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# Spherically Symmetric Quantum Horizons

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Isolated horizon conditions specialized to spherical symmetry can be imposed directly at the quantum level. This answers several questions concerning horizon degrees of freedom, which are seen to be related to orientation, and its fluctuations at the kinematical as well as dynamical level. In particular, in the absence of scalar or fermionic matter the horizon area is an approximate quantum observable. Including different kinds of matter fields allows to probe several aspects of the Hamiltonian constraint of quantum geometry that are important in inhomogeneous situations.

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Black holes are the main objects, besides cosmology, where the interface between classical gravity and expected properties of quantum gravity is strongest. Not only are they classically singular, which has to be resolved by quantum gravity [1], but also their horizons, which for massive black holes are far away from the strong curvature region around the singularity, have provided several puzzles that have been motivations for quantum gravity developments over several decades. Most influential was the observation that an entropy can be associated with a black hole horizon whose microscopic degrees of freedom cannot be accounted for classically. This problem has been solved, in the most general case of astrophysically relevant black holes, by detailed calculations in quantum geometry [2]. Isolating the relevant microscopic degrees of freedom responsible for black hole entropy was possible only by using new developments which replace the concept of an event horizon by the quasilocal definition of an isolated horizon [3]. A surface intersecting the isolated horizon was then used as an inner spatial boundary which carries degrees of freedom describing the horizon geometry that are matched to the bulk quantum geometry. The matching is non-trivial and provides a consistency test of the methods of quantum geometry.

There have been other expectations concerning horizons in addition to the fact that they should carry microscopic degrees of freedom. Partly motivated from possible explanations of the entropy, it has been suggested that the horizon in quantum theory would not be a sharp surface but would fluctuate. Also other aspects of quantum horizons, which cannot be tested when the horizon is introduced as a boundary, are of interest and necessary for a complete understanding of effects such as Hawking radiation. For this reason, one would like to 'quantize' the isolated (or even dynamical) horizon conditions and impose them on states at the quantum level. This is what we will do, in a first step, in this article. Our analysis is complementary to that in [2] in that we use the same ingredients — isolated horizons and quantum geometry — but impose all the horizon conditions at the quantum rather than some of them at the classical level. Since this, at the current stage of developments, is too complicated to do in the full theory, we do the analysis in spherical symmetry. Even though in this case the full machinery of isolated horizons would not seem to be necessary classically, we will see that those conditions are important to decide how a horizon is to be found at the quantum level. Despite the classical simplicity of the Schwarzschild solution, we are able to learn much about the quantum horizon structure such as the localization of the horizon, its degrees of freedom, and its area as an observable.

*Isolated horizon conditions.* There are three main parts to the definition of an isolated horizon $\Delta$ with spatial sections $S \cong S^2$ of given area $a_0$ [4, 5]:

i) The canonical fields $(A_a^i, E_i^a)$ on the horizon are completely described by a single field $W = \frac{1}{2}\iota^* A^i r_i$ on $S$ which is a U(1)-connection obtained from the pull-back of the Ashtekar connection to $S$. Here, $\iota\colon S \to \Sigma$ is the embedding of the horizon section $S$ into a spatial slice $\Sigma$ and $r_i$ an internal direction on the horizon chosen such that $W$ is a connection in the spin bundle on $S^2$ and $r^i E_i^a = \sqrt{\det q}\, r^a$ on the horizon with the internal metric $q$ on $S$ and the outward normal $r^a$ to $S$ in $\Sigma$.

ii) The intrinsic horizon geometry, given by the pull-back of the 2-form $\Sigma_{ab}^i := \epsilon_{abc} E_i^c$ to $S$, is determined by the curvature $F = \mathrm{d}W$ of $W$ by

$$F = -\frac{2\pi}{a_0}\iota^*\Sigma^i r_i\,. \tag{1}$$

iii) The constraints hold on $S$.

When the horizon is introduced as a boundary, condition i) is used to identify the horizon degrees of freedom represented by the field $W$. Condition ii) then shows that these degrees of freedom are fields of a Chern–Simons theory on the horizon. It is the main condition since it relates the horizon degrees of freedom to the bulk geometry, which after quantization selects the relevant quantum states to be counted. Condition iii), on the other hand, does not play a big role since an isolated horizon as boundary implies a vanishing lapse function on $S$ for the Hamiltonian constraint which then is to be imposed only in the bulk.

When we impose the conditions at the quantum level it is clear that the procedure will be very different. We will not be able to have an independent boundary theory which is then matched to the bulk, but would have to find the relevant degrees of freedom within the original quantum theory. More importantly, we cannot use the simplification of a vanishing lapse function since the horizon will no longer be regarded as a boundary. In particular, the Hamiltonian constraint will have to be imposed which in full generality is a daunting task. For this reason we specialize the situation to spherical symmetry which presents the simplest situation where horizons can occur.

*Spherical symmetry.* Spherically symmetric connections and densitized vector fields are given by

$$\begin{aligned} A &= A_x\tau_3\mathrm{d}x + A_\varphi\Lambda^A_\vartheta\mathrm{d}\vartheta + A_\varphi\Lambda^A_\varphi\sin\vartheta\mathrm{d}\varphi + \tau_3\cos\vartheta\mathrm{d}\varphi \\ E &= E^x\tau_3\sin\vartheta\partial_x + E^\varphi\Lambda^\vartheta_E\sin\vartheta\partial_\vartheta + E^\varphi\Lambda^\varphi_E\partial_\varphi \end{aligned} \quad (2)$$

in polar coordinates $(x,\vartheta,\varphi)$. We use generators $\tau_j = -\frac{i}{2}\sigma_j$ and choose a gauge in which the radial components are along $\tau_3$. The angular components then are in the $\tau_1$-$\tau_2$ plane and given by $\Lambda^A_\varphi = \cos\beta\tau_1+\sin\beta\tau_2$, $\Lambda^\varphi_E = \cos(\alpha+\beta)\tau_1+\sin(\alpha+\beta)\tau_2$, and $\Lambda^A_\vartheta$ and $\Lambda^\vartheta_E$ given by rotating the internal $\varphi$-directions by ninety degrees around $\tau_3$ (see [6] for details). All fields $A_x$, $E^x$, $A_\varphi$, $E^\varphi$, $\alpha$ and $\beta$ depend only on $x$, where $\beta$ is pure gauge.

To evaluate the isolated horizon conditions, we choose $r_i := \mathrm{sgn}(E^x)\delta_{i,3}$ such that in fact $r^iE^a_i = |E^x|\sin\vartheta\partial_x$ with the intrinsic horizon area element $|E^x|\sin\vartheta$. Thus, $W=\frac{1}{2}r_i\iota^*A^i = \frac{1}{2}\,\mathrm{sgn}(E^x)\cos\vartheta\mathrm{d}\varphi$ whose integrated curvature given by $\oint_S \mathrm{d}W = -2\pi\,\mathrm{sgn}(E^x)$ agrees with the Chern number of the spin bundle, depending on the orientation given by $\mathrm{sgn}(E^x)$.

Evaluating (1) first shows that in the spherically symmetric context it is not restrictive since we have $a_0 = 4\pi|E^x|$ and the right hand side given by $-\frac{1}{2}\,\mathrm{sgn}(E^x)$ equals $F$ for all $E$. This is not surprising since the spherically symmetric intrinsic geometry of $S$ is already given by the total area which is fixed from the outset. Now the first condition plays a major role. A further consequence of the isolated horizon conditions [5] is that the curvature $\mathcal{F}$ of the pull-back of $A^i_a$ to $S$ is given by the curvature of $W$: $r_i\mathcal{F}(\iota^*A^i) = 2\mathrm{d}W$. Since $\mathcal{F}(\iota^*A) = (A_\varphi^2-1)\tau_3\sin\vartheta\mathrm{d}\vartheta\wedge\mathrm{d}\varphi$, the condition requires $A_\varphi = 0$ which will be the main restriction we have to impose on quantum states in addition to the constraints.

This condition $A_\varphi=0$ selects 2-spheres in a spherically symmetric space-time corresponding to cross-sections of a horizon. Indeed, for the Schwarzschild solution we have $A_\varphi = \Gamma_\varphi$ since the extrinsic curvature vanishes. Moreover, computing the spin connection for a spherically symmetric co-triad $e_a$ yields the component $\Gamma_\varphi = e'_\varphi/e_x$ which for the Schwarzschild solution ($e_\varphi = x$, $e_x = |1-2M/x|^{-1/2}$) yields the correct condition $x=2M$.

*Quantization.* Gauge invariant states of spherically symmetric quantum geometry in the connection representation are given by [6]

$$T_{g,k,\mu} = \prod_e \exp\left(\tfrac{1}{2}ik_e \textstyle\int_e (A_x+\beta')\mathrm{d}x\right)\prod_v \exp(i\mu_v A_\varphi(v)) \quad (3)$$

where $g$ is a graph in the radial manifold with edges $e$ and vertices $v$ labeled by $k_e\in\mathbb{Z}$ and $\mu_v\in\mathbb{R}$. Connection components act as multiplication operators, while spatial geometry is encoded in the derivative operators

$$\hat{E}^x(x)T_{g,k,\mu} = \frac{\gamma\ell_{\mathrm{P}}^2}{8\pi}\frac{k_{e^+(x)}+k_{e^-(x)}}{2}T_{g,k,\mu} \quad (4)$$

$$\int_{\mathcal{I}}\hat{P}^\varphi T_{g,k,\mu} = \frac{\gamma\ell_{\mathrm{P}}^2}{4\pi}\sum_{v\in\mathcal{I}}\mu_v T_{g,k,\mu} \quad (5)$$

where the momentum $P^\varphi = 2E^\varphi\cos\alpha$ of $A_\varphi$ is integrated over intervals $\mathcal{I}$ in the radial manifold since it is a density. Here, $e^+(v)$ and $e^-(v)$ are the edges neighboring the vertex $v$ at larger and smaller $x$, respectively.

Geometrical operators can be obtained from the derivative operators. The area operator [7] for a sphere $S$ is simply proportional to (4): $\hat{A}(S) = 4\pi|\hat{E}^x(S)|$. The volume operator is more complicated since the volume element depends on $E^\varphi$ which is a rather complicated function of $P^\varphi$ and $\alpha$. Nevertheless, it can be quantized and its full spectrum is known [8]. Just as in the full theory, its action is non-zero only in vertices and its spectrum is discrete.

*Quantum horizons.* We are now ready to impose the isolated horizon condition on spherically symmetric states and to draw conclusions for the quantum structure of horizons. As derived above, the main condition is $A_\varphi = 0$ which can only be satisfied at a vertex. If the condition is required strictly at a vertex $S$, the $A_\varphi$-dependence at $S$ must be a delta function $\delta(A_\varphi) = \sum_\mu \exp(i\mu A_\varphi)$ which defines the expansion in spin network states (3). Such a state is not normalizable in the kinematical inner product, and it is not known what the situation would be for the physical inner product. Fortunately, we can proceed without a detailed knowledge of the state and just use the fact that a state having a quantum horizon at a vertex $S$ is peaked on small values of $A_\varphi(S)$. From semiclassical considerations, which must hold at the horizons of large black holes, it follows that $A_\varphi=0$ cannot be imposed sharply, for otherwise $P^\varphi$ and the volume of regions around the horizon would not behave classically. The diffeomorphism constraint, which just acts by moving the vertices along the radial manifold, can be averaged as usually without changing the structure of the horizon vertex. We will discuss the Hamiltonian constraint later.

It is immediate to see that fixing the horizon area, as usually done in considerations of black hole horizon properties and their thermodynamics, is consistent also at the kinematical quantum level. Imposing the horizon condition just restricts the $A_\varphi$-dependence at the vertex, but leaves even the neighboring edges and labels completely free. We can thus assume that our state is an eigenstate of the area operator at $S$ and satisfies the horizon condition there. Classically, this corresponds to the fact that $A_\varphi$ and $E^x$ have vanishing Poisson bracket.

The situation is different for the volume since it depends on $P^\varphi$. In fact, volume eigenstates have an $A_\varphi$-dependence in vertices given by Legendre functions [8] which is incompatible with the horizon condition. Thus, the volume of shells around the horizon will not be sharp in quantum gravity.

This observation allows to answer the question whether the horizon will be a sharply localized surface. Looking at the state, the horizon will be localized at a sharply defined vertex $S$, but this just refers to localization in the background manifold. Moreover, after solving the diffeomorphism constraint by group averaging the position of the vertex will not be defined at all. As for physical localization, we have to refer to a suitable measurement process. This can easily be done by measuring the radial distance from an observer in an asymptotic region at large $x$ to the horizon. The radial distance is obtained by integrating the volume element divided by the area element of spheres along the radial manifold, which is easily quantized to an operator which has the same eigenstates as the volume and area operator and acts as $\hat{L}(R)T_g = \sum_{v\in R\cap V(g)} V_v/A_v T_g$ where the sum is over vertices of the graph $g$ in the region $R$ between the horizon and the observer, and $V_v$ and $A_v$ are the volume and area eigenvalues, respectively, in a vertex $v$.

We now assume that the geometry in the asymptotic region up to regions close to the horizon is semiclassical. Vertices outside the horizon will then yield sharp contributions to the distance. But at the horizon itself the volume cannot be sharply defined and thus the location of the horizon itself as measured from outside is unsharp, confirming older expectations of a fluctuating horizon corresponding to a smeared-out region. Note that this occurs in a way which is consistent with treatments of the horizon as a sharp boundary as in entropy calculations. The boundary refers to the background manifold according to which the location is indeed sharp (at the boundary only tangential diffeomorphisms generate gauge transformations).

While the structure of the horizon can well be analyzed in spherical symmetry, the symmetry is too strong to preserve the microscopic degrees of freedom. Detailed calculations of [5] show that horizon degrees of freedom are given by a U(1)-connection $W$, which in spherical symmetry does not have free components. In fact, the intrinsic geometry induced on a non-rotating isolated horizon as a boundary by bulk quantum geometry is not spherically symmetric but characterized by a finite set of punctures which endow the 2-sphere with area. Thus, there are many more configurations describing a non-rotating horizon than a spherically symmetric one. This has also been indicated by attempts to derive black hole entropy from reduced phase space quantizations, which have to introduce degeneracies by additional arguments.

Here we see that there is only one binary degree of freedom to a spherically symmetric isolated horizon, given by $\mathrm{sgn}(E^x)$. All other components of the fields on the horizon are fixed either by the required area $a_0$ or by the horizon condition $A_\varphi = 0$. The situation remains the same in quantum theory: there are no new quantum degrees of freedom. In particular, the reduction to spherical symmetry removes almost all degrees of freedom counted in the black hole entropy calculations of [2].

Still, there are surprising similarities to earlier considerations in quantum gravity. First, there is one binary degree of freedom for an exactly spherically symmetric horizon. If one imagines a non-spherical horizon to be approximated as composed of spherically symmetric patches this agrees with Wheeler's 'It from Bit' picture [9] (which has been generalized in the full calculation [10]). Moreover, the binary degree of freedom is given by $\mathrm{sgn}(E^x) = \mathrm{sgn}\det E^a_i$ which determines the orientation of geometry at the horizon. This confirms the ideas of [11] where the orientation of patches has been proposed to provide gravitational horizon degrees of freedom. Note that these are indeed *horizon degrees of freedom* since for an arbitrary surface the pull back $A_\varphi$ of $A$ would provide a further, continuous parameter.

*Dynamics.* We have seen that from the kinematical point of view the horizon area even of a quantum black hole can be fixed without contradicting the horizon conditions. However, the dynamical point of view is more complicated since now, with the horizon not being a boundary, the Hamiltonian constraint is non-trivial. We have to check whether the area of an isolated horizon commutes with the Hamiltonian constraint at least approximately. Even using simplifications due to the symmetry the constraint is quite lengthy, consisting of several terms. As in the full theory [12] it is built from holonomies of $A$-components in (2) some of which appear in commutators with the volume operator $\hat{V}$.

Fortunately, the full expression simplifies under the isolated horizon condition $A_\varphi(S) = 0$. Terms with $\sin A_\varphi$, which appears in holonomies, then annihilate states on which the horizon condition is imposed sharply. As discussed above, the condition will not be sharp in general, but still angular holonomies acting on a state can be ignored compared to radial holonomies which depend on the unrestricted $A_x$. Remaining terms are then of the form

$$\hat{H} \sim \sin(\tfrac{1}{2}\textstyle\int A_x)\hat{V}\cos(\tfrac{1}{2}\int A_x) - \cos(\tfrac{1}{2}\int A_x)\hat{V}\sin(\tfrac{1}{2}\int A_x) \tag{6}$$

where we wrote just one edge holonomy.

First, we can observe that the approximated constraint does not change the $A_\varphi$-dependence of a state and thus preserves the horizon condition. With this result, we can then check if also the horizon area is preserved, as expected from the classical vacuum behavior. (Moreover, the Euclidean analysis of [13] shows that $\sqrt{|E^x|}(1-A_\varphi^2)$ is constant along the radial line and an observable proportional to the ADM mass. At the horizon where $A_\varphi=0$ this specializes to $E^x$.) Area is given by the labels $k_e$ in (3), which are changed by radial holonomies appearing in (6). However, as in [14] it can easily be checked that they appear only in combinations which leave the area eigenvalue invariant. Thus, the area operator commutes with (6) which, taking into account that we ignored terms using the horizon condition, implies that *the horizon area is an approximate quantum observable of spherically symmetric vacuum gravity.* Even though expected classically, this result about the *quantum observable* is non-trivial and depends on aspects of the Hamiltonian constraint operator. Using all terms in the constraint shows that some of the neglected ones change the area, such that the *quantum horizon area fluctuates dynamically.*

When a cosmological constant or an electromagnetic Hamiltonian [15] is added there are no new area-changing terms and the result still holds true, again agreeing with classical expectations. Coupling scalar or fermionic matter can introduce terms which change the area eigenvalue such that here the horizon can grow or shrink.

*Conclusions.* At first sight, there apparently are several possibilities to implement horizon conditions in spherically symmetric situations. For instance, one can try to quantize $x=2M$ by using the area operator for $x$ and the ADM mass for $M$. The drawbacks are that this requires a mass operator (which, for instance, would be sensitive to asymptotic flatness conditions) and that one part of the condition would be quantized at the horizon, the other at infinity. Thus, one would need a complete solution to the constraint in order to connect both parts of the condition. (Expressions for the horizon mass provided by the isolated horizon framework would work locally but make the condition an identity.) Moreover, this procedure would not work with matter.

The isolated horizon framework provides an unambiguous condition which is local at the horizon. This makes it possible to impose the condition without full knowledge of physical solutions, which to our knowledge results in the first *implementation of horizon conditions fully at the quantum level.* It is this isolated horizon condition which leads to strong simplifications in the quantum Hamiltonian constraint exploited here.

Our results verify some of the earlier expectations concerning fluctuating horizons and make them more detailed. Moreover, we can show that the horizon area is an approximate quantum observable in the sense that it commutes with the dominant contribution to the Hamiltonian constraint. These calculations test several aspects of the constraint operator, in particular those which did not play a role in homogeneous models [14, 16, 17]. As we have seen, going to the horizon simplifies the analysis of some aspects of quantum observables since a horizon is much easier to impose on quantum states than an asymptotic regime where one could test the ADM mass.

The framework introduced here allows, e.g., to answer questions related to black hole evaporation [1]. There are several new possibilities not yet studied when matter Hamiltonians are coupled: First, the horizon conditions need to be generalized to dynamical horizons [18], and whether or not the Hamiltonian constraint will again simplify at the horizon depends on the precise form of the conditions. A detailed analysis of the general situation is yet to be undertaken, but at least for slowly evolving horizons [19] $A_\varphi$ will be small: a horizon slowly evolving at a rate $\epsilon$ (as defined in [19]) has $A_\varphi\sim\epsilon$. Similar simplifications as used here will then remain to hold true approximately in the slowly evolving case which opens the prospect to investigate how quantum horizons grow when matter falls in and shrink from Hawking radiation.

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[1] A. Ashtekar and M. Bojowald, in preparation.
[2] A. Ashtekar, J. C. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998); A. Ashtekar, J. C. Baez, and K. Krasnov, Adv. Theor. Math. Phys. **4**, 1 (2001).
[3] A. Ashtekar and B. Krishnan, gr-qc/0407042.
[4] A. Ashtekar and J. Lewandowski, Class. Quantum Grav. **21**, R53 (2004).
[5] A. Ashtekar, A. Corichi, and K. Krasnov, Adv. Theor. Math. Phys. **3**, 419 (2000).
[6] M. Bojowald, Class. Quantum Grav. **21**, 3733 (2004).
[7] M. Bojowald and H. A. Kastrup, Class. Quantum Grav. **17**, 3009 (2000), (hep-th/9907043; Section 5 of [15]).
[8] M. Bojowald and R. Swiderski, Class. Quantum Grav. **21**, 4881 (2004).
[9] J. Wheeler, in *Sakharov Memorial Lectures on Physics* (Nova Science, New York, 1992), Vol. 2.
[10] M. Domagala and J. Lewandowski, gr-qc/0407051; K. A. Meissner, gr-qc/0407052.
[11] H. A. Kastrup, Ann. Phys. (Leipzig) **9**, 503 (2000); gr-qc/9906104.
[12] T. Thiemann, Class. Quantum Grav. **15**, 839 (1998).
[13] T. Thiemann and H. A. Kastrup, Nucl. Phys. B **399**, 211 (1993); H. A. Kastrup and T. Thiemann, Nucl. Phys. B **425**, 665 (1994).
[14] M. Bojowald, Class. Quantum Grav. **19**, 2717 (2002); Class. Quantum Grav. **20**, 2595 (2003).
[15] M. Bojowald and H. A. Kastrup, Class. Quantum Grav. **17**, 3009 (2000).
[16] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001).
[17] M. Bojowald, G. Date, and K. Vandersloot, Class. Quantum Grav. **21**, 1253 (2004).
[18] A. Ashtekar and B. Krishnan, Phys. Rev. D **68**, 104030 (2003); Phys. Rev. Lett. **89**, 261101 (2002).
[19] I. Booth and S. Fairhurst, Phys. Rev. Lett. **92**, 011102 (2004).