Fortschr. Phys. **53**, No. 5 – 6, 610 – 614 (2005) / **DOI** 10.1002/prop.200510227

Three-loop test of the dilatation operator and integrability in $\mathcal{N} = 4$ SYM

B. Eden[1], **C. Jarczak**[2], and **E. Sokatchev**[2,*]

[1] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Golm, Germany

[2] Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, B.P. 110, 74941 Annecy-le-Vieux, France**

Received 16 January 2005, accepted 18 February 2005
Published online 27 April 2005

We report on a three-loop calculation of the anomalous dimension of the Konishi operator and of the BMN operator of dimension three. Our results confirm the unique form of the $\mathcal{N} = 4$ SYM dilatation operator compatible with BMN scaling.

1 Introduction

During the last couple of years, following the important work of BMN [1], spectacular progress has been made in the understanding of the integrable structure underlying $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [2–4], on the one hand, and its correspondence to string theory [5–7], on the other. An important part of this program is the development of efficient tools for calculating anomalous dimensions of gauge invariant composite operators and then comparing to the string theory predictions.

One such remarkable tool is the so-called "dilatation operator" of $\mathcal{N} = 4$ SYM [3,8]. In this approach one is able to predict the anomalous dimensions of large classes of operators. The form of the dilatation operator can be determined by combining various symmetry arguments with some field theory input. In [3] the notion of one-loop integrability was extended to higher loops. Based on this and on the additional assumption of BMN scaling, the planar dilatation operator of the $SU(2)$ sector of $\mathcal{N} = 4$ SYM was elaborated up to three loops. Later on, a comprehensive treatment of the dilatation operator for the $SU(2|3)$ sector was given in [9]. It was shown that the two-loop dilatation operator is determined up to one free constant, which can be fixed from the known value of the two-loop anomalous dimension of the Konishi operator. At three loops, after taking into account the known quantum symmetries, a two-parameter freedom remains. The higher-loop integrability conjectured in [3] was shown to follow from conformal supersymmetry and the dynamics of the theory, keeping the above mentioned parameters arbitrary.

In [9] Beisert used the extra assumption of BMN scaling to fix the remaining freedom. The so-called BMN operators are gauge invariant composite operators made out two complex scalars Z and Φ. They form an $SU(2)$ doublet (the hypermultiplet subset of the $SU(3)$ triplet of scalars of the $\mathcal{N} = 4$ theory); one of them (Z) is charged under $U(1) \subset SU(2)$, the other (Φ) is not. Schematically, a BMN operator of charge $J + 2$ and naive dimension $J + 4$ is given by

$$\mathcal{O}^J = \mathrm{Tr}\,(ZZ \cdots Z\Phi\Phi)$$

* Corresponding author E-mail: emeri.sokatchev@cern.ch
** UMR 5108 associée à l'Université de Savoie

Here one sees $J + 2$ factors of Z and two "impurities" Φ which can be placed in different ways among the Zs. In the quantum theory such operators acquire anomalous dimension $\gamma(g, N, J)$ where g is the gauge coupling and N determines the rang of the gauge group $SU(N)$. String theory arguments lead to a prediction about the anomalous dimension in the so-called BMN (double scaling) limit

$$J \to \infty, \qquad N \to \infty, \qquad \frac{J^2}{N} \text{ fixed}$$

Introduced the coupling

$$\lambda' = \frac{g^2 N}{J^2}$$

BMN predicted that $\gamma_{\mathcal{O}^J}$ must be an analytic function of λ' around $\lambda' = 0$. This means that $\gamma_{\mathcal{O}^J}$ admits a power expansion in λ' and hence in g^2. Thus, the theory enters the perturbative regime and it becomes possible to test the string predictions by quantum field theory means.

In this context, "experimental data" in the form of the three-loop anomalous dimensions of any two BMN operators, obtained by a direct perturbative calculation, would unambiguously determine the three-loop dilatation operator. This would also provide an indirect check on the hypothesis of BMN scaling at this level of perturbation theory.

Thus, the necessity of such a three-loop perturbative "experiment" appears quite clear. Rather recently, a whole series of three-loop anomalous dimensions of operators of twist two were computed in [10, 11], using sophisticated QCD methods. The simplest among these operators is the Konishi scalar, which is also the lowest state in the BMN series. The value of its anomalous dimension obtained in [11] is in perfect agreement with the prediction of [3, 9] under the assumption of BMN scaling. However, it is not clear if the method of [10, 11] generalizes to operators of higher twist, such as the scalar BMN operators with non-vanishing charge.

In this contribution we report on a three-loop calculation of the anomalous dimension [12] of the first two operators in the BMN series (with two impurities), the Konishi one (i.e., twist two or $J = 0$) and the twist three (or $J = 1$) one. In the Konishi case we reproduce the result of [11]. The result for the $J = 1$ case is new. Most remarkably, the two values

$$J = 0 : \gamma_3 = \frac{21}{4} \frac{N^3}{(4\pi^2)^3}$$

$$J = 1 : \gamma_3 = \frac{17}{8} \frac{N^3}{(4\pi^2)^3}$$

are exactly those predicted by the dilatation operator whose form is compatible with BMN scaling!

2 The method of the three-loop calculation

The idea of the method, first proposed in [13] and successfully used at two loops in [14] consists in reducing the $O(g^6)$ computation to $O(g^4)$ by means of (super)space differentiation. Consider the simplest example of the Konishi "current". Its conformal two-point function has the form

$$\langle\, k_\mu(x)\, k^\dagger_\nu(0)\,\rangle \;=\; c(g^2)\, \frac{x^2\eta_{\mu\nu} - 2x_\mu x_\nu}{(x^2)^{4+\gamma(g^2)}}\,. \tag{1}$$

Here $\gamma(g^2) = \gamma_1 g^2 + \gamma_2 g^4 + \ldots$ is the anomalous dimension. Its perturbative origin is in the ultraviolet divergences of this two-point function. In the free field theory, where $g = 0$ and $\gamma(g^2) = 0$, this current is conserved, $\partial^\mu k_\mu = 0$. However, the classical interaction violates this conservation, $\partial^\mu k_\mu = gb(x)$ where b is often called the "classical anomaly". Differentiating (1) we obtain

$$\langle\, \partial^\mu k_\mu(x)\, \partial^\nu k^\dagger_\nu(0)\,\rangle_{g\neq 0} \;=\; -8\, c(g^2)\, \frac{\gamma(\gamma+2)}{(x^2)^{4+\gamma(g^2)}}\,. \tag{2}$$

Now, consider the ratio

$$\frac{\langle\, \partial^\mu k_\mu(x)\, \partial^\nu k^\dagger_\nu(0)\,\rangle}{\langle\, k^\rho(x)\, k^\dagger_\rho(0)\,\rangle} \;=\; -4\,\frac{\gamma(\gamma+2)}{x^2} \sim O(g^2)\,. \tag{3}$$

Anselmi computed the *one-loop* γ_1 from the ratio of two *tree level* two-point functions

$$\frac{g^2\langle\, b\, b\,\rangle_{g^0}}{\langle\, k^\rho\, k^\dagger_\rho\,\rangle_{g^0}} \;=\; -4\,\frac{g^2\gamma_1}{x^2} + O(g^4)\,. \tag{4}$$

Trying to generalize this trick to higher loops, one encounters a problem. The Konishi current has not only a “classical", but also a “quantum" anomaly. This is best understood in the context of the $\mathcal{N}=4$ SYM theory formulated in terms of $\mathcal{N}=1$ superfields. The matter is described by chiral superfields $\Phi^I = \phi^I(x) + \theta^\alpha \psi^I_\alpha(x)$ $(I=1,2,3)$, $\bar\nabla_{\dot\alpha}\Phi^I = 0$ satisfying the field equations

$$\bar\nabla_{\dot\alpha}\bar\nabla^{\dot\alpha}\bar\Phi_I = g\epsilon_{IJK}[\Phi^J,\Phi^K] \tag{5}$$

The $\mathcal{N}=1$ Konishi multiplet is defined by $\mathcal{K} = \mathrm{Tr}\left(e^{gV}\bar\Phi_I e^{-gV}\Phi^I\right) = \ldots + \theta\sigma^\mu\bar\theta k_\mu(x) + \ldots$ (V is the gauge potential). It is a scalar superfield of dimension two containing the Konishi current $k_\mu(x)$. The superfield analog of the non-conservation $\partial^\mu k_\mu = gb(x)$ is

$$\bar D_{\dot\alpha}\bar D^{\dot\alpha}K = g\epsilon_{IJK}\mathrm{Tr}\left(\Phi^I[\Phi^J,\Phi^K]\right) \equiv gB \tag{6}$$

Thus, the “super-Anselmi trick" becomes

$$\frac{\langle\, \bar D^2 K\; D^2 K\,\rangle}{\langle\, K\, K\,\rangle} \;=\; \frac{\gamma(\gamma+2)}{x^2} + \theta\ \text{terms}\,. \tag{7}$$

Now, the problem is that the classical equation $\bar D^2 K = gB$ gets modified by the so-called “quantum Konishi anomaly" [15]. In the quantum theory one should write

$$Z_K\bar D^2 K = g(Z_B B + Z_F F) \equiv gK_{10} \tag{8}$$

where $F = \mathrm{Tr}(W^\alpha W_\alpha)$ and $W_\alpha = \lambda_\alpha(x + \theta^\beta(\sigma_{\mu\nu})_{\beta\alpha}F^{\mu\nu}(x)$ is the chiral SYM field strength. The constants Z are renormalization and mixing factors which can be determined by perturbative calculations. For instance, at one loop

$$Z_K = Z_B = 1 + \frac{g^2}{\epsilon}\gamma_1\,, \qquad Z_F = \frac{Ng}{32\pi^2} \tag{9}$$

and Z_F is the“quantum Konishi anomaly". One can view Eq. (8) as the definition of a new operator K_{10}, a mixture of the bare operators B and F. It is a member (“descendant") of the $\mathcal{N}=4$ SYM Konishi supermultiplet belonging to a $\underline{10}$ of $SU(4)$.

So, the correct quantum version of Eq. (7) is

$$\frac{\langle\, K_{10}\; K^\dagger_{10}\,\rangle}{\langle\, K\, K\,\rangle} \;=\; \frac{\gamma(\gamma+2)}{x^2} + \theta\ \text{terms}\,. \tag{10}$$

In this equation it is essential to know the finite normalizations of the quantum operators K and K_{10} which are not easy to find out in the perturbative theory. A way out was proposed in [14]. It consists in considering the chain of $\mathcal{N}=4$ descendants

$$K \;\to\; K_{10} \;\to\; K_{84} \tag{11}$$

The first step is done by an off-shell $\mathcal{N} = 1$ supersymmetry transformation and it creates an anomaly; the second descendant is obtained by an on-shell $\mathcal{N} = 4$ transformation which is not anomalous. The reason is that this second step transforms both $B \to gY$ and $F \to g^2Y$ into the same single-trace dimension four operator $Y = \mathrm{Tr}\,\big([\Phi^1, \Phi^2][\Phi^1, \Phi^2]\big)$:

$$K_{10} \;\to\; K_{84} \equiv Z_B Y + g Z_F Y \tag{12}$$

Apart from K_{10}, the bare operators B and F form another, “protected" (or “short") combination:

$$O_{10} = F - gB \;\to\; g^2Y - g^2Y = 0 \tag{13}$$

This operator is a descendant of the so-called stress-tensor multiplet, the BPS (or CPO) operator $O_{20'} = \mathrm{Tr}\Phi^1\Phi^1$. It receives no quantum corrections and hence has no anomalous dimension,

$$\langle\, O\, O^\dagger \,\rangle = \langle\, O\, O^\dagger \,\rangle_{g=0} \tag{14}$$

As a consequence, the operators K_{10} and O_{10} must be orthogonal,

$$\langle\, K_{10}\, O_{10}^\dagger \,\rangle = \langle\, Z_B B + Z_F F|\, (F - gB)^\dagger \,\rangle = 0 \tag{15}$$

This condition fixes the mixing coefficients $Z_{B,F}$ up to a finite overall normalization.

The crucial point now is that in the ratio

$$\frac{\langle\, K_{84}\, K_{84}^\dagger \,\rangle}{\langle\, K_{10}\, K_{10}^\dagger \,\rangle} \sim 2\gamma_1 + (\gamma_1^2 + 2\gamma_2)g^2 + 2(\gamma_1\gamma_2 + \gamma_3)g^4 + O(g^6) \tag{16}$$

the finite normalization drops out. Further, from (16) it is clear that we only need the two-point functions up to order g^4. Thus, we have achieved the reduction “3 loops → 2 loops". This “2 loop" calculation can be further simplified by exploiting the similarity between the Konishi descendant K_{84} and another chiral operator, the protected primary CPO O_{105}, which leads to a massive cancellation of graphs. The most serious technical problem is the evaluation of a 4-loop momentum space integral[1] in dimensional regularization which was done by D. Kazakov [16].

Acknowledgements ES would like to thank the organizers of the Ahrensoop 2004 meeting for the kind invitation.

References

[1] D. Berenstein, J. M. Maldacena, and H. Nastase, JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].
[2] J.A. Minahan and K. Zarembo, JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].
[3] N. Beisert, C. Kristjansen, and M. Staudacher, Nucl. Phys. B **664**, 131 (2003) [arXiv:hep-th/0303060].
[4] N. Beisert and M. Staudacher, Nucl. Phys. B **670**, 439 (2003) [arXiv:hep-th/0307042].
[5] S. Frolov and A. A. Tseytlin, JHEP **0206**, 007 (2002) [arXiv:hep-th/0204226].
[6] N. Beisert, J.A. Minahan, M. Staudacher, and K. Zarembo, JHEP **0309**, 010 (2003) [arXiv:hep-th/0306139].
[7] G. Arutyunov and M. Staudacher, JHEP **0403**, 004 (2004) [arXiv:hep-th/0310182]; arXiv:hep-th/0403077.
[8] N. Beisert, C. Kristjansen, J. Plefka, and M. Staudacher, Phys. Lett. B **558**, 229 (2003) [arXiv:hep-th/0212269].
[9] N. Beisert, Nucl. Phys. B **682**, 487 (2004) [arXiv:hep-th/0310252].
[10] A. Vogt, S. Moch, and J.A. M. Vermaseren, Nucl. Phys. B **688**, 101 (2004) [arXiv:hep-ph/0403192].
[11] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, and V.N. Velizhanin, Phys. Lett. B **595**, 521 (2004) [arXiv:hep-th/0404092].

[1] Note the mismatch in the usual loop counting and the perturbative order: The two-point functions in (16) are of order g^4 but they involve 4-loop integrals

[12] B. Eden, C. Jarczak, and E. Sokatchev, arXiv:hep-th/0409009.
[13] D. Anselmi, Nucl. Phys. B **541**, 369 (1999) [arXiv:hep-th/9809192].
[14] B. Eden, Nucl. Phys. B **681**, 195 (2004) [arXiv:hep-th/0307081].
[15] T. E. Clark, O. Piguet, and K. Sibold, Nucl. Phys. B **143**, 445 (1978); K. Konishi, Phys. Lett. B **135**, 439 (1984).
[16] D. Kazakov, private communication.