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Gradient Representations and Affine Structures in AE_n

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Abstract

We study the indefinite Kac–Moody algebras AE_n, arising in the reduction of Einstein's theory from $(n+1)$ space-time dimensions to one (time) dimension, and their distinguished maximal regular subalgebras $A_{n-1} \equiv \mathfrak{sl}_n$ and $A_{n-2}^{(1)}$. The interplay between these two subalgebras is used, for $n = 3$, to determine the commutation relations of the 'gradient generators' within AE_3. The low level truncation of the geodesic σ-model over the coset space $AE_n/K(AE_n)$ is shown to map to a suitably truncated version of the $SL(n)/SO(n)$ non-linear σ-model resulting from the reduction Einstein's equations in $(n+1)$ dimensions to (1+1) dimensions. A further truncation to diagonal solutions can be exploited to define a one-to-one correspondence between such solutions, and null geodesic trajectories on the infinite-dimensional coset space $\mathfrak{H}/K(\mathfrak{H})$, where $\mathfrak{H}$ is the (extended) Heisenberg group, and $K(\mathfrak{H})$ its maximal compact subgroup. We clarify the relation between $\mathfrak{H}$ and the corresponding subgroup of the Geroch group.

1 Introduction

Infinite-dimensional symmetries in gravity were first discovered by Geroch [1] in the context of (3+1)-dimensional general relativity after dimensional reduction to (1+1) dimensions (see [2, 3] for an introduction and further references). The group structure was later shown to be associated with the affine extension of Ehlers's $SL(2,\mathbb{R})$ symmetry [4, 5], and similar affinizations of hidden symmetries were discovered for other (super-)gravity theories [4, 5, 6]. Evidence for the emergence of an even larger symmetry corresponding to the Kac–Moody theoretic over-extension AE_3 of the original $SL(2,\mathbb{R})$ was presented in [7], following earlier suggestions of [4, 8]. For gravity in $(n+1)$ space-time dimensions, the conjectured symmetry is AE_n, while for eleven-dimensional supergravity it is E_{10}, which contains AE_{10} as a subalgebra governing the gravitational sector of this theory.

In [9] this conjecture was re-examined using the insights from the study of cosmological billiards (reviewed in [10]). Besides a remarkable dynamical match of a certain 'geodesic' one-dimensional σ-model with the gravity theory, the conjecture was made that the Kac–Moody algebra[1] allows for a re-emergence of the dependence on the coordinates along which the theory had been reduced. This would entail a 'dimensional transmutation', in the sense that the evolution of the geometrical data of a higher-dimensional theory, usually governed by a set of partial differential equations, can be mapped to a *one-dimensional null-geodesic motion on some infinite-dimensional coset space*. This conjecture was based on the observation that the hyperbolic algebra E_{10} contains a set of generators possessing the correct structure for higher order gradients in the suppressed directions. These, and their analogs in AE_n, will be called 'gradient generators' in the present paper. Further progress in the study of the one-dimensional σ-model is partly (but not only) hindered by the lack of known commutation relations. Although the irreducible representations appearing in level expansions of these algebras w.r.t. to their $\mathfrak{sl}_n$ subalgebras can be determined rather efficiently on the computer [11], the commutators are much harder to obtain[2]. Some progress on this front was reported recently in [12] where an algorithm for computing commutation relations in a Borel subalgebra was outlined. Different aspects of the one-dimensional model were studied in [13, 14, 15].

In this paper we study the σ-model based on AE_n, extending previous results of [10], in order to examine aspects of the general picture explained above. Our focus is on AE_n rather than E_{10}, since the core difficulties which one encounters in matching the one-dimensional σ-model and the higher dimensional field equations, appear right away at levels ± 1 in a graded decomposition of AE_n under its $\mathfrak{sl}_n$

[1] We often use the acronyms 'KMA' for 'Kac–Moody Algebra' and 'CSA' for 'Cartan subalgebra'.
[2] See also remarks after (2.18) to appreciate the challenge.

subalgebra. In other words, the key problem of elevating the linear duality of free spin-2 theories to the non-linear level must be faced already in the first step, whereas for E_{10}, the difficulties become only visible at $\ell = 3$ and beyond, because the duality relating the 3- and 6-form fields is still a *linear* one, modulo metric factors, just like in Maxwell theory.

As we will show, the 'gradient representations' are intimately linked to the affine subalgebra $A_{n-2}^{(1)}$ of AE_n. Exploiting the interplay of this affine subalgebra with the finite-dimensional $A_{n-1} \equiv \mathfrak{sl}_n$ subalgebra we are able to derive an infinite new set of structure constants for AE_3. After restriction of the AE_n σ-model to the affine $A_{n-2}^{(1)}$ subsector we exhibit a map between special solutions of the σ-model and solutions of the gravitational field equations with $(n-1)$ commuting spacelike Killing vectors, corresponding to a reduction from $(n+1)$ dimensions to $(1+1)$ dimensions, but with a restricted space dependence. These results are analogous to previous ones in [9, 10, 13, 14, 15], but permit us to to expose the remaining discrepancies in the simplest possible context. In order to focus on these difficulties, we truncate the affine σ-model further to a 'Heisenberg coset model' $\mathfrak{H}/K(\mathfrak{H})$, which can be solved exactly, and whose gravitational counterpart corresponds to diagonal metric solutions with two commuting (spacelike) Killing vectors (known as 'polarised Gowdy cosmologies' [16, 17]). We will then exhibit an explicit one-to-one correspondence between these two models, by showing how the general initial data of the Heisenberg σ-model and the null geodesic trajectories on $\mathfrak{H}/K(\mathfrak{H})$ which they generate, can be mapped to a general space- and time-dependent diagonal metric configuration satisfying Einstein's equation. In this way, we are able to validate the 'gradient hypothesis', and thereby the main conjecture of [9], at least in this simplified context.

We also elucidate the relation between the standard action of the (restricted) Geroch group on diagonal solutions, and the action of the Heisenberg group $\mathfrak{H}$ on the null geodesics. More specifically, we will show that the action of these two groups coincides on the domain where the Geroch group acts non-trivially. Let us recall that only 'half' of the Geroch group acts non-trivially in the standard realisation [2, 5], whereas the other half merely shifts the integration constants arising in the definition of the higher order 'dual potentials', and has no effect on the physical metric. For this reason, the standard Geroch group affects only part of the initial data (the full initial data of the off-diagonal degrees of freedom, but only the 'initial coordinates' of the scale factors). By contrast, the realization proposed here is such that Heisenberg subgroup of the Geroch group acts non-trivially on *all* initial data.

Other attempts to generate space-time dependence through algebraic constructions based on E_{11} [18] and similar 'very-extended' algebras [19] have been developed and discussed in [20, 21, 22]. In [20] this was achieved by including a certain irreducible representation of the relevant very-extended algebra containing translation

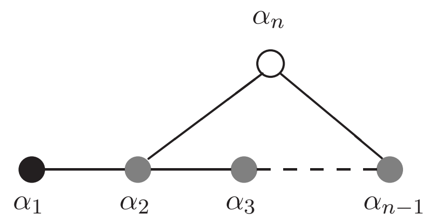


Figure 1: *The Dynkin diagram of AE_n with labelling of the simple roots shown. Solid nodes (black and grey) belong to $A_{n-1} \equiv \mathfrak{sl}_n$ which gets enhanced to $\mathfrak{gl}_n$. Grey nodes mark the common 'horizontal' $A_{n-2} \equiv \mathfrak{sl}_{n-1}$ subalgebra.*

generators and (infinitely many) other generalized central charges, whereas in [21] space-time is thought to occur through an auxiliary parameter and gradient representation in restricted models. A key difference between the present approach and [18] is that the correspondence exhibited here works *only after the gauge degrees of freedom on both sides have been eliminated by suitable gauge choices* : on the σ-model side this implies the complete elimination of the degrees of freedom associated with the maximal compact subgroups, whereas on the gravitational side, it involves not only gauge-fixing the vielbein (as discussed in section 5.1), but also solving the canonical constraints. As a consequence, the global symmetry relates solutions which are *physically distinct*. By contrast, the proposal of [18, 20] seeks a 'covariant formulation'. This means, that the symmetry is actually much larger than the relevant very-extended Kac–Moody group, as it must contain general coordinate transformations (for instance, via the closure with the conformal group [23]) and other gauge transformations; hence, one must disentangle transformations relating gauge equivalent configurations from those which generate physically inequivalent solutions. While it is not possible to really discriminate between the different proposals by analysing low level degrees of freedom, the issue will be decided by whether and how the higher level fields can be fitted into the scheme. The present paper is a first step in this direction.

This article is structured as followed. First we define and analyse the KMA AE_n and its two distinguished maximal regular subalgebras $A_{n-2}^{(1)}$ and A_{n-1} and define the gradient representations in this context in section 2. In section 3, we compute infinitely many new structure constants involving the gradient generators in the special case of AE_3 by combining these two subalgebras. The σ-model based on AE_n is defined in section 4 where we study its restriction to $A_{n-2}^{(1)}$. (The restriction to A_{n-1} is studied in appendix B where we also study gravity coupled to p-forms.) The map from (parts of) the affine model to the two-dimensional reduction of gravity is deduced in section 5. Finally, by using the Heisenberg model we examine the relation of the σ-model to the Geroch group in section 6. Appendix A contains a proof of the fact that the compact subalgebra of a (split) KMA is *not* a KMA.

2 Distinguished maximal regular subalgebras of AE_n

2.1 Definition of AE_n

The indefinite Kac–Moody algebras AE_n [3] (for $n \geq 3$) are defined in the usual way via the Chevalley–Serre presentation [24] associated with the Dynkin diagram displayed in figure 1. The simple positive and negative generators are denoted by e_a and f_a respectively, the Cartan subalgebra (CSA) generators by h_a, where $a = 1, 2, \ldots, n$. We consider the algebra over $\mathbb{R}$ in split real form. AE_n is hyperbolic for $n \leq 9$; this means that all Dynkin subdiagrams obtained by removing one or more nodes are either affine or finite. We will also need to make use of the Chevalley involution ω, defined on the simple generators by

$$\omega(e_a) = -f_a, \qquad \omega(f_a) = -e_a, \qquad \omega(h_a) = -h_a, \tag{2.1}$$

and the generalized transposition

$$x^T := -\omega(x). \tag{2.2}$$

The 'maximal compact' subalgebra $K(AE_n)$ of AE_n is then defined as the invariant algebra w.r.t. the involution ω, viz.

$$K(AE_n) := \{x \in AE_n \,|\, x = \omega(x)\} \tag{2.3}$$

and consists of the 'antisymmetric' Lie algebra elements in view of (2.2).

In this section we shall consider two distinguished maximal regular subalgebras of AE_n, namely $\mathfrak{sl}_n$ and the affine $A_{n-2}^{(1)}$, obtained by removing the nodes labeled n and 1, respectively. The first generates the group of special linear transformations acting on the spatial n-bein of gravity in $(n+1)$ space-time dimensions, and can be enlarged to $\mathfrak{gl}_n$ by inclusion of the CSA generator associated with the white node in figure 1. The second subalgebra corresponds to the generalization of the Geroch group that is obtained after reduction of pure gravity from $(n+1)$ dimensions to $(1+1)$ dimensions.

[3] We will designate by AE_n *both* the group *and* the Lie algebra, as it should be clear from the context which is meant.

2.2 $A_{n-1} \equiv \mathfrak{sl}_n$ Subalgebra and level decomposition of AE_n

The regular subalgebra of type $A_{n-1} \equiv \mathfrak{sl}_n$ is generated by considering only commutators of the simple generators associated with nodes 1 up to $(n-1)$ in figure 1. By including the Cartan subalgebra element h_n one can extend $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$. Its generators are denoted by $K^a{}_b$ $(a, b = 1, 2, \ldots, n)$ and obey the standard commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b. \tag{2.4}$$

For these generators, the transposition (2.2) reduces to $(K^T)^a{}_b = K^b{}_a$. (2.4) entails the following identification of the $\mathfrak{gl}_n$ elements with the Chevalley-Serre generators of AE_n for $i = 1, 2, \ldots, n-1$

$$e_i = K^i{}_{i+1}, \qquad f_i = K^{i+1}{}_i, \qquad h_i = K^i{}_i - K^{i+1}{}_{i+1}. \tag{2.5}$$

Regularity of the $\mathfrak{gl}_n$ subalgebra means that the standard invariant bilinear form on AE_n, which is defined by

$$\langle e_i | f_j \rangle = \delta_{ij} \quad ; \qquad \langle h_i | h_j \rangle = A_{ij} \tag{2.6}$$

coincides with the usual Cartan–Killing form on $\mathfrak{gl}_n$ when restricted to the latter subalgebra (A_{ij} is the Cartan matrix of AE_n, which is simply laced). Explicitly, the scalar product on $\mathfrak{gl}_n$ is given by

$$\langle K^a{}_b | K^c{}_d \rangle = \delta^c_b \delta^a_d - \delta^a_b \delta^c_d. \tag{2.7}$$

It is straightforward to check that this is indeed consistent with (2.6). Let us also express the trace $K = \sum_{a=1}^n K^a{}_a$ in $\mathfrak{gl}_n$ in terms of the CSA generators h_a; we have

$$K = -(n-1)h_1 - (2n-2)h_2 - (2n-3)h_3 - \ldots - n\, h_n. \tag{2.8}$$

Solving for h_n gives

$$h_n = -K^1{}_1 - K^2{}_2 + K^n{}_n = -K + K^3{}_3 + K^4{}_4 + \ldots + K^{n-1}{}_{n-1} + 2K^n{}_n \tag{2.9}$$

The adjoint representation of AE_n can be decomposed under the (adjoint) action of $\mathfrak{gl}_n$ into an infinite tower of $\mathfrak{sl}_n$ representations. The $\mathfrak{sl}_n$ level ℓ, or simply the *level*, of a given representation counts the number of occurrences of the simple root α_n in the corresponding AE_n root α, *i.e.*

$$\alpha = \ell \alpha_n + \sum_{j=1}^{n-1} m^j \alpha_j \tag{2.10}$$

This level is left invariant by the action of $\mathfrak{sl}_n$ and provides an elliptic slicing of the forward lightcone in the AE_n root lattice. Level $\ell = 0$ contains the adjoint of $\mathfrak{gl}_n$. At level $\ell = 1$ we have the representation

$$[0, 1, 0, 0, \ldots, 0, 1] \longleftrightarrow E^{a_1 \ldots a_{n-2}, a_{n-1}} \tag{2.11}$$

associated with the Young tableau

$$\begin{array}{|c|c|} \hline a_1 & a_{n-1} \\ \hline a_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_{n-2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

This tensor is therefore antisymmetric in its first $(n-2)$ indices $[a_1 \ldots a_{n-2}]$, and obeys

$$E^{[a_1 \ldots a_{n-2}, a_{n-1}]} \;\; = \;\; 0. \tag{2.12}$$

Under Chevalley transposition we obtain

$$F_{a_1 \ldots a_{n-2}, a_{n-1}} \equiv -\omega(E^{a_1 \ldots a_{n-2}, a_{n-1}}). \tag{2.13}$$

Under $\mathfrak{gl}_n$ the tensors $E^{a_1 \ldots a_{n-2}, a_{n-1}}$ and $F_{a_1 \ldots a_{n-2}, a_{n-1}}$ transform contravariantly or covariantly, as indicated by the position of indices, for instance

$$[K^a{}_b, E^{c_1 \ldots c_{n-2}, c_{n-1}}] = \delta_b^{c_1} E^{a c_2 \ldots c_{n-2}, c_{n-1}} + \cdots + \delta_b^{c_{n-1}} E^{c_1 \ldots c_{n-2}, a} \tag{2.14}$$

Hence the level is counted by the operator $\frac{1}{n-1} K$.

The identification of the AE_n Chevalley–Serre basis is completed by relating the generators e_n and f_n to the level $\ell = \pm 1$ generators via

$$e_n = E^{3\,4 \ldots n, n}, \qquad f_n = F_{3\,4 \ldots n, n}. \tag{2.15}$$

The commutator $[(\ell{=}1), (\ell{=}{-1})]$ is best written using an auxiliary (dummy) tensor $X_{a_1, \ldots a_{n-2}, a_{n-1}}$ (with the same Young symmetries as F) in the form

$$\begin{aligned} \big[X_{a_1 \ldots a_{n-2}, a_{n-1}} E^{a_1 \ldots a_{n-2}, a_{n-1}}, F_{b_1 \ldots b_{n-2}, b_{n-1}}\big] &= -(n-2)! \Big(X_{b_1 \ldots b_{n-2}, b_{n-1}} K \\ &- X_{b_1 \ldots b_{n-2}, e} K^e{}_{b_{n-1}} - (n-2) K^e{}_{[b_{n-2}} X_{b_1 \ldots b_{n-3}] e, b_{n-1}} \Big). \end{aligned} \tag{2.16}$$

This is consistent with the normalisation[4]

$$\begin{aligned}&\langle E^{a_1\dots a_{n-2},n-1}|F_{b_1\dots b_{n-2},b_{n-1}}\rangle \\ &\quad= -\frac{n-2}{n-1}(n-2)!\Big(\delta^{a_1\dots a_{n-2}}_{b_1\dots b_{n-2}}\delta^{a_{n-1}}_{b_{n-1}} + \delta^{[a_1}_{b_{n-1}}\delta^{a_2\dots a_{n-2}]}_{[b_2\dots b_{n-2}}\delta^{a_{n-1}}_{b_1]}\Big),\end{aligned} \tag{2.17}$$

corresponding to the standard normalisation for the Chevalley generators

$$\langle e_n|f_n\rangle = 1. \tag{2.18}$$

All higher levels in the $\mathfrak{sl}_n$ decomposition can be obtained by taking multiple commutators of the level $\ell = 1$ generator $E^{a_1\dots a_{n-2},a_{n-1}}$. For AE_3, the decomposition into irreducible representations of $\mathfrak{sl}_3$ is known up to level $\ell = 56$ [11]. Counting outer multiplicities, the total number of $\mathfrak{sl}_3$ representations up to that level is [25]

$$\#\big(\text{representations for } \ell \le 56\big) = 20\,994\,472\,770\,550\,672\,476\,591\,949\,725\,720 \tag{2.19}$$

Consequently, the complete table of structure constants up to that level will already contain more than 10^{62} entries! Hence, the 'gradient representations' that we will consider below constitute only a tiny subsector (but not a subalgebra) of the full Lie algebra.

2.3 Affine $A^{(1)}_{n-2}$ Subalgebra

Figure 1 also shows that AE_n has a regular affine subalgebra $A^{(1)}_{n-2} \equiv \widehat{\mathfrak{sl}_{n-1}} \oplus \mathbb{R}\hat{c} \oplus \mathbb{R}\hat{d}$ generated by nodes 2 up to n (the circular subdiagram in figure 1).[5] We write the corresponding *traceless* generators as $\bar{K}^{\alpha}_{m\,\beta}$ with $m \in \mathbb{Z}$, and Greek indices $\alpha, \beta, \dots \in \{2, \dots, n\}$. The generators which belong to both this affine subalgebra and the $\mathfrak{sl}_{n-1}$ constitute what we call the *horizontal algebra* $\mathfrak{sl}_{n-1}$ corresponding to the nodes 2 up to $n-1$, with generators

$$\bar{K}^{\alpha}_{0\,\beta} = K^{\alpha}{}_{\beta} - \frac{1}{n-1}\delta^{\alpha}_{\beta}K^{\gamma}{}_{\gamma} \tag{2.20}$$

[4]Symmetrization and antisymmetrization is defined with 'strength one' throughout this paper, such as for instance in

$$\delta^{a_1a_2}_{b_1b_2} := \frac{1}{2}\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \frac{1}{2}\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} \quad , \qquad \bar{\delta}^{a_1a_2}_{b_1b_2} := \frac{1}{2}\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + \frac{1}{2}\delta^{a_1}_{b_2}\delta^{a_2}_{b_1}.$$

where a bar over δ denotes symmetrisation, and no bar means antisymmetrisation.

[5]We follow the convention that $\widehat{\mathfrak{sl}_{n-1}}$ denotes the loop algebra based on $A_{n-2} \equiv \mathfrak{sl}_{n-1}$. The non-twisted affine algebra $A^{(1)}_{n-2}$ is obtained by adjoining the central element $\hat{c}$ and the derivation $\hat{d}$ to the Cartan subalgebra, which then has dimension n [24]. (The rank of $A^{(1)}_{n-2}$ is $n-1$.)

in the $\mathfrak{gl}_n$ from above. The relevant nodes in figure 1 are marked in grey.

The affine commutation relations are

$$[\bar{K}^\alpha_{m\beta}, \bar{K}^\gamma_{n\delta}] = \delta^\gamma_\beta \bar{K}^\alpha_{m+n\delta} - \delta^\alpha_\delta \bar{K}^\gamma_{m+n\beta} + m\delta_{m,-n}\left(\delta^\alpha_\delta \delta^\gamma_\beta - \frac{1}{n-1}\delta^\alpha_\beta\delta^\gamma_\delta\right)\hat{c}. \quad (2.21)$$

The central element $\hat{c}$ and the derivation $\hat{d}$ required for the n-dimensional CSA of $A^{(1)}_{n-2}$ can be identified in the CSA of the rank n KMA AE_n. Requiring the affine level counting operator $\hat{d}$ to obey

$$[\hat{d}, e_n] = -e_n, \quad , \quad [\hat{d}, e_i] = 0 \quad (i = 2, \dots, n-1), \quad \langle \hat{d}|\hat{d}\rangle = 0, \quad (2.22)$$

results in the following expressions for $\hat{c}$ and $\hat{d}$ in terms of the $\mathfrak{gl}_n$ elements

$$\hat{c} = \sum_{i=2}^{n} h_i = -K^1{}_1, \quad (2.23)$$

$$\hat{d} = \frac{n-2}{2(n-1)}K^1{}_1 - \frac{1}{n-1}K^\alpha{}_\alpha, \quad (2.24)$$

such that

$$\begin{aligned} K = K^a{}_a &= K^1{}_1 + K^\alpha{}_\alpha = -(n-1)\hat{d} - \frac{n}{2}\hat{c}, \quad (2.25) \\ &\Longrightarrow K^\alpha{}_\alpha = -(n-1)\,\hat{d} - \frac{n-2}{2}\,\hat{c}. \end{aligned}$$

The CSA elements $\hat{c}$ and $\hat{d}$ are normalised according to

$$\langle \hat{c}|\hat{d}\rangle = -1 \quad , \quad \langle \hat{d}|\hat{d}\rangle = \langle \hat{c}|\hat{c}\rangle = 0, \quad (2.26)$$

and the other non-vanishing inner product is

$$\left\langle \bar{K}^\alpha_{m\beta}|\bar{K}^\gamma_{n\delta}\right\rangle = \delta_{m,-n}\left(\delta^\alpha_\delta\delta^\gamma_\beta - \frac{1}{n-1}\delta^\alpha_\beta\delta^\gamma_\delta\right), \quad (2.27)$$

which is compatible with (2.6).

Alternatively, the adjoint of AE_n can be decomposed under the action of $A^{(1)}_{n-2}$. In this case, the grading is labeled by the *affine level*, which is equal to the coefficient of the root α_1 (now providing a parabolic slicing of the forward lightcone in the root lattice). This decomposition was introduced and studied in [26] for AE_3, but we will not require these results here.[6]

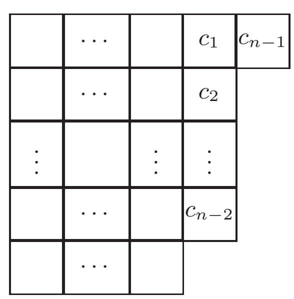
[6] Beyond the so-called basic representation on level one there are already infinitely many $A^{(1)}_{n-2}$ representations on affine level two, which can be calculated from modularity and an appropriate implementation of the Serre relations [26]. This is true not only for AE_3 but also for AE_n.

2.4 Horizontal $\mathfrak{sl}_{n-1}$ and Gradient Representations

Writing the Kac–Moody algebra AE_n as a graded representation of its $\mathfrak{sl}_n$ subalgebra is not independent of making use of the affine $A_{n-2}^{(1)}$ subalgebra. This is due to the identification of common generators in the horizontal algebra $\mathfrak{sl}_{n-1} \equiv A_{n-2} = A_{n-1} \cap A_{n-2}^{(1)}$. We repeat that this is the A_{n-2} subalgebra of AE_n generated by the $n-2$ (grey) nodes 2 up to $n-1$ in diagram 1, with the generators given in eq. (2.20). In particular, the affine generator $\bar{K}^{\alpha}_{m\beta}$ on affine level m, transforming in the adjoint of the horizontal $\mathfrak{sl}_{n-1}$, also transforms under $\mathfrak{sl}_n \supset \mathfrak{sl}_{n-1}$, and it is natural to ask to which $\mathfrak{sl}_n$ representation it belongs. This is most easily determined by considering the lowest weight vector and its Dynkin labels with respect to $\mathfrak{sl}_n$. The lowest weight vector lies in the root space of the level-m AE_n root (assuming $m \geq 1$ from now on)

$$\alpha = m\alpha_n + (m-1)\sum_{j=2}^{n-1}\alpha_j \tag{2.28}$$

corresponding to $\mathfrak{sl}_n$ Dynkin labels $[m-1, 1, 0, \ldots, 0, 1]$. We see that this is consistent with the common horizontal $\mathfrak{sl}_{n-1}$: In terms of the Dynkin labels restricting to the horizontal A_{n-2} of $A_{n-2}^{(1)}$ means dropping the first entry of $[m-1, 1, 0, \ldots, 0, 1]$ and the resulting representation is indeed simply the adjoint $[1, 0, \ldots, 0, 1]$ of the horizontal $\mathfrak{sl}_{n-1}$ as anticipated. The generator corresponding to the $[m-1, 1, 0, \ldots, 0, 1]$ representation of $\mathfrak{sl}_n$ is therefore associated with the following Young tableau



This level-m element of AE_n thus has $(m-1)$ symmetric sets of $(n-1)$ antisymmetric indices and then $(n-1)$ indices with the structure of the representation on $\mathfrak{sl}_n$ level $\ell = 1$, cf. (2.11)–(2.12). Because this is rather cumbersome to write out, it is convenient to treat this tensor as a representation of $\mathfrak{sl}_n$, and to dualise the sets of antisymmetric indices by means of the $\mathfrak{sl}_n$ ε-symbol. The result is an $\mathfrak{sl}_n$ tensor with $(m-1)$ *lower* indices

$$E_{b_1\ldots b_{m-1}}{}^{c_1\ldots c_{n-2}, c_{n-1}} \tag{2.29}$$

which is *symmetric* in these indices, while the upper indices belong to the level-one representation (2.11). There is a similar expression for the transposed F generator. We stress that (2.4) and (2.29) are equivalent *only* as representations of $\mathfrak{sl}_n$, but not $\mathfrak{gl}_n$, as they carry different $\mathfrak{gl}_1$ charges. The commutation relations must therefore be amended by a correction term for the transformation under the trace K in order to account for this charge. We will come back to this point below for AE_3.

As noted in [9] the infinite tower of representations (2.29) has precisely the right structure corresponding to the *spatial gradients* of the low level fields, such that each index b_i becomes associated with a gradient operator $\partial/\partial x^{b_i}$, and the irreducibility condition of the above Young tableau translates to vanishing divergence. More specifically, for AE_n, the relevant degree of freedom is the *dual graviton*, corresponding to the above tableau with only the right two columns as in (2.11), and appears already at level $\ell = 1$ (for E_{10} it appears only at level $\ell = 3$).[7] For this reason, we will refer to these representations as *gradient representations*. Let us emphasize that, except for the lowest levels, this is so far only a kinematical correspondence.[8]

The other main feature of the gradient representations is that they contain the affine generators. More specifically, the latter are obtained by performing a 'dimensional reduction', restricting the gradient indices (*i.e.* the lower indices in (2.29)) to the single value $b_1 = \cdots = b_{m-1} = 1$, and the upper indices to the remaining values $\alpha, \beta, \cdots = 2, \ldots, n$. This truncation corresponds precisely to a dimensional reduction of Einstein's theory from $(n+1)$ to (1+1) spacetime dimensions, with spacelike Killing vectors $\partial/\partial x^2, \ldots, \partial/\partial x^n$, and $x \equiv x^1$ as the surviving spatial coordinate. The precise identification of $\bar{K}^{\alpha}_{m\beta}$ (for $m > 0$) as part of the $\mathfrak{sl}_n$ tensor (2.29) is

$$\bar{K}^{\alpha}_{m\beta} = \frac{1}{(n-2)!}\varepsilon_{\beta\gamma_1\ldots\gamma_{n-2}} E_{\underbrace{1\ldots1}_{(m-1)}}{}^{\gamma_1\ldots\gamma_{n-2},\alpha}, \tag{2.30}$$

where the indices $\alpha, \beta, \cdots = 2, \ldots, n$ have been restricted to $\mathfrak{sl}_{n-1}$. For affine level $m < 0$ the identification is

$$\bar{K}^{\alpha}_{m\beta} = \frac{1}{(n-2)!}\varepsilon^{\alpha\gamma_1\ldots\gamma_{n-2}} F^{\overbrace{1\ldots1}^{(m-1)}}{}_{\gamma_1\ldots\gamma_{n-2},\beta}. \tag{2.31}$$

The correctness of these identifications is easy to check at levels $|m| \leq 1$.

[7] The dual graviton representation has been discussed previously in [27, 28, 29, 9, 30, 31, 32, 33].

[8] We note that there are generators resembling a k-th spatial derivative (on level $\ell + k$) of any given generator (on level ℓ). These will generally occur with exponentially increasing outer multiplicity as they belong to a standard lowest weight representation the affine algebra. The gradient representations (2.29) are distinguished by being generated from the adjoint representation of the affine algebra and hence have constant outer multiplicity, equal to 1.

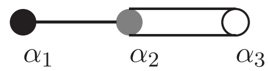


Figure 2: *The Dynkin diagram of AE_3 with labelling of the simple roots. Solid nodes belong to $\mathfrak{sl}_3$ which gets enhanced to $\mathfrak{gl}_3$. The grey node marks the horizontal $A_1 \equiv \mathfrak{sl}_2$.*

One can now exploit the interplay between $\mathfrak{sl}_n$ and $A_{n-2}^{(1)}$ to determine the complete commutators of the 'gradient generators' modulo non-gradient representations. This is achieved by writing down the general ansatz for a commutator in $\mathfrak{sl}_n$ covariant form, and then restricting to the horizontal $\mathfrak{sl}_{n-1}$ and using the the affine relations (2.21). In the next section, we exemplify this procedure for AE_3.

3 AE_3 and its gradient commutators via $\mathfrak{sl}_3$ and $A_1^{(1)}$

3.1 AE_3 in $\mathfrak{sl}_3$ form

The Dynkin diagram of AE_3 is displayed in figure 2. Before computing the contribution of the gradient generators to commutators for AE_3 we give the explicit formulæ for the $\mathfrak{sl}_3$ generators and the appropriate low $\mathfrak{sl}_3$ level commutators of AE_3.

In this case, the general identifications (2.8) and (2.9) read

$$h_3 = -K + 2K^3{}_3, \qquad K = -2h_1 - 4h_2 - 3h_3. \tag{3.1}$$

The $\mathfrak{gl}_3$ commutation relations and inner products are identical in form to (2.4) but with index ranges now restricted to $a = 1, 2, 3$. The decomposition of AE_3 under its $\mathfrak{sl}_3$ subalgebra results in table 1, containing levels $\ell = 1, \ldots, 5$.

Generators belonging to the class of gradient generators have been marked with an asterisk. Our notation for representations of $\mathfrak{gl}_3$ is the one inherited from $\mathfrak{sl}_3$. The general $\ell = 1$ generator (2.11) reduces to a generator of the type

$$E^{b_1 b_2}, \tag{3.2}$$

where we have dropped the separating comma. The Young irreducibility constraint (2.12) now implies that this tensor is symmetric,

$$E^{b_1 b_2} = E^{b_2 b_1}. \tag{3.3}$$

For $\mathfrak{sl}_3$ one can write any representation with Dynkin labels $[p, q]$ as a tensor with p lower and q upper indices, after lowering the p pairs of antisymmetric indices with ε

ℓ	$[p_1, p_2]$	α	α^2	multα	μ	Gradient
1	[0,2]	(0,0,1)	2	1	1	*
2	[1,2]	(0,1,2)	2	1	1	*
3	[2,2]	(0,2,3)	2	1	1	*
	[1,1]	(1,3,3)	-4	3	1	
4	[3,2]	(0,3,4)	2	1	1	*
	[2,1]	(1,4,4)	-6	5	2	
	[1,0]	(2,5,4)	-10	11	1	
	[0,2]	(2,4,4)	-8	7	1	
	[1,3]	(1,3,4)	-2	2	1	
5	[4,2]	(0,4,5)	2	1	1	*
	[3,1]	(1,5,5)	-8	7	3	
	[2,0]	(2,6,5)	-14	22	3	
	[0,1]	(3,6,5)	-16	30	2	
	[0,4]	(2,4,5)	-6	5	2	
	[1,2]	(2,5,5)	-12	15	4	
	[2,3]	(1,4,5)	-4	3	2	

Table 1: *Adjoint of AE_3 decomposed under regular $\mathfrak{sl}_3$ subalgebra. The charge of an element X on level ℓ under the trace $K = K^a{}_a$ of $\mathfrak{gl}_3$ is $[K, X] = 2\ell X$ as explained in the text.*

of $\mathfrak{sl}_3$. The irreducibility constraints are then symmetry in lower and upper indices separately and vanishing of any contraction. The totally antisymmetric tensor ε_{abc} used for lowering the indices, however, is *not* invariant under $\mathfrak{gl}_3$ as can be checked easily. That means that while the $\ell = 2$ tensor with natural indices $E^{c_1c_2,b_1b_2}$ transforms under $\mathfrak{gl}_3$ as the indices suggest, *i.e.*

$$[K^a{}_b, E^{c_1c_2,b_1b_2}] = \delta_b^{c_1} E^{ac_2,b_1b_2} + \delta_b^{c_2} E^{c_1a,b_1b_2} + \delta_b^{b_1} E^{c_1c_2,ab_2} + \delta_b^{b_2} E^{c_1c_2,b_1a}, \quad (3.4)$$

the corresponding tensor with lowered indices does *not* transform in the obvious way. Rather one has to add a compensating term for the transformation under the trace $K = K^a{}_a$. Generally,

$$[K^c{}_d, E_{a_1\dots a_k}{}^{b_1b_2}] = -k\delta^c_{(a_1} E_{a_2\dots a_k)d}{}^{b_1b_2} + 2\delta_d^{(b_1} E_{a_1\dots a_k}{}^{b_2)c} + k\delta_d^c E_{a_1\dots a_k}{}^{b_1b_2}. \quad (3.5)$$

The last term is the necessary correction: The charge of the original tensor on level $k+1$ with $2k+2$ upper indices under the trace K is $2k+2$, and this charge has to be maintained. Similar remarks apply to the Chevalley transposed generators

$$F^{a_1\dots a_p}{}_{b_1\dots b_q} = (E_{a_1\dots a_p}{}^{b_1\dots b_q})^T. \quad (3.6)$$

We now present the commutation of the AE_3 generators for levels $|\ell| \leq 2$ in $\mathfrak{sl}_3$ form. These can be deduced from table 1 after the generators have been normalised in the standard bilinear form.[9] For $\ell = 1, 2$ we demand

$$\begin{aligned} \langle E^{b_1 b_2} | F_{d_1 d_2} \rangle &= \bar{\delta}^{b_1 b_2}_{d_1 d_2} := \delta^{(b_1}_{d_1} \delta^{b_2)}_{d_2}, &(3.7)\\ \langle E_a{}^{b_1 b_2} | F^c{}_{d_1 d_2} \rangle &= P_a{}^{b_1 b_2}|^c{}_{d_1 d_2} := \delta^c_a \bar{\delta}^{b_1 b_2}_{d_1 d_2} - \frac{1}{2} \delta^{(b_1}_a \delta^{b_2)}_{(d_1} \delta^c_{d_2)}. &(3.8) \end{aligned}$$

The commutation relations are then

$$\begin{aligned} [E^{b_1 b_2}, F_{d_1 d_2}] &= -\bar{\delta}^{b_1 b_2}_{d_1 d_2} K + 2 \delta^{(b_1}_{(d_1} K^{b_2)}{}_{d_2)}, &(3.9)\\ [E^{b_1 b_2}, E^{c_1 c_2}] &= 2 \varepsilon^{b_1 c_1 a} E_a{}^{b_2 c_2}, &(3.10)\\ [F_{d_1 d_2}, F_{c_1 c_2}] &= -2 \varepsilon_{d_1 c_1 a} F^a{}_{d_2 c_2}, &(3.11)\\ [E_a{}^{b_1 b_2}, F_{d_1 d_2}] &= 2 \varepsilon_{ae(d_1} \delta^{(b_1}_{d_2)} E^{b_2)e}, &(3.12)\\ [F^c{}_{d_1 d_2}, E^{b_1 b_2}] &= 2 \varepsilon^{ce(b_1} \delta^{b_2)}_{(d_1} F_{d_2)e}, &(3.13)\\ [E_a{}^{b_1 b_2}, F^c{}_{d_1 d_2}] &= -P_a{}^{b_1 b_2}|^c{}_{d_1 d_2} K + 2 \delta^c_a \delta^{(b_1}_{(d_2} K^{b_2)}{}_{d_2)} \\ &\quad - \bar{\delta}^{b_1 b_2}_{d_1 d_2} K^c{}_a - \frac{1}{2} \delta^{(b_1}_a K^{b_2)}{}_{(d_1} \delta^c_{d_2)}. &(3.14) \end{aligned}$$

Here, symmetrisation over lower or upper non-contracted indices is implicit.

3.2 $A_1^{(1)}$ and gradient commutators

The identification of the affine $A_1^{(1)} \subset AE_3$ subalgebra is achieved through (cf. (2.23))

$$\begin{aligned} \hat{c} &= h_2 + h_3 = -K^1{}_1 \;\;, \quad \hat{d} = h_1 + \frac{5}{4} h_2 + \frac{3}{4} h_3 = \frac{1}{4} K^1{}_1 - \frac{1}{2}(K^2{}_2 + K^3{}_3) \\ K &= K^a{}_a = K^1{}_1 + K^\gamma{}_\gamma = -2\hat{d} - \frac{3}{2}\hat{c}, \quad \Longrightarrow K^\gamma{}_\gamma = -2\hat{d} - \frac{1}{2}\hat{c}. &(3.15) \end{aligned}$$

The conventional choice for AE_3 would be to take node 3 to be the horizontal node instead of 2 as we did, since this is more natural from the extension point of view. This choice would correspond to the Ehlers $SL(2, \mathbb{R})$, and it is also the choice taken by [26]. However, it does not reduce to the desired horizontal $\mathfrak{sl}_2 \subset \mathfrak{sl}_3$ subalgebra, and for this reason we here make another choice, corresponding to the 'Matzner–Misner' $SL(2, \mathbb{R})$ in the physical interpretation. The gradient generators (2.29) on

[9] In general, there remain signs ± 1 which need to be fixed consistently. Up to $|\ell| \leq 2$ our choice is consistent.

level ℓ have Dynkin labels $[\ell-1,2]$ and after lowering the sets of antisymmetric indices take the form

$$E_{a_1\dots a_{\ell-1}}{}^{b_1b_2}. \tag{3.16}$$

We now determine the contribution of the gradient generators to AE_3 commutators by writing down for each commutator the most general $\mathfrak{sl}_3$ covariant ansatz compatible with the level decomposition, and then matching this ansatz with the affine commutation relations (2.21). Let us first note down the projector onto a representation with Dynkin labels $[\ell,2]$:

$$\begin{aligned} P_{a_1\dots a_\ell}{}^{b_1b_2}|^{c_1\dots c_\ell}{}_{d_1d_2} &= \bar{\delta}^{c_1\dots c_\ell}_{a_1\dots a_\ell}\bar{\delta}^{b_1b_2}_{d_1d_2} - \frac{2\ell}{3+\ell}\bar{\delta}^{c_1\dots c_{\ell-1}}_{a_1\dots a_{\ell-1}}\delta^{(b_1}_{a_\ell}\delta^{b_2)}_{(d_1}\delta^{c_\ell}_{d_2)} \\ &\quad + \frac{\ell(\ell-1)}{(3+\ell)(2+\ell)}\bar{\delta}^{c_1\dots c_{\ell-2}}_{a_1\dots a_{\ell-2}}\bar{\delta}^{b_1b_2}_{a_{\ell-1}a_\ell}\bar{\delta}^{c_{\ell-1}c_\ell}_{d_1d_2}, \end{aligned} \tag{3.17}$$

where the right hand side also has to be symmetrized over the a and c indices. Here, $\bar{\delta}$ denotes the strength one symmetrizing projector. The second and third term are only present when $\ell \geq 1$ or $\ell \geq 2$ respectively. The projector has the unusual property that it also serves as a projector on the representation $[2,\ell]$ when acting from the right.[10]

The restriction to those elements in $E_{a_1\dots a_{\ell-1}}{}^{b_1b_2}$ belonging strictly to the affine subalgebra gives an identification of $\bar{K}_\ell^\alpha{}_\beta$ as explained above. Here, the identification of the affine subalgebra within AE_3 for all levels $\ell > 0$ reads

$$\bar{K}_\ell^\alpha{}_\beta = \varepsilon_{\beta\gamma}E_{\underbrace{1\dots1}_{(\ell-1)}}{}^{\gamma\alpha} \tag{3.18}$$

and for $-\ell < 0$

$$\bar{K}_{-\ell}^\alpha{}_\beta = \varepsilon^{\alpha\gamma}F^{\overbrace{1\dots1}^{(\ell-1)}}{}_{\gamma\beta}. \tag{3.19}$$

These formulas specialize (2.30) and (2.31) to the case $n=3$. Note that we use Euclidean conventions resulting in $\varepsilon^{23}=\varepsilon_{23}=1$. It is easy to check that indeed ($\ell \geq 1$)

$$\big[\hat{d}, E_{\underbrace{1\dots1}_{(\ell-1)}}{}^{\alpha\beta}\big] = -\ell E_{\underbrace{1\dots1}_{(\ell-1)}}{}^{\alpha\beta} \tag{3.20}$$

[10]That this is not naturally so can be seen, for instance, from considering the projector on the Riemann tensor symmetries.

and

$$\left[\hat{c}, E_{\underbrace{1\ldots1}_{(\ell-1)}}{}^{\alpha\beta}\right] = 0, \tag{3.21}$$

Here the correction term in (3.5) is essential in order to obtain a vanishing result.

Upon restriction to the affine subalgebra the projectors (3.17) simplify to

$$P_{\underbrace{1\ldots1}_{\ell}}{}^{\beta_1\beta_2}|^{c_1\ldots c_{\ell-1}}{}_{d_1d_2} = \bar{\delta}^{c_1\ldots c_\ell}_{1\ldots1}\bar{\delta}^{\beta_1\beta_2}_{d_1d_2} \tag{3.22}$$

for positive ℓ and for negative ℓ accordingly. The identification (3.18) together with the affine relations (2.21) then determines the commutators of the gradient representations. Note that the identification also fixes the normalisation of the gradient tensor in the AE_3 invariant form to be

$$\langle E_{a_1\ldots a_\ell}{}^{b_1b_2}|F^{c_1\ldots c_k}{}_{d_1d_2}\rangle = \begin{cases} P_{a_1\ldots a_\ell}{}^{b_1b_2}|^{c_1\ldots c_\ell}{}_{d_1d_2} & \text{for } \ell = k \\ 0 & \text{for } \ell \neq k \end{cases}, \tag{3.23}$$

by using eq. (2.27).

The commutator of two E generators on levels ℓ and k is then

$$\begin{aligned}[E_{a_1\ldots a_{\ell-1}}{}^{b_1b_2}, E_{c_1\ldots c_{k-1}}{}^{d_1d_2}] &= -E_{a_1\ldots a_{\ell-1}c_1\ldots c_{k-1}e}{}^{d_1(b_1}\varepsilon^{b_2)d_2e} \\ &\quad -E_{a_1\ldots a_{\ell-1}c_1\ldots c_{k-1}e}{}^{d_2(b_1}\varepsilon^{b_2)d_1e} + \ldots.\end{aligned} \tag{3.24}$$

The dots on the right hand side indicate possible other AE_3 generators appearing in this commutators which will however vanish upon restriction to the affine subalgebra $A_1^{(1)}$. The term on the right is fixed up to normalisation from the tensor symmetries and the normalisation is fixed from the affine subalgebra. A similar expression holds for $[F, F]$.

For $\ell > k$ we obtain

$$\begin{aligned}[E_{a_1\ldots a_\ell}{}^{b_1b_2}, F^{c_1\ldots c_k}{}_{d_1d_2}] &= -2P_{a_1\ldots a_\ell}{}^{b_1b_2}|^{e_1\ldots e_\ell}{}_{f_1f_2}P_{g_1\ldots g_k}{}^{h_1h_2}|^{c_1\ldots c_k}{}_{d_1d_2} \\ &\quad \cdot\varepsilon_{e_\ell h_1 i}\delta^{f_1}_{h_2}\bar{\delta}^{g_1\ldots g_k}_{e_1\ldots e_k}E_{e_{k+1}\ldots e_{\ell-1}}{}^{f_2i} + \ldots.\end{aligned} \tag{3.25}$$

Finally, level $\ell+1 > 0$ commutes with level $-\ell-1$ into the adjoint of $\mathfrak{gl}_3$

$$\begin{aligned}[E_{a_1\ldots a_\ell}{}^{b_1b_2}, F^{c_1\ldots c_\ell}{}_{d_1d_2}] &= P_{a_1\ldots a_\ell}{}^{b_1b_2}|^{e_1\ldots e_\ell}{}_{f_1f_2}P_{g_1\ldots g_\ell}{}^{h_1h_2}|^{c_1\ldots c_\ell}{}_{d_1d_2} \\ &\cdot\left(\delta^{g_1}_{e_1}\cdots\delta^{g_\ell}_{e_\ell}\delta^{f_1}_{h_1}\delta^{f_2}_{h_2}K - 2\delta^{g_1}_{e_1}\cdots\delta^{g_\ell}_{e_\ell}\delta^{f_1}_{h_1}K^{f_2}{}_{h_2} + \ell\delta^{f_1}_{h_1}\delta^{f_2}_{h_2}\delta^{g_1}_{e_1}\cdots\delta^{g_{\ell-1}}_{e_{\ell-1}}K^{g_\ell}{}_{e_\ell}\right).\end{aligned} \tag{3.26}$$

This commutator is exact since there are no other Lie algebra elements besides those of $\mathfrak{gl}_3$ at level $\ell = 0$. The omitted terms (=dots) in the other AE_3 commutators are necessary for the Jacobi identities. This phenomenon first occurs at level $\ell = 3$, since this is the first time a non-gradient representation is present (cf. table 1). Checking the Jacobi identities using solely the gradient generators shows a violation which is resolved when one also takes into account the $\ell = 3$ representation with Dynkin labels $[1, 1]$.

4 Affine truncation of the $AE_n/K(AE_n)$ σ-model

Having clarified the embeddings of the subgroups and having determined the gradient commutators, we next turn to the one-dimensional 'geodesic' σ-model over the coset space $AE_n/K(AE_n)$. This model is governed by the Lagrangian [10]

$$L = \frac{1}{2n}\langle \mathcal{P}|\mathcal{P}\rangle, \tag{4.1}$$

where all quantities depend only on the affine parameter ('time') t, and $n(t)$ is the lapse function required for reparametrisation invariance $t \to \tilde{t}(t)$. The quantity $\mathcal{P}(t)$ and the $K(AE_n)$ gauge connection $\mathcal{Q}(t)$ are determined from the Cartan form

$$\partial_t \mathcal{V}\mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P} \tag{4.2}$$

with $\mathcal{V}(t) \in AE_n/K(AE_n)$. By construction, $\mathcal{Q}$ belongs to the compact subalgebra $K(AE_n)$ (fixed by the Chevalley involution, cf. eq. (2.3)), and $\mathcal{P}$ to its complement in AE_n. With these definitions and notations, the equations of motion assume the simple looking form [14]

$$n\partial_t(n^{-1}\mathcal{P}) = [\mathcal{Q}, \mathcal{P}], \tag{4.3}$$

To get these equations into the standard second order form, we would have to choose coordinates on the $AE_n/K(AE_n)$ coset manifold and to substitute them into (4.2), but we will skip this step here. Although these objects are highly formal constructs at this point, readers need not worry about our lack of knowledge of what the 'group' AE_3 really is, because the truncated level expansion provides an algorithm which is such that all operations take place in the Lie algebra, and involve only a finite number of steps if one truncates at finite level. As we will show these steps remain well-defined for the *affine truncation* where one retains an infinite number of Lie algebra elements.

Varying the lapse n, we get the (Hamiltonian) constraint

$$\langle \mathcal{P}|\mathcal{P}\rangle = 0 \tag{4.4}$$

Connsequently, the 'trajectory' described by (4.3) on the coset manifold is a *null geodesic*.

The system described by this model is formally integrable [9]. For every solution $\mathcal{V}$ giving rise to $\mathcal{Q}$ and $\mathcal{P}$ satisfying (4.3) we can define a conserved (Noether) charge

$$\mathcal{J} = n^{-1}\mathcal{V}^{-1}\mathcal{P}\mathcal{V}, \qquad \partial_t \mathcal{J} = 0 \tag{4.5}$$

taking values in the Lie algebra and obeying a null condition

$$\langle \mathcal{J} | \mathcal{J} \rangle = 0. \tag{4.6}$$

Similarly, we can define a (symmetric) 'metric' $\mathcal{M}$ associated with the ∞-bein $\mathcal{V}$ by $\mathcal{M} = \mathcal{V}^T\mathcal{V}$, which is related to $\mathcal{J}$ via

$$\mathcal{J} = \frac{1}{2} n^{-1}\mathcal{M}^{-1}\partial_t\mathcal{M} \tag{4.7}$$

Eqn. (4.5) implies the following constraint on $\mathcal{J}$, and hence on the initial data,

$$\mathcal{J}^T = \mathcal{M}\mathcal{J}\mathcal{M}^{-1} \tag{4.8}$$

The general solution for $\mathcal{M}$ can be (formally) written as

$$\mathcal{M}(t) = \exp\big(\nu(t)\mathcal{J}^T\big)\mathcal{M}(0)\exp\big(\nu(t)\mathcal{J}\big). \tag{4.9}$$

where

$$\nu(t) := \int_{t_0}^{t} n(t')dt', \tag{4.10}$$

with t_0 some arbitrary initial time. Translated back to $\mathcal{V}(t)$, the general solution is

$$\mathcal{V}(t) = k(t)\mathcal{V}(0)\exp\big(\nu(t)\mathcal{J}\big). \tag{4.11}$$

Here, $k(t)$ belongs to the compact subgroup $K(AE_n)$ and is not determined by the equations of motion. This indeterminacy corresponds to the freedom of choosing a gauge for $\mathcal{V}(t)$. We can now work out the above equations of motion (4.3) level by level, following the low level results of [10].

4.1 Restriction to $A_{n-2}^{(1)}$

We will now restrict the one-dimensional geodesic σ-model (4.1), to its *affine subsector*; this is a consistent truncation. As we explained, the affine sector originates from the zeroth level in the $A_{n-2}^{(1)}$ analysis. To this end we define

$$\begin{aligned} S_m^{\alpha\beta} &= \frac{1}{2}(\bar{K}^{\alpha}_{m\beta} + \bar{K}^{\beta}_{-m\alpha}), \\ J_m^{\alpha\beta} &= \frac{1}{2}(\bar{K}^{\alpha}_{m\beta} - \bar{K}^{\beta}_{-m\alpha}), \end{aligned}$$

which obey the symmetry relations

$$S_m^{\alpha\beta} = S_{-m}^{\beta\alpha}, \qquad J_m^{\alpha\beta} = -J_{-m}^{\beta\alpha}. \tag{4.12}$$

The generators $S_m^{\alpha\beta}$, $J_m^{\alpha\beta}$ for $m \geq 0$, together with $\hat{c}$ and $\hat{d}$, define a basis of $A_{n-2}^{(1)}$. The commutation relations of $A_{n-2}^{(1)}$ in this basis are

$$[\hat{d}, S_m^{\alpha\beta}] = -mJ_m^{\alpha\beta} \quad , \qquad [\hat{d}, J_m^{\alpha\beta}] = -mS_m^{\alpha\beta} \tag{4.13}$$

and, for $m, n \geq 0$,

$$\begin{aligned}
\left[J_m^{\alpha\beta}, S_n^{\gamma\delta}\right] &= \frac{1}{2}\delta^{\beta\gamma} S_{m+n}^{\alpha\delta} + \frac{1}{2}\delta^{\beta\delta} S_{m-n}^{\alpha\gamma} - \frac{1}{2}\delta^{\alpha\gamma} S_{m-n}^{\delta\beta} - \frac{1}{2}\delta^{\alpha\delta} S_{m+n}^{\gamma\beta} \\
&\quad + \frac{1}{2} m\hat{c}\delta_{m,n}\delta^{\alpha\gamma}\delta^{\beta\gamma}, && (4.14) \\
\left[J_m^{\alpha\beta}, J_n^{\gamma\delta}\right] &= \frac{1}{2}\delta^{\beta\gamma} J_{m+n}^{\alpha\delta} - \frac{1}{2}\delta^{\beta\delta} J_{m-n}^{\alpha\gamma} + \frac{1}{2}\delta^{\alpha\gamma} J_{m-n}^{\delta\beta} - \frac{1}{2}\delta^{\alpha\delta} J_{m+n}^{\gamma\beta}, && (4.15) \\
\left[S_m^{\alpha\beta}, S_n^{\gamma\delta}\right] &= \frac{1}{2}\delta^{\beta\gamma} J_{m+n}^{\alpha\delta} + \frac{1}{2}\delta^{\beta\delta} J_{m-n}^{\alpha\gamma} - \frac{1}{2}\delta^{\alpha\gamma} J_{m-n}^{\delta\beta} - \frac{1}{2}\delta^{\alpha\delta} J_{m+n}^{\gamma\beta}. && (4.16)
\end{aligned}$$

Observe that the central charge drops out in the $[J, J]$ and $[S, S]$ commutators, but is present in the 'mixed' commutators $[J, S]$. The elements J_m (with $m \geq 0$) generate the (centerless) 'maximal compact' subalgebra $K(A_{n-2}^{(1)}) \equiv K(\widehat{\mathfrak{sl}_{n-1}})$.[11]

Next, we expand the Cartan form in this basis as in (4.2), with

$$\begin{aligned}
\mathcal{Q}(t) &= Q_{\alpha\beta}^{(0)} J_0^{\alpha\beta} + \sum_{m\geq 1} Q_{\alpha\beta}^{(m)} J_m^{\alpha\beta}, \\
\mathcal{P}(t) &= P_{\alpha\beta}^{(0)} S_0^{\alpha\beta} + \sum_{m\geq 1} P_{\alpha\beta}^{(m)} S_m^{\alpha\beta} + \partial_t\hat{\rho}\,\hat{\rho}^{-1}\,\hat{d} + \partial_t\sigma\,\hat{c}. && (4.17)
\end{aligned}$$

The quantity $\hat{\rho}$ will be seen to be directly related to the 'dilaton' field (=volume density for the internal manifold), while e^{σ} is related to the conformal factor in the dimensional reduction of Einstein's theory to two dimensions. Observe that the position of indices no longer matters, as the tensors appearing on the r.h.s. are to be regarded as $SO(n-1)$ tensors only. Also,

$$Q_{\alpha\beta}^{(0)} = -Q_{\beta\alpha}^{(0)} \quad , \qquad P_{\alpha\beta}^{(0)} = +P_{\beta\alpha}^{(0)} \tag{4.18}$$

whereas the higher modes are not subject to any such symmetry restrictions.

[11] That the latter is *not* a Kac–Moody algebra is demonstrated in appendix A. Note also that the additional CSA elements $\hat{c}$ and $\hat{d}$ do not survive the projection to the compact subalgebra and hence $K(A_{n-2}^{(1)})$ equals the compact subalgebra of the loop algebra $\widehat{\mathfrak{sl}_{n-1}}$.

In this general, non-gauge-fixed form, the equations of motion (4.3) read

$$\begin{aligned} n\partial_t\big(n^{-1}\hat\rho^{-1}\partial_t\hat\rho\big) &= 0, \\ n\partial_t\big(n^{-1}\partial_t\sigma\big) &= \frac{1}{2}\sum_{m=1}^{\infty} mQ^{(m)}_{\alpha\beta}P^{(m)}_{\alpha\beta} \quad , \\ n\partial_t\big(n^{-1}P^{(0)}_{\alpha\beta}\big) &= 2Q^{(0)}_{(\alpha|\gamma}P^{(0)}_{\gamma|\beta)} + \frac{1}{2}\sum_{m=1}^{\infty}\big(Q^{(m)}_{(\alpha|\gamma|}P^{(m)}_{\beta)\gamma} - Q^{(m)}_{\gamma(\beta}P^{(m)}_{|\gamma|\alpha)}\big) \end{aligned} \tag{4.19}$$

for the level-zero degrees of freedom, where the vertical bars on the r.h.s. of the third equation indicate that symmetrization should be performed only over the indices α and β. At levels $\ell \geq 1$, we obtain (recall that there is no symmetry under the exchange of α and β)

$$\begin{aligned} n\partial_t\big(n^{-1}P^{(\ell)}_{\alpha\beta}\big) &= \ell\hat\rho^{-1}\partial_t\hat\rho\, Q^{(\ell)}_{\alpha\beta} + \frac{1}{2}\sum_{m=0}^{\ell}\Big(Q^{(m)}_{\alpha\gamma}P^{(\ell-m)}_{\gamma\beta} - Q^{(m)}_{\gamma\beta}P^{(\ell-m)}_{\alpha\gamma}\Big) \\ &+\frac{1}{2}\sum_{m=0}^{\infty}\Big(Q^{(\ell+m)}_{\alpha\gamma}P^{(m)}_{\beta\gamma} - Q^{(\ell+m)}_{\gamma\beta}P^{(m)}_{\gamma\alpha} + Q^{(m)}_{\beta\gamma}P^{(\ell+m)}_{\alpha\gamma} - Q^{(m)}_{\gamma\alpha}P^{(\ell+m)}_{\gamma\beta}\Big). \end{aligned} \tag{4.20}$$

Finally, the Hamiltonian constraint in a general gauge is

$$\langle\mathcal{P}|\mathcal{P}\rangle = -2\,\hat\rho^{-1}\partial_t\hat\rho\,\partial_t\sigma + P^{(0)}_{\alpha\beta}P^{(0)}_{\alpha\beta} + \frac{1}{2}\sum_{m\geq 1}P^{(m)}_{\alpha\beta}P^{(m)}_{\alpha\beta} = 0. \tag{4.21}$$

The first equation in (4.19) can be integrated straightforwardly, with the result

$$n(t) = \partial_t \ln\hat\rho(t) \quad\Rightarrow\quad \nu(t) = \ln\hat\rho(t) \tag{4.22}$$

where integration constants have been chosen conveniently. Hence, the choice of the function $\hat\rho(t)$ can be viewed as a choice of gauge for the lapse $n(t)$; this function is only subject to the requirement that it be a monotonic function of the affine parameter t. This result can be plugged into the general solution (4.11) to give

$$\mathcal{V}(t) = k(t)\mathcal{V}(0)\exp\big(\ln\hat\rho(t)\mathcal{J}\big). \tag{4.23}$$

Although the above equations of motion constitute a consistent truncation of the full $AE_n/K(AE_n)$ σ-model, one may ask how they embed into the latter. Within $AE_n/K(AE_n)$, the above equations of motion may receive new contributions from those 'non-gradient fields' whose associated Lie algebra elements 'commute back' into the affine subalgebra. The role and significance of these extra contributions is not clear.

We emphasize that the expression (4.17) is the most general, because we have not chosen a gauge for the local subgroup $K(A^{(1)}_{n-2})$. In other words, *no matter which gauge is chosen, the equations of motion can always be cast into the form (4.19), (4.20) and (4.21).*

4.2 Triangular gauge

The equations of motion derived above can now be considered in various gauges, thereby fixing the factor $k(t)$ in (4.11). A convenient choice, and one that has been used almost exclusively in previous work, is the *triangular gauge*, for which the ∞-bein $\mathcal{V}(t)$ depends only on the level $\ell \geq 0$ fields. In this gauge, one obtains

$$\partial_t \mathcal{V}\mathcal{V}^{-1} = Q^{(0)}_{\alpha\beta} J_0^{\alpha\beta} + P^{(0)}_{\alpha\beta} S_0^{\alpha\beta} + \sum_{m\geq 1} P^{(m)}_{\alpha\beta} \bar{K}^{\alpha}_{m\beta} + \partial_t\hat{\rho}\,\hat{\rho}^{-1}\,\hat{d} + \partial_t\sigma\,\hat{c}. \tag{4.24}$$

Consequently,

$$Q^{(m)}_{\alpha\beta} = P^{(m)}_{\alpha\beta} \qquad \text{for } m \geq 1. \tag{4.25}$$

Note that the second term on the r.h.s. of (4.20) vanishes for this choice of gauge, in agreement with previous calculations [14]. While $P^{(0)}_{\alpha\beta}$ is symmetric, $P^{(m)}_{\alpha\beta}$ contains both symmetric and antisymmetric parts for $m > 0$, and we can therefore decompose it into irreducible $\mathfrak{so}_{n-1}$ representations:

$$P^{(m)}_{\alpha\beta} = \bar{P}^{(m)}_{\alpha\beta} + \bar{Q}^{(m)}_{\alpha\beta} \tag{4.26}$$

with

$$\bar{P}^{(m)}_{\alpha\beta} = \bar{P}^{(m)}_{\beta\alpha}, \qquad \bar{Q}^{(m)}_{\alpha\beta} = -\bar{Q}^{(m)}_{\beta\alpha}, \tag{4.27}$$

remembering that $\bar{P}^{(m)}_{\alpha\beta}$ is traceless. With these definitions the dynamical equations (4.19) and (4.20) become

$$n\partial_t\big(n^{-1}\hat{\rho}^{-1}\partial_t\hat{\rho}\big) = 0, \tag{4.28}$$

$$n\partial_t\big(n^{-1}\partial_t\sigma\big) = \frac{1}{2}\sum_{m\geq 1} m\big(\bar{Q}^{(m)}_{\alpha\beta}\bar{Q}^{(m)}_{\alpha\beta} + \bar{P}^{(m)}_{\alpha\beta}\bar{P}^{(m)}_{\alpha\beta}\big), \tag{4.29}$$

$$n\partial_t\big(n^{-1}\bar{P}^{(0)}_{\alpha\beta}\big) = 2\sum_{m\geq 0} \bar{Q}^{(m)}_{(\alpha|\gamma}\bar{P}^{(m)}_{\gamma|\beta)}\,, \tag{4.30}$$

$$n\hat{\rho}^{\ell}\partial_t\big(n^{-1}\hat{\rho}^{-\ell}\bar{P}^{(\ell)}_{\alpha\beta}\big) = 2\sum_{m\geq 0}\big(\bar{Q}^{(\ell+m)}_{(\alpha|\gamma}\bar{P}^{(m)}_{\gamma|\beta)} + \bar{Q}^{(m)}_{(\alpha|\gamma}\bar{P}^{(\ell+m)}_{\gamma|\beta)}\big)\,, \tag{4.31}$$

$$n\hat{\rho}^{\ell}\partial_t\big(n^{-1}\hat{\rho}^{-\ell}\bar{Q}^{(\ell)}_{\alpha\beta}\big) = 2\sum_{m\geq 0}\Big(-\bar{Q}^{(\ell+m)}_{\gamma[\alpha}\bar{Q}^{(m)}_{\beta]\gamma} + \bar{P}^{(\ell+m)}_{\gamma[\alpha}\bar{P}^{(m)}_{\beta]\gamma}\Big), \tag{4.32}$$

where the last two equations hold for $\ell \geq 1$. The Hamiltonian constraint (4.21) now reads

$$\langle \mathcal{P}|\mathcal{P}\rangle = -2\,\hat{\rho}^{-1}\partial_t\hat{\rho}\,\partial_t\sigma + \bar{P}^{(0)}_{\alpha\beta}\bar{P}^{(0)}_{\alpha\beta} + \frac{1}{2}\sum_{m\geq 1}\left(\bar{Q}^{(m)}_{\alpha\beta}\bar{Q}^{(m)}_{\alpha\beta} + \bar{P}^{(m)}_{\alpha\beta}\bar{P}^{(m)}_{\alpha\beta}\right) = 0. \quad (4.33)$$

Are there other viable gauge choices? While the triangular or Borel parametrisation is by no means the only possibility for finite-dimensional matrices, the situation is more subtle for infinite dimensional groups. In fact, it appears that so far the triangular parametrization is the only manageable one for indefinite KMAs. Nevertheless, indications were found recently [34] that the triangular parametrization must be relaxed if one is to include M theoretic corrections, and to extend the 'dictionary' beyond the very first few levels. Here we simply emphasize again, that independently of the gauge, the equations of motion can always be written in the form (4.19), (4.20) and (4.21). The only difference is that the equality (4.25) will fail to hold in a non-triangular gauge.

5 Comparison with two-dimensional reduction

5.1 Relation to higher dimensional vielbein

The infinite 'matrix' $\mathcal{V}(t) \in AE_n/K(AE_n)$ can be thought of as an ∞-bein analogous to the vielbein of general relativity. Indeed, it contains the (spatial) vielbein as a finite-dimensional 'submatrix', if one restricts attention to the level zero sector, *i.e.* upon restriction of the full coset to the finite-dimensional coset space $GL(n)/SO(n)$ (note that we keep the $GL(1)$ factor here as a remnant of the full algebra). As is well known, pure gravity in $(n+1)$ spacetime dimensions exhibits a hidden $SL(n-1)$ symmetry after reduction to three dimensions. This symmetry is obtained through an enlargement of the manifest $SL(n-2)$ symmetry upon dualization of $(n-2)$ Kaluza Klein vectors to $(n-2)$ scalar fields. After further reduction to two dimensions the system becomes integrable with a Lax pair, and admits an $A^{(1)}_{n-2}$ symmetry as a 'solution generating group' [4, 5, 6]; for $n = 3$ this group is known as the Geroch group [1, 2]. Together with the $GL(n)$ acting on the spatial n-bein, we thus recover precisely the subgroups of AE_n discussed in the foregoing section.

Let us therefore spell out this correspondence in a little more detail, in order to facilitate the comparison between the one-dimensional geodesic σ-model introduced in the foregoing section, and the $D = 2$ theory to be discussed in the following subsection. For this purpose we need to fix gauges such that

$$E_M{}^A = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & e_m{}^a \end{array}\right). \quad (5.34)$$

This equation displays the decomposition of the $(n+1)$-bein into a lapse factor N, and a spatial n-bein $e_m{}^a$; the shift variables (which enforce the spatial diffeomorphism constraint) have been set to zero, as required for the comparison with the Kac–Moody σ-model. As we said already, the spatial n-bein can be viewed as the level-0 sector of the ∞-bein $\mathcal{V}$, and thereby of the coset $AE_3/K(AE_3)$. As is well known [10], the restriction of the of the σ-model Lagrangian (4.1) to the $\ell = 0$ sector coincides precisely with the dimensional reduction of Einstein's theory to one time dimension.

In the following section 5.2, we will compare the one-dimensional σ-model equations of motion with those obtained in the reduction of Einstein's theory to (1+1) dimension (which depend on time and one extra spatial coordinate). For this comparison, one further step is required, namely the split of the spatial n-bein in (5.34) according to

$$E_M{}^A = \left(\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & \lambda & 0 \\ 0 & 0 & \rho^{1/(n-1)} \bar{e}_{\bar{m}}{}^{\bar{a}} \end{array} \right). \tag{5.35}$$

Here we have singled out one (the first) spatial direction, and decomposed the remaining $(n-1)$-bein into a unimodular part, and its determinant. So $\det(\bar{e}_{\bar{m}}{}^{\bar{a}}) = 1$, and ρ measures the volume of the $(n-1)$-dimensional internal space ('internal' from the (1+1)-dimensional perspective, of course). In addition, we have adopted the conformal gauge for the zweibein, with the conformal factor λ, thus tying the lapse to the (11) component of the spatial metric. Finally, setting the remaining elements of the first column and the first row of $e_m{}^a$ to zero is related to the hypersurface orthogonality of the Killing vectors usually assumed in the reduction to (1+1) dimensions [2, 5].

The special parametrisation (5.35) results in the following non-vanishing components of the coefficients of anholonomicity $\Omega_{ABC} \equiv 2E_A{}^M E_B{}^N \partial_{[M} E_{N]C}$ (with $\partial_x \equiv \partial/\partial x^1$)

$$\Omega_{0\bar{b}\,\bar{c}} = \frac{1}{n-1} \lambda^{-1} \delta_{\bar{b}\bar{c}} \partial_t \rho \rho^{-1} + \lambda^{-1} \bar{e}_{\bar{b}}{}^{\bar{m}} \partial_t \bar{e}_{\bar{m}\bar{c}}, \tag{5.36}$$

$$\Omega_{1\bar{b}\,\bar{c}} = \frac{1}{n-1} \lambda^{-1} \delta_{\bar{b}\bar{c}} \partial_x \rho \rho^{-1} + \lambda^{-1} \bar{e}_{\bar{b}}{}^{\bar{m}} \partial_x \bar{e}_{\bar{m}\bar{c}}, \tag{5.37}$$

$$\Omega_{01\,1} = \lambda^{-2} \partial_t \lambda, \tag{5.38}$$

$$\Omega_{01\,0} = -\lambda^{-2} \partial_x \lambda. \tag{5.39}$$

The spatial components $\Omega_{ab\,c}$ of the anholonomicity are related by duality to the first gradient representation (2.11), which we called dual graviton in 2.4. The irreducibility constraint (2.12) on this representation is equivalent to the tracelessness condition $\Omega_{ab\,b} = 0$ [9, 14]. In the present two-dimensional context this condition reduces to

$$\Omega_{1b\,b} \;\; = \;\; \Omega_{1\bar{b}\,\bar{b}} = \lambda^{-1}\partial_x\rho\rho^{-1} = 0. \tag{5.40}$$

Therefore the irreducibility condition $\Omega_{ab\,b} = 0$ holds if and only if $\rho = \rho(t)$. In the next section, we will arrive at the same conclusion by a somewhat different route.

The parametrisation (5.35) also yields the determinant of the original $(n+1)$-bein to be

$$\det(E) = N\sqrt{g} = \lambda^2\rho \tag{5.41}$$

where $\sqrt{g} \equiv \det e_m{}^a$. Using the identification between the original lapse N and the σ-model lapse n derived in [13, 14], we can relate the σ-model lapse to the internal volume density ρ via

$$n = Ng^{-1/2} = \rho^{-1}. \tag{5.42}$$

This result will be useful below.

To deduce the relation between the field σ and the conformal factor λ, we compute the contribution from (5.35) to the diagonal part of $\partial_t e_a{}^m e_m{}^b$, viz.

$$\partial_t e_a{}^m e_m{}^b = -\lambda^{-1}\partial_t\lambda\, K^1{}_1 - \frac{1}{n-1}\rho^{-1}\partial_t\rho\, K^\alpha{}_\alpha + \dots \tag{5.43}$$

where the dots stand for the contributions from the off-diagonal degrees of freedom. Rewriting this in terms of the CSA elements $\hat{c}$ and $\hat{d}$, cf. (2.23), we obtain

$$\partial_t[\rho^{(n-2)/2(n-1)}\lambda]\,[\rho^{(n-2)/2(n-1)}\lambda]^{-1}\,\hat{c} + \partial_t\rho\rho^{-1}\,\hat{d}. \tag{5.44}$$

Comparing (5.44) and (4.17) we conclude that the conformal factor should be identified as[12]

$$e^\sigma \equiv \lambda\rho^{\frac{n-2}{2(n-1)}} \tag{5.45}$$

while $\hat{\rho}$ indeed agrees the internal volume density ρ. This is consistent with the result from the irreducibility constraint (5.40) above that ρ is a function of time only.

[12] The same correction to the conformal factor is also obtained (for $n = 3$) in the Matzner–Misner coset formulation of gravity reduced to two space-time dimensions [5], see also [35].

5.2 σ-model equations of motion in (1+1) dimensions

Next we write out the equations of motion for the $D = 2$ theory as obtained from an $SL(n-1)/SO(n-1)$ σ-model coupled to gravity in two dimensions, and then compare them to the equations of motion of the one-dimensional σ-model derived in the foregoing section. This will not only allow us to extend previous results on the matching, but, more importantly, to exhibit those terms which do *not* match, and where the dictionary needs to be modified if one is to include higher order spatial gradients. The main advantage of the (1+1) theory is that the mismatch assumes the simplest possible form, and can therefore be scrutinized in full detail.

The basic object of the (1+1)-dimensional theory is a matrix $\mathtt{V}(t,x)$, which is an element of the coset space $SL(n-1)/SO(n-1)$, and which is the analog of $\mathcal{V}(t)$ of the previous section. The corresponding sector of the theory is governed by the standard σ-model Lagrangian on the worldsheet in a gravitational background. The corresponding Cartan form belongs to the horizontal $\mathfrak{sl}_{n-1}$

$$\partial_\mu \mathtt{V}\mathtt{V}^{-1} = \mathtt{Q}_\mu + \mathtt{P}_\mu \equiv \mathtt{Q}_{\mu\,\alpha\beta} J_0^{\alpha\beta} + \mathtt{P}_{\mu\,\alpha\beta} S_0^{\alpha\beta} \tag{5.46}$$

with the gauge field $\mathtt{Q}_\mu \in \mathfrak{so}_{n-1}$ and $\mathtt{P}_\mu \in \mathfrak{sl}_{n-1} \ominus \mathfrak{so}_{n-1}$, and $\mu, \nu, \dots = t, x$. The equations of motion read

$$D_\mu(\rho\sqrt{-\gamma}\gamma^{\mu\nu}\mathtt{P}_\nu) = 0, \tag{5.47}$$

where $D_\mu \mathtt{P}_\nu \equiv \partial_\mu \mathtt{P}_\nu - [\mathtt{Q}_\mu, \mathtt{P}_\nu]$ is the $\mathfrak{so}_{n-1}$ covariant derivative, and $\gamma_{\mu\nu}$ is the worldsheet metric, with inverse $\gamma^{\mu\nu}$ and determinant γ. The additional dependence on the 'dilaton' ρ in this equation is a remnant of the reduction from higher dimensions. In the conformal gauge $\gamma_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$ (cf. (5.35)), this equation simplifies to

$$D^\mu(\rho \mathtt{P}_\mu) = 0 \tag{5.48}$$

where indices are now to be raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. Writing out these equations in terms of t and x components, we get

$$\rho^{-1}\partial_t(\rho \mathtt{P}_t) - \rho^{-1}\partial_x(\rho \mathtt{P}_x) = [\mathtt{Q}_t, \mathtt{P}_t] - [\mathtt{Q}_x, \mathtt{P}_x]. \tag{5.49}$$

In addition, from (5.46), we have the integrability conditions

$$\partial_t \mathtt{P}_x - \partial_x \mathtt{P}_t = [\mathtt{Q}_t, \mathtt{P}_x] - [\mathtt{Q}_x, \mathtt{P}_t] \quad , \quad \partial_t \mathtt{Q}_x - \partial_x \mathtt{Q}_t = [\mathtt{Q}_t, \mathtt{Q}_x] + [\mathtt{P}_t, \mathtt{P}_x]. \tag{5.50}$$

with the gauge connection $\mathtt{Q}_t, \mathtt{Q}_x \in \mathfrak{so}_{n-1}$. Here we have written out explicitly all covariantizations in order to facilitate the comparison with the equations of motion (4.28) following from the one-dimensional σ-model.

In conformal gauge, the 'dilaton' ρ obeys a free field equation, viz.

$$(\partial_t^2 - \partial_x^2)\rho(t,x) = 0. \tag{5.51}$$

This equation is satisfied in particular if ρ equals one of the two-dimensional coordinates or a linear combination thereof (Weyl canonical coordinates). Since we are interested in cosmological applications, for which ρ is a timelike coordinate, we choose[13]

$$\rho(t,x) = t. \tag{5.52}$$

To see that this choice matches with the one-dimensional σ-model, we substitute (5.42) into the first equation of (4.28), which gives

$$\rho^{-1}\partial_t^2\rho = 0 \tag{5.53}$$

and indeed agrees with (5.51) if ρ is independent of x. Furthermore, independence of ρ of x is consistent with the Young irreducibility constraint in the σ-model as explained above in (5.40).

The first order equations for the conformal factor λ are just the Hamiltonian and diffeomorphism constraints, respectively, and read

$$\begin{aligned} \partial_t\rho\,\lambda^{-1}\partial_t\lambda + \partial_x\rho\,\lambda^{-1}\partial_x\lambda &= \frac{1}{2}\rho\,\mathrm{Tr}\left(\mathtt{P}_t^2 + \mathtt{P}_x^2\right), && (5.54)\\ \partial_t\rho\,\lambda^{-1}\partial_x\lambda + \partial_x\rho\,\lambda^{-1}\partial_t\lambda &= \rho\,\mathrm{Tr}\,\mathtt{P}_t\mathtt{P}_x. && (5.55) \end{aligned}$$

With (5.52) they simplify to

$$\lambda^{-1}\partial_t\lambda = \frac{t}{2}\,\mathrm{Tr}\left(\mathtt{P}_t^2 + \mathtt{P}_x^2\right) \quad , \quad \lambda^{-1}\partial_x\lambda = t\,\mathrm{Tr}\,\mathtt{P}_t\mathtt{P}_x. \tag{5.56}$$

There is also a second order equation for the conformal factor, which reads, for $\rho = t$,

$$t^{-1}\partial_t(t\lambda^{-1}\partial_t\lambda) - \partial_x(\lambda^{-1}\partial_x\lambda) = \mathrm{Tr}\,\mathtt{P}_x\mathtt{P}_x. \tag{5.57}$$

In order to compare the equations (5.49)–(5.57) with the equations of the one-dimensional affine σ-model (4.28)–(4.32), we must truncate both models appropriately. On the side of the one-dimensional model we do this by restricting the expansion to levels $|\ell| \le 1$. Making use of the equality (5.42) and the triangular gauge

[13] For spacelike ρ, we would obtain a variant of the so-called Einstein-Rosen waves, see e.g. [3].

(4.27), the equations of motion (4.28)–(4.32) are thus truncated to

$$\begin{aligned}
\rho^{-1}\partial_t\Big(\rho P^{(0)}_{\alpha\beta}\Big) &= 2\bar{Q}^{(0)}_{\gamma[\alpha}P^{(0)}_{\beta]\gamma} + 2\bar{Q}^{(1)}_{\gamma[\alpha}\bar{P}^{(1)}_{\beta]\gamma} \\
\partial_t\bar{P}^{(1)}_{\alpha\beta} &= 2\bar{Q}^{(0)}_{\gamma[\alpha}\bar{P}^{(1)}_{\beta]\gamma} + 2\bar{Q}^{(1)}_{\gamma[\alpha}\bar{P}^{(0)}_{\beta]\gamma} \\
\partial_t\bar{Q}^{(1)}_{\alpha\beta} &= -2\bar{Q}^{(1)}_{\gamma[\alpha}\bar{Q}^{(0)}_{\beta]\gamma} + 2\bar{P}^{(1)}_{\gamma[\alpha}\bar{P}^{(0)}_{\beta]\gamma} \qquad (5.58)\\
\rho^{-1}\partial_t(\rho\partial_t\sigma) &= \frac{1}{2}\left(\bar{Q}^{(1)}_{\alpha\beta}\bar{Q}^{(1)}_{\alpha\beta} + \bar{P}^{(1)}_{\alpha\beta}\bar{P}^{(1)}_{\alpha\beta}\right) \qquad (5.59)
\end{aligned}$$

which must now be matched to (5.49) and (5.50). On the side of the (1+1) theory, we must restrict the x-dependence of the two-dimensional quantities such that $\mathtt{P}_t$ and $\mathtt{P}_x$ become independent of x (which is tantamount to keeping only first order spatial gradients). Then the σ-model equations of motion match upon the identification

$$\bar{P}^{(0)}(t) \equiv \mathtt{P}_t(t, x_0), \qquad \bar{P}^{(1)}(t) \equiv \mathtt{P}_x(t, x_0), \qquad (5.60)$$

$$\bar{Q}^{(0)}(t) \equiv \mathtt{Q}_t(t, x_0), \qquad \bar{Q}^{(1)}(t) \equiv -\mathtt{Q}_x(t, x_0), \qquad (5.61)$$

where x_0 is some fixed, but arbitrarily chosen spatial point. The ‘dictionary’ (5.60) must be supplemented by the relations (5.42) and (5.45), already derived before.

While many terms thus do match, there remain several discrepancies between these equations and those of the geodesic σ-model. First of all, there is a mismatch in the equation of motion for the conformal factor, which is similar to the one encountered in previous work, and which in particular involves a contribution $\propto Q^{(1)}Q^{(1)}$, which has no analog involving the *gauge-variant* expression $\mathtt{Q}_x\mathtt{Q}_x$ in (5.56) above. We interpret this mismatch as another indication of the impossibility to reconcile the higher dimensional gauge invariance with the desired correspondence. Likewise, (5.57) has a term $\partial_x^2\sigma$ which has no analogue in (4.29). Furthermore, the spatial gradients $\partial_x\mathtt{P}_t, \partial_x\mathtt{P}_x$ and $\partial_x\mathtt{Q}_t$ in (5.49) are absent in the one-dimensional model. In view of these mismatches, we will now simplify the affine model yet further to an exactly solvable model.

6 Restriction to the ‘Heisenberg coset’ $\mathfrak{H}/K(\mathfrak{H})$

6.1 The $\mathfrak{H}/K(\mathfrak{H})$ σ-model

We consider the (prototype) affine coset $A_1^{(1)}/K(A_1^{(1)})$ restricted to its ‘Heisenberg subspace’ $\mathfrak{H}/K(\mathfrak{H})$. In this way, we are able to define a one-to-one correspondence between a very limited, albeit non-trivial, class of solutions of Einstein’s equations (diagonal metrics with two commuting Killing vectors), and the null geodesic motion on the infinite-dimensional manifold $\mathfrak{H}/K(\mathfrak{H})$. To this end we restrict the affine algebra $A_1^{(1)}$ to its Heisenberg subalgebra

$$\mathrm{Lie}(\mathfrak{H}) := \widehat{\mathfrak{gl}_1} \oplus \mathbb{R}\hat{c} \oplus \mathbb{R}\hat{d} \tag{6.1}$$

Note that our terminology is slightly unusual in that the last summand is usually not considered to be part of the Heisenberg algebra, but required here in order to obtain a non-degenerate (indefinite) metric on $\mathrm{Lie}(\mathfrak{H})$. Define

$$H_m = \frac{1}{\sqrt{2}}\big(\bar{K}^2_{m\,2} - \bar{K}^3_{m\,3}\big) \tag{6.2}$$

for all $m \in \mathbb{Z}$. Then the non-vanishing commutators of the Heisenberg algebra are

$$\big[H_m, H_n\big] = m\delta_{m,-n}\hat{c}, \qquad \big[\hat{d}, H_m\big] = -mH_m. \tag{6.3}$$

The symmetric and antisymmetric combinations are

$$\begin{aligned} S_m &:= \frac{1}{2}(H_m + H_{-m}) \quad \text{for } m \geq 0, \\ J_m &:= \frac{1}{2}(H_m - H_{-m}) \quad \text{for } m \geq 1. \end{aligned} \tag{6.4}$$

The J_m generate the maximal compact subgroup $K(\mathfrak{H})$ of the extended Heisenberg group $\mathfrak{H}$. The commutation relations (6.3) in this basis are (recall that $m, n \geq 0$)

$$\begin{aligned} \big[S_m, S_n\big] &= 0, \qquad \big[J_m, J_n\big] = 0, \\ \big[J_m, S_n\big] &= \frac{1}{2}m\delta_{m,n}\hat{c}, \\ \big[\hat{d}, S_m\big] &= -mJ_m, \qquad \big[\hat{d}, J_m\big] = -mS_m. \end{aligned} \tag{6.5}$$

Parametrising the coset space $\mathfrak{H}/K(\mathfrak{H})$ in triangular gauge by

$$\mathcal{V}(t) = \exp[\ln\hat{\rho}(t)\,\hat{d}] \cdot \exp[\sigma(t)\,\hat{c}] \cdot \exp\left(\sum_{m\geq 0} \phi_m(t) H_m\right), \tag{6.6}$$

we find

$$\partial_t \mathcal{V}\mathcal{V}^{-1} = \sum_{m\geq 0} \hat{\rho}^{-m}\partial_t\phi_m H_m + \hat{\rho}^{-1}\partial_t\hat{\rho}\,\hat{d} + \partial_t\sigma\,\hat{c}. \tag{6.7}$$

Defining $P_m(t) \equiv \hat{\rho}^{-m}\partial_t\phi_m$, we have

$$\begin{aligned} \mathcal{P}(t) &= \sum_{m\geq 0} P_m(t) S_m + \hat{\rho}^{-1}\partial_t\hat{\rho}(t)\,\hat{d} + \partial_t\sigma(t)\,\hat{c} \\ \mathcal{Q}(t) &= \sum_{m\geq 1} P_m(t) J_m \end{aligned} \tag{6.8}$$

Using the commutation relations (6.5) we get

$$[\mathcal{Q}, \mathcal{P}] = \sum_{m\geq 0} mP_m \hat{\rho}^{-1}\partial_t\hat{\rho}\, S_m + \frac{1}{2}\sum_{m\geq 1} mP_m^2\,\hat{c}. \tag{6.9}$$

Therefore the equations of motion of our model read

$$\begin{aligned} n\partial_t(n^{-1}\hat{\rho}^{-1}\partial_t\hat{\rho}) &= 0, \\ n\partial_t(n^{-1}\partial_t\sigma) &= \frac{1}{2}\sum_{m\geq 1} m\hat{\rho}^{-2m}(\partial_t\phi_m)^2, \\ n\partial_t(n^{-1}\hat{\rho}^{-m}\partial_t\phi_m) &= m\hat{\rho}^{-(m+1)}\partial_t\hat{\rho}\,\partial_t\phi_m. \end{aligned} \tag{6.10}$$

The Hamiltonian constraint takes the form

$$-2\hat{\rho}^{-1}\partial_t\hat{\rho}\partial_t\sigma + P_0^2 + \frac{1}{2}\sum_{m\geq 1} P_m^2 = 0. \tag{6.11}$$

Using the insights from the preceding section we solve the first equation of (6.10) by setting $n^{-1} = \hat{\rho} = t$. Then the null geodesic trajectory in $\mathfrak{H}/K(\mathfrak{H})$ is explicitly parametrized by

$$\begin{aligned} &\left.\begin{aligned} \phi_0(t) &= p_0\ln t + q_0 \\ \phi_m(t) &= \tfrac{1}{2m}p_m t^{2m} + q_m \quad (m>0) \end{aligned}\right\} \quad \Rightarrow \quad P_m(t) = p_m t^{m-1} \ \ (m\geq 0)\,, \\ &\sigma(t) = \frac{1}{2}p_0^2\ln t + \sum_{m\geq 1}\frac{1}{8m}p_m^2 t^{2m} + \sigma_0. \end{aligned} \tag{6.12}$$

where the coefficient of $\ln t$ in the last line is determined by imposing the Hamiltonian constraint (6.11), which, however, does not fix σ_0. Because the model is explicitly solvable, we see in particular how the solution $\mathcal{V}(t)$ depends on the most general initial data, *i.e.* the initial 'coordinates' q_m and 'momenta' p_m for $m \geq 0$.

The conserved current is, from (4.5),

$$\begin{aligned} \mathcal{J} &= n^{-1}\mathcal{V}^{-1}\mathcal{P}\mathcal{V} = n^{-1}\Bigg[\Big(\partial_t\sigma - \frac{1}{2}\sum_{m\geq 1} m\hat{\rho}^{-m}\phi_m P_m\Big)\,\hat{c} + \hat{\rho}^{-1}\partial_t\hat{\rho}\,\hat{d} \\ &\quad + P_0H_0 + \frac{1}{2}\sum_{m\geq 1}\big(-2m\hat{\rho}^{-1}\partial_t\hat{\rho}\phi_m + \hat{\rho}^m P_m\big)H_m + \frac{1}{2}\sum_{m\geq 1}\hat{\rho}^{-m}P_m H_{-m}\Bigg]. \end{aligned} \tag{6.13}$$

Plugging the solution (6.12), as well as $n^{-1}(t) = t$, into this expression yields

$$\begin{aligned} \mathcal{J} &= \left(\frac{1}{2}p_0^2 - \frac{1}{2}\sum_{m\geq 1} mp_m q_m\right)\hat{c} + \hat{d} \\ &\quad + \frac{1}{2}\sum_{m\geq 1} p_m H_{-m} + p_0 H_0 - \sum_{m\geq 1} mq_m H_m, \end{aligned} \tag{6.14}$$

which is evidently conserved since it depends only on the initial data. Observe that neither q_0 nor σ_0 appear in $\mathcal{J}$. In agreement with the general analysis of [10], the initial momenta appear in the *lower*, and the initial coordinates in the *upper* triangular half of Lie($\mathfrak{H}$).

Under global $\mathfrak{H}$ transformations g, $\mathcal{J}$ changes as $\mathcal{J} \to g\mathcal{J}g^{-1}$. Let

$$g(\alpha, \beta, \omega) := \exp(\ln \alpha\, \hat{c}) \cdot \exp(\ln \beta\, \hat{d}) \cdot \exp\left(\sum_{n\in\mathbb{Z}} \omega_n H_n\right), \tag{6.15}$$

then

$$\begin{aligned} g\mathcal{J}g^{-1} \;&=\; \left(\frac{1}{2}p_0^2 - \frac{1}{2}\sum_{m\geq 1} m(p_m - 2m\omega_{-m})(q_m - \omega_m)\right)\hat{c} + \hat{d} \\ &\quad - \sum_{m\geq 1} m\beta^{-m}(q_m - \omega_m)H_m + p_0 H_0 + \frac{1}{2}\sum_{m\geq 1}\beta^m(p_m - 2m\omega_{-m})H_{-m}. \end{aligned} \tag{6.16}$$

From this we immediately read off the transformation of the initial coordinates q_m and momenta p_m (for $m \geq 1$) under the action of $\mathfrak{H}$:

$$\left.\begin{array}{lcl} q_m & \to & \beta^{-m}(q_m - \omega_m) \\ p_m & \to & \beta^m(p_m - 2m\omega_{-m}) \end{array}\right\} \qquad (m \geq 1). \tag{6.17}$$

On $\mathcal{V}$ the corresponding transformation is $\mathcal{V} \to k\mathcal{V}g^{-1}$ where $k(t)$ is the (local) compensator required to restore triangular gauge, which does not contribute to the transformation of $\mathcal{J}$. By considering $\mathcal{V}$ transformations we find constant shifts (which drop out in $g\mathcal{J}g^{-1}$)

$$\sigma_0 \to \sigma_0 - \alpha \quad , \qquad q_0 \to q_0 - \omega_0. \tag{6.18}$$

Hence, all the constants are shifted except the (Kasner) coefficient p_0. All fields except σ are inert under a transformation associated with the central term. This is no longer the case for the full AE_n model where $\hat{c}$ ceases to be central.

6.2 Relation to polarised Gowdy cosmologies

Remarkably, there is a one-to-one correspondence between our model, and the 'polarised' Gowdy type cosmological model with diagonal metrics depending on two coordinates (t, x) [14]. Apart from possible reparametrisations of the time parameter t, *this correspondence works only after complete elimination of the gauge degrees*

[14] See also [36] for a discussion of such solutions in the framework of gravitational solitons.

of freedom on both sides. In particular, the conformal factor must be treated as a *dependent* degree of freedom via the constraints (6.21) below. The relevant line elements can be written in the form (see e.g. [3])

$$ds^2 = \lambda^2 e^{-Z}(-dt^2 + dx^2) + t^2 e^{-Z} dy^2 + e^Z dz^2 \tag{6.19}$$

where the function $Z = Z(t, x)$ is subject to the two-dimensional wave equation

$$\partial_t \big[t \partial_t Z(t, x) \big] = t \partial_x^2 Z(t, x) \tag{6.20}$$

The conformal factor can be determined by straightforward integration from the first order equations

$$\lambda^{-1} \partial_t \lambda = \frac{t}{4} \big[(\partial_t Z)^2 + (\partial_x Z)^2 \big] \quad , \qquad \lambda^{-1} \partial_x \lambda = \frac{t}{2} \partial_t Z \partial_x Z \tag{6.21}$$

which are compatible if (6.20) is satisfied. They give rise to the second order evolution equation for λ

$$\partial_t (t \lambda^{-1} \partial_t \lambda) - t \partial_x (\lambda^{-1} \partial_x \lambda) = \frac{t}{2} (\partial_x Z)^2. \tag{6.22}$$

The general solution of (6.20) can be written as [15]

$$Z(t, x) = \int_{-\infty}^{\infty} G(t, x - w) \tilde{q}(w) dw + \int_{-\infty}^{\infty} H(t, x - w) p(w) dw \tag{6.23}$$

with the Green's functions

$$G(t, x) := \frac{1}{\pi} \int_0^\infty \cos(kx) J_0(kt) dk \quad , \quad H(t, x) := \frac{1}{\pi} \int_0^\infty \cos(kx) Y_0(kt) dk \tag{6.24}$$

where $J_0(z)$ and $Y_0(z) = J_0(z) \ln(z) + \ldots$ are the standard Bessel functions (see e.g. [38]). Note that the second integral in (6.24) is well defined (as a distribution) for all $t > 0$ despite the logarithmic singularity of the integrand. Near the singular hypersurface $t = 0$, the general solution admits the expansion [16, 17]

$$Z(t, x) = q(x) + p(x) \ln t + F(t, x) \tag{6.25}$$

[15]For *regular* initial data, *i.e.* $p(x) = 0$, this formula can be written in the completely explicit (and manifestly causal) form [37]

$$Z(t, x) = \frac{1}{\pi} \int_{x-t}^{x+t} \frac{q(w) dw}{\sqrt{t^2 - (x - w)^2}}.$$

which also follows directly from (6.24) (we thank V. Moncrief for a discussion on this point).

where $q(x) \neq \tilde{q}(x)$ unless $p(x) = 0$, and $\lim_{t\to 0} F(t,x) = 0$. The conformal factor expands as

$$\ln \lambda(t,x) = \frac{1}{4} p^2(x) \ln t + \dots \tag{6.26}$$

The functions

$$q(x) = q_0 + q_1 x + q_2 x^2 + \dots \quad , \quad p(x) = p_0 + p_1 x + p_2 x^2 + \dots \tag{6.27}$$

can be viewed as the *initial values of the coordinates and momenta on the 'big bang' hypersurface* $t = 0$. $p(x)$ correspond to the canonical momenta (rather than the velocities) since we know from section 5.2 that the Lagrange density contains the kinetic term $\mathcal{L} = \rho \mathsf{P}_t \mathsf{P}_t$. With $\rho(t) = t$ the conjugate momenta are therefore

$$\Pi(t,x) = t \partial_t Z(t,x) \quad \stackrel{t\to 0}{\longrightarrow} \quad p(x). \tag{6.28}$$

We emphasize that we are here working locally in a fixed coordinate chart — the Taylor expansion could be equivalently performed about any other spatial point x_0.

Evidently, the *Kasner solution* corresponds to constant $q(x) = q_0$ and $p(x) = p_0$ in (6.25) and (6.26). In this case we have a direct correspondence between this particular solution of Einstein's equations, and the corresponding solution (6.12) of the $\mathfrak{H}/K(\mathfrak{H})$ model with $p_m = q_m = 0$ for $m \geq 1$. In view of (6.17) and (6.18), different orbits under $\mathfrak{H}$ are thus labelled by p_0. If $p(x) \equiv 0$, the solution is no longer of Kasner type; instead of a singularity, it may exhibit other features such as Cauchy horizons [16]. In the following we will not consider this case.

When the higher modes (and the higher order terms in (6.25)) are switched on, the relation is less transparent. *We now define an explicit one-to-one map between the null geodesic trajectories $\mathcal{V}(t)$ on $\mathfrak{H}/K(\mathfrak{H})$ characterized by (6.12), and the solutions of (6.20) characterized by the initial values (6.27), by associating the solutions with the same initial data $q_0, q_1, \dots$ and $p_0, p_1, \dots$ and σ_0.* We emphasize again that such an association makes sense *only after* gauges have been fixed; in particular, all possible coordinate transformations in (t,x) are 'used up' when the four-dimensional line element is cast into the form (6.19), and the diffeomorphism constraint is imposed on the initial data by solving (6.21). The action of the 'solution generating group' $\mathfrak{H}$ on the solutions of (6.20) is likewise *defined* by the corresponding action on the solution $\mathcal{V}(t)$ in (6.16). We next examine how this action compares to the standard realisation of the Geroch group on such solutions.

6.3 Geroch vs. Heisenberg

To elucidate the relation between the transformations (6.17) on the initial data (6.27) on the one hand, and the action of the corresponding subgroup of the Geroch group on solutions of (6.20) on the other, we briefly recall how the latter is usually realized. The integrability of general relativity reduced to (1+1) dimensions [39, 40] is usually expressed by means of a linear system (or Lax pair) whose integrability condition is equivalent to the reduced Einstein equations [5]. In the standard formulation this is achieved by promoting the matrix $\mathtt{V}(t,x)$ appearing in (5.46) of section 5 to an element of the associated *loop group* by introducing an extra dependence on a spectral parameter γ, viz.

$$\mathtt{V}(t,x) \rightarrow \hat{\mathtt{V}}(t,x;\gamma), \tag{6.29}$$

This matrix satisfies the linear system (Lax pair) equations [5, 6]

$$\partial_\mu \hat{\mathtt{V}}\hat{\mathtt{V}}^{-1} = \mathtt{Q}_\mu + \frac{1+\gamma^2}{1-\gamma^2}\mathtt{P}_\mu + \frac{2\gamma}{1-\gamma^2}\epsilon_{\mu\nu}\mathtt{P}^\nu. \tag{6.30}$$

The parameter γ is to be interpreted as the spectral parameter of the loop algebra over $\mathfrak{sl}_{n-2}$. The integrability of (6.30) implies the equations of motion (5.49) if

$$\gamma = \gamma(t,x;w) = \frac{(x-w) \pm \sqrt{(x-w)^2 - t^2}}{t} \quad \Leftrightarrow \quad w = -\frac{t}{2}\left(\gamma + \frac{1}{\gamma}\right) + x \tag{6.31}$$

where w is sometimes called the 'constant spectral parameter'. The appearance of *two* spectral parameters γ and w, one of which depends on the space-time coordinates, is a consequence of the coupling the σ-model to gravity, and the characteristic feature which distinguishes this model from flat space σ-models. It is precisely the coordinate dependence of the spectral parameter γ which allows the Geroch group to generate space-dependent solutions, as we will now explain for the polarised Gowdy cosmologies.

Diagonal solutions can be obtained by starting from the following w-dependent matrix [16] considered as an element of the loop group $\widehat{GL(1)} \subset A_1^{(1)}$

$$\hat{\mathtt{V}}(w) = \begin{pmatrix} \exp(G(w)) & 0 \\ 0 & \exp(-G(w)) \end{pmatrix} \tag{6.32}$$

To generate space-time dependent solutions from this matrix, one follows the general procedure of [5], by first expressing w as a function of t, x and γ by virtue of (6.31), and then removing the singularity at $\gamma = 0$ by a compensating transformation. The corresponding solution of the field equations is then obtained from $\hat{\mathtt{V}}$ by setting $\gamma = 0$, as $\hat{\mathtt{V}}$ is holomorphic at $\gamma = 0$ after removal of the poles. In infinitesimal form, these steps are summarized in the combined transformation

$$\delta\hat{\mathtt{V}}(t,x;\gamma) = \delta h(t,x;\gamma)\hat{\mathtt{V}}(t,x;\gamma) - \hat{\mathtt{V}}(t,x;\gamma)\delta g(w) \tag{6.33}$$

[16] The following considerations are based on unpublished work with T. Damour [41].

where $\delta g(w) \in \widehat{\mathfrak{gl}_1}$, and δh is the compensating transformation in $K(\widehat{\mathfrak{gl}_1})$ which removes the poles at $\gamma = 0$. (6.33) shows that *only half of* $\widehat{\mathfrak{gl}_1}$ has an actual effect on the solution $\mathtt{V}(t,x)$, namely those $\delta g(w) \propto w^n$ with $n > 0$. The other half ($n < 0$) yields expressions in $\delta\hat{\mathtt{V}}$ for which $\delta\mathtt{V} = \lim_{\gamma\to 0} \delta\hat{\mathtt{V}}(\gamma) = 0$. Hence, the latter transformations merely shift the integration constants arising in the definition of the higher order dual potentials, and have no effect on the physical solution.

All solutions of the type (6.25) can be generated from [41]

$$G(w) = f(w) + g(w) \ln w \tag{6.34}$$

with regular functions $f(w)$ and $g(w)$ (which are directly related to, but not identical with, the functions $q(w)$ and $p(w)$ in (6.27)). The presence of a $\ln w$ term in (6.34), which is necessary to obtain a non-trivial Kasner coefficient $p(x)$ in (6.25), signals that we are not dealing with the standard loop group in w. The necessity of the $\ln w$ term is another manifestation of the fact that the standard realisation of the Geroch group affects only one half of the initial data, and must therefore be 'amended'.

From (6.31) and (6.34), we can deduce the leading contributions to $\delta Z(t,x)$ for $t \sim 0$ from a given $\delta g(w)$, following the steps described above, with the result

$$\begin{aligned} \delta g(w) &= w^n \qquad &&\Rightarrow \qquad \delta Z(t,x) \propto x^n, \\ \delta g(w) &= w^n \ln w \qquad &&\Rightarrow \qquad \delta Z(t,x) \propto x^n \ln t, \end{aligned} \tag{6.35}$$

for $n \geq 0$. This shows that a regular (non-logarithmic) affine level n element of the Geroch group induces a spatial dependence $\propto x^n$ in the regular initial data in a Taylor expansion around $x = 0$, but that a logarithmic dependence on the loop parameter w is required to change the singular initial data (the Kasner coefficient function $p(x)$). The action of the Witt–Virasoro algebra, and hence that of infinitesimal variations along $\hat{d}$, were discussed in [42]. The central charge acts on the conformal factor λ by scaling transformations [4, 5, 6], in agreement with the results derived at the end of section 6.1.

Eqs. (6.35) illustrate the main advantage of the affine coset model and our new realization of (a subgroup of) the Geroch group in comparison with the linear system approach: In the Heisenberg model, the *full* algebra generates non-trivial transformations on the initial data — the positive half shifts the initial coordinates q_m, and the negative half shifts the initial momenta p_m, see (6.17). By contrast, for the standard realisation of the Geroch group only *half* the transformations $\propto w^n$ for $n > 0$, act non-trivially and change the initial coordinates encoded in $q(x)$; in this sector, the action of the Geroch group agrees with the action of the upper half of $\mathfrak{H}$. On the other hand, in the standard approach, we must extend the Geroch transformations by allowing $w^n \ln w$ terms in order to shift the initial momenta $p(x)$. Moreover, this rather *ad hoc* extension of the loop group fails at the non-linear level, *i.e.* for non-diagonal metrics because it would force us to admit *arbitrary* positive powers of $\ln w$ [41]. This difficulty is avoided altogether in our new coset approach, where the 'solution generating group' acts on *all* initial data, except p_0, and which furthermore does not require modifying the loop group by logarithimic terms to generate the most general solution.

The difference between the two realisations of the affine symmetry is also evident from way the (dynamical) fields are encoded in $\hat{\mathtt{V}}$ and $\mathcal{V}$ respectively. Since the Chevalley involution acts by [5]

$$\omega\big(\hat{\mathtt{V}}(t,x;\gamma)\big) = \big(\hat{\mathtt{V}}^T\big)^{-1}\left(t,x;\frac{1}{\gamma}\right) \tag{6.36}$$

the r.h.s. of (6.30) is invariant, whence

$$\partial_\mu \hat{\mathtt{V}}\hat{\mathtt{V}}^{-1} \in K(A_{n-2}^{(1)}). \tag{6.37}$$

By contrast, for the geodesic σ-model of section 4, all the dynamics is contained in the coset components, see (4.17). Hence, we conclude again that the linear system (6.30) is not immediately suitable for the comparison with the results of the foregoing sections.

7 Discussion

There is an alternative formulation of the Lax pair for two-dimensional gravity [43], which is somewhat more similar to (4.24), in that its r.h.s. belongs to a coset subalgebra, and involves the CSA generators $\hat{c}$ and $\hat{d}$ explicitly. In that formulation the Cartan form reads (using light-cone coordinates)

$$\partial_\pm \mathcal{V}\mathcal{V}^{-1}(t,x) = \pm\rho^{-1}\partial_\pm\rho(L_0 - L_{\pm 1}) + \frac{1}{2}Q_{\pm\alpha\beta}J^{\alpha\beta} + \frac{1}{2}P_{\pm\alpha\beta}K^{\alpha}_{\pm 1\beta} \mp \partial_\pm\sigma\hat{c}. \tag{7.1}$$

Here, L_0 and $L_{\pm 1}$ belong to the Möbius subalgebra of the Virasoro algebra (which can be embedded in the enveloping algebra of the affine algebra with $L_0 = \hat{d}$). In contradistinction to the linear system (6.30) the expansion of (7.1) is truncated at affine level one, but involves the Möbius generators. (This has an effect on the set of allowed dressing tranformations [43].)

In the above linear system, the generator L_{-1} acts like a derivative operator on a vertex representation, and belongs to a Witt–Virasoro algebra acting on the affine algebra via a semi-direct product. This suggests that there might exist a similar, but larger, subalgebra inside the enveloping algebra of AE_n, which would contain derivative operators in all spatial directions, and contain the standard Witt–Virasoro algebra as a subalgebra. In addition, the affine algebra might be embedded in a *toric algebra* (see e.g. [44] and references therein). Such an algebra is spanned by generators $T^A_{\vec{m}}$ (where $\vec{m}$ designates a vector in some vector space), with commutation relations of the form

$$\left[T^A_{\vec{m}}, T^B_{\vec{n}}\right] = f^{AB}{}_C T^C_{\vec{m}+\vec{n}} + \text{central terms.} \tag{7.2}$$

However, looking at the relation for the gradient fields derived for AE_3 in section 3, we see that the naive identification of $\vec{m}$ with the three-vector of the multiplicities of the indices $1,2,3$ among the 'gradient indices' $a_1, \dots a_{\ell-1}$ of the generator $E_{a_1,\dots,a_{\ell-1}}{}^{b_1 b_2}$ will fail since the commutator (3.24) will always *add* one index and so destroy the vector space structure on such $\vec{m}$. Moreover we have seen in section 3 that the gradient representations by themselves do not close into a subalgebra of AE_3. We thus conclude that there is no toric algebra inside AE_n or its enveloping algebra.

The interpretation of the affine coset proposed in section 6 provides a more favourable realisation of the Geroch group since it can incorporate transformations of both the initial coordinates and the initial velocities in the general case without extending the set of allowed transformations. We anticipate that similar results hold for the σ-model realisation of the affine group $A_1^{(1)}$ in comparison with the standard realisation of the full Geroch group. However, the more important challenge at this point is now to find out how the hyperbolic AE_3 coset model can generate the most general dependence on four-dimensional space and time coordinates, and to understand the significance of the fact that, unlike the affine elements, the gradient generators no longer form a closed subalgebra of AE_n.

It is also interesting to consider higher levels of the affine coset model by including as a next step the basic representation [26] of the affine algebra. This will show new aspects of the correspondence characteristic for the KMA σ-model crucial for the programme of [9].

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A The Compact Subalgebra is not a KMA

In this appendix we prove that the 'maximal compact' invariant subalgebra of an infinite-dimensional KMA is *not* of Kac–Moody type. This applies in particular to the algebras AE_n and E_{10}, and their invariant subalgebras.

Proposition 1 *Let $\mathfrak{g}$ be any infinite-dimensional split real Kac–Moody algebra $\mathfrak{g}$, which is not the direct sum of (infinitely many) finite dimensional algebras. Then the infinite-dimensional subalgebra $\mathfrak{k} \subset \mathfrak{g}$ fixed by the Chevalley involution ω is not a Kac–Moody algebra.*

Proof: The main observation is that the contravariant Hermitian form on the Kac–Moody algebra

$$(x|y) := -\langle x|\omega(y)\rangle \qquad \text{for}\, x, y, \in \mathfrak{g} \tag{A.1}$$

is not positive definite everywhere on the split real KMA if the latter is infinite-dimensional [24]: in the CSA, and only in the CSA, there exist elements of both positive and negative norm squared, because the Cartan matrix is not positive definite. However, since the CSA is not invariant under ω, the compact subalgebra $\mathfrak{k}$ does not contain any such elements, and the contravariant form on $\mathfrak{k}$ (inherited from $\mathfrak{g}$) is therefore positive definite on $\mathfrak{k}$ [45]. Because $\mathfrak{k}$ is infinite-dimensional, it can thus be a KMA if and only if it is the (orthogonal) sum of infinitely many finite-dimensional algebras of KM type.

We will prove that this cannot happen by exhibiting a contradiction if the assumption were true. Therefore assume that $\mathfrak{k}$ is the infinite direct sum of some finite-dimensional algebras. Now consider some (regular) level decomposition $\mathfrak{g} = \bigoplus_{\ell\in\mathbb{Z}} \mathfrak{g}_\ell$ of $\mathfrak{g}$ such that $0 < \dim \mathfrak{g}_\ell < \infty$ for all $\ell \in \mathbb{Z}$. Then $\mathfrak{g}$ is generated by taking multiple commutators of $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Since the Chevalley involution ω acts according to $\omega(\mathfrak{g}_\ell) = \mathfrak{g}_{-\ell}$, the invariant subalgebra admits the decomposition (not a grading!)

$$\mathfrak{k} = \bigoplus_{\ell\geq 0} \mathfrak{k}_\ell \quad \text{with} \quad \mathfrak{k}_0 \subset \mathfrak{g}_0, \qquad \mathfrak{k}_\ell \subset \mathfrak{g}_{-\ell} \oplus \mathfrak{g}_\ell \quad \Rightarrow \quad [\mathfrak{k}_0, \mathfrak{k}_\ell] \subset \mathfrak{k}_\ell \tag{A.2}$$

Like $\mathfrak{g}$, the infinite-dimensional invariant subalgeba $\mathfrak{k}$ is generated by taking multiple commutators of $\mathfrak{k}_0$ and $\mathfrak{k}_1$. Since all $\mathfrak{g}_\ell$ are of finite dimension, it follows in particular that $\mathfrak{k}_0$ and $\mathfrak{k}_1$ are also finite-dimensional. According to our assumption they must therefore belong to a *finite* sum of finite-dimensional subalgebras of $\mathfrak{k}$. Hence, the algebra they generate is also finite-dimensional, in contradiction with the fact that $\mathfrak{k}$ is infinite-dimensional. □

The proposition unfortunately does not give an answer to the important question what kind of Lie algebra $\mathfrak{k}$ is, and, more pressingly, what its representation theory is. It is expected that the representation theory of $\mathfrak{k}$ is associated with the supersymmetric extension of the one-dimensional σ-model [46, 13].

B AE_n σ-model in $\mathfrak{sl}_n$ decomposition and p-forms

In this appendix, we analyse the σ-model based on AE_n of section 4 from its $\mathfrak{sl}_n$ subalgebra. We will also add p-forms to the Einstein equation of the model to make contact with other oxidized theories.

B.1 Pure Gravity

We first study the σ-model and then compare it to the reduction of the Einstein equation.

The Cartan form in triangular gauge in $\mathfrak{sl}_n$ form now is

$$\partial_t \mathcal{V}\mathcal{V}^{-1} = P_{ab}S^{ab} + Q_{ab}J^{ab} + \frac{1}{(n-2)!}P_{a_1\dots a_{n-2},a_{n-1}}E^{a_1\dots a_{n-2},a_{n-1}} + \dots \tag{B.1}$$

with indices a, b taking values $1, \dots, n$. The resulting σ-model equation on level $\ell = 0$, in the truncation to $|\ell| \leq 1$ can be deduced from the results of section 2.2, with the result

$$n\partial_t(n^{-1}P_{ab}) = 2P_{c(a}Q_{b)c} + \frac{1}{2(n-2)!}\Big(\delta_{ab}P_{c_1\dots c_{n-2},c_{n-1}}P_{c_1\dots c_{n-2},c_{n-1}} - P_{c_1\dots c_{n-2},a}P_{c_1\dots c_{n-2},b} - (n-2)P_{ac_1\dots c_{n-3},c_{n-2}}P_{bc_1\dots c_{n-3},c_{n-2}}\Big). \tag{B.2}$$

In the corresponding gravity theory, the space-space components of reduced Einstein equations in flat components are

$$R_{ab} = R_{ab}^{\text{temp}} + R_{ab}^{\text{spat}} = 0, \tag{B.3}$$

where we defined, following [14],

$$R_{ab}^{\text{temp}} = \partial_0\omega_{ab0} + \omega_{cc0}\omega_{ab0} - 2\omega_{0c(a}\omega_{b)c0}, \tag{B.4}$$

$$R_{ab}^{\text{spat}} = \frac{1}{4}\tilde{\Omega}_{cd\,a}\tilde{\Omega}_{cd\,b} - \frac{1}{2}\tilde{\Omega}_{ac\,d}\tilde{\Omega}_{bc\,d} - \frac{1}{2}\tilde{\Omega}_{ac\,d}\tilde{\Omega}_{bd\,c} - \frac{1}{2}\partial_c\tilde{\Omega}_{c(a\,b)}. \tag{B.5}$$

Here, $\tilde{\Omega}_{ab\,c}$ is the tracefree part of the anholonomy $\Omega_{ab\,c} = 2e_{[a}{}^m e_{b]}{}^n\partial_m e_{nc}$ and $\omega_{a\,bc} = (\Omega_{ab\,c} + \Omega_{ca\,b} - \Omega_{bc\,a})/2$ is the spin connection. We use the same decomposition (5.34) of the vielbein as in section 5.1.

By comparing eq. (B.2) with eq. (B.3), we see that almost all terms match upon identifying

$$\begin{aligned} n(t) &\equiv Ne^{-1}(t,x_0), && \text{(B.6)}\\ P_{ab}(t) &\equiv \omega_{a\,bt}(t,x_0), && \text{(B.7)}\\ Q_{ab}(t) &\equiv -\omega_{t\,ab}(t,x_0), && \text{(B.8)}\\ P_{a_1\dots a_{n-2},a_{n-1}}(t) &\equiv \frac{N}{2}\varepsilon_{a_1\dots a_{n-2}bc}\tilde{\Omega}_{bc\,a_{n-1}}(t,x_0) && \text{(B.9)} \end{aligned}$$

for some fixed spatial point x_0. The terms which do not match are the second spatial derivatives and the cross-term $\tilde{\Omega}_{ac\,d}\tilde{\Omega}_{bd\,c}$ in (B.5). However, the second spatial derivatives are not expected to match in this approximation and the cross-term disappears upon using a conformal gauge for the zweibein.

B.2 Inclusion of p-forms

We now add a single p-form field to the Einstein action via their kinetic terms[17]

$$S = \int d^{n+1}x \det(E)\left(R - \frac{1}{2\,(p+1)!}F_{M_1\dots M_{p+1}}F^{M_1\dots M_{p+1}}\right). \qquad \text{(B.10)}$$

The generalisation to several p-forms is straight-forward and we do not consider dilaton terms here for simplicity.

The Einstein equation resulting from (B.10) can be written as

$$R_{MN} = \frac{1}{2\,p!}F_{MP_1\dots P_p}F_N{}^{P_1\dots P_p} - \frac{p}{2\,(p+1)!\,(n-1)}G_{MN}F_{P_1\dots P_{p+1}}F^{P_1\dots P_{p+1}}. \qquad \text{(B.11)}$$

Converting into flat indices and restricting to the spatial directions, we write

$$R^{\text{temp}}_{ab} + R^{\text{spat}}_{ab} = -N^{-2}T^{\text{el}}_{ab} + T^{\text{magn}}_{ab}, \qquad \text{(B.12)}$$

with the energy momentum tensor split into ‘electric’ and ‘magnetic’ contributions according to

$$\begin{aligned} T^{\text{el}}_{ab} &= \frac{p}{2\,p!}F_{tac_1\dots c_{p-1}}F_{tbc_1\dots c_{p-1}} - \frac{p}{2\,p!\,(n-1)}\delta_{ab}F_{tc_1\dots c_p}F_{tc_1\dots c_p}, && \text{(B.13)}\\ T^{\text{magn}}_{ab} &= \frac{1}{2\,p!}F_{ac_1\dots c_p}F_{bc_1\dots c_p} - \frac{p}{2\,(p+1)!\,(n-1)}\delta_{ab}F_{c_1\dots c_{p+1}}F_{c_1\dots c_{p+1}}. && \text{(B.14)} \end{aligned}$$

[17] We need not concern ourselves with possible Chern–Simons-like terms since these do not affect the Einstein equation. However, they are crucial for the gauge field equation of motions and Bianchi identities.

The factor $-N^{-2}$ in (B.12) stems from lowering the t index. Similarly, we split the Ricci tensor into temporal and spatial derivatives as before.

On the algebra side, we need to include additional generators to account for the electric and magnetic contributions to the Einstein equation. Consider the expansion of the Cartan form to be

$$\begin{aligned} \partial_t \mathcal{V}\mathcal{V}^{-1} &= P_{ab}S^{ab} + Q_{ab}J^{ab} + \frac{1}{(n-2)!}P_{a_1\dots a_{n-2},a_{n-1}}E^{a_1\dots a_{n-2},a_{n-1}} \\ &\quad + \frac{1}{p!}P^{\mathrm{el}}_{a_1\dots a_p}E^{a_1\dots a_p} + \frac{1}{(n-p-1)!}P^{\mathrm{magn}}_{a_1\dots a_{n-p-1}}E^{a_1\dots a_{n-p-1}} + \dots . \end{aligned} \tag{B.15}$$

In the first line of (B.15), we have the contribution of the AE_n fields as above. The second line now belongs to additional generators we have introduced into the algebra. That such generators are present in the relevant Kac–Moody algebra for the oxidized theory follows from the results of [47]. Demanding the normalizations

$$\langle E^{a_1\dots a_p}|F_{b_1\dots b_p}\rangle = p!\,\delta^{a_1\dots a_p}_{b_1\dots b_p}, \tag{B.16}$$

$$\langle E^{a_1\dots a_{n-p-1}}|F_{b_1\dots b_{n-p-1}}\rangle = (n-p-1)!\,\delta^{a_1\dots a_{n-p-1}}_{b_1\dots b_{n-p-1}}, \tag{B.17}$$

leads to the commutators of the new generators

$$\left[K^a{}_b, E^{c_1\dots c_p}\right] = (-1)^p\, p\, \delta^{[c_1}_b E^{c_2\dots c_p]a}, \tag{B.18}$$

$$\left[E^{a_1\dots a_p}, F_{b_1\dots b_p}\right] = -\frac{p}{n-1}p!\,\delta^{a_1\dots a_p}_{b_1\dots b_p}K + p\cdot p!\,\delta^{[a_1\dots a_{p-1}}_{[b_1\dots b_{p-1}}K^{a_p]}{}_{b_p]}, \tag{B.19}$$

and similar expressions for the magnetic generator with p replaced by $(n-p-1)$. From this, together with the commutator (2.16), we can deduce the following $\ell = 0$ equation of motion for the σ-model in this approximation

$$\begin{aligned} n\partial_t(n^{-1}P_{ab}) &= 2P_{c(a}Q_{b)c} + \frac{1}{2(n-2)!}\Big(\delta_{ab}P_{c_1\dots c_{n-2},c_{n-1}}P_{c_1\dots c_{n-2},c_{n-1}} \\ &\quad - P_{c_1\dots c_{n-2},a}P_{c_1\dots c_{n-2},b} - (n-2)P_{ac_1\dots c_{n-3},c_{n-2}}P_{bc_1\dots c_{n-3},c_{n-2}}\Big) \\ &\quad + \frac{p}{2\,p!}P^{\mathrm{el}}_{ac_1\dots c_{p-1}}P^{\mathrm{el}}_{bc_1\dots c_{p-1}} - \frac{p}{2\,p!(n-1)}\delta_{ab}P^{\mathrm{el}}_{c_1\dots c_p}P^{\mathrm{el}}_{c_1\dots c_p} \\ &\quad + \frac{n-p-1}{2\,(n-p-1)!}P^{\mathrm{magn}}_{ac_1\dots c_{n-p-2}}P^{\mathrm{magn}}_{bc_1\dots c_{n-p-2}} - \frac{p}{2\,p!(n-1)}\delta_{ab}P^{\mathrm{magn}}_{c_1\dots c_{n-p-1}}P^{\mathrm{magn}}_{c_1\dots c_{n-p-1}}. \end{aligned} \tag{B.20}$$

By comparing (B.12) and (B.20) we find agreement if we identify the new terms by

$$P^{\mathrm{el}}_{a_1\dots a_p}(t) \equiv F_{ta_1\dots a_p}(t, x_0), \tag{B.21}$$

$$P^{\mathrm{magn}}_{a_1\dots a_{n-p-1}}(t) \equiv \frac{N}{(p+1)!}\varepsilon_{a_1\dots a_{n-p-1}b_1\dots b_{p+1}}F_{b_1\dots b_{p+1}}(t, x_0). \tag{B.22}$$

in addition to (B.6). There are sign ambiguities here, since the fields appear quadratically in (B.20). These will be fixed from the equations of motion and Bianchi identities for the p-form fields.

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