Intermediate inflation and the slow-roll approximation

Alan D. Rendall
Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm, Germany

Abstract

It is shown that spatially homogeneous solutions of the Einstein equations coupled to a nonlinear scalar field and other matter exhibit accelerated expansion at late times for a wide variety of potentials V. These potentials are strictly positive but tend to zero at infinity. They satisfy restrictions on V'/V and V''/V' related to the slow-roll approximation. These results generalize Wald's theorem for space-times with positive cosmological constant to those with accelerated expansion driven by potentials belonging to a large class.

1 Introduction

In recent years astronomical observations have shown the necessity of incorporating accelerated expansion into the description of the evolution of our universe. Unfortunately there is no clear understanding of the mechanism leading to the acceleration. The simplest possibilities are a cosmological constant or a nonlinear minimally coupled scalar field with a potential. There is now a large and growing literature on nonlinear scalar fields with different potentials and many more exotic models. The observational data available at present only gives weak criteria for deciding between different models. For instance it is not even clear that it can rule out the case of a cosmological constant (cf. [1]). It is to be hoped that this situation will improve due the accumulation of more data in the next few years. In the meantime it makes sense to advance our theoretical understanding of the models available.

The present paper is concerned with the dynamics of spatially homogeneous solutions of the Einstein equations in the presence of a minimally coupled nonlinear scalar field and other matter. The latter represents ordinary baryonic matter and non-baryonic dark matter. In the literature on models with accelerated cosmological expansion one or more of the following effects are often ignored: shear, ordinary matter, spatial curvature. The usual intuitive picture says that they will be negligible under suitable circumstances in a phase of accelerated expansion. A rigorous proof that this is true was given by Wald [33] in the case the acceleration is caused by a cosmological constant. The first main result of the present paper is that Wald's theorem can be generalized to the case where the cosmological constant is replaced by a scalar field belonging to a large class. This class apparently includes all of the potentials considered in the literature for which accelerated expansion is expected to continue indefinitely.

A concept which is frequently used when studying the dynamics of models with accelerated expansion is that of the slow-roll approximation. It says that under certain circumstances the equation of motion of the scalar field can be replaced by a simpler equation while retaining the essential qualitative features of the evolution. The precise meaning of the slow-roll approximation has been investigated in [17]. It was shown that if a solution approaches the 'slow-roll attractor' many conclusions follow. The second main result of the present paper is a proof that for a wide class of potentials all solutions approach the slow-roll attractor.

Since the potential of the scalar field leading to accelerated expansion is not known it is useful to try to analyse the dynamics of spacetimes with a nonlinear scalar field for large classes of potentials assumed only to satisfy some qualitative restrictions. This programme was carried out in [22] for potentials with a strictly positive lower bound and solutions of Bianchi types I-VIII. The spacetimes analysed may contain matter satisfying the dominant and strong energy conditions in addition to the scalar field and the results are a direct generalization of Wald's theorem [33] on spacetimes with positive cosmological constant. As pointed out in [22] the next natural case to look at in order of increasing difficulty is that where the potential is everywhere positive but may tend to zero as the scalar field ϕ tends to $\pm\infty$. Furthermore it should be assumed that the fall-off at infinity is not too fast, since otherwise accelerated expansion may not occur.

To see the need for a restriction on the fall-off of the potential it is useful to look at explicit solutions given by Halliwell [12]. These solutions are homogeneous and isotropic and spatially flat. Their only matter content is a scalar field with an exponential potential. In notation which is convenient for the following the potential is given by $V(\phi) = V_0 e^{-\sqrt{8\pi}\lambda\phi}$, where λ is a positive constant. If these solutions are written in Gauss coordinates the spatial metric is of the form $g_{ij}(t) = a^2(t)\delta_{ij}$ where the scale factor $a(t)$ is proportional to t^{2/λ^2}. The expansion is accelerated iff the power occurring is greater than one, i.e. iff $\lambda < \sqrt{2}$. Thus if the aim is to consider models with accelerated expansion an assumption should be made on the potential which rules out too rapid exponential decay. This can be done by means of a restriction on V'/V.

Acceleration due to an exponential potential is often called power-law inflation since the scale factor has a power-law behaviour in the simplest cases. A class of models where the scale factor behaves in a way intermediate between a power law and the exponential expansion resulting from a positive cosmological constant is associated with name intermediate inflation [6]. The specific case considered in [6] is where the scale factor is proportional to $\exp(t^f)$ for $0 < f < 1$. This can be obtained from a potential which looks asymptotically like a negative power but is not exactly a power. Note that if a cosmological constant is thought of as a constant potential the potential in intermediate inflation is intermediate between the cosmological constant and the potential for power-law inflation as far as its asymptotic behaviour for large ϕ is concerned. The aim of this paper is to analyse the case of general potentials which are positive, tend to zero as $\phi \to \infty$ and are such that $-V'/V$ satisfies an upper bound which rules out exponential potentials with $\lambda \geq \sqrt{2}$.

Under an additional assumption on V''/V' further information is obtained on the asymptotic behaviour of the solutions. These properties are related to the slow-roll approximation [17]. It is shown that the results obtained are general enough to apply to many potentials considered in the literature.

After some necessary background material has been presented in section 2, the slow-roll approximation is discussed in section 3 for spatially flat isotropic models with a non-linear scalar field as the only matter content. Section 4 deals with the generalization to all models of Bianchi types I-VIII and to the case where arbitrary matter fields satisfying the dominant and strong energy conditions are present in addition to the scalar field. In section 5 it is shown that the results which have been obtained are applicable to many potentials considered in the literature. The late-time dynamics of perfect fluids and collisionless matter is treated in section 6. Section 7 sums up what has been achieved and indicates some avenues for future progress.

2 Background

This section recalls the equations which are needed in the following and some basic facts which were proved in [22]. Consider a spacetime with vanishing cosmological constant which contains a nonlinear scalar field and some other matter. Suppose that the other matter satisfies the dominant and strong energy conditions. The energy momentum tensor is

$$T_{\alpha\beta} = T^M_{\alpha\beta} + \nabla_\alpha\phi\nabla_\beta\phi - \left[\frac{1}{2}\nabla^\gamma\phi\nabla_\gamma\phi + V(\phi)\right] g_{\alpha\beta}. \tag{1}$$

where $T^M_{\alpha\beta}$ is the energy-momentum tensor of the matter other than the scalar field. Assume that the potential V is C^2 and non-negative. Restricting to a Bianchi spacetime leads to the following basic equations:

$$\frac{dH}{dt} = -4\pi\dot{\phi}^2 - \frac{1}{2}\sigma_{ab}\sigma^{ab} + \frac{1}{6}R - 4\pi(\rho^M + \frac{1}{3}\mathrm{tr}S^M) \tag{2}$$

$$\ddot{\phi} = -3H\dot{\phi} - V'(\phi) \tag{3}$$

$$H^2 = \frac{4\pi}{3}[\dot{\phi}^2 + 2V(\phi)] + \frac{1}{6}(\sigma_{ab}\sigma^{ab} - R) + \frac{8\pi}{3}\rho^M \tag{4}$$

Here H is the Hubble parameter, σ_{ab} is the tracefree part of the second fundamental form, R is the scalar curvature of the spatial metric and ρ^M and $\mathrm{tr}S^M$ are projections of the energy-momentum tensor $T^M_{\alpha\beta}$. For Bianchi types I to VIII the inequality $R \leq 0$ holds. Assuming that $H > 0$ at some time (so that the model is expanding at that time) it follows that $H > 0$ at all times. For more details see [22].

If H is positive at some time then it is everywhere positive and non-increasing and so tends to some limit $H_1 \geq 0$. In Theorem 1 of section 3 of [22] some general results on late-time dynamics for the above equations were proved unter the following three assumptions:

1. $V(\phi) \geq V_0$ for a constant $V_0 > 0$
2. V' is bounded on any interval on which V is bounded

3. V' tends to a limit, finite or infinite, as ϕ tends to ∞ or $-\infty$.

Suppose for a moment that the first of these assumptions is dropped but the other two are retained. The first part of the proof of Theorem 1 of [22] does not use assumption 1. and so it can still be applied. The conclusion is that if a spacetime of the type described above exists globally in the future then $\dot{\phi} \to 0$ and $V(\phi)$ tends to a limit $V_1 \geq 0$ as $t \to \infty$. If $H_1 > 0$ then the rest of the proof of the theorem also goes through and the same conclusions are obtained as in that theorem. In particular the expansion is accelerated at late times, the model isotropizes and the contribution of matter and spatial curvature to the field equations becomes negligible. Moreover $3H^2/8\pi V \to 1$ and $V'(\phi) \to 0$ as $t \to \infty$.

It remains to consider the case $H_1 = 0$. In that case it follows that $\dot{\phi} \to 0$ and $V(\phi) \to 0$ as $t \to \infty$ without having to make the additional assumptions 2. and 3. When $H_1 = 0$ it follows that $\sigma^{ab}\sigma_{ab}$, R and ρ^M converge to zero as $t \to \infty$. The quantity

$$Z = 9H^2 - 24\pi[\dot{\phi}^2/2 + V(\phi)] = \frac{3}{2}(\sigma_{ab}\sigma^{ab} - R) + 24\pi\rho^M \tag{5}$$

satifies the inequality $dZ/dt \leq -2HZ$. In [22] this was combined with the fact that H has a positive lower bound to show that Z decayed exponentially. In the present more general case this is no longer true but the differential inequality can be used in a different way. In fact $H^2 \geq Z/9$ and so $dZ/dt \leq -\frac{2}{3}Z^{3/2}$. It follows that $Z = O(t^{-2})$ and this gives decay rates for $\sigma^{ab}\sigma_{ab}$, R and ρ^M. Note that the conclusions up to this point also hold in the absence of a scalar field. Assume now that V is positive. (This assumption will be maintained for the rest of the paper.) Then $V(\phi(t)) \to 0$ and so $\phi \to \infty$ or $\phi \to -\infty$ as $t \to \infty$. These cases can be transformed into each other by reversing the sign of ϕ and so it can be assumed without loss of generality that $\phi \to \infty$. In this case $V(\phi) \to 0$ as $\phi \to \infty$.

3 The slow-roll approximation

In this section we specialize to the isotropic and spatially flat case with a nonlinear scalar field as the only matter. This is the situation most frequently considered in the literature. In this case $H = \dot{a}/a$ where a is the scale factor. It will be shown that certain assumptions on the potential V imply accelerated expansion at late times. The assumptions are

1. $V(\phi) > 0$ with $V(\phi) \to 0$ as $\phi \to \infty$
2. $V'(\phi) < 0$
3. $V'(\phi)/V(\phi) \to 0$ as $\phi \to \infty$.

Consider a situation where, as in the last section, $\phi \to \infty$ as $t \to \infty$. Note that $\dot{\phi} > 0$ for t sufficiently large. For $\dot{\phi}$ must be positive at some time. After that it must remain positive since $\dot{\phi} = 0$ and assumption 2. above imply that $\ddot{\phi} > 0$. When it is known that $\dot{\phi} > 0$ the assumptions of the last section can be weakened while maintaining the same conclusions. Assumption 2. only needs to be required for intervals whose left endpoint is finite. In assumption 3. only the existence of the limit for $t \to \infty$ need be required. These weakened versions of assumptions 2. and 3. of the last section follow from the assumptions 1.-3. of this section. The following argument was inspired by a paper of Muslimov [19] on the isotropic case but we will not use the complicated transformation of variables of that paper directly. Since ϕ is monotone we can regard functions of t as functions of ϕ. The key relation is the following:

$$\frac{d}{d\phi}\left(\frac{3H^2}{8\pi V}\right) = \frac{3H^2}{8\pi V}\left(-\frac{8\pi\dot{\phi}}{H} - \frac{V'}{V}\right) \tag{6}$$

Theorem 1 Consider a spatially flat homogeneous and isotropic solution of the Einstein equations coupled to a nonlinear scalar field with potential V of class C^2 satisfying conditions 1. - 3. above. Suppose that the solution is initially expanding ($H > 0$) and that $\dot{\phi} > 0$ at some time. If the solution exists globally to the future then $3H^2/8\pi V(\phi) \to 1$ as $t \to \infty$ and $\ddot{a} > 0$ for t sufficiently large.

Proof In order to control the first term on the right hand side of (6) note that

$$\frac{\dot{\phi}}{H} = \sqrt{\frac{3}{4\pi} - \frac{2V(\phi)}{H^2}} \tag{7}$$

and that the right hand side is an increasing function of $3H^2/8\pi V$. Moreover it tends to zero as $3H^2/8\pi V \to 1$. Let $C_1 > 8\pi/3$ be a constant and let $C_2 = 8\pi\sqrt{3/4\pi - 2/C_1}$. Since $V'/V \to 0$ as $\phi \to \infty$ there exists some ϕ_1 such that $-V'(\phi)/V(\phi) \le C_2/2$ for all $\phi \ge \phi_1$. At any point where $\phi \ge \phi_1$ and $H^2/V \ge C_1$ the expression in brackets on the right hand side of (6) is bounded above by $-C_2/2$. Hence H^2/V, if it is ever greater than C_1, reaches

C_1 at some greater value ϕ_2 of ϕ and remains below that value for all $\phi \geq \phi_2$. Because of the freedom to choose C_1, and the fact that $3H^2/8\pi V \geq 1$, it follows that $3H^2/8\pi V \to 1$ as $\phi \to \infty$ and hence as $t \to \infty$. Thus the first conclusion of the theorem has been proved.

The inequality $\dot{H} + H^2 \geq 0$ is equivalent to $\ddot{a} > 0$ and is thus the criterion for accelerated expansion. Substituting for H and $\dot{H}$ using the field equations shows that it is equivalent to the condition that $3H^2/8\pi V(\phi) < 3/2$, which is fulfilled at late times, giving the remaining conclusion of the theorem.

Remark The inequality $\dot{H} + H^2 \geq 0$ implies that $H(t) \geq (t+C)^{-1}$ for some $C > 0$ and t large and that $\int_{t_0}^{\infty} H(t)dt = \infty$ for any t_0.

An analogous result can be obtained in the case that assumption 3. above is replaced by the weaker condition that $\limsup(-V'/V) \leq \alpha$ for a suitable constant α.

Theorem 2 Consider a spatially flat homogeneous and isotropic solution of the Einstein equations coupled to a nonlinear scalar field with potential V of class C^2 satisfying conditions 1. - 2. above with $\alpha = \limsup(-V'/V) < 4\sqrt{\pi}$. Suppose that the solution is initially expanding ($H > 0$) and that $\dot{\phi} > 0$ at some time. If the solution exists globally to the future then $\ddot{a} > 0$ for t sufficiently large.

Proof Suppose that $C_1 > \beta$ where

$$\frac{2}{\beta} = \frac{3}{4\pi} - \frac{\alpha^2}{64\pi^2} \tag{8}$$

and define C_2 in terms of C_1 as above. Then $C_2 > \alpha$. There exists some ϕ_1 such that $-V'/V \leq (C_2 + \alpha)/2$ for $\phi > \phi_1$. If $\phi \geq \phi_1$ and $H^2/V \geq C_1$ then $-8\pi\dot{\phi}/H - V'/V \leq (\alpha - C_2)/2 < 0$. It follows that, as in the previous case, $(H(\phi))^2/V(\phi)$, if it is ever greater than C_1, reaches C_1 at some greater value ϕ_2 of ϕ and remains below that value for all $\phi \geq \phi_2$. Hence in the limit $\phi \to \infty$ we have $\limsup 3H^2/8\pi V \leq 3\beta/8\pi$. Thus the condition for accelerated expansion is $3\beta/8\pi < 3/2$, i.e. $\alpha < 4\sqrt{\pi}$. In particular the inequality for accelerated expansion in the case of an exponential potential with $\lambda = \alpha/\sqrt{8\pi}$ is recovered.

Remark For an exponential potential it follows from [16] that $3H^2/8\pi V \to 3\beta/8\pi$.

In the case that $V'/V \to 0$ as $t \to \infty$ it is possible to obtain further information on the asymptotic behaviour of the solutions which relates to the so-called slow-roll approximation [17]. This says that under the assumption on the potential that V''/V' is bounded the term $\ddot{\phi}$ in the equation of motion of the scalar field becomes negligible at late times. Equivalently $3H\dot{\phi}/V' \to -1$. In the terminology of [17] this shows that for a potential satisfying this assumption all solutions approach the slow-roll attractor.

Theorem 3 Let the assumptions of Theorem 1 hold and assume in addition that V''/V' is bounded as $\phi \to \infty$. Then $3H\dot{\phi}/V' \to -1$ and $\ddot{\phi}/3H\dot{\phi} \to 0$ as $t \to \infty$.

Proof The desired limiting behaviour can be proved using the relation

$$\frac{d}{dt}\left(\frac{H\dot{\phi}}{V'}\right) = -H\left[\left(1+\frac{3H\dot{\phi}}{V'}\right)+\frac{H\dot{\phi}}{V'}\left(\frac{4\pi\dot{\phi}^2}{H^2}+\frac{\dot{\phi}}{H}\frac{V''}{V'}\right)\right] \qquad (9)$$

Let $C_1 > 0$ be a constant. There exists a time t_1 such that for $t \geq t_1$ the inequality $(\dot{\phi}/H)(V''/V') \geq -C_1/(1+C_1)$ holds. If $3H\dot{\phi}/V' \leq -1 - C_1$ at some time $t \geq t_1$ then it follows from (9) that

$$\frac{d}{dt}\left(\frac{H\dot{\phi}}{V'}\right) \geq -\frac{2C_1 H}{(1+C_1)}\left(\frac{H\dot{\phi}}{V'}\right). \qquad (10)$$

Since H is not integrable it follows that $1 + 3H\dot{\phi}/V'$ must reach $-C_1$ after finite time and cannot become less than this value again. Using the freedom to choose C_1 shows that $\liminf(1 + H\dot{\phi}/V') \geq 0$. In particular, since V' is negative, it follows that $H\dot{\phi}/V'$ is bounded. Now choose t_2 such that the modulus of the second term in the square bracket in (9) is less than $C_1/2$ for $t \geq t_2$. Suppose that at some time $t \geq t_2$ the inequality $1 + 3H\dot{\phi}/V' \geq C_1$ holds. Then $1 + 3H\dot{\phi}/V'$ is decreasing there. Moreover the rate of decrease is such it must reach C_1 in finite time. It follows that $1 + 3H\dot{\phi}/V' \to 0$ as $t \to \infty$. The equation of motion for ϕ implies that $\ddot{\phi}/3H\dot{\phi} \to 0$.

Remark For an exponential potential it follows from [16] that $3H\dot{\phi}/V' \to -(1-\lambda^2/6)^{-1}$ as $t \to \infty$.

4 The anisotropic case with matter

Here we generalize the results of the last section to anisotropic solutions with matter.

Theorem 4 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a nonlinear scalar field with potential V of class C^2 satisfying

conditions 1. - 3. of section 3 and other matter satisfying the dominant and strong energy conditions. Suppose that the solution is initially expanding ($H>0$) and that $\dot\phi>0$ at some time. Then
a) if the solution exists globally to the future then $3H^2/8\pi V(\phi)\to 1$, $\sigma_{ab}\sigma^{ab}/H^2$, R/H^2, and ρ^M/H^2 tend to zero as $t\to\infty$ and $\dot H+H^2\ge 0$ for t sufficiently large and
b) if in addition V''/V' is bounded then $3H\dot\phi/V'\to -1$ as $t\to\infty$.
Proof Equation (6) is replaced by

$$\begin{aligned}\frac{d}{d\phi}\left(\frac{3H^2}{8\pi V}\right) &= \frac{3H^2}{8\pi V}\Bigg[-\frac{8\pi\dot\phi}{H}-\frac{V'}{V}\\ &\quad -\frac{2}{H\dot\phi}\left(\frac{1}{2}\sigma_{ab}\sigma^{ab}-\frac{1}{6}R+4\pi\left(\rho^M+\frac{1}{3}\mathrm{tr}S^M\right)\right)\Bigg] \qquad (11)\end{aligned}$$

while (7) is replaced by

$$\frac{\dot\phi}{H}=\left[\frac{3}{4\pi}-\frac{2V(\phi)}{H^2}-\frac{1}{8\pi}\frac{\sigma_{ab}\sigma^{ab}-R}{H^2}-\frac{2\rho^M}{H^2}\right]^{1/2} \qquad (12)$$

It will be shown that (11) implies that $\dot\phi/H\to 0$ as $\phi\to\infty$. Given a constant $C_1>0$ there exists ϕ_1 such that $V'/V\le C_1/2$ for all $\phi\ge\phi_1$. Hence if $\dot\phi/H\ge C_1/8\pi$ and $\phi\ge\phi_1$ then $d/d\phi(3H^2/8\pi V)\le -(C_1/2)(3H^2/8\pi V)$. Thus eventually $\dot\phi/H$ reaches the value C_1 and does not return. This proves the desired result. Now

$$\begin{aligned}&-\frac{8\pi\dot\phi}{H}-\frac{2}{H\dot\phi}\left(\frac{1}{2}\sigma_{ab}\sigma^{ab}-\frac{1}{6}R+4\pi\left(\rho^M+\frac{1}{3}\mathrm{tr}S^M\right)\right)\\ &\le -\sqrt{48\pi}+\sqrt{48\pi}\left(\frac{8\pi V(\phi)}{3H^2}\right)+\frac{\sigma_{ab}\sigma^{ab}}{H^2}\left(\frac{1}{6}\sqrt{48\pi}-\left(\frac{\dot\phi}{H}\right)^{-1}\right) \qquad (13)\\ &-\frac{R}{H^2}\left(\frac{1}{6}\sqrt{48\pi}-\frac{1}{3}\left(\frac{\dot\phi}{H}\right)^{-1}\right)+\frac{8\pi}{3H^2}\left(\sqrt{48\pi}\rho^M-\left(\frac{\dot\phi}{H}\right)^{-1}\left(3\rho^M+\mathrm{tr}S^M\right)\right)\end{aligned}$$

Provided $\dot\phi/H\le 1/\sqrt{12\pi}$ then

$$\frac{d}{d\phi}\left(\frac{3H^2}{8\pi V}\right)\le -\left[\sqrt{48\pi}\left(1-\frac{8\pi V(\phi)}{3H^2}\right)-\frac{V'}{V}\right]\left(\frac{3H^2}{8\pi V}\right) \qquad (14)$$

Thus arguing as in the isotropic case shows that $3H^2/8\pi V \to 1$ as $\phi \to \infty$, i.e. as $t \to \infty$. It follows that $\sigma_{ab}\sigma^{ab}/H^2$, R/H^2 and ρ^M/H^2 tend to zero as $t \to \infty$. In particular $\dot{H} + H^2 \geq 0$ at late times and there is accelerated expansion.

The argument to obtain the slow-roll approximation works just as in the isotropic case. Equation (9) is changed by replacing $4\pi\dot{\phi}^2$ in the last bracket by

$$H^{-2}\left[4\pi\dot{\phi}^2 + \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{6}R + 4\pi\left(\rho^M + \frac{1}{3}\mathrm{tr}S^M\right)\right] \tag{15}$$

The extra terms which are added have the right sign and decay as $t \to \infty$ and this is all that is needed.

With some more work this result can be strengthened. Let $\epsilon > 0$. It follows from the theorem that

$$\dot{H} + \epsilon H^2 = H^2(\epsilon - 4\pi\dot{\phi}^2/H^2 + o(1)) \tag{16}$$

Using what is known about $\dot{\phi}/H$ shows that for t large the right hand side is positive. Thus a differential inequality is obtained which implies that $H \geq \epsilon^{-1}(C+t)^{-1}$ for a constant $C > 0$. Thus H falls off slower than any positive multiple of t^{-1}. It follows from the inequality $dZ/dt \leq -2HZ$ that Z decays faster than any power of t. Hence Z/H^2 decays faster than any power of t. In particular $\sigma_{ab}\sigma^{ab}/H^2$ decays faster than any power of t.

5 Applications to potentials in the literature

In the literature on inflation and quintessence many choices of potential have been considered and we will not attempt to give a comprehensive survey. Large classes of potentials where the theorems of this paper are applicable will be identified and a number of examples in the literature belonging to these classes will be pointed out. Then it will be shown by example how the theorems can be used to prove asymptotic expansions for the solutions.

Consider the following class of potentials:

$$V(\phi) = V_0(\log\phi)^p\phi^n\exp(-\lambda\phi^m) \tag{17}$$

The case $p = 0$ was studied in [20] while the cases $m = 0$ and $n = 0$, $m = 1$ were studied in [7]. The requirement that $V \to 0$ as $\phi \to \infty$ leads to some restrictions. Suppose first that $\lambda < 0$. Then it must be assumed that $m \le 0$. If $m = 0$ it must further be assumed that $n \le 0$ and if $n = 0$ it must be assumed that $p < 0$. If $\lambda > 0$ it must be assumed that $m > 0$ or that n and p satisfy the restrictions already listed. If $\lambda = 0$ then n and p must also satisfy those restrictions. To compare with the other hypotheses of the theorems the first and second derivatives of V must be computed.

$$V'(\phi) = \left(\frac{p}{\phi \log \phi} + \frac{n}{\phi} - \lambda m \phi^{m-1}\right) V \tag{18}$$

$$V''(\phi) = \left[\frac{p(p-1)}{\phi^2 (\log \phi)^2} + \frac{p(2n-1)}{\phi^2 \log \phi} + \frac{n(n-1)}{\phi^2} - \lambda m \left(\frac{2p}{\phi \log \phi} + \frac{m+2n-1}{\phi}\right) \phi^{m-1} + \lambda^2 m^2 \phi^{2(m-1)}\right] V \tag{19}$$

Evidently V'/V tends to zero as $\phi \to \infty$ iff $m < 1$. When $m = 1$ the quantity $-V'/V$ converges to a positive limit. It can easily be checked that under the restrictions already made V' is negative for ϕ sufficiently large. In the case that $0 < m < 1$ and $\lambda > 0$ it follows that V''/V' behaves asymptotically like $-\lambda m \phi^{m-1}$ and so Theorem 4 is applicable. In the case where $m \le 0$, $n < 0$ the quantity V''/V' behaves asymptotically like $(n-1)/\phi$ and so Theorem 4 again applies. The case where $m \le 0$, $n = 0$ and $p < 0$ is similar, since there V''/V' behaves asymptotically like $(p-1)/\phi \log \phi$. When $m = 1$ the assumptions of Theorem 4 are not satisfied but it follows from Theorem 2 that in the spatially flat isotropic case with scalar field alone the models exhibit accelerated expansion.

Suppose now that W is a potential satisfying some of the assumptions of the theorems and let V be a potential for which $V = W(1 + o(1))$, $V' = W'(1+o(1))$ and $V'' = W''(1+o(1))$ as $\phi \to \infty$. Then V satisfies the corresponding assumptions. In particular $\limsup(-V'/V) = \limsup(-W'/W)$. As an example, consider the potentials for intermediate inflation given in [6]. These are of the form:

$$V(\phi) = V_0(\phi - \phi_0)^n + V_1(\phi - \phi_0)^{n-2} \tag{20}$$

for constants n, ϕ_0, V_0 and V_1 with n negative and V_0 positive. Taking $W = V_0 \phi^n$ allows the above observation to be applied in this case to see that Theorem 4 applies. Another example where this procedure works is in the case $V = V_0(\exp M_p/\phi - 1)$ [34]. Take $W = V_0 M_p/\phi$. The case where

$\limsup(-V'/V)$ is positive gives a proof of late-time accelerated expansion for spatially flat isotropic models with scalar field alone in a number of cases. There follow some examples taken from a list of potentials in [26]. If $V = V_0(\cosh\lambda\phi - 1)^p$ [27] with $-4\sqrt{\pi} < p\lambda < 0$ then take $W = (V_0/2^p)e^{p\lambda\phi}$. If $V = V_0\sinh^{-\alpha}(\lambda\phi)$ [32] with $0 < \alpha\lambda < 4\sqrt{\pi}$ take $W = (2^\alpha V_0)e^{-\alpha\lambda\phi}$. If $V = V_0(e^{\alpha\kappa\phi}+e^{\beta\kappa\phi})$ [5] with $-4\sqrt{\pi} < \alpha\kappa < 0$ and $\beta\kappa < \alpha\kappa$ then take $W = V_0 e^{\alpha\kappa\phi}$. If $V = V_0[(\phi-B)^\alpha + A]e^{-\lambda\phi}$ [2] and $\lambda > 0$ then take $W = V_0\phi^\alpha e^{-\lambda\phi}$ for $\alpha > 0$ and $W = V_0 A e^{-\lambda\phi}$ for $\alpha < 0$.

In [20] asymptotic expansions were written down for solutions corresponding to certain potentials. With the theorems of this paper in hand these asymptotic expansions can be given an interpretation which can be proved rigorously. For simplicity only the case $p = 0$, $n = 0$, $0 < m < 1$, $\lambda = 1$ of (17) will be discussed. Under the hypotheses of Theorem 4, with this choice of potential, $3H^2/8\pi V \to 1$ and $3H\dot{\phi}/V' \to -1$ as $t \to \infty$. It can be concluded that the solutions satisfy

$$\dot{\phi} = -\frac{V'}{\sqrt{24\pi V}} + o\left(\frac{V'}{\sqrt{V}}\right) \tag{21}$$

as $t \to \infty$. This should be compared with equation (2.9) of [20]. Substituting the specific form of the potential gives

$$\dot{\phi} = m\sqrt{\frac{V_0}{24\pi}}\phi^{m-1}\exp\left(-\frac{1}{2}\phi^m\right) + o\left(\phi^{m-1}\exp\left(-\frac{1}{2}\phi^m\right)\right) \tag{22}$$

This can integrated using the identity

$$\frac{d}{dx}(2m^{-1}x^{2-2m}e^{x^m/2}) = x^{1-m}e^{x^m/2}\left(1 + \frac{4-4m}{m}x^{-m}\right) \tag{23}$$

to give

$$t = \frac{2}{m^2}\sqrt{\frac{24\pi}{V_0}}\phi^{2-2m}\exp\left(\phi^m/2\right)(1 + o(1)). \tag{24}$$

This implies that

$$\phi^m = 2\log t - (4-4m)\log\phi + 2\log\left(\frac{m^2}{2}\sqrt{\frac{V_0}{24\pi}}\right) + o(1) \tag{25}$$

Taking logarithms gives

$$\log\phi = \frac{1}{m}(\log\log t + \log 2) + o(1) \tag{26}$$

Substituting this back into the expression for ϕ^m gives

$$\phi^m = 2\log t - \frac{4(1-m)}{m}(\log\log t + \log 2) + 2\log\left(\frac{m^2}{2}\sqrt{\frac{V_0}{24\pi}}\right) + o(1) \quad (27)$$

Now the asymptotic form of the potential can be computed.

$$V(\phi(t)) = V_0\left[\log\left(\frac{m^2}{2}\sqrt{\frac{V_0}{24\pi}}\right)\right]^2 t^{-2}(2\log t)^{4(1-m)/m}(1+o(1)) \quad (28)$$

From this the asymptotic form of H can immediately be computed H.

$$H(t) = (8\pi V_0/3)^{1/2}\log\left(\frac{m^2}{2}\sqrt{\frac{V_0}{24\pi}}\right) t^{-1}(2\log t)^{2(1-m)/m}(1+o(1)) \quad (29)$$

To end the section an example will be presented of a potential which does not satisfy the assumptions of Theorem 4. This is given by:

$$V(\phi) = \phi^{-1} + \phi^{-7/2}\sin(\phi^2) \quad (30)$$

In this case $V'(\phi) < 0$ for ϕ large, $V'/V \to 0$ as $t \to \infty$, $V''/V \to 0$ as $t \to \infty$ but V''/V' is unbounded as $t \to \infty$.

6 Specific matter models - perfect fluids and collisionless matter

All the theorems of the previous sections have been obtained under the assumption that a solution exists globally in the future. If a suitable well-behaved model is chosen for the matter other than the scalar field then global existence for arbitrary initial data can be proved. In the case of a perfect fluid with a linear equation of state this has been done in [22]. In the case of collisionless matter described by the Vlasov equation it has been done in [16].

Once global existence has been shown it is possible to obtain detailed information about the asymptotics of the matter fields. Consider an untilted perfect fluid with linear equation of state $p = (\gamma - 1)\rho$. The state of the fluid is described entirely by its energy density μ, which satisfies $d\mu/dt =$

$-3\gamma H\mu$. Let $l = (\det g)^{1/6}$. In the isotropic case l agrees with the scale factor up to a multiplicative constant. Now $dl/dt = Hl$. Hence $d/dt(l^{3\gamma}\mu) = 0$ and the density behaves like $l^{-3\gamma}$. From an asymptotic expression for H we can obtain a corresponding expression for l and hence an asymptotic expression for μ. For example, for the solutions whose asymptotic behaviour was analysed in the last section the leading order term in l is proportional to $\exp((\log t)^{(2-m)/m})$.

Consider now the case of collisionless matter with a potential satisfying the conditions of part a) of Theorem 4.

$$\frac{d}{dt}\left(\frac{g_{ij}}{l^2}\right) = -2\left(\frac{g_{is}}{l^2}\right)\sigma^s{}_j \tag{31}$$

Now $\sigma_{ij}\sigma^{ij}$ decays faster than any power of t and is, in particular, integrable in t. It is possible to argue just as in [15] to conclude first that g_{ij}/l^2 and its inverse are bounded and then that g_{ij}/l^2 converges as $t \to \infty$ to some quantity g^0_{ij}. It follows that

$$g_{ij} = l^2(g_{ij} + h_{ij}) \tag{32}$$

where $h_{ij} = O(t^{-n})$ for any n. It is then possible to proceed as in [15] to obtain asymptotics for geodesics, i.e. for the characteristics of the Vlasov equation. The results are strictly analogous to those in [15] except that e^{-Ht} is replaced by l^{-n} in the error terms in [15], where n is arbitrary. For a specific potential where the form of l can be determined these error estimates can be made sharper. If V_i are the spatial components of a tangent vector to a timelike geodesic then they converge faster than any power of t to a constant as $t \to \infty$. The energy density decays like l^{-3} as in the case of dust and all other components of the energy-momentum tensor in an orthonormal frame decay faster.

7 Conclusion

Combining the results of this paper with previous work gives a rather detailed picture of the dynamics of spatially homogeneous spacetimes with accelerated expansion driven by a minimally coupled nonlinear scalar field with potential. Potentials with a positive lower bound are covered in [22], potentials which are positive and tend to zero at infinity slower than exponentially are treated in this paper. Exponential decay with sufficiently small exponent has been handled in [13], [14] and [16]. In the borderline case where the potential asymptotically approaches an exponential one but is not exactly exponential it would be desirable to get more information. In this paper accelerated expansion was demonstrated for this case for a scalar field alone in a spatially flat isotropic model but isotropization in more general homogeneous models, which is presumably true, was not. If the potential is zero somewhere then the expected behaviour is very different. In that case accelerated expansion is a transient phenomenon and the late time behaviour must be expected to depend in a complicated way on the Bianchi type. Cf. the discussion in the last section of [22].

The questions of global existence and the asymptotics of matter fields have been answered in some special cases. A treatment of more features of their asymptotic behaviour and of wider classes of matter fields would be desirable. Another direction in which the results should be generalized is to other mechanisms for producing accelerated expansion such as k-essence [4]. See also the discussion in [23].

For potentials which do lead to late-time accelerated expansion an obvious next step is to look at inhomogeneous models. Expansions for spacetimes without symmetry were written down for the case of a cosmological constant by Starobinsky [28] and in the case of exponential potentials in [18]. In [24] the results of [28] were given a rigorous mathematical interpretation in terms of formal series and the existence of a large class of solutions (depending on the maximum number of free functions) was demonstrated in the vacuum case. Some of these results have been generalized to the case of a potential with a positive minimum by Bieli [9]. In the vacuum case it was shown with the help of results of [10] that all solutions evolving from small perturbations of de Sitter initial data have late-time asymptotic expansions of the form given in [28]. Presumably a corresponding result could be obtained in any even spacetime dimension using the results of [3]

For certain classes of spacetimes with high symmetry (spherical, plane and hyperbolic symmetry) it has been shown in [29] and [30] that for many solutions of the Einstein-Vlasov system with positive cosmological constant there is late-time accelerated expansion and the geometry has leading-order asymptotics as in [28]. It would be interesting to see to what extent these results can be generalized to nonlinear scalar fields with potentials which give accelerated expansion in the homogeneous case. The results of [31] on the Einstein-Vlasov system coupled to a linear scalar field are a first step in this direction. Note that the use of collisionless matter in these results is not an accident. If dust or perfect fluid with pressure were used then it is to be expected that there would be no global existence of classical solutions. Cf. the results of [21] and [25]. The effect of nonlinear scalar fields on cosmological expansion in the inhomogeneous case has been studied heuristically and numerically in [11]. One issue of particular interest which has no analogue in the homogeneous case is the formation of domain walls for potentials with more than one minimum.

The way in which information about inhomogeneities in the universe is compared with theoretical models is via linear perturbation theory. These calculations are at present out of reach of rigorous mathematical theorems. There is not a single result proving that an inhomogeneous solution of the full nonlinear equations is well approximated by the expressions coming from linear perturbation theory.

Cosmological solutions of the Einstein equations with accelerated expansion are a rich source of problems for mathematical relativity. It is to be hoped that the mathematical theory will soon have reached the point where it can contribute insights for the astrophysical applications of these solutions.

References

[1] Alam, U., Sahni, V., Saini, T. D. and Starobinsky, A. A. 2004 Is there supernova evidence for dark energy metamorphosis? Mon. Not. Roy. Astron. Soc. 354, 275-291.

[2] Albrecht, A. and Skordis, C. 2000 Phenomenology of a realistic accelerating universe using only Planck-scale physics. Phys. Rev. Lett. 84, 2076-2079.

[3] Anderson, M. 2004 Existence and stability of even dimensional asymptotically de Sitter spaces. Preprint gr-qc/0408072.

[4] Armendariz-Picon, C., Mukhanov, V. and Steinhardt, P. 2001 Essentials of k-essence. Phys. Rev. D63, 103510.

[5] Barreiro, T, Copeland, E. J. and Nunes, N. J. 2000 Quintessence arising from exponential potentials. Phys. Rev. D61, 127301.

[6] Barrow, J. D. 1990 Graduated inflationary universes. Phys. Lett. 235B, 40-43.

[7] Barrow, J. D. and Parsons, P. 1995 Inflationary models with logarithmic potentials. Phys. Rev. D52, 5576-5587.

[8] Barrow, J. D. and Saich, P. 1990 The behaviour of intermediate inflationary universes. Phys. Lett. 249B, 406-410.

[9] Bieli, R. 2004 Unpublished.

[10] Friedrich, H. 1986 Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant. J. Geom. Phys. 3, 101-117.

[11] Goldwirth, D. and Piran, T. 1992 Initial conditions for inflation. Physics Reports 214, 223-292.

[12] Halliwell, J. J. 1987 Scalar fields in cosmology with an exponential potential. Phys. Lett. 185B, 341-344.

[13] Kitada, Y. and Maeda, K. 1992 Cosmic no-hair theorem in power-law inflation. Phys. Rev. D45, 1416-1419.

[14] Kitada, Y. and Maeda, K. 1993 Cosmic no-hair theorem in homogeneous spacetimes I. Bianchi models. Class. Quantum Grav. 10, 703-734.

[15] Lee, H. 2004 Asymptotic behaviour of the Einstein-Vlasov system with a positive cosmological constant. Math. Proc. Camb. Phil. Soc. 137, 495-509.

[16] Lee, H. 2004 The Einstein-Vlasov system with a scalar field. Preprint gr-qc/0404007.

[17] Liddle, A. R., Parsons, P. and Barrow, J. D. 1994 Formalising the slow-roll approximation in inflation. Phys. Rev. D50, 7222-7232.

[18] Müller, V, Schmidt, H.-J. and Starobinsky, A. A. 1990 Power-law inflation as an attractor solution for inhomogeneous cosmological models. Class. Quantum Grav. 7, 1163-1168.

[19] Muslimov, A. G. 1990 On the scalar field dynamics in a spatially flat Friedmann universe. Class. Quantum Grav. 7, 231-237.

[20] Parsons, P. and Barrow, J. 1995 Generalised scalar field potentials and inflation. Phys. Rev. D51, 6757-6763.

[21] Rendall, A. D. 1997 Existence and non-existence results for global constant mean curvature foliations. Nonlinear Analysis, Theory, Methods and Applications 30, 3589-3598.

[22] Rendall, A. D. 2004 Accelerated cosmological expansion due to a scalar field whose potential has a positive lower bound. Class. Quantum Grav. 21, 2445-2454.

[23] Rendall, A. D. 2004 Mathematical properties of cosmological models with accelerated expansion. Preprint gr-qc/0408053.

[24] Rendall, A. D. 2004 Asymptotics of solutions of the Einstein equations with positive cosmological constant. Ann. H. Poincaré 5, 1041-1064.

[25] Rendall, A. D. and Ståhl, F. 2005 Shock waves in plane symmetric spacetimes. Unpublished.

[26] Sahni, V. 2004 Dark matter and dark energy. Preprint astro-ph/0403324.

[27] Sahni, V. and Wang, L. 2000 A new cosmological model of quintessence and dark matter. Phys. Rev. D62, 103517

[28] Starobinsky, A. A. 1983 Isotropization of arbitrary cosmological expansion given an effective cosmological constant. JETP Lett. 37, 66-69.

[29] Tchapnda, B. and Noutchegueme, N. 2003 The surface-symmetric Einstein-Vlasov system with cosmological constant. Preprint gr-qc/0304098.

[30] Tchapnda, B. and Rendall, A. D. 2003 Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant. Class. Quantum Grav, 20, 3037-3049.

[31] Tegankong, D., Noutchegueme, N. and Rendall, A. D. 2004 Local existence and continuation criteria for solutions of the Einstein-Vlasov-scalar field system with surface symmetry. J. Hyp. Diff. Eq. 1,

[32] Ureña-López, L. A. and Matos, T. 2000 New cosmological tracker solutions for quintessence. Phys. Rev D62, 081302.

[33] Wald, R. 1983 Asymptotic behaviour of homogeneous cosmological models with cosmological constant. Phys. Rev. D 28, 2118-2120.

[34] Zlatev, I, Wang, L. and Steinhardt, P. J. 1999 Quintessence, cosmic coincidence, and the cosmological constant. Phys. Rev. Lett. 82, 896-899.