

# Input-output relations for a three-port grating coupled Fabry-Perot cavity

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We analyze an optical three-port reflection grating by means of a scattering matrix formalism. Amplitude and phase relations among the three ports, i.e., the three orders of diffraction, are derived. Such a grating can be used as an all-reflective, low-loss coupler to Fabry-Perot cavities. We derive the input-output relations of a three-port grating coupled cavity and find distinct properties that are not present in two-port coupled cavities. The cavity relations further reveal that the three-port coupler can be designed such that the additional cavity port interferes destructively. In this case the all-reflective, low-loss, single-end Fabry-Perot cavity becomes equivalent to a standard transmissive, two-port coupled cavity. © 2005 Optical Society of America

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In a recent experiment a three-port reflection grating coupled Fabry-Perot cavity with high finesse was demonstrated.<sup>1</sup> The experiment was motivated by the idea that a three-port reflection grating should be able to provide two important features for advanced interferometry: low overall optical loss and no light transmission through optical substrates.<sup>2</sup> In advanced interferometers, such as in gravitational-wave detectors, these couplers might be crucial for achieving the optimal combination of extremely high-power laser fields, materials with a high mechanical quality factor for suspended optics, and cryogenic temperatures to reduce optics and suspension thermal noise.<sup>3</sup> Previously, a different concept for all-reflective linear Fabry-Perot cavities based on a two-port reflection grating was experimentally demonstrated.<sup>4</sup> In this approach the reflection grating was used in a first-order Littrow mount where the input-output relations of the cavity are analogous to those of a conventional cavity with transmissive mirrors. The major disadvantage of this concept is, however, that it relies on high first-order diffraction efficiency requiring deep grating structures that are associated with high scattering losses. Contrary to this, the concept demonstrated in Ref. 1 used a second-order Littrow mount and relies on low first-order diffraction efficiency that can be achieved by very shallow grating structures with smaller scattering losses. The latter approach is therefore better suited for low-loss couplers to high-finesse cavities, a stringent requirement in high-power laser interferometry. A grating used in a second-order Littrow mount, however, has three coupled ports in contrast with mirrors in which one input port is only coupled to two output ports. Knowledge of the phase relations of the three ports is essential for derivation of the input-output relations of the cavity.

In this Letter we derive the amplitude and phase relations of an optical three-port device by means of the scattering matrix formalism. We restrict ourselves to symmetric coupling between port 2 and the other two ports 1 and 3 described by  $\eta_1$  (see Fig. 1). Generally, optical devices such as mirrors and beam splitters can be described by a complex-valued  $n \times n$

scattering matrix  $\mathbf{S}$ ,<sup>5</sup> where  $n$  input ports are represented by a vector  $\mathbf{a}$  with components  $a_i$  that are the complex amplitudes of the incoming waves at the  $i$ th port. The outgoing amplitudes  $b_i$  are represented by vector  $\mathbf{b}$ . The coupling of input and output ports is given by

$$\mathbf{b} = \mathbf{S} \times \mathbf{a}. \quad (1)$$

For a lossless device  $\mathbf{S}$  must be unitary. Reciprocity of the device demands that  $|S_{ij}| = |S_{ji}|$ , where  $S_{ij}$  denotes an element of matrix  $\mathbf{S}$ . The magnitudes of the scattering coefficients are unique for a given device. The phase angles of the matrix elements, however, can be changed by choosing different reference planes in the various input and output arms. One can therefore derive different scattering matrices for the same device. Nevertheless, certain phase relationships between the different coefficients must be maintained. Transmissive mirrors are commonly used to couple light into Fabry-Perot cavities. The input-output relations of such cavities are well understood. Essential for their derivation is the knowledge of the phase relations at the mirrors for the reflected and transmitted beams. A conventional two-coupled-port mirror with amplitude reflectance  $\rho$  and transmittance  $\tau$ , for example, is generally described by

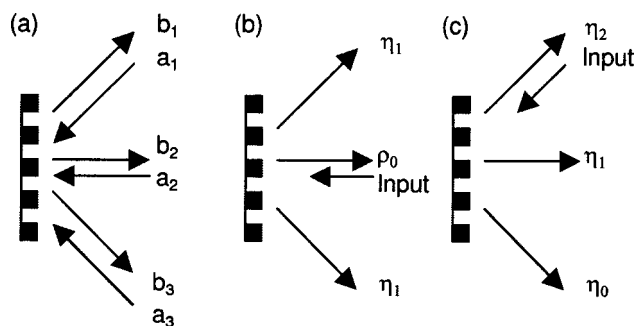


Fig. 1. Three-port reflection grating: (a) labeling of the input and output ports, (b) amplitudes of reflection coefficients for normal incidence, (c) amplitudes of reflection coefficients for second-order Littrow incidence.

$$\mathbf{S}_{2p} = \begin{bmatrix} \rho & \tau \\ \tau & -\rho \end{bmatrix}, \quad \mathbf{S}_{2p} = \begin{bmatrix} \rho & i\tau \\ i\tau & \rho \end{bmatrix}. \quad (2)$$

Using either one of these matrices, one can derive the amplitude reflectance  $r_{\text{FP}}$  and transmittance  $t_{\text{FP}}$  of a cavity consisting of two partially transmitting mirrors with reflectivities  $\rho_0, \rho_1$ . The length of the cavity is expressed by tuning parameter  $\phi = \omega L/c$ , where  $\omega$  is the angular frequency of the light and  $c$  is the speed of light, thus one obtains

$$r_{\text{FP}} = [\rho_0 - \rho_1 \exp(2i\phi)]d, \quad (3)$$

$$t_{\text{FP}} = -\tau_0\tau_1 \exp(-i\phi)d, \quad (4)$$

where  $\rho_{0,1}$  and  $\tau_{0,1}$  denote the reflectance and transmittance of the two cavity mirrors, respectively, and we introduce the resonance factor

$$d = [1 - \rho_0\rho_1 \exp(2i\phi)]^{-1}. \quad (5)$$

The power gain  $g_{\text{FP}}$  inside the cavity is given by

$$g_{\text{FP}} = |\tau_0 d|^2. \quad (6)$$

The three-port coupler used in Ref. 1 can be represented by the following scattering matrix:

$$\mathbf{S}_{3p} = \begin{bmatrix} \eta_2 \exp(i\phi_2) & \eta_1 \exp(i\phi_1) & \eta_0 \exp(i\phi_0) \\ \eta_1 \exp(i\phi_1) & \rho_0 \exp(i\phi_0) & \eta_1 \exp(i\phi_1) \\ \eta_0 \exp(i\phi_0) & \eta_1 \exp(i\phi_1) & \eta_2 \exp(i\phi_2) \end{bmatrix}. \quad (7)$$

As stated above, the grating is assumed to be symmetrical with respect to the grating normal. The grating period and the wavelength of light are chosen such that for normal incidence only the zeroth- and first-order diffraction are present. The magnitudes of their amplitude reflection coefficients are denoted  $\rho_0$  and  $\eta_1$ , respectively. For incidence at the second-order Littrow angle the zeroth, first, and second diffraction orders are present with the magnitudes of reflection coefficients  $\eta_0, \eta_1$ , and  $\eta_2$ , as depicted in Fig. 1. From the unitarity condition of  $\mathbf{S}$  we find the energy-conservation law:

$$\rho_0^2 + 2\eta_1^2 = 1, \quad (8)$$

$$\eta_0^2 + \eta_1^2 + \eta_2^2 = 1. \quad (9)$$

We denote the phase shift associated with the zeroth, first, and second diffraction orders as  $\phi_0, \phi_1$ , and  $\phi_2$ , respectively. As for the mirrors, the values of the phases are not unique. Reflection from a mirror is equivalent to zeroth-order diffraction of a grating. In analogy to the right-hand matrix of Eqs. (2) we demand no phase shift for zeroth-order diffraction and therefore set  $\phi_0 = 0$ . From the unitarity requirement of  $\mathbf{S}$  the remaining phases can be calculated, yielding the following possible set of phases:

$$\phi_0 = 0, \quad (10)$$

$$\phi_1 = -(1/2)\arccos[(\eta_1^2 - 2\eta_0^2)/(2\rho_0\eta_0)], \quad (11)$$

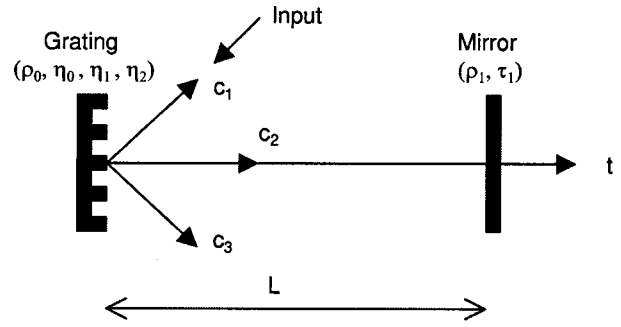


Fig. 2. Fabry-Perot cavity with a three-port grating coupler and a conventional end mirror. The amplitudes of the fields of interest ( $c_1, c_2, c_3, t$ ) are indicated by arrows.

$$\phi_2 = \arccos[-\eta_1^2/(2\eta_2\eta_0)]. \quad (12)$$

We emphasize that phases  $\phi_1$  and  $\phi_2$  are functions of the diffraction efficiencies and therefore vary depending on the properties of the grating. This contrasts with the properties of mirrors, where the phase shift between transmitted and reflected beams is independent of the transmittance and reflectance coefficients. Since phase  $\phi_2$  is a real number, the modulus of the argument of the arccos in Eq. (12) must be smaller than or equal to 1 and the following upper and lower limits for  $\eta_0$  and  $\eta_2$  for a given reflectivity  $\rho_0$  can be derived:

$$\eta_{0,\min}^{\max} = \eta_{2,\min}^{\max} = (1 \pm \rho_0)/2. \quad (13)$$

Note that these limits are fundamental in that a reflection grating can only be designed and manufactured with diffraction efficiencies within these boundaries. Equations (8)–(13) provide a full set of three-port coupling relations.

Knowledge of scattering matrix  $\mathbf{S}$  in Eq. (7) permits the calculation of input-output relations of interferometric topologies. Here we consider a three-port grating coupled Fabry-Perot cavity. The grating cavity is formed by placing a mirror with amplitude reflectivity  $\rho_1$  at a distance  $L$  parallel to the grating surface as is illustrated in Fig. 2. To characterize the cavity, amplitudes  $c_1, c_3$  for the two waves reflected from the cavity and intracavity amplitude  $c_2$  are calculated as a function of the cavity length. Assuming unity input and no input at port 3, the cavity is described by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{S}_{3p} \times \begin{pmatrix} 1 \\ \rho_1 c_2 \exp(2i\phi) \\ 0 \end{pmatrix}. \quad (14)$$

Solving for the amplitudes yields

$$c_1 = \eta_2 \exp(i\phi_2) + \eta_1^2 \exp[2i(\phi_1 + \phi)]d, \quad (15)$$

$$c_2 = \eta_1 \exp(i\phi_1)d, \quad (16)$$

$$c_3 = \eta_0 + \eta_1^2 \exp[2i(\phi_1 + \phi)]d, \quad (17)$$

$$t = i\tau_1 c_2 \exp(i\phi). \quad (18)$$

where  $\phi = \omega L/c$  is the tuning parameter,  $d$  is given according to Eq. (5), and  $t$  is the amplitude of the light transmitted through the cavity. The light power at the different ports is proportional to the squared moduli of the amplitudes. The power gain inside the cavity is given by  $|c_2|^2 = |\eta_1 d|^2$ , analogous to Eq. (6) for a conventional cavity. In contrast with the power gain, the power in the two reflecting ports  $|c_1|^2$  and  $|c_3|^2$  depends on  $\eta_2$  and  $\eta_0$ . Figure 3 illustrates how the power out of the backreflecting port varies as a function of  $\eta_2$  and the tuning  $\phi$  of the cavity. For simplicity a cavity with a perfect end mirror  $\rho_1 = 1$  is assumed. For a coupler with  $\eta_2 = \eta_{2,\max}$ , the cavity does not reflect any light back to the laser for a tuning of  $\phi = 0$ . This corresponds to an impedance-matched cavity that transmits all the light on resonance. For a coupler with  $\eta_{2,\min}$ , the situation is reversed and all the light is reflected back to the laser. For all other values of  $\eta_2$  the backreflected intensity has intermediate values and is significantly different from conventional cavities: the intensity as a function of cavity tuning is no longer symmetric to the  $\phi = 0$  axis.

Finally, we investigate the influence of loss in the cavity for a coupler with  $\eta_{2,\min}$ . Figure 4 illustrates the effect of an end mirror with transmittance  $\tau_1 > 0$  on the power of the two reflecting ports of the cavity on resonance. As a result, apart from the intracavity field, losses affect mainly the backreflecting port (dotted-dashed curve). The effect on the dark port (solid curve) is minor, as it stays essentially dark as long as the loss  $\tau_1^2$  is small compared with the coupling  $\eta_1^2$ .

In conclusion, we have investigated a three-port reflection grating and derived its coupling relations. A

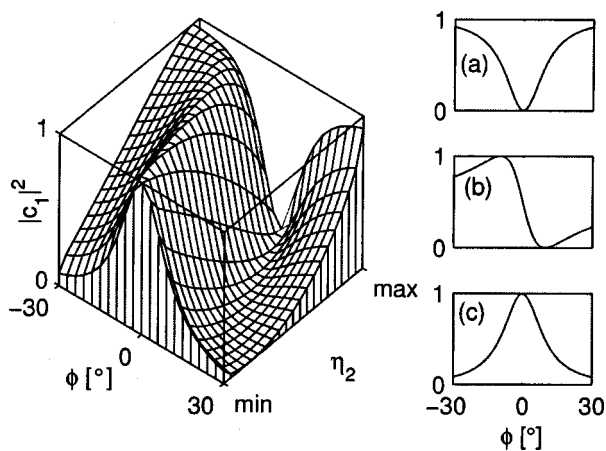


Fig. 3. Power  $|c_1|^2$  of cavity backreflecting port for gratings of different values of  $\eta_2$ . Left, power as a function of  $\phi$  and  $\eta_2$ , right, power as a function of  $\phi$  for (a)  $\eta_2 = \eta_{2,\max}$ , (b)  $\eta_2 = [(\eta_{2,\max}^2 + \eta_{2,\min}^2)/2]^{1/2}$ , (c)  $\eta_2 = \eta_{2,\min}$ . Cavity parameters:  $\rho_0 = 0.5$ ,  $\rho_1 = 1$ .

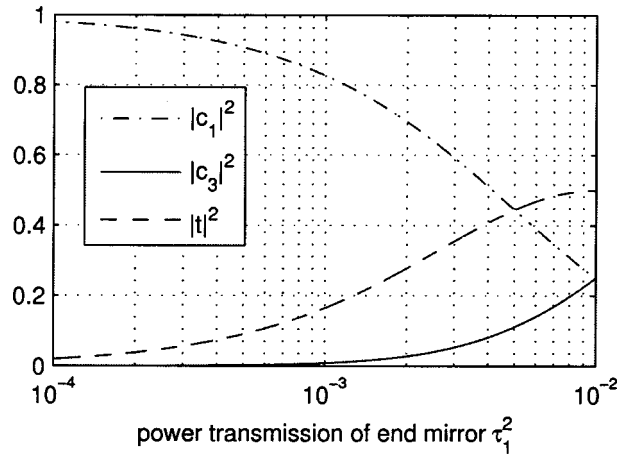


Fig. 4. Powers of the two reflected ports and the transmitting port as a function of end mirror transmittance  $\tau_1^2$  for a coupler with  $\rho_0^2 = 0.99$  and  $\eta_2 = \eta_{2,\min}$  for a tuning of  $\phi = 0$ .

three-port device can be used to couple light into a Fabry–Perot cavity. The input–output relations of such a three-port coupled cavity have revealed substantial differences from a conventional cavity. A grating with minimal  $\eta_2$  is suitable for a coupler to an arm cavity (single-ended cavity) of a gravitational-wave Michelson interferometer. On resonance all power is reflected back to the beam splitter of the interferometer. Hence no power is lost to the additional port. This makes possible power recycling that is used in all first- and probably also in second- and third-generation detectors. Furthermore we can calculate the phase signals carried by the fields in Eqs. (15) and (17) when cavity length  $L$  is changed and find that the additional port splits a cavity strain signal. However, the complete strain signal is still accessible to detection.

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## References

1. A. Bunkowski, O. Burmeister, P. Beyersdorf, K. Danzmann, R. Schnabel, T. Clausnitzer, E.-B. Kley, and A. Tünnermann, *Opt. Lett.* **29**, 2342 (2004).
2. R. W. P. Drever, in *Proceedings of the Seventh Marcel Grossman Meeting on General Relativity*, M. Keiser and R. T. Jantzen, eds. (World Scientific, Singapore, 1995), p. 1401.
3. S. Rowan, R. Byer, M. Fejer, R. Route, G. Cagnoli, D. Crooks, J. Hough, P. Sneddon, W. Winkler, *Proc. SPIE* **4856**, 292 (2003).
4. K.-X. Sun and R. L. Byer, *Opt. Lett.* **23**, 567 (1997).
5. A. Siegman, *Lasers* (University Science, Sausalito, Calif., 1986).