IJ} - \Upsilon_0^{\pm IJ}\big) + \tfrac{1}{2}\big(\Upsilon_1^{\mp A} - \Upsilon_1^{\pm A}\big)\,\rho^{-1}\partial_\pm\rho \;, \qquad (5.14)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
D_\pm \Upsilon_{2n}^{\pm IJ} &= \tfrac{1}{2}\,\Gamma^{IJ}_{AB} P^B_\pm\, \Upsilon_{2n+1}^{\pm A} + \big((1+n)\Upsilon_{2n+2}^{\pm IJ} - n\Upsilon_{2n}^{\pm IJ}\big)\,\rho^{-1}\partial_\pm\rho \;, \qquad (n\geq 1) \\
D_\pm \Upsilon_0^{\pm IJ} &= \tfrac{1}{2}\,\Gamma^{IJ}_{AB} P^B_\pm\, \big(\Upsilon_1^{\pm A} - \Xi_0^A\big) + \Upsilon_2^{\pm IJ}\,\rho^{-1}\partial_\pm\rho \;, \\
D_\pm \Upsilon_{2n+1}^{\pm A} &= \tfrac{1}{4}\,\Gamma^{IJ}_{AB} P^B_\pm\, \Upsilon_{2n}^{\pm IJ} + \big((n+\tfrac{3}{2})\Upsilon_{2n+3}^{\pm A} - (n+\tfrac{1}{2})\Upsilon_{2n+1}^{\pm A}\big)\,\rho^{-1}\partial_\pm\rho \;. \qquad (5.15)
\end{aligned}
$$

Note that only the relations for $\Upsilon_0^{\pm IJ}$, $\Upsilon_1^{\pm A}$ are not chiral.

It is then convenient to express the transformation of the fields (5.11), (5.12) in terms of these expansion coefficients. For the bosons we obtain

$$
\begin{aligned}
\delta_\Lambda\, \mathcal{V} &= -\mathcal{V}\, \Xi_0^A\, Y^A \;, \qquad \delta_\Lambda\, Q_\pm^{IJ} \;=\; \tfrac{1}{2}\,\Gamma^{IJ}_{AB}\, \Xi_0^A\, P_\pm^B \;, \\
\delta_\Lambda\, P_\pm^A &= \tfrac{1}{4}\,\Gamma^{IJ}_{AB}\, P^B_\pm\, \Upsilon_0^{\pm IJ} \;+\; \tfrac{1}{2}\,\rho^{-1}\partial_\pm\rho\, \Upsilon_1^{\pm A} \;, \\
\delta_\Lambda\, (\partial_\pm\sigma) &= \tfrac{1}{2}\, P^A_\pm\, \Upsilon_1^{\pm A} \;. \qquad (5.16)
\end{aligned}
$$

The second of these formulas tells us that $P_\pm^A$ indeed transforms as a component of a $K(\mathrm{E}_9)$ connection, cf. (4.13); the same can be verified for $Q_\pm^{IJ}$ by use of the second relation in (5.15). The transformation for the derivative of the conformal factor is in accord with the conformal constraints (2.17). For the fermions, we get

$$
\begin{aligned}
\delta_\Lambda\, \psi^I_{2\,\pm} &= \psi^J_{2\,\pm}\, \Upsilon_0^{\pm IJ} \;, \\
\delta_\Lambda\, \chi^{\dot A}_\pm &= \tfrac{1}{4}\Gamma^{IJ}_{\dot A\dot B}\, \chi^{\dot B}_\pm\, \Upsilon_0^{\pm IJ} \mp \tfrac{1}{2}\,\Gamma^I_{A\dot A}\, \psi^I_{2\,\pm}\, \Upsilon_1^{\pm A} \;, \\
\delta_\Lambda\, \psi^I_\pm &= \psi^J_\pm\, \Upsilon_0^{\pm IJ} + \tfrac{1}{2}\,\Gamma^I_{A\dot B}\, \chi^{\dot B}_\pm\, \Upsilon_1^{\pm A} \mp \psi^J_{2\,\pm}\, \Upsilon_2^{\pm IJ} \;. \qquad (5.17)
\end{aligned}
$$

The differential relations (5.14), (5.15) are crucial to check the invariance of the equations of motion under these transformations.

We have thus realized $K(\mathrm{E}_9)$ as a group of symmetry transformations on finitely many fields in such a way that the variations of the chiral components of the bosons and fermions are expressed in terms of the lowest coefficients of the transformation parameter $h(\gamma)$ about the two branch points $\gamma = \pm 1$. This representation of $K(\mathrm{E}_9)$, however, is *not* faithful, because the variations are only sensitive to the terms up to second order. This is even more evident for the redefined fermion fields

$$\begin{aligned}
\widetilde{\psi}^I_{2\pm} &\equiv \psi^I_{2\pm} \;, \\
\widetilde{\chi}^{\dot{A}}_\pm &\equiv \chi^{\dot{A}}_\pm \mp \frac{1}{\rho^{-1}\partial_\pm\rho}\,\Gamma^I_{A\dot{A}}P^A_\pm\widetilde{\psi}^I_{2\pm} \;, \\
\widetilde{\psi}^I_\pm &\equiv \psi^I_\pm \pm \frac{1}{\rho^{-1}\partial_\pm\rho}\,\Big(\rho^{-1}\,D_\pm(\rho\widetilde{\psi}^I_{2\pm}) + \partial_\pm\sigma\widetilde{\psi}^I_{2\pm} \mp \Gamma^I_{A\dot{A}}P^A_\pm\widetilde{\chi}^{\dot{A}}_\pm\Big) \;.
\end{aligned} \tag{5.18}$$

It is straightforward to check that these new fermion fields transform separately under $K(\mathrm{E}_9)$ and 'see' only the zero modes:

$$\delta_\Lambda\,\widetilde{\psi}^I_{2\,\pm} = \Upsilon_0^{\pm IJ}\,\widetilde{\psi}^J_{2\,\pm} \;, \qquad \delta_\Lambda\,\widetilde{\chi}^{\dot{A}}_\pm = \tfrac{1}{4}\Gamma^{IJ}_{\dot{A}\dot{B}}\,\Upsilon_0^{\pm IJ}\,\widetilde{\chi}^{\dot{B}}_\pm \;, \qquad \delta_\Lambda\,\widetilde{\psi}^I_\pm = \Upsilon_0^{\pm IJ}\,\widetilde{\psi}^J_\pm \;. \tag{5.19}$$

Because $\widetilde{\psi}^I_\pm$ is essentially the supersymmetry constraint (2.20) (the change of sign in the $\partial_\pm\sigma$ term is due to the change from χ to $\widetilde{\chi}$), the last of these formulas implies a similar formula for the supersymmetry constraints, viz.

$$\delta_\Lambda\,\mathcal{S}^I_\pm = \Upsilon_0^{\pm IJ}\,\mathcal{S}^J_\pm \;, \tag{5.20}$$

in agreement with the results of section 5.3 of [10]. This looks like the action of a chiral $\mathrm{Spin}(16)_+ \times \mathrm{Spin}(16)_-$ symmetry [11], but, as we have already emphasized, the latter is not a subgroup of $K(\mathrm{E}_9)$, and hence not a symmetry of the reduced theory.

6 A spinor representation of $K(\mathrm{E}_9)$ at low orders

As we have just shown, the contour integral expressions derived in [10] and describing the induced on shell action of $K(\mathrm{E}_9)$ on all fields, can be brought to the rather more simple form given in (5.16) and (5.17). The crucial point about these formulas is that they express the variations in terms of the coefficient functions $\Upsilon^\pm_n$ obtained by expanding about the branch points $\gamma = \pm 1$ as in (5.13), and that these expansions are in accord with the chiral decomposition of the world-sheet bosons and fermions. In this way, the positive and negative chiralities become 'attached' to the fixed points of the involution $\mathcal{I}$. We now propose to combine all the fermionic fields into a single object, an 'on shell' $K(\mathrm{E}_9)$ spinor made up of the fields of $N = 16$ supergravity. This spinor is expected to be part of an as yet unknown 'off shell' spinorial representation of $K(\mathrm{E}_9)$ with *infinitely many* components. We will argue that the local supersymmetry parameters should similarly be enlarged to a spinor representation of $K(\mathrm{E}_9)$ in an off shell version of the theory.

To this aim, we return to the expansion (5.13) in terms of local coordinates $u_\pm$. In order to combine the corresponding transformations on the fermions into a single formula, we introduce $u_\pm$ (and, of course, x-) dependent spinors $\Psi^I_\pm(x,u_\pm)$ and $\mathcal{X}^{\dot A}_\pm(x,u_\pm)$ in the local patches coordinatized by $u_\pm$, subject to the parity constraints

$$\Psi^I_\pm(u_\pm) = \Psi^I_\pm(-u_\pm) \; , \qquad \mathcal{X}^{\dot A}_\pm(u_\pm) = -\mathcal{X}^{\dot A}_\pm(-u_\pm) \; , \tag{6.1}$$

and in such a way that the $N=16$ supergravity fields appear as the lowest components, viz.

$$\begin{aligned} \Psi^I_\pm(u_\pm) &= (u_\pm^{-2}-1)\,\psi^I_{2\pm} \,\mp\, \psi^I_\pm \,+\, \mathcal{O}(u_\pm^2) \; , \\ \mathcal{X}^{\dot A}_\pm(u) &= \mp u_\pm^{-1}\,\chi^{\dot A} \,+\, \mathcal{O}(u_\pm) \; . \end{aligned} \tag{6.2}$$

It is important here that, as for the linear system, but unlike for the $K(\mathrm{E}_9)$ transformations, we do admit singular terms, and that in this way the fermions of $N=16$ supergravity become associated to the poles in this expansion, in analogy with the first order poles at $u_\pm = 0$ multiplying the scalar currents $P^A_\pm$ in the bosonic linear system. It is then straightforward to check that the transformations (5.17) are recovered from the $\mathcal{O}(u^n_\pm)$ terms for $n=-2,-1,0$ by expanding

$$\begin{aligned} \delta\Psi^I_\pm(u_\pm) &= \Upsilon^{IJ}_\pm(u_\pm)\Psi^J_\pm(u_\pm) + \tfrac{1}{2}\,\Gamma^I_{A\dot A}\Upsilon^A_\pm(u_\pm)\mathcal{X}^{\dot A}_\pm(u_\pm) \; , \\ \delta\mathcal{X}^{\dot A}_\pm(u_\pm) &= \tfrac{1}{4}\Gamma^{IJ}_{\dot A\dot B}\Upsilon^{IJ}_\pm(u_\pm)\mathcal{X}^{\dot B}_\pm(u_\pm) + \tfrac{1}{2}\,\Gamma^I_{A\dot A}\Upsilon^A_\pm(u_\pm)\Psi^I_\pm(u_\pm) \; , \end{aligned} \tag{6.3}$$

in $u_\pm$, making use of (5.13) and (6.2). Because the action of $K(\mathrm{E}_9)$ mixes gravitinos and matter fermions, one must consider $\{\Psi,\mathcal{X}\}$ as a *single object*. Furthermore, this action is double-valued because of the double-valuedness of the $Spin(16)$ representations $\mathbf{16}$ and $\mathbf{128_c}$ w.r.t. to the $Spin(16)$ subgroup of $K(\mathrm{E}_9)$ [18]. For this reason, we are indeed dealing with the beginnings of a *spinorial representation* of $K(\mathrm{E}_9)$.

However, (6.3) cannot be the complete answer, and thus the $K(\mathrm{E}_9)$ spinor must presumably contain further components beyond those indicated in (6.2). The obvious reason for this is the failure of the transformations (6.3) to close into the $\mathfrak{ke}_9$ algebra on the components of $\mathcal{X}^{\dot A}_\pm(u_\pm)$ other than the lowest one [7]. However, they do close properly on all components of $\Psi^I_\pm(u_\pm)$, and in particular on all the components written out in (6.2), i.e. precisely on the fermions of $N=16$ supergravity — as required by the consistency of the induced $K(\mathrm{E}_9)$ transformations on the $N=16$ supergravity multiplet. Extra fermionic fields will only appear at higher orders in $u_\pm$, but not introduce further singular terms (which we would have to associate with new fermionic physical degrees of freedom).

[7] Technically speaking, the main obstacle here is the fact that $\Gamma^I\Gamma^{(6)}\Gamma^I = 4\Gamma^{(6)} \neq 0$, whereas $\Gamma^{IJ}\Gamma^{(6)}\Gamma^{IJ}=0$ implying the closure of the transformations (4.2) (after an $\mathrm{SO}(16)$ Fierz rearrangement). The corresponding term in the expansion of $[\delta_1,\delta_2]\mathcal{X}^{\dot A}_\pm(u_\pm)$ is not yet visible at $\mathcal{O}(u_\pm^{-1})$.

At low orders in $u_\pm$ the ansatz (6.2) can be corroborated in several different ways. First of all, we can show that to the order indicated above, the fermionic equations of motion (Eq. (2.12) of [10]) can be combined into a single $K(\mathrm{E}_9)$ covariant 'Dirac equation', with the linear system as the $\mathfrak{ke}_9$ gauge connection. Namely, to linear order in the fermions, all fermionic equations of motion are contained in

$$\begin{aligned}(\mathcal{D}_\mp\Psi_\pm)^I(u_\pm) &\equiv \tilde{\mathcal{D}}_\mp\Psi^I_\pm(u_\pm)+\tfrac{1}{2}\Gamma^I_{A\dot{A}}\tilde{\mathcal{P}}^A_\mp(u_\pm)\mathcal{X}^{\dot{A}}_\pm(u_\pm) \;=\; 0\;,\\ (\mathcal{D}_\mp\mathcal{X}_\pm)^{\dot{A}}(u_\pm) &\equiv \tilde{\mathcal{D}}_\mp\mathcal{X}^{\dot{A}}_\pm(u_\pm)+\tfrac{1}{2}\Gamma^I_{A\dot{A}}\tilde{\mathcal{P}}^A_\mp(u_\pm)\Psi^I_\pm(u_\pm) \;=\; 0\;,\end{aligned} \tag{6.4}$$

expanded to lowest orders in $u_\pm$, where the connection components $\tilde{\mathcal{Q}}^{IJ}_\pm$ and $\tilde{\mathcal{P}}^A_\pm$ are to be taken from (4.9). In particular, the terms containing $\rho^{-1}\partial_\pm\rho$ are reproduced correctly from the derivatives acting on $u_\pm$ via the formula (2.10). The equations of motion tie together the expansions in u_+ and u_-; for instance, the terms proportional to $P_\mp\chi_\pm$ and $P_\mp\psi_{2\pm}$ have the correct pole order by virtue of the relation $u_+u_-=1$. In accordance with (6.3) and the transformation properties of the linear system, this equation is indeed $K(\mathrm{E}_9)$ covariant to lowest orders in $u_\pm$.

Evidently the local supersymmetry transformation parameters should similarly belong to some spinor representation of $K(\mathrm{E}_9)$, such that the local supersymmetry parameter of $N=16$ supergravity is but the lowest component of an infinite tower of local supersymmetries. We denote this representation by $\mathcal{E}$, and proceed from the hypothesis that it is the same as the one for the fermions, except that we require $\mathcal{E}$ to be regular at $u_\pm=0$. The reason for this assumption is that this appears to be the only possibility that will eventually allow us to gauge away all components of $\{\Psi^I_\pm,\mathcal{X}^{\dot{A}}_\pm\}$ other than the singular ones, which we wish to associate with the fermions of $N=16$ supergravity. We are thus led to introduce new x and $u_\pm$-dependent spinor parameters

$$\mathcal{E}^I_\pm(x,u_\pm)=\mathcal{E}^I_\pm(x,-u_\pm)\quad,\qquad \mathcal{E}^{\dot{A}}_\pm(x,u_\pm)=-\mathcal{E}^{\dot{A}}_\pm(x,-u_\pm)\;, \tag{6.5}$$

which are holomorphic in $u_\pm$ around the branch points

$$\mathcal{E}^I_\pm(x,u_\pm) \;=\; \varepsilon^I_{0\pm}(x)\;+\;\mathcal{O}(u^2_\pm)\;,\quad \mathcal{E}^{\dot{A}}_\pm(x,u_\pm)\;=\;u_\pm\varepsilon^{\dot{A}}_{1\pm}(x)\;+\;\mathcal{O}(u^3_\pm)\;. \tag{6.6}$$

We postulate that the $N=16$ local supersymmetry parameter $\varepsilon^I_\pm$ should appear at lowest order in the expansion of $\mathcal{E}^I_\pm(u_\pm)$, such that all other components correspond to new local supersymmetries. The superconformal constraints on the former [10] lead us to impose the constraint

$$\begin{aligned}(\mathcal{D}_\mp\mathcal{E}_\pm)^I(u_\pm) &\equiv \tilde{\mathcal{D}}_\mp\mathcal{E}^I_\pm(u_\pm)+\tfrac{1}{2}\Gamma^I_{A\dot{A}}\tilde{\mathcal{P}}^A_\mp(u_\pm)\mathcal{E}^{\dot{A}}_\pm(u_\pm) \;=\; 0\;,\\ (\mathcal{D}_\mp\mathcal{E}_\pm)^{\dot{A}}(u_\pm) &\equiv \tilde{\mathcal{D}}_\mp\mathcal{E}^{\dot{A}}_\pm(u_\pm)+\tfrac{1}{2}\Gamma^I_{A\dot{A}}\tilde{\mathcal{P}}^A_\mp(u_\pm)\mathcal{E}^I_\pm(u_\pm) \;=\; 0\;,\end{aligned} \tag{6.7}$$

on the generalized supersymmetry parameters. Again, these equations are $K(\mathrm{E}_9)$ covariant in linear order in $u_\pm$. Up to this order, they are solved by

$$\varepsilon^I_{0\pm} \;\equiv\; \varepsilon^I_\pm \quad \text{with } D_\mp\varepsilon^I_\pm=0\;,\qquad \varepsilon^{\dot{A}}_{1\pm}\,\rho^{-1}\partial_\pm\rho \;\equiv\; \Gamma^I_{A\dot{A}}\,P^A_\pm\,\varepsilon^I_\pm\;. \tag{6.8}$$

The first of these is the expected superconformal constraint on the $N = 16$ supersymmetry parameter. To verify the second relation for the 'new' supersymmetry parameter $\varepsilon_\pm^{\dot A}$ requires use of the integrability constraint $D_- P_+^A = D_+ P_-^A$ and the equation of motion $D_-(\rho P_+^A) + D_+(\rho P_-^A) = 0$ (neglecting cubic fermionic terms). The formula for $\varepsilon_\pm^{\dot A}$ can be viewed as resulting from a consistent truncation of an infinite number of local supersymmetries to a finite number; the fact that it can be expressed in terms of known quantities of $N = 16$ supergravity again reflects the fact that we are dealing with an 'on shell' realization of $K(\mathrm{E}_9)$ only.

Finally, also the generalized supersymmetry variations can be cast into the form

$$
\begin{aligned}
(\delta\Psi)^I_\pm(u_\pm) &= (\mathcal{D}_\pm\mathcal{E}_\pm)^I(u_\pm) \equiv u_\pm^2\mathcal{D}_\pm(u_\pm^{-2}\mathcal{E}^I_\pm(u_\pm)) + \tfrac{1}{2}\,\Gamma^I_{A\dot A}\mathcal{P}^A_\pm(u_\pm)\mathcal{E}^{\dot A}_\pm(u_\pm)\;, \\
(\delta\mathcal{X})^{\dot A}_\pm(u_\pm) &= (\mathcal{D}_\pm\mathcal{E}_\pm)^{\dot A}(u_\pm) \equiv u_\pm^2\mathcal{D}_\pm(u_\pm^{-2}\mathcal{E}^{\dot A}_\pm(u_\pm)) + \tfrac{1}{2}\Gamma^I_{A\dot A}\mathcal{P}^A_\pm(u_\pm)\mathcal{E}^I_\pm(u_\pm)\;, \qquad (6.9)
\end{aligned}
$$

again $K(\mathrm{E}_9)$ covariant in lowest orders of $u_\pm$, now with $\mathcal{Q}_\pm$, $\mathcal{P}_\pm$ from (4.7). With a little algebra and using formulas (2.10) one shows that these formulas yield the correct supersymmetry variations of the fermionic fields of $N = 16$ supergravity (cf. Eq. (2.18) of [10]), when expanded in powers of $u_\pm$. For instance, to lowest order in $u_\pm$ the r.h.s. of the first equation in (6.9) yields

$$
\begin{aligned}
u_\pm^2\hat D_\pm(u_\pm^{-2}\varepsilon^I_{0\pm}) + \tfrac{1}{2}\,\Gamma^I_{A\dot A}\hat P^A_\pm\varepsilon^{\dot A}_{1\pm} \quad &= \\
&= (u_\pm^{-2}-1)\,\rho^{-1}\partial_\pm\rho\,\varepsilon^I_\pm + D_\pm\varepsilon^I_\pm + \tfrac{1}{2}(\rho^{-1}\partial_\pm\rho)^{-1}P^A_\pm P^A_\pm\,\varepsilon^I_\pm \\
&= (u_\pm^{-2}-1)\,\rho^{-1}\partial_\pm\rho\,\varepsilon^I_\pm + D_\pm\varepsilon^I_\pm + \partial_\pm\sigma\,\varepsilon^I_\pm \\
&= (u_\pm^{-2}-1)\,\delta_\varepsilon\,\psi^I_{2\,\pm} \mp \delta_\varepsilon\,\psi^I_\pm\;,
\end{aligned}
$$

where for the second equality we have used the bosonic part of the conformal constraint (2.17). Likewise, the r.h.s. of the second equation in (6.9) combines contributions from both terms in order $u_\pm^{-1}$. The second condition (6.8) is thus crucial for our scheme to work.

When checking (6.4) against the fermionic equations of motion of $N = 16$ supergravity to linear order, it is sufficient to use the bosonic linear system (2.15). At higher order in the fermions, the fermionic bilinears in (2.18) will become relevant, and modify the r.h.s. of (6.4) by cubic fermionic terms. It is tempting to speculate that these are precisely the higher order fermionic terms in the fermionic field equations; however, the corresponding corrections appear only at constant or higher order in the $u_\pm$.

Further confirmation for the correctness of the ansatz (6.2) comes from the fact that the singular fermionic contributions in the full connection of the linear system (2.19) all arise from expanding the fermionic bilinear

$$
\Big(\Psi^I_\pm(u_\pm)\Psi^J_\pm(u_\pm) - \tfrac{1}{4}\Gamma^{IJ}_{\dot A\dot B}\,\mathcal{X}^{\dot A}(u_\pm)\mathcal{X}^{\dot B}(u_\pm)\Big)\,X^{IJ} - 2\Gamma^I_{A\dot A}\Psi^I_\pm(u_\pm)\mathcal{X}^{\dot A}_\pm(u_\pm)\,Y^A\;, \qquad (6.10)
$$

in lowest order. This means that the product of two spinorial representations contains the adjoint representation of $K(\mathrm{E}_9)$.

7 Outlook

Much attention has been devoted recently to the possible emergence of the indefinite Kac Moody algebras E_{10} [19, 20] and E_{11} [21, 22] and their relevance to the bosonic sector of M theory. Although no similar treatment exists for the fermionic sector, it is natural to conjecture that the fermionic degrees of freedom of M theory should consequently transform as spinors (i.e. as double-valued representations) under the maximal compact subgroups of these Kac Moody groups, in accordance with the chain of embeddings of 'generalized R symmetries'

$$\ldots \subset \mathrm{Spin}(16) \subset K(\mathrm{E}_9) \subset K(\mathrm{E}_{10}) \subset \ldots \tag{7.1}$$

We believe our results strengthen the evidence that these groups are indeed the correct R symmetry groups not only of dimensionally reduced supergravity, but possibly even of M theory itself. To work out the relevant spinor representations for $K(\mathrm{E}_{10})$ (and also for $K(\mathrm{E}_{11})$) will be no easy task; a recursive approach based an expansion under the respective subgroups $\mathrm{Spin}(10)$ and $\mathrm{Spin}(1,10)$ is expected to generate infinite towers of similar complexity as those in [23]. Still, whatever these representations are, it is clear that the $K(\mathrm{E}_9)$ spinor representations studied here must be embeddable in these bigger representations.

In recent work [24, 25, 26], an alternative chain of finite dimensional 'generalized holonomy groups' [8]

$$\ldots \subset \mathrm{SO}(16) \ \subset\ \mathrm{SO}(16)_+ \times \mathrm{SO}(16)_- \ \subset\ \mathrm{SO}(32) \ \subset\ \mathrm{SL}(32,\mathbb{R})\ , \tag{7.2}$$

was proposed to arise in the reduction of M theory to $d = 2, 1, 0$ dimensions (with analogous chains for spacelike and null reductions). This embedding chain is suggested by the fact that the $D = 11$ Γ-matrices generate $\mathrm{SL}(32,\mathbb{R})$, and therefore all of the subgroups listed above. For the case $d = 2$, we have found that the chiral split of $K(\mathrm{E}_9)$ with regard to the branch points $\gamma = \pm 1$ does seem to suggest a hidden $\mathrm{Spin}(16)_+ \times \mathrm{Spin}(16)_-$ symmetry [9]. Indeed, the transformation of the supersymmetry constraint in (5.20) shows how to realize this group via an unfaithful and on shell realization of $K(\mathrm{E}_9)$. However, the two chains no longer match because $\mathrm{Spin}(16)_+ \times \mathrm{Spin}(16)_-$ is not a subgroup of $K(\mathrm{E}_9)$.

The situation is much less clear for the conjectured holonomy groups $\mathrm{SO}(32)$ and $\mathrm{SL}(32,\mathbb{R})$. Although the 32-component Majorana spinor parameter of $D=11$ supergravity can be assigned to the $\mathbf{32}$ representation of $\mathrm{SO}(32)$ or $\mathrm{SL}(32,\mathbb{R})$, no such assignment is possible for the gravitino: neither the $\mathbf{288}$ of $\mathrm{Spin}(10)$ nor the $\mathbf{320}$ of $\mathrm{Spin}(1,10)$ can be 'lifted' to a representation of $\mathrm{SO}(32)$ or $\mathrm{SL}(32,\mathbb{R})$, respectively. Moreover, as shown in [18], these groups do not lead

[8] The group $\mathrm{SL}(32,\mathbb{R})$ was already suggested as a symmetry in [27].

[9] Or a hidden $\mathrm{SO}(16,\mathbb{C})$ for the reduction with one timelike Killing vector, as suggested in [24], and in agreement with the fact that the branch points are located at $\gamma = \pm i$ for a Euclidean worldsheet. Different real forms of $K(\mathrm{E}_9)$ have been recently studied in [28].

to the required double-valued representations when oxidized back to $d > 2$ dimensions. This is obvious for $\mathrm{SL}(32,\mathbb{R})$, which does not possess finite dimensional double valued representations, but it is also easy to see that no representation of $\mathrm{SO}(32)$ can ever give rise to a spinorial representation under its diagonally embedded $\mathrm{SO}(16)$ subgroup. Similar comments apply concerning the relation of $\mathrm{SO}(32)$ and $\mathrm{SL}(32,\mathbb{R})$ to the involutory subgroups $K(\mathrm{E}_{10})$ and $K(\mathrm{E}_{11})$; although, at levels $\ell \leq 4$ the latter contain all the requisite $\mathrm{SO}(10)$ and $\mathrm{SO}(1,10)$ representations arising in the decomposition of $\mathrm{SO}(32)$ and $\mathrm{SL}(32,\mathbb{R})$, respectively, these do not close into finite subalgebras of either $K(\mathrm{E}_{10})$ or $K(\mathrm{E}_{11})$.

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