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Non-perturbative contributions to the plane-wave string mass matrix

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Abstract: D-instanton contributions to the mass matrix of arbitrary excited string states of type IIB string theory in the maximally supersymmetric plane-wave background are calculated to leading order in the string coupling using a supersymmetric light-cone boundary state formalism. The explicit non-perturbative dependence of the mass matrix on the complex string coupling, the plane-wave mass parameter and the mode numbers of the excited states is determined.

Keywords: D-instanton, plane-wave, AdS/CFT.

Contents

1. Introduction

The conjectured correspondence between the BMN sector of $\mathcal{N}=4, d=4$ supersymmetric Yang-Mills [1] and type IIB string theory in the maximally supersymmetric plane-wave background [2] has been examined in some detail at the perturbative level [3, 4, 5]. However, the understanding of non-perturbative aspects of the correspondence has been very limited (although D-branes were constructed on the string theory side in [6, 7]). Such non-perturbative effects are well-studied in the context of the AdS/CFT correspondence where Yang–Mills instanton effects in $\mathcal{N}=4$ supersymmetric Yang-Mills correspond closely to D-instanton effects in type IIB superstring theory in $AdS_5 \times S^5$ [8]-[12]. A natural question to ask is whether there is a similar relationship between non-perturbative effects in plane-wave string theory and the BMN limit of the gauge theory.

According to the correspondence between plane-wave string theory and the BMN limit of $\mathcal{N}=4$ Yang–Mills theory, the light-cone gauge string theory mass matrix is related to the gauge theory conformal dimension. More precisely, the conformal dimension, Δ, and R-charge, J, are given by

$$\frac{H^{(2)}}{\mu} = \Delta - J\,, \tag{1.1}$$

where $H^{(2)}$ denotes the two-particle hamiltonian and μ is the constant background RR (Ramond–Ramond) five-form flux. The two-particle hamiltonian is the sum of two pieces

$$H^{(2)} = H^{(2)}_{\rm pert} + H^{(2)}_{\rm nonpert}\,. \tag{1.2}$$

The perturbative part, $H^{(2)}_{\rm pert}$, is a power series in the string coupling, g_s. These perturbative contributions to the mass spectrum have been analysed in some detail on both the string side and in the BMN limit of the gauge theory using the technology developed in [13]. There have been no calculations of non-perturbative corrections due to D-instanton effects, which contribute to $H^{(2)}_{\rm nonpert}$. It is the purpose of this paper to fill the gap in the existing literature and open the possibility of examining the proposed BMN/plane-wave correspondence at the non-perturbative level.

The single D-instanton sector has a measure that is proportional to

$$e^{2i\pi\tau}\, g_s^{7/2}\,, \tag{1.3}$$

where $\tau = \tau_1 + i\tau_2 \equiv C^{(0)} + ie^{-\phi}$ ($C^{(0)}$ is the Ramond–Ramond pseudoscalar, ϕ is the dilaton and $g_s = e^{\phi}$). The factor $g_s^{7/2}$ can be extracted from the form of certain higher derivative interactions that enter into the type IIB effective action at $O(1/\alpha')$ [14][1]. Although (1.3) is exponentially small, it is the leading contribution with the phase factor $e^{2\pi i C^{(0)}}$. It is therefore of interest to understand how the mass matrix is modified by these contributions. In the following we will calculate such D-instanton contributions to mass matrix elements to leading order in the string coupling. These can be compared with the gauge theory instanton contributions to the corresponding two-point functions in the BMN limit, which will be the subject of a separate paper [15].

We will use the light-cone boundary state description of the D-instanton [7, 16] to evaluate elements of the mass matrix for arbitrary string states. Equivalently, the leading contribution to the two-point function of string states in the D-instanton background will be associated with a world-sheet that is the product of two disks, with one closed-string state attached to each, and with Dirichlet boundary conditions. These boundary conditions impose the condition that the D-instanton has a position given by the bosonic moduli x_0^I, x_0^+, x_0^-. The sixteen broken kinematical and dynamical supersymmetries lead to the presence of sixteen fermionic moduli, ϵ^a, $\eta^{\dot a}$, which are included by attaching a total of sixteen fermionic open-string states to the boundaries of the disks. All the moduli are then integrated. This procedure will be implemented by use of the boundary state formalism.

This paper is organised as follows. Some of the notation and conventions of free light-cone plane-wave string theory are reviewed in section 2.1 and the structure of the

[1] More precisely, the complete dilaton dependence of any such process is a modular form that contains specific multi D-instanton contributions, from which one can read off the measure, including the $g_s^{7/2}$ factor.

D-instanton boundary state is reviewed in section 2.2. In section 2.3 the dependence on the fermionic moduli is obtained by applying the eight broken kinematical supersymmetries and the eight broken dynamical supersymmetries to the boundary state. This results in a 'dressed' boundary state. In order to evaluate the D-instanton contribution to two-point functions of string states this dressed boundary state is generalised to a composite two-string boundary state that is the product of two single string boundary states. Matrix elements of this state with physical two-string states give the D-instanton contribution to the elements of the mass matrix to leading order in the string coupling. Equivalently, we are evaluating the product of two disks with a single physical closed-string state attached to each and a total of sixteen open fermionic strings (representing the fermionic moduli) attached to the disk boundaries, which satisfy Dirichlet boundary conditions in all space-time directions. In section 2.3.1 we will see that there are no D-instanton corrections to the mass matrix with external supergravity states, as expected. The contributions of a single D-instanton to the mass matrix of massive string states created by the action of string creation modes are considered in sections 3 and 4. In section 3, states with two oscillator excitations (two impurities) are considered in detail. We will see that the leading contributions to $H^{(2)}_{\rm nonpert}/\mu$ behave as $g_s^{7/2}m^3$ when m is large, independent of the mode number of the oscillators. States with four bosonic string oscillator excitations are considered in section 4. The elements of the mass matrix between such states behave as $g_s^{7/2}m^7/r^2s^2$, where r, $-r$, s, $-s$ are the mode numbers of the oscillators on each of the strings[2]. States with fermionic excitations and states with larger numbers of impurities will also be discussed in section 4. We conclude with a summary and discussion of the implications for the BMN limit of $\mathcal{N}=4$ supersymmetric Yang–Mills theory in section 5.

2. Couplings of the plane-wave D-instanton

2.1 Plane-wave string theory

In this subsection and the appendix we will review some properties of free plane-wave light-cone gauge string theory [2]. The expressions for the supercharges and the hamiltonian are given in the appendix in terms of integrals over the usual superstring world-sheet fields, x^I, S and $\tilde{S}$. Here we will summarise the mode expansions of these expressions.

The free quantum mechanics hamiltonian [2] is given by[3]

$$\begin{aligned} 2p_-\,h &= m\left(a^{\dagger\,I}a^I + iS_0^a\Pi_{ab}\tilde{S}_0^b + 4\right) + \sum_{k=1}^{\infty}\left[\alpha^I_{-k}\alpha^I_k + \tilde{\alpha}^I_{-k}\tilde{\alpha}^I_k + \omega_k\left(S^a_{-k}S^a_k + \tilde{S}^a_{-k}\tilde{S}^a_k\right)\right] \\ &= m\left(a^{\dagger\,I}a^I + \theta^a_L\bar{\theta}^a_L + \bar{\theta}^a_R\theta^a_R\right) + \sum_{k=1}^{\infty}\left[\alpha^I_{-k}\alpha^I_k + \tilde{\alpha}^I_{-k}\tilde{\alpha}^I_k + \omega_k\left(S^a_{-k}S^a_k + \tilde{S}^a_{-k}\tilde{S}^a_k\right)\right]\,, \end{aligned} \tag{2.1}$$

[2]These arise in pairs of equal magnitude and opposite sign. They are also equal on the two strings as a result of the conservation of light-cone energy.

[3]We here use the lower case symbol h to distinguish the first quantised hamiltonian from the lowest order contribution to the string field theory hamiltonian.

where $m = \mu p_- \alpha'$, $\omega_n = \text{sign}(n)\sqrt{m^2+n^2}$. The matrix Π is defined in terms of $SO(8)$ γ matrices by

$$\Pi = \gamma^1\gamma^2\gamma^3\gamma^4 \,, \tag{2.2}$$

and its presence in (2.1) implies that h is invariant under $SO(4)\times SO(4)$ rather than $SO(8)$. The choice of the matrix Π in (2.2) reflects the choice of specific directions for the constant RR five-form flux which defines the background.

The modes of the left-moving and right-moving bosonic and fermionic world-sheet fields, α, $\tilde{\alpha}$, S and $\tilde{S}$, satisfy the (anti)commutation relations,

$$\begin{aligned} &[\alpha_m, \alpha_n] = \omega_n \delta_{m+n}\,, \quad [\tilde{\alpha}_m, \tilde{\alpha}_n] = \omega_n \delta_{m+n}\,, \\ &\{S^a_m, S^b_n\} = \delta^{ab}\delta_{m+n}\,, \quad \{\tilde{S}^a_m, \tilde{S}^b_n\} = \delta^{ab}\delta_{m+n}\,, \end{aligned} \tag{2.3}$$

while all other (anti)commutators of these variables vanish. The zero-mode fermionic variables θ_R, $\bar{\theta}_R$, θ_L and $\bar{\theta}_L$ in (2.1) are four-component spinors with $SO(4)\times SO(4)$ chiralities $(+,+)$ and $(-,-)$, defined in terms of S^a_0 and $\tilde{S}^a_0$ by

$$\begin{aligned} \theta_R &= \frac{1}{2\sqrt{2}}(1+\Pi)\,(S^a_0 + i\tilde{S}^a_0)\,, \qquad \bar{\theta}_R = \frac{1}{2\sqrt{2}}(1+\Pi)\,(S^a_0 - i\tilde{S}^a_0)\,, \\ \theta_L &= \frac{1}{2\sqrt{2}}(1-\Pi)\,(S^a_0 + i\tilde{S}^a_0)\,, \qquad \bar{\theta}_L = \frac{1}{2\sqrt{2}}(1-\Pi)\,(S^a_0 - i\tilde{S}^a_0)\,. \end{aligned} \tag{2.4}$$

The non-zero anticommutation relations between these spinors are

$$\{\bar{\theta}_R, \theta_R\} = \frac{(1+\Pi)}{2}\,, \qquad \{\bar{\theta}_L, \theta_L\} = \frac{(1-\Pi)}{2}\,. \tag{2.5}$$

The transverse position and momentum operators pair together to form harmonic oscillator creation and annihilation operators,

$$a^{\dagger\, I} = \frac{1}{\sqrt{2|m|}}(p^I_0 + i|m|x^I_0), \quad a^I = \frac{1}{\sqrt{2|m|}}(p^I_0 - i|m|x^I_0), \tag{2.6}$$

satisfying

$$[a^I, a^{\dagger\, J}] = \delta^{IJ}\,. \tag{2.7}$$

The operators a and $a^\dagger$ are referred to as the zero mode bosonic oscillators. The presence of the fermion mass-term in the hamiltonian explicitly breaks the $SO(8)$ symmetry to $SO(4)\times SO(4)$.

Let us briefly review the massless sector of the theory. Recall that it is usual to take the BMN vacuum state, $|0\rangle_h$, to be the bottom state of a 'massless' supermultiplet. It is natural to use a Fock space description for the fermions based on $|0\rangle_h$ as the ground state satisfying

$$\bar{\theta}_L|0\rangle_h = 0 = \theta_R|0\rangle_h\,. \tag{2.8}$$

In this basis the operators θ_L and $\bar{\theta}_R$ are creation operators and are used to create the other states in the multiplet. The state $|0\rangle_h$ is a nondegenerate bosonic state with $p_+ = 0$. All the other states have positive p_+ with equal numbers of degenerate bosons and fermions.

The lowest lying levels of the string are generated by acting with the zero bosonic and fermionic modes on the ground state. This generates towers of supergravity states that include infinite numbers of Kaluza–Klein-like excitations. It will prove convenient in the following to refer to a different basis in which the ground state is the complex dilaton, $|0\rangle_D$, that is annihilated by θ_L and θ_R. Acting on this state with $\bar\theta_R$ and $\bar\theta_L$ generates the 256 BPS states in a short supermultiplet. For example, acting with four powers of $\bar\theta_L$ gives the BMN ground state,

$$\frac{1}{4!}\epsilon_{a_1a_2a_3a_4}\bar\theta_L^{a_1}\bar\theta_L^{a_2}\bar\theta_L^{a_3}\bar\theta_L^{a_4}|0\rangle_D = |0\rangle_h\,, \tag{2.9}$$

where the 0 indicates that the state is the ground state of the zero mode bosonic harmonic oscillators. Similarly the conjugate of the BMN ground state is obtained by acting with $\bar\theta_R$,

$$\frac{1}{4!}\epsilon_{a_1a_2a_3a_4}\bar\theta_R^{a_1}\bar\theta_R^{a_2}\bar\theta_R^{a_3}\bar\theta_R^{a_4}|0\rangle_D = |0\rangle_{\bar h}\,. \tag{2.10}$$

In making contact with the boundary state description of the D-instanton we will often consider matrix elements between dilaton ground states, $|0\rangle_D$. In particular, the only non-zero matrix element in the space of the zero-mode fermions (the θ's) is

$${}_{\bar D}\langle 0|0\rangle_D \;= {}_D\langle 0|\prod_{a=1}^{4}(\bar\theta_L^a\bar\theta_R^a)|0\rangle_D\,. \tag{2.11}$$

We will use a convention in which $p_- > 0$ for incoming states and $p_- < 0$ for outgoing states. After integration over the instanton modulus x_0^-, p_- is conserved, which means that for any process involving M incoming and N outgoing states

$$\sum_{r=1}^{M} p_{r-} + \sum_{s=1}^{N} p_{s-} = 0\,. \tag{2.12}$$

The background preserves 32 supersymmetries. Sixteen of these are kinematical and do not commute with the hamiltonian, while sixteen are dynamical and commute with the hamiltonian. The kinematical supersymmetry generators are proportional to the θ's and $\bar\theta$'s. They will be denoted by q and $\bar q$, defined by

$$\begin{aligned} q_R &= e(p_-)\sqrt{|p_-|}\theta_R, \qquad & q_L &= e(p_-)\sqrt{|p_-|}\theta_L \\ \bar q_R &= \sqrt{|p_-|}\bar\theta_R, \qquad & \bar q_L &= \sqrt{|p_-|}\bar\theta_L\,, \end{aligned} \tag{2.13}$$

where $e(p_-) = \mathrm{sign}(p_-)$. The generators q and $\bar q$ satisfy the standard anti-commutation relations $\{q, \bar q\} = p_-$.

The dynamical supersymmetry generators, Q and $\tilde Q$, are given by

$$\begin{aligned} \sqrt{2|p_-|}\,Q_{\dot a} &= p_0^I\gamma^I_{\dot ab}S_0^b - |m|x_0^I\left(\gamma^I\Pi\right)_{\dot ab}\tilde S_0^b \\ &\quad + \sum_{n=1}^{\infty}\left(c_n\gamma^I_{\dot ab}(\alpha^I_{-n}S_n^b + \alpha^I_nS^b_{-n}) + \frac{im}{2\omega_nc_n}\left(\gamma^I\Pi\right)_{\dot ab}(\tilde\alpha^I_{-n}\tilde S^b_n - \tilde\alpha^I_n\tilde S^b_{-n})\right)\,, \end{aligned} \tag{2.14}$$

$$\sqrt{2|p_-|}\,\tilde Q_{\dot a} = p_0^I\gamma^I_{\dot ab}\tilde S_0^b + |m|x_0^I\left(\gamma^I\Pi\right)_{\dot ab}S_0^b$$

$$
+ \sum_{n=1}^{\infty} \left(c_n \gamma^I_{\dot{a}b} (\tilde{\alpha}^I_{-n} \tilde{S}^b_n + \tilde{\alpha}^I_n \tilde{S}^b_{-n}) - \frac{im}{2\omega_n c_n} \left(\gamma^I \Pi\right)_{\dot{a}b} (\alpha^I_{-n} S^b_n - \alpha^I_n S^b_{-n}) \right) , \tag{2.15}
$$

with $c_n = \sqrt{(\omega_n + n)/2\omega_n}$. The combinations $Q^\pm = \frac{1}{\sqrt{2}}(Q \pm i\tilde{Q})$ satisfy the anti-commutation relations

$$
\begin{aligned}
\{Q^+, Q^-\} &= 2h + m(\gamma^{ij}\Pi) J^{ij} + m(\gamma^{i'j'}\Pi) J^{i'j'} , \\
\{Q^+, Q^+\} &= \frac{1}{|p_-|}(N - \tilde{N}) \sim 0 , \\
\{Q^-, Q^-\} &= \frac{1}{|p_-|}(N - \tilde{N}) \sim 0 ,
\end{aligned} \tag{2.16}
$$

where J^{IJ} is the generator of angular momentum and $N, \tilde{N}$ are the left and right moving number operators, defined by

$$
N = \sum_{n=1}^{\infty} \left(\frac{n}{\omega_n} \alpha^I_{-n} \alpha^I_n + n\, S^a_{-n} S^a_n \right) , \qquad \tilde{N} = \sum_{n=1}^{\infty} \left(\frac{n}{\omega_n} \tilde{\alpha}^I_{-n} \tilde{\alpha}^I_n + n\, \tilde{S}^a_{-n} \tilde{S}^a_n \right) . \tag{2.17}
$$

In the above and in what follows, a capital index $I, J, \ldots$ labels an $SO(8)$ vector, an unprimed lower case index $i, j, \ldots$ labels a vector in one of the $SO(4)$ subgroups, while a primed lower case index $i', j', \ldots$ labels a vector in the other $SO(4)$ subgroup. The symbol $\sim$ indicates that $N - \tilde{N}$ vanishes for physical states, which satisfy the level-matching condition.

2.2 Review of the D-instanton boundary state

The D-instanton boundary state of [7] preserves eight kinematical and eight dynamical supersymmetries and is given by

$$
\|\boldsymbol{z}\rangle\!\rangle = \mathcal{N}_{(0,0)} \exp\left(\sum_{k=1}^{\infty} \frac{1}{\omega_k} \alpha^I_{-k} \tilde{\alpha}^I_{-k} - i\eta S_{-k} M_k \tilde{S}_{-k} \right) \|\boldsymbol{z}\,\rangle\!\rangle_0 , \tag{2.18}
$$

where $\|\boldsymbol{z}\,\rangle\!\rangle_0$ is the ground state of all the oscillators of non-zero mode number. The coordinate $\boldsymbol{z}^I$ is the eigenvalue of the position operator,

$$
x^I_0 = \frac{a^{\dagger\, I} - a^I}{i\sqrt{2|m|}} , \tag{2.19}
$$

constructed from the zero mode oscillators, $a^{\dagger\, I}$ and a^I. The parameter η is equal to ± 1, depending on whether the state describes a D-instanton or an anti D-instanton. From here on we shall choose $\eta = 1$. The normalisation constant in (2.18) is given by

$$
\mathcal{N}_{(0,0)} = (4\,\pi\, m)^2 . \tag{2.20}
$$

The matrix

$$
M_k = \frac{1}{k}(\omega_k \mathbb{1} - m\Pi) \tag{2.21}
$$

satisfies

$$M_k M_{-k}^T = \mathbb{1}\,. \tag{2.22}$$

The zero-mode part of the state is given by

$$\|\boldsymbol{z}\rangle\!\rangle_0 = (|I\rangle|I\rangle + i|\dot{a}\rangle|\dot{a}\rangle)\; e^{-m\boldsymbol{z}^2/2} e^{i\sqrt{2m}\boldsymbol{z}\cdot\boldsymbol{a}^\dagger} e^{\frac{1}{2}\boldsymbol{a}^\dagger\cdot\boldsymbol{a}^\dagger}|0\rangle_D\,, \tag{2.23}$$

where $|0\rangle_D$ is the ground state of all the oscillators in the basis in which θ_L and θ_R are annihilation modes.

The boundary state was shown in [7] to satisfy the conditions

$$\left(S_n^a + i\, M_n^{ab}\, \tilde{S}_{-n}^b\right) \|\boldsymbol{z}\,\rangle\!\rangle = 0\,. \tag{2.24}$$

It will be convenient to decompose the $SO(8)$ spinors S_n and $\tilde{S}_n$ into spinors of definite $SO(4)$ chiralities by defining

$$S_n^+ = \frac{1}{2}(1+\Pi)S_n\,, \qquad S_n^- = \frac{1}{2}(1-\Pi)S_n\,, \tag{2.25}$$

so that $\Pi S_n^\pm = \pm S_n^\pm$, and similarly for $\tilde{S}_n$. Then (2.24) can be rewritten as

$$\left(S_n^\pm + i\, M_n^\pm \tilde{S}_{-n}^\pm\right) \|\boldsymbol{z}\rangle\!\rangle = 0\,, \tag{2.26}$$

where

$$M_n^\pm = \frac{\omega_n \mp m}{n} = \sqrt{\frac{\omega_n \mp m}{\omega_n \pm m}}\,. \tag{2.27}$$

Note also that

$${}_S\langle 0|S_m^\pm S_{-n}^\pm|0\rangle_S = \delta_{mn}\,, \qquad {}_S\langle 0|S_m^\pm S_{-n}^\mp|0\rangle_S = 0\,, \tag{2.28}$$

where $|0\rangle_S$ is the ground state of S and $\tilde{S}$ for $n \neq 0$.

The $n = 0$ condition in (2.24) ensures that the state preserves half the kinematical supersymmetries,

$$q_L\|\boldsymbol{z}\,\rangle\!\rangle = 0 = q_R\|\boldsymbol{z}\,\rangle\!\rangle\,. \tag{2.29}$$

Likewise, it preserves a linear combination of dynamical supersymmetries,

$$Q^+\|\boldsymbol{z}\,\rangle\!\rangle = 0\,, \tag{2.30}$$

where $\sqrt{2}Q^+ = Q + i\tilde{Q}$. Applying the eight broken kinematical supersymmetries, $\bar{q}_L$ and $\bar{q}_R$, and the eight broken dynamical supersymmetries, $\sqrt{2}Q^- = Q - i\tilde{Q}$, to the boundary state generates sixteen fermionic moduli.

2.3 Two states coupling to a D-instanton

In this subsection we will calculate the contribution to the mass matrix of two-string states coupling to the D-instanton. To leading order in the string coupling this process is determined by the product of the one-point functions of closed strings coupling to a disk world-sheet, as shown in figure 1. It will be crucial to include the sixteen fermionic moduli associated with open strings coupling to the boundaries of the disks, which were not discussed in [7].

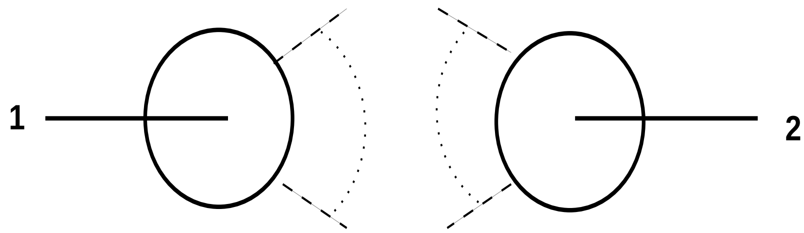

Figure 1: Leading order contribution to the two-point function. Dashed lines indicate the sixteen fermionic moduli, while solid lines indicate external states. The complete process is a sum of such contributions with the sixteen fermionic moduli distributed between the two disks in all possible ways.

Each disk is defined in a separate Fock space labelled 1 and 2, so that the D-instanton state in this space is given by

$$\|\hat{V}_2^{(0)}, \boldsymbol{z}\rangle\!\rangle = \|\boldsymbol{z}\rangle\!\rangle_1 \otimes \|\boldsymbol{z}\rangle\!\rangle_2 \, e^{ix_0^+(p_{1+}+p_{2+})} e^{ix_0^-(p_{1-}+p_{2-})}\,, \tag{2.31}$$

where the superscript (0) indicates that the fermionic moduli are not yet included.

Conservation of p_- follows upon integrating over the modulus x_0^- so that there is a factor of $\delta(p_{1-}+p_{2-})$ in the two-point function. We will take $p_{1-} \equiv p_- = -p_{2-} > 0$, which means that $m \equiv \mu\alpha' p_- > 0$ on disk 1 and $m < 0$ on disk 2. After integration over the light-cone time modulus, x_0^+, the process will preserve the p_+ component of momentum, which means that there is a factor of $\delta(h_i + h_f)$, where h_i, h_f are the light-cone hamiltonians for the incoming and outgoing states, respectively.

The dependence of the two-boundary state (2.31) on the transverse position modulus, $\boldsymbol{z}$, is given by the product of a factor (2.23) for each disk. Therefore, making use of the gaussian integral

$$\int d^8 z e^{-m\boldsymbol{z}^2} e^{-i\sqrt{2m}(\boldsymbol{a}_1^\dagger+\boldsymbol{a}_2^\dagger)\cdot\boldsymbol{z}} e^{(\boldsymbol{a}_1^{\dagger 2}+\boldsymbol{a}_2^{\dagger 2})/2}|0\rangle_1 \otimes |0\rangle_2 = \pi^4 m^{-4} e^{-\boldsymbol{a}_1^\dagger\cdot\boldsymbol{a}_2^\dagger}|0\rangle_1 \otimes |0\rangle_2\,, \tag{2.32}$$

the two-particle vertex integrated over the bosonic moduli (still ignoring fermionic moduli) has the form

$$\begin{aligned}\|\hat{V}_2^{(0)}\rangle\!\rangle &\equiv \int d^8 z\, \|\hat{V}_2^{(0)}, \boldsymbol{z}\rangle\!\rangle \\ &= (2\pi)^8 \exp\left(\sum_{k=1}^{\infty} \frac{1}{\omega_k}\alpha_{-k}^{(1)I}\tilde{\alpha}_{-k}^{(1)I} - iS_{-k}^{(1)}M_k\tilde{S}_{-k}^{(1)}\right. \\ &\qquad \left. + \frac{1}{\omega_k}\alpha_{-k}^{(2)I}\tilde{\alpha}_{-k}^{(2)I} - iS_{-k}^{(2)}M_k\tilde{S}_{-k}^{(2)}\right) e^{-\boldsymbol{a}_1^\dagger\cdot\boldsymbol{a}_2^\dagger}\|(0,0)\rangle\!\rangle_1 \otimes \|(0,0)\rangle\!\rangle_2\,. \end{aligned} \tag{2.33}$$

Note, in particular, that the factor of m^4 in the normalization of the boundary state cancels after integration over $\boldsymbol{z}$.

We will now consider the effect of applying the broken supersymmetries acting on the D-instanton in order to determine the dependence on the supermoduli. The broken kinematic supersymmetries multiply the vertex by the factor

$$(\bar{\epsilon}_R(\bar{q}_{1R}+\bar{q}_{2R}))^4 \quad (\bar{\epsilon}_L(\bar{q}_{1L}+\bar{q}_{2L}))^4\,. \tag{2.34}$$

while the broken dynamical supersymmetries give the factor

$$(\eta(Q_1^- + Q_2^-))^8 \,. \tag{2.35}$$

The spinors $\bar\epsilon_R^a$ and $\bar\epsilon_L^a$ are parameters of the kinematical supersymmetries with opposite $SO(4)\times SO(4)$ chiralities, so the index a has effectively four components for each chirality (which will be labelled by a_L and a_R). The dynamical supersymmetry parameter $\eta^{\dot a}$ has eight components.

Applying the broken kinematic supersymmetries (2.34) to the original D-instanton state and integrating over $\bar\epsilon_R$ and $\bar\epsilon_L$ gives the state

$$\|\hat V_2\rangle\!\rangle = \epsilon_{a_{L_1}a_{L_2}a_{L_3}a_{L_4}}\,\epsilon_{b_{L_1}b_{L_2}b_{L_3}b_{L_4}}\,\prod_{r=1}^{4}(\bar\theta_{2L}+\bar\theta_{1L})^{a_{L_r}}(\bar\theta_{2R}+\bar\theta_{1R})^{b_{R_r}}\,\|\hat V_2^{(0)}\rangle\!\rangle \,. \tag{2.36}$$

The products involving $\bar\theta_L$'s and $\bar\theta_R$'s can be interpreted as follows. The boundary state is located at a particular value of $\boldsymbol{z}, \epsilon_L, \epsilon_R$ in superspace. The two disks are therefore associated with factors

$$\delta^4(\bar\theta_{1L}+\bar\epsilon_L)\delta^4(\bar\theta_{2L}+\bar\epsilon_L)$$

and a similar one involving $\bar\theta_R$'s. Integrating over $\bar\epsilon_L$ and $\bar\epsilon_R$ gives the factor

$$\delta^4(\bar\theta_{2L}+\bar\theta_L)\delta^4(\bar\theta_{2R}+\bar\theta_{1R}) = \prod_{r=1}^{4}(\bar\theta_{2L}+\bar\theta_{1L})^{a_{L_r}}(\bar\theta_{2R}+\bar\theta_{1R})^{b_{R_r}} \,.$$

The vertex $\|\hat V_2\rangle\!\rangle$ satisfies the conditions

$$\begin{aligned}(\boldsymbol{a}_1+\boldsymbol{a}_2^\dagger)\|\hat V_2\rangle\!\rangle &= 0 = (\boldsymbol{a}_1^\dagger+\boldsymbol{a}_2)\|\hat V_2\rangle\!\rangle\\ (\bar\theta_{2L}+\bar\theta_{1L})\|\hat V_2\rangle\!\rangle &= 0 = (\bar\theta_{2R}+\bar\theta_{1R})\|\hat V_2\rangle\!\rangle \,.\end{aligned} \tag{2.37}$$

The remaining supermoduli are the $\eta^{\dot a}$ associated with the broken dynamical supersymmetries, Q^-. Applying the broken dynamical supersymmetries to the state $\|\hat V_2\rangle\!\rangle$ produces an additional prefactor, resulting in the complete boundary state,

$$\|V_2\rangle\!\rangle = \int d^8\eta\,(\eta(Q_1^-+Q_2^-))^8\|\hat V_2\rangle\!\rangle \,, \tag{2.38}$$

which couples to any pair of physical closed-string states subject to them preserving p_+ and p_-.

We now need to show that the unbroken supersymmetries,

$$\epsilon_R^{a_R}(q_{1R}+q_{2R})^{a_R} \qquad \epsilon_L^{a_L}(q_{1L}+q_{2L})^{a_L}\,, \qquad \tilde\eta^{\dot a}(Q_1^+-Q_2^+)^{\dot a}\,, \tag{2.39}$$

annihilate the vertex, $\|V_2\rangle\!\rangle$, so the state preserves half the supersymmetries. The vertex $\|\hat V_2^{(0)}\rangle\!\rangle$ is automatically annihilated by these supersymmetries, so the issue is whether they continue to do so in the presence of the prefactors in (2.36) and (2.38). So we need to show that the commutators of the unbroken supersymmetries with these prefactors all vanish. Although the conserved kinematical supersymmetry, $\epsilon_R(q_{1R}+q_{2R})$, does not commute with the prefactor $\big(\eta(Q_1^- + Q_2^-)\big)^8$, the commutator is proportional to $(\boldsymbol{a}_1 + \boldsymbol{a}_2^\dagger)$, which vanishes when acting on the vertex, $\|\hat{V}_2\rangle\rangle$ (similarly, the commutator of the conserved kinematical supersymmetry, $\epsilon_L(q_{1L} + q_{2L})$, is proportional to $(\boldsymbol{a}_2 + \boldsymbol{a}_1^\dagger)$, which also vanishes on the vertex). The conserved dynamical supersymmetry, $\eta\,(Q_1^+ - Q_2^+)$, also does not commute with the prefactor in (2.38), but the commutator is proportional to the sum, $p_{1+} + p_{2+}$, which vanishes since the light-cone energy is conserved in the on-shell two-point function. Finally, the conserved dynamical supersymmetry does not commute with the prefactor $\prod_{r=1}^4 (\bar\theta_{2L} + \bar\theta_{1L})^{a_{Lr}} (\bar\theta_{2R} + \bar\theta_{1R})^{b_{Rr}}$ in the vertex $\|\hat{V}_2\rangle\rangle$, but its commutator is proportional to $(\boldsymbol{a}_1 + \boldsymbol{a}_2^\dagger)$ or $(\boldsymbol{a}_1^\dagger + \boldsymbol{a}_2)$, each of which vanishes when acting on the vertex.

Elements of the mass matrix for external states χ_1 and χ_2 are proportional to matrix elements of the form

$$e^{2i\pi\tau}\, g_s^{7/2}\, {}_1\langle\chi_1| \otimes {}_2\langle\chi_2\|V_2\rangle\rangle\,. \tag{2.40}$$

The analysis of the integral over the eight components of η is very different in the zero mode sector (the supergravity sector) from the non zero-mode sector. We will therefore first analyse the mass matrix for the supergravity sector before considering more general stringy effects.

2.3.1 Decoupling of supergravity modes

The piece of Q^- that depends on zero modes is given by (see appendix)

$$\begin{aligned}\sqrt{2|p_-|}Q^-_{0\dot a} &= (p_0^I\gamma^I - i|m|x_0^I\gamma^I\Pi)_{\dot a b}(S_0 - i\tilde S_0)^b \\ &= \sqrt{2}(p_0^I\gamma^I - i|m|x_0^I\gamma^I\Pi)_{\dot a b}(\bar\theta_R + \bar\theta_L)^b \\ &= 2\sqrt{|m|}\,\Big(\boldsymbol{a}\cdot\gamma\bar\theta_R + \boldsymbol{a}^\dagger\cdot\gamma\bar\theta_L\Big)_{\dot a}\,,\end{aligned} \tag{2.41}$$

where we have used $\Pi\bar\theta_R = +\bar\theta_R$ and $\Pi\bar\theta_L = -\bar\theta_L$. We also note that the part of the conserved dynamical supersymmetries, Q^+, that depends on zero modes is given by

$$\sqrt{2|p_-|}Q^+_{0\dot a} = 2\sqrt{|m|}(\boldsymbol{a}\cdot\gamma\theta_L + \boldsymbol{a}^\dagger\cdot\gamma\theta_R)_{\dot a}\,. \tag{2.42}$$

The commutation relations between the a's and Q^- are given by

$$[Q^-, a^I] = \sqrt{2\mu}\gamma^I\bar\theta_L, \quad [Q^-, a^I] = \sqrt{2\mu}\gamma^I\bar\theta_R\,, \tag{2.43}$$

and

$$\{Q^-, \bar\theta_L\} = 0, \quad \{Q^-, \bar\theta_R\} = 0\,. \tag{2.44}$$

The expressions for the zero-mode parts of the broken dynamical supersymmetries that enter in (2.38) are[4]

$$Q_1^- = \sqrt{2\mu}\gamma\cdot\Big(\boldsymbol{a}_1\bar\theta_{1R} + \boldsymbol{a}_1^\dagger\bar\theta_{1L}\Big)\,, \qquad Q_2^- = -\sqrt{2\mu}\gamma\cdot\Big(\boldsymbol{a}_2^\dagger\bar\theta_{2R} + \boldsymbol{a}_2\bar\theta_{2L}\Big) \tag{2.45}$$

[4]Note that according to our conventions an incoming state has positive momentum while an outgoing state has negative momentum.

According to the above, the matrix elements of the mass matrix between states $\langle\chi_1|$ and $\langle\chi_2|$ have the form

$$ {}_1\langle\chi_1|\otimes{}_2\langle\chi_2\|V_2\rangle\!\rangle = \int d^8\eta\ {}_1\langle\chi_1|\otimes{}_2\langle\chi_2|\prod_{r=1}^{8}(\eta^{\dot{a}_r}\,(Q_1^-+Q_2^-)^{\dot{a}_r})\|\hat{V}_2\rangle\!\rangle\,. \tag{2.46} $$

The last factor in $\prod\eta\,(Q_1^-+Q_2^-)$ acts on the boundary state. In this factor one can substitute $\boldsymbol{a}_2^\dagger=-\boldsymbol{a}_1$ and $\boldsymbol{a}_1=-\ \boldsymbol{a}_2^\dagger$ (using (2.37)). This factor is then of the form

$$ \gamma\cdot\left(-\boldsymbol{a}_2^\dagger(\bar{\theta}_{1R}+\bar{\theta}_{2R})+\boldsymbol{a}_1^\dagger(\bar{\theta}_{1L}+\bar{\theta}_{2L})\right)\,, \tag{2.47} $$

which vanishes when multiplying the prefactor $\prod_{r=1}^{4}(\bar{\theta}_{1L}+\bar{\theta}_{2L})^{a_{Lr}}(\bar{\theta}_{1R}+\bar{\theta}_{2R})^{b_{Rr}}$ in $\|\hat{V}_2\rangle\!\rangle$. We therefore see that the matrix elements involving two supergravity states of arbitrary excitation number vanish, as expected.

When one of the states has no stringy excitations and the other does the matrix element also vanishes. This follows from the fact that the two external states have to have same value of p_+ (or level number) after integration over the x_0^+ modulus of the D-instanton.

When both of the states have stringy excitations the matrix element does not vanish.

3. Matrix elements between two-impurity single-string states

We now turn to consider the matrix elements of states with non-zero mode number excitations, which are generically non-vanishing. One general feature of these matrix elements follows from the fact that, apart from the prefactor associated with the broken supersymmetries, the boundary state is simply an exponential of a scalar quadratic form in the excited oscillators. In the absence of the prefactor the boundary state would factorise into the product of two scalar operators and therefore would only couple to states in which the impurities are combined into $SO(4)\times SO(4)$ singlets. Only the prefactor $(\eta\,(Q_1^-+Q_2^-))^8$ couples the $SO(4)\times SO(4)$ spin between the two disks. In the first instance we will restrict ourselves to states with only two impurities.

The complete list of two-impurity states based on the BMN vacuum at a given mass level is given by

$$ \begin{aligned} NS-NS\quad &: \alpha_{-n}^{(I}\tilde{\alpha}_{-n}^{J)}|0\rangle_h\,, \qquad \alpha_{-n}^{[I}\tilde{\alpha}_{-n}^{J]}|0\rangle_h\,, \qquad \alpha_{-n}^{I}\tilde{\alpha}_{-n}^{I}|0\rangle_h\,,\\ RR\quad &: S_{-n}^{a}\tilde{S}_{-n}^{a}|0\rangle_h\,, \qquad S_{-n}^{a}\gamma_{ab}^{IJ}\tilde{S}_{-n}^{b}|0\rangle_h\,, \quad S_{-n}^{a}\gamma_{ab}^{IJKL}\tilde{S}_{-n}^{b}|0\rangle_h\,,\\ NS-R\quad &: \alpha_{-n}^{I}\tilde{S}_{-n}^{a}|0\rangle_h\,,\\ R-NS\quad &: S_{-n}^{a}\tilde{\alpha}_{-n}^{I}|0\rangle_h\,, \end{aligned} \tag{3.1} $$

where NS and R indicate the Neveu–Schwarz and Ramond sectors, respectively. The masses of each of these states is $2\omega_n$ in the free string theory, while our aim is to evaluate the one-instanton mass matrix that corrects the masses of such states. Although the list (3.1) is labelled with SO(8) transverse vector indices for economy of presentation, the boundary state only respects the $SO(4)\times SO(4)$ subgroup. We will first consider states in the $NS-NS$ sector that have two bosonic impurities.

3.1 Two bosonic impurities

In this case we will take the external states to be stringy excitations of the BMN ground state. In this case the bra vectors in (2.46) are given by

$$ {}_1\langle\chi_1| \otimes {}_2\langle\chi_2| = \frac{1}{\omega_p\omega_q}\, t^{(1)}_{IJ}\, t^{(2)}_{KL}\, {}_h\langle 0|\alpha^{(1)I}_p\, \tilde\alpha^{(1)J}_p \otimes {}_h\langle 0|\alpha^{(2)K}_q\, \tilde\alpha^{(2)L}_q\,. \tag{3.2} $$

The normalisation has been chosen so that each external state has unit norm if the wave functions satisfy $t^{(1)}_{IJ}\, t^{(1)}_{IJ} = 1 = t^{(2)}_{IJ}\, t^{(2)}_{IJ}$.

We now proceed to evaluate the matrix element (2.46). The non-zero modes enter into Q^- in the following manner

$$ \begin{aligned} \sqrt{2|p_-|}Q^-_{\dot a\; n\neq 0} = &\sum_{n=1}^{\infty}(\gamma^I N_+)_{\dot a b}(\alpha^I_n S^b_{-n} - i\,\tilde\alpha^I_n \tilde S^b_{-n}) \\ &+ (\gamma^I N_-)_{\dot a b}(\alpha^I_{-n} S^b_n - i\,\tilde\alpha^I_{-n}\tilde S^b_n)\,, \end{aligned} \tag{3.3} $$

where we have defined

$$ (N_\pm)_{ab} = (c_n \mathbb{1} \pm \frac{m}{2\omega_n c_n}\Pi)_{ab}\,. \tag{3.4} $$

The matrices N_+ and N_- satisfy

$$ N_+^2 = \frac{n}{\omega_n}M_{-n}\,, \qquad N_+N_- = \frac{n}{\omega_n}\,, \qquad N_-^2 = \frac{n}{\omega_n}M_n\,. \tag{3.5} $$

We will make use of the following commutation relations that are valid for $p_- > 0$,

$$ \sqrt{2|p_-|}\left[Q^-_{\dot a}, \alpha^J_{-p}\right] = \omega_p \gamma^I N_{+\dot a b} S^b_{-p} \tag{3.6} $$

$$ \sqrt{2|p_-|}\left[Q^-_{\dot a}, \tilde\alpha^J_{-p}\right] = -i\omega_p \gamma^I N_{+\dot a b} \tilde S^b_{-p} \tag{3.7} $$

$$ \sqrt{2|p_-|}\left\{Q^-_{\dot a}, S^b_{-p}\right\} = \gamma^I N_{-\dot a b}\alpha^I_{-p} \tag{3.8} $$

$$ \sqrt{2|p_-|}\left\{Q^-_{\dot a}, \tilde S^b_{-p}\right\} = -i\gamma^I N_{-\dot a b}\tilde\alpha^I_{-p}\,. \tag{3.9} $$

For $p_- < 0$ the sign of m changes and the matrices N_+ and N_- are interchanged.

The vertex $\|V_2\rangle\rangle$ in (2.46) contains the prefactor

$$ (\eta(Q_1^- + Q_2^-))^8 = \sum_{p=0}^{8} C^8_p(\eta Q_1^-)^p(\eta Q_2^-)^{8-p}\,, \tag{3.10} $$

where C^8_p are binomial coefficients. The different terms in the sum in (3.10) generate couplings between external states in different $SO(4)\times SO(4)$ representations when they act on the boundary state $\|\hat V_2\rangle\rangle$.

With external states of the form (3.2) an even number of Q^-'s must be distributed between the disks so only the terms with even p contribute. For each value of p each disk couples to an external string state that lies in symmetric, antisymmetric or trace irreducible representations of $SO(4)\times SO(4)$ which are shown in table 1.

p	Disk 1	Disk 2	$SO(4)\times SO(4)$ reps.
0	$(Q^-)^0$	$(Q^-)^8$	$(i,i)+(j',j')$
2	$(Q^-)^2$	$(Q^-)^6$	$[i,j]\,,[i',j']\,,[i,j']$
4	$(Q^-)^4$	$(Q^-)^4$	$(i,j)_t\,,(i',j')_t\,,(i,i)\,,(j',j')\,,(i,j')$
6	$(Q^-)^6$	$(Q^-)^2$	$[i,j]\,,[i',j']\,,[i,j']$
8	$(Q^-)^8$	$(Q^-)^0$	$(i,i)+(j',j')$

Table 1: Distribution of the eight Q^-'s between the two disks leads to couplings between pairs of $SO(4)\times SO(4)$ irreducible representations listed in the last column. The symbol $[a,b]$ indicates an antisymmetric representation, $(a,b)_t$ indicates a symmetric traceless irreducible representation while (a,a) indicates a singlet of either $SO(4)$.

3.1.1 Matrix elements of symmetric tensor and singlet states

We will first consider the $p=4$ case, in which there is a pre-factor of Q^4. This contributes to the $SO(4)\times SO(4)$ representations, (i,j), (i',j'), (i,i), (j',j') and $(i,j')+(j',i)$. In this case we need to include the terms with binomial coefficients C^8_4 in (3.10).

Since the external states are bra vectors containing two excited annihilation oscillators, and since each factor of $(Q^-_1+Q^-_2)$ in the prefactor contains the sum of products of one creation mode and one annihilation mode, we need only keep the bilinear terms in the expansion of the exponential factor in the boundary ket state. The matrix elements therefore have the form

$$
{}_1\langle\chi_1|\otimes{}_2\langle\chi_2|\quad(\eta(Q^-_1+Q^-_2))^8|\left(\sum_{k=1}^{\infty}\frac{1}{\omega_k}\alpha^{(1)I}_{-k}\tilde\alpha^{(1)I}_{-k}-iS^{(1)}_{-k}M_k\tilde S^{(1)}_{-k}\right)
\left(\sum_{l=1}^{\infty}\frac{1}{\omega_l}\alpha^{(2)J}_{-l}\tilde\alpha^{(2)J}_{-l}-iS^{(2)}_{-l}M_l\tilde S^{(2)}_{-l}\right)\|(0,0)\rangle\!\rangle_1\otimes\|(0,0)\rangle\!\rangle_2\,. \tag{3.11}
$$

We will evaluate (3.11) by commuting factors of $\eta(Q^-_1+Q^-_2)$ to the right, noting that the commuted part annihilates the ground-state ket vector. We therefore pick up factors of $[(\eta Q^-)^r,\alpha^I]$ or $[(\eta Q^-)^r,S^a]$ for various values of the integer r. These factors are summarised in table 2 (in which $N^I_\pm\equiv\gamma^I\,N_\pm)$[5]. Overall powers of $|p_-|$ are omitted in this list since these cancel with factors of $|p_-|$ coming from the kinematic supersymmetries once both the disks are included.

[5]Recall that the sign of m in Q^-_2 is reversed relative to that in Q^-_1.

	α^I_{-n}	S^a_{-n}
$\sqrt{2\lvert p_-\rvert}Q^{-\dot a_1}$	$\omega_n(N^I_+)_{\dot a_1 c}S^c_{-n}$	$(N^I_-)_{\dot a_1 a}\alpha^I_{-n}$
$\prod_{r=1}^2\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$n\gamma^{IJ}_{\dot a_1\dot a_2}\alpha^J_{-n}$	$\omega_n(N^I_-)_{\dot a_1 a}(N^I_+S_{-n})_{\dot a_2}$
$\prod_{r=1}^3\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$n\omega_n\gamma^{IJ}_{\dot a_1\dot a_2}(N^J_+S_{-n})_{\dot a_3}$	$n(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}\alpha^J_{-n}$
$\prod_{r=1}^4\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$n^2\gamma^{IJ}_{\dot a_1\dot a_2}\gamma^{JK}_{\dot a_3\dot a_4}\alpha^K_{-n}$	$n\omega_n(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}(N^J_+S_{-n})_{\dot a_4}$
$\prod_{r=1}^5\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$\omega_n n^2\gamma^{IJ}_{\dot a_1\dot a_2}\gamma^{JK}_{\dot a_3\dot a_4}(N^K_+S_{-n})_{\dot a_5}$	$n^2(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}\gamma^{JK}_{\dot a_4\dot a_5}\alpha^K_{-n}$
$\prod_{r=1}^6\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$n^3\gamma^{IJ}_{\dot a_1\dot a_2}\gamma^{JK}_{\dot a_3\dot a_4}\gamma^{KL}_{\dot a_5\dot a_6}\alpha^L_{-n}$	$\omega_n n^2(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}\gamma^{JK}_{\dot a_4\dot a_5}(N^K_+S_{-n})_{\dot a_6}$
$\prod_{r=1}^7\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$\omega_n n^3\gamma^{IJ}_{\dot a_1\dot a_2}\gamma^{JK}_{\dot a_3\dot a_4}\gamma^{KL}_{\dot a_5\dot a_6}(N^L_+S_{-n})_{\dot a_7}$	$n^3(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}\gamma^{JK}_{\dot a_4\dot a_5}\gamma^{KL}_{\dot a_6\dot a_7}\alpha^L_{-n}$
$\prod_{r=1}^8\sqrt{2\lvert p_-\rvert}Q^{-\dot a_r}$	$n^4\gamma^{IJ}_{\dot a_1\dot a_2}\gamma^{JK}_{\dot a_3\dot a_4}\gamma^{KL}_{\dot a_5\dot a_6}\gamma^{LM}_{\dot a_7\dot a_8}\alpha^M_{-n}$	$\omega_n n^3(N^I_-)_{\dot a_1 a}\gamma^{IJ}_{\dot a_2\dot a_3}\gamma^{JK}_{\dot a_4\dot a_5}\gamma^{KL}_{\dot a_6\dot a_7}(N^L_+S_{-n})_{\dot a_8}$

Table 2: Action of broken supersymmetries on the left-moving (untilded) oscillators with $p_- > 0$. The result for the right-movers (tilded) oscillators is given by inserting an extra factor of $(-i)^r$. When $p_- < 0$ the matrices N_- and N_+ are interchanged. The matrices $N^I_\pm$ are defined by $N^I_\pm \equiv \gamma^I\, N_\pm$.

Since the external states are made of bosonic oscillators only, we will need to act with an even number of Q^-'s on each bosonic oscillator and with an odd number on each fermionic oscillator. In other words, we need to evaluate an expression of the form,

$$\frac{1}{\omega}[(\eta Q^-)^4,\alpha)]\tilde\alpha+\frac{1}{\omega}\alpha[(\eta Q^-)^4),\tilde\alpha]+6\frac{1}{\omega}[(\eta Q^-)^2,\alpha]\ [(\eta Q^-)^2,\tilde\alpha] \\ +4[(\eta Q^-)^3,S]\ [\eta Q^-,\tilde S]+4[\eta Q^-,S]\ [(\eta Q^-)^3,\tilde S]\,, \tag{3.12}$$

where we have suppressed the index structure. The various numerical factors in front of each term are the binomial coefficients C^4_n.

Using table 1 and the following identity

$$N_-M_nN_-+\frac{n}{\omega_n}=2M_n\,, \tag{3.13}$$

we obtain a contribution to the first disk of the form

$$|\text{Disk 1}\rangle\!\rangle\equiv\sum_{n=1}^{\infty}n\left[(\eta\gamma^R\gamma^K\eta)\left(\eta\gamma^K(\frac{\omega_n}{n}\mathbb{1}-\frac{m}{n}\Pi)\gamma^S\eta\right)\right]\alpha^{(1)(R}_{-n}\tilde\alpha^{(1)S)}_{-n}\|(0,0)\rangle\!\rangle_0\,. \tag{3.14}$$

For the second disk, the result can be obtained from equation (3.14) by replacing m by $-m$, giving

$$|\text{Disk 2}\rangle\!\rangle\equiv\sum_{n=1}^{\infty}n\left[(\eta\gamma^P\gamma^L\eta)\left(\eta\gamma^L(\frac{\omega_n}{n}\mathbb{1}+\frac{m}{n}\Pi)\gamma^Q\eta\right)\right]\alpha^{(2)(P}_{-n}\tilde\alpha^{(2)Q)}_{-n}\|(0,0)\rangle\!\rangle_0\,. \tag{3.15}$$

The mass matrix is obtained by evaluating

$$\frac{1}{\omega_p\omega_q}t^{(1)}_{(IJ)}t^{(2)}_{(KL)}\int d^8\eta\,\langle 0|\alpha^{(1)I}_p\tilde\alpha^{(1)J}_p|\text{Disk 1}\rangle\!\rangle\times\langle 0|\alpha^{(2)K}_q\tilde\alpha^{(2)L}_q|\text{Disk 2}\rangle\!\rangle\,, \tag{3.16}$$

This expression involves the Grassmann integral,

$$\int d^8\,\eta\eta\gamma^{IK}\eta\,\eta\gamma^K M_n^-\gamma^J\eta\,\eta\gamma^{PL}\eta\,\eta\gamma^L M_n^+\gamma^Q\eta\,. \tag{3.17}$$

Integration over the η's can be efficiently expressed in terms of the four-component chiral $SO(4)\times SO(4)$ spinors

$$\eta^\pm = \frac{1}{2}(1\pm\Pi)\,\eta\,. \tag{3.18}$$

The integrals we will meet later are all of the form

$$\int d^4\eta^\pm\eta^\pm\gamma^{ij}\eta^\pm\,\eta^\pm\gamma^{kl}\eta^\pm = (-\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}\pm\epsilon^{ijkl})\equiv\mathcal{T}^\pm_{ijkl}\,, \tag{3.19}$$

$$\int d^4\eta^\pm\eta^\pm\gamma^{i'j'}\eta^\pm\,\eta^\pm\gamma^{k'l'}\eta^\pm = (\delta^{i'k'}\delta^{j'l'}-\delta^{i'l'}\delta^{j'k'}\pm\epsilon^{i'j'k'l'})\equiv-\mathcal{T}^\mp_{i'j'k'l'}\,, \tag{3.20}$$

$$\int d^4\eta^\pm\eta^\pm\gamma^{ij}\eta^\pm\,\eta^\pm\gamma^{k'l'}\eta^\pm = 0\,, \tag{3.21}$$

or can be converted to this form by making use of the Fierz transformation

$$\eta^\pm_{\dot a}\eta^\pm_{\dot b} = \frac{1}{16}\left((\gamma^{kl}P^\pm)_{\dot a\dot b}\eta\gamma^{kl}\eta+(\gamma^{k'l'}P^\pm)_{\dot a\dot b}\eta\gamma^{k'l'}\eta\right)\,, \tag{3.22}$$

where $P^\pm=\frac{1}{2}(\mathbb{1}\pm\Pi)$.

The tensors $\mathcal{T}^\pm_{ijkl}$, defined in (3.19), have the property that they are left invariant under the interchange of the first and last pair of indices and they satisfy the (anti)self-duality condition

$$\epsilon^{j_1j_2p_3p_4}\,\mathcal{T}^\pm_{j_1j_2j_3j_4} = \pm2\mathcal{T}^\pm_{p_3p_4j_3j_4}\,. \tag{3.23}$$

We also note the property

$$\mathcal{T}^+_{ispj}\mathcal{T}^-_{rijq} = 2(\delta^{rq}\delta^{ps}+\delta^{sq}\delta^{rp})-\delta^{pq}\delta^{rs}\,, \tag{3.24}$$

which is invariant under $p\leftrightarrow q, r\leftrightarrow s$.

Mass matrix elements with $(I,J)=(i,j)$

Up to now the indices I,J,K,L have labelled $SO(8)$ vectors which can take values in either of the $SO(4)$ factors in the $SO(4)\times SO(4)$ subgroup. At this point we will specialise to the representation in which the vector indices (I,J) are in one of the $SO(4)$ subgroups of $SO(4)\times SO(4)$, so that $(I,J)\to(i,j)$. In this case the integration over the fermionic moduli, η, in (3.16) involves evaluation of the Grassmann integral

$$\int d^8\,\eta\eta\gamma^{iK}\eta\,\eta\gamma^K M_n^-\gamma^j\eta\,\eta\gamma^{PL}\eta\,\eta\gamma^L M_n^+\gamma^Q\eta\,, \tag{3.25}$$

where

$$M_n^\pm = \frac{\omega_n}{n}1\pm\frac{m}{n}\Pi\,. \tag{3.26}$$

We will only consider the leading power of m in the $m\to\infty$ limit, which is relevant to the comparison with the gauge theory. The only non-zero matrix elements arise when the indices (P,Q) are in the first $SO(4)$ factor, so that $(P,Q)=(p,q)$. In this case (3.22) and (3.19) lead to (in the limit $m\to\infty$)

$$\begin{aligned}&\frac{n^2}{m^2}\int d^8\,\eta\,\eta\gamma^{iK}\eta\,\eta\gamma^K M_n^-\gamma^j\eta\,\eta\gamma^{pL}\eta\,\eta\gamma^L M_n^+\gamma^q\eta\\&\to(\mathcal{T}^+_{ikkj}\mathcal{T}^-_{pllq})+(9\mathcal{T}^-_{iklq}\mathcal{T}^+_{kjpl})=9(\delta^{pq}\delta^{ij})+18(\delta^{pi}\delta^{qj}+\delta^{pj}\delta^{iq}-\frac{1}{2}\delta^{pq}\delta^{ij})\,.\end{aligned}\tag{3.27}$$

This expression couples the symmetric traceless wave functions, $t^{(1)}_{(ij)_t}$ and $t^{(2)}_{(pq)_t}$ as well as the $SO(4)$ singlets $t^{(1)}_{(ii)}$ and $t^{(2)}_{(pp)}$. Matrix elements with $(I,J)=(i,j)$ and $(P,Q)=(p',q')$ vanish. Of course, there is an expression similar to (3.27) with primed indices replacing the unprimed ones.

To summarise, the matrix elements between states with two bosonic impurities with symmetrised indices in the same $SO(4)$ factor are proportional to

$$e^{2\pi i\tau}\,g_s^{7/2}\,m^4\,t^{(1)}_{ij}\,t^{(2)}_{pq}\,(\delta^{pi}\delta^{qj}+\delta^{pj}\delta^{iq})\,.\tag{3.28}$$

Mass matrix elements with $(I,J)=(i,j')$

The disk with four Q^-'s attached couples to the symmetric combination $(I,J)\to(i,j')\oplus(j',i)$. Explicitly, the η-dependence for disk 1 appears in the combination

$$\frac{1}{2}\eta\gamma^{iK}\eta\eta\gamma^K(1-\Pi)\gamma^{j'}\eta=\eta^+\gamma^{ik}\eta^+\eta^+\gamma^{kj'}\eta^-+\eta^-\gamma^{ik}\eta^-\eta^+\gamma^{kj'}\eta^-+2\eta^+\gamma^{ik'}\eta^-\eta^-\gamma^{k'j'}\eta^-\,,\tag{3.29}$$

which only has odd powers of η^+ and η^-. The analogous factor for disk 2 is

$$\begin{aligned}\frac{1}{2}\eta\gamma^{PL}\eta\eta\gamma^L(1+\Pi)\gamma^Q\;&=\eta^+\gamma^{pl}\eta^+\eta^+\gamma^{lq'}\eta^-+\eta^-\gamma^{pl}\eta^-\eta^-\gamma^{lq'}\eta^+\\&+2\eta^-\gamma^{pl'}\eta^+\eta^+\gamma^{l'q'}\eta^++2\eta^+\gamma^{p'l}\eta^-\eta^-\gamma^{lq}\eta^-\\&+\eta^+\gamma^{p'l'}\eta^+\eta^+\gamma^{l'q}\eta^-+\eta^-\gamma^{p'l'}\eta^-\eta^+\gamma^{l'q}\eta^-\,.\end{aligned}\tag{3.30}$$

Multiplying the expressions (3.29) and (3.30) gives three types of terms,

$$\begin{aligned}&\int d^4\eta^+d^4\eta^-\Big[(\eta^+\gamma^{ik}\eta^+\eta^-\gamma^{kj'}\eta^+\eta^+\gamma^{l'q}\eta^-\eta^-\gamma^{p'l'}\eta^-)\\&+(\eta^-\gamma^{ik}\eta^-\eta^+\gamma^{p'l'}\eta^+\eta^+\gamma^{kj'}\eta^-\eta^+\gamma^{l'q}\eta^-)\Big]=-\frac{1}{8}(\mathcal{T}^+_{ikkq}\mathcal{T}^-_{p'l'l'j'}+\mathcal{T}^-_{ikkq}\mathcal{T}^+_{p'l'l'j'})\,,\end{aligned}\tag{3.31}$$

$$2\int d^4\eta^+d^4\eta^-\eta^-\gamma^{ik}\eta^-\eta^+\gamma^{l'q'}\eta^+\eta^-\gamma^{kj'}\eta^-\eta^-\gamma^{pl'}\eta^+=-\frac{1}{4}\mathcal{T}^-_{ikkp}\mathcal{T}^+_{q'l'l'j'}\,,\tag{3.32}$$

$$2\int d^4\eta^+d^4\eta^-\eta^-\gamma^{k'j'}\eta^-\eta^+\gamma^{pl}\eta^+\eta^-\gamma^{ik'}\eta^-\eta^+\gamma^{lq'}\eta^-=-\frac{1}{4}\mathcal{T}^-_{j'k'k'q'}\mathcal{T}^+_{plli}\,.\tag{3.33}$$

It is straightforward to show from these expressions that the matrix elements of this type are of the form

$$e^{2\pi i\tau}\,g_s^{7/2}\,m^4\,t^{(1)}_{ij'}\,t^{(2)}_{p'q}\left(\delta_{iq}\delta_{p'j'}+\delta_{q'j'}\delta_{pi}\right)\,.\tag{3.34}$$

It is of interest to compare the above matrix elements with those deduced from $\mathcal{N}=4$ Yang–Mills theory in the BMN limit. The gauge theory parameters relevant to this limit are expressed in terms of those of string theory by the relations [1, 17]

$$\frac{J^2}{N} \equiv g_2 = 4\pi g_s m^2 \,, \qquad g^2_{YM}\frac{N}{J^2} \equiv \lambda' = \frac{1}{m^2}, \tag{3.35}$$

and the light-cone string theory hamiltonian is expressed in terms of J and Δ by $H^{(2)}/\mu = \Delta - J$. The presence of the D-instanton contribution to the two-particle hamiltonian in string theory therefore implies that there should be a corresponding contribution to the two-point function of the corresponding BMN operators in $\mathcal{N}=4$ Yang–Mills gauge theory of the form

$$e^{i\theta - 8\pi^2/g^2_{YM}}\, g_2^{7/2}\, \lambda'^2\,.$$

We will see in a separate paper [15] that this dependence on the coupling constant does indeed emerge from the gauge theory although it proves very difficult to evaluate the instanton contribution in detail in the two-impurity case[6].

3.1.2 Matrix elements of $SO(8)$ antisymmetric tensors in the $NS-NS$ sector

The two-string state describing strings in the antisymmetric $NS-NS$ representation is

$$\frac{1}{\omega_p\omega_q}\, t^{(1)}_{[IJ]}\, t^{(2)}_{[KL]}\, {}_h\langle 0|\alpha^{(1)I}_p\, \tilde\alpha^{(1)J}_p \otimes {}_h\langle 0|\alpha^{(2)K}_q\, \tilde\alpha^{(2)L}_q. \tag{3.36}$$

In this case the boundary state contribution comes from terms in (3.10) with $p=2$ and $p=6$, where two Q^-'s are distributed on one disk and six on the other. Specialising again to the case in which the vector indices in (3.36) lie in one of the $SO(4)$ subgroups of $SO(8)$ the matrix element turns out to be proportional to $(\mathcal{T}^+_{ijkl}+\mathcal{T}^-_{ijkl})$. To leading order in m it is proportional to

$$e^{2\pi i\tau}\, g_s^{7/2}\, m^4\, t^{(1)}_{ij} t^{(2)}_{kl}\, (\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\,, \tag{3.37}$$

where $t^{(1)}$ and $t^{(2)}$ are the antisymmetric tensor wave functions of the two states. The result (3.37) has the same dependence on the parameters as in the symmetric case considered earlier.

A similar result follows when the external states have vector indices in the other $SO(4)$.

It is also easy to see that there is no mixing of the $[i,j]$ states with the $[i',j']$ states. Furthermore a state with one index in each of the $SO(4)$ factors, $[i,j']$, only mixes with a similar state, again resulting in a dependence on the parameters of the form $e^{2\pi i\tau}\, g_s^{7/2}\, m^4$.

3.1.3 Matrix elements of $SO(8)$ singlets in the $NS-NS$ sector

As remarked earlier, the singlet $SO(8)$ representation is the direct sum of $SO(4)$ singlets in $SO(4)\times SO(4)$. This is denoted by $(ii)+(j'j')$ in table 1. In this case the wave functions are $t^{(1)}_{II}$ and $t^{(2)}_{II}$ and the result turns out to be proportional to $g_s^{7/2}m^2$. This is suppressed by a power of m^{-2} relative to the matrix elements of $(ii)-(j'j')$, which accounts for the earlier observation that this matrix element vanishes in the $m\to\infty$ limit. In this case the gauge theory result should be proportional to g^6_{YM} rather than g^4_{YM}.

[6] It turns out that in the gauge theory, the four-impurity case is under better control.

3.2 Matrix elements of excited RR states

We will only briefly consider matrix elements of pairs of states with two R impurities. For simplicity let us consider the $SO(4) \times SO(4)$ RR singlet that comes from the $SO(8)$ RR four-form. In this case the leading contribution arises when each disk has four Q^-'s. The dominant contribution in the large m limit arises when two Q^-'s act on each S and $\tilde{S}$ on each disk. Consider the case in which the external state on disk 1 (with $p_- > 0$) is of the form $S^+\tilde{S}^+$. This disk contributes at order m^3. Now consider the case that the external state on the disk 2 (with $p_- < 0$ so that $m < 0$) is of the form $S^+\tilde{S}^+$. This is suppressed by $1/m^2$ relative to the first disk and the combined power of m is m^4, as in the $NS - NS$ sector. Note that this is an example of a two-point function which also gets perturbative contributions, as do the $NS - NS$ two-point functions considered earlier. However, if the external state is of the form $S^-\tilde{S}^-$ the result is proportional to m^3 on each disk and the net power of m is m^6. In this example there are no perturbative contributions, which follows from the fact that $\langle S^+ S^- \rangle = 0$.

3.2.1 Other matrix elements of two-impurity states

The last example illustrates a general feature of states involving fermionic excitations, namely, that the two-point functions of states with a large number of fermionic excitations can have a high positive power of m. On the gauge theory side, this corresponds to a large negative power of λ'. Of course, since the instanton contributions under consideration have a prefactor of $e^{-8\pi^2/g_{YM}^2} = e^{-8\pi^2 N/\lambda' J^2}$ the $\lambda' \to 0$ limit is not divergent. We will see in [15] that this qualitatively matches the behaviour of the analogous Yang–Mills instanton contributions to the gauge theory in the BMN limit.

Fermionic states can be analysed in a similar manner. They require an odd number of Q^-'s on each disk.

4. Matrix elements between states with four bosonic impurities

In this section we consider external states made from four bosonic oscillators. It will turn out that the comparison of matrix elements for these states with the anomalous dimensions of corresponding operators in the gauge theory is under better control than in the case of two impurity operators. We will only consider the case in which all the vector indices on the bosonic oscillators are in one of the $SO(4)$ factors. In this case the two-particle state has the form

$$
\begin{aligned}
{}_1\langle\chi_1| \otimes {}_2\langle\chi_2| = & \frac{1}{\omega_r^2\omega_s^2}\, t^{(1)}_{j_1j_2j_3j_4}\, t^{(2)}_{p_1p_2p_3p_4} \\
& {}_h\langle 0|\alpha_r^{(1)j_1}\, \tilde{\alpha}_r^{(1)j_2}\, \alpha_s^{(1)j_3}\, \tilde{\alpha}_s^{(1)j_4} \otimes {}_h\langle 0|\alpha_r^{(2)p_1}\, \tilde{\alpha}_r^{(2)p_2}\, \alpha_s^{(2)p_3}\, \tilde{\alpha}_s^{(2)p_4}.
\end{aligned} \tag{4.1}
$$

Here, we have used the property of the D-instanton boundary state that the only non-zero matrix elements are those in which each α_p mode is accompanied by a $\tilde{\alpha}_p$ with the same mode number, p. Also, conservation of p_+ requires that the mode numbers of state (1) are the same as those of state (2).

To leading order in the $m \to \infty$ limit the contribution to matrix elements, ${}_1\langle\chi_1| \otimes {}_2\langle\chi_2|V_2\rangle\rangle$, is obtained by expanding the boundary state and retaining the term involving the product of four fermion bilinears on each disk. These contributions are dominant since the fermion bilinear in the exponent of the boundary state operator comes with an explicit factor of m in the large-m limit. Therefore, in order to get a non-vanishing overlap with states made of bosonic oscillators, there must be 4 Q^-'s on each disk with one broken supersymmetry acting on each of the fermions. To leading order in m this leads to the following instantonic contribution to $H^{(2)}$

$$t_{j_1\tilde{j}_2j_3\tilde{j}_4}t_{p_1\tilde{p}_2p_3\tilde{p}_4}e^{2\pi i\tau}g_s^{7/2}m^8\frac{1}{r^2s^2}$$
$$\int d^8\eta\,\eta\gamma^{j_1}(\mathbb{1}-\Pi)\gamma^{j_2}\eta\,\eta\gamma^{j_3}(\mathbb{1}-\Pi)\gamma^{j_4}\eta\,\eta\gamma^{p_1}(\mathbb{1}+\Pi)\gamma^{p_2}\eta\,\eta\gamma^{p_3}(\mathbb{1}+\Pi)\gamma^{p_4}\eta\,, \quad (4.2)$$

where the tilded index is associated with $\tilde{\alpha}$ and the untilded with α and there is level matching for the first and last pair of indices. Considering all the indices on the external wavefunctions to belong to the first $SO(4)$ yields for the second line in equation (4.2)

$$\int d^8\eta\,\eta^+\gamma^{j_1\tilde{j}_2}\eta^+\,\eta^+\gamma^{j_3\tilde{j}_4}\eta^+\,\eta^-\gamma^{p_1\tilde{p}_2}\eta^-\,\eta^-\gamma^{p_3\tilde{p}_4}\eta^-\,, \quad (4.3)$$

which using equation (3.19) reduces to

$$\begin{aligned}\mathcal{T}^+_{j_1\tilde{j}_2j_3\tilde{j}_4}\mathcal{T}^-_{p_1\tilde{p}_2p_3\tilde{p}_4} &= \left(\epsilon_{j_1\tilde{j}_2j_3\tilde{j}_4}+\delta_{j_1j_3}\delta_{\tilde{j}_2\tilde{j}_4}-\delta_{j_1\tilde{j}_4}\delta_{\tilde{j}_2j_3}\right)\\ &\quad\times\left(\epsilon_{p_1\tilde{p}_2p_3\tilde{p}_4}-\delta_{p_1p_3}\delta_{\tilde{p}_2\tilde{p}_4}+\delta_{p_1\tilde{p}_4}\delta_{\tilde{p}_2p_3}\right). \end{aligned} \quad (4.4)$$

Thus, the matrix elements of two states with four bosonic impurities (with all impurities in one of the $SO(4)$ factors) is given by

$$t_{j_1\tilde{j}_2j_3\tilde{j}_4}t_{p_1\tilde{p}_2p_3\tilde{p}_4}e^{2\pi i\tau}g_s^{7/2}m^8\frac{1}{r^2s^2}\,\mathcal{T}^+_{j_1\tilde{j}_2j_3\tilde{j}_4}\mathcal{T}^-_{p_1\tilde{p}_2p_3\tilde{p}_4}\,. \quad (4.5)$$

Note that the tensors $\mathcal{T}^\pm_{j_1j_2j_3j_4}=\pm\epsilon_{j_1j_2j_3j_4}+\delta_{j_1j_3}\delta_{j_2j_4}-\delta_{j_1j_4}\delta_{j_2j_3}$ are self-dual, *i.e.*,

$$\epsilon_{j_1j_2p_3p_4}\mathcal{T}^\pm_{j_1j_2j_3j_4}=\pm2\mathcal{T}^\pm_{p_3p_4j_3j_4}\,. \quad (4.6)$$

Recalling that $4\pi g_s m^2 = g_2 = J^2/N$, we see that the string result predicts that the corresponding four impurity operators on the gauge theory side should receive instanton contributions to anomalous dimensions at order $J^7/N^{7/2}$. The rest of the possible four-impurity states are suppressed by powers of m compared to this leading result. For example, if the external states are of the type ${}_h\langle 0|\alpha_r^{[j_1}\tilde{\alpha}_r^{j_2]}\alpha_s^{[j_3}\tilde{\alpha}_s^{j_4]}$, then the result will be proportional to m^4 rather than m^8. The corresponding gauge theory result would appear at order $J^3g_{YM}^4/N^{3/2}$.

5. Discussion

We have evaluated the leading single D-instanton contribution to the mass matrix elements of certain states of maximally supersymmetric plane-wave string theory. These are contributions that are exponentially suppressed by a factor $e^{-2\pi\tau_2}$, but are uniquely specified by the characteristic instanton phase, $e^{2\pi i\tau_1}$. Our specific results concern states with up to four impurities with non-zero mode numbers (a state with zero mode number is a protected supergravity state and has vanishing mass matrix element with all other states). The structure of the boundary state makes it obvious that the only states that couple with a D-instanton are those with an even number of impurities. Furthermore, when there are $4+2n$ impurities the only states that couple to the D-instanton are those in which n pairs form $SO(4) \times SO(4)$ singlets. We also saw in section 3.2 that mass matrix elements of states with a large number of fermionic excitations can have a high positive power of m (multiplying the factor of $e^{2\pi i\tau}$). This only arises for matrix elements which do not get any contributions in string perturbation theory.

It is of interest to see how the string mass matrix elements translate into statements concerning contributions of Yang–Mills instantons to the anomalous dimensions of states in the BMN limit of $\mathcal{N}=4$ superconformal gauge theory. Since we have not diagonalized the mass-matrix the best we can do is to compare the individual matrix elements. Furthermore, in order to remain in the perturbative regime of the string theory, $g_s \to 0$, as well as in the Yang–Mills theory, $\lambda' \to 0$, we have concentrated on the limit $m \to \infty$. To leading order as $m \to \infty$ the matrix elements between two-impurity states given in section 3 have the form $e^{i\theta - 8\pi^2/g_{YM}^2}\, g_2^{7/2}\, \lambda'^2$, when expressed in terms of the parameters of the Yang–Mills theory in the BMN limit. In particular, they are independent of the mode numbers of the states. In the case of four impurities the mass-matrix elements have the form $e^{i\theta - 8\pi^2/g_{YM}^2}\, g_2^{7/2}(rs)^{-2}$, where r and s are the two independent mode numbers that label either of the states. In a separate paper [15] we are examining if these dependences can be reproduced from Yang-Mills instanton contributions to anomalous dimensions of the corresponding four impurity operators.

However, the issue of whether there should be a precise match between the string theory and gauge theory calculations is called into question by recent perturbative calculations. An analysis in [4] suggests that the one-loop string calculation in the literature is incomplete and may be incorrect. This raises questions about the claimed precise match with the gauge theory analysis, which is also incomplete. Furthermore, perturbative calculations in 'near-BMN sectors', both in string theory [18] and in the dual gauge theory [19], have shown deviations from BMN scaling. Explicit tests show that in the strict BMN case scaling is respected up to three loops [20], with indications from a related matrix model calculation [21] that this may break down at four loops. In view of these issues in the perturbative sector, it is not obvious to what extent one should expect agreement between the non-perturbative effects considered in this paper with corresponding effects in $\mathcal{N}$=4 SYM. However, such a comparison should help to shed further light on the corresponence.

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A. World-sheet generators of plane-wave string theory

In this appendix we summarise the expressions for the generators of the plane-wave algebra in the representation furnished by string theory in the light-cone gauge [2].

In the light-cone gauge the 32 supersymmetries of the plane-wave background are described by 16 kinematic and 16 dynamical supersymmetries (only the latter receive corrections due to string interactions). The kinematical supersymmetries, which satisfy the anticommutation relations $\{q,\bar{q}\} = p_-$, are given by

$$q = \frac{e(p_-)}{2\sqrt{2}\pi}\int_0^{2\pi|p_-|} d\sigma (S + i\tilde{S}) \equiv \int_0^{2\pi|p_-|} d\sigma \frac{1}{2\sqrt{2}\pi}\,\bar{\theta}\,, \tag{A.1}$$

$$\bar{q} = \frac{1}{2\sqrt{2}\pi}\int_0^{2\pi|p_-|} d\sigma (S - i\tilde{S}) \equiv \int_0^{2\pi|p_-|} d\sigma \frac{e(p_-)}{2\sqrt{2}\pi}\,\theta\,. \tag{A.2}$$

The field S satisfies $\{S(\sigma), S(\sigma')\} = 2\pi\delta(\sigma - \sigma')$ with a similar relation for $\tilde{S}$. $e(p_-) = \mathrm{sign}(p_-)$. In (A.2) we have defined

$$e(p_-)\frac{S + i\tilde{S}}{2\sqrt{2}\pi} = \frac{\bar{\theta}}{2\pi\sqrt{2}}\,, \qquad e(p_-)\frac{S - i\tilde{S}}{2\sqrt{2}\pi} = \frac{\theta}{2\pi\sqrt{2}}\,. \tag{A.3}$$

The dynamical supercharges are defined as $Q^+ = \frac{1}{\sqrt{2}}(Q + i\tilde{Q})$ and $Q^- = \frac{1}{\sqrt{2}}(Q - i\tilde{Q})$ with

$$Q^+ = \int_0^{2\pi|p_-|} d\sigma \left[p^I\gamma_I\bar{\theta} - \frac{ie(p_-)}{4\pi}\partial_\sigma x^I\gamma_I\theta - \frac{ie(p_-)}{4\pi}\mu x^I\gamma_I\Pi\,\bar{\theta}\right]\,, \tag{A.4}$$

$$Q^- = \int_0^{2\pi|p_-|} d\sigma \left[e(p_-)p^I\gamma_I\theta + \frac{i}{4\pi}\partial_\sigma x^I\gamma_I\bar{\theta} + \frac{i}{4\pi}\mu x^I\gamma_I\Pi\,\theta\right]\,. \tag{A.5}$$

In terms of these variables, the first quantised hamiltonian is given by

$$\begin{aligned} h = {} & \frac{e(p_-)}{2}\int_0^{2\pi|p_-|} d\sigma \left[4\pi p^2 + \frac{1}{4\pi}((\partial_\sigma x)^2 + \mu^2 x^2)\right] \\ & + e(p_-)\left[-4\pi\lambda\partial_\sigma\lambda + \frac{1}{4\pi}\theta\partial_\sigma\theta + 2\mu(\lambda\Pi\theta)\right]\,. \end{aligned} \tag{A.6}$$

The various mode expansions are as follows

$$\begin{aligned} x(\sigma,\tau=0) &= x_0 + i\sum_{n\neq 0}\frac{1}{\omega_n}\left(e^{\frac{in\sigma}{|p_-|}}\alpha_n + e^{\frac{-in\sigma}{|p_-|}}\tilde{\alpha}_n\right)\,, \\ |p_-|p(\sigma,\tau=0) &= p_0 + \sum_{n\neq 0}\left(e^{\frac{in\sigma}{|p_-|}}\alpha_n + e^{\frac{-in\sigma}{|p_-|}}\tilde{\alpha}_n\right)\,, \\ \sqrt{|p_-|}S(\sigma,\tau=0) &= S_0 + \sum_{n\neq 0} c_n\left[S_n e^{\frac{in\sigma}{|p_-|}} + \frac{i}{m}(\omega_n - n)\Pi\tilde{S}_n e^{\frac{-in\sigma}{|p_-|}}\right]\,, \\ \sqrt{|p_-|}\tilde{S}(\sigma,\tau=0) &= \tilde{S}_0 + \sum_{n\neq 0} c_n\left[\tilde{S}_n e^{\frac{-in\sigma}{|p_-|}} - \frac{i}{m}(\omega_n - n)\Pi S_n e^{\frac{in\sigma}{|p_-|}}\right]\,, \end{aligned} \tag{A.7}$$

where $m = \mu p_-\alpha'$ and the non-zero commutation relations are given by

$$[\alpha_k, \alpha_l] = \omega_k\delta_{k,-l}\,, \quad \{S_k, S_l\} = \delta_{k,-l}\,, \tag{A.8}$$

with similar relations for the right-movers. The quantity c_n is defined by

$$c_n = \frac{m}{\sqrt{2\omega_n(\omega_n - n)}}\,. \tag{A.9}$$

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