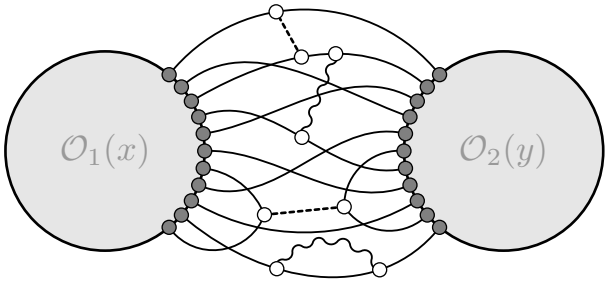
arXiv:hep-th/0407277 v2 9 Aug 2004

The Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory and Integrability

Dissertation
Eingereicht an der Humboldt-Universität zu Berlin
Überarbeitete Fassung



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9. August 2004

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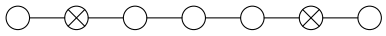
AEI-2004-057



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Zusammenfassung

Der Dilatationsoperator der $\mathcal{N} = 4$ Super Yang-Mills Theorie und Integrabilität

Der Dilatationsoperator mißt Skalendimensionen von lokalen Operatoren in einer konformen Feldtheorie. In dieser Dissertation betrachten wir ihn am Beispiel der maximal supersymmetrischen Eichtheorie in vier Raumzeit-Dimensionen. Wir entwicken und erweitern Techniken um den Dilatationsoperator abzuleiten, zu untersuchen und anzuwenden. Diese Werkzeuge sind ideal geeignet um Präzisionstests der dynamischen AdS/CFT-Vermutung anzustellen. Insbesondere wurden er im Zusammenhang mit Stringtheorie auf dem *plane-waves* Hintergrund (ebenfrontige planare Wellen) und dem Thema *spinning strings* erfolgreich angewendet.

Wir konstruieren den Dilatationsoperator ausschließlich mittels algebraischer Methoden: Indem wir die Symmetriealgebra und strukturelle Eigenschaften von Feynman-Diagrammen ausnützen, können wir aufwendige, feldtheoretische Berechnungen auf höheren Schleifen umgehen. Auf diese Weise erhalten wir den kompletten ein-schleifen Dilatationsoperator und die planare drei-schleifen Deformation in einem interessanten Untersektor. Diese Resultate erlauben es uns auf das Thema Integrabilität in vier-dimensionalen planaren Eichtheorien einzugehen: Wir beweisen, daß der komplette Dilatationsoperator auf einer Schleife integrabel ist, und präsentieren den dazugehörigen Bethe-Ansatz. Weiterhin argumentieren wir, daß die Integrabilität sich bis drei Schleifen und darüber hinaus fortsetzt. Unter der Annahme der Integrabilität konstruieren wir schließlich ein neuartiges Spinketten-Modell auf fünf Schleifen und schlagen einen Bethe-Ansatz vor, der sogar auf beliebig vielen Schleifen gültig sein mag!

Wir veranschaulichen den Nutzen unserer Methoden in zahlreichen Beispielen und stellen zwei wichtige Anwendungen im Rahmen der AdS/CFT-Korrespondenz vor: Wir leiten aus dem Dilatationsoperator den Hamiltonoperator der *plane-wave* String-Feldtheorie her und berechnen damit die Energieverschiebung auf dem Torus. Weiterhin wenden wir den Bethe-Ansatz an, um Skalendimensionen von Operatoren mit großen Quantenzahlen zu finden. Der Vergleich mit der Energie von *spinning strings* Konfigurationen zeigt eine erstaunliche Übereinstimmung.

Abstract

The dilatation generator measures the scaling dimensions of local operators in a conformal field theory. In this thesis we consider the example of maximally supersymmetric gauge theory in four dimensions and develop and extend techniques to derive, investigate and apply the dilatation operator. These tools are perfectly suited for precision tests of the dynamical AdS/CFT conjecture. In particular, they have been successfully applied in the context of strings on plane waves and spinning strings.

We construct the dilatation operator by purely algebraic means: Relying on the symmetry algebra and structural properties of Feynman diagrams we are able to bypass involved, higher-loop field theory computations. In this way we obtain the complete one-loop dilatation operator and the planar, three-loop deformation in an interesting subsector. These results allow us to address the issue of integrability within a planar four-dimensional gauge theory: We prove that the complete dilatation generator is integrable at one-loop and present the corresponding Bethe ansatz. We furthermore argue that integrability extends to three-loops and beyond. Assuming that it holds indeed, we finally construct a novel spin chain model at five-loops and propose a Bethe ansatz which might be valid at arbitrary loop-order!

We illustrate the use of our technology in several examples and also present two key applications for the AdS/CFT correspondence: We derive the plane-waves string field theory Hamiltonian from the dilatation operator and compute the energy shift on the torus. Furthermore, we use the Bethe ansatz to find scaling dimensions of operators with large quantum numbers. A comparison to the energy of spinning strings shows an intricate functional agreement.

Contents

Introduction

Probably the two most important advances in the deeper understanding of our world in terms of theoretical physics were made at the beginning of the twentieth century: The *theory of general relativity* and *quantum mechanics*. On the one hand, Einstein's theory of general relativity (GR) has replaced Newton's theory of gravity and, in its original form, is still the most accurate theory to describe forces between massive bodies. It brought about a major change in the notion of space and time. Both were unified into spacetime which, in addition, is curved by the masses that propagate on it. Two of the most important conceptual improvements of GR are *symmetry* and *locality*. The symmetry of GR is called diffeomorphism-invariance and allows to label points of spacetime in an arbitrary way, the equations of GR do not depend on this. Furthermore, GR is a local field theory, there is no action at a distance, but instead, just like in Maxwell's Electrodynamics, forces are mediated by a field.

On the other hand, there is Quantum Mechanics. It proposes a completely new notion of particles and forces, both of which should be considered as two manifestations of the same object. It also departed from a deterministic weltanschauung; a measurement is inevitably probabilistic and moreover must be considered as an action which influences the outcome of future measurements. Many aspects of quantum mechanics seem odd at first and second sight and truly make sense only in a quantum field theory (QFT). QFT introduced the notion of particle creation and annihilation, an essential element for local interactions. The price that has to be paid are spurious divergencies due to particles being created and annihilated at the same place and instant. It required some effort to understand, *regularise* and *renormalise* the divergencies in order to obtain finite, physical results.

The first fully consistent physical QFT was Quantum Electrodynamics (QED), the quantum counterpart of Electrodynamics. A guiding principle in the construction of QED is, again, *symmetry*. Here, the symmetry is given by gauge transformations; they allow to change some unphysical degrees of freedom of the theory by an arbitrary amount. Consequently, QED is termed a *gauge theory* and, in particular, it has an Abelian U(1) gauge group. Amplitudes in QED can be expanded in a coupling constant g related to the fundamental charge of an electron. This perturbative treatment leads to Feynman diagrams which describe interactions in a rather intuitive fashion. Besides electromagnetism, two other interactions between particles have been observed in particle accelerators: the weak and the strong interactions. Let us discuss the second kind. The strong nuclear force is responsible for the binding of nucleons to nuclei, which would otherwise disperse due to their electromagnetic charge. One of the earlier candidates for a description of these interactions was a *string theory*, a theory of string-like extended objects instead of point-like particles. It explained some qualitative aspects of particle (excitation) spectra

correctly; yet, soon it was found that it embodies some insurmountable theoretical as well as phenomenological shortcomings and interest in it declined.

In the meantime, an alternative description of strong (and weak) interactions had emerged. Like QED, it is based on a gauge theory, the so-called Quantum Chromodynamics (QCD). The gauge group for strong interactions is SU(3), which may, for instance, be inferred indirectly from the spectrum of hadrons. Here, symmetry is important for several reasons. First of all, and even more so for QCD than for QED, symmetry is essential for the theoretical consistency of the model. Furthermore, the particular gauge group of QCD leads to a feature called asymptotic freedom/confinement. It implies that QCD is effectively weak at very short distances, but becomes infinitely strong at larger dimensions (on the scale of nucleons). On a qualitative level, this may be understood as follows: The attraction/repulsion between two charges is mediated by flux lines. As opposed to QED, in QCD flux lines attract each other and will form a small tube stretching between the charges. The tube effectively behaves like a string with tension and binds the particles irrespective of their distance. This explains why it is not possible to observe an individual charged particle and leads us to confinement, which allows only uncharged particles to propagate freely.

A peculiarity of generic gauge theories with gauge group U(N), which we will make heavy use of, was observed by 't Hooft [1]: He derived a relationship between the topological structure of a Feynman graph and its N-dependence. When $1/N$ is interpreted as a coupling constant, he observed that the perturbative expansion in $1/N$ is very similar in nature to the perturbative genus expansion in a generic interacting string theory (string field theory).

Not only due to their mathematical beauty, the theory of General Relativity and Quantum Mechanics/QFT have become the foundations of modern physics, but mainly because of the accuracy to which they describe the world. On the one hand, gravity is a very weak force and it requires a large amount of matter to feel its effects. Consequently, GR describes the world at very large scales. For instance, GR was first confirmed when the aberration of light near the perimeter of the sun was investigated. On the other hand, the remaining three forces described by QFT's are incomparably stronger. Therefore quantum field theories chiefly describe the microcosm. In particular, the standard model of particle physics, the union of the above three gauge theories, has led to some non-trivial predictions which have been confirmed with unprecedented accuracy, e.g. the electric moment of the electron and muon.

One of the major open problems of theoretical physics is to understand what happens when an enormous amount of matter is concentrated on a very small region of space. For example, this situation arises at the singularity of a black hole or shortly after the big bang. To describe such a situation, we would need to combine General Relativity with the concepts of quantum field theory and consider *quantum gravity* (QG). Despite the better part of a century of research, such a unification correctly describing our world has not yet been found. The main obstacle for the direct construction of a quantum theory of gravity are the divergencies mentioned above, which cannot be renormalised in this case and render the quantum theory meaningless.

Currently, the most favoured theory for a consistent quantisation of gravity is *superstring theory*. It is a refinement of the (bosonic) string theory found in connection with strong interactions and involves an additional symmetry which relates fermions and

bosons, namely *supersymmetry*. Supersymmetry makes string theory very appealing to theorists: It overcomes several of the shortcomings of bosonic string theory and restricts the form such that there are only five types of string theories (IIA,IIB,I,HO,HE), which were, moreover, argued to be equivalent via duality. This is a very good starting point for a *theory of everything*, given that string theory not only naturally incorporates gravity, but also gauge theories, the type of theory on which the standard model is based.

With the advent of superstring theory, supersymmetry has been applied to field theories as well, giving rise to beautiful structures. One important aspect is that many of the divergencies observed in ordinary QFT's are absent in supersymmetric ones. Indeed, this is the case for the unique, maximally supersymmetric gauge theory in four spacetime dimensions, $\mathcal{N} = 4$ *super Yang-Mills theory* ($\mathcal{N} = 4$ SYM) [2]. This remarkable feature [3], allows the theory to be conformally invariant, even at the quantum level! Conformal symmetry is a very constraining property in field theory. Most importantly, two-point and three-point correlation functions are completely determined by the scaling dimensions and structure constants of the involved local operators. For instance, the two-point function of a scalar operator $\mathcal{O}$ of dimension D must be of the form

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = \frac{M}{|x-y|^{2D}}\,, \tag{1}$$

where M is an unphysical normalisation constant. In two dimensions, conformal symmetry is even more powerful, it makes a theory mathematically quite tractable and leads to a number of exciting phenomena such as integrability. Consequently, it plays a major role in the world-sheet description of string theory and was thoroughly investigated. In four dimensions, however, conformal invariance appeared to be more of a shortcoming at first sight: It makes the model incompatible with particle phenomenology, which might be the reason why $\mathcal{N} = 4$ SYM was abandoned soon after its discovery.

New interest in this theory was triggered by the *AdS/CFT correspondence*. Inspired by the studies of string/string dualities and D-branes, Maldacena conjectured that IIB string theory on the curved background[1] $AdS_5 \times S^5$ should be equivalent to $\mathcal{N} = 4$ SYM [4–6] (see [7] for comprehensive reviews of the subject) and thus substantiated the gauge/string duality proposed earlier by 't Hooft. The correspondence is supported by the well-known fact that the symmetry groups of both theories, PSU$(2,2|4)$, match. Consequently, the representation theory of the *superconformal algebra* $\mathfrak{psu}(2,2|4)$ [8] was investigated more closely [9, 10], and numerous non-renormalisation theorems were derived (see e.g. [11]). In addition, some unexpected non-renormalisation theorems, which do not follow from $\mathfrak{psu}(2,2|4)$ representation theory, were found [12]. Once thought to be somewhat boring, it gradually became clear that conformal $\mathcal{N} = 4$ gauge theory is an extremely rich and non-trivial theory with many hidden secrets; eventually, the correspondence has helped in formulating the right questions to discover some of them.

Yet, the conjecture goes beyond kinematics and claims the full dynamical agreement of both theories. For example, it predicts that the spectrum of scaling dimensions D in the conformal gauge theory should coincide with the spectrum of energies E of string states

$$\{D\} = \{E\}. \tag{2}$$

[1] This manifold consists of the five-sphere and the five-dimensional anti-de Sitter spacetime, which is an equivalent of hyperbolic space but with Minkowski signature.

Unfortunately, like many dualities, Maldacena's conjecture is of the strong/weak type: The weak coupling regime of gauge theory maps to the strong coupling (i.e. tensionless) regime in string theory and vice versa. The precise correspondence is given by

$$g_{\rm YM}^2 N = \lambda = \frac{R^4}{\alpha'^2}\,, \qquad \frac{1}{N} = \frac{4\pi g_{\rm s}}{\lambda}\,, \tag{3}$$

where $g_{\rm YM}$ is the Yang-Mills coupling constant and α' is the inverse string tension.[2] Furthermore, N is the rank of the U(N) gauge group of Yang-Mills theory, λ is the effective 't Hooft coupling constant in the large N limit, $g_{\rm s}$ is the topological expansion parameter in string theory and R is the radius of the $AdS_5 \times S^5$ background. It is not known how to fully access the strong coupling regime in either theory, let alone how to rigorously quantise string theory on the curved background. Therefore, the first tests of the AdS/CFT correspondence were restricted to the infinite tension regime of string theory which is approximated by supergravity and corresponds to the strong coupling regime on the gauge theory side. Gauge theory instanton calculations of four-point functions of operators which are protected by supersymmetry were shown to agree with the supergravity results see e.g. [13].

Despite a growing number of confirmations of the conjecture in sectors protected by symmetry, the fundamental problem of a strong/weak duality remained. For example, the AdS/CFT correspondence predicts that the scaling dimensions D of generic, unprotected operators in gauge theory should scale as

$$D \sim \lambda^{1/4} \tag{4}$$

for large λ, but how could this conjecture be tested? It was Berenstein, Maldacena and Nastase (BMN) who proposed a limit where this generic formula may be evaded [14]: In addition to a large λ, consider local operators with a large charge J on S^5, whose scaling dimension D is separated from the charge J by a finite amount only. More explicitly, the limit proposed by BMN is

$$\lambda, J \longrightarrow \infty \qquad \text{with} \qquad \lambda' = \frac{\lambda}{J^2} \quad \text{and} \quad D - J \quad \text{finite.} \tag{5}$$

In this limit, the $AdS_5 \times S^5$ background effectively reduces to a so-called *plane-wave* background [15] on which the spectrum of string modes can be found exactly and the theory can be quantised [16]. Remarkably, the light-cone energy $E_{\rm LC}$ of a string-mode excitation

$$E_{\rm LC} = \sqrt{1 + \lambda' n^2} = 1 + \tfrac{1}{2}\lambda' n^2 + \ldots \tag{6}$$

has a perturbative expansion at a small effective coupling constant λ'. As the light-cone energy corresponds to the combination $D - J$ in gauge theory, suddenly the possibility of a quantitative comparison for unprotected states had emerged! Indeed, BMN were able to show the agreement at first order in λ' for a set of operators. Their seminal article [14] has sparked a long list of further investigations and we would like to refer the reader to [17] for reviews. Let us only comment on one direction of research: In its original form, the BMN limit was proposed only for non-interacting strings and gauge theory in the planar

[2] The actions are inversely related to these constants, $S_{\rm YM} \sim 1/g_{\rm YM}^2$ and $S_{\rm string} \sim 1/\alpha'$. Therefore, quantum effects are suppressed at small $g_{\rm YM}$ and small α' in the respective theories.

limit. Soon after the BMN proposal, it was demonstrated that also non-planar corrections can be taken into account in gauge theory [18, 19], they correspond to energy shifts due to string interactions [20]. In gauge theory, the effective genus counting parameter in the so-called *double-scaling limit* is $g_2 = J^2/N$. The first order correction in λ' and g_2^2 was computed in [21, 22] and was argued to agree with string theory [23, 24]. This is yet another confirmation of the AdS/CFT correspondence, but for the first time within an interacting string theory!

In the study of the BMN correspondence, the attention has been shifted away from lower dimensional operators to operators with a large number of constituent fields [18, 19, 21, 22, 25]. There, the complications are mostly of a combinatorial nature. It was therefore desirable to develop efficient methods to determine anomalous dimensions without having to deal with artefacts of the regularisation procedure. This was done in various papers, on the planar [14, 25–28] and non-planar level [18, 19, 21, 22, 29, 30], extending earlier work on protected half-BPS [31–33] and quarter-BPS operators [34]. In [35] it was realised, following important insights in [36, 37], that these well-established techniques can be considerably simplified and extended by considering the *Dilatation Operator*. The dilatation operator $\mathfrak{D}$ is one of the generators of the conformal algebra and it measures the scaling dimension D of a local operator[3] $\mathcal{O}$

$$\mathfrak{D}\,\mathcal{O} = D\,\mathcal{O}. \tag{7}$$

In general, there are many states and finding the scaling dimension is an eigenvalue problem which requires to resolve the mixing of states. Once the dilatation operator has been constructed, it will generate the matrix of scaling dimensions for any set of local operators of a conformal field theory in a purely algebraic way (in App. A we present an introductory example of how to apply the dilatation operator). What is more, scaling dimensions can be obtained exactly for all gauge groups and, in particular, for the group U(N) with finite N [38]. Even two or higher-loop calculations of anomalous dimensions, which are generically plagued by multiple divergencies, are turned into a combinatorial exercise! Using the dilatation operator techniques, many of the earlier case-by-case studies of anomalous dimensions [33, 39–46] were easily confirmed [38]. They furthermore enabled a remarkable all-genus comparison between BMN gauge theory and plane-wave string theory [47]. The subject of this dissertation is the construction and investigation of the dilatation operator in $\mathcal{N} = 4$ SYM, a conformal quantum field theory, in perturbation theory.

Classical scaling dimensions of states are easily found by counting the constituent fields weighted by their respective scaling dimensions. It is just as straightforward to construct the classical dilatation operator to perform this counting. Scaling dimensions in a field theory generally receive quantum corrections, $D = D(g)$ and consequently the dilatation operator must receive radiative corrections $\mathfrak{D} = \mathfrak{D}(g)$, too. In the path integral framework there will be no natural way to obtain quantum corrections to the dilatation operator; we will have to derive them from correlators, for example from two-point functions. Now what is the benefit in considering the dilatation operator if a conventional calculation uses two-point correlators as well? There are two major advantages: Firstly, the dilatation generator is computed once and for all, while a two-point function will have to be evaluated

[3] To avoid confusion, we will later speak of 'states' instead of local operators.

for each pair of states (unless one makes use of some effective vertex e.g. [21,22]). Secondly, the dilatation operator computes only the scaling dimension $D(g)$. The two-point function also includes a contribution $M(g)$ from the normalisation of states. These two quantities will have to be disentangled before the scaling dimension can be read off from the two-point function (1). Here, a complicating issue is that in general the normalisation coefficient $M(g)$ obtained in field theory is divergent.

A radiative correction to the dilatation operator in the context of $\mathcal{N} = 4$ SYM has first been computed in [48, 35].[4] This one-loop correction was restricted to the sector of states composed from the six scalar fields of the theory only, the so-called $\mathfrak{so}(6)$ subsector, on which the one-loop dilatation operator closes.

However, there is nothing special about the scalar fields, except maybe their conceptual simplicity. Generic local operators can as well consist of fermions or gauge fields (in the guise of a field strength). What is more, we can also apply an arbitrary number of (covariant) derivatives to the basic constituent fields. In principle, one could now compute the one-loop dilatation operator for all fields (we shall denote a generic field with derivatives by the symbol $\mathcal{W}$). This is feasible, but certainly much more involved than the calculations for the $\mathfrak{so}(6)$ subsector due to infinitely many types of fields $\mathcal{W}$ and a complicated structure of spacetime indices in the expected conformal two-point function, see e.g. [30, 28, 49].

In [38] a different approach to obtain contributions to the dilatation generator has been proposed: Just as in field theory, all contributing diagrams to a two-point function are written down. The most complicated part of their computation is to evaluate the spacetime integrals due to vertices of the Feynman diagram. Nevertheless, the structural result of the integrals is known; it is some power of the distance $|x-y|^a$ of the local operators multiplied to some function $f(\epsilon)$ of the regulator.[5] The power a can be inferred by matching dimensions, but the function $f(\epsilon)$ is a genuine result of the integral. The crucial idea is not to compute the function, but to assume the most general singular behaviour when the regulator is removed, e.g. $f(\epsilon) = c_{-1}/\epsilon + c_0 + c_1\epsilon + \dots$. This allows to write down the contributions to the dilatation operator in terms of the unknown coefficients c_k. Now one can investigate the structure of the dilatation generator to simplify and combine the contributions. Usually, it turns out that there are only a few independent coefficients which actually contribute to anomalous dimensions. The proposed trick is to make use of known results or other constraints to determine these coefficients.

To derive the complete one-loop dilatation operator, it is useful to consider its symmetry. A common practice in physics is to derive some result only for one component of a multiplet of objects; symmetry will then ensure that the result applies to all components of the same multiplet. The same simplification can be applied to the one-loop dilatation operator: It was shown in [50] that superconformal symmetry considerably reduces the number of independent coefficients to just a single infinite sequence. This sequence was subsequently evaluated in field theory. Furthermore, it was conjectured that this last step might be unnecessary and making full use the symmetry algebra would *constrain the complete one-loop dilatation operator uniquely* up to an overall constant (the coupling constant). This is indeed the case as we shall prove in this work. Put differently, su-

[4] Note that the correction is precisely given by the effective vertices found earlier in [19, 21].

[5] For integrals with open spacetime indices the result is a linear combination of such terms with spacetime indices on $(x-y)_\mu$ or $\eta_{\mu\nu}$.

perconformal symmetry and some basic facts from field theory (i.e. the generic structure of a one-loop contribution) completely determine all two-point functions at the one-loop level! To outline the form of the dilatation operator, let us just note that the radiative correction acts on two fields at a time. The contribution $\mathfrak{D}_{12}$ from a pair of fields depends on their 'total spin'[6] j; it is proportional to the harmonic number

$$\mathfrak{D}_{12} \sim h(j) = \sum_{k=0}^{j} \frac{1}{k} \,. \tag{8}$$

Inspired by the strongly constraining nature of the superconformal algebra at one-loop, it is natural to expect it to be very powerful at higher-loops as well. This is a very exciting prospect, since direct higher-loop computations are exceedingly labourious and not much is known beyond the one-loop level. Although one might think that one-loop accuracy is sufficient for many purposes, one should keep in mind that it is only the first non-trivial order. Easily one can imagine some unexpected behaviour at next-to-leading order and, indeed, we shall encounter an example of a mismatch starting only at three-loops. Furthermore, taken that the one-loop dilatation operator is completely constrained, there is hardly any freedom for the quantum theory to decide in either direction. Therefore, a one-loop computation does not provide much information about the *quantum* theory itself.

The trick of writing down the most general structure for the dilatation operator with a number of undetermined coefficients can be used at higher-loops as well. We will, however, not try to generalise the complete dilatation operator to higher-loops. The derivation of the one-loop computation depends heavily on a particular feature of perturbation theory which allows us to restrict to *classical* superconformal invariance. Unfortunately, it does not apply at higher-loops and we would be left with a very large number of independent coefficients to be fixed. To obtain some higher-loop results with as little work as possible, we may restrict to a subsector. The $\mathfrak{so}(6)$ subsector of scalar fields, however, is not suitable, there will be mixing with states involving fermions and other fields; only at one-loop it happens to be closed. To proceed to higher-loops, one could therefore restrict to an even smaller subsector. This so-called $\mathfrak{su}(2)$ subsector consists of only two charged scalar fields (which we shall denote by $\mathcal{Z}$ and ϕ) and charge conservation protects the states from mixing with more general states. Here we can derive the two-loop dilatation operator by employing some known results without performing a full-fledged two-loop field theory computation [38].

We cannot go much further at the moment because there are no known results besides a few basic facts from representation theory. Symmetry is not very constraining in the $\mathfrak{su}(2)$ sector because the dilatation operator is abelian and not part of a bigger algebra. A better choice is the $\mathfrak{su}(2|3)$ subsector: It consists of only five fields and the symmetry algebra includes the dilatation generator. These properties make it both, convenient to handle and sufficiently constraining. Furthermore, not only the dilatation generator, but also the other generators of the algebra receive radiative corrections, a generic feature of the higher-loop algebra. In [51] this subsector was investigated in the planar limit and

[6] The total spin is a quantity of the representation theory of the superconformal symmetry similar to the total spin of the rotation group.

up to three-loops with an astonishing result: Although there are *hundreds* of independent coefficients at three-loops, closure of the symmetry algebra

$$\big[\mathfrak{J}_M(g), \mathfrak{J}_N(g)\big\} = \mathfrak{F}^P_{MN}\, \mathfrak{J}_P(g) \tag{9}$$

constrains nearly all of them in such a way that only a *handful* remain. Moreover, all of them can be related to symmetries of the defining equations. Again, symmetry in combination with basic field theory provides a *unique* answer.

Spectral studies of all the above radiative corrections to the dilatation operator reveal a property with tremendous importance: One finds a huge amount of *pairs* of states $\mathcal{O}_\pm$ whose scaling dimensions are exactly degenerate in the planar limit

$$D_+ = D_-. \tag{10}$$

This would not be remarkable if there was an obvious symmetry to relate those states. This symmetry, however, cannot be superconformal symmetry (or any subalgebra) for two reasons. Firstly, the degeneracy is actually broken by non-planar corrections while superconformal symmetry is exact. Secondly, the degenerate states have a different *parity* which is preserved by superconformal transformations. Here, as in the remainder of this thesis, parity refers to complex conjugation of the SU(N) gauge group. To explain the degeneracy we need some generator $\mathcal{Q}$ which inverts parity and commutes with the dilatation generator.

This curiosity of the spectrum is merely the tip of an iceberg; it will turn out that the conjectured generator $\mathcal{Q}$ is part of an infinite set of commuting charges due to *integrability*. Integrability of a planar gauge theory will be the other major topic of this dissertation. The statement of integrability is equivalent to the existence of an unlimited number of commuting scalar charges $\mathcal{Q}_r$

$$[\mathcal{Q}_r, \mathcal{Q}_s] = [\mathfrak{J}, \mathcal{Q}_r] = 0. \tag{11}$$

The planar dilatation operator $\delta\mathfrak{D} = g^2\mathcal{Q}_2$ is related to the second charge $\mathcal{Q}_2$. It turns out that the odd charges are parity odd, therefore the existence of the charge $\mathcal{Q} = \mathcal{Q}_3$ explains the pairing of states. Only a few states have no partner and are unpaired.

Integrable structures play a crucial role in two dimensional field theories. One of the many intriguing features of two-dimensional CFT's is that they are intimately connected to integrable 2+0 dimensional lattice models in statistical mechanics or, equivalently, to 1+1 dimensional quantum spin chains. The infinite set of charges is directly related to the infinite-dimensional conformal (Virasoro) algebra in $D = 2$. Given the huge success in understanding CFT's in *two* dimensions, one might hope that at least some of the aspects allowing their treatment might fruitfully reappear in *four* dimensions. One might wonder about standard no-go theorems that seem to suggest that integrability can never exist above $D = 2$. These may be potentially bypassed by the fact that there appears to be a hidden 'two-dimensionality' in U(N) gauge theory when we look at it at large N where Feynman diagrams can be classified in terms of two-dimensional surfaces.

The first signs of integrability in $\mathcal{N} = 4$ gauge theory were discovered by Minahan and Zarembo [48]. They found that the planar one-loop dilatation operator in the $\mathfrak{so}(6)$ sector is isomorphic to the Hamiltonian of a $\mathfrak{so}(6)$ integrable quantum spin chain. The

analogy between planar gauge theory and spin chains is as follows: In the strict large N limit, the structure of traces within local operators cannot be changed and therefore we may consider each trace individually or, for simplicity, only single-trace states. We then interpret the trace as a cyclic spin chain and the fields within the trace are the spin sites. For example, the $\mathfrak{su}(2)$ sector with two fields $\mathcal{Z}, \phi$ maps directly to the Heisenberg spin chain, in which the spin at each site can either point up ($\mathcal{Z}$) or down (ϕ). For the $\mathfrak{so}(6)$ sector one considers a more general spin chain for which the spin can point in six distinct abstract directions. The spin chain Hamiltonian alias the planar one-loop dilatation generator acts on the spin chain and returns a linear combination of states. The action is of a nearest-neighbour type, it can only modify two adjacent spins at a time. Likewise the higher charges $\mathcal{Q}_r$ act on r adjacent spins and are therefore local (along the spin chain).

Integrable spin chains had appeared before in four-dimensional gauge theories through the pioneering work of Lipatov on high energy scattering in planar QCD [52]. The model was subsequently identified as a Heisenberg $\mathfrak{sl}(2)$ spin chain of non-compact spin zero [53]. More recently, and physically closely related to the present study, further integrable structures were discovered in the computation of planar one-loop anomalous dimensions of various types of operators in QCD [54] (see also the review [55]).[7]

The full symmetry algebra of SYM is neither $\mathfrak{so}(6)$ nor $\mathfrak{sl}(2)$, but the full superconformal algebra $\mathfrak{psu}(2,2|4)$. If the discovered integrable structures are not accidental, we should expect that the $\mathfrak{so}(6)$ results of [48] and the $\mathfrak{sl}(2)$ results suggested from one-loop QCD [54,55] (see also [43,57]) can be combined and 'lifted' to a full $\mathfrak{psu}(2,2|4)$ super spin chain. Indirect evidence can be obtained by the investigation of the spectrum of anomalous dimensions. As we have mentioned above, the occurrence of pairs of states hints at the existence of at least one conserved charge. Indeed, the spectrum of the complete one-loop planar dilatation operator displays many such pairs. Obviously, they are found in the $\mathfrak{so}(6)$ and $\mathfrak{sl}(2)$ subsectors where integrability is manifest, but also generic states do pair up. Subsequently, it was shown in [58] that the complete one-loop planar dilatation operator is isomorphic to a $\mathfrak{psu}(2,2|4)$ supersymmetric spin chain.

Integrability is not merely an academic issue, for it opens the gates for very precise tests of the AdS/CFT correspondence. It is no longer necessary to compute and diagonalise the matrix of anomalous dimensions. Instead, one may use the Bethe ansatz (c.f. [59] for a pedagogical introduction) to obtain the one-loop anomalous dimensions directly [48,58]. In the thermodynamic limit of very long spin chains, which is practically inaccessible by conventional methods, the algebraic Bethe equations turn into integral equations. With the Bethe ansatz at hand, it became possible to compute anomalous dimensions of operators with large spin quantum numbers [60].

Via the AdS/CFT correspondence, these states correspond to highly spinning string configurations. Even though quantisation of string theory on $AdS_5 \times S^5$ is an open problem, these spinning strings can be treated in a classical fashion, c.f. [61,62], when interested in the leading large spin behaviour. It was shown by Frolov and Tseytlin [63,64] that quantum $(1/\sqrt{\lambda})$ corrections in the string theory sigma model are suppressed by powers of $1/J$, where J is a large spin on the five-sphere S^5. In direct analogy to the

[7] While QCD is surely not a conformally invariant quantum field theory [56], it still behaves like one as far as one-loop anomalous dimensions are concerned.

plane-wave limit, one obtains an effective coupling constant

$$\lambda' = \frac{\lambda}{J^2}\,. \tag{12}$$

What makes the low-energy spinning string configurations very appealing is that their energies permit an expansion in integer powers of λ' around $\lambda' = 0$ [65]. Just as in the case of the plane-wave/BMN limit one can now compare to perturbative gauge theory in a quantitative fashion. It was found that indeed string energies and gauge theory scaling dimensions agree at first order in λ' [66, 67]. Moreover, the comparison is not based on a single number, but on a function of the ratio of two spins. Except in a few special cases, this function is very non-trivial; it involves solving equations of elliptic or even hyperelliptic integrals. The agreement can also be extended to the commuting charges in (11), c.f. [68–70]. These are merely tests of the spinning string correspondence and there have been two recent proposals to prove the equivalence of classical string theory and perturbative gauge theory in the thermodynamic limit. The proposal of Kruczenski is based on comparing the string Hamiltonian to the dilatation operator [71–73] (see also the related work [74]) while Kazakov, Marshakov, Minahan and Zarembo find a representation of string theory in terms of integral equations and compare them to the Bethe ansatz [75]. For a review of the topic of spinning strings please refer to [76].

We have argued that integrability of the planar gauge theory is, on the one hand, an interesting theoretical aspect of $\mathcal{N} = 4$ SYM and, on the other hand, it allows for precision tests of the AdS/CFT correspondence. So far, however, integrability is only a firm result at the one-loop level. At higher-loops, it may seem to be inhibited for the following simple reason: The Hamiltonian of an integrable spin chain is usually of *nearest-neighbour* type (as for one-loop gauge theories) or, at least, involves only two, non-neighbouring spins at a time (as for the Haldane-Shastry and Inozemtsev integrable spin chains [77, 78]). This structure may appear to be required by the elastic scattering properties in integrable models. In contrast, higher-loop corrections to the dilatation generator require interactions of *more than two fields*. Moreover, the *number of fields is not even conserved* in general (as in the $\mathfrak{su}(2|3)$ subsector). Nevertheless, there are two major reasons to believe in higher-loop integrability: Firstly, the observation of pairing of states in the spectrum of anomalous dimensions has been shown to extend to at least three-loops in the $\mathfrak{su}(2|3)$ subsector [51] (see [79] for the related issue of integrability in the BMN matrix model)

$$D_+(g) = D_-(g). \tag{13}$$

At one-loop this degeneracy is explained by integrability, but there is no obvious reason why it should extend to higher-loops unless integrability does.[8] Moreover, it is possible to construct a four-loop correction in the $\mathfrak{su}(2)$ sector with this property [38, 81]. Secondly, one might interpret the AdS/CFT correspondence as one important indication of the validity of integrability: The classical world sheet theory, highly non-trivial due to the curved $AdS_5 \times S^5$ background, is integrable [82–84] (for the simpler but related case of plane-wave backgrounds see also [85, 86]).

[8] Pairing may appear to be a weaker statement, but there are some indications that it is sufficient to ensure integrability, see e.g. [80, 38].

It seems that spin chains with interactions of many spins or *dynamic* spin chains with a fluctuating number of spin sites have not been considered so far.[9] Yet, their apparent existence [38, 79, 81, 51] is fascinating. The novelty of such a model, however, comes along with a lack of technology to investigate it. For instance, we neither know how to construct higher commuting charges or even prove integrability, nor is there an equivalent of the Bethe ansatz to push the comparison with spinning strings to higher loops.

A first step to overcome those difficulties has been taken by Serban and Staudacher who found a way to match the Inozemtsev integrable spin chain [78] to the three-loop results in gauge theory [87]. The Bethe ansatz for the Inozemtsev spin can thus be used to obtain exact planar three-loop anomalous dimensions in gauge theory. They have furthermore pushed the successful comparison of [66] to higher-loops and found that the agreement persists at two-loops. The agreement was subsequently generalised to a matching of integral equations or Hamiltonians in [75, 72].

However, at three-loops the *string theory prediction turned out not to agree with gauge theory*. This parallels a discrepancy starting at three-loops which has been observed earlier in the near plane-wave/BMN correspondence [88]. These puzzles have not been resolved at the time this work was written and we shall comment on some possible explanations, such as *an order of limits problem* and *wrapping interactions*, in the main text. Here we mention only one, even if unlikely: The AdS/CFT correspondence might not be exact after all. Irrespective of the final word on this issue, we have learned that it is not always sufficient to restrict to the leading, one-loop order, but there are interesting and relevant effects to be found at higher-loops.

To deepen our understanding of the string/gauge correspondence, whether or not exact, it would be useful to know the quantitative difference. Unfortunately, starting at four-loops, the Inozemtsev spin chain has a scaling behaviour in the thermodynamic limit which does not agree with the one of string theory; consequently it makes no sense to compare beyond three-loops. However, there is a proposal for an integrable spin chain with the correct scaling behaviour even at four-loops [81]. In [89] a Bethe ansatz is presented which accurately reproduces the spectrum of the four-loop (and even five-loop) spin chain. What is more, the Bethe ansatz has a natural generalisation to all-loops, which incidentally reproduces the BMN energy formula (6). In principle, this allows to compute scaling dimensions as a true function of the coupling constant[10] and thus overcome some of the handicaps of perturbation theory. One may hope that the ansatz gives some insight into gauge theory away from the weak coupling regime.

Note added: This work is based on the author's PhD thesis, which was submitted to Humboldt University, Berlin.

[9] The higher charges of an integrable spin chain are indeed of non-nearest neighbour type. Nevertheless, they cannot yield higher-loop corrections because they commute among themselves, whereas the higher-loop corrections in general do not.

[10] The ansatz cannot deal with short states correctly, it should only be trusted when the number of constituent fields is larger than the loop order.

Overview

This thesis is organised as follows: The main text is divided into six chapters, in the first two we investigate generic aspects of the dilatation operator and in the remaining four we will explicitly construct one-loop and higher-loop corrections and investigate their integrability.

1. We start by presenting the $\mathcal{N} = 4$ supersymmetric field theory and review some useful results concerning the representation theory of the superconformal algebra $\mathfrak{psu}(2,2|4)$ on which we will base the investigations of the following chapters.
2. We will then investigate some scaling dimensions and introduce the dilatation operator as a means to measure them. Most of the chapter is devoted to the discussion of various aspects of the dilatation operator and its structure. These include the behaviour in perturbation theory and how one can consistently restrict to certain subsectors of states in order to reduce complexity. From an explicit and a conceptual computation of two-point functions in a subsector we shall learn about the structure of quantum corrections to the dilatation generator. Finally, we will investigate the planar limit and introduce some notation.
3. Having laid the foundations, we will now turn towards explicit algebraic constructions. In this chapter we will derive the complete one-loop dilatation operator of $\mathcal{N} = 4$ SYM. The derivation is similar to the one presented in the article [50], but here we improve it by replacing the field theory calculations by algebraic constraints.
4. Next, we introduce the notion of integrability and a framework to investigate integrable quantum spin chains. We will then prove the integrability of the just derived dilatation generator in the planar limit. We extend the results of the article [58] by a proof of a Yang-Baxter equation. This allows us to write down the Bethe ansatz for the corresponding supersymmetric quantum spin chain.
5. At this point, the investigations of one-loop scaling dimensions is complete and we proceed to higher-loops. For simplicity we will restrict to a subsector with finitely many fields and the planar limit. We demand the closure of the pertinent symmetry algebra, determine its most general three-loop deformations [51] and find an essentially unique result. An interesting aspect of the deformations is that they do not conserve the number of component fields within a state.
6. In the final chapter we consider integrability at higher-loops and argue why it should apply to planar $\mathcal{N} = 4$ SYM. We will then construct deformations to the Heisenberg spin chain to model higher-loop interactions; they turn out to be unique even at five-loops. Finally, we present an all-loop Bethe ansatz which reproduces the energies of this model.

The developed techniques are illustrated by several sample calculations at various places in the text. In particular, we will present two important computations of scaling dimensions in the context of the AdS/CFT correspondence. In Sec. 3.6 we shall compute the genus-one energy shift of two-excitation BMN operators to be compared to strings on plane waves. The agreement represents the first dynamical test including string interactions. In Sec. 4.6 we consider classical spinning strings on $AdS_5 \times S^5$ and compare them to states with a large spin of $\mathfrak{so}(6)$ to find an intricate functional agreement.

We then conclude and present a list of interesting open questions. To expand on the main text we present some miscellaneous aspects in the appendices:

A. An example to illustrate the application of the dilatation operator, at finite N or in the planar limit.

B. Spinor identities in four, six and ten dimensions.

C. A short review of the ten-dimensional supersymmetric gauge theory, either in superspace or in components.

D. The algebra $\mathfrak{u}(2,2|4)$, its commutation relations and the oscillator representation.

E. Some `Mathematica` functions to deal with planar interactions in the $\mathfrak{su}(2)$ subsector which can be used in the application and construction of the dilatation operator.

F. The harmonic action to compute one-loop scaling dimensions in a more convenient fashion than by using the abstract formula (8).

Chapter 1

Field Theory and Symmetry

In this chapter we will discuss various, loosely interrelated aspects of $\mathcal{N} = 4$ super Yang-Mills theory, the superconformal algebra and its representation theory. We lay the foundations for the investigations of the following chapters and introduce our notation, conventions as well as important ideas.

We will start with a review of classical $\mathcal{N} = 4$ SYM in Sec. 1.1 and its path-integral quantisation in Sec. 1.2. In the following two sections we consider the gauge group (a generic group in Sec. 1.3 or a group of large rank in Sec. 1.4) in a quantum field theory. In Sec. 1.5 we introduce the superconformal algebra, a central object of this thesis. The remainder of this chapter deals with representation theory. Firstly, we present our notion of fields and local operators and relate it to the algebra in Sec. 1.6. In Sec. 1.7,1.8 we consider generic highest-weight modules and special properties of multiplets close the unitarity bounds. The multiplet of fields and the current multiplet is investigated in Sec. 1.9,1.11. Finally, in Sec. 1.10 we review correlation functions in a conformal field theory.

1.1 $\mathcal{N} = 4$ Super Yang-Mills Theory

We start by defining the field theory on which we will focus in this work, $\mathcal{N} = 4$ maximally supersymmetric gauge theory in four dimensions [2].[1] It consists of a covariant derivative $\mathcal{D}$ constructed from the gauge field $\mathcal{A}$, four spinors Ψ as well as six scalars Φ to match the number of bosonic and fermionic on-shell degrees of freedom. We will collectively refer to the fields by the symbol $\mathcal{W}$ [2]

$$\mathcal{W}_{\mathcal{A}} = (\mathcal{D}_{\mu}, \Psi_{\alpha a}, \dot{\Psi}^{a}_{\dot{\alpha}}, \Phi_{m}). \tag{1.1}$$

Our index conventions are as follows: Greek letters refer to spacetime $\mathfrak{so}(4) = \mathfrak{su}(2)\times\mathfrak{su}(2)$ symmetry.[3] Spacetime vector indices $\mu, \nu, \ldots$ take four values, spinor indices $\alpha, \beta, \ldots$ of one $\mathfrak{su}(2)$ and spinor indices $\dot{\alpha}, \dot{\beta}, \ldots$ of the other $\mathfrak{su}(2)$ take values $1, 2$. Latin indices

[1] It is convenient to derive the four-dimensional theory with $\mathcal{N} = 4$ supersymmetry from a ten-dimensional theory with $\mathcal{N} = 1$ supersymmetry. In App. C we shall present this ancestor theory.

[2] Of course, the covariant derivative $\mathcal{D}$ is not a field. Instead of the gauge field $\mathcal{A}$, we shall place it here so that all 'fields' $\mathcal{W}$ have uniform gauge transformation properties.

[3] As we are dealing with algebras only, global issues such as the difference between a group and its double covering need not concern us.

signature	$\eta^{\mu\nu}$	η^{mn}	spacetime sym.	internal sym.
physical	$(3,1)$	$(6,0)$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{su}(4)$
Euclidian	$(4,0)$	$(5,1)$	$\mathfrak{sp}(1)\times\mathfrak{sp}(1)$	$\mathfrak{sl}(2,\mathbb{H})$
Minkowski, non-compact	$(3,1)$	$(4,2)$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{su}(2,2)$
maximally non-comapct	$(2,2)$	$(3,3)$	$\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(4,\mathbb{R})$
complex	4	6	$\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sl}(4,\mathbb{C})$

Table 1.2: Possible signatures of spacetime, internal space and symmetry algebras.

belong to the internal $\mathfrak{so}(6)=\mathfrak{su}(4)$ symmetry; internal vector indices $m,n,\dots$ take six values whereas spinor indices $a,b,\dots$ take values $1,2,3,4$. Calligraphic indices $\mathcal{A},\mathcal{B},\dots$ label the fundamental fields in $\mathcal{W}$.

Let us comment on the signature of the field theory and the algebras. In order to write down a real-valued Lagrangian, the signatures of spacetime and internal space must be correlated, we have listed the possible choices in Tab. 1.2. The physical choice has Minkowski signature and a positive-definite norm for internal space. The other choices require an internal metric of indefinite signature and possibly a spacetime with two time-like directions. As far as perturbation theory and Feynman diagrams are concerned, the signature is irrelevant because we can perform Wick rotations at any point of the investigation. It may therefore be convenient to work with the maximally non-compact signature which leads to a completely real theory and where conjugation does not play a role. Alternatively, we can use a complexified spacetime and algebra. In the following we will not pay much attention to signatures and assume either the maximally non-compact or complex version of the algebra.

We define the covariant derivative

$$\mathcal{D}_\mu=\partial_\mu-ig\mathcal{A}_\mu,\qquad \mathcal{D}_\mu\mathcal{W}:=[\mathcal{D}_\mu,\mathcal{W}]=\partial_\mu\mathcal{W}-ig\mathcal{A}_\mu\mathcal{W}+ig\mathcal{W}\mathcal{A}_\mu, \tag{1.2}$$

where we have introduced a dimensionless coupling constant g. Later on, in the quantum theory, g will be an important parameter; however, on a classical level, we can absorb it completely by rescaling the fields, this corresponds to $g=1$. Throughout this work we will assume the gauge group to be $\mathrm{SU}(N)$ or $\mathrm{U}(N)$ and represent all adjoint fields $\mathcal{W}$ by (traceless) hermitian $N\times N$ matrices. Under a gauge transformation $U(x)\in\mathrm{U}(N)$ the fields transform canonically according to

$$\mathcal{W}\mapsto U\mathcal{W}U^{-1},\qquad \mathcal{A}_\mu\mapsto U\mathcal{A}_\mu U^{-1}-ig^{-1}\,\partial_\mu U\,U^{-1}. \tag{1.3}$$

The gauge field $\mathcal{A}$ transforms differently from the other fields to compensate for the non-covariant transformation of the partial derivative within $\mathcal{D}$. The covariant derivative $\mathcal{D}$ is not truly a field, it must always act on some other field. Nevertheless we can construct a field from the gauge connection alone, the field strength $\mathcal{F}$. Together with the associated Bianchi identity, it is given by

$$\mathcal{F}_{\mu\nu}=ig^{-1}[\mathcal{D}_\mu,\mathcal{D}_\nu]=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu-ig[\mathcal{A}_\mu,\mathcal{A}_\nu],\qquad \mathcal{D}_{[\rho}\mathcal{F}_{\mu\nu]}=0. \tag{1.4}$$

After these preparations we can write down the Lagrangian of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. It is

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}}[\mathcal{W}]&=\tfrac14\operatorname{Tr}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\tfrac12\operatorname{Tr}\mathcal{D}^\mu\Phi^n\mathcal{D}_\mu\Phi_n-\tfrac14g^2\operatorname{Tr}[\Phi^m,\Phi^n][\Phi_m,\Phi_n]\\&\quad+\operatorname{Tr}\dot\Psi^a_{\dot\alpha}\sigma^{\dot\alpha\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a}-\tfrac12ig\operatorname{Tr}\Psi_{\alpha a}\sigma^{ab}_m\varepsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta b}]-\tfrac12ig\operatorname{Tr}\dot\Psi^a_{\dot\alpha}\sigma^m_{ab}\varepsilon^{\dot\alpha\dot\beta}[\Phi_m,\dot\Psi^b_{\dot\beta}].\end{aligned} \tag{1.5}$$

In addition to the standard kinetic terms for the gauge field, spinors and scalars, there is a quartic coupling of the scalars and a cubic coupling of a scalar and two spinors. The symbols ε are the totally antisymmetric tensors of $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$. The matrices σ^μ and σ^m are the chiral projections of the gamma matrices in four or six dimensions, respectively. They have the symmetry properties $\sigma^\mu_{\dot\alpha\beta} = \sigma^\mu_{\beta\dot\alpha}$, $\sigma^m_{ab} = -\sigma^m_{ba}$ and satisfy the relations[4]

$$\sigma^{\{\mu}\sigma^{\nu\}} = \eta^{\mu\nu}, \quad \sigma^{\{m}\sigma^{n\}} = \eta^{mn}, \tag{1.6}$$

when considered as matrices which are summed over a pair of alike upper and lower intermediate indices. Please refer to App. B for a number of useful identities and conventions. The equations of motion which follow from this action are

$$\begin{aligned}
\mathcal{D}_\nu\mathcal{F}^{\mu\nu} &= ig[\Phi_n, \mathcal{D}^\mu\Phi^n] - ig\sigma^{\dot\alpha\beta}_\mu\{\dot\Psi^a_{\dot\alpha}, \Psi_{\beta a}\},\\
\mathcal{D}_\nu\mathcal{D}^\nu\Phi^m &= -g^2[\Phi_n,[\Phi^n,\Phi^m]] + \tfrac{1}{2}ig\sigma^{m,ab}\varepsilon^{\alpha\beta}\{\Psi_{\alpha a}, \Psi_{\beta b}\} + \tfrac{1}{2}ig\sigma^m_{ab}\varepsilon^{\dot\alpha\dot\beta}\{\dot\Psi^a_{\dot\alpha}, \dot\Psi^b_{\dot\beta}\},\\
\sigma^{\dot\alpha\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a} &= ig\varepsilon^{\dot\alpha\dot\beta}\sigma^m_{ab}[\Phi_m, \dot\Psi^b_{\dot\beta}],\\
\sigma^{\alpha\dot\beta}_\mu\mathcal{D}^\mu\dot\Psi^a_{\dot\beta} &= ig\varepsilon^{\alpha\beta}\sigma^{ab}_m[\Phi^m, \Psi_{\beta b}].
\end{aligned} \tag{1.7}$$

It can be shown that the action and the equations of motion are invariant under the $\mathcal{N}=4$ super Poincaré algebra. It consists of the manifest Lorentz and internal rotation symmetries $\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}$ of $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{su}(4)$ as well as the (super)translations $\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$. The (super)translation variations are parameterised by the fermionic and bosonic shifts $\epsilon^\alpha_a, \dot\epsilon^{\dot\alpha a}$ and e^μ

$$\delta_{\epsilon,\dot\epsilon,e} = \epsilon^\alpha_a\mathfrak{Q}^a{}_\alpha + \dot\epsilon^{\dot\alpha a}\dot{\mathfrak{Q}}_{\dot\alpha a} + e^\mu\mathfrak{P}_\mu. \tag{1.8}$$

The action of the variation on the fundamental fields $\delta_{\epsilon,\dot\epsilon,e}\mathcal{W} := [\delta_{\epsilon,\dot\epsilon,e}, \mathcal{W}]$ is given by

$$\begin{aligned}
\delta_{\epsilon,\dot\epsilon,e}\mathcal{D}_\mu &= ig\epsilon^\alpha_a\varepsilon_{\alpha\beta}\sigma^{\beta\dot\gamma}_\mu\dot\Psi^a_{\dot\gamma} + ig\dot\epsilon^{a\dot\alpha}\varepsilon_{\dot\alpha\dot\beta}\sigma^{\dot\beta\gamma}_\mu\Psi_{\gamma a} + ige^\nu\mathcal{F}_{\mu\nu},\\
\delta_{\epsilon,\dot\epsilon,e}\Phi_m &= \epsilon^\alpha_a\sigma^{ab}_m\Psi_{\alpha b} + \dot\epsilon^{a\dot\alpha}\sigma_{m,ab}\dot\Psi^b_{\dot\alpha} + e^\mu\mathcal{D}_\mu\Phi_m,\\
\delta_{\epsilon,\dot\epsilon,e}\Psi_{\alpha a} &= -\tfrac{1}{2}\sigma^\mu_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma}\sigma^\nu_{\dot\gamma\delta}\epsilon^\delta_a\mathcal{F}_{\mu\nu} + \tfrac{1}{2}ig\sigma^m_{ab}\sigma^{bc}_n\varepsilon_{\alpha\beta}\epsilon^\beta_c[\Phi_m,\Phi^n]\\
&\quad + \sigma^n_{ab}\sigma^\mu_{\alpha\dot\beta}\dot\epsilon^{b\dot\beta}\mathcal{D}_\mu\Phi_n + e^\mu\mathcal{D}_\mu\Psi_{\alpha a},\\
\delta_{\epsilon,\dot\epsilon,e}\dot\Psi^a_{\dot\alpha} &= -\tfrac{1}{2}\sigma^\mu_{\dot\alpha\beta}\varepsilon^{\beta\gamma}\sigma^\nu_{\gamma\dot\delta}\dot\epsilon^{a\dot\delta}\mathcal{F}_{\mu\nu} + \tfrac{1}{2}ig\sigma^{ab}_m\sigma^n_{bc}\varepsilon_{\dot\alpha\dot\beta}\dot\epsilon^{c\dot\beta}[\Phi^m,\Phi_n]\\
&\quad + \sigma^{ab}_n\sigma^\mu_{\dot\alpha\beta}\epsilon^\beta_b\mathcal{D}_\mu\Phi^n + e^\mu\mathcal{D}_\mu\dot\Psi^a_{\dot\alpha}.
\end{aligned} \tag{1.9}$$

The algebra of supertranslations resulting from these variations is given by

$$\begin{aligned}
\{\mathfrak{Q}^a{}_\alpha, \mathfrak{Q}^b_\beta\} &= -2ig\epsilon_{\alpha\beta}\sigma^{ab}_m\Phi^m, &\qquad [\mathfrak{P}_\mu, \mathfrak{Q}^a{}_\alpha] &= -ig\varepsilon_{\alpha\beta}\sigma^{\beta\dot\gamma}_\mu\dot\Psi^a_{\dot\gamma},\\
\{\dot{\mathfrak{Q}}_{\dot\alpha a}, \dot{\mathfrak{Q}}_{\dot\beta b}\} &= -2ig\epsilon_{\dot\alpha\dot\beta}\sigma^m_{ab}\Phi_m, &\qquad [\mathfrak{P}_\mu, \dot{\mathfrak{Q}}_{\dot\alpha a}] &= -ig\varepsilon_{\dot\alpha\dot\beta}\sigma^{\dot\beta\gamma}_\mu\Psi_{\gamma a},\\
\{\mathfrak{Q}^a{}_\alpha, \dot{\mathfrak{Q}}_{b\dot\beta}\} &= 2\delta^a_b\sigma^\mu_{\alpha\dot\beta}\mathfrak{P}_\mu, &\qquad [\mathfrak{P}_\mu, \mathfrak{P}_\nu] &= -ig\mathcal{F}_{\mu\nu},
\end{aligned} \tag{1.10}$$

up to terms proportional to the equations of motion.[5] Note that the action of the generators $\mathfrak{J}$ on a combination of fields X should be read as $[\mathfrak{J}, X]$. When X is a covariant

[4] The brackets $\{\ldots\}$ at index level indicate a symmetric projection of enclosed indices. Likewise $[\ldots]$ and $(\ldots)$ correspond to a antisymmetric and symmetric-traceless projection with respect to the metric η.

[5] It is a common feature of supersymmetric theories that the algebra closes only on-shell. Here, it is related to the fact that the equations of motion (C.9) are implied by the constraint (C.7) which is used in the reduction of superspace fields to their top level components. For theories with less supersymmetry one can introduce auxiliary fields or work in superspace to achieve off-shell supersymmetry.

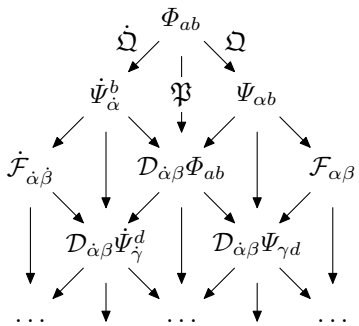

Figure 1.2: Classical supertranslation variations of the fields in spinor notation. Left, vertical and right arrows correspond to generators $\dot{\mathfrak{Q}}$, $\mathfrak{P}$ and $\mathfrak{Q}$, respectively. We have dropped all commutators of fields which are suppressed for a vanishing coupling constant g.

combination of fields, the above commutators therefore yield $[\mathcal{W}, X]$, where $\mathcal{W}$ is the field which appears on the right-hand side of (1.10). For a gauge invariant combination X all fields drop out and only the momentum generator $\mathfrak{P}$ acts non-trivially $[\mathfrak{P}_\mu, X] = \partial_\mu X$.

As a more unified notation, it is possible to replace all vector indices $\mu, \nu, \ldots, m, n, \ldots$ by a pair of spinor indices by contracting with the σ symbols

$$\begin{aligned}
\mathcal{D}_\mu &\sim \sigma_\mu^{\dot\alpha\beta}\mathcal{D}_{\dot\alpha\beta},\\
\mathcal{F}_{\mu\nu} &\sim \sigma_\mu^{\alpha\dot\gamma}\varepsilon_{\dot\gamma\dot\delta}\sigma_\nu^{\dot\delta\beta}\mathcal{F}_{\alpha\beta} + \sigma_\mu^{\dot\alpha\gamma}\varepsilon_{\gamma\delta}\sigma_\nu^{\delta\dot\beta}\dot{\mathcal{F}}_{\dot\alpha\dot\beta},\\
\Phi_m &\sim \sigma_m^{ba}\Phi_{ab}.
\end{aligned} \tag{1.11}$$

In this notation Φ_{ab} is antisymmetric while $\mathcal{F}_{\alpha\beta}$ and $\dot{\mathcal{F}}_{\dot\alpha\dot\beta}$ are both symmetric. Using identities in App. B, one can remove all explicit σ's from the action and equations of motion and replace them by totally antisymmetric ε tensors of $\mathfrak{su}(2), \mathfrak{su}(2), \mathfrak{su}(4)$. We will not do this explicitly here, but note that the set of fields (together with the covariant derivative) is given by

$$\mathcal{W} = (\mathcal{D}_{\dot\alpha\beta}, \Phi_{ab}, \Psi_{\alpha b}, \dot\Psi^b_{\dot\alpha}, \mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot\alpha\dot\beta}), \tag{1.12}$$

all of which are bi-spinors. For Minkowski signature the dotted fields would be related to the undotted ones by complex conjugation. Here we will consider them to be independent and real as for a spacetime of signature $(2,2)$. The structure of supersymmetry transformations of these fields, depicted in Fig. 1.2, is of elegant simplicity. Generators $\mathfrak{Q}$ simply change a $\mathfrak{su}(4)$ index into an undotted $\mathfrak{su}(2)$ index, whereas generators $\dot{\mathfrak{Q}}$ add both a $\mathfrak{su}(4)$ index and a dotted $\mathfrak{su}(2)$ index. The momentum generator adds both an undotted $\mathfrak{su}(2)$ index and a dotted $\mathfrak{su}(2)$ index. We will come back to this in Sec. 1.9, where we will represent the fields and generator in terms of a set of harmonic oscillators.

The $\mathcal{N} = 4$ gauge theory is pure in the sense that it consists only of the superspace gauge field, c.f. App. C.1. As such it must be a massless theory and enjoys an enhancement of Poincaré symmetry to *conformal* symmetry. Even more, conformal symmetry and super(translation)symmetry join to form *superconformal* symmetry. We will discuss this symmetry in detail in Sec. 1.5, here we only note that it yields additional special conformal generators or boosts. The (super)boosts $\mathfrak{S}, \dot{\mathfrak{S}}, \mathfrak{K}$ are essentially the conjugate transformations of (super)translations $\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$.

1.2 The Quantum Theory

There are various ways to quantise a field theory, we will consider only the path integral approach. The path integral measures the expectation value of some operator functional $\mathcal{O}[\mathcal{W}]$ by summing over all field configuration weighted by the exponential of the action[6]

$$\big\langle \mathcal{O}[\mathcal{W}] \big\rangle := \int D\mathcal{W}\, \mathcal{O}[\mathcal{W}]\, \exp\big(-S[\mathcal{W}]\big). \tag{1.13}$$

We assume the path integral to be normalised, $\langle 1 \rangle = 1$. The Yang-Mills action S is the spacetime integral of the gauge theory Lagrangian (1.5)

$$S[\mathcal{W}] = \frac{2}{g_{\text{YM}}^2} \int d^4x\, \mathcal{L}_{\text{YM}}[\mathcal{W}, g = 1], \tag{1.14}$$

where we have used the common definition of the Yang-Mills coupling constant g_{YM}. For a reason to be explained in Sec. 1.4, it will be more convenient to work with a different coupling constant

$$g^2 := \frac{g_{\text{YM}}^2 N}{8\pi^2}\,, \tag{1.15}$$

where N is the rank of the gauge group $\mathrm{U}(N)$. We can easily recast the action in the following form

$$S[\mathcal{W}] = \frac{N}{4\pi^2} \int d^4x\, \mathcal{L}_{\text{YM}}[\mathcal{W}/g, g = \sqrt{g_{\text{YM}}^2 N/8\pi^2}\,]. \tag{1.16}$$

This form yields a convenient normalisation for spacetime correlators when the fields $\mathcal{W}$ are rescaled by g. The rescaling can be absorbed into the normalisation of the path integral and we obtain the action to be used in this work

$$S[\mathcal{W}] = N \int \frac{d^4x}{4\pi^2}\, \mathcal{L}_{\text{YM}}[\mathcal{W}]. \tag{1.17}$$

There are various expectation values which one might wish to compute, let us state a few: A frequent application is scattering of particles. Particles are represented by fields with well-prepared momenta p_i and spins ϵ_i. One inserts these into the path integral

$$F(p_i, \epsilon_i) = \big\langle \epsilon_1 \cdot \Psi(p_1)\, \Phi(p_2)\, \ldots \big\rangle \tag{1.18}$$

and obtains the scattering function F which describes the scattering process of the involved particles. Another possibility is to insert Wilson loops $\mathcal{O}[\gamma]$

$$F[\gamma] = \big\langle \mathcal{O}[\gamma] \big\rangle. \tag{1.19}$$

Wilson loops are operators which are supported on a curve $x = \gamma(\tau)$ in spacetime. The function $F[\gamma]$ can, for example, be used to describe the potential between two heavy charged objects. In this work we shall consider local operators $\mathcal{O}(x)$, objects supported at a single point x in spacetime, and their correlators

$$F(x_i) = \big\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \ldots \big\rangle. \tag{1.20}$$

[6] We assume the signature of spacetime to be Euclidean. For Minkowski signature the weight would be $\exp iS$.

In particular we will focus on two-point functions

$$F(x_1, x_2) = \big\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \big\rangle, \tag{1.21}$$

which are used to measure some generic properties of the local operators in question. They describe how a particle which is created/annihilated by that operator propagates through spacetime. Local operators will be discussed in detail in Sec. 1.6.

The symmetries of the theory will be reflected by the correlation functions F. For example, due to translation invariance, the Wilson-loop expectation value will not depend on a global shift of the contour, $F[\gamma + c] = F[\gamma]$. For the same reason two-point functions can only depend on the distance of the two points $F(x_1, x_2) = F(x_1 - x_2)$. There are further constraints on two-point functions due to superconformal symmetry which will be discussed in Sec. 1.10.

However, there is a possible catch about symmetries: Classical symmetries of the action might not survive in the quantum theory. In the path integral formalism such *anomalies* arise when it is impossible to consistently define a measure $D\mathcal{W}$ which obeys the symmetry. In particular, conformal symmetry usually is anomalous. When quantising a field theory, it is necessary to regularise it first in order to remove divergencies; this inevitably requires the introduction of a mass scale μ. In the regularised theory μ breaks conformal symmetry for which scale invariance is indispensable. When, after quantisation, the regulator is removed, the correlation functions F usually still depend on the scale μ. Of course, a physically meaningful result must not depend on the arbitrary scale. This apparent puzzle is resolved by assuming that the parameters of the quantum theory also depend on the scale μ in such a way that the explicit and implicit dependence cancel out. In the case at hand, the only parameter is the coupling constant g and its dependence on the scale is described by the beta function

$$\beta = \mu \frac{\partial g}{\partial \mu}\,. \tag{1.22}$$

The appearance of the beta function is related to the breakdown of scale invariance and conformal symmetry in a massless gauge theory. For $\mathcal{N} = 4$ SYM, however, the beta function is believed to vanish to all orders in perturbation theory as well as non-perturbatively [3]

$$\beta = 0. \tag{1.23}$$

In other words, (super)conformal symmetry is preserved even at the quantum level! This does not imply, however, that there are no divergencies in $\mathcal{N} = 4$ SYM; it merely means that, once the operators are properly renormalised, all divergencies and scale dependencies drop out in physically meaningful quantities.

Let us evaluate the expectation value $\langle \mathcal{O}[\mathcal{W}] \rangle$ in perturbation theory. Using standard path integral methods we find the generator of Feynman diagrams

$$\big\langle \mathcal{O}[\mathcal{W}] \big\rangle = \big(\exp(W_0[\partial/\partial\mathcal{W}]) \exp(-S_{\text{int}}[g, \mathcal{W}])\, \mathcal{O}[\mathcal{W}] \big)_{\mathcal{W}=0}, \tag{1.24}$$

where we have split up the action $S(g) = S_0 + S_{\text{int}}(g)$ into the free part, quadratic in the fields, and the interacting part, which is (at least) cubic.[7] The free connected generating

[7] Apart from the trivial vacuum, in which all fields are identically zero, other classical solutions to the

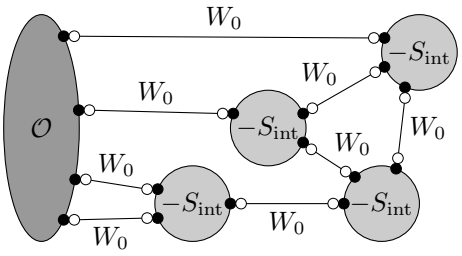

Figure 1.4: A contribution to the quantum expectation value of the operator $\mathcal{O}$ (Feynman graph).

functional is given by

$$W_0[\mathcal{J}] = \frac{1}{N}\int dx\, dy\, \tfrac{1}{2}\,\mathrm{Tr}\,\mathcal{J}(x)\Delta(x,y)\mathcal{J}(y). \tag{1.25}$$

Here, $\Delta(x,y)$ is the free propagator which is the inverse of the kinetic term in the free action S_0. The source fields $\mathcal{J}$ will usually be replaced by variations $\partial/\partial\mathcal{W}$. The expression (1.24) can be read as follows, see also Fig. 1.4: There are arbitrarily many propagators W_0 and arbitrarily many vertices $S_{\rm int}$. Each propagator connects two fields $\mathcal{W}$ within the vertices or the operator $\mathcal{O}$. In the end, all fields must be saturated.

For a perturbative treatment of a quantum gauge theory one must modify the action slightly. Firstly, the divergencies which appear in a QFT need to be regularised. A convenient scheme which preserves most of the symmetries is dimensional regularisation. In this scheme the number of spacetime dimensions is not fixed to four, but rather assumed to be $4-2\epsilon$ with a regularisation parameter ϵ. Correlators are thus analytic functions of ϵ and divergencies become manifest as poles at $\epsilon = 0$. The other issue is gauge fixing: Gauge invariance leads to non-propagating modes of the gauge field and a naive gauge field propagator is ill-defined. We need to fix a gauge and a consistent treatment may require the introduction of *ghosts*. The ghosts are auxiliary fermionic fields which interact with the gauge fields at a cubic vertex. They are an artefact of the quantisation procedure and can appear only in the bulk of Feynman graphs; they are forbidden in external states (operators). These two issues are important for a consistent quantisation; they will however hardly affect our investigations which are algebraic in nature. We will merely have to assume that the perturbative contributions can be obtained consistently.

Let us comment on the counting of quantum loops. For simplicity, we will assume only cubic interactions. In gauge theories there are also quartic interactions, but these may be represented by two cubic interactions connected by an auxiliary field. This fits well with the fact that cubic interactions are suppressed by one power of the coupling constant and quartic ones by two. A Feynman graph can then be characterised by the number of vertices V, propagators I, fields within the operator E and connected components C. As the number of fields $\mathcal{W}$ and variations $\partial/\partial\mathcal{W}$ must match exactly, we have $3V+E=2I$. Counting of momentum integrals L (loops) furthermore implies $L = I-V-E+C$: Each propagator introduces one new momentum variable, while each vertex and external field

equations of motion exist. For example, there are instantonic vacua with non-trivial topological charge and non-conformal vacua in which some of the scalar fields have a constant value. One can also expand around *these* configurations which leads to qualitatively very different results. For simplicity we shall only consider the trivial vacuum in this work.

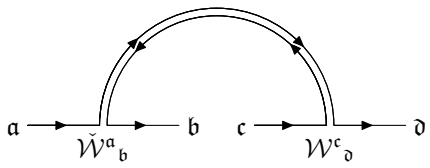

Figure 1.6: The contraction of a matrix-valued variation $\check{\mathcal{W}}$ and field $\mathcal{W}$.

introduces a constraint. Due to momentum conservation the external momenta within each component must add up to zero, reducing the number of constraints by one for each component. In total we can write

$$V = 2L + 2(E/2 - C). \tag{1.26}$$

In the free theory, there are neither vertices nor loops. Therefore we have $C_0 = E/2$ independent pairwise contractions of fields. In the interacting theory $E/2 - C = C_0 - C$ gives the number of components that are now connected due to interactions. The above formula states that it takes two vertices to construct a loop or to connect two components. The number of vertices is important because it gives the order in perturbation theory g^V. We will consider a graph of order $g^{2\ell}$ in perturbation theory to be an 'ℓ-loop' graph

$$\text{`}\ell\text{-}loop\text{'}: \quad \mathcal{O}(g^{2\ell}). \tag{1.27}$$

Note that these 'loops' are not the momentum-loops counted by L. The motivation for this terminology is that, when working in position space, connecting two components of a graph may produce the same kind of divergency as adding a loop. This is quite different in momentum space, where divergencies can only arise from true loops in the graph. At any rate, the counting scheme is different there, as one usually considers only connected graphs with external propagators removed. For Wilson loops the counting is again different, because each external leg also contributes one power of g.

1.3 The Gauge Group

In the following we will present some useful notation to deal with the matrix-valued fields $\mathcal{W}_\mathcal{A}$. For a start, let us introduce explicit matrix indices for the fields $(\mathcal{W}_\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}}$. For variations with respect to these fields we introduce the notation $\check{\mathcal{W}}^\mathcal{A}$, see also Fig. 1.6,[8]

$$(\check{\mathcal{W}}^\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}} := \frac{\delta}{\delta(\mathcal{W}_\mathcal{A})^{\mathfrak{b}}{}_{\mathfrak{a}}}, \qquad (\check{\mathcal{W}}^\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}}(\mathcal{W}_\mathcal{B})^{\mathfrak{c}}{}_{\mathfrak{d}} = \delta^\mathcal{A}_\mathcal{B}\delta^{\mathfrak{a}}_{\mathfrak{d}}\delta^{\mathfrak{c}}_{\mathfrak{b}}. \tag{1.28}$$

When, for the gauge group SU(N), the matrices are traceless $\mathcal{W}^{\mathfrak{a}}{}_{\mathfrak{a}} = 0$, the trace of the variation must vanish as well and we define the variation by

$$(\check{\mathcal{W}}^\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}}(\mathcal{W}_\mathcal{B})^{\mathfrak{c}}{}_{\mathfrak{d}} = \delta^\mathcal{A}_\mathcal{B}\delta^{\mathfrak{a}}_{\mathfrak{d}}\delta^{\mathfrak{c}}_{\mathfrak{b}} - N^{-1}\delta^\mathcal{A}_\mathcal{B}\delta^{\mathfrak{a}}_{\mathfrak{b}}\delta^{\mathfrak{c}}_{\mathfrak{d}}. \tag{1.29}$$

We furthermore introduce normal ordering $:\ldots:$ which suppresses all possible contractions between fields and variations by moving all variations to the right, for example

$$:\ldots(\check{\mathcal{W}}^\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}}\ldots(\mathcal{W}_\mathcal{B})^{\mathfrak{c}}{}_{\mathfrak{d}}\ldots: := \ldots(\mathcal{W}_\mathcal{B})^{\mathfrak{c}}{}_{\mathfrak{d}}\ldots(\check{\mathcal{W}}^\mathcal{A})^{\mathfrak{a}}{}_{\mathfrak{b}}. \tag{1.30}$$

[8] In a canonical quantisation scheme, $\mathcal{W}$ and $\check{\mathcal{W}}$ correspond to creation and annihilation operators.

For all practical purposes we need not write out the matrix indices writing simply

$$\check{\mathcal{W}}^{\mathcal{A}} := \frac{\delta}{\delta \mathcal{W}_{\mathcal{A}}}\,. \tag{1.31}$$

It is useful to write down the action of a variation on a field (1.28) when both are inserted within traces. There are two cases to be considered: The variation and field might be within different traces or within the same; these are the fusion and fission rules, respectively

$$\begin{aligned} \operatorname{Tr} X\check{\mathcal{W}}^{\mathcal{A}} \operatorname{Tr} Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}} \operatorname{Tr} XY, \\ \operatorname{Tr} X\check{\mathcal{W}}^{\mathcal{A}} Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}} \operatorname{Tr} X \operatorname{Tr} Y. \end{aligned} \tag{1.32}$$

Clearly, $\check{\mathcal{W}}$ also acts on further fields $\mathcal{W}$ within Y in the same way. For the case of a gauge group $\mathrm{SU}(N)$, the fusion and fission rules following from (1.29) are

$$\begin{aligned} \operatorname{Tr} X\check{\mathcal{W}}^{\mathcal{A}} \operatorname{Tr} Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}}(\operatorname{Tr} XY - N^{-1} \operatorname{Tr} X \operatorname{Tr} Y), \\ \operatorname{Tr} X\check{\mathcal{W}}^{\mathcal{A}} Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}}(\operatorname{Tr} X \operatorname{Tr} Y - N^{-1} \operatorname{Tr} XY). \end{aligned} \tag{1.33}$$

Commonly, variations will appear within commutators only. The appropriate rules are

$$\begin{aligned} \operatorname{Tr} X[Z,\check{\mathcal{W}}^{\mathcal{A}}] \operatorname{Tr} Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}} \operatorname{Tr} X[Z,Y], \\ \operatorname{Tr} X[Z,\check{\mathcal{W}}^{\mathcal{A}}] Y\mathcal{W}_{\mathcal{B}} &= \delta^{\mathcal{A}}_{\mathcal{B}}(\operatorname{Tr} XZ \operatorname{Tr} Y - \operatorname{Tr} X \operatorname{Tr} ZY), \end{aligned} \tag{1.34}$$

which are valid for both, $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ (the abelian trace does not contribute in commutators). Note that when normal ordering expressions, it is sometimes impossible to simply move all variations to the right in this notation. Instead, the possible contractions have to be removed by hand, for example

$$:\operatorname{Tr} \mathcal{W}_{\mathcal{A}}\check{\mathcal{W}}^{\mathcal{B}}\mathcal{W}_{\mathcal{C}}\check{\mathcal{W}}^{\mathcal{D}}: = \operatorname{Tr} \mathcal{W}_{\mathcal{A}}\check{\mathcal{W}}^{\mathcal{B}}\mathcal{W}_{\mathcal{C}}\check{\mathcal{W}}^{\mathcal{D}} - \delta^{\mathcal{B}}_{\mathcal{C}} N \operatorname{Tr} \mathcal{W}_{\mathcal{A}}\check{\mathcal{W}}^{\mathcal{D}}. \tag{1.35}$$

This notation is convenient to express, for example, gauge transformations $\mathcal{W} \mapsto U\mathcal{W}U^{-1}$, which are generated infinitesimally by

$$\delta_\epsilon \mathcal{W} = i[\epsilon, \mathcal{W}]. \tag{1.36}$$

Using our notation for matrix-valued variations this becomes

$$\delta_\epsilon = \operatorname{Tr} \epsilon \mathfrak{j}, \quad \text{where} \quad \mathfrak{j} = i:[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{A}}]:. \tag{1.37}$$

We can also consider a more general gauge group. We will start with the gauge theory Lagrangian as defined in (1.5) for $\mathrm{SU}(N)$. Let us parameterise the fields using $\mathrm{SU}(N)$ generators $\mathfrak{t}_{\mathfrak{m}}$

$$\mathcal{W}_{\mathcal{A}} = \mathcal{W}^{\mathfrak{m}}_{\mathcal{A}}\mathfrak{t}_{\mathfrak{m}}. \tag{1.38}$$

We assume the generators and structure constants $\mathfrak{f}^{\mathfrak{p}}_{\mathfrak{m}\mathfrak{n}}$ to be normalised in a way such that

$$\operatorname{Tr} \mathfrak{t}_{\mathfrak{m}}\mathfrak{t}_{\mathfrak{n}} = \mathfrak{g}_{mn}, \qquad [\mathfrak{t}_{\mathfrak{m}}, \mathfrak{t}_{\mathfrak{n}}] = i\mathfrak{f}^{\mathfrak{p}}_{\mathfrak{m}\mathfrak{n}}\mathfrak{t}_{\mathfrak{p}}. \tag{1.39}$$

The more general variations will be defined as

$$\check{\mathcal{W}}^{\mathcal{A}} := \mathfrak{t}_{\mathfrak{m}}\mathfrak{g}^{\mathfrak{m}\mathfrak{n}}\frac{\delta}{\delta \mathcal{W}^{\mathfrak{n}}_{\mathcal{A}}}, \qquad \frac{\delta}{\delta \mathcal{W}^{\mathfrak{m}}_{\mathcal{A}}}\mathcal{W}^{\mathfrak{n}}_{\mathcal{B}} = \delta^{\mathcal{A}}_{\mathcal{B}}\delta^{\mathfrak{n}}_{\mathfrak{m}}. \tag{1.40}$$

This allows us to rewrite the gauge theory and all our results purely in terms of the metric $\mathfrak{g}^{\mathfrak{mn}}$ and the structure constants $\mathfrak{f}^{\mathfrak{p}}_{\mathfrak{mn}}$. In that form the results generalise to arbitrary gauge groups. Nevertheless the matrix notation is most convenient and we will stick to it in this work. On rare occasions we shall use generators $\mathfrak{t}_{\mathfrak{m}}$ to write down expressions valid for generic groups; for example, it is better to write instead of (1.35)

$$:\mathrm{Tr}\, \mathcal{W}_{\mathcal{A}} \check{\mathcal{W}}^{\mathcal{B}} \mathcal{W}_{\mathcal{C}} \check{\mathcal{W}}^{\mathcal{D}}: = \mathrm{Tr}\, \mathcal{W}_{\mathcal{A}} \check{\mathcal{W}}^{\mathcal{B}} \mathcal{W}_{\mathcal{C}} \check{\mathcal{W}}^{\mathcal{D}} - \delta^{\mathcal{B}}_{\mathcal{C}} \mathfrak{g}^{\mathfrak{mn}}\, \mathrm{Tr}\, \mathcal{W}_{\mathcal{A}} \mathfrak{t}_{\mathfrak{m}} \mathfrak{t}_{\mathfrak{n}} \check{\mathcal{W}}^{\mathcal{D}}. \tag{1.41}$$

For a unitary group we can define a *parity operation*. It replaces a matrix by its negative transpose

$$\text{`}\textit{parity operation}\text{'}: \quad \mathfrak{p}\mathcal{W} \mapsto -\mathcal{W}^{\mathsf{T}}. \tag{1.42}$$

For hermitian matrices the conjugate equals the transpose, therefore this parity is equivalent to charge conjugation. Its eigenvalues ± 1 will be denoted by the letter P. It is easily seen that the Lagrangian (1.5) is invariant under this operation. Therefore parity is an exact symmetry of U(N) or SU(N) gauge theory. Note that this parity is a unique feature of the unitary groups, it does not generalise to the orthogonal or symplectic groups.

1.4 The 't Hooft Limit

A field theory with U(N) gauge symmetry has remarkable properties when N is interpreted as an additional coupling constant: In the article [1] 't Hooft realised that, in the large-N limit, for any Feynman graph there is an associated two-dimensional surface. The N-dependence of a graph is given by the Euler characteristic (genus) of the corresponding surface. This makes the large-N field theory very similar to a string field theory whose coupling constant also counts the genus of the world sheet.

Let us consider a gauge invariant Feynman graph and investigate its structure in terms of the gauge group. The structure may be represented graphically using 'double lines' or a 'fat graph'. For that purpose we represent every upper (lower) U(N) vector index of a field within the operators or vertices by a black (white) dot. Consequently, every (adjoint) field provides one black and one white dot. For all external contractions between two vector indices, i.e. those in the operator or in the vertex, draw an arrow from the black to the white dot. For a gauge invariant graph all indices must be contracted, so there are no unconnected dots. Now we perform the contractions generated by the path integral, i.e. due to the propagators (1.25)

$$W_0[\mathcal{J}] = \frac{1}{N} \int dx\, dy\, \tfrac{1}{2} \mathcal{J}^{\mathfrak{a}}{}_{\mathfrak{b}}(x) \Delta(x,y) \mathcal{J}^{\mathfrak{b}}{}_{\mathfrak{a}}(y), \tag{1.43}$$

where we made the gauge indices visible. The propagator connects two fields, we should now connect the corresponding two pairs of dots by antiparallel arrows along the propagator. Here, we will draw the arrows from white dot to black dot. In this way all propagators of the Feynman graph are represented by two parallel lines or, alternatively, a fat line, c.f. Fig. 1.8. In this representation it is easy to perform all U(N) contractions: All dots have exactly one incoming and one outgoing arrow. The arrows thus form closed circles, which are known as index loops. Each index loop provides $\delta^{\mathfrak{a}}_{\mathfrak{a}} = \mathrm{Tr}\, 1 = N$ and furthermore there are explicit factors of N in the action and propagators (1.17,1.43). The result is

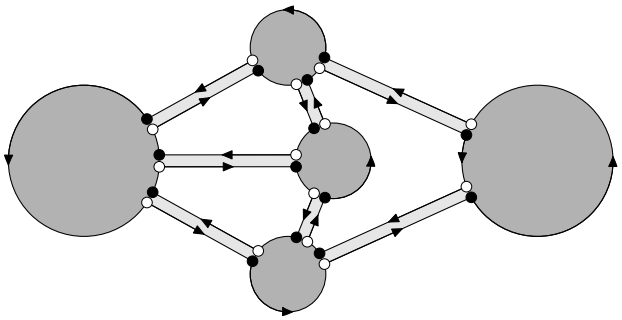

Figure 1.8: A fat graph. The circles are U(N) traces of the vertices and operators and the fat lines are propagators. For each face of the graph there is a closed empty trace $\mathrm{Tr}\,1 = N$.

given by N^{F+V-I} where F is the number of closed (index) loops, V the number of vertices and I the number of (double) lines. Assume now that the graph has C components and that each component will be drawn on a surface of minimal genus without crossing of any lines. Let G be the total genus of all component surfaces and T the number of traces within the local operators. Then Euler's theorem relates these numbers as follows $T + V + F = I + 2C - 2G$. In total the N-dependence is given by[9]

$$\mathcal{O}(N^{2C-2G-T}) = \mathcal{O}(N^{\chi}). \tag{1.44}$$

Note that the coupling constant g^2 as defined in (1.15) is proportional to the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$

$$g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} = \frac{\lambda}{8\pi^2}\,. \tag{1.45}$$

Note also that (1.44) gives precisely the Euler characteristic $\chi = 2C - 2G - T$ of the set of surfaces. This led 't Hooft to his conjecture of the emergence of a string field theory in the large N limit of a gauge theory: In string field theory, an amplitude on a world sheet with Euler characteristic χ is proportional to $g_{\mathrm{s}}^{-\chi}$, where g_{s} is the string coupling constant. This matches with the N-dependence in gauge theory when we identify

$$g_{\mathrm{s}} \sim \frac{1}{N}\,. \tag{1.46}$$

With some additional work, the large N limit can also be taken for gauge groups $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$ as well as fields in the fundamental representation. Then, also unoriented surfaces as well as surfaces with boundaries[10] may appear.

1.5 The Superconformal Algebra

The Lagrangian (1.5) and action (1.14) of $\mathcal{N} = 4$ SYM in four spacetime dimensions do not involve any dimensionful coupling constants. Therefore the action is invariant under the scale transformation

$$x \mapsto c^{-1}x, \quad \mathcal{A} \mapsto c\mathcal{A}, \quad \Psi \mapsto c^{3/2}\Psi. \tag{1.47}$$

[9] A contribution N^V which commonly appears at $V/2$ quantum loops has already been absorbed into the definition of the coupling constant (1.15).

[10] Here we mean boundaries which are not associated to an operator insertion.

For a gauge theory this implies also conformal invariance and, in the case of a supersymmetric theory, also superconformal invariance. This symmetry is especially important for $\mathcal{N} = 4$ SYM, because it is believed to be an exact symmetry even in the quantum theory, where the beta function (1.23) is apparently exactly zero.

The super Poincaré symmetry algebra consisting of Lorentz rotations $\mathfrak{L}, \dot{\mathfrak{L}}$, internal rotations $\mathfrak{R}$ and (super)translations $\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$ is enlarged by (super)conformal boosts $\mathfrak{S}, \dot{\mathfrak{S}}, \mathfrak{K}$ and the scaling operator $\mathfrak{D}$ which is also known as

$$\text{`}\textit{The Dilatation Operator}\text{'}\ \mathfrak{D}. \tag{1.48}$$

The boosts are essentially the conjugate transformations of the translations. The action of the translations and boosts on the fields in the free theory is depicted in Fig. 1.2. The arrows correspond to momenta whereas boosts act in the inverse direction of the arrows.

The conformal symmetry algebra in four spacetime dimensions is $\mathfrak{so}(4,2) = \mathfrak{su}(2,2)$, the superconformal algebra is $\mathfrak{su}(2,2|\mathcal{N})$. In the case of maximal $\mathcal{N} = 4$ supersymmetry, the algebra $\mathfrak{su}(2,2|4)$ is reducible and the superconformal algebra is considered to be only the irreducible part $\mathfrak{psu}(2,2|4)$. Let us, for the moment, consider the supermatrix algebra $\mathfrak{u}(2,2|4)$ and later restrict to $\mathfrak{psu}(2,2|4)$. It consists of the generators

$$\mathfrak{J} \in \{\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}, \mathfrak{P}, \mathfrak{K}, \mathfrak{D}, \mathfrak{B}, \mathfrak{C} | \mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{S}, \dot{\mathfrak{S}}\}. \tag{1.49}$$

These are the $\mathfrak{su}(2), \mathfrak{su}(2), \mathfrak{su}(4)$ rotations $\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}$, the (super)translations $\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$, the (super)boosts $\mathfrak{S}, \dot{\mathfrak{S}}, \mathfrak{K}$ as well as the dilatation generator $\mathfrak{D}$, hypercharge $\mathfrak{B}$ and central charge $\mathfrak{C}$. Please refer to App. D for details of this superalgebra and its commutation relations. The signature of spacetime will not be important here; for algebraic purposes we can safely assume to work with a complexified algebra. In the irreducible superconformal algebra $\mathfrak{psu}(2,2|4)$, the generators $\mathfrak{B}, \mathfrak{C}$ are absent: The $\mathfrak{u}(1)$ hypercharge $\mathfrak{B}$ of $\mathfrak{pu}(2,2|4) = \mathfrak{u}(1) \ltimes \mathfrak{psu}(2,2|4)$ is an external automorphism which consistently assigns a charge to all the generators of $\mathfrak{psu}(2,2|4)$. The $\mathfrak{u}(1)$ central charge $\mathfrak{C}$ of $\mathfrak{su}(2,2|4) = \mathfrak{psu}(2,2|4) \ltimes \mathfrak{u}(1)$ must vanish to be able to reduce to $\mathfrak{psu}(2,2|4)$. The Lorentz algebra $\mathfrak{so}(3,1) = \mathfrak{su}(2) \times \mathfrak{su}(2)$ is formed by $\mathfrak{L}, \dot{\mathfrak{L}}$. Together with $\mathfrak{P}, \mathfrak{K}, \mathfrak{D}$ one gets the conformal algebra $\mathfrak{so}(4,2) = \mathfrak{su}(2,2)$.

Note that only the Lorentz and internal symmetries, $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and $\mathfrak{su}(4)$, are manifestly realised in the quantum theory; the other generators receive radiative corrections, i.e. they depend on the coupling constant g, see (1.9). In particular the dilatation generator $\mathfrak{D}$ receives loop corrections. As we shall see, it makes sense to define an operator which measures the *classical dimension*, the

$$\text{`}\textit{Classical Dilatation Operator}\text{'}\quad \mathfrak{D}_0, \tag{1.50}$$

even in the quantum theory. The shift of scaling dimensions by quantum effects, the *anomalous dimension*, is measured by the

$$\text{`}\textit{Anomalous Dilatation Operator}\text{'}\quad \delta\mathfrak{D} = \mathfrak{D} - \mathfrak{D}_0. \tag{1.51}$$

This is a $\mathfrak{u}(1)$ abelian generator. It is also reasonable to identify $\delta\mathfrak{D}$ with a '*Hamiltonian*' $\mathcal{H}(g)$ in the following way, c.f. Sec. 2.2.4

$$\text{`}\textit{The Hamiltonian}\text{'}\quad \mathcal{H} = g^{-2}\,\delta\mathfrak{D}. \tag{1.52}$$

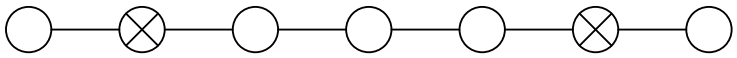

Figure 1.10: The Dynkin diagram of $\mathfrak{psu}(2,2|4)$ convenient for $\mathcal{N}=4$ SYM.

Its eigenvalues, the *energies*, will be denoted by the letter E.

For a bosonic, semi-simple Lie algebra the Dynkin diagram is unique. In the case of superalgebras, however, there is some freedom to distribute the simple fermionic roots. Different choices of fermionic roots correspond to different assignments of positive and negative roots. In the context of $\mathcal{N}=4$ SYM one particular choice of Dynkin diagram turns out to be very useful [90], see Fig. 1.10.[11] For this particular Dynkin diagram the generators associated to positive and negative roots and elements of the Cartan subalgebra $\mathfrak{J}^+, \mathfrak{J}^-, \mathfrak{J}^0$ are given by

$$\begin{aligned}
\mathfrak{J}^+ &\in \{\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{S}^\alpha{}_b, \dot{\mathfrak{S}}^{a\dot\beta}, \mathfrak{L}^\alpha{}_\beta\,(\alpha<\beta), \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}\,(\dot\alpha<\dot\beta), \mathfrak{R}^a{}_b\,(a<b)\},\\
\mathfrak{J}^0 &\in \{\mathfrak{L}^\alpha{}_\beta\,(\alpha=\beta), \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}\,(\dot\alpha=\dot\beta), \mathfrak{R}^a{}_b\,(a=b), \mathfrak{D}, \mathfrak{B}, \mathfrak{C}\},\\
\mathfrak{J}^- &\in \{\mathfrak{P}_{\dot\alpha\beta}, \mathfrak{Q}^a{}_\beta, \dot{\mathfrak{Q}}_{\dot\alpha b}, \mathfrak{L}^\alpha{}_\beta\,(\alpha>\beta), \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}\,(\dot\alpha>\dot\beta), \mathfrak{R}^a{}_b\,(a>b)\}.
\end{aligned} \tag{1.53}$$

All the elements of the Cartan subalgebra, spanned by $\{\mathfrak{J}^0\}$, commute among each other. One can therefore find simultaneous eigenstates with respect to all its elements, the eigenvalues are the charges or '*labels*' of that state. There are many useful bases for the Cartan subalgebra which give rise to different labellings of states, we will use two of them.

Let us first note the Dynkin labels corresponding to the diagram in Fig. 1.10 of $\mathfrak{su}(2,2|4)$, see also [91][12]

$$w = [s_1; r_1; q_1, p, q_2; r_2; s_2]; \tag{1.54}$$

these are defined as the following linear combinations of the eigenvalues $L^\alpha{}_\beta, \dot L^{\dot\alpha}{}_{\dot\beta}, R^a{}_b$ of Cartan generators $\mathfrak{L}^\alpha{}_\beta, \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}, \mathfrak{R}^a{}_b$ $(\alpha=\beta, \dot\alpha=\dot\beta, a=b)$

$$\begin{aligned}
s_1 &= L^2{}_2 - L^1{}_1, & s_2 &= \dot L^2{}_2 - \dot L^1{}_1,\\
r_1 &= \tfrac{1}{2}D - \tfrac{1}{2}C - L^1{}_1 + R^1{}_1, & r_2 &= \tfrac{1}{2}D + \tfrac{1}{2}C - \dot L^1{}_1 - R^4{}_4,\\
q_1 &= R^2{}_2 - R^1{}_1, & q_2 &= R^4{}_4 - R^3{}_3,\\
p &= R^3{}_3 - R^2{}_2, & r &= -D + L^1{}_1 + \dot L^1{}_1.
\end{aligned} \tag{1.55}$$

The charges $[q_1, p, q_2]$ are the Dynkin labels of the $\mathfrak{su}(4)$ subalgebra. Equivalently $[s_1, s_2]$ are the Dynkin labels of the Lorentz algebra $\mathfrak{so}(3,1) = \mathfrak{su}(2)\times\mathfrak{su}(2)$. Together with r they give the Dynkin labels $[s_1, r, s_2]$ of the conformal algebra $\mathfrak{su}(2,2)$. Note that we shall always use integer valued Dynkin labels s_1, s_2 instead of the more common half-integer valued spin labels for $\mathfrak{su}(2)$, i.e. s_1, s_2 equal *twice* the spin. The labels q_1, q_2 will also be integers, only for the labels r_1, r_2, r we have to allow irrational numbers corresponding to anomalous dimensions. For the $\mathfrak{su}(2,2|4)$ labels (1.54) we do not need the label r which is given by $r = -r_1 - q_1 - p - q_2 - r_2$.

[11] One might be tempted to denote the superconformal algebra by $\mathfrak{psu}(2|4|2)$.

[12] There is no obvious choice for the sign of the odd labels r_1, r_2. Our choice implies, e.g., for the product of two fundamental representations $[1;0;\ldots]\times[1;0;\ldots] = [2;0;\ldots] + [0;-1;\ldots]$.

Often, we will consider states of the classical theory at $g = 0$.[13] To label them we will use a notation which is based on more physical quantities

$$w = (D_0; s_1, s_2; q_1, p, q_2; B, L). \tag{1.56}$$

Here, $[q_1, p, q_2]$ and $[s_1, s_2]$ are as above. The label D_0 is the classical scaling dimension as measured by $\mathfrak{D}_0$. These are the six labels for a weight of the algebra $\mathfrak{psu}(2,2|4)$ of rank six. All of them are non-negative integers except D_0, which can take half-integer values.

We further introduce labels B, C, L which do not belong to $\mathfrak{psu}(2,2|4)$; they are important to describe a state of the classical theory. The hypercharge B, measured by $\mathfrak{B}$ of $\mathfrak{pu}(2,2|4)$, is half-integer valued and is defined via Tab. 1.4. The '*length*' L, measured by the operator $\mathcal{L}$, counts the number of constituent fields of a state and is therefore a positive integer. The central charge C, measured by $\mathfrak{C}$ of $\mathfrak{su}(2,2|4)$, will always be zero. The anomalous dimension $\delta D = D - D_0$ is not part of this set of labels, it will be given elsewhere.

1.6 Fields and States

In this work we will be interested in the properties of local, gauge invariant operators, which we will also refer to as '*states*' and which will usually be denoted by the symbol '$\mathcal{O}$'. Local operators are constructed as linear combinations of products of the fundamental fields and their derivatives[14]

$$\text{'}states\text{'}: \quad \mathcal{O}(x) = \Phi_*(x)\cdot\Phi_*(x)\cdot\mathcal{D}_*\mathcal{D}_*\Psi_*(x)\cdot\mathcal{D}_*\mathcal{F}_*(x)\ldots + \ldots . \tag{1.57}$$

Note that the coordinate x is the same for all the fields and hence we can drop it altogether.[15] We demand states to be gauge invariant. Due to the inhomogeneous gauge transformation ($X \mapsto UXU^{-1} + \ldots$) of the gauge field $\mathcal{A}$ and partial derivative ∂, these cannot be used in the construction of states. Instead, we can use field strengths $\mathcal{F}$ and covariant derivatives $\mathcal{D}$, which transform homogeneously ($\mathcal{W} \mapsto U\mathcal{W}U^{-1}$). In the case of gauge groups $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$ and adjoint fields, a basis for the space of states is given by the multi-trace operators

$$\text{'}state\ basis\text{'}: \quad \big\{\mathrm{Tr}\,\mathcal{W}_*\cdots\mathcal{W}_* \;\; \mathrm{Tr}\,\mathcal{W}_*\cdots\mathcal{W}_* \;\; \ldots\big\}, \tag{1.58}$$

where each $\mathcal{W}$ is one of the *fields*

$$\text{'}field\ basis\text{'}: \quad \mathcal{W}_{\mathcal{A}} \in \big\{\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k\dot{\Psi}, \mathcal{D}^k\mathcal{F}\big\}. \tag{1.59}$$

Here we slightly change the definition of the symbol $\mathcal{W}_{\mathcal{A}}$: By including arbitrarily many derivatives, in the sense of a Taylor expansion $\mathcal{W}(x) = \mathcal{W}_0 + x\mathcal{W}_1 + \frac{1}{2}x^2\mathcal{W}_2 + \ldots$, we trade in spacetime-dependence for infinitely many components.[16] Roughly speaking, the index

[13] To use this set of labels makes sense even in the interacting theory: There, the labels should be defined as the labels in the limit $g \to 0$.

[14] The $*$'s refer to some unspecified indices

[15] Moreover, we consider fields and states as abstract objects which are not positioned at some point in space. They are merely elements of the space of fields or states, respectively.

[16] This is analogous to moving from a superspace to ordinary spacetime when one trades in the dependence on fermionic coordinates θ for component fields.

field	D_0	$\mathfrak{su}(2)\times\mathfrak{su}(2)$	$\mathfrak{su}(4)$	B	L
$\mathcal{D}^k\mathcal{F}$	$k+2$	$[k+2,k\quad]$	$[0,0,0]$	$+1$	1
$\mathcal{D}^k\Psi$	$k+\frac{3}{2}$	$[k+1,k\quad]$	$[0,0,1]$	$+\frac{1}{2}$	1
$\mathcal{D}^k\Phi$	$k+1$	$[k\quad,k\quad]$	$[0,1,0]$	± 0	1
$\mathcal{D}^k\dot{\Psi}$	$k+\frac{3}{2}$	$[k\quad,k+1]$	$[1,0,0]$	$-\frac{1}{2}$	1
$\mathcal{D}^k\dot{\mathcal{F}}$	$k+2$	$[k\quad,k+2]$	$[0,0,0]$	-1	1

Table 1.4: Basis fields $\mathcal{W}_\mathcal{A}$ of the $\mathcal{N}=4$ SYM field strength multiplet $\mathcal{V}_{\text{F}}$.

$\mathcal{A}$ now comprises also the coordinate x, but in a fashion more suitable for local operators. The basis (1.58) is in general over-complete. One reason is the Bianchi identity (1.4) which defines the field strength

$$[\mathcal{D},\mathcal{D}]X \sim g[\mathcal{F},X], \tag{1.60}$$

which tells us that the left hand side is reducible, i.e. it can be written as a product of fields. As products of fields appear naturally within the basis states (1.58), there is no reason to use reducible fields $\mathcal{W}$. By avoiding them, one eliminates obscure identities between basis states (1.58) and reduces the ambiguity. Consequently, for an irreducible field, all derivatives in (1.59) should be totally symmetrised. Furthermore, the Bianchi identity (1.4)

$$\mathcal{D}\wedge\mathcal{F}=0 \tag{1.61}$$

implies that none of the derivatives can be antisymmetrised with the field strength. Finally, the equations of motion (1.7)

$$\mathcal{D}\cdot\mathcal{D}\Phi,\mathcal{D}\cdot\Psi,\mathcal{D}\cdot\dot{\Psi},\mathcal{D}\cdot\mathcal{F}=\ldots, \tag{1.62}$$

lead to further reducible terms $\Phi^3,\Phi\mathcal{D}\Phi,\Psi^2,\ldots$.[17] Therefore, contractions between indices are not allowed for irreducible fields. In total these constraints lead to the set of *irreducible fields* as presented in Tab. 1.4. In the table we have split up the field strengths into their chiral and antichiral parts according to (1.11). We will use index letters $\mathcal{A},\mathcal{B},\ldots$ to label precisely the elements of this set of irreducible fields. For the rest of this work we will consider only irreducible fields and speak of '*fields*' for short.

Matrix identities are another source of linear dependencies between the basis states (1.58) at finite N. These involve traces of $L>N$ fields and become irrelevant when N is sufficiently large, e.g., in the large N limit. Note also that traces are cyclic and states related by cyclic permutations within the traces are to be identified.

An alternative way of representing local operators is to use the state-operator map for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$, which is conformally equivalent to flat $\mathbb{R}^4$. When decomposing the fundamental fields into spherical harmonics on S^3 one gets precisely the same spectrum of fields $\mathcal{W}$ as in Tab. 1.4.

1.7 Highest-Weight Modules and Representations

There are various types of representations of $\mathfrak{u}(2,2|4)$; for example the defining one $\mathbf{4}|\mathbf{4}$ or the adjoint $\mathbf{30}|\mathbf{32}+\mathbf{1}+\mathbf{1}$ have finitely many components. In the context of field theory

[17] In a quantum theory the equations of motion might be modified, but the modifications are again reducible.

we have to deal mainly with a different kind, namely non-compact or infinite-dimensional highest-weight representations. A (Verma) module, i.e. the vector space on which the representation acts, is characterised by its *highest-weight* or *primary* state $|0\rangle$. In field theory this corresponds to a field or local operator,[18] for example the primary field $\mathcal{Z}$ or the primary Konishi state $\mathcal{K}$

$$|\mathcal{Z}\rangle = \Phi_{34} \quad \text{or} \quad |\mathcal{K}\rangle = \varepsilon^{abcd} \operatorname{Tr} \Phi_{ab}\Phi_{cd} \sim \eta^{mn} \operatorname{Tr} \Phi_m \Phi_n. \tag{1.63}$$

A highest-weight state $|0\rangle$ is defined as being annihilated by all raising operators $\mathfrak{J}^+$ in (1.53)

$$\mathfrak{J}^+|0\rangle = 0, \qquad \mathfrak{J}^0|0\rangle = w|0\rangle; \tag{1.64}$$

the action of the Cartan subalgebra $\mathfrak{J}^0$ in (1.53) measures the charge vector w of the highest weight, see (1.54,1.56). Application of the lowering operators yields new states

$$|0\rangle, \quad \mathfrak{J}^-|0\rangle, \quad \mathfrak{J}^-\mathfrak{J}^-|0\rangle, \quad \dots, \tag{1.65}$$

which belong to the highest-weight module. These are called *descendants*. For example

$$\mathfrak{Q}^3{}_2|\mathcal{Z}\rangle = \Psi_{24} \quad \text{or} \quad \mathfrak{P}_{22}\mathfrak{P}_{22}|\mathcal{K}\rangle = 2\varepsilon^{abcd}\big(\operatorname{Tr} \mathcal{D}_{22}\Phi_{ab}\, \mathcal{D}_{22}\Phi_{cd} + \operatorname{Tr} \Phi_{ab}\, \mathcal{D}_{22}\mathcal{D}_{22}\Phi_{cd}\big) \tag{1.66}$$

belong to the modules with highest weights $|\mathcal{Z}\rangle$ or $|\mathcal{K}\rangle$, respectively. See Fig. 1.2 for an illustration of the module with highest weight $|\mathcal{Z}\rangle$.

In general one can apply any number of lowering operators to the highest weight and obtain an infinite multiplet of states. The space spanned by the states is a module of $\mathfrak{u}(2,2|4)$ because applying any of the generators yields another element: For lowering operators this is obvious while raising and Cartan generators have to be commuted all the way to the vacuum $|0\rangle$ first. In the most general case, the obtained module is irreducible. However, for very special highest weights, one will find further states which are annihilated by all the raising operators. In that case the multiplet is reducible and the irreducible module which contains $|0\rangle$ is called short, see Sec. 1.8 for the cases relevant to $\mathcal{N} = 4$ SYM. Finite-dimensional representations are just extremely short: By repeatedly applying lowering operators to the highest weight, one will inevitably leave the irreducible module that belongs to the highest weight $|0\rangle$. In a conformal field theory, the modules will commonly be very short, only a few of the shortening conditions are not satisfied. This means that, when one considers a fairly large subalgebra, here $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$, the modules will split into (infinitely many) finite-dimensional modules of the subalgebra.

Let us demonstrate this feature in terms of a simple example using the algebra of $\mathfrak{sl}(2)$ spanned by $\mathfrak{J}^\pm, \mathfrak{J}^0$. The algebra of generators is

$$[\mathfrak{J}^0, \mathfrak{J}^\pm] = \pm 2\mathfrak{J}^\pm, \qquad [\mathfrak{J}^+, \mathfrak{J}^-] = J^0. \tag{1.67}$$

We specify a highest-weight state $|s\rangle$ by

$$\mathfrak{J}^+|s\rangle = 0, \qquad \mathfrak{J}^0|s\rangle = s|s\rangle. \tag{1.68}$$

A module is spanned by the states

$$|s,k\rangle = (\mathfrak{J}^-)^k|s\rangle. \tag{1.69}$$

[18] A local operator can be viewed as an abstract object, i.e. not based at some point in spacetime.

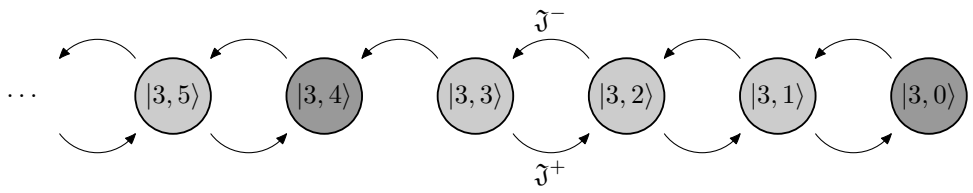

Figure 1.12: A reducible highest-weight module. All states can be obtained from the highest-weight state $|3,0\rangle$ by the action of $\mathfrak{J}^-$, but there is no way back from $|3,4\rangle$ to $|3,3\rangle$ using $\mathfrak{J}^+$. Consequently $|3,4\rangle$ is a highest-weight state.

Let us act with $\mathfrak{J}^+$ on some state $|s,k\rangle$, using the algebra relations we find

$$\mathfrak{J}^+|s,k\rangle = k(s+1-k)|s,k-1\rangle. \tag{1.70}$$

The state $|s,0\rangle = |s\rangle$ with $k=0$ is a highest-weight state by construction. However, if s is a non-negative integer, the state $|s,s+1\rangle$ is another highest weight, see Fig. 1.12. It has $\mathfrak{J}^0$ charge $s' = -2-s$, it is therefore equivalent to $|s'\rangle$

$$|s,s+1\rangle \mathrel{\widehat{=}} |-2-s\rangle. \tag{1.71}$$

The charge s' of this highest weight is negative and the 'lower' module therefore irreducible. By defining $|s,s+1\rangle = 0$ we can also make the 'upper' highest weight module irreducible. As one can see, this is compatible with the algebra (1.67).

1.8 Unitarity and Multiplet Shortenings

The real algebra $\mathfrak{psu}(2,2|4)$ of indefinite signature does not have any non-trivial finite-dimensional unitary representations. Unitary representations, which are relevant to quantum physics, are necessarily infinite-dimensional. These have been classified [8] and the following two bounds have been found[19]

$$\begin{aligned} & D \geq 2+s_1+p+\tfrac{3}{2}q_1+\tfrac{1}{2}q_2 \quad \text{or} \quad D = p+\tfrac{3}{2}q_1+\tfrac{1}{2}q_2, \quad s_1 = 0, \\ \text{and} \quad & D \geq 2+s_2+p+\tfrac{1}{2}q_1+\tfrac{3}{2}q_2 \quad \text{or} \quad D = p+\tfrac{1}{2}q_1+\tfrac{3}{2}q_2, \quad s_2 = 0. \end{aligned} \tag{1.72}$$

Unitary multiplets fall into different series, the first one is the 'half-BPS' series with highest weights given by[20]

$$w = (p;0,0;0,p,0;0,p). \tag{1.73}$$

These are the shortest physical multiplets. Multiplets which are of this type in the classical theory are protected, the scaling dimension of the classical theory is preserved exactly in the quantum theory, $D=p$. In field theory the highest weight state is composed only from fields $\mathcal{Z}$ and all states of this form are half-BPS.

Furthermore, there are two 'eighth-BPS' conditions[21]

$$\text{'}\textit{eighth-BPS}\text{'}: \quad \begin{aligned} \text{I}: \quad & D = p+\tfrac{3}{2}q_1+\tfrac{1}{2}q_2, \quad s_1 = 0, \\ \text{II}: \quad & D = p+\tfrac{1}{2}q_1+\tfrac{3}{2}q_2, \quad s_2 = 0, \end{aligned} \tag{1.74}$$

[19] Using the fermionic labels r_1, r_2 (D.11) these conditions simplify to $r_i - s_i \geq 1$ or $r_i = s_i = 0$.
[20] Alternatively $w = [0;0;0,p,0;0;0]$.
[21] Alternatively $r_i = s_i = 0$.

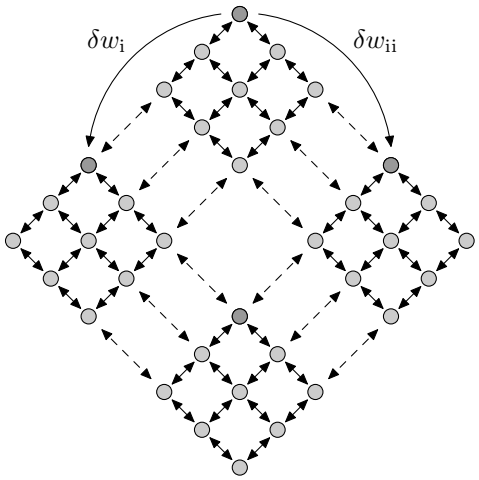

Figure 1.14: A long multiplet can split in up to four short submultiplets at the unitarity bounds. Short arrows correspond to $\mathcal{O}(1)$ generators whereas dashed arrows correspond to $\mathcal{O}(\sqrt{D-D_0})$ generators.

and two shortening[22] conditions[23]

$$\text{`}\textit{short}\text{'}: \qquad \begin{array}{ll} \text{i}: & D = 2 + s_1 + p + \frac{3}{2}q_1 + \frac{1}{2}q_2, \\ \text{ii}: & D = 2 + s_2 + p + \frac{1}{2}q_1 + \frac{3}{2}q_2. \end{array} \tag{1.75}$$

These condition can be combined into a quarter-BPS condition (I+II), two short-eighth-BPS conditions (i+II,I+ii) and a doubly-short condition (i+ii).

In perturbation theory multiplets close to the unitarity bound have some special features, see for example [8, 10, 92]. Consider a multiplet whose classical dimension D_0 saturates one of the bounds in (1.72) and whose anomalous dimension δD is non-zero. When we send the coupling constant to zero, the anomalous dimension δD vanishes and the highest-weight multiplet becomes short (1.75). Nevertheless, the remaining states of the interacting long multiplet cannot disappear, instead they form an additional short highest-weight multiplet, see Fig. 1.14. For $s_{1,2} > 0$ the highest weight of the submultiplet is offset from the highest weight of the long multiplet by

$$\begin{aligned} \delta w_{\rm i} &= (+0.5; -1, 0; +1, 0, 0; -0.5, +1), \\ \delta w_{\rm ii} &= (+0.5; 0, -1; 0, 0, +1; +0.5, +1). \end{aligned} \tag{1.76}$$

The new highest-weight multiplet is also of short type. For $s_{1,2} = 0$ the above shift would lead to a negative spin; then the new highest weight is shifted further by in total

$$\begin{aligned} \delta w_{\rm I} &= (+1.0; 0, 0; +2, 0, 0; 0.0, +1), \\ \delta w_{\rm II} &= (+1.0; 0, 0; 0, 0, +2; 0.0, +1); \end{aligned} \tag{1.77}$$

this multiplet is of eighth-BPS type.

[22] These are usually called semi-shortening conditions. Here we shall distinguish between *short* and *BPS* multiplets.

[23] Alternatively $r_i - s_i = 1$.

Naively, if one of the eighth-BPS conditions is satisfied in the classical theory, one might think that the dimension is protected because an anomalous dimension would violate unitarity (1.72). However, the eighth-BPS multiplet may join, in similarity with the Higgs mechanism, with short multiplets and form a generic, long multiplet. Note that the shift $\delta w_{\text{I,II}}$ yields multiplets with $q_{1,2} \geq 2$, therefore eighth-BPS multiplets with $q_{1,2} = 0, 1$ are indeed protected.

1.9 The Field-Strength Multiplet

Let us now reconsider the fields $\mathcal{W}_{\mathcal{A}}$ and understand their transformation properties. For that purpose we have another look at Tab. 1.4. All representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ are symmetric tensor products of the fundamental representation, while the representations of $\mathfrak{su}(4)$ are antisymmetric. Using two bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ with $\alpha, \dot\alpha = 1, 2$ and one fermionic oscillator $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ with $a = 1, 2, 3, 4$ we can thus write [93][24]

$$\begin{aligned}
\mathcal{D}^k\mathcal{F} &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k+2}(\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^0|0\rangle,\\
\mathcal{D}^k\Psi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^1|0\rangle,\\
\mathcal{D}^k\Phi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^2|0\rangle,\\
\mathcal{D}^k\dot\Psi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+1}(\mathbf{c}^\dagger)^3|0\rangle,\\
\mathcal{D}^k\dot{\mathcal{F}} &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+2}(\mathbf{c}^\dagger)^4|0\rangle.
\end{aligned} \tag{1.78}$$

Each of the oscillators $\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger_a$ carries one of the $\mathfrak{su}(2), \mathfrak{su}(2), \mathfrak{su}(4)$ spinor indices of the fields, for example

$$\mathcal{D}_{\dot\alpha\beta}\,\dot\Psi^a_{\dot\gamma} \sim \varepsilon^{abcd}\mathbf{a}^\dagger_\beta\mathbf{b}^\dagger_{\dot\alpha}\mathbf{b}^\dagger_{\dot\gamma}\mathbf{c}^\dagger_b\mathbf{c}^\dagger_c\mathbf{c}^\dagger_d|0\rangle. \tag{1.79}$$

The statistics of the oscillators automatically symmetrises the indices in the desired way as explained in Sec. 1.6. We will further assume the commutation relations

$$[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta, \qquad [\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \qquad \{\mathbf{c}^a, \mathbf{c}^\dagger_b\} = \delta^a_b. \tag{1.80}$$

Finally, the oscillators $\mathbf{a}^\alpha, \mathbf{b}^{\dot\alpha}, \mathbf{c}^a$ are taken to annihilate the state $|0\rangle$.

Using oscillators we can construct a representation of the unitary superalgebra. We assemble the oscillators $(\mathbf{a}, \mathbf{b}^\dagger|\mathbf{c})$ into a $\mathbf{4}|\mathbf{4}$-dimensional superoscillator $\mathbf{A}$, whereas $\mathbf{A}^\dagger$ consists of $(\mathbf{a}^\dagger, -\mathbf{b}|\mathbf{c}^\dagger)$. The generators of $\mathfrak{u}(2,2|4)$ are then given by[25]

$$\mathfrak{J}_A{}^B = \mathbf{A}^\dagger_A\mathbf{A}^B, \qquad \left[\mathbf{A}^B, \mathbf{A}^\dagger_A\right\} = \delta^B_A. \tag{1.81}$$

It is straightforward to verify that the generators satisfy the commutation relations of a unitary superalgebra. In App. D.4 we split up the generators into $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ notation.

Using the expressions in App. D.4 one can see that the set of fields (1.78) is closed under the action of the generators, the fields thus span a module of $\mathfrak{u}(2,2|4)$. This module will be denoted by $\mathcal{V}_{\text{F}}$ and is spanned by all the fields $\mathcal{W}_{\mathcal{A}}$

$$\text{`\textit{field-strength module}'}: \quad \mathcal{V}_{\text{F}} = [\mathcal{W}_{\mathcal{A}}] = [\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k\dot\Psi, \mathcal{D}^k\mathcal{F}, \mathcal{D}^k\dot{\mathcal{F}}]. \tag{1.82}$$

[24] In a complex algebra we can assume the oscillators $(\mathbf{a}^\alpha, \mathbf{b}^{\dot\alpha}, \mathbf{c}^a)$ and $(\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger_a)$ to be independent.

[25] $[A, B\}$ is the graded commutator: It equals $[A, B]$ if A or B is bosonic and $\{A, B\}$ if both A and B are fermionic.

This module is also called the *singleton*. It is furthermore a module of $\mathfrak{psu}(2,2|4)$ because the central charge vanishes for all fields

$$C = 1 - \tfrac{1}{2}n_{\mathbf{a}} + \tfrac{1}{2}n_{\mathbf{b}} - \tfrac{1}{2}n_{\mathbf{c}} = 0. \tag{1.83}$$

Let us now identify the highest weight. The supercharge $\mathfrak{S}$ transforms an $\mathbf{a}^\dagger$ into a $\mathbf{c}^\dagger$. Annihilation of a state by all $\mathfrak{S}$'s requires there to be no excitation of type $\mathbf{a}^\dagger$ or the maximum number of 4 excitations of type $\mathbf{c}^\dagger$. Conversely, annihilation by $\dot{\mathfrak{S}}$ requires there to be no excitation of type $\mathbf{b}^\dagger$ or no excitation of type $\mathbf{c}^\dagger$. Among the fields (1.78), these conditions are satisfied only by the scalar fields $\Phi \mathrel{\widehat{=}} (\mathbf{c}^\dagger)^2|0\rangle$. Annihilation by $\mathfrak{R}^a{}_b$ with $a < b$ picks out

$$\Phi_{34} \mathrel{\widehat{=}} \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger|0\rangle = |\mathcal{Z}\rangle \tag{1.84}$$

as the highest weight state or superconformal primary field. The field $|\mathcal{Z}\rangle$ is a scalar $[s_1, s_2] = [0, 0]$, an $\mathfrak{so}(6)$ vector $[q_1, p, q_2] = [0, 1, 0]$ and has dimension $D = 1$ as well as central charge $C = 0$. The highest weight is therefore

$$w_{\text{F}} = (1; 0, 0; 0, 1, 0; 0, 1) = [0; 0; 0, 1, 0; 0; 0], \tag{1.85}$$

where we have defined the hypercharge $B = 0$ and length $L = 1$.

The vacuum state $|0\rangle$ is invariant under $\mathfrak{su}(4)$, but it is not physical. Conversely, the highest weight state $|\mathcal{Z}\rangle$ is physical, but superficially breaks $\mathfrak{su}(4)$ to $\mathfrak{su}(2) \times \mathfrak{su}(2)$. When dealing with physical states it is convenient to employ a notation suited for $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariance. We define the oscillator $\mathbf{d}^{\dot{a}}$ with index $\dot{a} = 1, 2$

$$\mathbf{d}_1^\dagger = \mathbf{c}^4, \quad \mathbf{d}_2^\dagger = \mathbf{c}^3, \qquad \mathbf{d}^1 = \mathbf{c}_4^\dagger, \quad \mathbf{d}^2 = \mathbf{c}_3^\dagger. \tag{1.86}$$

The benefit of this notation is that now the highest weight state $|\mathcal{Z}\rangle = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger|0\rangle$, see (1.84), is annihilated by $\mathbf{a}_{1,2}, \mathbf{b}_{1,2}, \mathbf{c}_{1,2}, \mathbf{d}_{1,2}$. The drawback is that the notation breaks the $\mathfrak{su}(4)$ invariant notation and the expressions for the $\mathfrak{u}(2,2|2+2)$ generators thus complicate. Let us also state the central charge constraint

$$C = \tfrac{1}{2}(n_{\mathbf{b}} + n_{\mathbf{d}}) - \tfrac{1}{2}(n_{\mathbf{a}} - n_{\mathbf{c}}) = 0, \tag{1.87}$$

i.e. the number of $\mathbf{a}^\dagger, \mathbf{c}^\dagger$'s must equal the number of $\mathbf{b}^\dagger, \mathbf{d}^\dagger$'s.

In this context it is useful to know how to represent a state in terms of excitations of the oscillators. We introduce a multi-particle vacuum operator $|\mathcal{Z}, L\rangle$ which is the tensor product of L vacua $|\mathcal{Z}\rangle$. The oscillators $\mathbf{a}_{p,\alpha}^\dagger, \mathbf{b}_{p,\dot{\alpha}}^\dagger, \mathbf{c}_{p,a}^\dagger, \mathbf{d}_{p,\dot{a}}^\dagger$ now act on a site specified by p and commutators of two oscillators vanish unless they act on the same site. Equivalently, we define the unphysical multi-particle vacuum state $|0, L\rangle$. A generic state is written as

$$(\mathbf{a}^\dagger)^{n_{\mathbf{a}}}(\mathbf{b}^\dagger)^{n_{\mathbf{b}}}(\mathbf{c}^\dagger)^{n_{\mathbf{c}}}(\mathbf{d}^\dagger)^{n_{\mathbf{d}}}|\mathcal{Z}, L\rangle \qquad \text{or} \qquad (\mathbf{a}^\dagger)^{n_{\mathbf{a}}}(\mathbf{b}^\dagger)^{n_{\mathbf{b}}}(\mathbf{c}^\dagger)^{n_{\mathbf{c}}}|0, L\rangle. \tag{1.88}$$

The individual oscillator excitation numbers n for a state with given weight can be found in App. D.4.

1.10 Correlation Functions

In a conformal field theory, correlation functions of local operators obey certain relations due to conformal symmetry. These are especially tight for two-point and three-point

functions: For example, two-point functions are allowed only between multiplets of equal labels and involve only one free parameter. A similar result holds for superconformal symmetry, but we will not consider it explicitly because it would require working in superspace. For example, let us consider scalar primary or highest-weight operators $\mathcal{O}_{1,2,3}$ with scaling dimensions $D_{1,2,3}$ at points $x_{1,2,3}$. For two-point functions the dimensions must agree exactly $D = D_1 = D_2$, the correlator is

$$\big\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \big\rangle = \frac{M_{12}}{|x_{12}|^{2D}}\,, \tag{1.89}$$

where x_{ij} is the distance $x_i - x_j$. Three-point functions are constrained to

$$\big\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \big\rangle = \frac{C_{123}}{|x_{12}|^{D_1+D_2-D_3}\, |x_{23}|^{D_2+D_3-D_1}\, |x_{31}|^{D_3+D_1-D_2}}\,. \tag{1.90}$$

The structural uniqueness of those correlators can be understood by the fact that all configurations of two or three (non coinciding) points can be transformed to a standard set, say $\{0, 1\}$ and $\{-1, 0, 1\}$, by means of conformal transformations. The value of the correlator for this configuration determines the value for all configurations when the points are shifted back to $\{x_1, x_2\}$ or $\{x_1, x_2, x_3\}$.

For non-scalar primary operators the story is similar, but we have to take care of spacetime indices. Although we consider flat $\mathbb{R}^4$, from the point of view of conformal symmetry, spacetime is not flat; it is rather the coset space of the conformal group by the Poincaré group and dilatations. As such we cannot simply compare the tangent spaces at two different points, but we must introduce a connection. For spinor indices the connection is

$$J_{12}^{\dot\alpha\beta} = \frac{x_{12}^\mu \sigma_\mu^{\dot\alpha\beta}}{|x_{12}|}\,. \tag{1.91}$$

A vector may be represented as a bi-spinor and the vector connection is

$$J_{12}^{\mu\nu} = -\tfrac{1}{2}\sigma^\mu_{\dot\alpha\beta} J_{12}^{\dot\alpha\delta} J_{12}^{\beta\dot\gamma} \sigma^\nu_{\dot\gamma\delta} = \eta^{\mu\nu} - 2\,\frac{x_{12}^\mu x_{12}^\nu}{|x_{12}|^2}\,. \tag{1.92}$$

The two-point function for primary vector operators is thus

$$\big\langle \mathcal{O}_1^\mu(x_1)\, \mathcal{O}_2^\nu(x_2) \big\rangle = \frac{M_{12}\, J_{12}^{\mu\nu}}{|x_{12}|^{2D}}\,. \tag{1.93}$$

In addition to primary operators there are also descendant operators

$$\mathcal{O}'_{\mu\nu\ldots} = \mathfrak{P}_\mu \mathfrak{P}_\nu \ldots \mathcal{O}. \tag{1.94}$$

Although correlators of descendants follow immediately from the corresponding correlators of primaries by differentiation

$$\big\langle \mathcal{O}'_{\mu\nu\ldots}(x) \ldots \big\rangle = \partial_\mu \partial_\nu \ldots \big\langle \mathcal{O}(x) \ldots \big\rangle, \tag{1.95}$$

it is sometime hard to distinguish between primaries and descendants when mixing occurs. Therefore it is useful to know the difference in correlation functions explicitly. For example, the two point function of descendants of a scalar operator of dimension $D-1$ is

$$\big\langle \mathcal{O}'^\mu_1(x_1)\, \mathcal{O}'^\nu_2(x_2) \big\rangle = \partial_1^\mu \partial_2^\nu \frac{M_{12}}{|x_{12}|^{2D-2}} = \frac{2(D-1) M_{12} \big(J_{12}^{\mu\nu} - 2(D-1)\, x_{12}^\mu x_{12}^\nu / x_{12}^2 \big)}{|x_{12}|^{2D}}\,. \tag{1.96}$$

Up to normalisation this is similar to (1.93) but for the extra piece in the numerator. If mixing between primaries and descendants has not been resolved, one will see traces of the extra piece in all correlators.

Starting with four-point functions, conformal invariants appear, e.g.

$$s = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}, \quad t = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}. \tag{1.97}$$

Naturally, four-point functions may depend on s, t and their form is not fully restricted. However, in a conformal field theory one may expect to have an operator product expansion (OPE), which enables one to write products of two local operators at sufficiently close point x, $x + \delta x$ as a sum of local operators at point x

$$\mathcal{O}_1(x)\,\mathcal{O}_2(x+\delta x) = \sum\nolimits_k F_{12}^k\, e_{12,k}(\delta x)\,\mathcal{O}_k(x), \tag{1.98}$$

where $e_{12,k}(\delta x)$ is the conformal partial wave corresponding to the involved operators. The structure constants F_{12}^k can be determined by inserting this expression in the three-point function and comparing to the two-point function; roughly speaking one obtains

$$F_{kl}^n M_{nm} \sim C_{kln}. \tag{1.99}$$

Equivalently, one obtains for a four-point function where two pairs of points are close

$$\langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_3\,\mathcal{O}_4\rangle \sim \sum\nolimits_{kl} F_{12}^k F_{34}^l\, e_{12,k}(\delta x_{12}) e_{34,l}(\delta x_{34})\,\langle \mathcal{O}_k\,\mathcal{O}_l\rangle. \tag{1.100}$$

1.11 The Current Multiplet

Superconformal symmetry is an exact global symmetry. As such it should give rise to one conserved current $\mathcal{Q}_\mu$ for each of its generators $\mathfrak{J}$,

$$\partial^\mu \mathcal{Q}_\mu = 0. \tag{1.101}$$

In the Hamiltonian picture, a conserved charge Q is obtained as the integral of the time component $\mathcal{Q}_t$ over a time slice at t_0

$$Q = \int d^3x\, \mathcal{Q}_t(t_0, x). \tag{1.102}$$

The charge is indeed independent of the time slice t_0 due to the conservation of $\mathcal{Q}_\mu$. It acts as a symmetry generator $\mathfrak{J}$ when inserted within Poisson brackets $\mathfrak{J} = \{Q, \ldots\}$. Furthermore, the currents satisfy an algebra $\{\mathcal{Q}, \mathcal{Q}\} \sim \mathcal{Q}$, the *current algebra*. This gives rise to the symmetry algebra at the level of charges $\{Q, Q\} = \mathfrak{F}\, Q$. All this naturally translates into a quantum field theory in canonical quantisation.

The transformation properties of the current $\mathcal{Q}_{\mu\ldots}$ translate into the transformation properties of the generator $\mathfrak{J}$. For example, the conserved current associated to the momentum generator $\mathfrak{P}_\nu$ is the stress-energy tensor $\mathcal{Q}_{\mu\nu}$. It has canonical dimension 4, one for $\mathfrak{P}_\nu$ and three for d^3x in (1.102). The stress-energy tensor also gives rise to the currents corresponding to the other generators of the conformal algebra[26]

$$\mathcal{Q}^{\mathfrak{P}}_{\mu\nu} \sim \mathcal{Q}_{\mu\nu}, \quad \mathcal{Q}^{\mathfrak{L}\rho}_{\mu\ \ \sigma} \sim \eta^{\rho\nu}\mathcal{Q}_{\mu[\nu}x_{\sigma]}, \quad \mathcal{Q}^{\mathfrak{D}}_{\mu} \sim \mathcal{Q}_{\mu\nu}x^\nu, \quad \mathcal{Q}^{\mathfrak{K}}_{\mu\nu} \sim x^2\mathcal{Q}_{\mu\nu} - 2\mathcal{Q}_{\mu\rho}x^\rho x_\nu. \tag{1.103}$$

[26]Note that the stress energy tensor is part of the reducible module with highest weight $[1,0,1]$, the adjoint of $\mathfrak{su}(4)$, as described in Sec. 1.7

D	$\mathfrak{su}(2,2) \times \mathfrak{su}(4)$
2	$[0,-2,0] \times [0,2,0]$
2.5	$[1,-3,0] \times [0,1,1] + [0,-3,1] \times [1,1,0]$
3	$[2,-4,0] \times [0,1,0] + [1,-4,1] \times [1,0,1] + [0,-4,2] \times [0,1,0] +$
	$[0,-3,0] \times [0,0,2] + [0,-3,0] \times [2,0,0]$
3.5	$[2,-5,1] \times [1,0,0] + [1,-5,2] \times [0,0,1] +$
	$[1,-4,0] \times [0,0,1] + [0,-4,1] \times [1,0,0]$
4	$[2,-6,2] \times [0,0,0] + [0,-4,0] \times [0,0,0] + [0,-4,0] \times [0,0,0]$

Table 1.6: The supercurrent multiplet decomposed in $\mathfrak{su}(2,2) \times \mathfrak{su}(4)$.

One easily verifies that all currents $\mathcal{Q}_\mu$ are indeed conserved despite the appearance of x_ν. However, conservation of $\mathcal{Q}^{\mathfrak{D}}_\mu$ and $\mathcal{Q}^{\mathfrak{K}}_{\mu\nu}$ requires the trace of the stress-energy tensor to vanish

$$\eta^{\mu\nu}\mathcal{Q}_{\mu\nu} = 0. \tag{1.104}$$

In a quantum theory it is often impossible to construct a stress-energy tensor $\mathcal{Q}_{\mu\nu}$ which obeys (1.101) and (1.104) at the same time. This indicates the breakdown of conformal symmetry; only Poincaré symmetry remains because conservation of $\mathcal{Q}^{\mathfrak{L}\rho}_{\mu\ \sigma}$ and $\mathcal{Q}^{\mathfrak{P}}_{\mu\nu}$ is independent of the tracelessness of $\mathcal{Q}_{\mu\nu}$. In fact, the trace of the stress-energy tensor is related to the beta function (1.22).

Before we proceed to superconformal symmetry we note that it is useful to write the conserved currents in spinor notation. The stress-energy tensor becomes $\mathcal{Q}_{\dot\alpha\dot\gamma\beta\delta}$ which is symmetric in both pairs of indices. Now it is straightforward to construct the currents by contracting the indices of $\mathcal{Q}_{\dot\alpha\dot\gamma\beta\delta}$ by $x^\mu\sigma_\mu$, for example

$$\mathcal{Q}^{\mathfrak{D}}_{\dot\alpha\beta} \sim x^\mu\sigma_\mu^{\dot\gamma\delta}\mathcal{Q}_{\dot\alpha\dot\gamma\beta\delta}, \qquad \mathcal{Q}^{\mathfrak{K}\ \dot\gamma\delta}_{\dot\alpha\beta} \sim x^\mu x^\nu\sigma_\mu^{\dot\gamma\lambda}\sigma_\nu^{\delta\dot\kappa}\mathcal{Q}_{\dot\alpha\dot\kappa\beta\lambda}. \tag{1.105}$$

The currents are conserved due to the symmetry of the indices.

For superconformal symmetry there are four conserved currents

$$\mathcal{Q}_{\dot\alpha\beta}{}^c{}_d, \quad \mathcal{Q}^c_{\dot\alpha\beta\delta}, \quad \mathcal{Q}_{\dot\alpha\dot\gamma\beta d}, \quad \mathcal{Q}_{\dot\alpha\dot\gamma\beta\delta}. \tag{1.106}$$

The first three correspond to rotations $\mathfrak{R}^c{}_d$ and supertranslations $\mathfrak{Q}^c{}_\delta$ and $\dot{\mathfrak{Q}}_{\dot\gamma d}$. Furthermore, upon contraction with one x_μ the second and third ones correspond to $\dot{\mathfrak{S}}^{c\dot\delta}$ and $\mathfrak{S}^\gamma{}_d$. All of these currents are part of the supercurrent multiplet with highest weight

$$w_{\text{curr}} = (2;0,0;0,2,0;0,2) = [0;0;0,2,0;0;0]. \tag{1.107}$$

This multiplet, decomposed in terms of representations of $\mathfrak{su}(2,2) \times \mathfrak{su}(4)$, is presented in Tab. 1.6. The labels $[s_1, r, s_2]$ of $\mathfrak{su}(2,2)$ are the spins s_1, s_2 as well as $r = -D - \frac{1}{2}s_1 - \frac{1}{2}s_2$. It is worth noting that the two scalars at $D = 4$ are the on-shell Lagrangian $\mathcal{L}_{\text{YM}}$ and the topological charge density $\operatorname{Tr}\mathcal{F}\wedge\mathcal{F}$.

Chapter 2

The Dilatation Operator

The dilatation generator is a means to investigate scaling dimensions in a conformal field theory. We will start in Sec. 2.1 by comparing different methods of obtaining scaling dimensions. We will then go on to study aspects of the dilatation operator which will be useful in the following chapters. In Sec. 2.2 we will consider the symmetry algebra and states in perturbation theory. Sec. 2.3 contains an investigation of closed subsectors [50] and in Sec. 2.4 we will compute the one-loop dilatation operator within a subsector [38]. General perturbative contributions in field theory are investigated in Sec. 2.5. Finally, in Sec. 2.6 we will consider the planar limit and introduce the notation to be used in most parts of this dissertation.

2.1 Scaling Dimensions

There are many ways to calculate scaling dimensions for local operators in a conformal field theory. We will explain a few, paying special attention to two-point functions because their structure will be guiding us in the construction of the dilatation operator.

2.1.1 Two-Point Functions

In Sec. 1.10 we have described how scaling dimensions affect correlation functions. It seems that they appear in the most direct way within two-point functions, see (1.89). Let us make the dependence on the coupling constant in the two-point function explicit

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = \frac{M(g)}{|x_{12}|^{2D(g)}}\,. \tag{2.1}$$

We are aiming for a perturbative investigation and we can only expect to reproduce the form predicted by conformal symmetry in a series expansion in g. Here we pause and reconsider the above equation noting that $|x_{12}|$ is a dimensionful quantity and its exponent $-2D(g)$ depends on g. A perturbative expansion in g will lead to a formally meaningless expression involving $\log|x_{12}|$. This is related to the fact that the mass dimension of the operator changes with g. The only fully consistent way to treat this issue in a series expansion is to introduce an arbitrary scale μ and rescale $\mathcal{O}$ by $\mu^{-\delta D(g)}$ to a fixed mass

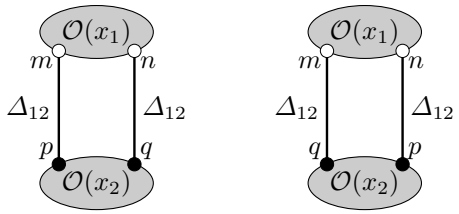

Figure 2.2: Tree-level contributions to the two-point function of $\mathcal{O}_{mn} = \operatorname{Tr} \Phi_m \Phi_n$.

dimension $D_0 = D(0)$. We can now expand and obtain

$$\mu^{-2\delta D(g)} \langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \frac{M_0}{|x_{12}|^{2D_0}} + g^2 \frac{M_2 + M_0 D_2 \log |\mu x_{12}|^{-2}}{|x_{12}|^{2D_0}} + \dots . \tag{2.2}$$

The very same problem occurs in perturbative quantum field theories and requires for the introduction of an auxiliary scale. Let us now go ahead and calculate the scaling dimension of the operator

$$\mathcal{O}_{mn} = \operatorname{Tr} \Phi_m \Phi_n. \tag{2.3}$$

Using the free generating functional (1.25) with scalar propagator

$$\Delta(x, y) = \frac{1}{(x-y)^2}\,, \tag{2.4}$$

the tree-level two point function is readily evaluated using SU(N) as gauge group and the diagrams in Fig. 2.2[1]

$$\begin{aligned} \langle \mathcal{O}_{mn}(x_1)\, \mathcal{O}_{pq}(x_2) \rangle &= \frac{\eta_{mp}\eta_{nq}\, \mathfrak{g}^{\mathfrak{mp}} \mathfrak{g}^{\mathfrak{nq}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{m}} \mathfrak{t}_{\mathfrak{n}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{p}} \mathfrak{t}_{\mathfrak{q}}}{N^2 |x_{12}|^4} + \frac{\eta_{mq}\eta_{np}\, \mathfrak{g}^{\mathfrak{mq}} \mathfrak{g}^{\mathfrak{np}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{m}} \mathfrak{t}_{\mathfrak{n}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{p}} \mathfrak{t}_{\mathfrak{q}}}{N^2 |x_{12}|^4} + \mathcal{O}(g) \\ &= \frac{2(1 - N^{-2})\, \eta_{m\{p}\eta_{q\}n}}{|x_{12}|^4} + \mathcal{O}(g). \end{aligned} \tag{2.5}$$

We can read off the classical dimension $D_0 = 2$ from this expression.

Trying to compute the one-loop correction we will inevitably fail and get a divergent result unless we first regularise the theory. We will chose the dimensional regularisation/reduction scheme in which we assume to have a $4 - 2\epsilon$ dimensional spacetime. The difference between the two schemes is that in dimensional regularisation we work with 6 internal directions, i.e. 6 flavours of scalars, whereas in dimensional reduction this number is assumed to be $6 + 2\epsilon$. The dimensional reduction scheme [94, 95] is convenient for regularising extended supersymmetric theories.[2] We will work with the action (see App. C.2 for the ten-dimensional Lagrangian)

$$S_{\text{DR}}[\mathcal{W}] = N \int \frac{d^{4-2\epsilon}x}{(2\pi)^{2-\epsilon}}\, \mathcal{L}_{\text{YM}}[\mathcal{W}, g\mu^{\epsilon}]. \tag{2.6}$$

[1] We have neglected contractions between fields at the same point. Their (divergent) contribution will have to be absorbed into the definition of $\mathcal{O}$. This is always possible and we will assume that this step has already been performed. In other words, the operator $\mathcal{O}$ is considered to be 'normal ordered'.

[2] Apparently, the dimensional reduction scheme proposed in [94] leads to problems at higher loops [96]. Certainly, at one-loop it is fine.

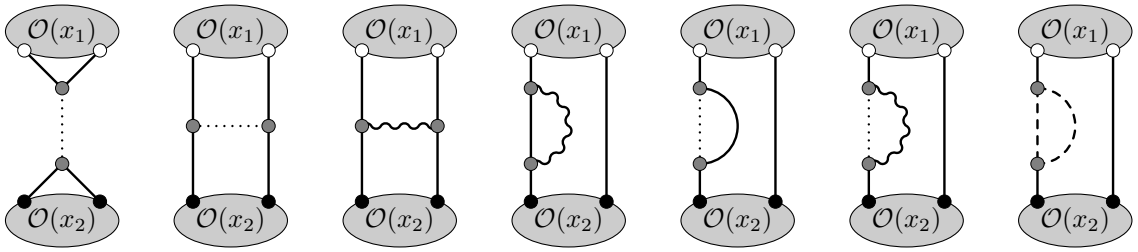

Figure 2.4: One-loop contributions to the two-point function. The solid, wiggly, dashed lines represent scalars, gluons, fermions, respectively. The dotted lines correspond to a non-propagating auxiliary field that represents a quartic interaction.

This action is dimensionless if the fields $\Phi, \mathcal{A}$ have canonical dimensions $1-\epsilon$ and Ψ has dimension $\frac{3}{2}-\epsilon$. The dimensionally regularised propagator is

$$\Delta(x,y) = \frac{2^{-\epsilon}\,\Gamma(1-\epsilon)}{|x-y|^{2-2\epsilon}}\,. \tag{2.7}$$

We need to evaluate a couple of diagrams, see Fig. 2.4, and find for the one-loop correlator

$$\begin{aligned}\big\langle \mathcal{O}_{mn}(x_1)\,\mathcal{O}_{pq}(x_2)\big\rangle &= 2(1-N^{-2})\,\eta_{m\{p}\eta_{q\}n}\,\Delta_{12}^2 \\ &\quad + (1-N^{-2})g^2\left(\tfrac{1}{2}\eta_{m\{p}\eta_{q\}n}\,\tilde{H}_{12,12} - \tfrac{1}{4}\eta_{mn}\eta_{pq}\,X_{1122}\right) + \mathcal{O}(g^3).\end{aligned} \tag{2.8}$$

The following functions and integrals appear at the one-loop level

$$\begin{aligned} I_{x_1x_2} &= \tfrac{1}{2}\Delta(x_1-x_2), \\ Y_{x_1x_2x_3} &= \mu^{2\epsilon}\int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1z}I_{x_2z}I_{x_3z}, \\ X_{x_1x_2x_3x_4} &= \mu^{2\epsilon}\int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1z}I_{x_2z}I_{x_3z}I_{x_4z}, \\ \tilde{H}_{x_1x_2,x_3x_4} &= \tfrac{1}{2}\mu^{2\epsilon}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)^2\int \frac{d^{4-2\epsilon}z_1\,d^{4-2\epsilon}z_2}{(2\pi)^{4-2\epsilon}}\,I_{x_1z_1}I_{x_2z_1}I_{z_1z_2}I_{z_2x_3}I_{z_2x_4}. \end{aligned} \tag{2.9}$$

where the shape of the letter represents the connections in terms of scalar propagators. In two-point functions they evaluate to [97]

$$Y_{112} = \frac{\xi\, I_{12}}{\epsilon(1-2\epsilon)}\,,\quad X_{1122} = \frac{2(1-3\epsilon)\kappa\,\xi\, I_{12}^2}{\epsilon(1-2\epsilon)^2}\,,\quad \tilde{H}_{12,12} = -\frac{2(1-3\epsilon)(\kappa-1)\,\xi\, I_{12}^2}{\epsilon^2(1-2\epsilon)}\,. \tag{2.10}$$

These involve two convenient combinations ξ, κ

$$\xi = \frac{\Gamma(1-\epsilon)}{\left|\frac{1}{2}\mu^2x_{12}^2\right|^{-\epsilon}}\,,\qquad \kappa = \frac{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)}{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)} = 1+6\zeta(3)\,\epsilon^3+\mathcal{O}(\epsilon^4). \tag{2.11}$$

At this point, it is useful to split up the operator into irreducible representations of $\mathfrak{so}(6)$. There are two, the symmetric-traceless $[0,2,0]$ and the singlet $[0,0,0]$

$$\mathcal{Q}_{mn} = \mathcal{O}_{(mn)} = \mathcal{O}_{mn} - \tfrac{1}{6+2\epsilon}\,\eta_{mn}\eta^{pq}\,\mathcal{O}_{pq},\qquad \mathcal{K} = \eta^{mn}\,\mathcal{O}_{mn}. \tag{2.12}$$

These have classical weights

$$w_{\mathcal{Q}} = (2;0,0;0,2,0;0,2), \qquad w_{\mathcal{K}} = (2;0,0;0,0,0;0,2), \tag{2.13}$$

which are indeed highest weights, essentially because there are no states of lower dimension. For the symmetric-traceless operator the correlator reduces to

$$\big\langle \mathcal{Q}_{mn}(x_1)\, \mathcal{Q}_{pq}(x_2)\big\rangle = 2(1-N^{-2})\, \eta_{m(p}\eta_{q)n}\, \big(\Delta_{12}^2 + \tfrac{1}{4}g^2\tilde{H}_{12,12}\big) + \mathcal{O}(g^3). \tag{2.14}$$

Here we can take the limit $\epsilon \to 0$, it turns out that the one-loop correction vanishes identically and we obtain precisely the tree-level result [31]. This remarkable cancellation is intimately related to the vanishing of the beta function. The operator $\mathcal{Q}$ is part of the half-BPS multiplet with weight (2.13), see Sec. 1.8. In fact it is part of the current multiplet of superconformal symmetry, see also Sec. 1.11, were it not protected, superconformal symmetry would be broken.

For the Konishi operator $\mathcal{K}$ the result is very different

$$\begin{aligned}\big\langle \mathcal{K}(x_1)\, \mathcal{K}(x_2)\big\rangle &= 4(1-N^{-2})(3+\epsilon)\, \big(\Delta_{12}^2 + \tfrac{1}{4}g^2\tilde{H}_{12,12} - \tfrac{1}{4}g^2(3+\epsilon)X_{1122}\big) + \mathcal{O}(g^3)\\ &= 4(1-N^{-2})(3+\epsilon)\Delta_{12}^2(1 - g^2\gamma\,\xi/\epsilon) + \mathcal{O}(g^3).\end{aligned} \tag{2.15}$$

We see that the coefficient

$$\gamma = \frac{2(1-3\epsilon)(\kappa-1)}{\epsilon(1-2\epsilon)} + \frac{2(3+\epsilon)(1-3\epsilon)\kappa}{(1-2\epsilon)^2} \to 6 \tag{2.16}$$

is finite in the limit $\epsilon \to 0$ and the correlator is thus ill-defined. We should first renormalise the operator in order to remove the $1/\epsilon$ pole

$$Z\mathcal{K} = (1 + \tfrac{1}{2}g^2\gamma\,\xi_0/\epsilon)\,\mathcal{K}, \quad \text{where} \quad \xi_0 = 2^{-\epsilon}\,\Gamma(1-\epsilon). \tag{2.17}$$

In a correlator of $Z\mathcal{K}$'s, this replaces ξ in (2.15) by $\xi - \xi_0$. We can now evaluate the one-loop term which is regular at $\epsilon \to 0$

$$\begin{aligned}-\,g^2\gamma \lim_{\epsilon\to 0}\frac{\xi-\xi_0}{\epsilon} &= -g^2\gamma \lim_{\epsilon\to 0}\frac{\big(|\mu x_{12}|^{-2}\big)^{-\epsilon}-1}{\epsilon}\,\xi_0\\ &= -g^2\gamma \lim_{\epsilon\to 0}\frac{\partial\big(|\mu x_{12}|^{-2}\big)^{-\epsilon}}{\partial\epsilon} = g^2\gamma\,\log|\mu x_{12}|^{-2}\end{aligned} \tag{2.18}$$

and take the limit. We find

$$\big\langle Z\mathcal{K}(x_1)\, Z\mathcal{K}(x_2)\big\rangle = \frac{12(1-N^{-2})}{|x_{12}|^4}\big(1 + 6g^2\log|\mu x_{12}|^{-2}\big) + \mathcal{O}(g^3). \tag{2.19}$$

By comparing to (2.2) we obtain the one-loop anomalous dimension $D_2 = 6$ or altogether, after inserting the definition $g^2 = g_{\text{YM}}^2 N/8\pi^2$ (1.14), in agreement with [39]

$$D = 2 + 6g^2 + \mathcal{O}(g^3) = 2 + \frac{3g_{\text{YM}}^2 N}{4\pi^2} + \mathcal{O}(g^3). \tag{2.20}$$

The calculation presented above resulted in the simplest non-trivial scaling dimension. In a generic computation one has to deal with more involved operator mixing and many more diagrams. We have seen only a glimpse of that here, luckily representation theory alone was sufficient to resolve the mixing.

2.1.2 Higher-Point Functions

There are other ways in which to obtain scaling dimensions. One could, for example, calculate three-point functions. They contain information not only about the scaling dimension of all three involved operators, but at the same time also about the coefficients C_{123}. These are interesting because they are related to the structure constants F_{12}^3 of the operator product expansion. The price one has to pay is the added difficulty due to the additional spacetime point in the Feynman diagrams. In practice three-point functions are rarely considered. More interesting are four-point functions, although they might appear even more difficult at first sight. The simplification comes about when one considers protected operators at all four points [41]. Using superspace techniques these correlators turn out to be manifestly finite without the need to regularise or renormalise [98, 41]. Furthermore, there are some constraints from superconformal field theory which can be used to reduce the complexity of the calculation [99]. Despite their simplicity, these four-point functions are interesting due to the OPE (c.f. Sec. 1.10) which allows for unprotected operators in the intermediate channel. A single four-point function can be shown to encode the information about scaling dimensions and also structure constants of infinitely many local operators [44]. A number of scaling dimensions, even at two-loops, have been obtained in this way [41, 42, 44–46].

2.1.3 Violation of Current Conservation

A completely different method to evaluate scaling dimensions due to Anselmi led to a few early results [40]. It is rather algebraic in nature and does not require quantum field theoretic computations as those presented above. It makes use of multiplet splitting at the unitarity bounds, see Sec. 1.8. Multiplet splitting occurs when the classical dimension D_0 is on one of the unitarity bounds. When δD is precisely zero, the multiplet splits up into several short multiplets. A superconformal generator which would usually translate between states of different submultiplets, must annihilate the state. Therefore, in the interacting theory the action of this generator is proportional to $\sqrt{\delta D} \sim g$ when g approaches zero.[3] When the states are properly normalised, the anomalous dimension δD can be read off from the action of the generator. In practice, to compute a one-loop anomalous dimension, this method requires to normalise the operators, i.e. their two-point functions, at tree level. For the generator one may use the semi-classical expression (1.9) which does involve the coupling constant. In principle, this trick allows also to obtain higher-loop anomalous dimensions from a field theory calculation at one loop below. However, one has to take into account modifications of the generators due to the Konishi anomaly [100].

2.1.4 The Dilatation Generator

The dilatation generator offers a different perspective on scaling dimensions. As described in Sec. 1.5, it measures the scaling dimension of states transforming under the superconformal algebra. In Sec. 1.6 we have emphasised that local operators can be viewed as such states in an abstract space. Therefore the dilatation generator $\mathfrak{D}$ should

[3] This square root explains why multiplet splitting takes place at the unitarity bound $\delta D \geq 0$: A negative δD would yield an imaginary action and thus violate unitarity.

yield the scaling dimension D when acting on an eigenstate $\mathcal{O}$. In particular, we have learned in Sec. 2.1.1 that

$$\mathfrak{D}\,\mathcal{Q}_{mn} = 2\,\mathcal{Q}_{mn}, \qquad \mathfrak{D}\,\mathcal{K} = (2+6g^2)\,\mathcal{K} + \mathcal{O}(g^3). \tag{2.21}$$

Clearly, the dilatation operator can act on any state, not just eigenstates. The action of $\mathfrak{D}$ on the mixed operator $\mathcal{O}_{mn}$ is

$$\mathfrak{D}\,\mathcal{O}_{mn} = (2\delta^p_m\delta^q_n + \eta_{mn}\eta^{pq}\,g^2)\,\mathcal{O}_{pq} + \mathcal{O}(g^3). \tag{2.22}$$

We obtain (2.21) when we project the indices to irreducible representations of $\mathfrak{so}(6)$.

So far not much is gained by considering the dilatation generator; we have merely rephrased the physical results of Sec. 2.1.1 into a single equation (2.22). Notice, however, that (2.22) describes the eigenoperators along with their scaling dimensions. In contrast, a two-point function also contains the normalisation coefficients. In practice, this fact is rather disadvantageous because the normalisation coefficients obscure the scaling dimension and their proper calculation usually involves a large amount of work. The dilatation generator clearly distinguishes between scaling dimensions and normalisation coefficients and thus avoids this complication.

To make true progress we need to find a way to obtain the action of the dilatation generator on the set of states in a more direct fashion. There are several ways in which this could be done. To compute classical scaling dimensions is a rather trivial task, we will describe how to implement this at the level of the classical dilatation operator $\mathfrak{D}_0$ in Sec. 2.2. Quantum corrections $\delta\mathfrak{D}$ to the dilatation generator are much harder to obtain. In Sec. 2.4 we will show how to extract some information from the calculation of a two-point function of abstract operators.

2.1.5 Canonical Quantisation

From the path integral point of view there seems to be no obvious way in which to represent the dilatation operator, but in the Hamiltonian formalism and its canonical quantisation there is. In that picture, the generators of the symmetry group correspond to conserved currents as explained in Sec. 1.11. In particular, the dilatation operator is given by

$$\mathfrak{D} = \int d^3x\, \mathcal{Q}_{t\mu}x^\mu, \tag{2.23}$$

where $\mathcal{Q}_{\mu\nu}$ is the stress energy tensor of $\mathcal{N}=4$ SYM. This we can apply to a local operator state $|\mathcal{O}(x)\rangle = \mathcal{O}^\dagger(x)|0\rangle$

$$\mathfrak{D}\,|\mathcal{O}(x)\rangle \tag{2.24}$$

and thus obtain its scaling dimension. However, it is questionable whether in practice this leads to a reasonable simplification as compared to Sec. 2.1.1. As the eigenvalues of the dilatation operator are finite, naively one might think that regularisation would be unnecessary. Unfortunately, the bare $\mathfrak{D}$ can only act on renormalised states $Z|\mathcal{O}\rangle$. When the dilatation operator is intended to act on bare states, we need to renormalise it instead

$$\mathfrak{D}_{\text{ren}} = Z^{-1}\,\mathfrak{D}_{\text{bare}}\,Z. \tag{2.25}$$

The renormalised $\mathfrak{D}_{\text{ren}}$ is finite and $\mathfrak{D}_{\text{bare}}$ and Z do not commute, therefore $\mathfrak{D}_{\text{bare}}$ is expected to diverge.

2.1.6 Matrix Quantum Mechanics

A nice representation for the dilatation operator is offered in gauge theory on the curved manifold $\mathbb{R} \times S^3$, which is conformally equivalent to flat $\mathbb{R}^4$. The map from $\mathbb{R}^4$ to $\mathbb{R} \times S^3$ is best described in radial coordinates on $\mathbb{R}^4$. The spherical coordinates map directly to S^3 whereas the radial coordinate r is mapped logarithmically to the coordinate t along $\mathbb{R}$ of $\mathbb{R} \times S^3$

$$(r, \theta, \phi, \psi) \mapsto (t, \theta, \phi, \psi) \quad \text{with} \quad t = \log r. \tag{2.26}$$

Therefore the dilatation generator, which generates scale transformations $r \mapsto cr$, maps straightforwardly to the Hamiltonian, i.e. the generator of time translations, on $\mathbb{R} \times S^3$. Spacetime rotations $\mathfrak{so}(4)$ naturally map to rotations of the sphere, whereas translations and boosts act on both, $\mathbb{R}$ and S^3. In this picture, it is natural to Kaluza-Klein decompose fields on a time-slice, S^3, in terms of spherical harmonics. For $\mathcal{N} = 4$ SYM this yields precisely the spectrum of fields as given in Tab. 1.4. The decomposition turns the quantum field theory into a quantum mechanical system of infinitely many matrices. This *matrix quantum mechanics* is equivalent to $\mathcal{N} = 4$ SYM and one could attempt to derive the dilatation operator in this system. Unfortunately, the Hamiltonian, which is derived as the Legendre transform of the Lagrangian, is not of the desired form, see Sec. 2.2.3. To perform the proposed diagonalisation might turn out to be very labourious in practice due to the infinite number of matrices.

A simpler model which appears to have a lot in common with $\mathcal{N} = 4$ SYM is the BMN matrix model [14]. It can be obtained from $\mathcal{N} = 4$ on $\mathbb{R} \times S^3$ by removing all non-singlet fields under one of the $\mathfrak{su}(2)$ symmetry algebras [101]. From the infinite set of fields in Tab. 1.4, only finitely many remain: $\mathcal{D}^0\Phi, \mathcal{D}^0\Psi, \mathcal{D}^0\mathcal{F}$ [101]. Explicit calculations up to a relatively high order in perturbation theory are feasible in this model and they show qualitative agreement with $\mathcal{N} = 4$ [101, 79], even if the results cannot agree in all cases due to the different multiplet structure.

2.2 Perturbation Theory

In this section we will investigate the corrections to the generators of the symmetry algebra in perturbation theory. Attention is payed to the dilatation operator which will take a special role.

2.2.1 Quantum Representations

The superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ is spanned by the generators $\mathfrak{J}$. They satisfy the algebra relations[4]

$$[\mathfrak{J}_{\mathfrak{A}}, \mathfrak{J}_{\mathfrak{B}}] = \mathfrak{F}^{\mathfrak{C}}_{\mathfrak{A}\mathfrak{B}}\, \mathfrak{J}_{\mathfrak{C}}, \tag{2.27}$$

where $\mathfrak{F}^{\mathfrak{C}}_{\mathfrak{A}\mathfrak{B}}$ are the structure constants of $\mathfrak{psu}(2,2|4)$. The generators can act on the set of states, or, more precisely, there is a representation which we shall also denote by $\mathfrak{J}$. When

[4] Although $\mathfrak{psu}(2,2|4)$ is a superalgebra, for the sake of presentation, we shall assume that all operators and fields are bosonic. Everything generalises to fermions in a straightforward fashion, but at the cost of obscure signs at various places.

quantum corrections are turned on, the transformation properties of states change. In other words, the representation $\mathfrak{J}(g)$ depends on the coupling constant g. For all values of g the generators must satisfy the $\mathfrak{psu}(2,2|4)$ algebra

$$[\mathfrak{J}_{\mathfrak{A}}(g), \mathfrak{J}_{\mathfrak{B}}(g)] = \mathfrak{F}^{\mathfrak{C}}_{\mathfrak{A}\mathfrak{B}}\, \mathfrak{J}_{\mathfrak{C}}(g). \tag{2.28}$$

The structure constants are, in particular, independent of the coupling constant. We will consider a perturbative quantum theory, therefore we shall expand the (representation of) generators in powers of the coupling constant

$$\mathfrak{J}(g) = \sum_{k=0}^{\infty} g^k\, \mathfrak{J}_k. \tag{2.29}$$

In perturbation theory the algebra relations can be written as

$$\sum_{k=0}^{l} [\mathfrak{J}_{\mathfrak{A},k}, \mathfrak{J}_{\mathfrak{B},l-k}] = \mathfrak{F}^{\mathfrak{C}}_{\mathfrak{A}\mathfrak{B}}\, \mathfrak{J}_{\mathfrak{C},l}. \tag{2.30}$$

Not all generators receive quantum corrections. The Lorentz and internal rotations $\mathfrak{su}(2)\times \mathfrak{su}(2), \mathfrak{su}(4)$ are manifest symmetries and thus independent of g. We do not intend to modify them

$$\mathfrak{L}(g)^{\alpha}{}_{\beta} = \mathfrak{L}^{\alpha}{}_{\beta}, \quad \dot{\mathfrak{L}}(g)^{\dot\alpha}{}_{\dot\beta} = \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}, \quad \mathfrak{R}(g)^{a}{}_{b} = \mathfrak{R}^{a}{}_{b}. \tag{2.31}$$

2.2.2 Tree-Level Algebra

Let us start by investigating the classical algebra spanned by $\mathfrak{J}_0 = \mathfrak{J}(0)$. In the classical theory the fields transform among themselves

$$\mathfrak{J}_0\, \mathcal{W}_{\mathcal{A}} = (\mathfrak{J}_0)_{\mathcal{A}}{}^{\mathcal{B}} \mathcal{W}_{\mathcal{B}}. \tag{2.32}$$

When interactions are turned off, none of the fields can feel the presence of the others in the state. Therefore it is natural for a state to transform in the tensor product representation of its composite fields. A generator $\mathfrak{J}_0$ of $\mathfrak{psu}(2,2|4)$ at tree-level can thus be written in terms of its action on a single field $\mathcal{W}_{\mathcal{A}}$ as

$$\mathfrak{J}_0\, \mathcal{W}_{\mathcal{A}} \cdots \mathcal{W}_{\mathcal{B}} = (\mathfrak{J}_0)_{\mathcal{A}}{}^{\mathcal{C}} \mathcal{W}_{\mathcal{C}} \cdots \mathcal{W}_{\mathcal{B}} + \ldots + (\mathfrak{J}_0)_{\mathcal{B}}{}^{\mathcal{C}} \mathcal{W}_{\mathcal{A}} \cdots \mathcal{W}_{\mathcal{C}}. \tag{2.33}$$

Using the notation of variations with respect to fields introduced in Sec. 1.3 we shall write this as

$$\mathfrak{J}_0 = (\mathfrak{J}_0)_{\mathcal{A}}{}^{\mathcal{B}}\, \mathrm{Tr}\, \mathcal{W}_{\mathcal{B}} \check{\mathcal{W}}^{\mathcal{A}}. \tag{2.34}$$

The variation will pick any of the fields within the state and replace it by the transformed field. In particular, the tree-level dilatation generator is

$$\mathfrak{D}_0 = \textstyle\sum_{\mathcal{A}} \dim(\mathcal{W}_{\mathcal{A}})\ \mathrm{Tr}\, \mathcal{W}_{\mathcal{A}} \check{\mathcal{W}}^{\mathcal{A}}. \tag{2.35}$$

This isolates any of the fields and returns the same state multiplied by the dimension of the field. When summed over all constituent fields, the dilatation operator returns the same state multiplied by the total dimension being the sum of constituent dimensions

$$\dim(\mathcal{W}_{\mathcal{A}} \cdots \mathcal{W}_{\mathcal{B}}) = \dim(\mathcal{W}_{\mathcal{A}}) + \ldots + \dim(\mathcal{W}_{\mathcal{B}}). \tag{2.36}$$

Similarly, we can determine the classical dimension of any operator X acting on the set of states

$$X = \mathcal{W}_{\mathcal{A}} \cdots \mathcal{W}_{\mathcal{B}} \check{\mathcal{W}}^{\mathcal{C}} \cdots \check{\mathcal{W}}^{\mathcal{D}}, \qquad [\mathfrak{D}_0, X] = \dim(X)\, X, \tag{2.37}$$

where the dimension is given by

$$\dim(X) = \dim(\mathcal{W}_{\mathcal{A}}) + \ldots + \dim(\mathcal{W}_{\mathcal{B}}) - \dim(\mathcal{W}_{\mathcal{C}}) - \ldots - \dim(\mathcal{W}_{\mathcal{D}}). \tag{2.38}$$

2.2.3 Pre-Diagonalisation

Our aim is to diagonalise the full dilatation operator $\mathfrak{D}(g)$. We cannot expect this to be possible at the level of generators. However, as a first step, we can obtain a dilatation generator $\mathfrak{D}(g)$ which commutes with the classical dimension

$$[\mathfrak{D}_0, \mathfrak{D}(g)] = 0. \tag{2.39}$$

This serves two purposes: On the practical side we will have to diagonalise $\mathfrak{D}(g)$ only on the subspace of states with equal classical dimension, which is most easily determined through (2.36). On the theoretical side, this removes the possibility of states decaying into the vacuum or being created from it. This would be an obstacle for the definition of a planar limit. For the rest of this work we will assume (2.39) to hold. This has an interesting side-effect, it specialises the dilatation generator $\mathfrak{D}$ with respect to the other generators $\mathfrak{J}$ of the superconformal algebra, see Sec. 2.2.4.

In dimensional regularisation we can take (2.39) for granted. If (2.39) does not hold from the beginning,[5] we can diagonalise $\mathfrak{D}(g)$ perturbatively with respect to $\mathfrak{D}_0$ by means of a similarity transformation

$$\mathfrak{J}(g) \mapsto T(g)\, \mathfrak{J}(g)\, T^{-1}(g). \tag{2.40}$$

This is possible on the operatorial level, i.e. without acting on explicit states, because all elementary interactions for the construction of $\mathfrak{D}_k$ have a definite dimension as given by (2.37,2.38). Let us state the resulting dilatation operator up to second order. Assume $\delta\mathfrak{D}(g) = \mathfrak{D}(g) - \mathfrak{D}_0$ decomposes as

$$\delta\mathfrak{D} = \sum\nolimits_d \delta\mathfrak{D}_d \quad \text{with} \quad \dim(\delta\mathfrak{D}_d) = d. \tag{2.41}$$

Then the transformation

$$T(g) = 1 + \sum\nolimits_{d\neq 0} \frac{1}{d}\, \delta\mathfrak{D}_d + \ldots \tag{2.42}$$

yields the diagonalised dilatation operator

$$\delta\mathfrak{D} \mapsto \delta\mathfrak{D}_0 + \sum\nolimits_{d\neq 0} \delta\mathfrak{D}_d\, \frac{1}{d}\, \delta\mathfrak{D}_{-d} + \ldots . \tag{2.43}$$

Note that this is merely standard perturbation theory: The first terms is the first order energy shift and the second term is the second order energy shift of two interactions connected by a propagator. For a given order in g the series terminates, because $\delta\mathfrak{D}$ is at least of first order.

[5] This is the case for the canonically quantised matrix quantum mechanics of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

2.2.4 The Hamiltonian

Conservation of classical dimensions by $\mathfrak{D}(g)$ also implies that the other interacting generators have a definite classical dimension

$$[\mathfrak{D}_0, \mathfrak{J}(g)] = \dim(\mathfrak{J})\,\mathfrak{J}(g), \tag{2.44}$$

which can be shown as follows: Let Π_d project to the states of classical dimension d. Then Π_d commutes with $\mathfrak{D}_m$ for arbitrary d, m due to (2.39). Now we project the algebra relation $[\mathfrak{D}(g), \mathfrak{J}(g)] = \dim(\mathfrak{J})\,\mathfrak{J}(g)$ to subspaces of dimension d, d' from the left and right, respectively, and expand in the coupling constant g. The contribution at $\mathcal{O}(g^l)$ reads

$$\sum_{k=1}^{l} \Pi_d[\mathfrak{D}_k, \mathfrak{J}_{l-k}]\Pi_{d'} = \big(\dim(\mathfrak{J}) - (d-d')\big)\Pi_d\,\mathfrak{J}_l\,\Pi_{d'}. \tag{2.45}$$

where we have moved the term with $k = 0$ from the left to the right hand side making use of $\Pi_d\mathfrak{D}_0 = \mathfrak{D}_0\Pi_d = d\Pi_d$. We assume that $[\mathfrak{D}_0, \mathfrak{J}_k] = \dim(\mathfrak{J})\,\mathfrak{J}_k$ for all $k < l$. This is equivalent to the statement $\Pi_d\,\mathfrak{J}_k\,\Pi_{d'} = 0$ for all $d-d' \neq \dim(\mathfrak{J})$. Choosing $d-d' \neq \dim(\mathfrak{J})$ in (2.45) we find that $\Pi_d\,\mathfrak{J}_l\,\Pi_{d'}$ must also vanish. The claim is proved by induction.

We can now combine (2.44) with the algebra relation (D.3)

$$[\mathfrak{D}(g), \mathfrak{J}(g)] = \dim(\mathfrak{J})\,\mathfrak{J}(g) \tag{2.46}$$

and infer that the anomalous dimension is conserved by the interacting algebra

$$[\mathfrak{J}(g), \delta\mathfrak{D}(g)] = 0. \tag{2.47}$$

Thus we have constructed a $\mathfrak{u}(1)$ charge $\delta\mathfrak{D}$ in addition to the superconformal algebra $\mathfrak{psu}(2,2|4)$. A very important consequence of (2.47) is that, *at leading order, the anomalous dilatation operator $\delta\mathfrak{D}$ must commute with the classical algebra $\mathfrak{J}_0$,*

$$[\mathfrak{D}_0, \mathfrak{D}_l] = 0. \tag{2.48}$$

We will see in Ch. 3 that the leading order is one-loop or g^2, i.e. $l = 2$. For some purposes, it will therefore turn out that $\mathfrak{D}_{k+l}$ should be treated on equal footing with $\mathfrak{J}_k$. To make this manifest we introduce the notion of the 'Hamiltonian' which is just the anomalous dilatation order shifted by $l = 2$ powers of g

$$\text{'}\textit{The Hamiltonian}\text{'}: \quad \mathcal{H}(g) = g^{-2}\delta\mathfrak{D}(g), \qquad [\mathfrak{J}(g), \mathcal{H}(g)] = 0. \tag{2.49}$$

The Hamiltonian is an invariant operator under superconformal symmetry. Note that its *leading order* is $\mathcal{H}_0$ and corresponds to *one-loop*, $\mathcal{H}_0 = \mathfrak{D}_2$. The eigenvalues of the Hamiltonian are called 'energies', E, and are related to the scaling dimension by

$$D(g) = D_0 + g^2 E(g). \tag{2.50}$$

2.2.5 Eigenstates

Let us investigate the eigenstates of the dilatation operator. For this purpose, we will introduce some basis of states $\mathcal{E}_{\mathcal{M}}$. We have seen that the classical dilatation operator $\mathfrak{D}_0$ commutes with $\mathfrak{D}(g)$. To find eigenstates of $\mathfrak{D}(g)$ we need to consider only a basis with fixed classical dimension D_0.[6] As there are only a finite number of fields with a dimension bounded from above, see Tab. 1.4, also the basis $\mathcal{E}_{\mathcal{M}}$ is finite. When we expand the operator in the basis as $\mathcal{O} = \mathcal{O}^{\mathcal{M}}\mathcal{E}_{\mathcal{M}}$, we can write the eigenstate equation in a finite matrix form. The matrix of scaling dimensions $D^{\mathcal{M}}{}_{\mathcal{N}}$ is obtained by acting with the dilatation operator on the basis

$$\mathfrak{D}_{\rm op}(g)\,\mathcal{E}_{\mathcal{M}} = \mathcal{E}_{\mathcal{N}}\,D^{\mathcal{N}}{}_{\mathcal{M}}(g). \tag{2.51}$$

We will often find such a basis and write down the action of the dilatation operator in matrix form. The eigenstate equation is turned into an eigenvector equation

$$D^{\mathcal{M}}{}_{\mathcal{N}}(g)\,\mathcal{O}^{\mathcal{N}}(g) = D_{\rm ev}(g)\,\mathcal{O}^{\mathcal{M}}(g). \tag{2.52}$$

In general the matrix depends on g and so should an eigenvector $\mathcal{O}^{\mathcal{M}}(g)$. We can expand the eigenstate equation in powers of the coupling constant, at l-th order we find

$$\sum_{k=0}^{l} D_k^{\mathcal{M}}{}_{\mathcal{N}}\,\mathcal{O}^{\mathcal{N}}_{l-k} = \sum_{k=0}^{l} D_{{\rm ev},k}\,\mathcal{O}^{\mathcal{M}}_{l-k}\,. \tag{2.53}$$

Note that we chose a basis of fixed classical dimension D_0, therefore $D_0^{\mathcal{M}}{}_{\mathcal{N}} = \delta^{\mathcal{M}}_{\mathcal{N}}\,D_0$. The equation for $l = 0$ naturally requires $D_{0,{\rm ev}} = D_0$. The equation at leading non-trivial order ($l = 2$) simplifies to

$$D_l^{\mathcal{M}}{}_{\mathcal{N}}\,\mathcal{O}_0^{\mathcal{N}} = D_{{\rm ev},l}\,\mathcal{O}_0^{\mathcal{M}}\,. \tag{2.54}$$

In general the eigenvalue problem is an algebraic equation which can only be solved numerically. Once that is done and the spectrum of $D_l^{\mathcal{M}}{}_{\mathcal{N}}$ happens to be non-degenerate, solving (2.53) for any value of l involves only linear algebra. If the leading order spectrum is degenerate, the diagonalisation of $D^{\mathcal{M}}_{l+1\mathcal{N}}$ in the degenerate subspace is again an eigenvalue problem. This continues as long as there are eigenvalues which are degenerate up to some order in perturbation theory.

The expansion of scaling dimensions is expected to be in even powers of g,

$$D_{\rm ev}(g) = D_{{\rm ev},0} + g^2 D_{{\rm ev},2} + g^4 D_{{\rm ev},4} + \dots\,, \tag{2.55}$$

even though $\mathfrak{D}_{\rm op}$ involves also odd powers

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \dots\,. \tag{2.56}$$

The odd powers play a special role as we will see in Ch. 5. This pattern may in principle be broken when there are degenerate eigenvalues at leading order. Similar problems occur in a double series expansion in g and $1/N$. There are cases in which the leading order degeneracy is lifted by both, higher-loop and higher genus effects. The eigenstates for expansion in g and $1/N$ are not expected to agree, consequently the double expansion will turn out to be inconsistent [38].

[6] The generators $\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}$ do not depend on g and commute with $\mathfrak{D}(g)$. Therefore one can also restrict to definite representations of $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$.

2.3 Subsectors

In principle it would be desirable to derive the dilatation operator which is valid for all fields of $\mathcal{N} = 4$ SYM. In most practical cases, however, this will turn out to be too involved. Therefore it is useful to know how to consistently restrict to subsectors of fields in such a way that $\mathfrak{D}(g)$ closes on the subsector. Within a subsector the number of fields as well as the symmetry algebra is reduced. This reduction of complexity leads to a simplification of the dilatation generator within the subsector. Thus, restricting to subsectors one can efficiently compute anomalous dimensions.

2.3.1 Construction of Subsectors

To construct subsectors, we note that the number of excitations in the oscillator picture, see Sec. 1.9,D.4 and Tab. D.2, naturally puts constraints on the weights of operators. Certainly, there cannot be negative excitations. Furthermore, the oscillators $\mathbf{c}^\dagger$ are fermionic, therefore there can only be one excitation on each site. In total we find twelve bounds

$$n_{\mathbf{a}} \geq 0, \quad n_{\mathbf{b}} \geq 0, \quad n_{\mathbf{c}} \geq 0, \quad n_{\mathbf{d}} = L - n_{\mathbf{c}} \geq 0. \tag{2.57}$$

All these excitation numbers will turn out to be conserved by $\mathfrak{D}(g)$ at the one-loop level (c.f. Sec. 3.1.3), i.e. they commute with $\mathfrak{D}_2$. This means that the action of the one-loop dilatation operator closes on operators with fixed excitation numbers. Therefore, we can construct 'one-loop subsectors' by considering operators for which several of the bounds are met and thus some of the oscillators are not excited. In some cases the subsectors remain closed even at higher loops. We will refer to these as 'closed subsectors'.

Let us investigate all closed subsectors. Using Tab. D.2 we can express the oscillator excitation numbers in terms of the charges $D_0, s_1, s_2, p, q_1, q_2, B, L$. We know that $\mathfrak{D}(g)$ commutes with the Cartan generators s_1, s_2, p, q_1, q_2 which are independent of the coupling constant. Also the classical dimension D_0 is preserved by construction, see Sec. 2.2.3. Only the charges B and L which are not part of $\mathfrak{psu}(2,2|4)$ do not commute with $D(g)$ in general. To construct a closed subsector we therefore need to find a positive linear combination of the bounds that is independent of B and L. Put differently, it must be independent of $L - B$ and $L + B$. The number of excitations $n_{\mathbf{a}}$ involves the combination $B - L$. This can only be cancelled by $L - B$ in $n_{\mathbf{c}}$. Therefore, we can remove oscillators of type $\mathbf{a}$ if and only if we also remove oscillators of type $\mathbf{c}$. Equivalently, we can remove oscillators of type $\mathbf{b}$ if and only if we also remove oscillators of type $\mathbf{d}$.

In the following we will construct all possible closed subsectors and determine the set of fields as well as the residual symmetry that transforms states within the subsector. Note that for local operators we can enhance the superconformal algebra by the anomalous dimension operator $\delta\mathfrak{D}(g) = \mathfrak{D}(g) - \mathfrak{D}_0$ and consider

$$\mathfrak{psu}(2,2|4) \times \mathfrak{u}(1) \tag{2.58}$$

as the full algebra.

2.3.2 The Half-BPS Subsector

Let us demonstrate how to obtain a rather trivial subsector. We will consider the subsector of operators with no oscillator excitations

$$n_{\mathbf{a}_1} = n_{\mathbf{a}_2} = n_{\mathbf{b}_1} = n_{\mathbf{b}_2} = n_{\mathbf{c}_1} = n_{\mathbf{c}_2} = n_{\mathbf{d}_1} = n_{\mathbf{d}_2} = 0. \tag{2.59}$$

Using Tab. D.2, the constraints (2.59) force the weight to be

$$w = (L; 0, 0; 0, L, 0; 0, L). \tag{2.60}$$

Here we have removed oscillators of all types, therefore the subsector is closed not only at one-loop but to all orders in perturbation theory. We can express the length in terms of a conserved charge, $L = p$, which implies that the length is protected even at higher loops. Equivalently, the hypercharge B is exactly zero.

In conventional language the operators within this subsector consist only of the highest weight of the field-strength multiplet

$$\mathcal{Z} = |\mathcal{Z}\rangle = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle = \Phi_{34}. \tag{2.61}$$

These are the half-BPS operators $\mathrm{Tr}\, \mathcal{Z}^L$ and its multi-trace cousins, the subsector will therefore be called 'half-BPS' subsector. The anomalous dilatation operator within this subsector vanishes identically, as required by protectedness of half-BPS operators. Note that almost all elements of a half-BPS multiplet are outside this subsector. The important point is that *every* half-BPS multiplet has one component, its highest weight, within this subsector. Due to superconformal invariance this is enough to obtain information about the complete supermultiplet.

The subalgebra of $\mathfrak{psu}(2,2|4) \times \mathfrak{u}(1)$ which closes on this subsector is $\mathfrak{psu}(2|2)^2 \times \mathfrak{u}(1)^3$. Effectively, however, the symmetry is only $\mathfrak{u}(1)$ which measures $p = D_0 = D = L$, the other factors act trivially. Therefore we will only consider

$$\mathfrak{u}(1) \tag{2.62}$$

as the residual symmetry.

2.3.3 Short Subsectors

Suppose we require either i or ii in

$$\begin{aligned} \text{i}: &\quad n_{\mathbf{a}_1} = n_{\mathbf{c}_1} = 0, \\ \text{ii}: &\quad n_{\mathbf{b}_1} = n_{\mathbf{d}_1} = 0, \end{aligned} \tag{2.63}$$

which is equivalent to

$$\begin{aligned} \text{i}: &\quad D_0 = s_1 + \tfrac{1}{2} q_2 + p + \tfrac{3}{2} q_1, \qquad L - B = D_0 - s_1, \\ \text{ii}: &\quad D_0 = s_2 + \tfrac{1}{2} q_1 + p + \tfrac{3}{2} q_2, \qquad L + B = D_0 - s_2. \end{aligned} \tag{2.64}$$

In perturbation theory ($D_0 \approx D$) the weight is beyond the unitarity bound (c.f. Sec. 1.8) and cannot be the highest weight state of a unitary multiplet of $\mathfrak{psu}(2,2|4)$.[7] However,

[7] The only exception is $s_{1,2} = 0$ and $\delta D = 0$ which is the highest weight of an eighth-BPS multiplet.

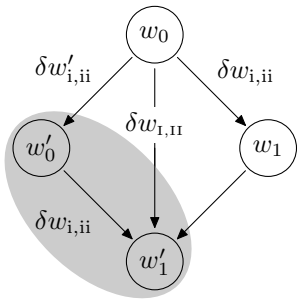

Figure 2.6: Structure of highest weights in short subsectors. The superconformal highest weight is w_0. It is at a unitary bound and w_1 is the highest weight of the splitting submultiplet. The short subsector is marked as shaded. The highest weight within is w'_0 and w'_1 is the highest weight of the submultiplet. For an eighth-BPS state at w_0, the multiplet at w'_0 is absent.

there is exactly one supersymmetry generator that decreases the combination $D - s_1 - \frac{1}{2}q_2 - p - \frac{3}{2}q_1$ and one that decreases $D - s_2 - \frac{1}{2}q_1 - p - \frac{3}{2}q_2$. These are $\mathfrak{Q}^1{}_2$ and $\dot{\mathfrak{Q}}_{24}$ and they shift a weight by

$$\begin{aligned} \delta w'_{\mathrm{i}} &= (+0.5; +1, 0; +1, 0, 0; +0.5, 0), & \delta(D - s_1 - \tfrac{1}{2}q_2 - p - \tfrac{3}{2}q_1) &= -2, \\ \delta w'_{\mathrm{ii}} &= (+0.5; 0, +1; 0, 0, +1; -0.5, 0), & \delta(D - s_2 - \tfrac{1}{2}q_1 - p - \tfrac{3}{2}q_2) &= -2. \end{aligned} \quad (2.65)$$

Due to the fermionic nature of the generators, the shift can only be applied once and the highest weight must be close to the unitarity bound. In the classical theory the dimensions are exactly at the unitarity bound and the multiplets become short. The subsectors i, ii will be called short subsectors, because all short multiplets of $\mathfrak{psu}(2,2|4)$ are represented by their highest weight shifted by the above $\delta w'_{\mathrm{i,ii}}$. Shortening also implies that the multiplet splits up, the weight of the additional submultiplet is reached from the highest weight by adding (1.76)[8]

$$\begin{aligned} \delta w_{\mathrm{i}} &= (+0.5; -1, 0; +1, 0, 0; -0.5, +1), & \delta(D_0 - s_1 - \tfrac{1}{2}q_2 - p - \tfrac{3}{2}q_1) &= 0, \\ \delta w_{\mathrm{ii}} &= (+0.5; 0, -1; 0, 0, +1; +0.5, +1), & \delta(D_0 - s_2 - \tfrac{1}{2}q_1 - p - \tfrac{3}{2}q_2) &= 0, \end{aligned} \quad (2.66)$$

which correspond to $\mathfrak{Q}^1{}_1$ and $\dot{\mathfrak{Q}}_{14}$.[9] An interesting aspect is that also the additional submultiplet has a descendant in the subsector. The descendants of the submultiplets in the subsector are thus related by $\mathfrak{Q}^1{}_1$ and $\dot{\mathfrak{Q}}_{14}$. In the classical theory these generators cannot act at all because the corresponding oscillators are disabled, see (2.63). In the quantum theory this changes and the submultiplets join. The relationship between the above highest weights is illustrated in Fig. 2.6.

The residual symmetry within this subsector is

$$\mathfrak{u}(1) \ltimes \mathfrak{psu}(1,2|3) \times \mathfrak{psu}(1|1) \ltimes \mathfrak{u}(1). \quad (2.67)$$

The $\mathfrak{psu}(1,2|3)$ subgroup classically transforms all oscillators except $\mathbf{a}_1, \mathbf{c}_1$ or $\mathbf{b}_1, \mathbf{d}_1$. The $\mathfrak{psu}(1|1)$ is associated to the supercharges which shift by $\pm\delta w_{\mathrm{i}}$ and relate the two submultiplets. The $\mathfrak{u}(1)$ charge $L \mp B$ and the $\mathfrak{u}(1)$ anomalous dimension δD are external automorphisms and central charges, respectively, for both $\mathfrak{psu}$ factors.

[8] In the case of $s_{1,2} = 0$ for the highest weight, the shift would lead to a negative $s_{1,2}$. In this particular case, $\delta w_{\mathrm{i,ii}} + \delta w'_{\mathrm{i,ii}}$ leads to the highest weight of the other submultiplet which is eighth-BPS.

[9] Note that the shifts have anomalous values $\delta B, \delta L$ and manifestly break the associated symmetries.

In addition to $n_{\mathbf{a}_1} = n_{\mathbf{c}_1} = 0$ we can further demand (similarly for the other subsector)

$$n_{\mathbf{c}_2} = 0, \quad n_{\mathbf{c}_2} = n_{\mathbf{c}_3} = 0 \quad \text{or} \quad n_{\mathbf{c}_2} = n_{\mathbf{c}_3} = n_{\mathbf{c}_4} = 0. \tag{2.68}$$

This restricts to states which have charges

$$q_1 = 0, \quad q_1 = p = 0 \quad \text{or} \quad q_1 = p = q_2 = 0 \tag{2.69}$$

and leads to even shorter subsectors with residual symmetries

$$\mathfrak{su}(1,2|2) \times \mathfrak{u}(1), \quad \mathfrak{su}(1,2|1) \times \mathfrak{u}(1) \quad \text{or} \quad \mathfrak{su}(1,2) \times \mathfrak{u}(1). \tag{2.70}$$

2.3.4 BPS Subsectors

Assume we now remove both oscillators of either type **a** or **b**

$$\begin{aligned} \text{I}: &\quad n_{\mathbf{a}_1} = n_{\mathbf{a}_2} = n_{\mathbf{c}_1} = 0, \\ \text{II}: &\quad n_{\mathbf{b}_1} = n_{\mathbf{b}_2} = n_{\mathbf{d}_1} = 0. \end{aligned} \tag{2.71}$$

Using Tab. D.2, these conditions are equivalent to

$$\begin{aligned} \text{I}: &\quad D_0 = \tfrac{1}{2}q_2 + p + \tfrac{3}{2}q_1, \quad s_1 = 0, \qquad L - B = D_0, \\ \text{II}: &\quad D_0 = \tfrac{1}{2}q_1 + p + \tfrac{3}{2}q_2, \quad s_2 = 0, \qquad L + B = D_0. \end{aligned} \tag{2.72}$$

For $D_0 = D$ these are precisely the eighth-BPS conditions, see Sec. 1.8, and therefore every eighth-BPS multiplet has components in these subsectors. Consequently we call them eighth-BPS sectors. In perturbation theory when $\delta D \approx 0$ the states are beyond the unitarity bound. As discussed below (2.66) we need to apply two supersymmetry generators $\varepsilon^{\alpha\beta}\mathfrak{Q}^1{}_\alpha\mathfrak{Q}^1{}_\beta$ or $\varepsilon^{\dot\alpha\dot\beta}\dot{\mathfrak{Q}}_{\dot\alpha 4}\dot{\mathfrak{Q}}_{\dot\beta 4}$ to reach any state within the subsector from the highest weight. The highest weight is shifted by

$$\begin{aligned} \delta w_{\text{I}} &= \delta w_{\text{i}} + \delta w'_{\text{i}} = (1;0,0;2,0,0;0,1), \\ \delta w_{\text{II}} &= \delta w_{\text{ii}} + \delta w'_{\text{ii}} = (1;0,0;0,0,2;0,1). \end{aligned} \tag{2.73}$$

The residual symmetry within this sector is

$$\mathfrak{su}(2|3) \times \mathfrak{u}(1), \tag{2.74}$$

where $\mathfrak{u}(1)$ corresponds to the anomalous dimension δD. Note that, as there are no oscillators of either type **a** or type **b**, we can only have a finite number of oscillator excitations for an elementary field. Therefore there are only finitely many fields within this subsector, for type II they are

$$\Phi_{a4} = \mathbf{c}^\dagger_a\mathbf{c}^\dagger_4|0\rangle, \qquad \Psi_{\alpha 4} = \mathbf{a}^\dagger_\alpha\mathbf{c}^\dagger_4|0\rangle, \tag{2.75}$$

with $a = 1,2,3$, $\alpha = 1,2$. These transform in the fundamental representation of $\mathfrak{su}(2|3)$. This sector will be discussed in detail in Ch. 5.

As opposed to Sec. 2.3.3 we cannot disable more of the **c**'s, here. Requiring $n_{\mathbf{a}_1} = n_{\mathbf{a}_2} = n_{\mathbf{c}_1} = n_{\mathbf{c}_2} = 0$ leads, via the central charge constraint, c.f. Sec. 1.9, to $n_{\mathbf{b}_1} = n_{\mathbf{b}_2} = n_{\mathbf{d}_1} = n_{\mathbf{d}_2} = 0$, i.e. the half-BPS sector of Sec. 2.3.2.

2.3.5 Combined Subsectors

We can also combine one of the restrictions on $\mathbf{a}, \mathbf{c}$ with a restriction on $\mathbf{b}, \mathbf{d}$. Let us denote the restrictions of Sec. 2.3.3 by $1, 2, 3, 4$ depending on how many of the $\mathbf{c}$'s or $\mathbf{d}$'s are removed. The BPS restriction of Sec. 2.3.4 will be denoted by 1^+. No restriction is denoted by 0. The possible subsectors are given by a pair of symbols (m, n). Not all combinations are possible, we cannot remove and fully excite one oscillator of the type $\mathbf{c}$ at the same time (fully exciting is equivalent to removing the corresponding oscillator of type $\mathbf{d}$). This yields the bound $m + n \leq 4$.

We find the following cases:

- The only subsector which does not fit this scheme is the half-BPS subsector $(2^+, 2^+)$ discussed in Sec. 2.3.2.
- We have already discussed all subsectors of type $(0, n)$ in Sec. 2.3.3 and Sec. 2.3.4.
- We can combine two eighth-BPS conditions $(1^+, 1^+)$ to the quarter-BPS subsector. We will discuss this one in Sec. 2.4.
- We can combine a short condition with an eighth-BPS condition in $(n, 1^+)$ for $n = 1, 2, 3$. The fields and residual symmetries are

$$\begin{aligned} (1,1^+) \qquad & \{\mathbf{c}_2^\dagger, \mathbf{c}_3^\dagger, \mathbf{a}_2^\dagger\}\mathbf{c}_4^\dagger|0\rangle, \qquad \mathfrak{su}(1|2) \times \mathfrak{u}(1|1), \\ (2,1^+) \qquad & \{\mathbf{c}_3^\dagger, \mathbf{a}_2^\dagger\}\mathbf{c}_4^\dagger|0\rangle, \qquad \mathfrak{u}(1|1) \times \mathfrak{u}(1), \\ (3,1^+) \qquad & \mathbf{a}_2^\dagger\mathbf{c}_4^\dagger|0\rangle, \qquad \mathfrak{u}(1) \times \mathfrak{u}(1). \end{aligned} \tag{2.76}$$

  In particular the sector $(1, 1^+)$ appears to be very interesting due to its high amount of supersymmetry in combination with only three fundamental fields. This might allow for higher loop calculations with a minimum amount of work, c.f. the treatment of the $(0, 1^+)$ sector in Ch. 5 of which this a subsector. The sector $(2, 1^+)$ has been investigated in [102] and found to be equivalent to free fermions in the one-loop approximation.
- There are four doubly-short sectors $(1, 1)$, $(2, 1)$, $(2, 2)$ and $(3, 1)$. We find the following fields and symmetries

$$\begin{aligned} (1,1) \quad & \{1, \mathbf{c}_2^\dagger\mathbf{d}_2^\dagger, \mathbf{a}_2^\dagger\mathbf{d}_2^\dagger, \mathbf{c}_2^\dagger\mathbf{b}_2^\dagger\}(\mathbf{a}_2^\dagger\mathbf{b}_2^\dagger)^n|\mathcal{Z}\rangle, \qquad \mathfrak{u}(1)^2 \ltimes \mathfrak{psu}(1,1|2) \times \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1), \\ (2,1) \quad & \{1, \mathbf{a}_2^\dagger\mathbf{d}_2^\dagger\}(\mathbf{a}_2^\dagger\mathbf{b}_2^\dagger)^n|\mathcal{Z}\rangle, \qquad \mathfrak{su}(1,1|1) \times \mathfrak{u}(1|1), \\ (3,1) \quad & \mathbf{a}_2^\dagger\mathbf{d}_2^\dagger(\mathbf{a}_2^\dagger\mathbf{b}_2^\dagger)^n|\mathcal{Z}\rangle, \qquad \mathfrak{su}(1,1) \times \mathfrak{u}(1|1), \\ (2,2) \quad & (\mathbf{a}_2^\dagger\mathbf{b}_2^\dagger)^n|\mathcal{Z}\rangle, \qquad \mathfrak{su}(1,1) \times \mathfrak{u}(1) \times \mathfrak{u}(1). \end{aligned} \tag{2.77}$$

  Of particular interest is the sector $(3, 1)$ which allows for a determination of the one-loop dilatation operator by purely algebraic means. It will be discussed in Sec. 3.2. The sector $(2, 2)$ is quite similar to $(3, 1)$ and also very useful, we will discuss it in Sec. 3.4. The sector $(2, 1)$ combines the two.

2.3.6 Excitation Subsector

Instead of removing oscillators of certain kinds, we can also fix the number of oscillator excitations to some value. Here we will consider only the total number of oscillator

excitations above the physical vacuum $|\mathcal{Z}, L\rangle$; this is an even number because oscillators can only be excited in pairs due to the central charge constraint. A state with $2M$ oscillator excitations will be said to have M *excitations*

$$\text{`}M \text{ excitations'}: \quad (\mathbf{A}^\dagger)^M(\dot{\mathbf{A}}^\dagger)^M|\mathcal{Z}, L\rangle, \tag{2.78}$$

where $\mathbf{A} = (\mathbf{a}, \mathbf{c})$ and $\dot{\mathbf{A}} = (\mathbf{b}, \mathbf{d})$. According to Tab. D.2 the excitation number is related to the charges by

$$M = \tfrac{1}{2}(n_{\mathbf{a}} + n_{\mathbf{b}} + n_{\mathbf{c}} + n_{\mathbf{d}}) = D_0 - p - \tfrac{1}{2}q_1 - \tfrac{1}{2}q_2, \tag{2.79}$$

it is thus exactly conserved by the dilatation operator. In other words, the sector of states with M excitations is closed. This type of sector is different from the above subsectors in that no type of oscillator is excluded. Instead, there is an upper bound on the number of excitations on a single field; this also leads to a simplification of the representation of generators.

The excitation subsectors are somewhat similar to the half-BPS subsector discussed in Sec. 2.3.2, which is in fact the sector with zero excitations. The residual symmetry in this type of subsector is

$$\big(\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \ltimes \mathfrak{u}(1)\big) \times \mathfrak{u}(1). \tag{2.80}$$

The generators of $\mathfrak{psu}(2|2)$ are given by a pair of $\mathfrak{su}(2)$ generators $\mathfrak{L}^\alpha{}_\beta, \mathfrak{R}^a{}_b$ and a pair of supercharges $\mathfrak{Q}^a{}_\alpha, \mathfrak{S}^\alpha{}_a$. Classically, they transform between oscillators $\mathbf{a}$ and $\mathbf{c}$. Equivalently, the other $\mathfrak{psu}(2|2)$ is given by $\dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}, \dot{\mathfrak{R}}^{\dot a}{}_{\dot b}, \dot{\mathfrak{Q}}^{\dot a}{}_{\dot\alpha}, \dot{\mathfrak{S}}^{\dot\alpha}{}_{\dot a}$. A $\mathfrak{u}(1)$ external automorphism for both $\mathfrak{psu}(2|2)$'s is given by $\mathfrak{D}_0$. The $\mathfrak{u}(1)$ central charge for both $\mathfrak{psu}(2|2)$'s is given by $M + \delta\mathfrak{D}$. Another central charge is $\delta\mathfrak{D}$. The four sets of $\mathfrak{su}(2)$ generators transform indices canonically. The non-vanishing anticommutators of supergenerators are given by

$$\begin{aligned}
\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\} &= \delta^b_a\mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta\mathfrak{R}^b{}_a + \tfrac{1}{2}\delta^b_a\delta^\alpha_\beta(M + \delta\mathfrak{D}),\\
\{\dot{\mathfrak{S}}^{\dot\alpha}{}_{\dot a}, \dot{\mathfrak{Q}}^{\dot b}{}_{\dot\beta}\} &= \delta^{\dot b}_{\dot a}\dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}_{\dot\beta}\dot{\mathfrak{R}}^{\dot b}{}_{\dot a} + \tfrac{1}{2}\delta^{\dot b}_{\dot a}\delta^{\dot\alpha}_{\dot\beta}(M + \delta\mathfrak{D}).
\end{aligned} \tag{2.81}$$

The Dynkin labels of a weight of one of the $\mathfrak{psu}(2|2)$'s are given by $(i = 1, 2)$

$$[s_i; r_i; q_i], \qquad r_i = \tfrac{1}{2}M + \tfrac{1}{2}\delta D + \tfrac{1}{2}s_i - \tfrac{1}{2}q_i. \tag{2.82}$$

For a unitary representation the highest weights should obey $r_i \geq s_i + 1$ or $s_i = r_i = 0$. A multiplet is short for $r_i = s_i + 1$ and BPS for $s_i = r_i = 0$. At the unitarity bound $r_i = s_i + 1$, a long multiplet $[s_i; r_i; q_i]$ splits off a short multiplet $[s_i - 1; r_i - 1; q_i + 1]$ or, when $s_i = 0$, a BPS multiplet $[0; r_i - 1; q_i + 2]$.

A subsector of this kind is suited perfectly to investigate plane-wave physics and BMN operators [14]. The number of excitations M equals the classical BMN energy $D_0 - J$ or *impurity* number. The residual symmetry in this sector maps directly to parts of the symmetries of the dual plane-wave string theory.

2.4 The $\mathfrak{su}(2)$ Quarter-BPS Sector

In this section we will demonstrate how to extract the dilatation generator from a perturbative calculation of the two-point function. We will restrict to the one-loop level and to the quarter-BPS subsector.

2.4.1 The $\mathfrak{su}(2)$ Subsector

First of all let us describe the subsector. The quarter-BPS subsector is obtained by combining both eighth-BPS conditions described in Sec. 2.3.4.

$$n_{\mathbf{a}_1} = n_{\mathbf{a}_2} = n_{\mathbf{b}_1} = n_{\mathbf{b}_2} = n_{\mathbf{c}_1} = n_{\mathbf{d}_1} = 0. \tag{2.83}$$

There are only two charged scalar fields in this subsector

$$\begin{aligned} \mathcal{Z} := \varphi_1 := \Phi_{34} &= \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle = |\mathcal{Z}\rangle, \\ \phi := \varphi_2 := \Phi_{24} &= \mathbf{c}_2^\dagger \mathbf{c}_4^\dagger |0\rangle = \mathbf{c}_2^\dagger \mathbf{d}_2^\dagger |\mathcal{Z}\rangle, \end{aligned} \tag{2.84}$$

therefore it is the smallest non-trivial subsector and we will often make use of it. The possible weights are

$$w = (L; 0, 0; K, L-2K, K; 0, L), \tag{2.85}$$

where K counts the number of ϕ's and L is the total number of fields. The residual symmetry is

$$\mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1). \tag{2.86}$$

The $\mathfrak{su}(2)$ factor transforms $\varphi_1 = \mathcal{Z}$ and $\varphi_2 = \phi$ in the fundamental representation, whereas the $\mathfrak{u}(1)$'s measure the classical dimension $D_0 = L$ and the anomalous dimension δD. With respect to $\mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$ a state is thus described by the charges

$$[L-2K], \quad L, \quad \delta D, \tag{2.87}$$

where $[L-2K]$ is the Dynkin label[10] of $\mathfrak{su}(2)$ corresponding to a third component of spin $L/2 - K$. In terms of the superconformal algebra, a state with $\delta D = 0$ is (at least) quarter-BPS, a generic state, however, will not be protected. In that case the weight w is beyond the unitarity bounds and cannot be primary. The highest weight within the subsector is obtained from the highest weight of the $\mathfrak{psu}(2,2|4)$ multiplet by a shift of

$$\delta w_{\mathrm{I+II}} = (2; 0, 0; 2, 0, 2; 0, 2). \tag{2.88}$$

The $\mathfrak{psu}(2,2|4)$ highest weight is on both unitarity bounds and has no spin.

2.4.2 Tree-Level

We will now compute the two-point function of states within the $\mathfrak{su}(2)$ sector. From the formal expression we will then extract the dilatation operator. Let us start at tree-level. The state $\mathcal{O}_\varphi$ at point x_1 is constructed from fields φ_i of the $\mathfrak{su}(2)$ subsector. Conversely, the other operator $\dot{\mathcal{O}}_{\dot\varphi}$ at point x_2 is constructed from fields $\dot\varphi^i$ of a conjugate $\mathfrak{su}(2)$ subsector. Note the charge conjugation requires us to use two different $\mathfrak{su}(2)$ subsectors. The operators are constructed as (not necessarily equal) products of traces of fields

$$\begin{aligned} \mathcal{O}_\varphi[\mathcal{W}] &= \operatorname{Tr} \varphi_* \cdots \varphi_* \operatorname{Tr} \varphi_* \cdots \varphi_* \ldots, \qquad \varphi_i = \varphi_i(x_1), \\ \dot{\mathcal{O}}_{\dot\varphi}[\mathcal{W}] &= \operatorname{Tr} \dot\varphi^* \cdots \dot\varphi^* \operatorname{Tr} \dot\varphi^* \cdots \dot\varphi^* \ldots, \qquad \dot\varphi^i = \dot\varphi^i(x_2). \end{aligned} \tag{2.89}$$

[10] The Dynkin label for $\mathfrak{su}(2)$ equals twice the spin.

Written in this way, the operators become abstract objects in the tensor product space of fields and are not necessarily based at some point in spacetime.

According to the path integral (c.f. Sec. 1.2) the two-point function at tree-level is given by

$$\langle \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi} \rangle = \exp(W_0[\partial/\partial\mathcal{W}]) \, \mathcal{O}_\varphi[\mathcal{W}] \, \dot{\mathcal{O}}_{\dot\varphi}[\mathcal{W}]\big|_{\mathcal{W}=0} + \mathcal{O}(g). \tag{2.90}$$

In fact, we do not need to work with generic x-dependent fields $\mathcal{W}$, but only the values of the scalar fields $\varphi, \dot\varphi$ at points $x_{1,2}$ are relevant. The correlator now becomes

$$\begin{aligned} \langle \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi} \rangle &= \exp\big(W_0(x_{12}, \partial/\partial\varphi, \partial/\partial\dot\varphi)\big) \, \mathcal{O}_\varphi(\varphi) \, \dot{\mathcal{O}}_{\dot\varphi}(\dot\varphi)\big|_{\varphi=\dot\varphi=0} + \mathcal{O}(g) \\ &= \exp\big(W_0(x_{12}, \check\varphi, \check{\dot\varphi})\big) \, \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0} + \mathcal{O}(g), \end{aligned} \tag{2.91}$$

where W_0 is the free generating functional of connected Graphs

$$W_0(x_{12}, \check\varphi, \check{\dot\varphi}) = N^{-1} \Delta_{12} \operatorname{Tr} \check\varphi_i \check{\dot\varphi}^i. \tag{2.92}$$

The scalar propagator $\Delta_{12} = \Delta(x_1, x_2)$ is defined in (2.7). Note that the second line in (2.91) merely involves performing ordinary derivatives $\check\varphi, \check{\dot\varphi}$ with respect to the matrices $\varphi, \dot\varphi$. In order for the result to be non-vanishing, all the fields φ in $\mathcal{O}_\varphi$ need to be contracted to fields $\dot\varphi$ in $\dot{\mathcal{O}}_{\dot\varphi}$ with propagators Δ_{12}. In particular, the numbers of fields of the two states must be equal.

2.4.3 One-Loop

To compute higher-loop correlators, we insert the interactions $S_{\text{int}}[g, \mathcal{W}]$ into the path integral

$$\langle \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi} \rangle = \exp(W_0[\partial/\partial\mathcal{W}]) \, \exp(-S_{\text{int}}[g, \mathcal{W}]) \, \mathcal{O}_\varphi[\mathcal{W}] \, \dot{\mathcal{O}}_{\dot\varphi}[\mathcal{W}]\big|_{\mathcal{W}=0}. \tag{2.93}$$

All the fields $\mathcal{W}$ in S_{int} need to be contracted to propagators before setting $\mathcal{W} = 0$, therefore we can combine the first two exponentials into one and write

$$\begin{aligned} \langle \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi} \rangle &= \exp(W[g, \partial/\partial\mathcal{W}]) \, \mathcal{O}_\varphi[\mathcal{W}] \, \dot{\mathcal{O}}_{\dot\varphi}[\mathcal{W}]\big|_{\mathcal{W}=0} \\ &= \exp\big(W(g, x_{12}, \check\varphi, \check{\dot\varphi})\big) \, \mathcal{O}_\varphi \, \mathcal{O}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0}. \end{aligned} \tag{2.94}$$

Again, it will be sufficient to evaluate the full generating functional of connected graphs $W[g, \mathcal{J}]$ only for fields $\varphi, \dot\varphi$ at points $x_{1,2}$.

Let us now consider the connected graphs at one-loop. There is no diagram at $\mathcal{O}(g)$ which conserves the charges. The $\mathcal{O}(g^2)$ connected Green functions are depicted in Fig. 2.8. To evaluate them we make use of the regularised $\mathcal{N} = 4$ SYM action in (2.6,1.5). The one-loop Green functions evaluate to

$$\begin{aligned} W_{2,a} &= \tfrac{1}{32} N^{-3} \, X_{1122} \operatorname{Tr}[\check{\dot\varphi}_i, \check{\dot\varphi}_j][\check\varphi^i, \check\varphi^j], \\ W_{2,b} &= \tfrac{1}{32} N^{-3} \, X_{1122} \operatorname{Tr}[\check{\dot\varphi}_i, \check\varphi^j][\check\varphi^i, \check{\dot\varphi}_j], \\ W_{2,c} &= \tfrac{1}{32} N^{-3} \big(-2\tilde H_{12,12} - 4 Y_{112} I_{12} + X_{1122}\big) \operatorname{Tr}[\check{\dot\varphi}_i, \check\varphi^i][\check{\dot\varphi}_j, \check\varphi^j], \\ W_{2,d} &= -\tfrac{1}{4} N^{-2} \, Y_{112} \, \mathfrak{g}^{\mathfrak{mn}} \operatorname{Tr}[\check{\dot\varphi}_i, \mathfrak{t}_{\mathfrak{m}}][\mathfrak{t}_{\mathfrak{n}}, \check\varphi^i] \end{aligned} \tag{2.95}$$

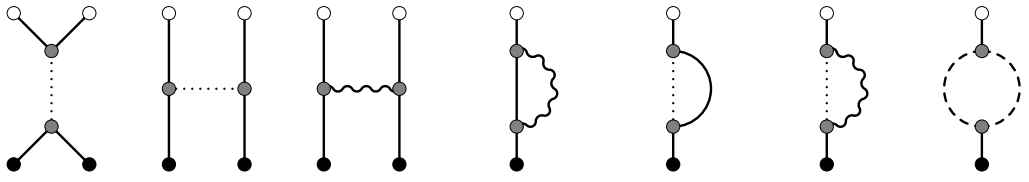

Figure 2.8: Connected graphs at one-loop. The solid, wiggly and dashed lines represent scalars, gluons and fermions, respectively. The dotted lines correspond to a non-propagating auxiliary field that represents a quartic interaction. The diagrams display the combinatorial structure with respect to the gauge group rather than their space-time configuration: The white and black dots are at the spacetime points x_1 and x_2, respectively.

with the integrals $X, Y, \tilde{H}$ defined in (2.9). We use a Jacobi identity to transform the second structure in $W_{2,b}$

$$\mathrm{Tr}\,[\check{\dot{\varphi}}_i, \check{\varphi}^j][\check{\varphi}^i, \check{\dot{\varphi}}_j] = \mathrm{Tr}\,[\check{\dot{\varphi}}_i, \check{\dot{\varphi}}_j][\check{\varphi}^i, \check{\varphi}^j] - \mathrm{Tr}\,[\check{\varphi}^i, \check{\dot{\varphi}}_i][\check{\varphi}^j, \check{\dot{\varphi}}_j] \tag{2.96}$$

and order the terms according to their spacetime integrals

$$\begin{aligned} W_{2,X} &= \tfrac{1}{16} N^{-3}\, X_{1122}\, \mathrm{Tr}\,[\check{\dot{\varphi}}_i, \check{\dot{\varphi}}_j][\check{\varphi}^i, \check{\varphi}^j], \\ W_{2,H} &= -\tfrac{1}{16} N^{-3}\, \tilde{H}_{12,12}\, \mathrm{Tr}\,[\check{\dot{\varphi}}_i, \check{\varphi}^i][\check{\dot{\varphi}}_j, \check{\varphi}^j], \\ W_{2,IY} &= -\tfrac{1}{8} N^{-3}\, I_{12} Y_{112} \big(\mathrm{Tr}\,[\check{\dot{\varphi}}_i, \check{\varphi}^i][\check{\dot{\varphi}}_j, \check{\varphi}^j] + N \Delta_{12}^{-1} \mathfrak{g}^{\mathfrak{mn}}\, \mathrm{Tr}\,[\check{\dot{\varphi}}_i, \mathfrak{t}_\mathfrak{m}][\mathfrak{t}_\mathfrak{n}, \check{\varphi}^i]\big). \end{aligned} \tag{2.97}$$

We refrain from evaluating these functions until later and insert them as they stand into the expression for the one-loop correlator

$$\big\langle \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi} \big\rangle = \exp\big(W_0(x_{12}, \check{\varphi}, \check{\dot{\varphi}})\big)\big(1 + g^2 W_2(x, \check{\varphi}, \check{\dot{\varphi}})\big)\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0} + \mathcal{O}(g^3). \tag{2.98}$$

We now change the argument $\check{\dot{\varphi}}$ of W_2 to $N\Delta_{12}^{-1}\varphi$. This can be done because the result vanishes unless every φ is removed by some $\check{\varphi}$ before the fields φ are set to zero. Here, the only possibility is to contract with W_0 which effectively changes $N\Delta_{12}^{-1}\varphi$ back to $\check{\dot{\varphi}}$. In doing so we need to make sure that no new contractions appear between the arguments φ and $\check{\varphi}$ of W_2. Formally, this is achieved by 'normal ordering'. The correlator becomes

$$\big\langle \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi} \big\rangle = \exp\big(W_0(x, \check{\varphi}, \check{\dot{\varphi}})\big)\big(1 + g^2 V_{2,\varphi}(x_{12})\big)\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0} + \mathcal{O}(g^3) \tag{2.99}$$

with the one-loop effective vertex

$$V_{2,\varphi}(x_{12}) = {:}W_2(x_{12}, \check{\varphi}, N\Delta_{12}^{-1}\varphi){:}. \tag{2.100}$$

We transform the explicit expressions for the connected graphs (2.97) and obtain

$$\begin{aligned} V_{2,X} &= \tfrac{1}{4} N^{-1}\, X_{1122} I_{12}^{-2}\, {:}\mathrm{Tr}\,[\varphi_i, \varphi_j][\check{\varphi}^i, \check{\varphi}^j]{:}, \\ V_{2,H} &= -\tfrac{1}{4} N^{-1}\, \tilde{H}_{12,12} I_{12}^{-2}\, {:}\mathrm{Tr}\,[\varphi_i, \check{\varphi}^i][\varphi_j, \check{\varphi}^j]{:}, \\ V_{2,IY} &= -\tfrac{1}{2} N^{-1} Y_{112} I_{12}^{-1} \big({:}\mathrm{Tr}\,[\varphi_i, \check{\varphi}^i][\varphi_j, \check{\varphi}^j]{:} + \mathfrak{g}^{\mathfrak{mn}}{:}\mathrm{Tr}\,[\varphi_i, \mathfrak{t}_\mathfrak{m}][\mathfrak{t}_\mathfrak{n}, \check{\varphi}^i]{:}\big). \end{aligned} \tag{2.101}$$

We can change the normal ordering in the first term of $V_{2,IY}$ in order to absorb the second, see (1.41)

$$V_{2,IY} = -\tfrac{1}{2} N^{-1} Y_{112} I_{12}^{-1}\ \mathrm{Tr}{:}[\varphi_i, \check{\varphi}^i]{::}[\varphi_j, \check{\varphi}^j]{:} = \tfrac{1}{2} N^{-1} Y_{112} I_{12}^{-1}\ \mathrm{Tr}\,\mathfrak{j}\mathfrak{j}. \tag{2.102}$$

We can thus write $V_{2,IY}$ in terms of the generator of gauge rotations $\mathfrak{j} = i{:}[\varphi_i, \check{\varphi}^i]{:}$ within the $\mathfrak{su}(2)$ subsector, see (1.37). Therefore $V_{2,IY}$ does not act on gauge invariant objects such as the states $\mathcal{O}$ and we can drop it altogether, $V_{2,IY} \mathrel{\widehat{=}} 0$.

Instead of replacing $\check{\dot{\varphi}}$ we could also have replaced $\check{\varphi}$ resulting in the effective vertex

$$\dot{V}_{2,\dot{\varphi}}(x_{12}) = {:}W_2(x_{12}, N\Delta_{12}^{-1}\dot{\varphi}, \check{\dot{\varphi}}){:}. \tag{2.103}$$

This shows that in (2.99) $V_{2,\varphi}$ is equivalent to $\dot{V}_{2,\dot{\varphi}}$

$$V_{2,\varphi} \mathrel{\widehat{=}} \dot{V}_{2,\dot{\varphi}}. \tag{2.104}$$

The form of this $\dot{V}_{2,\dot{\varphi}}$ is the same as in (2.101) upon conjugation of $\mathfrak{su}(2)$ indices. In other words, V_2 is self-adjoint with respect to the tree-level scalar product.

In a renormalised theory we should compute the correlator of renormalised states $Z\mathcal{O}$. At this point it is possible to guess the operator Z for the renormalisation of states

$$Z = 1 - \tfrac{1}{2}g^2 V_2(1/\mu) + \mathcal{O}(g^3). \tag{2.105}$$

We insert this into (2.99) and use the equivalence of $V_{2,\varphi}$ and $\dot{V}_{2,\dot{\varphi}}$ to find

$$\big\langle Z\mathcal{O}_\varphi\, \dot{Z}\dot{\mathcal{O}}_{\dot{\varphi}}\big\rangle = \exp\big(W_0(x_{12}, \check{\varphi}, \check{\dot{\varphi}})\big)\big(1 + g^2 V_{2,\varphi}(x_{12}) - g^2 V_{2,\varphi}(1/\mu)\big)\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot{\varphi}}\big|_{\varphi=\dot{\varphi}=0} + \mathcal{O}(g^3). \tag{2.106}$$

A closer look at $V_2(x_{12})$ reveals that the x_{12}-dependence is only through ξ as defined in (2.11). This is a manifest property of a renormalisable field theory in dimensional regularisation. We can thus write

$$V_2(x_{12}) = \xi V_2 = \frac{\Gamma(1-\epsilon)}{\big|\frac{1}{2}\mu^2 x_{12}^2\big|^{-\epsilon}}\, V_2. \tag{2.107}$$

We send the regulator to zero and find

$$\lim_{\epsilon\to 0}\big(V_2(x_{12}) - V_2(1/\mu)\big) = \log|\mu x_{12}|^{-2}\, D_2 \tag{2.108}$$

with

$$\mathfrak{D}_2 = -\lim_{\epsilon\to 0}\epsilon V_2. \tag{2.109}$$

In the case at hand, we obtain from (2.101)

$$\mathfrak{D}_2 = -\tfrac{1}{2}N^{-1}{:}\mathrm{Tr}\,[\phi_i, \phi_j][\check{\phi}^i, \check{\phi}^j]{:}, \tag{2.110}$$

where we have used the following expansion in ϵ, see (2.10), for the functions appearing in V_2

$$\begin{aligned} X_{00xx} I_{0x}^{-2}\xi^{-1} &= 2\epsilon^{-1} + 2 + \mathcal{O}(\epsilon^2), \\ \tilde{H}_{0x,0x} I_{0x}^{-2}\xi^{-1} &= -48\zeta(3)\,\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.111}$$

The final answer for the renormalised correlator at $\epsilon = 0$ is

$$\big\langle Z\mathcal{O}_\varphi\, \dot{Z}\dot{\mathcal{O}}_{\dot{\varphi}}\big\rangle = \exp(W_0)\exp\big(\log|\mu x_{12}|^{-2} g^2 \mathfrak{D}_{2,\varphi}\big)\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot{\varphi}}\big|_{\varphi=\dot{\varphi}=0} + \mathcal{O}(g^3), \tag{2.112}$$

in agreement with the form predicted by conformal field theory.[11] The operator $\mathfrak{D}_2$ is the one-loop correction to the dilatation generator. Furthermore, the coefficient of the correlator is given by its tree-level value. Notice that although we are interested in correlators of renormalised operators $Z\mathcal{O}$ as on the left-hand side of (2.112), we can work with bare operators $\mathcal{O}$ as on the right hand side of (2.112). In other words, we choose to renormalise the dilatation operator instead of the states.

2.4.4 Application

In the last section we have derived the one-loop dilatation generator (2.110) for the $\mathfrak{su}(2)$ subsector. When we write it in components $(\mathcal{Z},\phi)=(\varphi_1,\varphi_2)$ it reads

$$\mathfrak{D}_2 = -N^{-1}{:}\mathrm{Tr}\,[\mathcal{Z},\phi][\check{\mathcal{Z}},\check{\phi}]{:}. \tag{2.113}$$

Using the rules in Sec. 1.5 we can determine its action on any operator of the form $\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\phi\mathcal{Z}\phi\,\mathrm{Tr}\,\phi\ldots$.

Let us now apply $\mathfrak{D}_2$ to rederive the results of Sec. 2.1.1. The first observation is that D_2 acts on $\mathcal{Z}$ and ϕ simultaneously. If either of them is absent in the state, $\mathfrak{D}_2$ will annihilate it, therefore

$$\mathfrak{D}_2\ \mathrm{Tr}\,\mathcal{Z}\cdots\mathcal{Z}\ \mathrm{Tr}\,\mathcal{Z}\cdots\mathcal{Z}\ldots=0. \tag{2.114}$$

As emphasised in Sec. 2.3.2, these states are half-BPS and thus protected from quantum corrections. In particular, the operator $\mathcal{Q}_{mn}$ has one component, $\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}$, of this type. The other state discussed in Sec. 2.1, the Konishi operator $\mathcal{K}$, is not part of the $\mathfrak{su}(2)$ sector. However, it is on both unitarity bounds and has spin zero. Therefore, it has a descendant within the subsector whose weight is given by (2.13,2.88)

$$w'_{\mathcal{K}} = w_{\mathcal{K}} + \delta w_{\mathrm{I+II}} = (4;0,0;2,0,2;0,4). \tag{2.115}$$

This is a state of length $L=4$ with $K=2$ fields of type ϕ, see also [41]. Let us write down a basis for all such states in $\mathrm{SU}(N)$ gauge theory (the line separates single from double-trace states)

$$\mathcal{E}^{\intercal} = \left(\begin{array}{c} \mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\phi\phi \\ \mathrm{Tr}\,\mathcal{Z}\phi\mathcal{Z}\phi \\ \hline \mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\ \mathrm{Tr}\,\phi\phi \\ \mathrm{Tr}\,\mathcal{Z}\phi\ \mathrm{Tr}\,\mathcal{Z}\phi \end{array}\right). \tag{2.116}$$

We apply the one-loop dilatation operator to the basis, $\mathfrak{D}_2\mathcal{E}=\mathcal{E}D_2$, and obtain the matrix of anomalous dimensions

$$D_2 = \left(\begin{array}{cc|cc} +2 & -4 & +\frac{8}{N} & -\frac{4}{N} \\ -2 & +4 & -\frac{8}{N} & +\frac{4}{N} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{2.117}$$

Its eigenvectors are

$$(-2,2,0,0)^{\intercal},\quad (2,1,0,0)^{\intercal},\quad (0,0,1,2)^{\intercal},\quad (-\tfrac{2}{N},\tfrac{2}{N},1,-1)^{\intercal}. \tag{2.118}$$

[11] In fact, the mass dimension of the operators has not changed from its classical value, hence the residual μ-dependence. The fully renormalised operator to be inserted into the path integral would be $\mu^{\delta D(g)}Z\mathcal{O}$, but formally this cannot be expanded into a series as emphasised above.

The first one corresponds to the Konishi descendant

$$\mathcal{K}' = -2\,\mathrm{Tr}\,\mathcal{Z}\phi\mathcal{Z}\phi + 2\,\mathrm{Tr}\,\mathcal{Z}\phi\phi\mathcal{Z} = \mathrm{Tr}\,[\mathcal{Z},\phi][\mathcal{Z},\phi], \qquad D(g) = 2 + 6g^2 + \mathcal{O}(g^3) \quad (2.119)$$

with eigenvalue $D_2 = 6$. The other three states have vanishing anomalous dimension. The first two, $2\,\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\phi\phi + \mathrm{Tr}\,\mathcal{Z}\phi\mathcal{Z}\phi$ and $\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\,\mathrm{Tr}\,\phi\phi + 2\,\mathrm{Tr}\,\mathcal{Z}\phi\,\mathrm{Tr}\,\mathcal{Z}\phi$ are related to $\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\mathcal{Z}\mathcal{Z}$ and $\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\,\mathrm{Tr}\,\mathcal{Z}\mathcal{Z}$ by $\mathfrak{su}(2)$ rotations; this explains $D_2 = 0$. The last operator

$$\mathcal{O} = \mathrm{Tr}\,\mathcal{Z}\mathcal{Z}\,\mathrm{Tr}\,\phi\phi - \mathrm{Tr}\,\mathcal{Z}\phi\,\mathrm{Tr}\,\mathcal{Z}\phi + N^{-1}\,\mathrm{Tr}\,[\mathcal{Z},\phi][\mathcal{Z},\phi], \qquad D_2 = 0 \quad (2.120)$$

is indeed a highest weight state of $\mathfrak{psu}(2,2|4)$, as such it is, unlike $\mathcal{K}'$, quarter-BPS and protected [41, 42, 34].

2.5 Field Theoretic Considerations

In this section we will investigate the structure of the dilatation operator at higher orders in perturbation theory without actually computing it. This will yield important structural constraints for the algebraic construction pursued in the following chapters.

2.5.1 Two-Point Functions at Higher-Loops

Here, we would like to continue the investigation of the last section at higher loops and see how the dilatation operator can be extracted. We will show how to resolve some complications which appear starting at four loops and which are due to the fact that the various loop contributions to the dilatation operator do not commute with each other, e.g. $[\mathfrak{D}_2, \mathfrak{D}_4] \neq 0$. We will not compute higher-loop amplitudes explicitly.

To obtain the arbitrary loop correlator we insert all ℓ-loop connected Green functions $W_{2\ell}$ in the correlator

$$\langle \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi} \rangle = \exp(W_0) \exp\left(\textstyle\sum_{\ell=1}^\infty g^{2\ell} W_{2\ell}(x_{12}, \check\varphi, \check{\dot\varphi})\right) \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0}. \quad (2.121)$$

In analogy to (2.99) we change the argument $\check{\dot\varphi}$ of $W_{2\ell}(x, \check\varphi, \check{\dot\varphi})$ to $N\Delta_{12}^{-1}\varphi$

$$\langle \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi} \rangle = \exp(W_0)\, {:}\exp\left(\textstyle\sum_{\ell=1}^\infty g^{2\ell} W_{2\ell}(x_{12}, \check\varphi, N\Delta_{12}^{-1}\varphi)\right){:}\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0}. \quad (2.122)$$

Alternatively, we could change the argument $\check\varphi$ to $N\Delta_{12}^{-1}\dot\varphi$. We would then like to rewrite (2.122) in a convenient form for the conformal structure of the correlator:

$$\begin{aligned} \langle \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi} \rangle &= \exp(W_0) \exp\left(V_\varphi(x_{12})\right) \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0} \\ &= \exp(W_0) \exp\left(\dot V_{\dot\varphi}(x_{12})\right) \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot\varphi}\big|_{\varphi=\dot\varphi=0}, \end{aligned} \quad (2.123)$$

where[12]

$$V(x_{12}) = \sum_{\ell=1}^\infty g^{2\ell} V_{2\ell}(x_{12}) - \tfrac{1}{48} g^8 \big[V_2(x_{12}), [V_2(x_{12}), V_4(x_{12})]\big] + \dots. \quad (2.124)$$

[12] The commutator term was included for convenience, it could have been included in V_8, we will explain this issue below.

The terms $V_{2\ell}$ are defined by the equality of (2.122) and (2.123)

$$\exp\big(V(x_{12})\big) = :\exp\big(\textstyle\sum_{\ell=1}^{\infty} g^{2\ell} W_{2\ell}(x, \breve{\varphi}, N\Delta_{12}^{-1}\varphi)\big):, \tag{2.125}$$

which will have to be solved perturbatively. All the terms that arise due to normal ordering of the exponential and the commutator terms in (2.124) need to be absorbed into the definition of higher order vertices. For example, the two-loop effective vertex is

$$V_4(x) = :W_4(x_{12}, \breve{\varphi}, N\Delta_{12}^{-1}\varphi): - \tfrac{1}{2}\big(V_2(x_{12})V_2(x_{12}) - :V_2(x_{12})V_2(x_{12}):\big). \tag{2.126}$$

Let us introduce a transpose operation on a generator X by the definition

$$\exp(W_0)\, X(\varphi, \breve{\varphi}) = \exp(W_0)\, X^{\intercal}(\dot{\varphi}, \breve{\dot{\varphi}}). \tag{2.127}$$

In other words, letting X act on φ is equivalent to letting $X^{\intercal}$ act on $\dot{\varphi}$. The alternative forms of (2.123) lead to

$$V_{2\ell}^{\intercal}(x_{12}) = \dot{V}_{2\ell}(x_{12}). \tag{2.128}$$

In a real field theory $W_{2\ell}(x_{12}, \breve{\varphi}, \breve{\dot{\varphi}})$ must be hermitian in the arguments $\breve{\varphi}$ and $\breve{\dot{\varphi}}$. Therefore $\dot{V}_{2\ell}$ is indeed the complex conjugate of $V_{2\ell}$ and (2.128) shows that $V_{2\ell}$ is self-adjoint.

We renormalise the operators according to

$$\begin{aligned} Z &= \exp\Big(-\tfrac{1}{2}\textstyle\sum_{\ell=1}^{\infty} g^{2\ell} V_{2\ell}(1/\mu) + \tfrac{1}{24} g^6 \big[V_2(1/\mu), V_4(1/\mu)\big] + \ldots\Big), \\ \dot{Z} &= \exp\Big(-\tfrac{1}{2}\textstyle\sum_{\ell=1}^{\infty} g^{2\ell} \dot{V}_{2\ell}(1/\mu) + \tfrac{1}{24} g^6 \big[\dot{V}_2(1/\mu), \dot{V}_4(1/\mu)\big] + \ldots\Big). \end{aligned} \tag{2.129}$$

This gives

$$\big\langle Z\mathcal{O}_\varphi\, \dot{Z}\dot{\mathcal{O}}_{\dot{\varphi}}\big\rangle = \exp(W_0) \exp\big(V_\varphi(x)\big) Z_\varphi \mathcal{O}_\varphi\, \dot{Z}_{\dot{\varphi}} \dot{\mathcal{O}}_{\dot{\varphi}}\big|_{\phi=\varphi=0}. \tag{2.130}$$

We can commute objects that depend only on φ with objects that depend only on $\dot{\varphi}$ freely. Then we use the transpose operation (2.127) to make $\dot{Z}_{\dot{\varphi}}$ act on φ instead. We get

$$\big\langle Z\mathcal{O}_\phi\, Z\mathcal{O}_\varphi\big\rangle = \exp(W_0)\, \dot{Z}_\varphi^{\intercal} \exp\big(V_\varphi(x_{12})\big) Z_\varphi\, \mathcal{O}_\varphi\, \dot{\mathcal{O}}_{\dot{\varphi}}\big|_{\varphi=\dot{\varphi}=0}. \tag{2.131}$$

The vertices $V_{2\ell}(1/\mu)$ in Z are hermitian, (2.128), only the commutator in (2.129) requires special care, because V_2 and V_4 need to be transformed consecutively. This effectively inverts their order and flips the sign of the commutator:

$$\dot{Z}^{\intercal} = \exp\big(-\tfrac{1}{2}\textstyle\sum_{\ell=1}^{\infty} g^{2\ell} V_{2\ell}(1/\mu) - \tfrac{1}{24} g^6 \big[V_2(1/\mu), V_4(1/\mu)\big] + \ldots\big). \tag{2.132}$$

In a renormalisable field theory the dependence of $V_{2\ell}$ on x_{12} is determined, we write

$$V_{2\ell}(x_{12}) = \xi^\ell V_{2\ell}. \tag{2.133}$$

We combine the exponentials in (2.131) with the surrounding Z's into a single exponent

$$\sum_{\ell=1}^{\infty} (\xi^\ell - \xi_0^\ell) g^{2\ell} V_{2\ell,\varphi} - \tfrac{1}{48} g^8 (\xi - \xi_0)^4 \big[V_{2,\varphi}, [V_{2,\varphi}, V_{4,\varphi}]\big] + \ldots . \tag{2.134}$$

The ℓ-loop Green function $W_{2\ell}$ is expected to have multiple poles at $\epsilon = 0$. In a conformal field theory, however, these poles must have cancelled in the combination $V_{2\ell}$ as given by (2.124,2.125). If so, we can finally send the regulator to zero and find

$$\left\langle Z\mathcal{O}_\varphi \, \dot{Z}\dot{\mathcal{O}}_{\dot\varphi} \right\rangle = \exp(W_0) \exp\left(\log|\mu x_{12}|^{-2} \sum_{\ell=1}^{\infty} g^{2\ell} \mathfrak{D}_{2\ell,\varphi} \right) \mathcal{O}_\varphi \, \dot{\mathcal{O}}_{\dot\varphi} \Big|_{\varphi=\dot\varphi=0} \tag{2.135}$$

with

$$\mathfrak{D}_{2\ell} = -\ell \lim_{\epsilon\to 0} \epsilon V_{2\ell}. \tag{2.136}$$

Note that the commutator term in (2.134) vanishes due to four powers of ϵ from $(\xi - \xi_0)^4$ as opposed to only three powers of $1/\epsilon$ from the $V_{2\ell}$. For this cancellation to happen the commutator terms in (2.124) and (2.129) are necessary: We have investigated all possible terms that can arise in a four-loop computation. We find that precisely the commutator structure in (2.124) is required to obtain a finite, conformally covariant correlator.

Some comments about the renormalisation programme are in order. Firstly, the programme ensures that the coefficient of the two-point function is given by free contractions of the unrenormalised operators. Secondly, the effective vertices $V_{2\ell}$ are self-adjoint with respect to the scalar product induced by free contractions, see (2.128). The same holds for the dilatation generator which consequently has real eigenvalues. Notice that in some case there may appear to be complex eigenvalues. However, a more careful analysis will show that the corresponding eigenstate is zero. This may happen if the rank of the group is small compared to the size of the operators and group identities lead to non-trivial linear dependencies in the basis of operators.

2.5.2 Two-Point Functions of Non-Scalar Operators

Correlation functions of non-scalar operators are not as easy to handle as their scalar counterparts. This is due to their spacetime indices which can not only be contracted among themselves but also with x_{12}. Furthermore, there are qualitative differences between primaries and descendants, see Sec. 1.10. Therefore the form predicted by conformal symmetry is not as simple as (2.135). It certainly involves the symmetry generators $\mathfrak{K}$ and $\mathfrak{P}$ to be able to distinguish between primaries and descendants. These generators also receive quantum corrections, which would have to be found at the same time.

However, in some cases the dilatation generator may be obtained anyway without taking these complications into account. A crucial observation is that, although (1.93) and (1.96) are different, the difference is only in the part that multiplies $x_{12}^\mu x_{12}^\nu$. The 'direct' contraction via $\eta_{\mu\nu}$ is the same for both. If all contractions between the operator indices and x_{12} are dropped, the operators behave as though they were a set of scalars. In [50] this simplification made a computation of the one-loop dilatation operator possible within the non-scalar subsector $(2, 2)$.

Note that the covariant derivatives acting on a field are just ordinary partial derivatives at leading order. The appearance of the gauge connection should be treated as an interaction that takes place at the point of the field (boundary) and everywhere in spacetime (bulk). Algebraically, the structure of boundary interactions is the same as in the bulk, the gauge field couples to the field via the gauge group structure constants and one power of the coupling constant.

2.5.3 Feynman Diagrams

In Sec. 2.5.1 we have seen how the corrections to the dilatation operator arise from divergent Feynman diagrams. Here we would like to investigate the structure of interacting contributions to the dilatation operator and other generators of the symmetry algebra. This will be an important constraint for the constructions in the following chapters. The '*interactions*', i.e. the contributions to the group generators are constructed from fields $\mathcal{W}$, variations $\check{\mathcal{W}}$ and structure constants $\mathfrak{f}$ of the gauge group. Notice that due to the form of the Lagrangian (1.5) there is exactly one power of the coupling constant g for each $\mathfrak{f}$ (before gauge group identities are used).

Our first claim is that the generators $\mathfrak{J}(g)$ are *connected*. Here, connectedness refers to the gauge algebra. It means that all gauge group indices are contracted so that the symbols form a connected graph. The connectedness can be inferred from Sec. 2.5.1: The effective vertices $V_{2\ell}$ are connected diagrams. They are generated from the Green functions $W_{2\ell}$ by removing the normal ordering of an exponential (2.125) and adding commutators (2.124). One can easily convince oneself that these operations produce only connected diagrams. The same is true also for the dilatation generator $\mathfrak{D}$. Connectedness can also be seen in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. There the bare Hamiltonian is clearly connected, but it does not obey $[\mathcal{H}(0), \mathcal{H}(g)] = 0$ (2.39). In this work we will require this identity and therefore need to diagonalise the Hamiltonian first. This diagonalisation procedure described in Sec. 2.2.3 produces only commutator terms and thus connected diagrams. This can be seen in (2.43) by rewriting it as

$$\delta\mathfrak{D} \mapsto \delta\mathfrak{D}_0 + \sum\nolimits_{d>0} \frac{1}{d} [\delta\mathfrak{D}_d, \delta\mathfrak{D}_{-d}] + \dots . \tag{2.137}$$

Secondly, we can count the number of external legs E, i.e. the number of fields E_o plus the number of variations E_i. According to (1.26) this equals

$$E = V + 2 - 2L', \tag{2.138}$$

where V is the number of structure constants and L' is the number of adjoint index loops. For each structure constant there is precisely one power of g. A contribution of $\mathcal{O}(g^V)$ therefore has no more than $V + 2$ legs.[13] A useful basis for interactions which can be achieved by making use of Jacobi identities is

$$\text{'}\textit{interactions}\text{'}: \quad \mathfrak{J}(g) \sim g^{E_\mathrm{i}+E_\mathrm{o}+2L'-2}\, \mathfrak{f}^{E_\mathrm{i}+E_\mathrm{o}+2L'-2}_{\mathfrak{m}\dots\mathfrak{m}} (\mathfrak{g}^{\mathfrak{m}\mathfrak{m}})^{L'} (\mathcal{W}_*{}^{\mathfrak{m}})^{E_\mathrm{o}} (\check{\mathcal{W}}^{*\mathfrak{m}})^{E_\mathrm{i}}, \tag{2.139}$$

where the linear contraction of V structure constants $\mathfrak{f}^V_{\mathfrak{m}\dots\mathfrak{m}}$ is given by, see Fig. 2.10,

$$\mathfrak{f}^V_{\mathfrak{m}\dots\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}_1\mathfrak{n}_1} \mathfrak{f}^{\mathfrak{n}_1}{}_{\mathfrak{m}_2\mathfrak{n}_2} \mathfrak{f}^{\mathfrak{n}_2}{}_{\mathfrak{m}_3\mathfrak{n}_3} \cdots \mathfrak{f}^{\mathfrak{n}_V}{}_{\mathfrak{m}_{V+1}\mathfrak{m}_{V+2}} = i^V \operatorname{Tr} \mathfrak{t}_{\mathfrak{m}_1} [\mathfrak{t}_{\mathfrak{m}_2}, [\mathfrak{t}_{\mathfrak{m}_3}, [\dots, [\mathfrak{t}_{\mathfrak{m}_{V+1}}, \mathfrak{t}_{\mathfrak{m}_{V+2}}] \dots]]]. \tag{2.140}$$

These interactions preserve the parity operation defined in Sec. 1.3 as one can confirm easily. This is because they are only composed of structure constants which have positive parity.

Finally, there is a peculiar feature of maximal scalar diagrams which will be important to select the right terms later on. These are diagrams without index loops $L' = 0$ which

[13] The definition of the Hamiltonian $\mathcal{H}$ involves two powers of g, therefore $\mathcal{H}$ generically has two legs more at a given order of g.

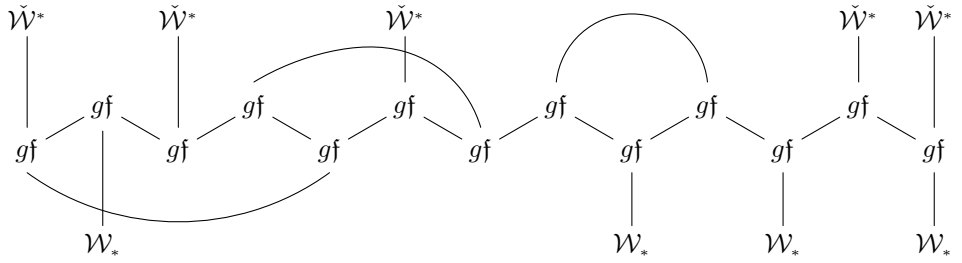

Figure 2.10: A generic interaction with $E_{\rm i} = 5$ variations, $E_{\rm o} = 4$ fields, $L' = 3$ index loops and $V = 13$ structure constants and powers of the coupling constant.

therefore have the maximum number of external scalar legs $V+2$ at order V in perturbation theory. We are interested in the flow of $\mathfrak{so}(6)$ vector indices across the diagram. As this is a tree diagram, the internal lines can only be scalars or gauge fields. Gauge fields are singlets of $\mathfrak{so}(6)$ therefore only scalar fields can support the flow. Only at quartic interactions of the scalars two lines of flow can cross. This shows that at order V there cannot be more than $V/2$ crossings of $\mathfrak{so}(6)$ vector flow lines.

2.6 The Planar Limit and Spin Chains

Generic interactions have a very complicated structure due to a large number of possible contractions between the indices in (2.139). Most of the time it is therefore useful to restrict to the planar limit, see Sec. 1.4.

2.6.1 States

In the large N limit, field theory diagrams are suppressed unless $2C-2G-T=0$, see (1.44). As each component requires at least two traces, one incoming and one outgoing (there are no vacuum diagrams), we need $T = 2C$ and $G = 0$. In other words, the diagrams may connect only two single trace operators and cannot have handles. Therefore it makes sense to consider only single trace states

$$\text{`}\textit{single-trace state}\text{'}: \quad |\mathcal{A}_1 \dots \mathcal{A}_L\rangle := \operatorname{Tr} \mathcal{W}_{\mathcal{A}_1} \cdots \mathcal{W}_{\mathcal{A}_L}. \tag{2.141}$$

The cyclicity of the trace gives rise to cyclic identifications

$$|\mathcal{A}_1 \dots \mathcal{A}_p \mathcal{A}_{p+1} \dots \mathcal{A}_L\rangle = (-1)^{(\mathcal{A}_1 \dots \mathcal{A}_p)(\mathcal{A}_{p+1} \dots \mathcal{A}_L)} |\mathcal{A}_{p+1} \dots \mathcal{A}_L \mathcal{A}_1 \dots \mathcal{A}_p\rangle. \tag{2.142}$$

The sign is due to statistics of the fields: $(-1)^{XY}$ equals -1 if both, X and Y, are fermionic and $+1$ otherwise. In particular, some states are incompatible with this symmetry

$$|\mathcal{A}_1 \dots \mathcal{A}_{L/2} \mathcal{A}_1 \dots \mathcal{A}_{L/2}\rangle = 0, \qquad \text{if } \mathcal{A}_1 \dots \mathcal{A}_{L/2} \text{ is fermionic.} \tag{2.143}$$

A generic state is a linear superposition of the above basis states

$$\mathcal{O} = c\,|\mathcal{A}_1 \dots \mathcal{A}_L\rangle + c'\,|\mathcal{A}'_1 \dots \mathcal{A}'_{L'}\rangle + \dots\,, \tag{2.144}$$

where mixing of states with different length is explicitly allowed.

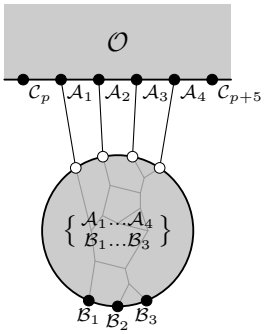

Figure 2.12: Insertion of a planar interaction. The black dots correspond to fields, the white dots to variations. Inside the blob there is some unspecified planar diagram that connects the dots.

2.6.2 Interactions

For planar interactions the precise structure of internal connections does not play a role as long as it is planar, c.f. Fig. 2.12. The only relevant structure is the order of external legs. In the planar limit it is therefore sufficient to consider interactions of the type

$$\text{`}\textit{planar interactions}\text{'}: \quad \left\{ \begin{smallmatrix} \mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \\ \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \end{smallmatrix} \right\} := N^{1-E_\mathrm{i}} \operatorname{Tr} \mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1}, \tag{2.145}$$

which searches for the sequence of fields $\mathcal{W}_{\mathcal{A}_1} \dots \mathcal{W}_{\mathcal{A}_{E_\mathrm{i}}}$ within a state and replaces it by the sequence $\mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}}$. More explicitly, the action on a state $|\mathcal{C}_1 \dots \mathcal{C}_L\rangle$ is

$$\sum_{p=1}^{L} (-1)^{(\mathcal{C}_1 \dots \mathcal{C}_{p-1})(\mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}})} \delta_{\mathcal{C}_p}^{\mathcal{A}_1} \dots \delta_{\mathcal{C}_{p+E_\mathrm{i}-1}}^{\mathcal{A}_{E_\mathrm{i}}} |\mathcal{C}_1 \dots \mathcal{C}_{p-1} \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \mathcal{C}_{p+E_\mathrm{i}} \dots \mathcal{C}_L\rangle. \tag{2.146}$$

A sample action is

$$\left\{ \begin{smallmatrix} \mathcal{A}\mathcal{B} \\ \mathcal{B}\mathcal{A} \end{smallmatrix} \right\} |12345\rangle = |21345\rangle \pm |13245\rangle \pm |12435\rangle \pm |12354\rangle \pm |52341\rangle. \tag{2.147}$$

The order of an interaction in perturbation theory is given by $V = E_\mathrm{i} + E_\mathrm{o} + 2L' - 2$, therefore $\mathfrak{J}(g) \sim g^{E_\mathrm{i}+E_\mathrm{o}+2L'-2} \left\{ \begin{smallmatrix} \mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \\ \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \end{smallmatrix} \right\}$. We see that for planar interactions, adding an index loop simply increases the loop order by one. At a fixed loop order this leads to diagrams with fewer external legs. To reduce the complexity, we can install a pair of legs by means of a gauge transformation. For that purpose we insert the generator of gauge rotations j into some interaction

$$\begin{aligned} 0 &\mathrel{\widehat{=}} -i \operatorname{Tr} \mathsf{j}\, \mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1} \\ &= \operatorname{Tr}{:}\mathcal{W}_{\mathcal{C}} \check{\mathcal{W}}^{\mathcal{C}}{:}\mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1} - \operatorname{Tr}{:}\check{\mathcal{W}}^{\mathcal{C}} \mathcal{W}_{\mathcal{C}}{:}\mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1} \\ &\mathrel{\widehat{=}} N \operatorname{Tr} \mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1} \mp \operatorname{Tr} \mathcal{W}_{\mathcal{C}} \mathcal{W}_{\mathcal{B}_1} \dots \mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}} \check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \dots \check{\mathcal{W}}^{\mathcal{A}_1} \check{\mathcal{W}}^{\mathcal{C}}. \end{aligned} \tag{2.148}$$

The equivalence in the last line is for planar insertions only. This means that adding a pair of legs to the left of the interaction has no effect. Equivalently, we can add a pair of legs to the right of the interaction

$$\left\{ \begin{smallmatrix} \mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \\ \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \end{smallmatrix} \right\} \mathrel{\widehat{=}} \left\{ \begin{smallmatrix} \mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \mathcal{C} \\ \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \mathcal{C} \end{smallmatrix} \right\} \mathrel{\widehat{=}} (-1)^{\mathcal{C}(\mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}})} \left\{ \begin{smallmatrix} \mathcal{C}\mathcal{A}_1 \dots \mathcal{A}_{E_\mathrm{i}} \\ \mathcal{C}\mathcal{B}_1 \dots \mathcal{B}_{E_\mathrm{o}} \end{smallmatrix} \right\}. \tag{2.149}$$

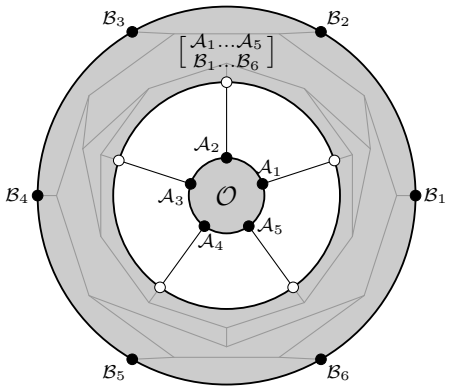

Figure 2.14: The action of a wrapping interaction. For a planar insertion, the wrapping interaction must surround the state and the number of fields L must match the number of variations E_i.

This is obvious because the additional pair of legs does not change the field at that position, it is only a *spectator*. We can now add L' pairs of spectator legs to an interaction and thus drop the index loop parameter[14]

$$\mathfrak{J}_k \sim \left\{\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right\}, \quad \text{with} \quad E_\mathrm{i}+E_\mathrm{o}=k+2. \tag{2.150}$$

2.6.3 Wrapping Interactions

This can, however, not be completely true: If the number of variations, E_i, equals the length of the state, L, we cannot add a spectator pair of legs. In fact, there is a subtlety in the second equivalence in (2.148): When in the second term the variation hits the field $\mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}}$ we get a '*wrapping diagram*'. It can be represented by the following symbol and trace structure

$$\text{'}\textit{wrapping interactions}\text{'}: \quad \left[\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right] := N^{-E_\mathrm{i}}\,\mathrm{Tr}\,\mathcal{W}_{\mathcal{B}_1}\ldots\mathcal{W}_{\mathcal{B}_{E_\mathrm{o}}}\,\mathrm{Tr}\,\check{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}}\ldots\check{\mathcal{W}}^{\mathcal{A}_1}. \tag{2.151}$$

Wrapping interactions remove the state as a whole and replace it by a new one. They are best understood graphically, see Fig. 2.14. Wrapping diagrams are generically non-planar, but when applied to a state of the minimally required length, the action becomes planar. This is because the diagram can be wrapped fully around the trace. If, however, some uncontracted fields remain within the trace, they are disconnected from the fields of the interaction and the action is non-planar. An improved version of (2.149) which takes states of finite length into account is

$$\left\{\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right\} = \left\{\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\mathcal{C}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\mathcal{C}\end{smallmatrix}\right\} + \left[\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right] = (-1)^{\mathcal{C}(\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}})}\left\{\begin{smallmatrix}\mathcal{C}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{C}\mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right\} + \left[\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right] \tag{2.152}$$

For wrapping diagrams the order in perturbation theory is given by

$$\mathfrak{J}_k \sim \left[\begin{smallmatrix}\mathcal{A}_1\ldots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\ldots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right] \quad \text{with} \quad E_\mathrm{i}+E_\mathrm{o}=k+2-2L', \quad L'\geq 1. \tag{2.153}$$

They act only on states with length $L=E_\mathrm{i}$ and need at least one index loop L'. Therefore they appear only at rather high loop orders, especially for long states. Unfortunately,

[14]Note that for $\mathcal{H}_k$ as defined in Sec. 2.2.4 there should be $E_\mathrm{i}+E_\mathrm{o}=k+4$ legs.

there are no obvious structural constraints on wrapping interactions as the one described at the end of Sec. 2.5.3. This makes them rather hard to handle and we will not make quantitative statements in this work.

2.6.4 Parity

In Sec. 1.3 we have defined a parity operation for a unitary gauge group. It replaces all fields by their negative transpose. Transposing all matrices within a trace simply reverses their order. We find that parity acts on a state as

$$\mathfrak{p}\,|\mathcal{A}_1\dots\mathcal{A}_L\rangle = (-1)^{L+f(f-1)/2}|\mathcal{A}_L\dots\mathcal{A}_1\rangle, \tag{2.154}$$

where f is the number of fermionic fields in the trace. Effectively we can use this definition of parity also for gauge groups $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$. There, however, parity must act trivially and only states of positive parity are allowed.

The parity operation for interactions is (f_i and f_o are the numbers of fermions in $A_1\dots A_{E_\mathrm{i}}$ and $B_1\dots B_{E_\mathrm{o}}$, respectively)

$$\mathfrak{p}\,\left\{\begin{smallmatrix}\mathcal{A}_1\dots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\dots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right\}\mathfrak{p}^{-1} = (-1)^{E_\mathrm{i}+E_\mathrm{o}+f_\mathrm{i}(f_\mathrm{i}-1)/2+f_\mathrm{o}(f_\mathrm{o}-1)/2}\left\{\begin{smallmatrix}\mathcal{A}_{E_\mathrm{i}}\dots\mathcal{A}_1\\ \mathcal{B}_{E_\mathrm{o}}\dots\mathcal{B}_i\end{smallmatrix}\right\}. \tag{2.155}$$

For the interactions within algebra generators $\mathfrak{J}(g)$ parity must be positive. Nevertheless, we will also make contact with generators of negative parity later on.

2.6.5 Scalar Product

Our investigations in this work are independent of the definition of a scalar product. Nevertheless, it is useful to know how to construct a meaningful norm because the dilatation operator will be self-adjoint with respect to this norm and thus have real eigenvalues. We will sketch how the norm should look like.

The construction in Sec. 2.5.1 shows that states can be renormalised in such a way as to preserve the classical scalar product. At tree-level the scalar product is given by pairwise contractions $\langle\mathcal{A}|\mathcal{B}\rangle$ between fields of both states. In the planar limit all contractions must be parallel. Therefore the planar scalar product of two states

$$\langle\mathcal{A}_1\dots\mathcal{A}_L|\mathcal{B}_1\dots\mathcal{B}_{L'}\rangle = \delta_{L=L'}\sum_{p'=1}^{L}(\pm1)\prod_{p=1}^{L}\langle\mathcal{A}_p|\mathcal{B}_{p'-p}\rangle \tag{2.156}$$

vanishes unless $L=L'$ and both states are related by a cyclic permutation. For generic overlapping states the elementary scalar products in (2.156) are non-zero for all p only for one very specific value of p'. However, for a state which can be written as $|(\mathcal{A}_1\dots\mathcal{A}_{L/n})^n\rangle$ with n as large as possible, there are n possible values for p'. The square norm for a state is thus proportional to $\pm n$

$$|(\mathcal{A}_1\dots\mathcal{A}_{L/n})^n\rangle \sim \sqrt{n}\,. \tag{2.157}$$

An adjoint operation for interactions compatible with the above scalar product for states should interchange the two rows in the interaction symbol[15]

$$\left\{\begin{smallmatrix}\mathcal{A}_1\dots\mathcal{A}_{E_\mathrm{i}}\\ \mathcal{B}_1\dots\mathcal{B}_{E_\mathrm{o}}\end{smallmatrix}\right\}^\dagger \sim \left\{\begin{smallmatrix}\mathcal{B}_{E_\mathrm{o}}\dots\mathcal{B}_1\\ \mathcal{A}_{E_\mathrm{i}}\dots\mathcal{A}_1\end{smallmatrix}\right\}. \tag{2.158}$$

[15]The reverse ordering of the adjoint is related to the scalar product. One could combine the adjoint with parity to define a different adjoint operation which only interchanges both rows.

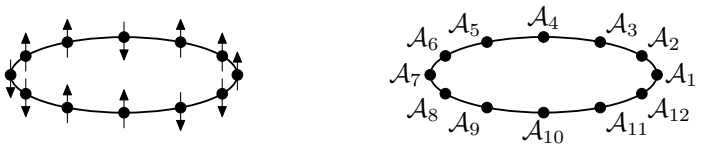

Figure 2.16: A single-trace operator as a spin chain. For the simplest spin chain, the spin can take two alignments, for $\mathcal{N} = 4$ the 'spin' can take infinitely many.

planar $\mathcal{N} = 4$ SYM	$\mathfrak{psu}(2,2\|4)$ spin chain
single trace operator	cyclic spin chain
field	spin site
anomalous dilatation operator $g^{-2}\delta\mathfrak{D}$	Hamiltonian $\mathcal{H}$
anomalous dimension $g^{-2}\delta D$	energy E
cyclicity constraint	zero-momentum condition $\mathcal{U} = 1$

Table 2.2: Dictionary for $\mathcal{N} = 4$ SYM and the spin chain picture.

Note, however, that the action of a self-adjoint interaction on a set of states is only equivalent to a hermitian matrix if all states are normalised to one with respect to (2.157). Otherwise the matrix is only self-adjoint with respect to the norm on the set of states. For example, this is the case for the asymmetric matrix (2.117).

2.6.6 Spin Chains

Single-trace local operators can be viewed as states of a dynamic, cyclic, quantum *spin chain* [48]. A cyclic spin chain is a set of L *spin sites* with a cyclic adjacency property. In a quantum spin chain, the *spin* at each site is a module of the symmetry algebra of the system and the Hilbert space is the tensor product of L spin modules. For a *dynamic* spin chain the number of sites L is not fixed [51]; the full Hilbert space is the tensor product of all Hilbert spaces of a fixed length. The basic quantum spin chain is the Heisenberg chain. Its symmetry group is $\mathfrak{su}(2)$ and all spins transform in the fundamental representation. A basis for the Hilbert space is given by those states for which the spin at each site points either 'up' or 'down'. The Hilbert space is thus $\mathbb{C}^{2^L}$. In a more general spin chain, the spin can point in more than just two directions, in most cases even infinitely many. Note that the cyclic identification (2.142) of field theory states is an additional constraint on cyclic spin chains. For example, the field theory Hilbert space corresponding to the Heisenberg chain is $\mathbb{C}^{2^L}/\mathbb{Z}_L$. Physical states are identified by a trivial shift operator, states with non-zero momentum are unphysical.

In the spin chain picture each field is identified with one site of the chain. The alignment of the spin at that site corresponds to the component of the multiplet of fields, c.f. Fig. 2.16. For $\mathcal{N} = 4$ SYM, the spin chain is a $\mathfrak{psu}(2,2|4)$ cyclic super spin chain with spins transforming in the representation $[0;0;0,1,0;0;0]$, see Sec. 1.9, [58]. When working in the planar limit, we will commonly make use of spin chain terminology. In particular, the quantum correction to the dilatation generator will be called the '*Hamiltonian*' $\mathcal{H} = g^{-2}\delta\mathfrak{D}$ and anomalous dimensions are synonymous for '*energies*' $E = g^{-2}\delta D$, see Tab. 2.2 for a small dictionary. For the other generators of the superconformal group we use the same symbols as in the non-planar case.

Chapter 3

One-Loop

In this chapter we will derive the complete one-loop dilatation operator of $\mathcal{N}=4$ Super Yang-Mills Theory. The text is based on the article [50], but we present a new derivation of the coefficients C_j. In [50] the coefficients have been obtained in a quantum field theory calculation, here we will merely employ the superconformal algebra. The spectral and plane-wave investigations have been compiled from the articles [35, 27, 38, 50, 103].

3.1 The Form of the Dilatation Generator

We start by investigating the general form of the one-loop dilatation generator. We will see that representation theory of the symmetry group as well as Feynman diagrammatics put tight constraints on the form. What remains is a sequence of undetermined coefficients C_j, one for each value of 'total spin'.

3.1.1 One-Loop as Leading Order

In Sec. 2.2.4 we have learned that the leading order anomalous dilatation operator D_l is invariant under classical superconformal transformations $\mathfrak{J}_0$. It is impossible to construct an invariant operator $\mathfrak{D}_1$ at first order of the coupling constant g, therefore the leading order is one-loop, $l=2$. We will come back to this point after having reviewed some representation theory at the end of Sec. 3.1.3. In what follows we will consider only the classical $\mathfrak{psu}(2,2|4)$ algebra of generators $\mathfrak{J}_0$; the one-loop anomalous dilatation generator $\mathfrak{D}_2$ will be considered an independent object; we will refer to it as the Hamiltonian $\mathcal{H}$,

$$\mathfrak{J}(g)=\mathfrak{J}+\mathcal{O}(g),\qquad \mathfrak{D}(g)=\mathfrak{D}+g^2\,\mathcal{H}+\mathcal{O}(g^3),\qquad [\mathfrak{J},\mathcal{H}]=0. \tag{3.1}$$

3.1.2 Generic Form

The Hamiltonian has the following generic form[1]

$$\begin{aligned}\mathcal{H}=&-N^{-1}(C_{\mathrm{a}})^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}}\ {:}\mathrm{Tr}\left[\mathcal{W}_{\mathcal{A}},\check{\mathcal{W}}^{\mathcal{C}}\right]\left[\mathcal{W}_{\mathcal{B}},\check{\mathcal{W}}^{\mathcal{D}}\right]{:}\\ &-N^{-1}(C_{\mathrm{b}})^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}}\ {:}\mathrm{Tr}\left[\mathcal{W}_{\mathcal{A}},\mathcal{W}_{\mathcal{B}}\right]\left[\check{\mathcal{W}}^{\mathcal{C}},\check{\mathcal{W}}^{\mathcal{D}}\right]{:}\\ &+N^{-1}(C_{\mathrm{c}})^{\mathcal{A}}_{\mathcal{B}}\ \mathfrak{g}^{\mathfrak{m}\mathfrak{n}}{:}\mathrm{Tr}\left[\mathcal{W}_{\mathcal{A}},\mathfrak{t}_{\mathfrak{m}}\right]\left[\mathfrak{t}_{\mathfrak{n}},\check{\mathcal{W}}^{\mathcal{B}}\right]{:}.\end{aligned} \tag{3.2}$$

[1] These expressions are valid for bosonic fields $\mathcal{W}_{\mathcal{A}}$ only. They do generalise to fermions, but only at the cost of obscure signs at various places.

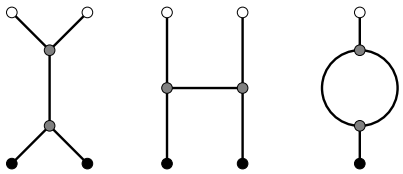

Figure 3.2: Algebraic structure of the one-loop diagrams contributing to the anomalous dimension. The lines correspond to any of the fundamental fields of the theory.

These terms correspond to the three basic types of divergent Feynman diagrams which arise at the one-loop level, see Fig. 3.2. As before, in Sec. 2.4.3, we can transform the term of type c by means of gauge invariance. The generator of gauge transformations (see Sec. 1.3) is $\mathfrak{j} = i{:}[\mathcal{W}_{\mathcal{C}}, \check{\mathcal{W}}^{\mathcal{C}}]{:}$ and it annihilates gauge invariant operators. Therefore we can write (note the change of normal orderings)

$$0 \mathrel{\widehat{=}} -i\,\mathrm{Tr}\,\mathfrak{j}\,{:}\big[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{B}}\big]{:} = {:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{C}}, \check{\mathcal{W}}^{\mathcal{C}}\big]\big[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{B}}\big]{:} + \mathfrak{g}^{\mathfrak{mn}}{:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{A}}, \mathfrak{t}_{\mathfrak{m}}\big]\big[\mathfrak{t}_{\mathfrak{n}}, \check{\mathcal{W}}^{\mathcal{B}}\big]{:}, \tag{3.3}$$

which allows us to write the term of type c as a term of type a. Furthermore the term of type b can be transformed by means of a Jacobi-identity

$${:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{A}}, \mathcal{W}_{\mathcal{B}}\big]\big[\check{\mathcal{W}}^{\mathcal{C}}, \check{\mathcal{W}}^{\mathcal{D}}\big]{:} = {:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{C}}\big]\big[\mathcal{W}_{\mathcal{B}}, \check{\mathcal{W}}^{\mathcal{D}}\big]{:} - {:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{D}}\big]\big[\mathcal{W}_{\mathcal{B}}, \check{\mathcal{W}}^{\mathcal{C}}\big]{:}. \tag{3.4}$$

We combine all coefficients into a single one of type a

$$C^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}} = -\big((C_{\mathrm{a}})^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}} + (C_{\mathrm{b}})^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}} - (C_{\mathrm{b}})^{\mathcal{A}\mathcal{B}}_{\mathcal{D}\mathcal{C}} + \tfrac{1}{2}\delta^{\mathcal{A}}_{\mathcal{C}}(C_{\mathrm{c}})^{\mathcal{B}}_{\mathcal{D}} + \tfrac{1}{2}(C_{\mathrm{c}})^{\mathcal{A}}_{\mathcal{C}}\delta^{\mathcal{B}}_{\mathcal{D}}\big). \tag{3.5}$$

The total Hamiltonian is

$$\mathcal{H} = -N^{-1}C^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}}\,{:}\mathrm{Tr}\,\big[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{C}}\big]\big[\mathcal{W}_{\mathcal{B}}, \check{\mathcal{W}}^{\mathcal{D}}\big]{:} \tag{3.6}$$

with some yet undetermined coefficient $C^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}}$.

3.1.3 Symmetry

The combined coefficient $C^{\mathcal{A}\mathcal{B}}_{\mathcal{C}\mathcal{D}}$ must be invariant under the classical superconformal algebra, it describes an *intertwining map* $\mathcal{V}_{\mathrm{F}} \times \mathcal{V}_{\mathrm{F}} \mapsto \mathcal{V}_{\mathrm{F}} \times \mathcal{V}_{\mathrm{F}}$. This requirement puts tight constraints on the coefficients, the independent components can be obtained by investigating the irreducible modules in the tensor product $\mathcal{V}_{\mathrm{F}} \times \mathcal{V}_{\mathrm{F}}$. The tensor product of two $\mathcal{V}_{\mathrm{F}}$ is given by (see e.g. [103])

$$\mathcal{V}_{\mathrm{F}} \times \mathcal{V}_{\mathrm{F}} = \sum_{j=0}^{\infty} \mathcal{V}_j, \tag{3.7}$$

where $\mathcal{V}_j$ are the modules with primary weights

$$\begin{aligned} w_0 &= (2; 0, 0; 0, 2, 0; 0, 2), \\ w_1 &= (2; 0, 0; 1, 0, 1; 0, 2), \\ w_j &= (j; j-2, j-2; 0, 0, 0; 0, 2). \end{aligned} \tag{3.8}$$

The module $\mathcal{V}_0$ is the half-BPS current multiplet, $\mathcal{V}_1$ is quarter-BPS and the other ones are doubly-short. As an aside, it is interesting to see that the quadratic Casimir (c.f. App. D.3) for these modules is given by

$$\mathfrak{J}_{12}^2\,\mathcal{V}_j = j(j+1)\,\mathcal{V}_j, \tag{3.9}$$

just as if we were dealing with $\mathfrak{sl}(2)$ alone. Due to invariance, $C_{\mathcal{CD}}^{\mathcal{AB}}$ must be of block-diagonal form: All states of a module $\mathcal{V}_j$ must be mapped to the same type of module $\mathcal{V}_j$ with equal coefficients. In our case, each module $\mathcal{V}_j$ appears only once in the tensor product, therefore we can assign only one coefficient C_j for each $\mathcal{V}_j$. Let $(\mathcal{P}_j)_{\mathcal{CD}}^{\mathcal{AB}}$ project two fields $\mathcal{W}_{\mathcal{A}}, \mathcal{W}_{\mathcal{B}}$ to the module $\mathcal{V}_j$. Then the most general invariant coefficients can be written as

$$C_{\mathcal{CD}}^{\mathcal{AB}} = \sum_{j=0}^{\infty} C_j\,(\mathcal{P}_j)_{\mathcal{CD}}^{\mathcal{AB}}. \tag{3.10}$$

Note that the decomposition (3.7) is also valid for the group $\mathfrak{pu}(2,2|4)$. Therefore, the hypercharge $\mathfrak{B}$ is preserved by $\mathcal{H}$. Obviously, also the length, measured by the operator $\mathcal{L}$, is conserved

$$[\mathcal{H}, \mathfrak{B}] = [\mathcal{H}, \mathcal{L}] = 0. \tag{3.11}$$

This will clearly not be the case for higher-loop corrections to the dilatation generator, which act on more than two fields at the same time. At higher loops, the Konishi anomaly [104] mixes operators of different hypercharges. The same points also hold for the length L of a state. Nevertheless, it makes perfect sense to speak of the leading order hypercharge and length to describe a state. Mixing with states of different hypercharges or lengths is sub-leading, because the one-loop dilatation generator conserves these.

At this point we can also exclude the possibility of a 'half-loop' contribution $\mathfrak{D}_1$ or a length non-preserving contribution to $\mathfrak{D}_2$ on algebraic grounds: There is no overlap between the irreducible modules in the in and out channels

$$\mathcal{V}_{\mathrm{F}}^0 \notin \mathcal{V}_{\mathrm{F}}^3, \quad \mathcal{V}_{\mathrm{F}} \notin \mathcal{V}_{\mathrm{F}}^2, \qquad \mathcal{V}_{\mathrm{F}}^0 \notin \mathcal{V}_{\mathrm{F}}^4, \quad \mathcal{V}_{\mathrm{F}} \notin \mathcal{V}_{\mathrm{F}}^3, \tag{3.12}$$

which can be seen by comparing the scaling dimensions. The only possible contributions up to second order in g are $\mathcal{V}_{\mathrm{F}}^2 \mapsto \mathcal{V}_{\mathrm{F}}^2$ or $\mathcal{V}_{\mathrm{F}} \mapsto \mathcal{V}_{\mathrm{F}}$ as assumed in (3.2).

3.1.4 Planar Limit

We can now take the planar limit of (3.6)

$$\mathcal{H} = \sum_{j=0}^{\infty} 2\,C_j\,(\mathcal{P}_j)_{\mathcal{CD}}^{\mathcal{AB}}\,\left\{{\mathcal{CD} \atop \mathcal{AB}}\right\} \tag{3.13}$$

in the notation introduced in Sec. 2.6.2. In this chapter we will use a slightly different notation which assumes that $\mathcal{H}$ acts on a spin chain of length L and transforms two adjacent fields[2]

$$\mathcal{H} = \sum_{p=1}^{L} \mathcal{H}_{p,p+1}, \qquad \mathcal{H}_{p,p+1} = \sum_{j=0}^{\infty} 2\,C_j \mathcal{P}_{p,p+1,j}. \tag{3.14}$$

[2]We assume cyclic site indices, i.e. $\mathcal{H}_{L,L+1} = \mathcal{H}_{L,1}$.

The symbol $\mathcal{P}_{p,p+1,j}$ projects the fields at positions $p, p+1$ to the module $\mathcal{V}_j$. We see that all coefficients C_j can be read off from this Hamiltonian. Therefore, the *Hamiltonian density* $\mathcal{H}_{12}$ generalises uniquely to the non-planar Hamiltonian $\mathcal{H}$ in (3.6). In what follows we can safely restrict ourselves to the investigation of $\mathcal{H}_{12}$ alone.

To simplify some expressions, we introduce the $\mathfrak{psu}(2,2|4)$ invariant total spin operator $\mathcal{J}_{12}$ by the implicit definition

$$\mathcal{J}_{12}\,\mathcal{V}_j = j\,\mathcal{V}_j. \tag{3.15}$$

We can now define a function $f(\mathcal{J}_{12})$ of this operator by

$$f(\mathcal{J}_{12}) = \sum_{j=0}^{\infty} f(j)\,\mathcal{P}_{12,j}. \tag{3.16}$$

Using the short notation the Hamiltonian density becomes simply

$$\mathcal{H}_{12} = 2\,C(\mathcal{J}_{12}). \tag{3.17}$$

3.2 The Fermionic $\mathfrak{su}(1,1) \times \mathfrak{u}(1|1)$ Subsector

It remains to determine the coefficients C_j. To accomplish this task we will consider the closed subsector $(1,3)$ of $\mathcal{N}=4$ SYM (c.f. Sec. 2.3.5) and show how to derive the Hamiltonian from the algebraic constraints.

3.2.1 Fields and States

The fields in this subsector consist only of the fermion $\Psi = \Psi_{42}$ with K derivatives $\mathcal{D} = \mathcal{D}_{22}$ acting on it, see Sec. 2.3.5. In the oscillator notation of Sec. 1.9 they can be written as

$$|k\rangle := \frac{1}{(k+1)!}\,(\mathbf{a}_2^\dagger \mathbf{b}_2^\dagger)^k |\Psi\rangle = \frac{1}{(k+1)!}\,(\mathbf{a}_2^\dagger \mathbf{b}_2^\dagger)^k \mathbf{a}_2^\dagger \mathbf{d}_2^\dagger |\mathcal{Z}\rangle. \tag{3.18}$$

States are constructed as tensor products of the fields

$$|k_1, \ldots, k_L\rangle \tag{3.19}$$

with the cyclic identifications (the sign is due to statistics)

$$|k_1, \ldots k_p, k_{p+1}, \ldots, k_L\rangle = (-1)^{p(L-p)} |k_{p+1}, \ldots, k_L, k_1, \ldots k_p\rangle. \tag{3.20}$$

The identifications exclude states of the form

$$|k_1, \ldots, k_{L/2}, k_1, \ldots, k_{L/2}\rangle = 0. \tag{3.21}$$

The weight of a state with a total number of K excitations is given by

$$w = (3L/2 + K; K + L, K; 0, 0, L; L/2, L). \tag{3.22}$$

This weight is beyond a unitarity bound of $\mathfrak{psu}(2,2|4)$ and cannot be primary. The generic shift from the highest superconformal weight to the highest weight within the subsector is given by

$$\delta w = \delta w'_{\mathrm{i}} + \delta w'_{\mathrm{ii}} + (1; -2, 0; -1, 0, 1; 1, 0) = (2; -1, +1; 0, 0, 2; 1, 0). \tag{3.23}$$

The shifts $\delta w'_{\mathrm{i,ii}}$ take the weight beyond the unitarity bound and should be omitted for quarter-BPS multiplets. The additional shift is related to the two additional conditions $n_{\mathbf{c}_2} = n_{\mathbf{c}_2} = 0$ in the definition of the subsector.

3.2.2 Symmetry

The subsector is invariant under an $\mathfrak{su}(1,1) \times \mathfrak{u}(1|1)$ subalgebra of the superconformal algebra. In the fully interacting theory, the $\mathfrak{su}(1,1)$ algebra consists of the generators[3]

$$\mathfrak{J}'_+(g) = \mathfrak{K}^{22}(g), \quad \mathfrak{J}'_-(g) = \mathfrak{P}_{22}(g), \quad \mathfrak{J}'_0(g) = \mathcal{L} - 2\mathfrak{D}_0 - \delta\mathfrak{D}(g). \tag{3.24}$$

Note that the dilatation generator $\mathfrak{D}$ is part of the algebra. At higher loops, one should keep in mind that only half of the anomalous piece appears. The $\mathfrak{su}(1,1)$ algebra is

$$[\mathfrak{J}'_+(g), \mathfrak{J}'_-(g)] = -\mathfrak{J}'_0(g), \qquad [\mathfrak{J}'_0(g), \mathfrak{J}'_\pm(g)] = \pm 2\mathfrak{J}'_\pm(g). \tag{3.25}$$

The $\mathfrak{u}(1|1)$ algebra is generated by

$$\mathcal{L}, \quad \mathfrak{Q}'_-(g) = \mathfrak{Q}^1{}_1(g), \quad \mathfrak{Q}'_+(g) = \mathfrak{S}^1{}_1(g), \quad \delta\mathfrak{D}(g) \tag{3.26}$$

and the non-zero commutators are[4]

$$[\mathcal{L}, \mathfrak{Q}'_\pm(g)] = \mp\mathfrak{Q}'_\pm(g), \qquad \{\mathfrak{Q}'_+(g), \mathfrak{Q}'_-(g)\} = \tfrac{1}{2}\delta\mathfrak{D}(g). \tag{3.27}$$

The generators of $\mathfrak{su}(1,1)$ and $\mathfrak{u}(1|1)$ commute with each other

$$[\mathfrak{J}'(g), \mathcal{L}] = [\mathfrak{J}'(g), \mathfrak{Q}'_\pm(g)] = [\mathfrak{J}'(g), \delta\mathfrak{D}(g)] = 0. \tag{3.28}$$

In the classical limit, the algebra $\mathfrak{u}(1|1)$ is trivial, $\mathfrak{Q}'_\pm(0) = 0$, it transforms between oscillators of type $\mathbf{a}_1^\dagger$ and $\mathbf{c}_1^\dagger$, both of which are absent in this subsector. In the interacting theory, however, the generators $\mathfrak{Q}'_\pm(g)$ must receive non-trivial corrections (see also Fig. 1.14) because they close on $\delta\mathfrak{D}(g)$. In particular, they must produce $\mathfrak{D}_2$ which is possible only if $\mathfrak{Q}'_{\pm,1} \neq 0$.

Let us now restrict to the leading orders of all generators as in Sec. 3.1.1

$$\mathfrak{J}'_\pm := \mathfrak{J}'_{\pm,0}, \quad \mathfrak{J}'_0 := \mathfrak{J}'_{0,0}, \quad \mathfrak{Q}'_\pm := \mathfrak{Q}'_{\pm,1}, \quad \mathcal{H}' := \mathfrak{D}_2. \tag{3.29}$$

The resulting non-trivial commutators of $\mathfrak{u}(1|1)$ are

$$[\mathcal{L}, \mathfrak{Q}'_\pm] = \mp\mathfrak{Q}'_\pm, \qquad \{\mathfrak{Q}'_+, \mathfrak{Q}'_-\} = \tfrac{1}{2}\mathcal{H}'. \tag{3.30}$$

3.2.3 Representations

The fields transform under $\mathfrak{su}(1,1)$ as

$$\mathfrak{J}'_-|k\rangle = (k+2)|k+1\rangle, \quad \mathfrak{J}'_+|k\rangle = k|k-1\rangle, \quad \mathfrak{J}'_0|k\rangle = -2(k+1)|k\rangle, \tag{3.31}$$

as can be inferred from the oscillator representation. All of the fields can be transformed into each other, they therefore span an irreducible module $\mathcal{V}'_{\text{F}}$ of $\mathfrak{su}(1,1)$. The Dynkin label of the highest weight $|0\rangle$, measured by $\mathfrak{J}'_0$, is

$$w'_{\text{F}} = [-2], \tag{3.32}$$

[3]The precise form of $\mathfrak{J}_3$ can be obtained from the commutator of $\mathfrak{P}_{22}$ and $\mathfrak{K}^{22}$ in App. D.1 noting that we can set $\mathfrak{L}^2{}_2 = \frac{1}{2}\mathfrak{D}_0 - \frac{1}{4}\mathcal{L}$ and $\dot{\mathfrak{L}}^2{}_2 = \frac{1}{2}\mathfrak{D} - \frac{3}{4}\mathcal{L}$ in this sector.

[4]Note that $\mathfrak{L}^1{}_1 = -\frac{1}{2}\mathfrak{D}_0 + \frac{1}{4}\mathcal{L}$ and $\mathfrak{R}^1{}_1 = -\frac{1}{4}\mathcal{L}$ in the sector.

in other words, the fields transform in the spin -1 irreducible representation.

The Hamiltonian density $\mathcal{H}'_{12}$ is $\mathfrak{su}(1,1)$ invariant and acts on two fields at a time. Of particular interest is therefore the tensor product of two $\mathcal{V}'_{\text{F}}$'s, by standard $\mathfrak{sl}(2)$ rules it splits into modules of spin $-1-j$, $j \geq 1$

$$\mathcal{V}'_{\text{F}} \times \mathcal{V}'_{\text{F}} = \sum_{j=1}^{\infty} \mathcal{V}'_j, \qquad \text{with} \quad w'_j = [-2-2j]. \tag{3.33}$$

All irreducible modules have multiplicity one and we can write the invariant Hamiltonian as

$$\mathcal{H}'_{12} = C'(\mathcal{J}'_{12}), \tag{3.34}$$

where the total spin operator $\mathcal{J}'_{12}$ is defined implicitly by

$$\mathcal{J}'_{12}\, \mathcal{V}'_j = j\, \mathcal{V}'_j. \tag{3.35}$$

3.2.4 Supercharges

In order to find the Hamiltonian $\mathcal{H}'$, it suffices to find the supercharges $\mathfrak{Q}'_{\pm}$; via the supersymmetry relation (3.30) we can generate $\mathcal{H}'$ later. The supercharges $\mathfrak{Q}'_{\pm}$ are of order g in perturbation theory, therefore they should have three legs (c.f. Sec. 2.5.3). We already know from (3.30) that $\mathfrak{Q}'_-$ increases the length by one and $\mathfrak{Q}'_+$ decreases it. Consequently, we make the ansatz[5]

$$\mathfrak{Q}'_-|m\rangle = \sum_{k=0}^{m-1} c^-_{m,k}|k, m-1-k\rangle. \tag{3.36}$$

The supercharge should commute with all generators $\mathfrak{J}'$ because they belong to distinct algebras. Therefore $\mathfrak{Q}'_-$ conserves the $\mathfrak{J}'_0$ charge and $\mathfrak{Q}'_-|m\rangle$ may only yield states of the form $|k, m-1-k\rangle$. The commutator of $\mathfrak{Q}'_-$ with $\mathfrak{J}'_-$ is

$$[\mathfrak{Q}'_-, \mathfrak{J}'_-]|m\rangle = \sum_{k=0}^{m} \big(c^-_{m+1,k}(n+2) - \delta_{k\neq 0} c^-_{m,k-1}(k+1) - \delta_{k\neq m} c^-_{m,k}(m-k+1)\big)|k, m-k\rangle. \tag{3.37}$$

The coefficients can be computed recursively and one easily confirms that the only possibility to make $[\mathfrak{Q}'_-, \mathfrak{J}'_-]$ vanish identically is $c^-_{m,k} = 0$. For that purpose, start with $m = k = 0$ and find that $c^-_{1,0} = 0$; then continue with $m = 1$, $k = 0, 1$ and so on. In terms of representation theory this is understood because $\mathcal{V}'_{\text{F}}$ and $\mathcal{V}'_{\text{F}} \times \mathcal{V}'_{\text{F}}$ have no irreducible modules in common. This might seem to be disastrous for it leads to $\mathfrak{Q}'_- = \mathcal{H}' = 0$. However, we do not need to require that $[\mathfrak{Q}'_-, \mathfrak{J}'_-]$ vanishes identically, but only that its action annihilates all states. In particular, this allows $[\mathfrak{Q}'_-, \mathfrak{J}'_-]$ to generate a gauge transformation which annihilates gauge invariant states. The only suitable gauge transformation to match $|m\rangle \mapsto |k, m-k\rangle$ is $\mathcal{D}^m\Psi \mapsto \{\Psi, \mathcal{D}^m\Psi\}$. Therefore we should merely require

$$[\mathfrak{Q}'_-, \mathfrak{J}'_-]|m\rangle = c_-|m, 0\rangle + c_-|0, m\rangle. \tag{3.38}$$

[5]Here, $|n\rangle$ is considered to be a field within a state. A single field should be annihilated by $\mathfrak{Q}'_-$.

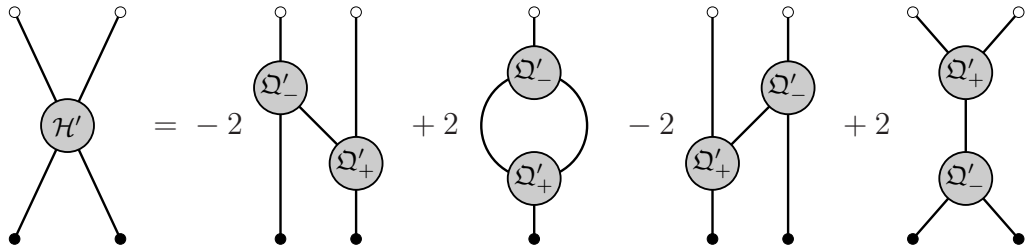

Figure 3.4: Diagrammatic representation of the commutator $\mathcal{H}' = 2\{\mathfrak{Q}'_+, \mathfrak{Q}'_-\}$. The diagrams are to be inserted to a state at the white dots.

It is not difficult to see that there is a unique solution to this equation, namely $c^-_{m,k} = c_-$.

By making a similar ansatz for $\mathfrak{Q}'_+$ we find in total

$$\begin{aligned} \mathfrak{Q}'_-|m\rangle &\sim \frac{1}{2}\sum_{k=0}^{m-1} |k, m-1-k\rangle, \\ \mathfrak{Q}'_+|k, m-k\rangle &\sim \left(\frac{1}{k+1} + \frac{1}{m-k+1}\right)|m+1\rangle. \end{aligned} \tag{3.39}$$

Again, $\mathfrak{Q}'_+$ commutes with $\mathfrak{J}'_+$ only up to a gauge transformation

$$[\mathfrak{Q}'_+, \mathfrak{J}'_+]|k, m-k\rangle \sim (\delta_{k=0} + \delta_{k=m})|m\rangle. \tag{3.40}$$

The other commutators $[\mathfrak{Q}'_+, \mathfrak{J}'_-]$ and $[\mathfrak{Q}'_-, \mathfrak{J}'_+]$ turn out to vanish identically.

3.2.5 The Hamiltonian

We are now ready to compute the Hamiltonian $\mathcal{H}' = 2\{\mathfrak{Q}'_+, \mathfrak{Q}'_-\}$. For definiteness, we will assume unit proportionality constants in (3.39). In total there are four types of diagrams to represent the anticommutator, see Fig. 3.4. To compute $\mathfrak{Q}'_+\mathfrak{Q}'_-$ there are three different ways in which $\mathfrak{Q}'_+$ could act. Let us therefore add labels to indicate the fields on which each generator acts. Firstly, $\mathfrak{Q}'_+$ could act on one of the fields generated by $\mathfrak{Q}'_-$ and one adjacent field (the sign is due to statistics).

$$\begin{aligned} \mathfrak{Q}'_{+,23}\mathfrak{Q}'_{-,1}|k, m-k\rangle &= -\frac{1}{2}\sum_{k'=0}^{k-1}\left(\frac{1}{k-k'} + \frac{1}{m-k+1}\right)|k', m-k'\rangle, \\ \mathfrak{Q}'_{+,12}\mathfrak{Q}'_{-,2}|k, m-k\rangle &= -\frac{1}{2}\sum_{k'=k+1}^{m}\left(\frac{1}{k+1} + \frac{1}{k'-k}\right)|k', m-k'\rangle. \end{aligned} \tag{3.41}$$

Secondly, it could act on both fields that result from the action of $\mathfrak{Q}'_-$

$$\mathfrak{Q}'_{+,12}\mathfrak{Q}'_{-,1}|k, m-k\rangle = \frac{1}{2}\sum_{k'=0}^{k-1}\left(\frac{1}{k'+1} + \frac{1}{k-k'}\right)|k, m-k\rangle = h(k)|k, m-k\rangle, \tag{3.42}$$

where the *harmonic numbers* $h(m)$ are defined as

$$h(m) := \sum_{k=1}^{m}\frac{1}{k}\,. \tag{3.43}$$

We want all interactions to act on two adjacent sites, therefore we should evenly distribute this contribution between both fields

$$\big(\tfrac{1}{2}\mathfrak{Q}'_{+,12}\mathfrak{Q}'_{-,1}+\tfrac{1}{2}\mathfrak{Q}'_{+,23}\mathfrak{Q}'_{-,2}\big)|k,m-k\rangle=\tfrac{1}{2}\big(h(k)+h(m-k)\big)|k,m-k\rangle. \tag{3.44}$$

Finally, $Q'_-Q'_+$ is easily computed

$$\mathfrak{Q}'_{1,-}\mathfrak{Q}'_{+,12}|k,m-k\rangle=\frac{1}{2}\bigg(\frac{1}{k+1}+\frac{1}{m-k+1}\bigg)\sum_{k'=0}^{m}|k',m-k'\rangle. \tag{3.45}$$

In total the action of the Hamiltonian density is

$$\mathcal{H}'_{12}=2\mathfrak{Q}'_{+,23}\mathfrak{Q}'_{-,1}+2\mathfrak{Q}'_{+,12}\mathfrak{Q}'_{-,2}+2\mathfrak{Q}'_{1,-}\mathfrak{Q}'_{+,12}+\mathfrak{Q}'_{+,12}\mathfrak{Q}'_{-,1}+\mathfrak{Q}'_{+,23}\mathfrak{Q}'_{-,2}; \tag{3.46}$$

summing the terms in (3.41,3.44,3.45) we get

$$\begin{aligned}\mathcal{H}'_{12}|k,m-k\rangle\sim&\sum_{k'=0}^{k-1}\bigg(\frac{1}{k+1}-\frac{1}{k-k'}\bigg)|k',m-k'\rangle\\&+\big(h(k+1)+h(m-k+1)\big)|k,m-k\rangle\\&+\sum_{k'=k+1}^{m}\bigg(\frac{1}{m-k+1}-\frac{1}{k'-k}\bigg)|k',m-k'\rangle.\end{aligned} \tag{3.47}$$

The $\mathfrak{su}(1,1)$ invariance of $\mathcal{H}'_{12}$ is inherited from $\mathfrak{Q}'_{\pm}$.

3.2.6 Eigenvalues of the Hamiltonian

In order to transform this expression into the concise form (3.17) we need to find the eigenvalue of the Hamiltonian on module $\mathcal{V}'_j$. The highest weight state of $\mathcal{V}'_j$, which is annihilated by $\mathfrak{J}'_{1,+}+\mathfrak{J}'_{2,+}$, is

$$|j\rangle=\sum_{k=0}^{j-1}\frac{(-1)^k(j-1)!}{k!(j-k-1)!}\,|k,j-k-1\rangle. \tag{3.48}$$

We know that $|j\rangle$ is an eigenstate of $\mathcal{H}'_{12}$ because $\mathcal{H}'_{12}$ is invariant under $\mathfrak{su}(1,1)$. Therefore we only need to compute the coefficient of $|0,j-1\rangle$ in $\mathcal{H}'_{12}|j\rangle$; we obtain

$$\mathcal{H}'_{12}|j\rangle\sim\bigg(1+h(j)+\sum_{k=1}^{j-1}\frac{(-1)^k(j-1)!}{k!(j-k-1)!}\bigg(\frac{1}{k+1}-\frac{1}{k}\bigg)\bigg)|0,j-1\rangle+\dots\,. \tag{3.49}$$

The first part of the sum is easily performed by extending the range and thus completing the binomial $(1-1)^j$

$$\sum_{k=1}^{j-1}\frac{(-1)^k(j-1)!}{(k+1)!(j-k-1)!}=\sum_{k=-1}^{j-1}\frac{(-1)^k(j-1)!}{(k+1)!(j-k-1)!}-1+\frac{1}{j}=-1+\frac{1}{j}\,. \tag{3.50}$$

For the second part we replace $1/k$ by $1/(j-1)+(j-k-1)/k(j-1)$ and get

$$-\sum_{k=1}^{j-1}\frac{(-1)^k(j-1)!}{k!(j-k-1)!k}=-\sum_{k=1}^{j-1}\frac{(-1)^k(j-2)!}{k!(j-k-1)!}-\sum_{k=1}^{j-1}\frac{(-1)^k(j-2)!}{k!(j-k-2)!k}\,. \tag{3.51}$$

The first of the resulting sums is done by completing $(1-1)^{j-1}$ and evaluates to $1/(j-1)$. The second sum is the same as above with j replaced by $j-1$. By induction we thus get

$$-\sum_{k=1}^{j-1}\frac{(-1)^k(j-1)!}{k!(j-k-1)!k}=\frac{1}{j-1}+\frac{1}{j-2}+\ldots+\frac{1}{1}=h(j-1). \tag{3.52}$$

Summing up we obtain

$$\mathcal{H}'_{12}|j\rangle\sim\big(1+h(j)-1+1/j+h(j-1)\big)|0,j-1\rangle+\ldots=2h(j)\,|j\rangle. \tag{3.53}$$

where we have reconstructed all other terms of $|j\rangle$ by means of invariance. We have therefore determined the Hamiltonian in this subsector up to an overall constant

$$\mathcal{H}'_{12}\sim 2h(\mathcal{J}'_{12}). \tag{3.54}$$

3.3 The Lift to $\mathfrak{psu}(2,2|4)$

Let us now compare the results of the preceding sections. This allows us to derive the complete one-loop dilatation operator of $\mathcal{N}=4$ supersymmetric gauge theory. The state $|j\rangle$ has length $L=2$ and $K=j-1$ excitations. According to (3.22,3.23) the highest weight of the superconformal multiplet that includes $|j\rangle$ is

$$w'_j=(j;j-2,j-2;0,0,0;0,2). \tag{3.55}$$

For $j=1$ the shift is exceptional, it excludes $\delta w'_{\rm i}+\delta w'_{\rm ii}$ in (3.23) because the multiplet is quarter-BPS. The corresponding highest superconformal weight is

$$w'_1=(2;0,0;1,0,1;0,2). \tag{3.56}$$

These are precisely the highest weights of the superconformal modules $\mathcal{V}_j$, $j\geq 1$ in (3.7). There is a one-to-one correspondence between the modules $\mathcal{V}_j$, $j\geq 1$ (3.8) and $\mathcal{V}'_j$

$$\mathcal{V}'_j\subset\mathcal{V}_j,\quad j\geq 1. \tag{3.57}$$

Using the fact that the two Hamiltonians must agree within the subsector we find

$$2\,C_j\,|j\rangle=\mathcal{H}_{12}\,|j\rangle=\mathcal{H}'_{12}\,|j\rangle\sim 2h(j)\,|j\rangle,\quad j\geq 1. \tag{3.58}$$

This leaves one overall constant and the coefficient C_0 to be determined. The multiplet $\mathcal{V}_0$ is half-BPS and thus protected, it cannot acquire an anomalous dimension; we must set

$$C_0=h(0)=0. \tag{3.59}$$

The overall constant cannot possibly be fixed by algebraic considerations, we need to match it to a field theory calculation. For example, we can use the anomalous dimension

of the Konishi operator, $D_2 = 6$ (2.20), as input. In the free theory, it is part of the multiplet $\mathcal{V}_2$. The Hamiltonian acting on a state of length $L = 2$ is $\mathcal{H} = \mathcal{H}_{12} + \mathcal{H}_{21}$. For the Konishi state we therefore get $D_2 = 4C_2$ and set $C_2 = \frac{3}{2} = h(2)$. The resulting 'Hamiltonian density' $\mathcal{H}_{12}$ for $\mathcal{N} = 4$ SYM is thus

$$\mathcal{H}_{12} = \sum_{j=0}^{\infty} 2h(j)\, \mathcal{P}_{12,j} = 2h(\mathcal{J}_{12}). \tag{3.60}$$

To conclude, the one-loop dilatation generator of $\mathcal{N} = 4$ can be written as

$$\mathfrak{D}(g) = \mathfrak{D}_0 - \frac{g_{\text{YM}}^2}{8\pi^2} \sum_{j=0}^{\infty} h(j)\, (\mathcal{P}_j)^{\mathcal{AB}}_{\mathcal{CD}}\, {:}\text{Tr}\left[\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{C}}\right]\left[\mathcal{W}_{\mathcal{B}}, \check{\mathcal{W}}^{\mathcal{D}}\right]{:} + \mathcal{O}(g^3), \tag{3.61}$$

where we have inserted the conventional Yang-Mills coupling constant $g_{\text{YM}}^2 = 8\pi^2 g^2/N$. This is the principal result of this chapter. The coefficients are the *harmonic numbers* $h(j)$, elements of the harmonic series

$$\text{'}\textit{harmonic numbers}\text{'}: \quad h(m) := \sum_{k=1}^{m} \frac{1}{k} = \Psi(m+1) - \Psi(1), \tag{3.62}$$

which can also be expressed in terms of the digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$. In App. F we will present the *harmonic action*, a practical implementation of the action of $\mathcal{H}_{12} = 2h(\mathcal{J}_{12})$.

3.4 The Bosonic $\mathfrak{su}(1,1)$ Subsector

In this section we will consider the closed subsector $(2,2)$; it is a nice sector, quite similar to the fermionic subsector $(3,1)$, see Sec. 3.2, and we will make use of it later. The fields in this subsector consist only of the field $\mathcal{Z} = \Phi_{34}$ with K derivatives $\mathcal{D} = \mathcal{D}_{22}$ acting on it. They can be written as

$$|k\rangle := \frac{1}{k!}\, (\mathbf{a}_2^\dagger \mathbf{b}_2^\dagger)^k |\mathcal{Z}\rangle = \frac{1}{k!}\, \mathcal{D}^k \mathcal{Z}. \tag{3.63}$$

States are constructed as tensor products of the fields modulo cyclicity of the trace

$$|k_1, \ldots, k_L\rangle = |k_{p+1}, \ldots, k_L, k_1, \ldots k_p\rangle. \tag{3.64}$$

The weight of a state with a total number of K excitations is given by

$$w = (L + K; K, K; 0, L, 0; 0, L). \tag{3.65}$$

The generic shift from the highest superconformal weight to the highest weight within the subsector is given by

$$\delta w = \delta w'_{\text{i}} + \delta w'_{\text{ii}} + (1; -1, -1; -1, +2, -1; 0, 0) = (2; -2, -2; 0, -2, 0; 0, 0). \tag{3.66}$$

The shifts $\delta w'_{\text{i,ii}}$ take the weight beyond the unitarity bound and should be omitted for quarter-BPS multiplets with $K = 1$. The additional shift is related to the two additional

conditions $n_{\mathbf{c}_2} = n_{\mathbf{d}_2} = 0$ in the definition of the subsector; it should be omitted for half-BPS multiplets with $K = 0$.

The subsector is invariant under an $\mathfrak{su}(1,1) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$ subalgebra of the superconformal algebra. The $\mathfrak{su}(1,1)$ algebra of generators $\mathfrak{J}''$ is exactly the same as in (3.24,3.25). The two $\mathfrak{u}(1)$ charges are the length L and anomalous dimension $\delta D(g)$.

The fields transform under $\mathfrak{su}(1,1)$ as

$$\mathfrak{J}''_-|k\rangle = (k+1)|k+1\rangle, \quad \mathfrak{J}''_+|k\rangle = k|k-1\rangle, \quad \mathfrak{J}''_0|k\rangle = -(2k+1)|k\rangle. \tag{3.67}$$

All fields can be transformed into each other, they therefore span an irreducible module $\mathcal{V}''_{\text{F}}$ of $\mathfrak{su}(1,1)$. The Dynkin label of the highest weight $|0\rangle$ is

$$w''_{\text{F}} = [-1], \tag{3.68}$$

in other words, the fields transform in the spin $-\frac{1}{2}$ irreducible representation.

The tensor product of two $\mathcal{V}''_{\text{F}}$ is as in (3.33), but here also the module $\mathcal{V}''_0$ with $j = 0$ appears. There is a one-to-one correspondence between the modules $\mathcal{V}''_j$ and the irreducible modules of the superconformal algebra

$$\mathcal{V}''_j \subset \mathcal{V}_j. \tag{3.69}$$

In [50] the Hamiltonian density was obtained from a direct field theory computation

$$\mathcal{H}''_{12}\,|m, m-k\rangle = \sum_{k'=0}^{m} \left(\delta_{k=k'} \big(h(k) + h(m-k) \big) - \frac{\delta_{k\neq k'}}{|k-k'|} \right) |k', m-k'\rangle, \tag{3.70}$$

It is straightforward to verify that $\mathcal{H}''_{12}$ is invariant under the generators $\mathfrak{J}''_{12}$. As in Sec. 3.2.6, one can show that (3.70) is equivalent to

$$\mathcal{H}''_{12} = 2h(\mathcal{J}''_{12}). \tag{3.71}$$

This result can be lifted to $\mathcal{N} = 4$ SYM as well to obtain the complete one-loop dilatation operator [50].

3.5 Planar Spectrum

In this section we will apply the planar, one-loop dilatation generator (Hamiltonian) to find some anomalous dimensions (energies).

3.5.1 Lowest-Lying States

In Tab. 3.2 we present the spectrum of lowest-lying states in $\mathcal{N} = 4$ SYM. For a given highest weight of the classical algebra we write the anomalous dimensions along with the parity P. The parity is defined such that for a SO(N) or Sp(N) gauge group the states with negative parity are projected out. Parity $P = \pm$ indicates a pair of states with opposite parity and degenerate energies. Furthermore, we have indicated states with conjugate representations for which the order of $\mathfrak{su}(2)^2$, $\mathfrak{su}(4)$ and $\mathfrak{psu}(2,2|4)$ labels as well as the hypercharge B are inverted. Generically, the one-loop energy shifts are not

D_0	$\mathfrak{su}(2)^2$	$\mathfrak{su}(4)$	B	L	$\mathfrak{psu}(2,2\|4)$	E^P
2	$[0,0]$	$[0,2,0]$	0	2	$[0;0;0,2,0;0;0]$	0^+
	$[0,0]$	$[0,0,0]$	0	2	$[0;1;0,0,0;1;0]$	6^+
3	$[0,0]$	$[0,3,0]$	0	3	$[0;0;0,3,0;0;0]$	0^-
	$[0,0]$	$[0,1,0]$	0	3	$[0;1;0,1,0;1;0]$	4^-
4	$[0,0]$	$[0,4,0]$	0	4	$[0;0;0,4,0;0;0]$	0^+
	$[0,0]$	$[0,2,0]$	0	4	$[0;1;0,2,0;1;0]$	$(10E-20)^+$
	$[0,0]$	$[1,0,1]$	0	4	$[0;1;1,0,1;1;0]$	6^-
	$[0,0]$	$[0,0,0]$	0	4	$[0;2;0,0,0;2;0]$	$(13E-32)^+$
	$[2,0]$	$[0,0,0]$	1	3	$[2;3;0,0,0;2;0]$	9^- + conj.
	$[1,1]$	$[0,1,0]$	0	3	$[1;2;0,1,0;2;1]$	$\frac{15}{2}^\pm$
	$[2,2]$	$[0,0,0]$	0	2	$[2;3;0,0,0;3;2]$	$\frac{25}{3}^+$
5	$[0,0]$	$[0,5,0]$	0	5	$[0;0;0,5,0;0;0]$	0^-
	$[0,0]$	$[0,3,0]$	0	5	$[0;1;0,3,0;1;0]$	$2^-, 6^-$
	$[0,0]$	$[1,1,1]$	0	5	$[0;1;1,1,1;1;0]$	$5^\pm$
	$[0,0]$	$[0,0,2]$	0	5	$[0;2;0,0,2;1;0]$	$(14E-36)^+$ + conj.
	$[2,0]$	$[0,0,2]$	1	4	$[2;3;0,0,2;1;0]$	10^- + conj.
	$[0,0]$	$[0,1,0]$	0	5	$[0;2;0,1,0;2;0]$	$10^-, 10^-, (10E-20)^-$
	$[2,0]$	$[0,1,0]$	1	4	$[2;3;0,1,0;2;0]$	$(16E-62)^+$ + conj.
	$[1,1]$	$[0,2,0]$	0	4	$[1;2;0,2,0;2;1]$	$6^\pm$
	$[1,1]$	$[1,0,1]$	0	4	$[1;2;1,0,1;2;1]$	$5^\pm, 10^\pm$
	$[1,1]$	$[0,0,0]$	0	4	$[1;3;0,0,0;3;1]$	$9^\pm$
	$[2,2]$	$[0,1,0]$	0	3	$[2;3;0,1,0;3;2]$	6^-
5.5	$[1,0]$	$[0,2,1]$	$\frac{1}{2}$	5	$[1;2;0,2,1;1;0]$	$8^\pm$ + conj.
	$[1,0]$	$[1,1,0]$	$\frac{1}{2}$	5	$[1;2;1,1,0;2;0]$	$(16E-62)^\pm$ + conj.
	$[1,0]$	$[0,0,1]$	$\frac{1}{2}$	5	$[1;3;0,0,1;2;0]$	$(\frac{35}{2}E-\frac{305}{4})^\pm$ + conj.
	$[2,1]$	$[0,1,1]$	$\frac{1}{2}$	4	$[2;3;0,1,1;2;1]$	$9^\pm$ + conj.
	$[2,1]$	$[1,0,0]$	$\frac{1}{2}$	4	$[2;3;1,0,0;3;1]$	$(\frac{37}{2}E-\frac{333}{4})^\pm$ + conj.
	$[3,2]$	$[0,0,1]$	$\frac{1}{2}$	3	$[3;4;0,0,1;3;2]$	$10^\pm$ + conj.

Table 3.2: All one-loop planar anomalous dimensions of primary operators with $D_0 \leq 5.5$. The label P refers to parity, $P = \pm$ indicates a degenerate pair of states. The label '+conj.' represents conjugate states with $\mathfrak{su}(2)^2, \mathfrak{su}(4), \mathfrak{psu}(2,2|4)$ labels reversed and opposite hypercharge B.

fractional numbers but solutions to some algebraic equations. We refrain from solving these numerically, but instead give the equations. In the table such states are represented by polynomials $X(E)$ of degree $k-1$. The true energies E are obtained as solutions to the equation

$$E^k = X(E). \tag{3.72}$$

For example, the quadratic polynomial for the state with weight $(4;0,0;0,2,0;0,4)$ is $X(E) = 10E - 20$. It translates to the energy (see also [45])

$$E^2 = 10E - 20, \qquad E = 5 \pm \sqrt{5}. \tag{3.73}$$

The table was computed as follows: A `C++` programme was used to determine all highest weight states up to and including classical dimension 5.5 as well as their descendants. In analogy to the sieve of Eratostene the algorithm [105] subsequently removes descendants from the set of all states. What remains, are the primary states. Please refer to [106] for details of the implementation of the sieve algorithm. For each multiplet we pick one state and compute the total excitation numbers using Tab. D.2. Here it is crucial to choose a descendant for which the mixing problem is minimised. This reduces the size of the energy matrix to be computed and diagonalised. For these purposes, a good descendant usually has as few different types of oscillators as possible.

In a `Mathematica` programme all states with a given set of oscillator excitations were collected: We spread the oscillators on the sites of the spin chain in all possible ways taking the central charge constraint at each site into account. Identical states with respect to cyclicity of the trace are dropped. In a second step, the harmonic action, c.f. App. F, was applied to all the states to determine the matrix of anomalous dimensions. For all the descendants which were removed in the sieve algorithm, we remove the corresponding energy eigenvalues. The remaining eigenvalues are the one-loop planar anomalous dimensions of highest weight states.

To go to higher canonical dimensions involves obtaining and diagonalising bigger and bigger matrices. One can reduce the complexity by going to certain subsectors. The smallest subsector is the $\mathfrak{su}(2)$ subsector, see Sec. 2.4. There are only two fields, $\mathcal{Z}, \phi$, which we might indicate in a planar notation as

$$|0\rangle = \mathcal{Z}, \quad |1\rangle = \phi. \tag{3.74}$$

The Hamiltonian density acts on two adjacent fields, within this sector it is (2.110)

$$\mathcal{H}'''_{12}\,|k_1, k_2\rangle = |k_1, k_2\rangle - |k_2, k_1\rangle. \tag{3.75}$$

Here there are far less states and it is much easier to compute the energy matrix. In Tab. 3.4 we show a complete table of states and energies up to classical dimension $D_0 \leq 9$. We have omitted the vacuum states with $K = 0$; there is one for each length L and its energy vanishes. The states and their energies can be obtained conveniently using a computer algebra system. In App. E we present a couple of `Mathematica` functions to deal with the $\mathfrak{su}(2)$ subsector. Similarly, we can obtain the spectrum for the bosonic $\mathfrak{su}(1,1)$ subsector. The expression (3.70) can be used to calculate any one-loop anomalous dimension within this subsector. We display our results in Tab. 3.6.

There are two points to be observed in the spectra in Tab. 3.2,3.4,3.6. Firstly, we note the appearance of pairs of states with degenerate energy and opposite parities $P = \pm$

$$\text{`}\textit{paired state}\text{'}: \quad E_+ = E_-. \tag{3.76}$$

L	K	E^P
4	2	6^+
5	2	4^-
6	2	$(10E-20)^+$
	3	6^-
7	2	$2^-, 6^-$
	3	$5^\pm$

L	K	E^P
8	2	$(14E^2-56E+56)^+$
	3	$4^\pm, 6^-$
	4	$(20E^2-116E+200)^+$
9	2	$(8E-8)^-, 4^-$
	3	$(17E^2-90E+147)^\pm$
	4	$5^\pm, (12E-24)^-$

Table 3.4: The lowest-lying states within the $\mathfrak{su}(2)$ subsector [38]. The weights of the corresponding primaries are $(L-2;0,0;K-2,L-2K,K-2;0,L-2)$.

D_0	L	K	E^P
4	2	2	6^+
5	3	2	4^-
6	4	2	$(10E-20)^+$
	3	3	$\frac{15}{2}^\pm$
	2	4	$\frac{25}{3}^+$
7	5	2	$2^-, 6^-$
	4	3	$6^\pm$
	3	4	6^-

D_0	L	K	E^P
8	6	2	$(14E^2-56E+56)^+$
	5	3	$(\frac{25}{2}E-\frac{147}{4})^\pm$
	4	4	$\frac{23}{3}^\pm, (\frac{73}{3}E^2-\frac{553}{3}E+\frac{1274}{3})^+$
	3	5	$\frac{35}{4}^\pm$
	2	6	$\frac{49}{5}^+$
9	7	2	$(8E-8)^-, 4^-$
	6	3	$(19E^2-\frac{459}{4}E+216)^\pm$
	5	4	$(13E-32)^-, (\frac{97}{6}E-\frac{2291}{36})^\pm$
	4	5	$(\frac{35}{2}E-\frac{665}{9})^\pm$
	3	6	$\frac{22}{3}^-, \frac{227}{20}^\pm$

Table 3.6: The first few states within the bosonic $\mathfrak{su}(1,1)$ subsector [50]. The weights of the corresponding primaries are $(L+K-2;K-2,K-2;0,L-2,0;0,L)$.

This will be an important issue for integrability discussed in Ch. 4. Secondly, we find some overlapping primaries in Tab. 3.4,3.6, clearly their energies do agree. What is more, we find that a couple of energies repeatedly occur. These are for example, $6, 10, 5, 9$, but also $10E-20$ and $13E-32$. As these states are primaries transforming in different representations, they cannot be related by $\mathfrak{psu}(2,2|4)$. Of course, these degeneracies could merely be a coincidence of small numbers. Nevertheless the reappearance of e.g. $13E-32$ is somewhat striking. This could hint at yet another symmetry enhancement of the planar one-loop Hamiltonian. It might also turn out to be a consequence of integrability. Furthermore, one might speculate that it is some remnant of the broken higher spin symmetry of the free theory, see e.g. [107] and references in [105].

3.5.2 Two Partons

A straightforward exercise is to determine the spectrum of states of length $L=2$. These so-called twist-two states can conveniently be written as

$$\mathcal{O}_{j,\mathcal{AB}} = (\mathcal{P}_j)^{\mathcal{CD}}_{\mathcal{AB}} \operatorname{Tr} \mathcal{W}_{\mathcal{C}} \mathcal{W}_{\mathcal{D}}. \tag{3.77}$$

Note that j must be even due to cyclicity of the trace. Using (3.60) we find[6]

$$E = 4h(j), \qquad \delta D = \frac{g_{\rm YM}^2 N}{2\pi^2}\, h(j) + \mathcal{O}(g^3) \tag{3.78}$$

in agreement with the results of [43, 44]. Twist-two states have positive parity.

3.5.3 Three Partons

For states of length $L = 3$ the following multiplets are found within a trace [103]

$$\mathrm{Tr}\, \mathcal{V}_{\rm F} \times \mathcal{V}_{\rm F} \times \mathcal{V}_{\rm F} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \big(\mathcal{V}^{-}_{2m,2n} + \mathcal{V}^{+}_{2m+1,2n} + c_n \mathcal{V}^{\pm}_{m,n+3}\big), \tag{3.79}$$

where $c_{0,1,2,3,4,5} = (1, 0, 1, 1, 1, 1)$ and $c_{n+6} = c_n + 1$. The modules $\mathcal{V}_{m,n}$ have highest weights

$$\begin{aligned}
w_{0,0} &= (3; 0, 0; 0, 3, 0; 0, 3),\\
w_{0,n} &= (n+1; n-2, n-2; 0, 1, 0; 0, 3),\\
w_{1,0} &= (3; 0, 0; 0, 0, 0; 0, 3),\\
w_{1,n} &= (n+5/2; n, n-1; 0, 0, 1; 1/2, 3),\\
w_{m,n} &= (n+2m; n+2m-2, n+m-2; 0, 0, 0; 1, 3)
\end{aligned} \tag{3.80}$$

and the conjugate $w_{-m,n}$ has reversed $\mathfrak{su}(2)^2, \mathfrak{su}(4)$ labels and opposite hypercharge. The multiplets $\mathcal{V}_{0,n}$ have components in the subsector $(2, 2)$, see (3.4), the multiplets $\mathcal{V}_{1,n}$ have components in the fermionic subsector $(3, 1)$, see (3.2), and all the other $\mathcal{V}_{m,n}$ are represented in the sector $(4, 0)$.

By inspecting the spectrum of lowest-lying states and their energies, we find that almost all of them form pairs with degenerate energies. We list the pairs in Tab. 3.8.[7] Concerning the unpaired states, there is one for each even n, it has parity $(-1)^{m+1}$. For the unpaired states one can observe a pattern in the table of energies, Tab. 3.10. We find that all energies agree with the formula

$$E = 2h(\tfrac{1}{2}m - \tfrac{1}{2}) + 2h(m + \tfrac{1}{2}n) + 2h(\tfrac{1}{2}m + \tfrac{1}{2}n) - 2h(-\tfrac{1}{2}). \tag{3.81}$$

In particular, for $m = 1$ the energies are

$$E = 2h(1 + \tfrac{1}{2}n) + 2h(\tfrac{1}{2} + \tfrac{1}{2}n) - 2h(-\tfrac{1}{2}) = 4h(n+2), \tag{3.82}$$

which agrees precisely with the energy (3.78) of the short twist-two multiplet $\mathcal{V}_{2n+2}$. Superconformal invariance requires this degeneracy so that the short multiplets can join to form a long multiplet. The cases $m = 0$ and $n = 0$ also seem interesting, we find $E = 4h(\frac{1}{2}n)$ and $E = 6h(m)$.

[6] Note that $\mathcal{H} = \mathcal{H}_{12} + \mathcal{H}_{21} = 2\mathcal{H}_{12} = 4h(\mathcal{J}_{12})$.

[7] The energies are all rational numbers because there is always just a single pair up to $n \leq 8$ (3.79). Starting from $n = 9$ there is more than one pair and the energies become irrational.

$n\backslash m$	0	1	2	3	4	5	6
3	$\frac{15}{2}$	10	$\frac{47}{4}$	$\frac{131}{10}$	$\frac{71}{5}$	$\frac{1059}{70}$	$\frac{4461}{280}$
5	$\frac{35}{4}$	$\frac{133}{12}$	$\frac{761}{60}$	$\frac{487}{35}$	$\frac{12533}{840}$	$\frac{39749}{2520}$	$\frac{13873}{840}$
6	$\frac{227}{20}$	$\frac{761}{60}$	$\frac{967}{70}$	$\frac{2069}{140}$	$\frac{39349}{2520}$	$\frac{2747}{168}$	$\frac{3929}{231}$
7	$\frac{581}{60}$	$\frac{179}{15}$	$\frac{3763}{280}$	$\frac{18383}{1260}$	$\frac{39133}{2520}$	$\frac{7543}{462}$	$\frac{94373}{5544}$
8	$\frac{5087}{420}$	$\frac{1403}{105}$	$\frac{18187}{1260}$	$\frac{38677}{2520}$	$\frac{49711}{3080}$	$\frac{2593}{154}$	$\frac{629227}{36036}$

Table 3.8: First few paired anomalous dimensions for $\mathcal{V}_{m,n}$ [103].

$n\backslash m$	0	1	2	3	4	5	6
0	0	6	9	11	$\frac{25}{2}$	$\frac{137}{10}$	$\frac{147}{10}$
2	4	$\frac{25}{3}$	$\frac{32}{3}$	$\frac{123}{10}$	$\frac{407}{30}$	$\frac{3067}{210}$	$\frac{542}{35}$
4	6	$\frac{49}{5}$	$\frac{71}{6}$	$\frac{929}{70}$	$\frac{72}{5}$	$\frac{9661}{630}$	$\frac{2259}{140}$
6	$\frac{22}{3}$	$\frac{761}{70}$	$\frac{191}{15}$	$\frac{8851}{630}$	$\frac{528}{35}$	$\frac{221047}{13860}$	$\frac{21031}{1260}$
8	$\frac{25}{3}$	$\frac{7381}{630}$	$\frac{202}{15}$	$\frac{101861}{6930}$	$\frac{6581}{420}$	$\frac{329899}{20020}$	$\frac{21643}{1260}$
10	$\frac{137}{15}$	$\frac{86021}{6930}$	$\frac{493}{35}$	$\frac{2748871}{180180}$	$\frac{20383}{1260}$	$\frac{203545}{12012}$	$\frac{122029}{6930}$

Table 3.10: First few unpaired anomalous dimensions for $\mathcal{V}_{m,n}$ [103]. The parity is $P = (-1)^{m+1}$.

3.5.4 Two Excitations

Instead of considering a fixed number of fields, one can also consider the L-particle vacuum state $|\mathcal{Z}, L\rangle$ and add a small number of excitations, see Sec. 2.3.6. A state without excitations is just the half-BPS vacuum and a state with one excitation is related to the vacuum by one of the lowering operators. The first interesting case is two excitations [14]. Assume we consider four oscillator excitations of type $\mathbf{a}_2^\dagger, \mathbf{a}_2^\dagger, \mathbf{b}_2^\dagger, \mathbf{b}_2^\dagger$. This corresponds to a state of the bosonic $\mathfrak{su}(1,1)$ sector with a total of two derivatives $\mathcal{D}$ acting on L fields $\mathcal{Z}$. A useful basis of states is thus

$$\mathcal{E}_1^L = \operatorname{Tr} \mathcal{D}\mathcal{D}\mathcal{Z}\, \mathcal{Z}^{L-1}, \qquad \mathcal{E}_p^L = \operatorname{Tr} \mathcal{D}\mathcal{Z}\, \mathcal{Z}^{p-2}\, \mathcal{D}\mathcal{Z}\, \mathcal{Z}^{L-p}. \tag{3.83}$$

Note that we should identify $\mathcal{E}_p^L$ and $\mathcal{E}_{L+2-p}^L$ due to cyclicity of the trace and consider a matrix with half the number of rows and columns. Equivalently, we may choose to restrict to vectors which are symmetric under $p \leftrightarrow L+2-p$. Using the Hamiltonian (3.70), we find the matrix of anomalous dimensions in this basis

$$H = \left(\begin{array}{c|cccccc} +2 & -1 & & & & & -1 \\ \hline -2 & +4 & -2 & & & & \\ & -2 & \ddots & \ddots & & & \\ & & \ddots & \ddots & -2 & & \\ & & & -2 & +4 & -2 & \\ -2 & & & & -2 & +4 \end{array}\right). \tag{3.84}$$

The bulk of the matrix has precisely the form of a second lattice derivative. The appropriate ansatz to diagonalise it, is a vector with elements $\cos(ap+b)$. The boundary contributions together with the symmetry determine the constants a and b. The matrix

(3.84) has the following exact eigenvectors

$$\mathcal{O}_n^L = \frac{1}{L}\sum_{p=1}^{L} \cos\left(\frac{\pi n(2p-1)}{L+1}\right)\mathcal{E}_p^L. \tag{3.85}$$

Note that $\mathcal{O}_n^L = \mathcal{O}_{-n}^L = -\mathcal{O}_{L+1-n}^L$. Thus the set of independent modes is given by the mode numbers $0 \le n < (L+1)/2$. The corresponding exact planar one-loop anomalous dimension is [27]

$$E_n^L = 8\sin^2\frac{\pi n}{L+1}, \qquad \delta D = \frac{g_{\rm YM}^2 N}{\pi^2}\,\sin^2\frac{\pi n}{L+1} + \mathcal{O}(g^3). \tag{3.86}$$

This is just one component of a multiplet of the residual symmetry $\mathfrak{psu}(2|2)\times\mathfrak{psu}(2|2)$ within the sector. The oscillators $\mathbf{A}^\dagger = (\mathbf{a}^\dagger, \mathbf{c}^\dagger)$ (c.f. Sec. 2.3.6) transform in the fundamental representation $[0;0;1]$ of one of the $\mathfrak{psu}(2|2)$'s. For two excitations we should consider the tensor product of two fundamental modules which is $[0;0;1]\times[0;0;1] = [0;0;2]_+ + [0;1;0]_-$. These two correspond to the symmetric and antisymmetric combination of two indices A, B. The same applies to the oscillators $\dot{\mathbf{A}}^\dagger = (\mathbf{b}^\dagger, \mathbf{d}^\dagger)$. In total we find four multiplets corresponding to the combined symmetrisations $++, +-, -+, --$:

$$\begin{aligned}
\mathcal{O}^L_{n,\{AB\}\{\dot C\dot D\}} &= \sum_{p=1}^{L} \cos\left(\frac{\pi n(2p-1)}{L+1}\right)\,\mathrm{Tr}\,\mathbf{A}^\dagger_{1,\{A}\mathbf{A}^\dagger_{p,B\}}\,\dot{\mathbf{A}}^\dagger_{1,\{\dot C}\dot{\mathbf{A}}^\dagger_{p,\dot D\}}\,|\mathcal{Z},L\rangle,\\
\mathcal{O}^{L+1}_{n,\{AB\}[\dot C\dot D\}} &= \sum_{p=2}^{L+1} \sin\left(\frac{\pi n(2p-2)}{L+1}\right)\,\mathrm{Tr}\,\mathbf{A}^\dagger_{1,\{A}\mathbf{A}^\dagger_{p,B\}}\,\dot{\mathbf{A}}^\dagger_{1,[\dot C}\dot{\mathbf{A}}^\dagger_{p,\dot D\}}\,|\mathcal{Z},L+1\rangle,\\
\mathcal{O}^{L+1}_{n,[AB\}\{\dot C\dot D\]} &= \sum_{p=2}^{L+1} \sin\left(\frac{\pi n(2p-2)}{L+1}\right)\,\mathrm{Tr}\,\mathbf{A}^\dagger_{1,[A}\mathbf{A}^\dagger_{p,B\}}\,\dot{\mathbf{A}}^\dagger_{1,\{\dot C}\dot{\mathbf{A}}^\dagger_{p,\dot D]}\,|\mathcal{Z},L+1\rangle,\\
\mathcal{O}^{L+2}_{n,[AB\}[\dot C\dot D\}} &= \sum_{p=2}^{L+2} \cos\left(\frac{\pi n(2p-3)}{L+1}\right)\,\mathrm{Tr}\,\mathbf{A}^\dagger_{1,[A}\mathbf{A}^\dagger_{p,B\}}\,\dot{\mathbf{A}}^\dagger_{1,[\dot C}\dot{\mathbf{A}}^\dagger_{p,\dot D\}}\,|\mathcal{Z},L+2\rangle.
\end{aligned} \tag{3.87}$$

All of these have the same energy E_n^L. This is related to the fact that all short multiplets join to form a long multiplet in the interacting theory [27] (unless $n = 0$). For $\mathfrak{psu}(2|2)$ the interacting multiplet $[0; 1+\frac{1}{2}\delta D; 0]$ is at the unitarity bound $r_1 \approx s_1 + 1$. When δD approaches zero, the interacting multiplet splits up into a short and a BPS multiplet $[0; 1+\frac{1}{2}\delta D; 0] \to [0;1;0] + [0;0;2]$. In total, the highest weight of the long multiplet is therefore

$$[0; 1+\tfrac{1}{2}\delta D; 0]\times[0; 1+\tfrac{1}{2}\delta D; 0]. \tag{3.88}$$

In $\mathfrak{psu}(2,2|4)$ the classical highest weight for states $n \neq 0$ is $(L;0,0;0,L-2,0;0,0) = [0;1;0,L-2,0;1;0]$. The protected states for $n = 0$ are part of the half-BPS multiplet $(L;0,0;0,L,0;0,0)$. The states have parity $(-1)^L$.

3.5.5 Three Excitations

Let us investigate the states with three excitations. We find that such states almost always form pairs with degenerate planar energies. The only exceptions from this rule are

states with weight

$$(2m+4;0,0;1,2m,1;0,2m+4) = [0;1;1,2m,1;1;0], \quad m \geq 0. \tag{3.89}$$

They have a descendant in the $\mathfrak{su}(2)$ subsector which is given by

$$\mathcal{O} = \sum_{k=1}^{2m+2} (-1)^k \operatorname{Tr} \phi\phi\mathcal{Z}^k\phi\mathcal{Z}^{2m+3-k}, \qquad E = 6, \qquad P = -1. \tag{3.90}$$

Interestingly, two of the excitations are always adjacent in this leading order approximation. Further states of this kind with more excitations can be found.

3.6 Plane Wave Physics

In this section we would like to demonstrate the use of the dilatation operator to find *non-planar corrections*, i.e. corrections in $1/N$, to the scaling dimensions. Here the dilatation operator brings about a major simplification [35] as opposed to the computation of correlation functions [18,19,108,21,22,36] because it allows to derive scaling dimensions independently of two-point normalisation constants. In particular we will derive the genus-one correction to the scaling dimension of two-excitation BMN operators in the BMN limit.

3.6.1 The BMN Limit

Berenstein, Maldacena and Nastase (BMN) [14] suggested to investigate operators of a large dimension D_0 and a nearly equally large charge J of $\mathfrak{su}(4)$

$$J = p - \tfrac{1}{2}q_1 - \tfrac{1}{2}q_2. \tag{3.91}$$

Then the relevant states constitute long strings of $\mathcal{Z}$-fields with $D_0 - J$ excitations or *impurities* scattered in[8]

$$\operatorname{Tr} \mathcal{Z} \ldots \mathcal{Z}\, \phi\, \mathcal{Z} \ldots \mathcal{Z}\, \mathcal{D}\mathcal{Z}\, \mathcal{Z} \ldots \mathcal{Z}\, \psi\, \mathcal{Z} \ldots \mathcal{Z}, \tag{3.92}$$

which became known as BMN operators. As particular examples, BMN investigated operators with zero, one and two excitations of scalar type. The operators with less than two excitations belong to half-BPS multiplets and are thus protected. Starting with two excitations there are states whose scaling dimension changes in the quantum theory. For large J one finds that the smallest one-loop planar anomalous dimensions scale as $1/J^2$,

$$E = \mathcal{O}(1/J^2). \tag{3.93}$$

as confirmed by the exact values in the case of two excitations (3.86). BMN proposed to absorb the dependence on J into an effective coupling constant λ'

$$\lambda' := \frac{\lambda}{J^2} = \frac{g_{\mathrm{YM}}^2 N}{J^2}\,, \tag{3.94}$$

[8] The excitation subsectors in Sec. 2.3.6 were constructed to describe states of this kind.

for our purposes it seems convenient to use the combination $\hat{g}$

$$\hat{g} := \frac{g}{J}, \qquad \lambda' = 8\pi^2 \hat{g}^2. \tag{3.95}$$

BMN conjectured that this would lead to finite planar eigenvalues for the BMN energy operator $\mathfrak{D} - J$ (as a function of $\hat{g}$) in the large J limit, even beyond one-loop.

Moreover, it was found [18, 19] that also the genus counting parameter $1/N$ can be renormalised in such a way as to obtain finite results for non-planar correlators

$$\hat{g}_{\rm s} = g_2 := \frac{J^2}{N}. \tag{3.96}$$

The non-planar BMN limit can be defined as the *double-scaling limit*

$$\text{'}\textit{BMN limit}\text{'}: \qquad J, N, g, \lambda \to \infty \quad \text{with} \quad \hat{g}, \hat{g}_{\rm s} \quad \text{fixed}. \tag{3.97}$$

The physical significance of the above lies in the *BMN correspondence*, which is a limit of the celebrated AdS/CFT correspondence. The statement of the correspondence is that BMN operators are dual to states of string theory on the *plane-wave* background. The scaling dimensions of BMN operators minus their charge J should match the light-cone energies of the corresponding string states. In an operatorial form, the correspondence can be written as

$$\text{'}\textit{BMN Correspondence}\text{'}: \qquad H_{\rm LC} = \mathfrak{D} - J + \mathcal{O}(1/J). \tag{3.98}$$

The planar limit corresponds to a non-interacting string theory and it is fairly easy to derive the light-cone energy eigenvalues [14]

$$E_{\rm LC} = \sum_{k=1}^{M} \sqrt{1 + \lambda' n_k^2} = \sum_{k=1}^{M} \sqrt{1 + 2\hat{g}^2 (2\pi n_k)^2}. \tag{3.99}$$

The numbers n_k are the mode numbers (positive, negative or zero) of M string oscillator excitations and are subject to the level matching constraint $\sum_{k=1}^{M} n_k = 0$. There are some indications that this all-loop prediction for gauge theory might indeed be true [25, 109].

There exists an exceedingly large literature on the BMN correspondence, see [17] for reviews.

3.6.2 Basis of States

Starting from here, we will only consider operators with two excitations. As shown in Sec. 3.5.4, it makes perfect sense to consider these operators also for arbitrary finite values of J. All two-excitation states form a single multiplet of superconformal symmetry, so we are free to choose a particular descendant to be used in our investigation. In particular there is one descendant in the $\mathfrak{su}(2)$ sector that can be written in terms of J fields $\mathcal{Z}$ and two excitations of type ϕ. Generic multi-trace operators with two excitations can have both excitations in one trace

$$\mathcal{E}_p^{J_0;J_1,\ldots,J_K} = \operatorname{Tr} \phi \mathcal{Z}^p \phi \mathcal{Z}^{J_0-p} \prod_{k=1}^{K} \operatorname{Tr} \mathcal{Z}^{J_k}, \tag{3.100}$$

or the two excitations separated in different traces

$$\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_K} = \mathrm{Tr}\,\phi\mathcal{Z}^{J_0}\ \mathrm{Tr}\,\phi\mathcal{Z}^{J_1}\prod_{k=2}^{K}\mathrm{Tr}\,\mathcal{Z}^{J_k}, \tag{3.101}$$

with $\sum_{k=0}^{K} J_k = J$. Both series of operators are symmetric under the interchange of sizes J_k of traces $\mathrm{Tr}\,\mathcal{Z}^{J_k}$, $\mathcal{E}$ is symmetric under $p \to J_0 - p$ and $\mathcal{Q}$ is symmetric under $J_0 \leftrightarrow J_1$.

3.6.3 The Action of the Dilatation Generator

The non-planar dilatation generator (2.113)

$$\mathcal{H} = \mathfrak{D}_2 = -N^{-1}{:}\mathrm{Tr}\,[\mathcal{Z},\phi][\check{\mathcal{Z}},\check{\phi}]{:} \tag{3.102}$$

can be seen to act as

$$\mathcal{H}\begin{pmatrix}\mathcal{E}_p & \mathcal{Q}\end{pmatrix} = \begin{pmatrix}\mathcal{E}_p & \mathcal{Q}\end{pmatrix}\begin{pmatrix}* & * \\ 0 & 0\end{pmatrix}, \tag{3.103}$$

i.e. operators of type $\mathcal{Q}$ are never produced. This follows from the fact that all produced objects will contain a commutator $[\mathcal{Z},\phi]$ in some trace and this trace will vanish unless it contains another ϕ. It immediately follows that for every $\mathcal{Q}$ there is one protected quarter-BPS operator. Its leading part is given by $\mathcal{Q}$ itself, plus a $1/N$ correction from the operators $\mathcal{E}_p$ [34, 110, 21, 22].[9] On the other hand, the operators $\mathcal{E}_p$ are in general not protected and we will investigate their spectrum of anomalous dimensions in what follows. From the form of the dilatation matrix we infer that operators of type $\mathcal{E}_p$ do not receive corrections from operators of type $\mathcal{Q}$; the latter therefore completely decouple as far as the consideration of the $\mathcal{E}_p$'s is concerned.

It is easy to write down the exact expression for $\mathcal{H}\mathcal{E}_p$. Let us define

$$\mathcal{H} = \mathcal{H}_0 + N^{-1}\mathcal{H}_+ + N^{-1}\mathcal{H}_-, \tag{3.104}$$

where $\mathcal{H}_0$ is trace conserving and $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively increases and decreases the number of traces by one. These three different contributions arise from three different contractions of the variations in the dilatation generator with the fields in the states, see Fig. 3.6. Contractions to adjacent fields within a trace lead to planar contributions. Contractions to non-adjacent fields will split up the trace. Contractions to different traces will join them. We find

$$\begin{aligned}
\mathcal{H}_0\,\mathcal{E}_p^{J_0;J_1,\ldots,J_K} &= -2\big(\delta_{p\neq J_0}\mathcal{E}_{p+1}^{J_0;J_1,\ldots,J_K} - (\delta_{p\neq J_0}+\delta_{p\neq 0})\mathcal{E}_p^{J_0;J_1,\ldots,J_K} + \delta_{p\neq 0}\mathcal{E}_{p-1}^{J_0;J_1,\ldots,J_K}\big)\,,\\
\mathcal{H}_+\,\mathcal{E}_p^{J_0;J_1,\ldots,J_K} &= \sum_{J_{K+1}=1}^{p-1} 2\big(\mathcal{E}_{p-J_{K+1}}^{J_0-J_{K+1};J_1,\ldots,J_{K+1}} - \mathcal{E}_{p-1-J_{K+1}}^{J_0-J_{K+1};J_1,\ldots,J_{K+1}}\big)\\
&\quad - \sum_{J_{K+1}=1}^{J_0-p-1} 2\big(\mathcal{E}_{p+1}^{J_0-J_{K+1};J_1,\ldots,J_{K+1}} - \mathcal{E}_{p}^{J_0-J_{K+1};J_1,\ldots,J_{K+1}}\big),
\end{aligned}$$

[9] We note that all quarter-BPS states in [34, 110] are annihilated by the operator $[\check{\mathcal{Z}},\check{\phi}]$ which is part of D_2. It is also part of a superboost which relates would-be quarter-BPS states with their partners in a long multiplet. For true quarter-BPS states, this must not happen and $[\check{\mathcal{Z}},\check{\phi}]$ annihilates them.

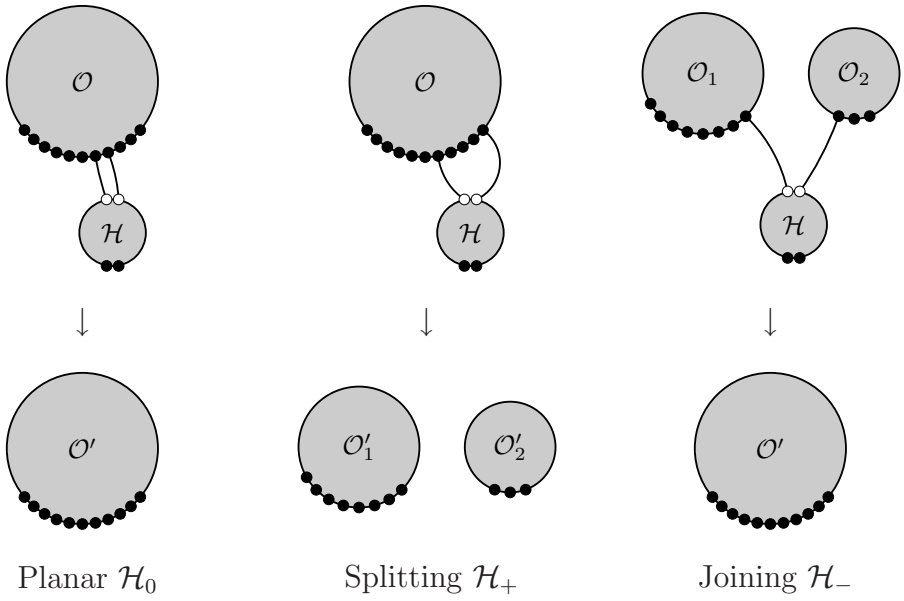

Figure 3.6: Topological structures of the action of the one-loop dilatation operator.

$$\begin{aligned}\mathcal{H}_-\,\mathcal{E}_p^{J_0;J_1,\ldots,J_K} = &\sum_{k=1}^{K} 2J_k\left(\mathcal{E}_{J_k+p}^{J_0+J_k;J_1,\ldots,\not{J}_k,\ldots,J_K} - \mathcal{E}_{J_k+p-1}^{J_0+J_k;J_1,\ldots,\not{J}_k,\ldots,J_K}\right)\\ &-\sum_{k=1}^{K} 2J_k\left(\mathcal{E}_{p+1}^{J_0+J_k;J_1,\ldots,\not{J}_k,\ldots,J_K} - \mathcal{E}_{p}^{J_0+J_k;J_1,\ldots,\not{J}_k,\ldots,J_K}\right).\end{aligned} \tag{3.105}$$

In view of the AdS/CFT and BMN correspondence this is very suggestive. The one-loop dilatation operator can either not change the structure of traces, split one trace into two, or join two into one. This is in qualitative agreement with string field theory when traces are interpreted as strings. Also the parameter $1/N$ appears in the right places to be interpreted as the string coupling constant.

3.6.4 The BMN Limit of Two Excitation Operators

With J being very large we can view $\hat{p} = p/J$ and $\hat{J}_k = J_k/J$ as continuum variables and replace the discrete set of states in equation (3.100) by a set of continuum states

$$\mathcal{E}_p^{J_0;J_1,\ldots,J_K} \to |\hat{p};\hat{J}_1,\ldots,\hat{J}_K\rangle = |\hat{J}_0-\hat{p};\hat{J}_1,\ldots,\hat{J}_K\rangle, \tag{3.106}$$

where

$$\hat{p}\in[0,\hat{J}_0], \qquad \hat{J}_0,\hat{J}_k\in[0,1] \qquad \text{and} \qquad \hat{J}_0 = 1-(\hat{J}_1+\ldots+\hat{J}_K). \tag{3.107}$$

It is understood that $|\hat{p};\hat{J}_1,\ldots,\hat{J}_K\rangle = |\hat{p};\hat{J}_{\pi(1)},\ldots,\hat{J}_{\pi(K)}\rangle$ with π an arbitrary permutation of K elements.

Absorbing the J-dependence into the definition of the Hamiltonian

$$\hat{\mathcal{H}} = J^2\mathcal{H} \quad \text{and} \quad \hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{g}_{\rm s}\hat{\mathcal{H}}_+ + \hat{g}_{\rm s}\hat{\mathcal{H}}_-, \tag{3.108}$$

we impose the BMN limit (3.97) and get a continuum version of (3.105)

$$\begin{aligned}
\hat{\mathcal{H}}_0\,|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle &= -2\partial_{\hat{p}}^2\,|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle, \qquad (3.109)\\
\hat{\mathcal{H}}_+\,|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle &= \int_0^{\hat{p}} d\hat{J}_{K+1}\,2\partial_{\hat{p}}|\hat{p}-\hat{J}_{K+1};\hat{J}_1,\dots,\hat{J}_{K+1}\rangle\\
&\quad - \int_0^{\hat{J}_0-\hat{p}} d\hat{J}_{K+1}\,2\partial_{\hat{p}}|\hat{J};\hat{J}_1,\dots,\hat{J}_{K+1}\rangle,\\
\hat{\mathcal{H}}_-|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle &= \sum_{k=1}^{K} 2\hat{J}_k\,\partial_{\hat{p}}|\hat{p}+\hat{J}_k;\hat{J}_1,\dots,\not{\hat{J}}_k,\dots,\hat{J}_K\rangle\\
&\quad - \sum_{k=1}^{K} 2\hat{J}_k\,\partial_{\hat{p}}\,|\hat{p};\hat{J}_1,\dots,\not{\hat{J}}_k,\dots,\hat{J}_K\rangle.
\end{aligned}$$

3.6.5 The Mode Decomposition

The $(K+1)$-trace eigenstates at $\hat{g}_{\rm s}=0$ are

$$|n;\hat{J}_1,\dots,\hat{J}_K\rangle = \frac{1}{\hat{J}_0}\int_0^{\hat{J}_0} d\hat{p}\,\cos\left(\tfrac{2\pi n}{\hat{J}_0}\hat{p}\right)|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle, \qquad n=0,1,2,\dots. \qquad (3.110)$$

This is of course in accordance with the nature of the exact eigenstates at finite J, c.f. Sec. 3.5.4. The inverse transformation of (3.110) reads

$$|\hat{p};\hat{J}_1,\dots,\hat{J}_K\rangle = |0;\hat{J}_1,\dots,\hat{J}_K\rangle + 2\sum_{n=1}^{\infty}\cos\left(\tfrac{2\pi n}{\hat{J}_0}\hat{p}\right)|n;\hat{J}_1,\dots,\hat{J}_K\rangle. \qquad (3.111)$$

In the basis (3.110), the action of the operator $\hat{\mathcal{H}}$ reads

$$\begin{aligned}
\hat{\mathcal{H}}_0\,|n;\hat{J}_1,\dots,\hat{J}_K\rangle &= 2\left(\tfrac{2\pi n}{\hat{J}_0}\right)^2|n;\hat{J}_1,\dots,\hat{J}_K\rangle, \qquad (3.112)\\
\hat{\mathcal{H}}_+\,|n;\hat{J}_1,\dots,\hat{J}_K\rangle &= \frac{16}{\hat{J}_0}\int_0^{\hat{J}_0} d\hat{J}_{K+1}\sum_{n'=1}^{\infty}\frac{\left(\frac{2\pi n'}{\hat{J}_0-\hat{J}_{K+1}}\right)^2\sin^2\left(\pi n\frac{\hat{J}_{K+1}}{\hat{J}_0}\right)}{\left(\frac{2\pi n'}{\hat{J}_0-\hat{J}_{K+1}}\right)^2-\left(\frac{2\pi n}{\hat{J}_0}\right)^2}\,|n';\hat{J}_1,\dots,\hat{J}_{K+1}\rangle,\\
\hat{\mathcal{H}}_-|n;\hat{J}_1,\dots,\hat{J}_K\rangle &= 16\sum_{k=1}^{K}\frac{\hat{J}_k}{\hat{J}_0}\sum_{n'=1}^{\infty}\frac{\left(\frac{2\pi n'}{\hat{J}_0+\hat{J}_k}\right)^2\sin^2\left(\pi n'\frac{\hat{J}_k}{\hat{J}_0+\hat{J}_k}\right)}{\left(\frac{2\pi n'}{\hat{J}_0+\hat{J}_k}\right)^2-\left(\frac{2\pi n}{\hat{J}_0}\right)^2}\,|n';\hat{J}_1,\dots,\not{\hat{J}}_k,\dots,\hat{J}_K\rangle.
\end{aligned}$$

In interacting plane-waves string theory similar expressions have been derived [20,111, 23,112]. The Hamiltonians of both theories should however not be compared directly, but only modulo a similarity transformation. A proposal for the change of basis was given in [24, 113] and applied in [47] to show the equivalence of the Hamiltonians in the one-loop approximation. Up to some assumptions [23] (which appear to be inconsistent [114]) regarding excitation number non-preserving amplitudes in string theory, it proves the BMN correspondence (3.98) at first order in $\hat{g}^2$ (one-loop) and all orders in $\hat{g}_{\rm s}$ (all-genus) for single trace states with two excitations. A similar statement for three excitation states was investigated in [115], but a generalisation to arbitrarily many excitations of arbitrary type has not been attempted yet.

3.6.6 The Genus-One Energy Shift

Now the scene is set for determining the spectrum of the full one-loop Hamiltonian order by order in $\hat{g}_{\rm s}$ by standard quantum mechanical perturbation theory. The leading non-planar correction to the energy $\hat{E}_{n,0} = 2(2\pi n)^2$ of a single trace state $|n\rangle$ is obtained by second-order perturbation theory

$$\hat{E}_{n,2}|n\rangle = \pi_n \,\hat{\mathcal{H}}_- \frac{1}{\hat{E}_{n,0} - \hat{\mathcal{H}}_0}\,\hat{\mathcal{H}}_+ \,|n\rangle, \tag{3.113}$$

where π_n projects to $|n\rangle$. We now insert (3.112) and get the genus-one (torus) correction to the energy

$$\hat{E}_{n,2} = \int_0^1 d\hat{J}_1 \sum_{n'=1}^{\infty} \frac{128\,\hat{J}_1 (\frac{2\pi n'}{1-\hat{J}_1})^2 (2\pi n)^2 \sin^4(\pi n \hat{J}_1)}{(1-\hat{J}_1)\big((\frac{2\pi n'}{1-\hat{J}_1})^2 - (2\pi n)^2\big)^3} = \frac{1}{6} + \frac{35}{4(2\pi n)^2}\,. \tag{3.114}$$

The total scaling dimension is thus

$$\begin{aligned} D &= J + 2 + 2(2\pi n)^2 \hat{g}^2 + \left(\frac{1}{6} + \frac{35}{4(2\pi n)^2}\right)\hat{g}^2 \hat{g}_{\rm s}^2 + \dots \\ &= J + 2 + \lambda' n^2 + \lambda' \hat{g}_{\rm s}^2 \left(\frac{1}{48\pi^2} + \frac{35}{128\pi^4 n^2}\right) + \dots\,. \end{aligned} \tag{3.115}$$

This genus-one result was first derived by computing gauge theory correlation functions [21, 22] and confirmed in string theory [23], see also [24]. Here it was assumed that one can restrict to excitation number preserving amplitudes in string theory, however, there are doubts that this assumption is consistent [114]. Subsequently, the formula (3.114) was rederived by Janik by considering matrix elements of the dilatation generator [37]. This lead to a great simplification of the calculation.

Chapter 4

Integrability

In Sec. 2.6 we have demonstrated how, in the planar limit, local operators can be interpreted as quantum spin chains. In that picture, the planar dilatation operator is represented by the spin chain Hamiltonian. Minahan and Zarembo realised that the one-loop dilatation operator of $\mathcal{N} = 4$ SYM for states composed from only scalar fields (the one-loop $\mathfrak{so}(6)$ sector) is precisely the Hamiltonian of an *integrable* spin chain [48]. This parallels earlier discoveries of integrable spin chains in generic, non-supersymmetric gauge theories at one-loop and in the large N limit when dealing with states composed mostly from covariant derivatives [52–54] (see also the review [55]).

In this chapter we will show how these two lines of development can be combined into a $\mathfrak{psu}(2,2|4)$ *supersymmetric spin chain* [58]. We will start by introducing the notion of integrable spin chains and later present the algebraic Bethe ansatz technique. As an application, we shall derive the one-loop anomalous dimension of a state dual to a macroscopic spinning string in $AdS_5 \times S^5$ and find a remarkable agreement [60, 66].

4.1 Integrable Spin Chains

A quantum integrable system is a quantum mechanical system with an infinite number[1] of mutually commuting scalar charges $\mathcal{Q}_r$

$$[\mathcal{Q}_r, \mathcal{Q}_s] = [\mathfrak{J}, \mathcal{Q}_r] = 0. \tag{4.1}$$

In other words, the naive symmetry algebra is enlarged by infinitely many abelian generators constituting the algebra $\mathfrak{u}(1)^\infty$. The Hamiltonian $\mathcal{H}$, a $\mathfrak{u}(1)$ generator invariant under the symmetry algebra, will turn out to be one of the charges, $\mathcal{H} = \mathcal{Q}_2$, and is absorbed into $\mathfrak{u}(1)^\infty$. The symmetry enhancement might thus be stated as

$$\mathfrak{u}(1) \longrightarrow \mathfrak{u}(1)^\infty. \tag{4.2}$$

In this section we will discuss the integrable structures found at the one-loop level.

[1] The precise counting is somewhat unclear in a quantum system. In a classical system one needs exactly half the number of phase-space dimensions. Here, the spin chains can be arbitrarily long which gives rise to an arbitrarily large number of conserved charges. This is what is meant by infinitely many.

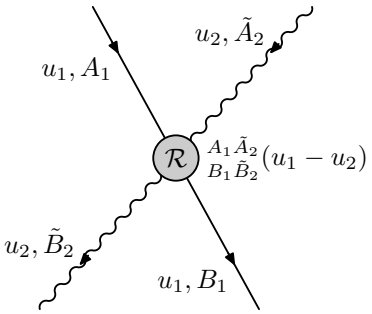

Figure 4.2: A scattering process of two particles is described by the R-matrix.

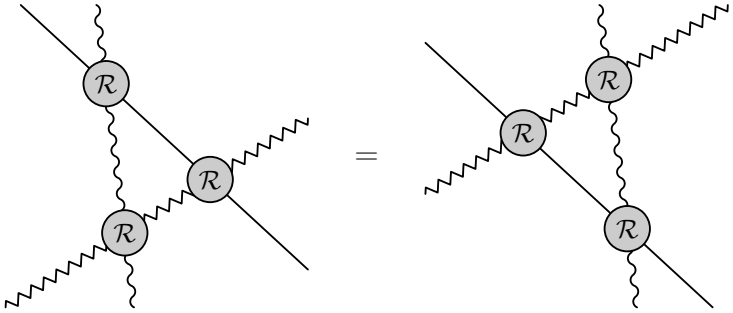

Figure 4.4: The Yang-Baxter equation.

4.1.1 The R-Matrix

A spin chain is composed from L modules transforming in some representation of a symmetry algebra. We will assume the symmetry algebra to be of unitary type, $\mathfrak{su}(M)$. To understand the integrable model, it makes sense to consider the individual spins as 'particles'. A particle $X_A(u)$ is thus defined as an element of a module of the symmetry group together with a spectral parameter u. The central object of the integrable model is the R-matrix, it describes the 'scattering' of particles. The R-matrix rotates two modules depending on their representations and difference of spectral parameters, c.f. Fig. 4.2

$$\tilde{X}'_{2,\tilde{B}_2}(u_2)\, X'_{1,B_1}(u_1) = \mathcal{R}^{A_1\tilde{A}_2}_{B_1\tilde{B}_2}(u_1-u_2)\, X_{1,A_1}(u_1)\, \tilde{X}_{2,\tilde{A}_2}(u_2). \tag{4.3}$$

The scattering is elastic in the sense that neither the representation nor the spectral parameters are changed; the only effect is a generalised phase shift described by the R-matrix. In an integrable system, the order in which particles scatter does not matter. For the scattering of three particles this fact is described by the Yang-Baxter equation

$$\begin{aligned}&\mathcal{R}^{A_1\tilde{A}_2}_{B_1\tilde{B}_2}(u_1-u_2)\, \mathcal{R}^{B_1\hat{A}_3}_{C_1\hat{B}_3}(u_1-u_3)\, \mathcal{R}^{\tilde{B}_2\hat{B}_3}_{\tilde{C}_2\hat{C}_3}(u_2-u_3)\\ &= \mathcal{R}^{\tilde{A}_2\hat{A}_3}_{\tilde{B}_2\hat{B}_3}(u_2-u_3)\, \mathcal{R}^{A_1\hat{B}_3}_{B_1\hat{C}_3}(u_1-u_3)\, \mathcal{R}^{B_1\tilde{B}_2}_{C_1\tilde{C}_2}(u_1-u_2)\end{aligned} \tag{4.4}$$

or $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ for short. The Yang-Baxter equation is most intuitively represented in a diagrammatic fashion, see Fig. 4.4; it implies that the particle lines can be moved around freely, even past other interactions. From this it follows that, also for a larger number of particles, the order of scatterings does not matter.

In addition to the Yang-Baxter equation, there is the relation, see Fig. 4.6

$$\mathcal{R}^{A_1\tilde{A}_2}_{B_1\tilde{B}_2}(u_1-u_2)\, \mathcal{R}^{\tilde{B}_2B_1}_{\tilde{C}_2C_1}(u_2-u_1) = \delta^{A_1}_{C_1}\delta^{\tilde{A}_2}_{\tilde{C}_2}, \tag{4.5}$$

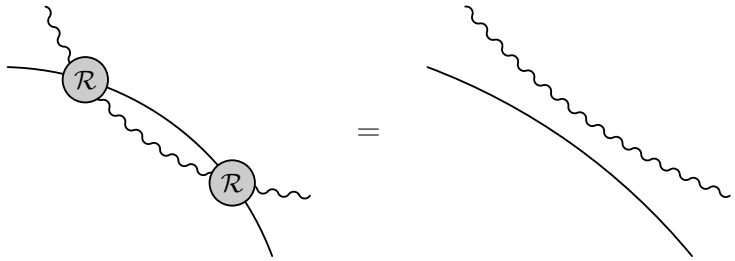

Figure 4.6: The R-matrix and its inverse.

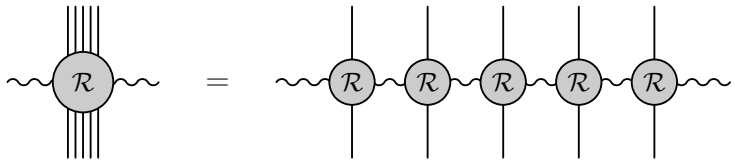

Figure 4.8: The monodromy matrix for a composite particle.

which defines the inverse of the R-matrix.

The case of all particles transforming in the fundamental representation is particularly easy to solve. The solution of the bi-fundamental Yang-Baxter equation is

$$\mathcal{R}^{ab}_{cd}(u) = \frac{u}{u+i}\,\delta^a_c\delta^b_d + \frac{i}{u+i}\,\delta^a_d\delta^b_c \qquad \text{or} \qquad \mathcal{R}_{12}(u) = \frac{u}{u+i}\,\mathcal{I}_{12} + \frac{i}{u+i}\,\mathcal{P}_{12}, \tag{4.6}$$

where $\mathcal{I}_{12}$ is the identity acting on particles $1,2$ and $\mathcal{P}_{12}$ is the permutation. It is useful to write this in a mixed notation where we keep one index manifest and suppress the other in a matrix notation

$$\mathcal{R}^a_b(u) = \frac{u+i/M}{u+i}\,\delta^a_b + \frac{i}{u+i}\,\mathfrak{J}^a{}_b, \tag{4.7}$$

where we make use the symmetry generator $(\mathfrak{J}^a{}_b)^c{}_d = \delta^a_d\delta^c_b - \delta^a_b\delta^c_d/M$ in the fundamental representation. The fundamental R-matrix, where one particle transforms in the fundamental representation and the other in an arbitrary one, is given by a similar expression as (4.7) using the symmetry generators.

4.1.2 Transfer Matrices

Several particles can be grouped into a composite particle $X_{A_1\dots A_L}(v_1,\dots,v_L)$.[2] The spin chain is just such a composite particle. For a composite particle one can define a composite R-matrix (monodromy matrix) by, see Fig. 4.8

$$\tilde{\mathcal{R}}^{\tilde{A},A_1\dots A_L}_{\tilde{B},B_1\dots B_L}(u) = \mathcal{R}^{\tilde{A}\ A_1}_{\tilde{C}_2B_1}(u-v_1)\,\mathcal{R}^{\tilde{C}_2A_2}_{\tilde{C}_3B_2}(u-v_2)\dots\mathcal{R}^{\tilde{C}_LA_L}_{\tilde{B}\ \ B_L}(u-v_L). \tag{4.8}$$

This R-matrix naturally satisfies the Yang-Baxter equation. Usually one suppresses most indices and spectral parameters $\mathcal{R}^A_B = (\mathcal{R}_1)^A_{C_2}\,(\mathcal{R}_2)^{C_2}_{C_3}\dots(\mathcal{R}_L)^{C_L}_B$. Let us write the monodromy matrix for the fundamental representation using (4.7) with all spectral parameters aligned $v_p = 0$

$$\mathcal{R}^a_b(u) = \left(\frac{u+i/M}{u+i}\,\delta^a_{c_2} + \frac{i}{u+i}\,\mathfrak{J}_1{}^a{}_{c_2}\right)\cdots\left(\frac{u+i/M}{u+i}\,\delta^{c_L}_b + \frac{i}{u+u_L}\,\mathfrak{J}_L{}^{c_L}{}_b\right). \tag{4.9}$$

[2]Commonly, all spectral parameters will be aligned $v_p = v$ and constitute a homogeneous chain. In Sec. 6.4.1 we will however encounter an inhomogeneous chain with different v_p's.

This expression reveals an interesting interpretation of the monodromy matrix for a spin chain: The generator of rotations, $\mathfrak{J}$, may be considered as the component of some gauge field in the direction of the spin chain. Then, the monodromy matrix has a great similarity to a Wilson line along the spin chain. In that picture, an elementary R-matrix is just the monodromy of the gauge field across one spin chain site. Moreover, there is not only a single gauge field, but a family of gauge fields, parameterised by the spectral parameter u. When viewed in this way, the integrable structure is very similar to the one found in string theory on $AdS_5 \times S^5$, see [82, 83, 86]. This similarity allowed the authors of [116] to promote the Yangian structure from string theory to gauge theory.

For a cyclic spin chain it is natural to close the Wilson line to a loop and take the trace. One obtains the *transfer matrix*

$$\tilde{\mathcal{T}}(u) = \tilde{\mathcal{R}}^{\tilde{A}}_{\tilde{A}}(u) = (\mathcal{R}_1)^{\tilde{C}_1}_{\tilde{C}_2}\,(\mathcal{R}_2)^{\tilde{C}_2}_{\tilde{C}_3}\,\ldots(\mathcal{R}_L)^{\tilde{C}_L}_{\tilde{C}_1}. \tag{4.10}$$

The transfer matrix can be taken for any representation circulating around the Wilson loop and for any spectral parameter. In this work, however, we shall restrict to the equal representations of the spins and the Wilson loop. The interesting aspect of transfer matrices is that all of them commute

$$[\tilde{\mathcal{T}}(u), \hat{\mathcal{T}}(v)] = 0. \tag{4.11}$$

This statement can be shown easily by inserting a R-matrix and its inverse into the traces, see Fig. 4.6. Using the Yang-Baxter equation, the R-matrix is commuted around the traces interchanging the order of monodromy matrices

$$\begin{aligned}\tilde{\mathcal{T}}(u)\,\hat{\mathcal{T}}(v) &= \tilde{\mathcal{R}}^{\tilde{A}}_{\tilde{A}}(u)\,\hat{\mathcal{R}}^{\hat{A}}_{\hat{A}}(v) = \mathcal{R}^{\hat{A}\tilde{A}}_{\hat{B}\tilde{B}}(v-u)\,\mathcal{R}^{\tilde{B}\hat{B}}_{\tilde{C}\hat{C}}(u-v)\,\tilde{\mathcal{R}}^{\tilde{C}}_{\tilde{A}}(u)\,\hat{\mathcal{R}}^{\hat{C}}_{\hat{A}}(v)\\ &= \mathcal{R}^{\hat{A}\tilde{A}}_{\hat{B}\tilde{B}}(v-u)\,\hat{\mathcal{R}}^{\hat{B}}_{\hat{C}}(v)\,\tilde{\mathcal{R}}^{\tilde{B}}_{\tilde{C}}(u)\,\mathcal{R}^{\tilde{C}\hat{C}}_{\tilde{A}\hat{A}}(u-v)\\ &= \hat{\mathcal{R}}^{\hat{B}}_{\hat{B}}(v)\,\tilde{\mathcal{R}}^{\tilde{B}}_{\tilde{B}}(u) \;= \hat{\mathcal{T}}(v)\,\tilde{\mathcal{T}}(u).\end{aligned} \tag{4.12}$$

Afterwards the R-matrix and its inverse cancel out and the transfer matrices are interchanged.

4.1.3 The Local Charges

There are many uses for monodromy and transfer matrices. A particular one is the Yangian, an associative Hopf algebra which enlarges the symmetry algebra, see e.g. [117]. The Yangian is an important object for integrable systems. The elements of the Yangian are given by the monodromy matrix in the fundamental representation (4.8). Commonly, the Yangian is expanded around $u = \infty$. In the leading two orders one finds the identity and the generators of the symmetry algebra, $\mathfrak{J}$, acting on the full spin chain. At the next order, the first non-trivial elements of the Yangian appear. They are bi-local along the spin-chain and can be used to generate all higher elements. In $\mathcal{N} = 4$ SYM we deal with cyclic spin chains and the open Wilson line of the Yangian breaks cyclic symmetry. At the moment it is not clear how to make direct use of the Yangian for the study of scaling dimension and we will not consider it further. See [116] for a treatment of the Yangian in $\mathcal{N} = 4$ SYM.

Here we would like to investigate the transfer matrices. These are closed Wilson loops and they preserve cyclic symmetry. The transfer matrix can be used as a generating function for the charges $\mathcal{U}, \mathcal{Q}_n$ when expanded in the spectral parameter

$$\mathcal{T}(u) = \mathcal{U} \exp \sum_{r=2}^{\infty} i u^{r-1} \mathcal{Q}_r. \tag{4.13}$$

All of these charges commute with each other due to commuting of the transfer matrices at different values of the spectral parameters (4.11).

For spin chains with equal spin representations at each site, it is useful to pick the same representation to circle around the Wilson loop as well. We will furthermore assume that all R-matrices for the construction of the transfer matrix are the same and have a specific value at $u = 0$

$$\mathcal{R}^{A_1 A_2}_{B_1 B_2}(0) = \delta^{A_1}_{B_2} \delta^{A_2}_{B_1}, \tag{4.14}$$

i.e. they permute the modules.[3] Let us now expand the transfer matrix in u. At $u = 0$ we find the cyclic shift operator

$$\mathcal{U}^{A_1 \dots A_L}_{B_1 \dots B_L} = \mathcal{T}^{A_1 \dots A_L}_{B_1 \dots B_L}(0) = \delta^{A_1}_{B_2} \delta^{A_2}_{B_3} \dots \delta^{A_{L-1}}_{B_L} \delta^{A_L}_{B_1}. \tag{4.15}$$

Expanding to first order in u we find that we have to insert a derivative $\mathcal{R}' = \partial \mathcal{R}/\partial u$ of the R-matrix into the shift operator and sum over all insertion points,

$$\mathcal{T}^{A_1 \dots A_L}_{B_1 \dots B_L}(u) = \mathcal{U}^{A_1 \dots A_L}_{B_1 \dots B_L} + u \sum_{p=1}^{L} \delta^{A_1}_{B_2} \dots \mathcal{R}'^{\,A_p A_{p+1}}_{B_{p+2} B_{p+1}}(0) \dots \delta^{A_L}_{B_1} + \mathcal{O}(u^2). \tag{4.16}$$

Let us define the charge density

$$\mathcal{Q}_{2,}{}^{A_1 A_2}_{B_1 B_2} = -i \mathcal{R}'^{\,A_1 A_2}_{B_2 B_1}(0) \quad \text{or} \quad \mathcal{Q}_{2,12} = -i\, \mathcal{P}_{12}\, \mathcal{R}'_{12}(0), \tag{4.17}$$

where the second form is short for the first. The permutation $\mathcal{P}_{12}$ interchanges the spins at two sites. According to the definition (4.13) we should absorb the cyclic shift in (4.16) into $\mathcal{U}$ and the second charge $\mathcal{Q}_2$ is simply

$$\mathcal{Q}_{2,}{}^{A_1 \dots A_L}_{B_1 \dots B_L} = \sum_{p=1}^{L} \delta^{A_1}_{B_1} \dots \mathcal{Q}_{2,}{}^{A_p A_{p+1}}_{B_p B_{p+1}} \dots \delta^{A_L}_{B_L} \quad \text{or} \quad \mathcal{Q}_2 = \sum_{p=1}^{L} \mathcal{Q}_{2,p,p+1}. \tag{4.18}$$

It is very suggestive to interpret this charge as the Hamiltonian, it has a nearest-neighbour type interaction as desired for a spin chain

$$\mathcal{H} = \mathcal{Q}_2 = \sum_{p=1}^{L} \mathcal{H}_{p,p+1}, \qquad \mathcal{H}_{12} = \mathcal{Q}_{2,12} = -i\, \mathcal{P}_{12}\, \mathcal{R}'_{12}(0). \tag{4.19}$$

Expanding $\mathcal{T}(u)$ to quadratic order in u we find many terms. There are some disconnected terms which should be absorbed into $-\frac{1}{2} u^2 \mathcal{U} \mathcal{Q}_2^2$ from the expansion of the exponential in (4.13). To complete the square $\mathcal{Q}_2^2$ we need the identity $\mathcal{R}''_{12}(0) = \mathcal{R}'_{12}(0) \mathcal{P}_{12} \mathcal{R}'_{12}(0)$ due to

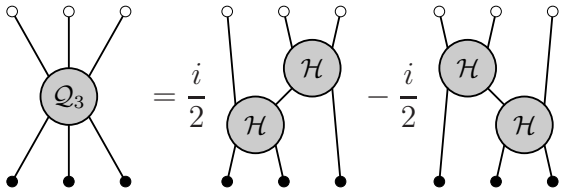

Figure 4.10: The third charge density is composed from two copies of the Hamiltonian density.

the Yang-Baxter equation. The remaining terms give rise to the third charge, see Fig. 4.10

$$\mathcal{Q}_3 = \sum_{p=1}^{L} \mathcal{Q}_{3,p,p+1,p+2}, \qquad \mathcal{Q}_{3,123} = \tfrac{i}{2}(\mathcal{H}_{12}\mathcal{H}_{23} - \mathcal{H}_{23}\mathcal{H}_{12}). \tag{4.20}$$

One can go on constructing the higher charges $\mathcal{Q}_r$ in this way and finds that they can all be written in terms of the Hamiltonian density $\mathcal{H}_{12}$.[4]

4.1.4 Parity and Pairs

Above we have constructed two charges of the spin chain, $\mathcal{H} = \mathcal{Q}_2$ and $\mathcal{Q}_3$. From (4.1,4.11) we know that they commute

$$[\mathcal{H}, \mathcal{Q}_3] = 0, \tag{4.21}$$

even though this statement is labourious to verify explicitly. Let us find out what happens when we invert the order of spins within the spin chain. This is equivalent to the parity operation $\mathfrak{p}$ defined in Sec. 1.3,2.6.4. up to a factor of $(-1)^L$. The Hamiltonian density will be assumed to have positive parity

$$\mathfrak{p}\,\mathcal{H}\,\mathfrak{p}^{-1} = \mathcal{H}, \qquad [\mathfrak{p}, \mathcal{H}] = 0. \tag{4.22}$$

From this it immediately follows that the third charge has negative parity (it has negative mirror symmetry with respect to the vertical axis, see Fig. 4.10)

$$\mathfrak{p}\,\mathcal{Q}_3\,\mathfrak{p}^{-1} = -\mathcal{Q}_3, \qquad \{\mathfrak{p}, \mathcal{Q}_3\} = 0. \tag{4.23}$$

Similarly one finds for the higher charges

$$\mathfrak{p}\,\mathcal{Q}_r\,\mathfrak{p}^{-1} = (-1)^r\mathcal{Q}_r. \tag{4.24}$$

A consequence of (4.22,4.23,4.24) is that the spectrum of $\mathcal{H}$ will display a degeneracy of states $|\pm\rangle$ with opposite parities [118]

$$\text{`}\textit{paired state}\text{'}: \qquad \big\{|+\rangle, |-\rangle\big\} \quad \text{with} \quad E_+ = E_-. \tag{4.25}$$

[3]One might have to redefine the R-matrix slightly exploiting the symmetries of the Yang-Baxter equation (4.4): Firstly, we can rescale the R-matrix by a function of the spectral parameter. Secondly, we can shift and rescale the spectral parameters by a constant. The bi-fundamental R-matrix in (4.7) is already in this from.

[4]A more efficient way is to use the boost operator $\mathcal{B} = \sum_{p=1}^{L} ip\,\mathcal{H}_{p,p+1}$. It generates the higher charges recursively via $[\mathcal{B}, \mathcal{Q}_r] = r\mathcal{Q}_{r+1}$. This can be deduced by assigning different spectral parameters to the individual spins, $u_p = u + p\epsilon$. Note however, that the boost leaves some undesired boundary terms which are, in particular, incompatible with the cyclic nature of the spin chain. These should be dropped.

This is a very non-trivial statement because $\mathcal{H}$ conserves parity and thus cannot relate states with opposite parities in any way. Assume we find a state of positive parity $|+\rangle$ and energy E_+. Then the state $|-\rangle = \mathcal{Q}_3|+\rangle$ has negative parity and energy E_+

$$\mathcal{H}\,|-\rangle = \mathcal{H}\,\mathcal{Q}_3\,|+\rangle = \mathcal{Q}_3\,\mathcal{H}\,|+\rangle = E_+\,\mathcal{Q}_3\,|+\rangle = E_+\,|-\rangle. \tag{4.26}$$

Of course, we cannot exclude that $\mathcal{Q}_3$ annihilates $|+\rangle$ and $|-\rangle = 0$. In this case the state of definite parity $P = +$ or $P = -$ is unpaired

$$\textit{'unpaired state':} \qquad |P\rangle \quad \text{with} \quad \mathcal{Q}_3|P\rangle = 0. \tag{4.27}$$

Such states exist when, for example, the numbers of positive and negative parity states do not agree.

We have seen that the third charge of the integrable spin chain has important consequences. It is thus natural to investigate the higher charges $\mathcal{Q}_r$. In contrast to $\mathcal{Q}_3$ we find that $\mathcal{Q}_4$ does not pair up operators, it simply assigns a number (charge) to each operator. This is in fact what might be expected. The reason why $\mathcal{Q}_3$ was interesting is that it anticommutes with $\mathfrak{p}$, while $\mathcal{H}$ commutes, thus giving rise to pairs. The next charge, $\mathcal{Q}_5$, does again anticommute with parity. This generator will relate the same pairs, only with different coefficients (charges).

4.2 One-Loop Integrability

In this section we derive the R-matrix for the integrable spin chain considered in this chapter. For this purpose we make use of a special subsector of the spin chain with residual $\mathfrak{su}(1,1)$ symmetry and show how to lift the universal $\mathfrak{sl}(2) = \mathfrak{su}(1,1)$ R-matrix to an $\mathfrak{psu}(2,2|4)$ invariant R-matrix. The derived Hamiltonian is shown to agree with the complete one-loop planar dilatation generator of $\mathcal{N} = 4$ SYM, thus proving the integrability of the latter.

4.2.1 Planar Parity Pairs

Let us have a look at the tables of one-loop planar spectra in Sec. 3.5. One observes a large number of degenerate pairs of states with opposite parity which are indicated by $\mathfrak{p} = \pm$. In fact, in no representation of the symmetry group unpaired states of both parities can be found. In other words, it appears that the only possibility for an unpaired state to exist, is the absence of a suitable partner. This picture is not expected to continue strictly at higher dimensions, but it shows that the pairing of states is very systematic and not merely a coincidence. A simple explanation for the pairing of states would be the existence of a conserved charge that anticommutes with parity, just like $\mathcal{Q}_3$, as explained in Sec. 4.1.4. Indeed, pairing of states is a useful criterion for integrability: The planar one-loop spectrum of $\mathcal{N} = 4$ displays pairing and is thus a candidate integrable system. Moreover, there is phenomenological evidence that paired spectra can only arise in an integrable system, see also [80]. We will discuss this point in Sec. 6.2.3. In this section we will use the methods of integrable spin chains introduced in Sec. 4.1 to find that planar one-loop $\mathcal{N} = 4$ is indeed integrable.

One may wonder whether the degeneracy also holds at the non-planar level. In an example we show that this is not the case. There are three unprotected multiplets with highest weights $w = (5; 0, 0; 1, 1, 1; 0, 5)$, two single-trace and one double-trace state. They are at both unitarity bounds and have zero spin. As such they have descendants in the quarter-BPS $\mathfrak{su}(2)$ sector. The states have length $L = 7$ and excitation number $K = 3$, i.e. they are of the form $\mathcal{Z}^4\phi^3$. Two have negative parity and one has positive parity, the line separates between them

$$\mathcal{E}^{\intercal} = \left(\begin{array}{c} 2\,\mathrm{Tr}\,\mathcal{Z}^4\phi^3 + 2\,\mathrm{Tr}\,\mathcal{Z}^2\phi\mathcal{Z}^2\phi^2 + 2\,\mathrm{Tr}\,\mathcal{Z}^2\phi\mathcal{Z}\phi\mathcal{Z}\phi - 3\,\mathrm{Tr}\,\mathcal{Z}^3\phi\{\phi,\mathcal{Z}\}\phi \\ \mathrm{Tr}\,\mathcal{Z}\phi\,\mathrm{Tr}\,\mathcal{Z}^2[\phi,\mathcal{Z}]\phi - \mathrm{Tr}\,\mathcal{Z}^2\,\mathrm{Tr}\,\mathcal{Z}[\phi,\mathcal{Z}]\phi^2 \\ \hline \mathrm{Tr}\,[\phi,\mathcal{Z}][\phi,\mathcal{Z}][\phi,\mathcal{Z}]\mathcal{Z}, \end{array} \right) . \tag{4.28}$$

The dilatation operator (2.113) acts on these as (note $\mathcal{H}\mathcal{E} = \mathcal{E}H$)

$$H = \left(\begin{array}{cc|c} 5 & \frac{10}{N} & 0 \\ \frac{4}{N} & 4 & 0 \\ \hline 0 & 0 & 5 \end{array} \right) . \tag{4.29}$$

This corresponds to the scaling dimensions exact for all values of N [34]

$$E_+ = 5, \qquad E_- = \tfrac{9}{2} \pm \sqrt{\tfrac{1}{4} + \tfrac{40}{N^2}}\,. \tag{4.30}$$

We find that the scaling dimensions of the two single-trace operators are degenerate at $N = \infty$. For finite N or in an expansion in powers of $1/N$ we find that the degeneracy is broken. Therefore integrability, as defined above, can only hold in the planar limit and breaks down when topological interactions take place. This is in agreement with the picture of a Wilson loop as a generating function for the charges. The Wilson loop of a flat connection can be moved around freely on the 'world-sheet' of the spin chain. It cannot, however, be moved past points of topological changes. This would require to cut open the loop and glue the ends in a different order, thus modifying the Wilson loop. Still, one may hope for some aspects of integrability to survive even when non-planar corrections are taken into account: The family of gauge connections (alias the R-matrix) underlying the Wilson loop is a local object and does not depend on the global structure of the world-sheet.

4.2.2 The Bosonic $\mathfrak{su}(1,1)$ Subsector

We will use a similar trick as in Sec. 3.3 to derive the R-matrix of the complete $\mathfrak{psu}(2,2|4)$ spin chain. Here, we shall use the Hamiltonian within the bosonic $\mathfrak{su}(1,1)$ subsector introduced in Sec. 3.4 to obtain an expression for the R-matrix which is subsequently lifted to the full theory.

The Hamiltonian density (3.71)

$$\mathcal{H}''_{12} = 2h(\mathcal{J}''_{12}) := \sum_{j=0}^{\infty} 2h(j)\,\mathcal{P}''_{12,j} \tag{4.31}$$

equals the one of the so-called Heisenberg $\mathrm{XXX}_{-1/2}$ spin chain.[5] Recall that the spins belong to $\mathcal{V}''_{\mathrm{F}}$ which is the highest-weight module $[-1]$ (spin $-1/2$) and the tensor product

[5] The integrable $\mathfrak{sl}(2)$ spin chain with fundamental spin representation $[s]$ (spin $s/2$) is called the 'Heisenberg $\mathrm{XXX}_{s/2}$ spin chain'

of two $\mathcal{V}''_{\rm F}$ decomposes into $\mathcal{V}''_j$ with highest weight $[-2-2j]$ (spin $-1-j$). The operator $\mathcal{P}''_{12,j}$ projects a two-spin state to the module $\mathcal{V}''_j$ and $\mathcal{J}''_{12}$ measures the label j of $\mathcal{V}''_j$. The function $h(j)$ gives the harmonic numbers.

Let us show that the above Hamiltonian is integrable. To accomplish this, we make use of the universal R-matrix of $\mathfrak{sl}(2)$ spin chains. This $\mathfrak{sl}(2)$ invariant operator can be decomposed into its irreducible components corresponding to the modules $\mathcal{V}''_j$

$$\mathcal{R}''_{12}(u) = \sum_{j=0}^{\infty} \mathcal{R}''_j(u)\, \mathcal{P}''_{12,j}. \tag{4.32}$$

The eigenvalues $\mathcal{R}''_j(u)$ of the $\mathfrak{sl}(2)$ universal R-matrix were determined in [119]. In a spin $-1-j$ representation the eigenvalue is

$$\mathcal{R}''_j(u) = (-1)^{j+1}\, \frac{\Gamma(-j-cu)}{\Gamma(-j+cu)}\, \frac{f(+cu)}{f(-cu)}\,. \tag{4.33}$$

The arbitrary function $f(u)$ and normalisation constant c reflect trivial symmetries of the Yang-Baxter equation. We choose the function and constant to be[6]

$$f(cu) = \Gamma(1+cu)\,, \quad c = -i. \tag{4.34}$$

This enables us to find rational expressions for $\mathcal{R}''_j$ and its derivative when j is integer

$$\mathcal{R}''_j(u) = \prod_{k=1}^{j} \frac{u-ik}{u+ik}\,, \qquad \frac{\partial \mathcal{R}''_j}{\partial u}(u) = \mathcal{R}''_j(u) \sum_{k=1}^{j} \frac{2ik}{k^2+u^2}\,. \tag{4.35}$$

We note that for even (odd) j the composite module $\mathcal{V}''_j$ is a (anti)symmetric combination of two $\mathcal{V}''_{\rm F}$, consequently the permutation acts as

$$\mathcal{P}_{12}\, \mathcal{V}''_j = (-1)^j\, \mathcal{V}''_j. \tag{4.36}$$

In other words, the R-matrix at $u=0$, whose elements equal $(-1)^j$, is a permutation

$$\mathcal{R}''_{12}(0) = \mathcal{P}_{12}. \tag{4.37}$$

We now obtain the induced Hamiltonian density using (4.19,4.35)

$$\mathcal{H}''_{12} = -i\, \mathcal{P}_{12}\, \frac{\partial \mathcal{R}''_j}{\partial u}(0) = 2h(\mathcal{J}''_{12})\, \mathcal{P}_{12}\, \mathcal{R}''_{12}(0) = 2h(\mathcal{J}''_{12}). \tag{4.38}$$

This proves the integrability of the Hamiltonian density $\mathcal{H}''_{12}$.

4.2.3 The Complete R-matrix

To derive an R-matrix for the full $\mathfrak{psu}(2,2|4)$ spin chain we will assume that for given representations of the symmetry algebra there exists a unique R-matrix which satisfies the Yang-Baxter equation (modulo the symmetries of the YBE). This claim [119] is supported

[6] The normalization for (4.6) uses $c=+i$. For non-compact representations it is however more convenient to use a different sign $c=-i$.

by the existence and uniqueness of the algebraic Bethe ansatz procedure in Sec. 4.3. Let $\mathcal{R}_{12}$ be this R-matrix for the $\mathfrak{psu}(2,2|4)$ integrable spin chain. The R-matrix is an invariant operator, thus it can be reduced to its irreducible components corresponding to the modules $\mathcal{V}_j$

$$\mathcal{R}_{12}(u) = \sum_{j=0}^{\infty} \mathcal{R}_j(u)\, \mathcal{P}_{12,j}. \tag{4.39}$$

The restriction $\mathcal{R}''$ of the R-matrix to the bosonic $\mathfrak{su}(1,1)$ sector must also satisfy the Yang-Baxter equation. The unique solution for the eigenvalues of $\mathcal{R}''$ is (4.33). Due to the one-to-one correspondence of modules $\mathcal{V}_j$ and $\mathcal{V}''_j$, c.f. Sec. 3.4, the eigenvalues of the unique $\mathfrak{psu}(2,2|4)$ R-matrix must be

$$\mathcal{R}_j(u) = \mathcal{R}''_j(u) = (-1)^{j+1}\, \frac{\Gamma(-j-cu)}{\Gamma(-j+cu)}\, \frac{f(+cu)}{f(-cu)}\,. \tag{4.40}$$

For the choice (4.34) of f and c, this R-matrix yields (4.38)

$$\mathcal{H}_{12} = 2h(\mathcal{J}_{12}). \tag{4.41}$$

This is just the one-loop Hamiltonian density of $\mathcal{N}=4$ SYM, c.f. Sec. 3.3, which in turn shows that the planar one-loop dilatation generator of $\mathcal{N}=4$ is integrable. Note, however, that this proof is based on the assumption of the existence of a unique R-matrix.

Let us verify that the R-matrix satisfies the Yang-Baxter equation involving two multiplets $\mathcal{V}_{\text{F}}$ and one fundamental module. We shall use the fundamental R-matrix

$$\mathcal{R}_p(u_p) = \frac{u_p}{u_p - i} - \frac{i}{u_p - i}\, \mathfrak{J}_p, \tag{4.42}$$

which obeys the Yang-Baxter equation with two fundamental particles and is similar to the bi-fundamental R-matrix (4.7).[7] The generator $\mathfrak{J}_p$ is a matrix of operators, the operators act on $\mathcal{V}_{\text{F}}$ at site p and the matrix is bi-fundamental. We now substitute this into the Yang-Baxter equation and expand (we suppress all indices)

$$\begin{aligned} 0 &= \mathcal{R}_{12}(u_1-u_2)\, \mathcal{R}_1(u_1)\, \mathcal{R}_2(u_2) - \mathcal{R}_2(u_2)\, \mathcal{R}_1(u_1)\, \mathcal{R}_{12}(u_1-u_2) \\ &= -\frac{i(u_1+u_2)}{2(u_1-i)(u_2-i)}\, [\mathcal{R}_{12}, \mathfrak{J}_{12}] - \frac{1}{2(u_1-i)(u_2-i)}\, [\mathcal{R}_{12}, Q'_{12}] \\ &\quad + \frac{i(u_1-u_2)}{2(u_1-i)(u_2-i)}\, [\mathcal{R}_{12}, q_{12}] - \frac{1}{2(u_1-i)(u_2-i)}\, \{\mathcal{R}_{12}, Q_{12}\}, \end{aligned} \tag{4.43}$$

where we have defined the bi-fundamental matrices of operators[8]

$$\mathfrak{J}_{12} = \mathfrak{J}_1 + \mathfrak{J}_2, \quad \mathcal{A}_{12} = \mathfrak{J}_1 - \mathfrak{J}_2, \quad \mathcal{B}_{12} = [\mathfrak{J}_1, \mathfrak{J}_2], \quad \mathcal{C}_{12} = \{\mathfrak{J}_1, \mathfrak{J}_2\}. \tag{4.44}$$

The action of these operators on the modules $\mathcal{V}_j$ was investigated in [116]. To understand how these operators act, it is useful to know their parity. It is straightforward to see

[7] For non-compact representations it is convenient to flip the sign of u.

[8] The commutator in Q_{12} does not vanish, because the operators $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are matrices.

that $\mathfrak{J}_{12}$ and $\mathcal{C}_{12}$ have positive parity, while $\mathcal{A}_{12}$ and $\mathcal{B}_{12}$ have negative parity. Therefore $\mathfrak{J}_{12}, \mathcal{C}_{12}$ map between modules of the same parity and $\mathcal{A}_{12}, \mathcal{B}_{12}$ invert the parity

$$\mathfrak{J}_{12}, \mathcal{C}_{12} : \mathcal{V}_j \to \mathcal{V}_{j+2n}, \qquad \mathcal{A}_{12}, \mathcal{B}_{12} : \mathcal{V}_j \to \mathcal{V}_{j+2n+1}. \tag{4.45}$$

Furthermore, all operators are invariant under $\mathfrak{psu}(2,2|4)$ if one simultaneously rotates the modules $\mathcal{V}_{\text{F}}$ and the bi-fundamental matrix. The bi-fundamental representation is just the adjoint, which can shift the highest weight of the module by not more than one step

$$\mathfrak{J}_{12}, \mathcal{A}_{12}, \mathcal{C}_{12}, \mathcal{B}_{12} : \mathcal{V}_j \to \mathcal{V}_{j-1}, \mathcal{V}_j, \mathcal{V}_{j+1}. \tag{4.46}$$

Together this teaches us that $\mathfrak{J}_{12}, \mathcal{C}_{12}$ do not change the spin j while $\mathcal{A}_{12}, \mathcal{B}_{12}$ change the spin j by one. We can immediately see that the first two commutators in (4.43) vanish[9] because $\mathcal{R}_{12}$ depends only on the total spin j.

We will now choose some state $|j\rangle$ from the module $\mathcal{V}_j$. Then $\mathcal{A}_{12}$ must change the spin by one $\mathcal{A}_{12}|j\rangle = |j+1\rangle + |j-1\rangle$ with some states $|j+1\rangle, |j-1\rangle$ from the modules $\mathcal{V}_{j+1}, \mathcal{V}_{j-1}$. We note a useful identity [116] to express $\mathcal{B}_{12}$ in terms of the quadratic Casimir $\mathfrak{J}_{12}^2$, c.f. App. D.3,

$$\mathcal{B}_{12} = -\tfrac{1}{2}[\mathcal{A}_{12}, \mathfrak{J}_{12}^2]. \tag{4.47}$$

Now we can compute $\mathcal{B}_{12}$ acting on $|j\rangle$ making use of $\mathfrak{J}_{12}^2|j\rangle = j(j+1)|j\rangle$, see (3.9),

$$\mathcal{B}_{12}|j\rangle = -\tfrac{1}{2}[\mathcal{A}_{12}, \mathfrak{J}_{12}^2]|j\rangle = (j+1)|j+1\rangle - j|j-1\rangle. \tag{4.48}$$

Let us now determine the remaining two terms in (4.43) with $u = u_1 - u_2$

$$\begin{aligned} 0 &= \big(u[\mathcal{R}_{12}, \mathcal{A}_{12}] + i\{\mathcal{R}_{12}, \mathcal{B}_{12}\}\big)|j\rangle \\ &= +\big((u + i(j+1))\mathcal{R}_{j+1} - (u - i(j+1))\mathcal{R}_j\big)|j+1\rangle \\ &\quad - \big((u+ij)\mathcal{R}_j - (u-ij)\mathcal{R}_{j-1}\big)|j-1\rangle. \end{aligned} \tag{4.49}$$

Due to (4.35), the R-matrix satisfies the recursion relation

$$\mathcal{R}_{j+1}(u) = \frac{u - i(j+1)}{u + i(j+1)}\,\mathcal{R}_j(u), \tag{4.50}$$

which completes the proof of the Yang-Baxter equation.

4.3 The Algebraic Bethe Ansatz

The Bethe ansatz determines the energy eigenvalues of a quantum integrable spin chain. It is very different from the direct diagonalisation of the Hamiltonian in that it does not involve finding a matrix representation for the Hamiltonian on some basis of states. Instead, it gives a set of algebraic equations whose solution directly leads to the energies as well as the eigenvalues of the higher charges.

[9] The commutator $[\mathcal{R}_{12}, \mathfrak{J}_{12}]$ is trivially zero by invariance of the R-matrix.

4.3.1 The Heisenberg Chain

Let us explain the Bethe ansatz in the simplest case of an $\mathfrak{sl}(2)$ chain, the so-called $\mathrm{XXX}_{s/2}$ Heisenberg chain. (For a very pedagogical introduction, see [59]). The results apply directly to the $\mathfrak{su}(2)$ subsector of Sec. 2.4 when $s = 1$ (spin 1/2) and the $\mathfrak{su}(1,1)$ subsectors of Sec. 3.2,3.4 when $s = -2$ or $s = -1$. Each eigenstate of the Hamiltonian is uniquely characterised by a set of complex Bethe roots u_k, $k = 1, \ldots, K$,

$$\text{`}\textit{Bethe roots}\text{'}: \quad \{u_1, \ldots, u_K\}, \qquad u_k \in \mathbb{C}. \tag{4.51}$$

These determine the energy E and eigenvalue U of the shift operator $\mathcal{U}$ of the state by[10]

$$E = \sum_{k=1}^{K} \frac{|s|}{u_k^2 + \frac{1}{4}s^2}, \qquad U = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}|s|}{u_k - \frac{i}{2}|s|}. \tag{4.52}$$

More generally, the matrix elements of the transfer matrix in a spin $t/2$ representation for a given set of roots are determined by

$$\begin{aligned} T_t(u) = \sum_{m=0}^{t} &\left(\frac{\Gamma(iu \operatorname{sign} s - \frac{1}{2}t + \frac{1}{2}s + m)}{\Gamma(iu \operatorname{sign} s - \frac{1}{2}t - \frac{1}{2}s + m)} \frac{\Gamma(iu \operatorname{sign} s - \frac{1}{2}t - \frac{1}{2}s)}{\Gamma(iu \operatorname{sign} s - \frac{1}{2}t + \frac{1}{2}s)} \right)^L \\ &\times \prod_{k=1}^{K} \left(\frac{u - u_k + \frac{i}{2}(-t) \operatorname{sign} s}{u - u_k + \frac{i}{2}(t - 2m) \operatorname{sign} s} \frac{u - u_k + \frac{i}{2}(t+2) \operatorname{sign} s}{u - u_k + \frac{i}{2}(t - 2m + 2) \operatorname{sign} s} \right). \end{aligned} \tag{4.53}$$

Here the upper limit of the sum should be extended to infinity whenever t is not a positive integer. From the transfer matrix $T(u) = T_s(u)$ in the spin representation, $t = s$, we can read off the higher charges Q_r via (4.13)

$$Q_r = \frac{i}{r-1} \sum_{k=1}^{K} \left(\frac{1}{(u_k + \frac{i}{2}|s|)^{r-1}} - \frac{1}{(u_k - \frac{i}{2}|s|)^{r-1}} \right), \quad T(u) = \prod_{k=1}^{K} \frac{u - u_k - \frac{i}{2}|s|}{u - u_k + \frac{i}{2}|s|} + \ldots . \tag{4.54}$$

The charges are only valid for $r \leq L$ due to the neglected terms in $T(u)$ with $m \neq 0$.

The Bethe roots are found by solving the Bethe equations for $k = 1, \ldots, K$

$$\text{`}\textit{Bethe equations}\text{'}: \qquad \left(\frac{u_k - \frac{i}{2}s}{u_k + \frac{i}{2}s} \right)^L = \prod_{\substack{l=1 \\ l \neq k}}^{K} \frac{u_k - u_l - i}{u_k - u_l + i}. \tag{4.55}$$

These equations should be solved subject to the constraint that no two roots coincide. Furthermore, roots at infinity correspond to descendants; for highest-weight states there are no roots at infinity. Note that the above Bethe equations follow from (4.53) by cancellation of poles in $T_t(u)$ at $u = u_k - \frac{i}{2}(t - 2m + 2) \operatorname{sign} s$.

Note that the Bethe ansatz conceptually agrees with the particle picture presented in Sec. 4.1.1: Each Bethe root can be considered as a particle. The right-hand side of the Bethe equations (4.55) corresponds to scattering of two particles, while the left-hand

[10] The absolute value for s is used for convenience; it makes the energy positive, but requires a redefinition of u_k when changing the sign of s.

side corresponds to the propagation of the particle across L spin chain sites. There is no interaction of more than two particles. The phase-shifts due to these interactions must agree for an eigenstate. The total energy (4.52) is just the sum of the energies of the particles within the system.

We will start with the simplest example: The $\mathfrak{su}(2)$ sector with spin representation $s = 1$ (spin 1/2) and states of the form

$$\operatorname{Tr} \mathcal{Z}^{L-K}\phi^K + \dots . \tag{4.56}$$

In this particular model, the spin at each site can either point up ($\mathcal{Z}$) or down (ϕ). The vacuum state with no excitations, $K = 0$, is the half-BPS state

$$|\mathcal{Z}, L\rangle = \mathcal{Z}^L, \tag{4.57}$$

with all spins aligned. This is the ferromagnetic ground state of the chain. The excitation number K, giving the total number of roots, counts the number of ϕ's or down-spins along the chain. Assuming excitations are generated by some creation operator $\mathcal{B}(u)$,[11] a generic state has the form

$$\{u_1, \dots, u_K\} \quad \leftrightarrow \quad \mathcal{B}(u_1) \dots \mathcal{B}(u_K)|\mathcal{Z}, L\rangle \quad \leftrightarrow \quad \mathcal{Z}^{L-K}\phi^K + \dots . \tag{4.58}$$

There is an additional constraint on the Bethe roots:

$$1 = U = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} . \tag{4.59}$$

For the spin chain, it means that we have periodic boundary conditions and we are only looking for zero-momentum states. In the gauge theory interpretation it expresses the cyclicity of the trace

$$\{u_1, \dots, u_K\} \text{ with (4.59)} \quad \leftrightarrow \quad \operatorname{Tr} \mathcal{B}(u_1) \dots \mathcal{B}(u_K)|\mathcal{Z}, L\rangle \quad \leftrightarrow \quad \operatorname{Tr} \mathcal{Z}^{L-K}\phi^K + \dots . \tag{4.60}$$

For the $s = 1$ Bethe ansatz, the exact eigenvalue of the transfer matrix (4.53) is

$$T(u) = \prod_{k=0}^{K} \frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \left(\frac{u}{u+i}\right)^L \prod_{k=0}^{K} \frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{i}{2}} . \tag{4.61}$$

Note that we can derive the Bethe equations from this expression by demanding that $(u+i)^L T(u)$ has no singularities.

The Dynkin label of a solution with K excitations is $[L - 2K]$. Therefore, one should consider only solutions with $K \leq L/2$, there are clearly no highest-weight states with more excited spin sites. However, the Bethe equations do have solutions also for $K > L/2$. It is interesting to see that for a solution with $K \leq L/2$, there exists also a mirror solution with $K' = L + 1 - K > L/2$.[12] In terms of Dynkin labels, the solutions are related by $s' = -s - 2$. There is an explanation for this behaviour in terms of multiplet shortening, see Sec. 1.7. We are considering a spin chain with a finite (short) $s = 1$ multiplet at

[11] The operator $\mathcal{B}$ is an element of the monodromy matrix in the fundamental representation.

[12] This solution has norm zero, thus it is not realised as a spin chain state.

each site, consequently also the eigenstates form finite multiplets. In the Bethe ansatz, shortening is not taken into account and all multiplets are assumed to be infinite (long). The relevant solutions are therefore highest weights of reducible multiplets which split into two irreducible components. Interestingly, the Bethe ansatz finds the highest weight states of both submultiplets and naturally the energies and changes must agree. In some cases this peculiarity can be made use of by solving for the mirror states.

The second simplest example concerns the bosonic $\mathfrak{su}(1,1)$ subsector with spin representation $s=-1$ (spin $-1/2$) and states of the form

$$\mathrm{Tr}(\mathcal{D}^{n_1}\mathcal{Z})\cdots(\mathcal{D}^{n_L}\mathcal{Z}). \tag{4.62}$$

Here the spins at each lattice site p may take any value $n_p=0,1,2,\ldots$, as we have an infinite $[-1]$ representation of $\mathfrak{sl}(2)$. Furthermore, the total excitation number $K=\sum n_k$ is not bounded as in the above example. The vacuum is still $\mathcal{Z}^L$. Again, the energies of the states (4.62) with momentum $\mathcal{U}=1$ are given via (4.52,4.55).

The third example is the fermionic $\mathfrak{su}(1,1)$ subsector with spin representation $s=-2$ (spin -1) and states of the form

$$\mathrm{Tr}(\mathcal{D}^{n_1}\Psi)\cdots(\mathcal{D}^{n_L}\Psi). \tag{4.63}$$

There are two chief differences as compared to the other subsectors: Firstly, the fermionic nature of the fields requires a modified cyclicity condition

$$(-1)^{L+1}=U=\prod_{k=1}^{K}\frac{u_k+i}{u_k-i}\,. \tag{4.64}$$

For example, the ground state exists only for odd L, as for even L we have $\mathrm{Tr}\,\Psi^L=0$. Secondly, the ground state does not have zero energy, but $E=2L$

$$E=2L+\sum_{k=1}^{K}\frac{2}{u_k^2+1}\,. \tag{4.65}$$

In particular the ground state $\mathrm{Tr}\,\Psi^3$ is a Konishi descendant with $E=6$.

4.3.2 Generic Algebras

In the above example the algebra was $\mathfrak{su}(2)$ and thus of rank one. There is a beautiful extension of the Bethe equations to an arbitrary symmetry algebra and arbitrary representation due to Reshetikhin and Ogievetsky, Wiegmann [120]. The general form also extends to the case of super algebras, see [121] and references therein, and is precisely what we need for $\mathcal{N}=4$ SYM at one-loop. There, we should expect Bethe equations for the superalgebra $\mathfrak{psu}(2,2|4)$ to generate the correct spectrum. The general equation is based on knowing the Dynkin diagram of the algebra. The Dynkin diagram of $\mathfrak{psu}(2,2|4)$ contains seven dots corresponding to a choice of seven simple roots. Consider a total of K excitations. For each of the corresponding Bethe roots u_k, $k=1,\ldots,K$, we specify by $j_k=1,\ldots,7$ which of the seven simple roots is excited. The Bethe equations for

$k = 1, \ldots, K$ can then be written in the compact form

$$\left(\frac{u_k - \frac{i}{2}V_{j_k}}{u_k + \frac{i}{2}V_{j_k}}\right)^L = \prod_{\substack{l=1 \\ l\neq k}}^{K} \frac{u_k - u_l - \frac{i}{2}M_{j_k,j_l}}{u_k - u_l + \frac{i}{2}M_{j_k,j_l}}\,. \tag{4.66}$$

Here, M is the Cartan matrix of the algebra and V are the Dynkin labels of the spin representation. Furthermore, we still consider a cyclic spin chain with zero total momentum. This gives the additional constraint[13]

$$1 = U = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}V_{j_k}}{u_k - \frac{i}{2}V_{j_k}}\,. \tag{4.67}$$

The energy of a configuration of roots that satisfies the Bethe equations is now given by[14]

$$E = \sum_{k=1}^{K} \frac{V_{j_k}}{u_k^2 + \frac{1}{4}V_{j_k}^2}\,. \tag{4.68}$$

Apparently, also the higher charges ($r \leq L$) and transfer matrix can be obtained [122]

$$Q_r = \frac{i}{r-1}\sum_{k=1}^{K}\left(\frac{1}{(u_k + \frac{i}{2}V_{j_k})^{r-1}} - \frac{1}{(u_k - \frac{i}{2}V_{j_k})^{r-1}}\right), \quad T(u) = \prod_{k=1}^{K}\frac{u - u_k - \frac{i}{2}V_{j_k}}{u - u_k + \frac{i}{2}V_{j_k}} + \ldots\,. \tag{4.69}$$

It is easily seen that restricting these equations to the Dynkin diagram of the algebra $\mathfrak{so}(6)$ reproduces the Bethe equations of [48]. It will turn out, see below, that these general equations, which are well known in the literature on integrable spin chains, indeed solve the entire problem of computing planar anomalous dimensions in $\mathcal{N} = 4$ SYM, once we (*i*) identify the correct representations of the fundamental fields on the lattice sites, and (*ii*) after resolving certain subtleties concerning Dynkin diagrams for superalgebras.

4.3.3 The Complete Bethe Ansatz

In Sec. 4.2 we have established that the planar one-loop dilatation operator of $\mathcal{N} = 4$ SYM is integrable. We therefore expect the general Bethe ansatz equations (4.66) to hold. However, for them to be useful, we still need to specify the Dynkin labels, the Cartan matrix and precise form of the energy (4.68). Furthermore, we will perform a check of the validity of this $\mathfrak{psu}(2,2|4)$ Bethe ansatz which goes beyond the $\mathfrak{so}(6)$ spin chain investigated in [48].

First, we need to specify the Cartan matrix, determined by the Dynkin diagram, and the Dynkin labels of the spin representation corresponding to the module $\mathcal{V}_{\rm F}$. For a classical semi-simple Lie algebra the Dynkin diagram is unique. In the case of superalgebras, however, there is some freedom to distribute the simple fermionic roots. For $\mathcal{N} = 4$ SYM the Dynkin diagram Fig. 1.10,4.12 turns out to be very convenient. On top of the Dynkin diagram Fig. 4.12 we have indicated the Dynkin labels of the spin representation.

[13] For a fermionic vacuum there is an additional sign as in (4.64).

[14] In fact, the Bethe equations determine the energy only up to scale c and a shift eL as in (4.65).

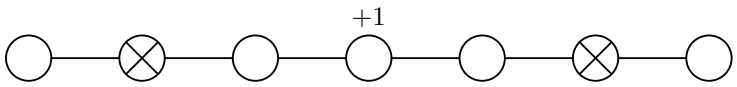

Figure 4.12: Dynkin diagram and spin representation vector for the $\mathfrak{psu}(2,2|4)$ Bethe ansatz.

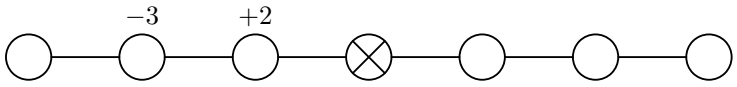

Figure 4.14: A different Dynkin diagram and spin representation vector for the $\mathfrak{psu}(2,2|4)$ Bethe ansatz.

We write the Cartan matrix corresponding to this choice of Dynkin diagram and the representation vector as[15]

$$M = \left(\begin{array}{c|c|ccccc|c} -2 & +1 & & & & & & \\ \hline +1 & & -1 & & & & & \\ \hline & -1 & +2 & -1 & & & & \\ & & -1 & +2 & -1 & & & \\ & & & -1 & +2 & -1 & & \\ \hline & & & & -1 & & +1 & \\ \hline & & & & & +1 & -2 & \end{array}\right), \qquad V = \left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \\ 1 \\ 0 \\ \hline 0 \\ \hline 0 \end{array}\right). \tag{4.70}$$

There exist other choices of Dynkin diagrams. E.g. the 'distinguished' one is depicted in Fig. 4.14. We have indicated the Dynkin labels of $\mathcal{V}_{\mathrm{F}}$ on top. The energy is given by (4.68), except for a vacuum energy shift of $3L$. The ansatz is rather odd and appears hardly helpful in terms of physics. Nevertheless, it was investigated in [58] and shown to yield the same spectrum by means of example, a good confirmation of the validity of the Bethe ansatz methods.

4.3.4 Excitation Numbers

Finally, we need to obtain the number of excitations K_j, $j = 1, \ldots, 7$, of the individual simple roots for a state with a given weight

$$w = (D_0; s_1, s_2; q_1, p, q_2; B, L). \tag{4.71}$$

This is most easily seen in the oscillator picture in Sec. 1.9 using the physical vacuum $|\mathcal{Z}, L\rangle$. We present the action of the generators corresponding to the simple roots in terms of creation and annihilation operators in Fig. 4.16. It is now clear that $K_1 = n_{\mathbf{a}_1}$,

[15] In fact, the Cartan matrix is obtained from this by inverting some lines. The Bethe equations are invariant under the inversion and it is slightly more convenient to work with a symmetric matrix M.

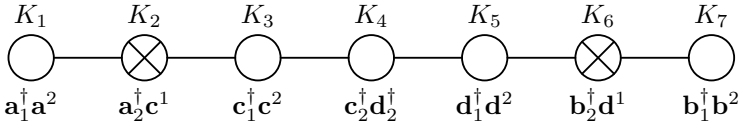

Figure 4.16: Excitation numbers for the Bethe roots and associated oscillator representation.

$K_2 = n_{\mathbf{a}_1} + n_{\mathbf{a}_2}$ and so on. Using the formulas in Tab. D.2 we write down the corresponding excitation numbers of the simple roots

$$K_j = \begin{pmatrix} \frac{1}{2}D_0 - \frac{1}{2}(L-B) - \frac{1}{2}s_1 \\ D_0 - (L-B) \\ D_0 - \frac{1}{2}(L-B) - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ D_0 - p - \frac{1}{2}q_1 - \frac{1}{2}q_2 \\ D_0 - \frac{1}{2}(L+B) - \frac{1}{2}p - \frac{1}{4}q_1 - \frac{3}{4}q_2 \\ D_0 - (L+B) \\ \frac{1}{2}D_0 - \frac{1}{2}(L+B) - \frac{1}{2}s_2 \end{pmatrix}. \tag{4.72}$$

Not all excitations of the simple roots correspond to physical states. Obviously, the excitation numbers of the oscillators must be non-negative, this gives the bounds

$$0 \leq K_1 \leq K_2 \leq K_3 \leq K_4 \geq K_5 \geq K_6 \geq K_7 \geq 0. \tag{4.73}$$

Furthermore, each fermionic oscillator cannot be excited more than once, this gives the bounds[16]

$$K_2 + 2L \geq K_3 + L \geq K_4 \leq K_5 + L \leq K_6 + 2L. \tag{4.74}$$

Certainly, we should obtain the $\mathfrak{so}(6)$ subsector studied by Minahan and Zarembo [48] when we remove the outer four simple roots from the Dynkin diagram in Fig. 4.12. When we restrict to the states of this subsector the number of excitations (4.72) of the outer four roots is trivially zero. They become irrelevant for the Bethe ansatz and can be discarded. Thus all solutions to the $\mathfrak{so}(6)$ Bethe equations are also solutions to the $\mathfrak{psu}(2,2|4)$ Bethe equations. What is more, we can apply this Bethe ansatz to a wider range of operators, in fact, to *all* single-trace operators of $\mathcal{N} = 4$ SYM.

4.3.5 Multiplet Splitting

Now we can write down and try to solve the Bethe equations for any state in $\mathcal{N} = 4$ SYM. Note, however, that the Bethe equations need to be solved only for highest weight states. All descendants of a highest weight state are obtained by adding Bethe roots at infinity, $u_k = \infty$. In other words, the solutions to the Bethe equations corresponding to highest weight states are distinguished in that they have no roots u_k at infinity. Nevertheless, there is one subtlety related to this point which can be used to our advantage. Namely this is multiplet splitting at the unitarity bounds as discussed in Sec. 1.8. We assume that the spin chain of L sites transforms in the tensor product of L spin representations. The corrections δD to the scaling dimension induced by the Hamiltonian $\mathcal{H}$ are not included in this picture. Thus, in terms of the spin chain, only the classical $\mathfrak{psu}(2,2|4)$ algebra applies where the scaling dimension is exactly D_0. The shortening conditions given in Sec. 1.8 can also be expressed in terms of excitations of simple roots,

[16] Superconformal primaries reside in the fundamental Weyl chamber defined by the bounds $-2K_1 + K_2 > -1$, $K_2 - 2K_3 + K_4 > -1$, $K_3 - 2K_4 + K_5 + L > -1$, $K_4 - 2K_5 + K_6 > -1$, $K_6 - 2K_7 > -1$. Together with (4.73) this implies, among other relations, (4.74). Solutions of the Bethe equations outside the fundamental domain apparently correspond to mirror images of primary states due to reflections at the chamber boundaries.

we find

$$\begin{aligned} \text{i}: &\quad K_1 + K_3 = K_2 + 1, \\ \text{ii}: &\quad K_7 + K_5 = K_6 + 1. \end{aligned} \tag{4.75}$$

The corresponding offsets translate into

$$\begin{aligned} \delta K_{\text{i}} &= (\ 0, -1, -1,\ 0,\ 0,\ 0,\ 0), \quad \delta L = 1, \qquad \text{for } K_2 > K_1, \\ \delta K_{\text{I}} &= (\ 0,\ 0, -1,\ 0,\ 0,\ 0,\ 0), \quad \delta L = 1, \qquad \text{for } K_2 = K_1, \\ \delta K_{\text{ii}} &= (\ 0,\ 0,\ 0,\ 0, -1, -1,\ 0), \quad \delta L = 1, \qquad \text{for } K_6 > K_7, \\ \delta K_{\text{II}} &= (\ 0,\ 0,\ 0,\ 0, -1,\ 0,\ 0), \quad \delta L = 1, \qquad \text{for } K_6 = K_7. \end{aligned} \tag{4.76}$$

We thus see that in the case of multiplet shortening, the primaries of higher submultiplets have less excitations. In an explicit calculation this may reduce the complexity of the Bethe equations somewhat as we shall see in an example below.

Multiplet splitting is an extremely interesting issue from the point of view of integrability. Let us consider some operator acting on a spin chain. Assume the operator is invariant under the classical algebra $\mathfrak{psu}(2,2|4)$. In the most general case, this operator can assign a different value to all irreducible multiplets of states. In particular this is so for the submultiplets of a long multiplet at the unitarity bound, see Sec. 1.8. Now, if we impose integrability on the operator, we obtain the one-loop planar correction to the dilatation operator of $\mathcal{N} = 4$ SYM. In $\mathcal{N} = 4$ SYM the submultiplets rejoin into a long multiplet and for consistency they must be degenerate (note that the momentum constraint $U = 1$ is crucial for this observation). A priori, from the point of view of the spin chain, this seems like a miracle, especially in view of the fact that the submultiplets have a different number of spin sites L! Why should integrability imply *this* degeneracy? For a simple manifestation of this fact, one may consider the fermionic $\mathfrak{su}(1,1)$ subsector discussed in Sec. 3.2. This subsector has an additional $\mathfrak{u}(1|1)$ supersymmetry, which relates states of different length and which was used to construct the complete one-loop dilatation generator. Obviously the solutions to the Bethe ansatz for this system, c.f. Sec. 4.3.1, must display this symmetry (note the momentum constraint). What is the origin of this symmetry (putting $\mathcal{N} = 4$ SYM aside)? It almost seems as if integrability selects the one invariant operator which is suitable as a consistent deformation of the dilatation generator! Then, clearly the miracle would turn into the condition for integrability.

4.3.6 Degenerate Pairs

The Bethe equations are invariant under the map

$$\{u_k\} \mapsto \{-u_k\}. \tag{4.77}$$

Also the energy and all even charges are invariant, the odd charges change sign,

$$Q_r \mapsto (-1)^r Q_r. \tag{4.78}$$

This operation is most naturally identified with parity $\mathfrak{p}$. Therefore, for every solution $\{u_k\}$ there is another solution $\{-u_k\}$ with degenerate energy and even charges, but negative odd charges

$$\text{`}\textit{paired states}\text{'}: \quad \{u_k\} \neq \{-u_k\}. \tag{4.79}$$

Unpaired states are such states for which

$$\text{`}\textit{unpaired state}\text{'}: \quad \{u_k\} = \{-u_k\}. \tag{4.80}$$

This is the manifestation of the findings of Sec. 4.1.4 within the Bethe ansatz.[17]

4.4 Spectrum

In order to illustrate the application of the Bethe ansatz, we shall repeat the investigation of the spectrum in Sec. 3.5 with the Bethe ansatz. We will see that, except in a few examples, it is rather tedious to find exact solutions to the Bethe equations. In the following section, however, we will investigate a class of states for which the Bethe ansatz is of tremendous importance.

4.4.1 Example

In the following, we will apply the complete Bethe ansatz to the two-parton state with highest weight (c.f. Sec. 3.5.2)

$$w = (4; 2, 2; 0, 0, 0; 0, 2). \tag{4.81}$$

Using (4.72) we find the excitation numbers and length

$$K_{0,j} = (0, 2, 3, 4, 3, 2, 0), \quad L_0 = 2. \tag{4.82}$$

This weight is on both unitarity bounds, c.f. (4.75), the excitation numbers of the highest submultiplet, c.f. (4.76), are

$$K_j = K_{0,j} + \delta K_{\mathrm{I},j} + \delta K_{\mathrm{II},j} = (0, 1, 2, 4, 2, 1, 0), \quad L = L_0 + 2\delta L = 4. \tag{4.83}$$

We therefore configure the simple roots as follows

$$j_k = (2, 3, 3, 4, 4, 4, 4, 5, 5, 6). \tag{4.84}$$

Now we note that twist-two states are unpaired states. Therefore the configurations of Bethe roots must be invariant under the symmetry $\{u_k\} \mapsto \{-u_k\}$. This tells us

$$u_1 = u_{10} = 0, \quad u_2 = -u_3, \quad u_4 = -u_5, \quad u_6 = -u_7, \quad u_8 = -u_9 \tag{4.85}$$

and the momentum constraint (4.67) is automatically satisfied. Furthermore the excitations (4.83) are invariant under flipping the Dynkin diagram, $K_j \mapsto K_{8-j}$. This suggests the ansatz

$$u_2 = u_8. \tag{4.86}$$

The Bethe equations (4.66) are then solved exactly by

$$u_2 = \sqrt{\frac{5}{7}}, \quad u_{4,6} = \sqrt{\frac{65 \pm 4\sqrt{205}}{140}}, \tag{4.87}$$

[17] The parity eigenvalue P seems to be determined by the number of Bethe roots at zero and the length.

which yields the energy (4.68)

$$E = \frac{25}{3}\,. \tag{4.88}$$

This is indeed the energy of the twist-two state at dimension four, c.f. Sec. 3.5.2 and [40].

Note that in some cases it may be more convenient to use a different Dynkin diagram from the one in Fig. 4.12. In this example, the distinguished Dynkin diagram Fig. 4.14 would require only two Bethe roots [58]. Alternatively, one might consider one of the $\mathfrak{su}(1,1)$ subsectors of Sec. 3.2,3.4 to simplify the investigation.

4.4.2 Two Excitations

States with two excitations, see Sec. 3.5.4, are the simplest solutions to the Bethe equations [48]. Let us consider the two-excitation state of the $\mathfrak{su}(2)$ subsector first. For this purpose, we can restrict to the Bethe ansatz for the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain. We should solve the Bethe equations for two roots $u_{1,2}$. Let us start with the momentum constraint

$$1 = U = \frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}} \frac{u_2 + \frac{i}{2}}{u_2 - \frac{i}{2}}\,, \tag{4.89}$$

this requires $u_2 = -u_1$. Now the Bethe equations for u_1 and u_2 collapse to the single equation

$$\left(\frac{u_1 - \frac{i}{2}}{u_1 + \frac{i}{2}}\right)^L = \frac{2u_1 - i}{2u_1 + i} \qquad \text{or} \qquad \left(\frac{u_1 - \frac{i}{2}}{u_1 + \frac{i}{2}}\right)^{L-1} = 1. \tag{4.90}$$

This equation has the solutions

$$u_{1,2} = \pm\frac{1}{2}\cot\frac{\pi n}{L-1} \tag{4.91}$$

for $0 \le n < (L-1)/2$. The energy of this solution is

$$E = 8\sin^2\frac{\pi n}{L-1}\,, \tag{4.92}$$

in agreement with Sec. 3.5.4. In addition to the energy, we can also compute the values of the higher charges

$$Q_r = \frac{\big(1+(-1)^r\big)2^r}{r-1}\,\sin\frac{\pi(r-1)n}{L-1}\,\sin^{r-1}\frac{\pi n}{L-1}\,. \tag{4.93}$$

The two-excitation multiplets are actually at both unitarity bounds and split up in the classical theory. As far as the one-loop Bethe ansatz is concerned, the classical symmetry algebra applies and we should be able to find further solutions corresponding to the three other submultiplets, Sec. 3.5.4. The highest weight of the top multiplet is $(L;0,0;0,L-2,0;0,L)$ According to (4.72), it all requires two excitations of type 4 and one excitation of types $3,5$ each. We will configure the Bethe roots as $j_k = (4,4,3,5)$. The solution to the Bethe equations is found straightforwardly

$$u_{1,2} = \pm\frac{1}{2}\cot\frac{\pi n}{L+1}\,, \quad u_3 = u_4 = 0, \qquad E = 8\sin^2\frac{\pi n}{L+1}\,. \tag{4.94}$$

Finally, the highest weights of the middle submultiplets are $(L;0,0;2,L-3,0;0,L)$ and its conjugate. The solutions to the root configuration $j_k=(4,4,3)$ or $j_k=(4,4,5)$ are

$$u_{1,2}=\pm\frac{1}{2}\cot\frac{\pi n}{L}\,,\quad u_3=0,\qquad E=8\sin^2\frac{\pi n}{L}\,. \tag{4.95}$$

Their energies agree precisely with the results of Sec. 3.5.4.

4.4.3 Three Excitations

In Sec. 3.5.5 we have investigated a peculiar set of states with three impurities in the $\mathfrak{su}(2)$ sector and found their exact planar one-loop energies and eigenstates. The energy of all states turned out to be the same for all states, $E=6$. This is best understood in terms of the Bethe equations of which they are very special solutions. The states are unpaired and therefore we should expect the Bethe roots to be invariant under $\{u_k\}\mapsto\{-u_k\}$. This requires that $u_3=0$ is one of the roots and $u_1=-u_2$. Unfortunately, this seems to imply $U=-1$ and violate the trace condition. However, the singular points $u_{1,2}=\pm\frac{i}{2}$ can invert the momentum once again.[18] Therefore the roots must be

$$u_{1,2}=\pm\tfrac{i}{2}\,,\qquad u_3=0. \tag{4.96}$$

The singularity needs to be regularised, e.g. the Bethe equations and the energy formula are naively divergent. It is best to consider the transfer matrix

$$\begin{aligned}T(u)&=\frac{u}{(u+i)}\,\frac{(u-i)}{u}\,\frac{(u-\frac{i}{2})}{(u+\frac{i}{2})}+\left(\frac{u}{u+i}\right)^L\frac{(u+2i)}{(u+i)}\,\frac{(u+i)}{u}\,\frac{(u+\frac{3i}{2})}{(u+\frac{i}{2})}\\&=\frac{(u-i)}{(u+i)}\,\frac{(u-\frac{i}{2})}{(u+\frac{i}{2})}+\left(\frac{u}{u+i}\right)^{L-1}\frac{(u+2i)}{(u+i)}\,\frac{(u+\frac{3i}{2})}{(u+\frac{i}{2})}\,.\end{aligned} \tag{4.97}$$

It is easy to confirm that $(u+i)^LT(u)$ has no poles for even L and thus $\{u_{1,2,3}\}$ is indeed a solution, even if the Bethe equations and energy formula appear divergent. From this expression it is also straightforward to derive the energy $E=6$ and higher charges Q_r

$$Q_r=\frac{\big((+i)^{r-2}+(-i)^{r-2}\big)\big(2^{r-1}+1\big)}{(r-1)}\,,\qquad r\leq L-2, \tag{4.98}$$

which can clearly seen to be independent of L for small r.

4.5 The Thermodynamic Limit

The BMN limit (c.f. Sec. 3.6) is very interesting for the AdS/CFT correspondence because it allows to make contact to (plane-wave) string theory on a quantitative level. In the BMN limit, the length of the spin chain approaches infinity, $J\sim L\to\infty$, while the number of excitations is fixed at a finite value. This requires some rescaling of energies.

The thermodynamic limit is a generalisation of the BMN limit in that the spin chain grows very long while focusing on the low energy spectrum. The difference to the BMN

[18] The singular roots lead to states with sticky excitations which are always on adjacent spin chain sites, c.f. (3.90).

limit is that the number of excitations is proportional to L and also approaches infinity. In this case, the Bethe equations turn into integral equations, similar to the ones found in matrix models. As in the BMN limit, one can make contact to string theory as will be seen in the following section. Here, we will lay the foundation for this comparison on a general level. For a beautiful review of the thermodynamic limit of the Bethe equations and the arising Riemann surfaces, see [75].

4.5.1 The Heisenberg Chain

Here we will outline the thermodynamic limit of the Bethe ansatz system of equations (4.52,4.55) for the case of the $\mathrm{XXX}_{s/2}$ Heisenberg spin chain with length L and K excitations (c.f. [123])

$$E = \sum_{k=1}^{K} \frac{|s|}{u_k^2 + \frac{1}{4}s^2}\,, \qquad 1 = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}|s|}{u_k - \frac{i}{2}|s|}\,, \qquad \left(\frac{u_k - \frac{i}{2}s}{u_k + \frac{i}{2}s}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{K} \frac{u_k - u_l - i}{u_k - u_l + i}\,. \tag{4.99}$$

For a large length L and solutions of a sufficiently low energy, we expect that the positions of the roots u_k scale as L, see (4.91) [48]. Let us therefore define $u_k = L\tilde{u}_k$.[19] We then take the logarithm of the equations (4.99) and obtain for large L

$$\tilde{E} = LE = \frac{1}{L}\sum_{k=1}^{K} \frac{|s|}{\tilde{u}_k^2}\,, \qquad 2\pi n = \frac{1}{L}\sum_{k=1}^{K} \frac{|s|}{\tilde{u}_k}\,, \qquad 2\pi n_k - \frac{s}{\tilde{u}_k} = \frac{1}{L}\sum_{\substack{l=1\\ l\neq k}}^{K} \frac{2}{\tilde{u}_l - \tilde{u}_k}\,. \tag{4.100}$$

The integer mode numbers n_k, n enumerate the possible branches of the logarithm. The rescaled energy $\tilde{E} = LE$ was defined such that there is one power of $1/L$ in front of the sum. This will cancel against the $\mathcal{O}(L)$ terms of the sum.[20] The total rescaled dimension is thus given by

$$\tilde{D} = D/L = \tilde{D}_0 + \tilde{g}^2\tilde{E} + \mathcal{O}(\tilde{g}^4) \tag{4.101}$$

where we have introduced the effective coupling constant $\tilde{g}^2$ similar to the BMN coupling $\hat{g}^2 \sim \lambda'$, see Sec. 3.6[21]

$$\tilde{g}^2 = \frac{g^2}{L^2} = \frac{\lambda}{8\pi^2 L^2}\,. \tag{4.102}$$

Likewise, the charges and transfer matrix in the thermodynamic limit are given by [68][22]

$$\tilde{Q}_r = L^{r-1}Q_r = \frac{1}{L}\sum_{k=1}^{K} \frac{|s|}{\tilde{u}_k^r}\,, \quad -i\log\tilde{T}(\tilde{u}) = -i\log T(\tilde{u}L) = \frac{1}{L}\sum_{k=1}^{K} \frac{|s|}{\tilde{u}_k - \tilde{u}} + \dots\,. \tag{4.103}$$

[19] Interestingly, we might include s in the rescaling and remove it completely from the equations.

[20] This is the chief difference to the BMN limit, where there are only finitely many excitations. Consequently, in the BMN limit one would define $\hat{E} = L^2 E$.

[21] The length L in this section corresponds to the combination $J+M$ from discussion of the BMN limit. Hence, we distinguish between the coupling constant $\tilde{g} = g/L$ for the thermodynamic limit and $\hat{g} = g/J$ from Sec. 3.6.

[22] For the BMN limit one would define $\hat{Q}_r = L^r Q_r$ and $\hat{T}(\hat{u}) = T(\hat{u}L)^L$ to account for the different scaling of the number of excitations.

We shall start by assuming that in the large L limit the Bethe roots accumulate on A smooth contours $\mathcal{C}_a$, the so-called 'Bethe-strings'

$$\text{`}\textit{Bethe-strings}\text{'}: \quad \mathcal{C}_1, \ldots, \mathcal{C}_A. \tag{4.104}$$

It is reasonable, therefore, to replace the discrete root positions $\tilde{u}_k$ by a smooth continuum variable $\tilde{u}$ and introduce a density $\rho(\tilde{u})$ describing the distribution of the roots in the complex $\tilde{u}$-plane:

$$\frac{1}{L}\sum_{k=1}^{K} \longrightarrow \int_{\mathcal{C}} d\tilde{u}\, \rho(\tilde{u}), \tag{4.105}$$

where $\mathcal{C}$ is the support of the density, i.e. the union of contours $\mathcal{C}_a$ along which the roots are distributed. The density is normalised to the filling fraction $\tilde{K} = K/L$,

$$\int_{\mathcal{C}} d\tilde{u}\, \rho(\tilde{u}) = \tilde{K}. \tag{4.106}$$

Moreover, we may specify solutions by the contour filling fractions, i.e. the numbers of roots $L\tilde{K}_a$ residing on each contour $\mathcal{C}_a$, by

$$\int_{\mathcal{C}_a} d\tilde{u}\, \rho(\tilde{u}) = \tilde{K}_a. \tag{4.107}$$

The Bethe equations (4.100) in the 'thermodynamic limit' then conveniently turn into singular integral equations:

$$\tilde{E} = |s| \int_{\mathcal{C}} \frac{d\tilde{u}\, \rho(\tilde{u})}{\tilde{u}^2}\,, \qquad 2\pi n = |s| \int_{\mathcal{C}} \frac{d\tilde{u}\, \rho(\tilde{u})}{\tilde{u}}\,, \qquad 2\pi n_{\tilde{u}} - \frac{s}{\tilde{u}} = 2 \fint_{\mathcal{C}} \frac{d\tilde{v}\, \rho(\tilde{v})}{\tilde{v} - \tilde{u}}\,, \tag{4.108}$$

where $n_{\tilde{u}}$ is the mode number n_k at point $\tilde{u} = \tilde{u}_k$. It is expected to be constant, $n_{\tilde{u}} = n_a$, along each contour $\mathcal{C}_a$ and contours are distinguished by their mode number. Here and in the following, the slash through the integral sign implies a principal part prescription. In addition, we have a consistency condition derived from the right of (4.108) by integrating both sides over $\mathcal{C}$ and using (4.107): $n = \operatorname{sign} s \sum_{a=1}^{A} \tilde{K}_a n_a$. Finally, we can compute the eigenvalues of the higher charges (4.69), they read [68]

$$\tilde{Q}_r = |s| \int_{\mathcal{C}} \frac{d\tilde{u}\, \rho(\tilde{u})}{\tilde{u}^r}\,, \qquad G(\tilde{u}) = |s| \int_{\mathcal{C}} \frac{d\tilde{v}\, \rho(\tilde{v})}{\tilde{v} - \tilde{u}}\,. \tag{4.109}$$

The resolvent $G(\tilde{u})$ is a central object of a solution. It is defined by

$$G(\tilde{u}) = \sum_{r=1}^{\infty} \tilde{u}^{r-1} \tilde{Q}_r \qquad \text{with} \quad Q_1 = -i \log U = 2\pi n, \tag{4.110}$$

so naively one might think $G(\tilde{u}) = -i \log \tilde{T}(\tilde{u})$. This is not quite true due to the omitted terms in the eigenvalue of the transfer matrix (4.54). The additional term is of $\mathcal{O}(u^L)$, so in the large L limit one might be tempted to drop it. However, let us see what happens for $s = 1$ for which we know the exact transfer matrix (4.61). The second term is multiplied by $u^L/(u+i)^L$. In the thermodynamic limit this becomes

$$\left(\frac{u}{u+i}\right)^L = \left(\frac{\tilde{u}L}{\tilde{u}L+i}\right)^L = \left(1 + \frac{i}{L\tilde{u}}\right)^{-L} \longrightarrow \exp\left(-\frac{i}{\tilde{u}}\right), \tag{4.111}$$

which is indeed non-zero despite the suppression by u^L. In total we obtain for the eigenvalue of the transfer matrix

$$\tilde{T}(\tilde{u}) = \exp\big(iG(\tilde{u})\big) + \exp\big(-iG(\tilde{u}) - i/\tilde{u}\big) = \exp\big(-i/2\tilde{u}\big)\, 2\cos\big(G_{\text{sing}}(\tilde{u})\big). \tag{4.112}$$

The exponential prefactor may now be absorbed into the definition of $\tilde{T}$ and we merely have $2\cos G_{\text{sing}}(\tilde{u})$ with the singular resolvent

$$G_{\text{sing}}(\tilde{u}) = G(\tilde{u}) + \frac{1}{2\tilde{u}}\,. \tag{4.113}$$

Gladly, the additional terms only change the form of the transfer matrix, all physically relevant information is encoded into the non-singular resolvent G. The resolvent G may therefore be obtained even in ignorance of the additional terms in $\tilde{T}$.

Note that the Bethe equation (4.100) can alternatively be obtained as a consistency condition on the transfer matrix $\tilde{T}(\tilde{u})$. The resolvent has many sheets, but $2\cos G'_{\text{sing}}(\tilde{u})$ must be single-valued on the complex $\tilde{u}$ plane. This requires

$$G_{\text{sing}}(\tilde{u} + i\epsilon) + G'_{\text{sing}}(\tilde{u} - i\epsilon) = 2\pi n_{\tilde{u}} \tag{4.114}$$

across a branch cut of G at $\tilde{u}$, which is an equivalent formulation of the Bethe equation (4.100).

4.5.2 Generic Algebras

Let us briefly state the generalisation of the thermodynamic limit for arbitrary groups with Cartan matrix M and representation labels V. In addition to the mode numbers n_a, here we have to specify for each contour $\mathcal{C}_a$ to which simple root j_a of the algebra it belongs. The energy, momentum constraint and Bethe equations are

$$\tilde{E} = \int \frac{d\tilde{u}\,\rho(\tilde{u})\,V_{\tilde{u}}}{\tilde{u}^2}\,, \qquad 2\pi n = \int \frac{d\tilde{u}\,\rho(\tilde{u})\,V_{\tilde{u}}}{\tilde{u}}\,, \qquad 2\pi n_{\tilde{u}} - \frac{V_{\tilde{u}}}{\tilde{u}} = \int \frac{d\tilde{v}\,\rho(\tilde{v})\,M_{\tilde{u},\tilde{v}}}{\tilde{v} - \tilde{u}}\,. \tag{4.115}$$

Here we have used the short notation $M_{\tilde{u},\tilde{v}} = M_{j_a,j_{a'}}$ or $V_{\tilde{u}} = V_{j_a}$ for the element of the Cartan matrix or representation vector corresponding to the simple roots of the contours $\tilde{u} \in \mathcal{C}_a, \tilde{v} \in \mathcal{C}_{a'}$. The higher charges (4.69) and resolvent (4.110) as their generating function, $G(\tilde{u}) \approx -i \log \tilde{T}(\tilde{u})$, are given by

$$\tilde{Q}_r = \int \frac{d\tilde{u}\,\rho(\tilde{u})\,V_{\tilde{u}}}{\tilde{u}^r}\,, \qquad G(\tilde{u}) = \int \frac{d\tilde{v}\,\rho(\tilde{v})\,V_{\tilde{v}}}{\tilde{v} - \tilde{u}}\,. \tag{4.116}$$

4.6 Stringing Spins

Following the work [61], Frolov and Tseytlin proposed a novel possibility for a quantitative comparison of string theory and gauge theory [65, 63], see [76] for a nice review of the subject. They suggested to investigate states with large charges (angular momenta) of both, the conformal symmetry $\mathfrak{su}(2,2)$ and the internal symmetry $\mathfrak{su}(4)$. In the case of string theory it was understood [61, 62] (see also [124]) that the string sigma model can be efficiently treated by semi-classical methods. On the gauge theory side, it was realised

that such states can be treated with the Bethe ansatz in the thermodynamic limit [60]. This lead to a remarkable agreement at the one-loop level [60, 66]. We shall use this example to illustrate the use of the Bethe ansatz. First, we will shortly review the string theory computations, then derive the gauge theory result and compare.

4.6.1 String Theory Details

We will investigate a folded string (a closed string, which is folded to a line) which stretches along a spatial direction of AdS_5. It rotates with angular momentum (spin) S around its centre of mass and moves in the time direction with energy D as well as on a great circle of S^5 with angular momentum (charge) J. For this string configuration, we would like to find the dependence of the energy D on the charges S, J

$$D = D(S, J). \tag{4.117}$$

We will make the ansatz that the embedding coordinates of the string world sheet, parameterised by τ, σ, are given by

$$t = \kappa\tau, \quad \phi_1 = \phi = \omega\tau, \quad \varphi_3 = \varphi = w\tau, \quad \rho = \rho(\sigma) = \rho(\sigma + 2\pi), \tag{4.118}$$

all the other coordinates are zero. In fact, the string moves only in a subspace $AdS_3 \times S^1$ of $AdS_5 \times S^5$. The relevant part of the $AdS_5 \times S^5$ metric is

$$ds^2 = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\phi^2 + d\varphi^2. \tag{4.119}$$

The string theory sigma model is given by the Polyakov action

$$S_{\text{string}} = \sqrt{\lambda} \int d\tau \int \frac{d\sigma}{2\pi} \tfrac{1}{2} G_{MN}(\dot{X}^M \dot{X}^N - X'^M X'^N) \tag{4.120}$$

together with the Virasoro constraints

$$G_{MN}(\dot{X}^M \dot{X}^N + X'^M X'^N) = G_{MN}\dot{X}^M X'^N = 0. \tag{4.121}$$

The equations of motion following from the action are

$$\frac{\partial}{\partial\tau}(G_{MN}\dot{X}^N) - \frac{\partial}{\partial\sigma}(G_{MN}X'^N) = 0. \tag{4.122}$$

The conserved charges D, S, J corresponding to t, ϕ, φ are determined using (4.120,4.119)

$$D = \kappa\sqrt{\lambda} \int \frac{d\sigma}{2\pi} \cosh^2\rho, \quad S = \omega\sqrt{\lambda} \int \frac{d\sigma}{2\pi} \sinh^2\rho, \quad J = w\sqrt{\lambda}\,. \tag{4.123}$$

From the prefactor of the action we can now infer that quantum loops are counted by $1/\sqrt{\lambda}$ if λ is large. Furthermore, we see that, as $J = w\sqrt{\lambda}$, quantum loops are effectively counted by $1/J$ if we fix w. Therefore, if we content ourselves with the leading order in an expansion with respect to $1/J$ and w fixed, we can neglect all quantum loops and consider the classical string theory.

In the classical model, the parameter λ can be absorbed into the definition of the charges, we use

$$\mathcal{D} = \frac{D}{\sqrt{\lambda}} = \kappa \int \frac{d\sigma}{2\pi} \cosh^2\rho, \quad \mathcal{S} = \frac{S}{\sqrt{\lambda}} = \omega \int \frac{d\sigma}{2\pi} \sinh^2\rho, \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}} = w. \tag{4.124}$$

The only non-trivial equation of motion is

$$\rho'' - (\kappa^2 - \omega^2) \sinh\rho \cosh\rho = 0. \tag{4.125}$$

and the non-trivial Virasoro constraint is

$$\rho'^2 - \kappa^2 \cosh^2\rho + \omega^2 \sinh^2\rho + w^2 = 0. \tag{4.126}$$

To solve the system, we will assume that $\rho(\sigma)$ is a periodic function stretching between $\pm\rho_0$. By inverting the function $\rho(\sigma)$ to $\sigma(\rho)$ (for half of the period) we can rewrite the worldsheet integrals as

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} = \frac{2}{2\pi} \int_{-\rho_0}^{\rho_0} \frac{d\rho}{\rho'}\,. \tag{4.127}$$

There are two points to be taken into account. Firstly, an integral without an integrand should yield 1 and secondly, ρ' must be zero at $\pm\rho_0$, this leads to two new constraints. We can now solve the Virasoro constraint for ρ' and compute the integrals $\mathcal{D}$ and $\mathcal{S}$. At this point we have five equations in total: the definition of the three charges $\mathcal{D}, \mathcal{S}, \mathcal{J}$ and two constraints from the change of parameters. It is now possible to solve three equations for κ, ω, w, see [66] for details which we omit here. The two remaining equations are

$$\left(\frac{\mathcal{J}}{\mathrm{K}(x)}\right)^2 - \left(\frac{\mathcal{D}}{\mathrm{E}(x)}\right)^2 = \frac{4}{\pi^2}\, x, \quad \left(\frac{\mathcal{S}}{\mathrm{K}(x) - \mathrm{E}(x)}\right)^2 - \left(\frac{\mathcal{J}}{\mathrm{K}(x)}\right)^2 = \frac{4}{\pi^2}\,(1-x), \tag{4.128}$$

where $x = -\sinh^2\rho_0$ is related to the end-points of the string. The functions $\mathrm{K}(x)$ and $\mathrm{E}(x)$ are the elliptic integrals of the first and second kind, respectively

$$\mathrm{K}(x) = \int_0^1 \frac{dy}{\sqrt{1-y^2}} \frac{1}{\sqrt{1-xy^2}}\,, \qquad \mathrm{E}(x) = \int_0^1 \frac{dy}{\sqrt{1-y^2}} \sqrt{1-xy^2}\,. \tag{4.129}$$

The first equation in (4.128) determines the energy $\mathcal{D}$ in terms of the charges $\mathcal{S}, \mathcal{J}$ and the parameter x. The parameter x is fixed by the second equation. In total we obtain the energy as a function of the charges as

$$\mathcal{D}(\mathcal{S}, \mathcal{J}) = \mathcal{D}(\mathcal{S}, \mathcal{J}, x(\mathcal{S}, \mathcal{J})). \tag{4.130}$$

Frolov and Tseytlin noticed that $\mathcal{D}$ admits an expansion in powers of $1/\mathcal{J}$ when we fix the ratio of the charges $\alpha = \mathcal{S}/\mathcal{J}$

$$\mathcal{D}(\mathcal{S}, \mathcal{J}) = \delta_0(\alpha)\, \mathcal{J} + \frac{\delta_1(\alpha)}{8\pi^2 \mathcal{J}} + \frac{\delta_2(\alpha)}{64\pi^4 \mathcal{J}^3} + \frac{\delta_3(\alpha)}{512\pi^6 \mathcal{J}^5} + \dots\,, \qquad \alpha = \frac{\mathcal{S}}{\mathcal{J}}\,. \tag{4.131}$$

Using the original charges D, S, J we can write

$$D(S, J, g) = J\left(\delta_0(\alpha) + \tilde{g}^2\, \delta_1(\alpha) + \tilde{g}^4\, \delta_2(\alpha) + \tilde{g}^6\, \delta_3(\alpha) + \dots\right), \qquad \alpha = \frac{S}{J}\,. \tag{4.132}$$

where we have used the effective coupling constant $\tilde{g}^2$, see (4.102), in the thermodynamic limit (note that $L = J$ in this case)

$$\tilde{g}^2 = \frac{g^2}{J^2} = \frac{\lambda}{8\pi^2 J^2} = \frac{1}{8\pi^2 \mathcal{J}^2}\,. \tag{4.133}$$

The expression (4.132) suggests that, when $g \sim \sqrt{\lambda}$ is assumed to be small, we can compare to perturbative gauge theory! Nevertheless, a word of caution is in order here: We have started out assuming that λ is indeed large. One may hope that, due to analyticity, the function $D(S, J, g)$ is valid even for small g, but there might be some additional terms which can be neglected for large g, but become relevant for small g [87, 75]. We will comment on this issue in Sec. 6.5.

Extracting the leading-order or 'one-loop' term δ_1 from the relations (4.128) is straightforward. For large $\mathcal{J}$ one sets $x = x_0 + x_1/\mathcal{J}^2 + \ldots$ and solves the resulting transcendental equation for x_0. One then finds

$$\delta_0(\alpha) = 1 + \alpha \tag{4.134}$$

and the parametric solution

$$\delta_1 = -16\,\mathrm{K}(x_0)\big(\mathrm{E}(x_0) - (1 - x_0)\mathrm{K}(x_0)\big), \qquad \alpha = \frac{S}{J} = \frac{\mathrm{E}(x_0)}{\mathrm{K}(x_0)} - 1. \tag{4.135}$$

4.6.2 Gauge Theory Details

Let us see whether we can obtain an expression for δ_1 from gauge theory. We shall be interested in a state with large charge J of $\mathfrak{su}(4)$, large charge (spin) S of $\mathfrak{su}(2,2)$ as well as a large dimension D. In the classical theory, the charges should obey the relations (4.134)

$$D_0 = J + S. \tag{4.136}$$

The weight of such a state is

$$w = (J + S; S, S; 0, J, 0; 0, J) \tag{4.137}$$

it belongs to the bosonic $\mathfrak{su}(1,1)$ sector, c.f. Sec. 3.4, and the state has the form $\mathrm{Tr}\,\mathcal{D}^S \mathcal{Z}^J$.

As the charges S, J in string theory are very large while the one-loop energy is small, we can use the thermodynamic limit of the Bethe equations as explained in Sec. 4.5. We expect the roots for the ground state to lie on the real axis (this may be verified by explicit solution of the exact Bethe equations for small values of J). Furthermore, we assume the distribution of roots to be symmetric, $d\tilde{u}\,\rho(\tilde{u}) = d\tilde{u}'\,\rho(\tilde{u}')$ with $\tilde{u}' = -\tilde{u}$, which implies $n = 0$. For the ground state we expect the support of the root density to split into two disjoint intervals $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$ with $\mathcal{C}_- = [-b, -a]$ and $\mathcal{C}_+ = [a, b]$, where $a < b$ are both real. The mode numbers should be $n_\pm = \mp 1$ on $\mathcal{C}^\pm$ and the filling fractions should be $\tilde{K}_\pm = S/2J$. The total filling filling fraction will be denoted by

$$\alpha = \tilde{K} = \frac{S}{J}\,. \tag{4.138}$$

For this distribution of roots, the Bethe equations (4.108) become

$$⨍_a^b \frac{d\tilde{v}\,\rho(\tilde{v})\,\tilde{u}^2}{\tilde{v}^2 - \tilde{u}^2} = \frac{1}{4} - \frac{\pi}{2}\,\tilde{u}\,, \qquad \tilde{E} = 2\int_a^b \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}^2}\,. \tag{4.139}$$

The solution of the integral equation (see, e.g., [125]), yielding the density $\rho(\tilde{u})$, may be obtained explicitly; it reads

$$\rho(\tilde{u}) = \frac{2}{\pi\tilde{u}} \int_a^b \frac{d\tilde{v}\,\tilde{v}^2}{\tilde{v}^2-\tilde{u}^2} \sqrt{\frac{(b^2-\tilde{u}^2)(\tilde{u}^2-a^2)}{(b^2-\tilde{v}^2)(\tilde{v}^2-a^2)}}\,. \tag{4.140}$$

This density may be expressed explicitly through standard functions:

$$\rho(\tilde{u}) = \frac{1}{2\pi b\tilde{u}} \sqrt{\frac{\tilde{u}^2-a^2}{b^2-\tilde{u}^2}} \Big(\frac{b^2}{a} - 4\tilde{u}^2 \Pi\Big(\frac{b^2-\tilde{u}^2}{b^2}, q\Big)\Big), \qquad q = \frac{b^2-a^2}{b^2}\,, \tag{4.141}$$

where we introduced the modulus q which is related to the end-points a, b of the 'strings' of Bethe roots; it plays the role of an auxiliary parameter. The function Π is the elliptic integral of the third kind

$$\Pi(m,q) = \int_0^1 \frac{dy}{\sqrt{1-y^2}} \frac{1}{(1-my^2)\sqrt{1-qy^2}}\,. \tag{4.142}$$

Furthermore, we may derive two conditions determining the interval boundaries a, b as a function of the filling fraction α:

$$\int_a^b \frac{d\tilde{u}\;\tilde{u}^2}{\sqrt{(b^2-\tilde{u}^2)(\tilde{u}^2-a^2)}} = \frac{1+2\alpha}{4} \qquad \text{and} \qquad \int_a^b \frac{d\tilde{u}}{\sqrt{(b^2-\tilde{u}^2)(\tilde{u}^2-a^2)}} = \frac{1}{4ab}\,. \tag{4.143}$$

The first is derived from the normalisation condition (4.106), while the second is a consistency condition assuring the positivity of the density. These may be reexpressed through standard elliptic integrals of, respectively, the second and the first kind (4.129). It is straightforward to eliminate the interval boundaries a, b from these equations via

$$a = \frac{1}{4\mathrm{K}(q)}\,, \qquad b = \frac{1}{4\sqrt{1-q}\,\mathrm{K}(q)}\,. \tag{4.144}$$

Furthermore, we can integrate the density and compute the energy $\tilde{E}$ from the right equation in (4.139)

$$\tilde{E} = -4\,\mathrm{K}(q)\big(2\mathrm{E}(q) - (2-q)\mathrm{K}(q)\big), \qquad \alpha = \frac{S}{J} = \frac{1}{2\sqrt{1-q}}\,\frac{\mathrm{E}(q)}{\mathrm{K}(q)} - \frac{1}{2}\,. \tag{4.145}$$

In total the scaling dimension of our solution is

$$D(S,J,g) = S + J + \frac{g^2}{J}\,\tilde{E}(\alpha) + \ldots = J\big(1+\alpha+\tilde{g}^2\,\tilde{E}(\alpha)+\ldots\big) + \ldots \tag{4.146}$$

with the effective coupling $\tilde{g}$ defined in (4.102,4.133).

Finally, we can compute the resolvent (4.110) [68][23]

$$-\,i\log\tilde{T}(\tilde{u}) \approx G(\tilde{u}) = \frac{1}{2\tilde{u}} \mp \frac{2a^2}{b\tilde{u}} \sqrt{-\frac{b^2-\tilde{u}^2}{\tilde{u}^2-a^2}}\,\Pi\bigg(\frac{q\,\tilde{u}^2}{\tilde{u}^2-a^2}, q\bigg) + 2\pi n. \tag{4.147}$$

The resolvent $G(\tilde{u})$ is a central object in the investigation of the solution. It is multivalued on the complex plane and has branch cuts with the discontinuity proportional to the density $\rho(\tilde{u})$. Furthermore, it encodes the values of all rescaled charges $\tilde{Q}_r$ when expanded around $\tilde{u} = 0$.

[23] Note that the first term in $G(\tilde{u})$ is due to the shift in $G_{\mathrm{sing}}(\tilde{u}) = G(\tilde{u}) + s/2\tilde{u}$, the generalisation of (4.113) to arbitrary spin s. The ambiguities $\pm$ and n drop out in $2\cos G_{\mathrm{sing}}(\tilde{u})$.

4.6.3 Comparison

Let us now compare the string theory system (4.135) for the classical energy and the gauge theory system (4.145) for the one-loop anomalous dimension. Both systems are parametric, i.e. finding energy/dimension as a function of spins involves elimination of auxiliary parameters. They look similar, but superficially they are not identical. However, if we relate the auxiliary parameters x_0 and q by

$$x_0 = -\frac{(1-\sqrt{1-q})^2}{4\sqrt{1-q}}\,, \tag{4.148}$$

one can show, using the elliptic integral modular transformation relations

$$\mathrm{K}(x_0) = (1-q)^{1/4}\mathrm{K}(q), \quad \mathrm{E}(x_0) = \tfrac{1}{2}(1-q)^{-1/4}\mathrm{E}(q) + \tfrac{1}{2}(1-q)^{1/4}\mathrm{K}(q)\,, \tag{4.149}$$

that the systems (4.135) and (4.145) are, in fact, exactly the same. As a result, their solutions $\delta_1(\alpha) = \tilde{E}(\alpha)$ do become identical! We have thus demonstrated the equivalence between the string theory and gauge theory results for a particular two-spin part of the spectrum at the full *functional* level. In [68] it was furthermore shown, that not only the energy, but also the set of all higher charges agrees with string theory!

Recently, Kazakov, Marshakov, Minahan and Zarembo have proposed a proof for the complete agreement between string theory and gauge theory at the one-loop (and two-loop) level in the dual case of strings spinning on S^5 instead of AdS_5, i.e. the $\mathfrak{su}(2)$ subsector [75]. We will comment on the comparison at higher-loops in Sec. 6.5. In another line of work initiated by Kruczenski similar statements can be made [71, 72]. These are based on a coherent state picture and independent of integrability.

Chapter 5

Higher-Loops

In Ch. 3 we have seen how to make use of the algebra to find the complete one-loop dilatation operator. and in the previous chapter we have seen that its integrability in the planar limit enables a precise comparison to string theory within the AdS/CFT correspondence. It would be exciting to see whether these ideas may be extended to higher loops. In this chapter we will aim at the construction of higher-loop corrections to the dilatation generator. Higher-loop integrability will be the subject of the next chapter.

At one-loop the analysis was simplified due to the preserved classical algebra; at higher-loops this is not the case and a derivation of the complete dilatation operator would require a large amount of work. We will therefore restrict to the $\mathfrak{su}(2|3)$ subsector of $\mathcal{N}=4$ SYM with a finite number of fundamental fields and a smaller supersymmetry algebra, which includes the dilatation operator.

Here we will find and investigate deformations $\mathfrak{J}(g)$ of the classical representation $\mathfrak{J}_0$ of the symmetry algebra on the space of states. These deformations are furnished in such a way that they are compatible (i) with the symmetry algebra and (ii) with $\mathcal{N}=4$ SYM field theory and its Feynman diagrams. The text is based on the article [51] and contains excerpts from [38, 81].

5.1 The $\mathfrak{su}(2|3)$ Eighth-BPS Sector

The model discussed in this chapter is the $(0,1^+)$ subsector with $\mathfrak{su}(2|3)\times\mathfrak{u}(1)$ symmetry, c.f. Sec. 2.3.4, in the planar limit. In the large N limit, the gauge theory turns into a quantum spin chain as described in Sec. 2.6.6 and we will use spin chain terminology. Note that the model is a subsector, not only of $\mathcal{N}=4$ SYM, but also of the BMN matrix model, which was briefly introduced in Sec. 2.1.6. Therefore, all results obtained in this chapter apply equally well to the BMN matrix model.

In the following, we shall describe the model in terms of the space of states, symmetry and how it is related to $\mathcal{N}=4$ gauge theory.

5.1.1 Fields, States and Interactions

The subsector consists of three complex scalars ϕ_a (Latin indices take the values $1,2,3$) and two complex fermions ψ_α (Greek indices take the values $1,2$)

$$\phi_a \quad (a=1,2,3), \qquad \psi_\alpha \quad (\alpha=1,2). \tag{5.1}$$

These can be combined into a supermultiplet $\mathcal{W}_A$ (capital indices range from 1 to 5) of fields

$$\mathcal{W}_{1,2,3} = \phi_{1,2,3}, \quad \mathcal{W}_{4,5} = \psi_{1,2}. \tag{5.2}$$

We shall use the notation introduced in Sec. 2.6 to describe (single-trace) states and interactions

$$|A_1 \dots A_L\rangle = \operatorname{Tr} \mathcal{W}_{A_1} \dots \mathcal{W}_{A_L}, \qquad \left\{ {A_1 \dots A_{E_i} \atop B_1 \dots B_{E_o}} \right\}. \tag{5.3}$$

To distinguish between bosons and fermions in the interaction symbols, we use Latin and Greek letters. For example, the interaction $\left\{ {\alpha bc \atop c\alpha b} \right\}$ searches for one fermion followed by two bosons within the trace. Wherever they can be found these three fields are permuted such that the last boson comes first, next the fermion and the other boson last. A sample action is

$$\left\{ {\alpha bc \atop c\alpha b} \right\} |142334452\rangle = |134234452\rangle + |242334415\rangle. \tag{5.4}$$

5.1.2 The Algebra

The fields $\mathcal{W}_A$ transform canonically in a fundamental $\mathbf{3|2}$ representation of $\mathfrak{su}(2|3)$. Let us start by defining this algebra. The $\mathfrak{su}(2|3) \times \mathfrak{u}(1)$ algebra consists of the generators

$$\mathfrak{J} = \{\mathfrak{L}^\alpha{}_\beta, \mathfrak{R}^a{}_b, \mathfrak{D}_0, \mathcal{H} | \mathfrak{Q}^a{}_\alpha, \mathfrak{S}^\alpha{}_a\}. \tag{5.5}$$

The bar separates bosonic from fermionic operators. The $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ generators $\mathfrak{L}^\alpha{}_\beta$ and $\mathfrak{R}^a{}_b$ are traceless, $\mathfrak{L}^\alpha{}_\alpha = \mathfrak{R}^a{}_a = 0$. The commutators are defined as follows: Under the rotations $\mathfrak{L}, \mathfrak{R}$, the indices of any generator $\mathfrak{J}$ transform canonically according to the rules

$$\begin{array}{ll} [\mathfrak{L}^\alpha{}_\beta, \mathfrak{J}_\gamma] = \delta^\alpha_\gamma \mathfrak{J}_\beta - \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}_\gamma, & [\mathfrak{L}^\alpha{}_\beta, \mathfrak{J}^\gamma] = -\delta^\gamma_\beta \mathfrak{J}^\alpha + \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma, \\ [\mathfrak{R}^a{}_b, \mathfrak{J}_c] = \delta^a_c \mathfrak{J}_b - \frac{1}{3}\delta^a_b \mathfrak{J}_c, & [\mathfrak{R}^a{}_b, \mathfrak{J}^c] = -\delta^c_b \mathfrak{J}^a + \frac{1}{3}\delta^a_b \mathfrak{J}^c. \end{array} \tag{5.6}$$

The commutators of the dilatation operator $\mathfrak{D}_0$ and the Hamiltonian $\mathcal{H}$ are given by

$$[\mathfrak{D}_0, \mathfrak{J}] = \dim(\mathfrak{J})\, \mathfrak{J}, \qquad [\mathcal{H}, \mathfrak{J}] = 0. \tag{5.7}$$

In other words, $\mathcal{H}$ is the central $\mathfrak{u}(1)$ generator and the non-vanishing dimensions are

$$\dim(\mathfrak{Q}) = -\dim(\mathfrak{S}) = \tfrac{1}{2}. \tag{5.8}$$

The supercharges anticommuting into rotations are given by

$$\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta (\tfrac{1}{3}\mathfrak{D}_0 + \tfrac{1}{2} g^2 \mathcal{H}). \tag{5.9}$$

This implies that the linear combination $\mathfrak{D}_0 + \frac{3}{2} g^2 \mathcal{H}$ belongs to the algebra $\mathfrak{su}(2|3)$. Furthermore, we demand a parity even algebra

$$\mathfrak{p}\, \mathfrak{J}\, \mathfrak{p}^{-1} = \mathfrak{J} \quad \text{or} \quad [\mathfrak{p}, \mathfrak{J}] = 0. \tag{5.10}$$

It is straightforward to find the fundamental $\mathbf{3|2}$ representation acting on the fundamental module (we will do this explicitly in Sec. 5.2)

$$\mathcal{V}_{\text{F}} = [\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5]. \tag{5.11}$$

As states are constructed from the fundamental fields $\mathcal{W}_A$ there is an induced representation on the space of states; this is simply a tensor product representation and we will denote it by $\mathfrak{J}_0$. The higher order corrections $\mathfrak{J}_k$ will act on more than one field at a time.

5.1.3 Representations

In terms of representation theory, a state is characterised by the charges

$$D_0, \quad s, \quad [q,p], \quad E, \tag{5.12}$$

where D_0 is the classical dimension, E is the energy (i.e. the eigenvalue of the Hamiltonian $\mathcal{H}$), s is *twice* the $\mathfrak{su}(2)$ spin and $[q,p]$ are the $\mathfrak{su}(3)$ Dynkin labels. These can be arranged into Dynkin labels[1] of $\mathfrak{su}(2|3)$

$$w = [s;r;q,p], \qquad r = \tfrac{1}{3}D_0 + \tfrac{1}{2}g^2E + \tfrac{1}{2}s - \tfrac{1}{3}p - \tfrac{2}{3}q. \tag{5.13}$$

Although it is sufficient to give either the dimension or the label r, we will usually state both for convenience. The labels s,q,p are integer-valued, whereas the fermionic label r can be any real number.[2] Representations are characterised by their highest weight. For instance, the highest weight of the fundamental module $\mathcal{V}_{\text{F}}$ is

$$w_{\text{F}} = [0;0;0,1]. \tag{5.14}$$

It is helpful to know how to construct a state with given charges D_0, s, p, q and length L from the fundamental fields $\phi_{1,2,3}, \psi_{1,2}$. The numbers of constituents of each kind are given by

$$n_\phi = n_{1,2,3} = \begin{pmatrix} L - \frac{2}{3}D_0 + \frac{2}{3}p + \frac{1}{3}q \\ L - \frac{2}{3}D_0 - \frac{1}{3}p + \frac{1}{3}q \\ L - \frac{2}{3}D_0 - \frac{1}{3}p - \frac{2}{3}q \end{pmatrix}, \qquad n_\psi = n_{4,5} = \begin{pmatrix} D_0 - L + \frac{1}{2}s \\ D_0 - L - \frac{1}{2}s \end{pmatrix}. \tag{5.15}$$

The following 'unitarity' bound[3] applies to multiplets of $\mathfrak{su}(2|3)$

$$D_0 + \tfrac{3}{2}g^2E \geq 3 + \tfrac{3}{2}s + p + 2q, \quad r - s \geq 1, \qquad \text{or} \qquad D_0 + \tfrac{3}{2}g^2E = p + 2q, \quad r = s = 0. \tag{5.16}$$

A (typical) multiplet of $\mathfrak{su}(2|3)$ away from the bound consists of

$$(32|32) \times (s+1) \times \tfrac{1}{2}(p+1)(q+1)(p+q+2) \tag{5.17}$$

components. However, under certain conditions on the dimension, the multiplet is shortened (atypical). We find three conditions relevant to the spin chain. The first one is the 'half-BPS'[4] condition

$$D_0 + \tfrac{3}{2}g^2E = p, \quad s = r = q = 0, \qquad n_2 = n_3 = n_4 = n_5 = 0, \tag{5.18}$$

where we have also displayed the condition in terms of the number of fields (5.15). Such a multiplet has $1 + p(p+1)|p(p+1)$ components. The second one is the 'quarter-BPS'[5] condition

$$D_0 + \tfrac{3}{2}g^2E = p + 2q, \quad s = r = 0, \qquad n_3 = n_4 = n_5 = 0. \tag{5.19}$$

[1] The sign of r was chosen such that $[1;0;0,0] \times [1;0;0,0] = [2;0;0,0] + [0;-1;0,0]$.

[2] Nevertheless we will usually write its value at $g = 0$ and state the irrational part $E(g)$ separately.

[3] We use the terminology of $\mathcal{N} = 4$ SYM even if some terms might be inappropriate.

[4] In fact, 4 out of 6 supercharges annihilate the state.

[5] In fact, 2 out of 6 supercharges annihilate the state. Multiplets of this kind have states belonging to the $\mathfrak{su}(2)$ subsector of just two complex bosonic fields $\phi_{1,2}$.

Although a quarter-BPS multiplet is beyond the unitarity bound, it can acquire a non-zero energy if it joins with another multiplet to form a long one. The last condition determines short (usually called semi-short) multiplets

$$D_0 + \tfrac{3}{2}g^2 E = 3 + \tfrac{3}{2}s + p + 2q, \quad r - s = 1, \qquad n_3 + n_5 = 1. \tag{5.20}$$

A long multiplet whose energy approaches the unitarity bound (5.16) splits in two at (5.20). If $s > 0$, the highest weight of the upper short submultiplet is shifted by

$$\delta D_0 = +\tfrac{1}{2}, \quad \delta w_{\rm i} = [-1; -1; +1, 0], \qquad \delta L = +1. \tag{5.21}$$

For $s = 0$ the upper submultiplet is quarter-BPS and its highest weight is shifted by

$$\delta D_0 = +1, \quad \delta w_{\rm I} = [0; -1; +2, 0], \qquad \delta L = +1. \tag{5.22}$$

Multiplet shortening will turn out to be important later on. This is because the generators which relate both submultiplets must act as $\mathcal{O}(g)$ so that the multiplet can indeed split at $g = 0$.

5.1.4 Fluctuations in Length

Note that all three bosons together have vanishing $\mathfrak{su}(3)$ charges and dimension 3. Similarly, both fermions have vanishing $\mathfrak{su}(2)$ spin and dimension 3, i.e. the same quantum numbers

$$\phi_{[1}\phi_2\phi_{3]} \sim \psi_{[1}\psi_{2]}. \tag{5.23}$$

Therefore one can expect fluctuations between these two configurations. In field theory these are closely related to the Konishi anomaly [104, 126]. A state composed from $n_1 \geq n_2 \geq n_3$ bosons and $n_4 \geq n_5$ fermions can mix with states

$$(n_1 - k, n_2 - k, n_3 - k; n_4 + k, n_5 + k), \qquad -n_5 \leq k \leq n_3. \tag{5.24}$$

Note that the length $L = n_1 + n_2 + n_3 + n_4 + n_5$ decreases by k. We will refer to this aspect of the spin chain as *dynamic*. The length can fluctuate by as much as $n_3 + n_5 = r - s$ units, a quantity directly related to the distance to the unitarity bound.

Length fluctuations are especially interesting for multiplet shortenings. The highest weight state of a half-BPS or quarter-BPS multiplet has fixed length due to $n_3 = n_5 = 0$. For short multiplets we have $n_3 + n_5 = 1$. This means that the length fluctuates by one unit for the highest weight state. Two of the six supercharges transform a ϕ_3 into a fermion $\psi_{1,2}$. Naively, both cannot act at the same time because there is only one ϕ_3 (we will always have $n_3 = 1$ for a long multiplet), and the multiplet becomes short. However, we could simultaneously replace the resulting $\psi_{[1}\psi_{2]}$ by $\phi_{[1}\phi_2\phi_{3]}$ and thus fill up the ϕ_3-hole. A suitable transformation rule is[6]

$$\mathfrak{Q}^3{}_2\, \psi_1 \sim g\, \phi_{[1}\phi_{2]}. \tag{5.25}$$

This is also the step between the two short submultiplets (5.21). This property was used in [100] to determine two-loop scaling dimensions for operators at the unitarity

[6] In fact, this is part of the 'classical' supersymmetry variation.

bounds from a one-loop field-theory calculation. Note that $\mathfrak{Q}^3{}_2$ annihilates the highest weight when $n_4 = 0$; we need to apply $\mathfrak{Q}^3{}_1$ first to produce a ψ_1. In this case the upper submultiplet is quarter-BPS (5.22). Furthermore note that when we apply $\mathfrak{Q}^3{}_1$ first, there are no more ϕ_3's and ψ_2's left and length fluctuations are ruled out. Therefore, in a BPS or short multiplet we can always find states with fixed length; fluctuations are frozen at the unitarity bound. In contrast, all states in a multiplet away from the unitarity bound (5.16) are mixtures of states of different lengths.

5.1.5 From $\mathcal{N} = 4$ SYM to $\mathfrak{su}(2|3)$

A state of free $\mathcal{N} = 4$ SYM is characterised by the classical dimension D_0, the $\mathfrak{su}(2)^2$ labels $[s, s_2]$, the $\mathfrak{su}(4)$ Dynkin labels $[q, p, q_2]$, the $\mathfrak{u}(1)$ hypercharge B as well as the length L. The $\mathfrak{su}(2|3)$ subsector is obtained by restricting to states with (c.f. Sec. 2.3.4)

$$D_0 = p + \tfrac{1}{2}q + \tfrac{3}{2}q_2, \qquad s_2 = 0. \tag{5.26}$$

This also implies $D_0 = B + L$. We write these as relations of the corresponding generators

$$(\mathfrak{R}_{\mathfrak{su}(4)})^4{}_4 = \tfrac{1}{2}\mathfrak{D}_0, \quad \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} = 0, \quad \mathfrak{D}_0 = \mathcal{L} + \mathfrak{B}. \tag{5.27}$$

Furthermore, we express the $\mathfrak{su}(4)$ generator $\mathfrak{R}_{\mathfrak{su}(4)}$ in terms of an $\mathfrak{su}(3)$ generator $\mathfrak{R}$

$$(\mathfrak{R}_{\mathfrak{su}(4)})^a{}_b = \mathfrak{R}^a{}_b - \tfrac{1}{6}\delta^a_b\mathfrak{D}_0. \tag{5.28}$$

Now we can reduce the $\mathfrak{psu}(2,2|4)$ algebra as given in App. D.1 to the $\mathfrak{su}(2|3)$ subsector and find precisely the $\mathfrak{su}(2|3)$ relations (c.f. Sec. 5.1.2) if the Hamiltonian $\mathcal{H}$ is identified with the anomalous dilatation generator as follows

$$\delta\mathfrak{D} = g^2\mathcal{H}. \tag{5.29}$$

As we would like to compare directly to $\mathcal{N} = 4$ SYM, we write one of the generators of $\mathfrak{su}(2|3)$ as $\mathfrak{D}_0 + \frac{3}{2}g^2\mathcal{H}$ instead of assigning a new letter.

We note that the states in this subsector are (classically) eighth-BPS in terms of $\mathcal{N} = 4$ SYM (in Sec. 5.6.4 we will present a true eighth-BPS state). Unprotected primary states of the subsector can therefore not be primary states of $\mathfrak{psu}(2,2|4)$. To shift from the corresponding superconformal primary to the highest weight in the subsector we have to shift by (c.f. Sec. 2.3.4)

$$\delta w_{\text{II}} = (+1; 0, 0; 0, 0, +2; 0, +1). \tag{5.30}$$

Note that in terms of the Dynkin labels $[s; r; q, p, q_2; r_2; s_2]$ the last two are zero in this subsector $r_2 = s_2 = 0$. We then simply restrict to the first four labels $[s; r; q, p]$.

5.2 Tree-Level

We would like to construct a representation $\mathfrak{J}(g)$ of $\mathfrak{su}(2|3) \times \mathfrak{u}(1)$ on the spin chain. The generators must satisfy the algebra relations

$$\big[\mathfrak{J}_M(g), \mathfrak{J}_N(g)\big\} = \mathfrak{F}^P_{MN}\,\mathfrak{J}_P(g) \tag{5.31}$$

with $\mathfrak{F}^P_{MN}$ the structure constants of the symmetry algebra as given in Sec. 5.1.2.

Let us illustrate the procedure for the generators at tree-level. At tree-level, composite states transform in tensor product representations of the fundamental representation $\mathbf{3}|\mathbf{2}$. The generators therefore act on one field at a time. We write down the most general form of generators that respects $\mathfrak{su}(3) \times \mathfrak{su}(2)$ symmetry

$$\begin{aligned}
\mathfrak{R}^a{}_b &= c_1 \left\{ {a \atop b} \right\} + c_2 \delta^a_b \left\{ {c \atop c} \right\}, \\
\mathfrak{L}^\alpha{}_\beta &= c_3 \left\{ {\alpha \atop \beta} \right\} + c_4 \delta^\alpha_\beta \left\{ {\gamma \atop \gamma} \right\}, \\
\mathfrak{D}_0 &= c_5 \left\{ {a \atop a} \right\} + c_6 \left\{ {\alpha \atop \alpha} \right\}, \\
(\mathfrak{Q}_0)^a{}_\alpha &= c_7 \left\{ {a \atop \alpha} \right\}, \\
(\mathfrak{S}_0)^\alpha{}_a &= c_8 \left\{ {\alpha \atop a} \right\}.
\end{aligned} \tag{5.32}$$

The algebra relations have two solutions. One is the trivial solution $c_k = 0$ corresponding to the trivial representation. The other solution requires

$$c_1 = c_3 = c_5 = 1, \quad c_2 = -\tfrac{1}{3}, \quad c_4 = -\tfrac{1}{2}, \quad c_6 = \tfrac{3}{2}, \quad c_7 = e^{i\beta_1}, \quad c_8 = e^{-i\beta_1}. \tag{5.33}$$

As expected, we find that the bosons and fermions have dimension 1 and $\frac{3}{2}$, respectively

$$\mathfrak{D}_0 = \left\{ {a \atop a} \right\} + \tfrac{3}{2} \left\{ {\alpha \atop \alpha} \right\}. \tag{5.34}$$

The appearance of a free parameter β_1 is related to a possible rescaling of the bosons and fermions. This can be represented in terms of a similarity transformation on the algebra

$$\mathfrak{J}_0 \mapsto \exp\left(2i\beta_1 \mathfrak{D}_0\right) \mathfrak{J}_0 \, \exp\left(-2i\beta_1 \mathfrak{D}_0\right). \tag{5.35}$$

Obviously, the algebra relations in Sec. 5.1.2 are invariant under such a transformation. The only other $\mathfrak{su}(3) \times \mathfrak{su}(2)$ invariant similarity transformation besides (5.35) is

$$\mathfrak{J}_0 \mapsto \exp\left(i\beta_2 \mathcal{L}\right) \mathfrak{J}_0 \, \exp\left(-i\beta_2 \mathcal{L}\right), \tag{5.36}$$

where $\mathcal{L}$ is the length operator

$$\mathcal{L} = \left\{ {a \atop a} \right\} + \left\{ {\alpha \atop \alpha} \right\} \qquad \text{or simply} \quad \mathcal{L} = \left\{ {\cdot \atop \cdot} \right\}. \tag{5.37}$$

The transformation (5.36) is trivial and does not give rise to a new parameter at tree-level because the length is conserved there

$$[\mathcal{L}, \mathfrak{J}_0] = 0. \tag{5.38}$$

5.3 One-Loop

In this section we construct deformations of the algebra generators $\mathfrak{J}(g)$ obeying the algebra relations in Sec. 5.1.2. Here, we will proceed up to $\mathcal{O}(g)$ for the deformations of the Hamiltonian $\mathcal{H}(g)$. This can still be done conveniently by hand without the help of computer algebra systems. This section is meant to illustrate the methods of this chapter in a simple context before we proceed to higher-loops in the sections to follow.

The most important one of the algebra relations is the invariance of the interaction Hamiltonian

$$[\mathfrak{J}_M(g), \mathcal{H}(g)] = 0. \tag{5.39}$$

Moreover we will assume the $\mathfrak{su}(2), \mathfrak{su}(3)$ rotation generators $\mathfrak{R}^a{}_b$ and $\mathfrak{L}^\alpha{}_\beta$ to receive no corrections. This is natural, for the rotation symmetries are preserved by the quantisation procedure.

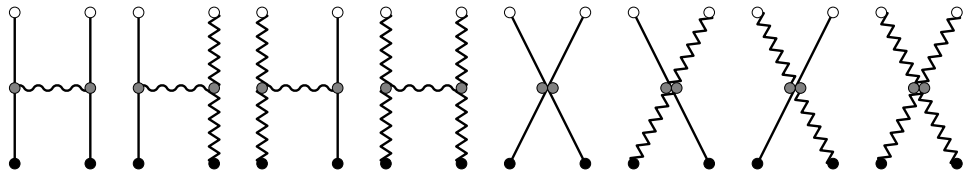

Figure 5.2: The structures for the construction of $\mathcal{H}_0$. Straight and zigzag lines correspond to bosons and fermions, respectively.

5.3.1 Pre-Leading Order

Let us restrict (5.39) to its leading order

$$[\mathfrak{J}_0, \mathcal{H}_l] = 0, \tag{5.40}$$

in other words, the leading order of the Hamiltonian at some $\mathcal{O}(g^l)$ is conserved by the classical algebra. The leading order for $\mathcal{H}$ will be $l = 0$ and we shall now exclude a correction to $\mathcal{H}$ at order $l = -1$ [7] by representation theory in analogy to Sec. 3.1.3: At this order the interactions have three legs and the possible ways to distribute them among the in and out channels are

$$\left\{ {\cdot \atop ABC} \right\}, \quad \left\{ {C \atop AB} \right\}, \quad \left\{ {BC \atop A} \right\}, \quad \left\{ {ABC \atop \cdot} \right\}. \tag{5.41}$$

The indices cannot be contracted fully, hence there is no invariant interaction at this order. In other words there is no common irreducible representation of the free algebra among the in and the out channel

$$\mathcal{V}_{\mathrm{F}}^0 \notin \mathcal{V}_{\mathrm{F}}^3, \quad \mathcal{V}_{\mathrm{F}}^1 \notin \mathcal{V}_{\mathrm{F}}^2. \tag{5.42}$$

5.3.2 Leading Order

A similar argument is used to show that at leading order we must evenly distribute the four fields among the in and out channel, i.e.

$$\mathcal{V}_{\mathrm{F}}^0 \notin \mathcal{V}_{\mathrm{F}}^4, \quad \mathcal{V}_{\mathrm{F}}^1 \notin \mathcal{V}_{\mathrm{F}}^3, \quad \text{but } \mathcal{V}_{\mathrm{F}}^2 = \mathcal{V}_{\mathrm{F}}^2. \tag{5.43}$$

The most general form of $\mathcal{H}_0$, expressed as an action on bosons (a, b) and fermions (α, β) is therefore

$$\begin{aligned} \mathcal{H}_0 = {} & c_1 \left\{ {ab \atop ab} \right\} + c_2 \left\{ {a\beta \atop a\beta} \right\} + c_2' \left\{ {\alpha b \atop \alpha b} \right\} + c_3 \left\{ {\alpha\beta \atop \alpha\beta} \right\} \\ & + c_4 \left\{ {ab \atop ba} \right\} + c_5 \left\{ {a\beta \atop \beta a} \right\} + c_5' \left\{ {\alpha b \atop b\alpha} \right\} + c_6 \left\{ {\alpha\beta \atop \beta\alpha} \right\}, \end{aligned} \tag{5.44}$$

see also Fig. 5.2. First of all we demand that $\mathcal{H}_0$ conserves parity (5.10),

$$\mathfrak{p}\, \mathcal{H}_0\, \mathfrak{p}^{-1} = \mathcal{H}_0. \tag{5.45}$$

As can be seen easily, this requires

$$c_2 = c_2', \quad c_5 = c_5'. \tag{5.46}$$

[7] Note that $\mathcal{H}$ is shifted by two orders in perturbation theory due to $\delta\mathfrak{D} = g^2\mathcal{H}$. It therefore makes sense to consider $\mathcal{H}_{-1} = \mathfrak{D}_1$.

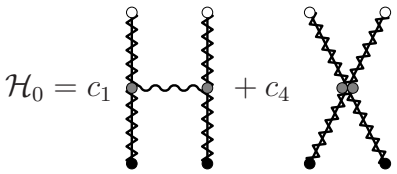

Figure 5.4: The structures of $\mathcal{H}_0$ which are compatible with $\mathfrak{su}(2|3)$ symmetry at leading order. A straight+zigzag line correspond to a supermultiplet.

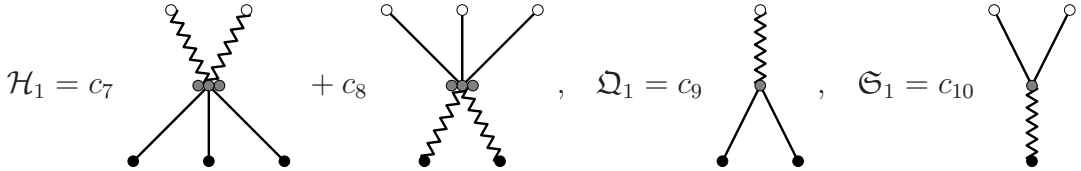

Figure 5.6: The structures for the construction of $\mathcal{H}_1, \mathfrak{Q}_1, \mathfrak{S}_1$. The number of spin sites is not conserved here.

We now commute $\mathfrak{Q}_0$ with $\mathcal{H}_0$ and find

$$\begin{aligned}[(\mathfrak{Q}_0)^a{}_\alpha, \mathcal{H}_0] = {} & e^{i\beta_1}(c_1 - c_2)\big(\left\{ {ab \atop \alpha b} \right\} + \left\{ {ba \atop b\alpha} \right\}\big) + e^{i\beta_1}(c_4 - c_5)\big(\left\{ {ba \atop \alpha b} \right\} + \left\{ {ab \atop b\alpha} \right\}\big) \\ & + e^{i\beta_1}(c_2 - c_3)\big(\left\{ {a\beta \atop \alpha\beta} \right\} - \left\{ {\beta a \atop \beta\alpha} \right\}\big) - e^{i\beta_1}(c_5 + c_6)\big(\left\{ {a\beta \atop \beta\alpha} \right\} - \left\{ {\beta a \atop \alpha\beta} \right\}\big).\end{aligned} \tag{5.47}$$

According to (5.40) this must vanish, so we set

$$c_1 = c_2 = c_3, \quad c_4 = c_5 = -c_6. \tag{5.48}$$

The commutator $[\mathfrak{S}_0, \mathcal{H}_0]$ leads to the same set of constraints. The two independent constants correspond to the two irreducible representations in the tensor product (see Fig. 5.4)

$$\mathcal{V}_{\mathrm{F}} \times \mathcal{V}_{\mathrm{F}} = [0;0;0,2]_+ + [0;0;1,0]_- \,. \tag{5.49}$$

More explicitly, $c_1 + c_4$ corresponds to the symmetric product $[0;0;0,2]$ which is half-BPS and $c_1 - c_4$ to the antisymmetric one $[0;0;1,0]$ which is quarter-BPS.

5.3.3 First Order

The virtue of a classically invariant interaction applies only to the leading order, for $\mathcal{H}_1$ we should break it. However, we do not wish to break classical $\mathfrak{su}(2|3)$ in the most general way, but assume that the classical $\mathfrak{su}(3) \times \mathfrak{su}(2)$ invariance is conserved. In field theory these correspond to symmetries compatible with the regularisation scheme.

The possible first order corrections involve the totally antisymmetric tensors of $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$, see Fig. 5.6:

$$\begin{aligned}\mathcal{H}_1 &= c_7\, \varepsilon_{\alpha\beta}\varepsilon^{abc}\left\{ {\alpha\beta \atop abc} \right\} + c_8\, \varepsilon_{abc}\varepsilon^{\alpha\beta}\left\{ {abc \atop \alpha\beta} \right\}, \\ (\mathfrak{Q}_1)^a{}_\alpha &= c_9\, \varepsilon_{\alpha\beta}\varepsilon^{abc}\left\{ {\beta \atop bc} \right\}, \\ (\mathfrak{S}_1)^\alpha{}_a &= c_{10}\, \varepsilon_{abc}\varepsilon^{\alpha\beta}\left\{ {bc \atop \beta} \right\}.\end{aligned} \tag{5.50}$$

With these expressions it is possible, yet tedious, to work out the commutators at first order by hand. It is useful to note a version of the gauge invariance identity (2.149)

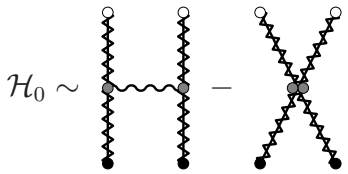

Figure 5.8: Closure of the algebra at $\mathcal{O}(g^3)$ fixes the relative coefficients within $\mathcal{H}_0$.

adapted to this particular situation

$$\left\{ {\beta d \atop bcd} \right\} + \left\{ {\beta\delta \atop bc\delta} \right\} = \left\{ {d\beta \atop dbc} \right\} - \left\{ {\delta\beta \atop \delta bc} \right\} = \left\{ {\beta \atop bc} \right\}. \tag{5.51}$$

Furthermore we will employ some identities of the totally antisymmetric tensors ε^{abc} and $\varepsilon_{\alpha\beta}$ and find for the commutator $[\mathfrak{Q}, \mathcal{H}]$ at $\mathcal{O}(g)$

$$\begin{aligned}[(\mathfrak{Q}_0)^a{}_\alpha, \mathcal{H}_1] + [(\mathfrak{Q}_1)^a{}_\alpha, \mathcal{H}_0] = {} & (c_4c_9 - e^{i\beta_1}c_7)\,\varepsilon^{bcd}\varepsilon_{\alpha\beta}\Big(\left\{ {a\beta \atop bcd} \right\} - \left\{ {\beta a \atop bcd} \right\}\Big) \\ & + (c_4c_9 - e^{i\beta_1}c_7)\,\varepsilon^{abc}\varepsilon_{\beta\gamma}\Big(-\left\{ {\beta\gamma \atop \alpha bc} \right\} + \left\{ {\beta\gamma \atop b\alpha c} \right\} - \left\{ {\beta\gamma \atop bc\alpha} \right\}\Big) \\ & - (c_1 + c_4)c_9\,\varepsilon^{abc}\varepsilon_{\alpha\beta}\left\{ {\beta \atop bc} \right\}. \end{aligned} \tag{5.52}$$

To satisfy (5.39) this must vanish. The commutator $[\mathfrak{S}, \mathcal{H}]$ gives similar constraints and closure of the algebra requires

$$c_1 = -c_4, \quad e^{i\beta_1}c_7 = c_4c_9, \quad e^{-i\beta_1}c_8 = c_4c_{10}. \tag{5.53}$$

Here, there are two types of constraints. The latter two fix the coefficients of $\mathfrak{Q}_1$ and $\mathfrak{S}_1$. The first one is more interesting, it fixes a coefficient of $\mathcal{H}_0$ from one order below, see Fig. 5.8. This is related to the fact that $\mathcal{H}_0$ was constructed to assign equal energies to all states of a multiplet of the free algebra. In a superalgebra, several atypical multiplets of the free theory can join to form one typical multiplet in the interacting theory, see Sec. 5.1.3 and Fig. 1.14. A consistency requirement for this to happen is that the energy shift of the submultiplets agree. In this case it is achieved by $c_1 = -c_4$. To ensure agreement of energies in terms of commutators, we need to consider one additional power of the coupling constant, which is required to move between the submultiplets. Furthermore, we note that the constraint $c_1 = -c_4$ assigns a zero eigenvalue to the representation $[0;0;0,2]$ in (5.49). This is essential, because $[0;0;0,2]$ is in fact half-BPS and must have zero energy. It is good to see though, that the protectedness of half-BPS states follows from the algebraic constraints; we will not have to impose it by hand.

5.3.4 Conclusions

We now set the remaining independent constants c_1, c_9, c_{10} to

$$c_1 = \alpha_1^2, \quad c_9 = \tfrac{1}{\sqrt{2}}\,\alpha_1\,e^{i\beta_1+i\beta_2}, \quad c_{10} = \tfrac{1}{\sqrt{2}}\,\alpha_1\,e^{-i\beta_1-i\beta_2'}. \tag{5.54}$$

In total we find the deformations at first order

$$\begin{aligned} \mathcal{H}_0 = {} & \alpha_1^2\left\{ {ab \atop ab} \right\} + \alpha_1^2\Big(\left\{ {a\beta \atop a\beta} \right\} + \left\{ {\alpha b \atop \alpha b} \right\}\Big) + \alpha_1^2\left\{ {\alpha\beta \atop \alpha\beta} \right\} \\ & - \alpha_1^2\left\{ {ab \atop ba} \right\} - \alpha_1^2\Big(\left\{ {a\beta \atop \beta a} \right\} + \left\{ {\alpha b \atop b\alpha} \right\}\Big) + \alpha_1^2\left\{ {\alpha\beta \atop \beta\alpha} \right\}, \\ \mathcal{H}_1 = {} & -\tfrac{1}{\sqrt{2}}\,\alpha_1^3\,e^{i\beta_2}\,\varepsilon_{\alpha\beta}\varepsilon^{abc}\left\{ {\alpha\beta \atop abc} \right\} - \tfrac{1}{\sqrt{2}}\,\alpha_1^3\,e^{-i\beta_2'}\,\varepsilon_{abc}\varepsilon^{\alpha\beta}\left\{ {abc \atop \alpha\beta} \right\}, \\ (\mathfrak{Q}_1)^a{}_\alpha = {} & \tfrac{1}{\sqrt{2}}\,\alpha_1\,e^{i\beta_1+i\beta_2}\,\varepsilon_{\alpha\beta}\varepsilon^{abc}\left\{ {\beta \atop bc} \right\}, \\ (\mathfrak{S}_1)^\alpha{}_a = {} & \tfrac{1}{\sqrt{2}}\,\alpha_1\,e^{-i\beta_1-i\beta_2'}\,\varepsilon_{abc}\varepsilon^{\alpha\beta}\left\{ {bc \atop \beta} \right\}. \end{aligned} \tag{5.55}$$

Let us discuss the free parameters. The parameter β_2' will in fact be determined by a constraint from fourth order, see the following section: $c_9 c_{10} = \frac{1}{2} c_1$ or

$$\beta_2' = \beta_2. \tag{5.56}$$

As shown in (5.35,5.36), the coefficients $\beta_{1,2}$ correspond to a similarity transformation of the algebra

$$\mathfrak{J}_0 \mapsto \exp\left(2i\beta_1 \mathfrak{D}_0 + i\beta_2 \mathcal{L}\right) \mathfrak{J}_0 \exp\left(-2i\beta_1 \mathfrak{D}_0 - i\beta_2 \mathcal{L}\right). \tag{5.57}$$

The algebra relations (5.31) are invariant under similarity transformations, so $\beta_{1,2}$ can take arbitrary values. For convenience, we might fix a gauge and set $\beta_1 = \beta_2 = 0$, but we will refrain from doing that here. Last but not least, the parameter α_1 corresponds to a rescaling of the coupling constant

$$g \mapsto \alpha_1 \, g. \tag{5.58}$$

The algebra relations (5.31) are also invariant under this redefinition.

In a real form of the algebra we get a few additional constraints. There, the algebra should be self-adjoint which imposes some reality constraint on $\alpha_1, \beta_{1,2}$. For a real $\mathfrak{su}(2|3)$ they have to be real and α_1^2 needs to be positive. This ensures positive planar energies as required by the unitarity bound.

In conclusion we have found that the deformations of the generators are uniquely fixed at one-loop. Note that $\mathcal{H}_0$ agrees with the complete one-loop dilatation operator found in Ch. 3. Here, it is understood that some parameters cannot be fixed due to symmetries of the algebra relations. In determining the coefficients we saw that $[\mathcal{H}(g), \mathfrak{J}(g)] = 0$ at order $\mathcal{O}(g^{2\ell-2})$ makes the ℓ-loop energy shift agree within short multiplets, whereas $\mathcal{O}(g^{2\ell-1})$ joins up short multiplets into long multiplets. Note that the anticommutator of supercharges at first order is trivially satisfied due to the flavours of incoming and outgoing fields

$$\{(\mathfrak{S}_1)^\alpha{}_a, (\mathfrak{Q}_0)^b{}_\beta\} + \{(\mathfrak{S}_0)^\alpha{}_a, (\mathfrak{Q}_1)^b{}_\beta\} = 0 = \tfrac{1}{2}\delta^b_a \delta^\alpha_\beta \mathcal{H}_{-1}. \tag{5.59}$$

5.4 Two-Loops

In this section we will discuss the restrictions from the algebra at two-loops, i.e. up to third order. The steps are straightforward, but involve very lengthy expressions. We have relied on the algebra system `Mathematica` to perform the necessary computations.

5.4.1 Structures

At second order we need to determine $\mathcal{H}_2, \mathfrak{Q}_2, \mathfrak{S}_2$. For $\mathcal{H}_2$ the $\mathfrak{su}(3) \times \mathfrak{su}(2)$ invariant interactions which preserve the dimension also preserve the number of fields, i.e. three fields are mapped into three fields. Similarly, for $\mathfrak{Q}_2, \mathfrak{S}_2$ we need two fields going into two fields

$$\mathcal{H}_2 \sim \left\{ {A_1 A_2 A_3 \atop B_1 B_2 B_3} \right\}, \quad \mathfrak{Q}_2, \mathfrak{S}_2 \sim \left\{ {A_1 A_2 \atop B_1 B_2} \right\}. \tag{5.60}$$

It is an easy exercise to count the number of structures in $\mathcal{H}_2, \mathfrak{Q}_2, \mathfrak{S}_2$. For $\mathcal{H}_2$ there $2^3 = 8$ ways to determine the statistics of $A_1 A_2 A_3$ and $3! = 6$ ways to permute the fields (each A must be contracted to one of the B's). In total there are $6 \cdot 8 = 48$ structures for $\mathcal{H}_2$

and 8 for $\mathfrak{Q}_2, \mathfrak{S}_2$ each. We now demand parity conservation. This restricts the number of independent structures to 28 and 4 for $\mathcal{H}_2$ and $\mathfrak{Q}_2, \mathfrak{S}_2$, respectively.

At third order we need to determine $\mathcal{H}_3, \mathfrak{Q}_3, \mathfrak{S}_3$. Like $\mathcal{H}_1, \mathfrak{Q}_1, \mathfrak{S}_1$, all of these involve the totally antisymmetric tensors for $\mathfrak{su}(3), \mathfrak{su}(2)$ and change the number of fields by one. Counting of independent structures is also straightforward, we find 48 for $\mathcal{H}_3$ and 12 for $\mathfrak{Q}_3, \mathfrak{S}_3$ each. Parity conservation halves each of these numbers.

5.4.2 Coefficients

Now we demand that energy shifts are conserved at third order

$$[\mathfrak{Q}^a{}_\alpha, \mathcal{H}] = [\mathfrak{S}^\alpha{}_a, \mathcal{H}] = \mathcal{O}(g^4). \tag{5.61}$$

This fixes the remaining coefficient β_2' at first order (5.56) and many coefficients at second and third order. The anticommutator of supercharges

$$\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta (\tfrac{1}{3}\mathfrak{D}_0 + \tfrac{1}{2} g^2 \mathcal{H}) + \mathcal{O}(g^4) \tag{5.62}$$

does not lead to additional constraints. The resulting deformations of the generators up to second order are presented in Tab. 5.2. In the remainder of this subsection we shall discuss the undetermined coefficients α, γ, δ and we shall find an explanation for each of them.

- Firstly, the constants $\delta_{1,2}$ multiply a structure which has a spectator leg on either side of the interaction

$$\left\{ {A_1 \ldots A_{E_\mathrm{i}} C \atop B_1 \ldots B_{E_\mathrm{o}} C} \right\} - (-1)^{C(A_1 \ldots A_{E_\mathrm{i}} B_1 \ldots B_{E_\mathrm{o}})} \left\{ {C A_1 \ldots A_{E_\mathrm{i}} \atop C B_1 \ldots B_{E_\mathrm{o}}} \right\}, \tag{5.63}$$

  such that both interactions cancel out in a cyclic state.

- Secondly, the constant δ_3 multiplies a structure which is zero due to an $\mathfrak{su}(2)$ identity. We cannot antisymmetrise more than two fundamental representations of $\mathfrak{su}(2)$

$$\left\{ {\alpha\beta\gamma \atop [\alpha\beta\gamma]} \right\} = 0. \tag{5.64}$$

- Thirdly, we can use a similarity transformation to modify the generators

$$\mathfrak{J}(g) \mapsto T(g)\, \mathfrak{J}(g)\, T(g)^{-1}. \tag{5.65}$$

  In Sec. 5.3.4, we have used a transformation which is independent of the coupling constant, here we consider a transformation $T(g) = 1 + g^2 T_2 + \ldots$ proportional to g^2. For consistency with the algebra, the transformation will have to be $\mathfrak{su}(3) \times \mathfrak{su}(2)$ invariant and preserve the dimension as well as parity. Also, according to Sec. 2.6.2, it should involve four fields. These are exactly the requirements for the form of $\mathcal{H}_0$, the 6 independent structures are given in (5.44,5.45). Out of these six, there are two special combinations: One of them is $\mathcal{H}_0$ itself and the other one is equivalent to the length operator

$$\mathcal{L} = \left\{ {ab \atop ab} \right\} + \left\{ {a\beta \atop a\beta} \right\} + \left\{ {\alpha b \atop \alpha b} \right\} + \left\{ {\alpha\beta \atop \alpha\beta} \right\} \tag{5.66}$$

$$
\begin{aligned}
\mathfrak{R}^a{}_b &= \left\{ {a \atop b} \right\} - \tfrac{1}{3}\delta^a_b \left\{ {c \atop c} \right\},\\
\mathfrak{L}^\alpha{}_\beta &= \left\{ {\alpha \atop \beta} \right\} - \tfrac{1}{2}\delta^\alpha_\beta \left\{ {\gamma \atop \gamma} \right\},\\
\mathfrak{D}_0 &= \left\{ {a \atop a} \right\} + \tfrac{3}{2}\left\{ {\alpha \atop \alpha} \right\},\\
\mathcal{H}_0 &= \alpha_1^2 \left\{ {ab \atop ab} \right\} + \alpha_1^2 \Big(\left\{ {a\beta \atop a\beta} \right\} + \left\{ {\alpha b \atop \alpha b} \right\} \Big) + \alpha_1^2 \left\{ {\alpha\beta \atop \alpha\beta} \right\}\\
&\quad - \alpha_1^2 \left\{ {ab \atop ba} \right\} - \alpha_1^2 \Big(\left\{ {a\beta \atop \beta a} \right\} + \left\{ {\alpha b \atop b\alpha} \right\} \Big) + \alpha_1^2 \left\{ {\alpha\beta \atop \beta\alpha} \right\},\\
\mathcal{H}_1 &= -\tfrac{1}{\sqrt{2}}\, \alpha_1^3\, e^{i\beta_2}\, \varepsilon_{\alpha\beta}\varepsilon^{abc} \left\{ {\alpha\beta \atop abc} \right\} - \tfrac{1}{\sqrt{2}}\, \alpha_1^3\, e^{-i\beta_2}\, \varepsilon_{abc}\varepsilon^{\alpha\beta} \left\{ {abc \atop \alpha\beta} \right\},\\
\mathcal{H}_2 &= (-2\alpha_1^4 + 2\alpha_1\alpha_3) \left\{ {abc \atop abc} \right\} + (-\tfrac{1}{2}\alpha_1^4 + 2\alpha_1\alpha_3 + \delta_3) \left\{ {\alpha\beta\gamma \atop \alpha\beta\gamma} \right\}\\
&\quad + (\tfrac{1}{2}\alpha_1^4 + 2\alpha_1\alpha_3 + 2\delta_2) \left\{ {a\beta c \atop a\beta c} \right\} + (-4\alpha_1^4 + 2\alpha_1\alpha_3 - 2\delta_2) \left\{ {\alpha b\gamma \atop \alpha b\gamma} \right\}\\
&\quad + (-\tfrac{11}{4}\alpha_1^4 + 2\alpha_1\alpha_3 - \delta_2) \Big(\left\{ {ab\gamma \atop ab\gamma} \right\} + \left\{ {\alpha bc \atop \alpha bc} \right\} \Big) + (2\alpha_1\alpha_3 + \delta_2) \Big(\left\{ {a\beta\gamma \atop a\beta\gamma} \right\} + \left\{ {\alpha\beta c \atop \alpha\beta c} \right\} \Big)\\
&\quad + (\tfrac{3}{2}\alpha_1^4 - \alpha_1\alpha_3) \Big(\left\{ {abc \atop bac} \right\} + \left\{ {abc \atop acb} \right\} \Big) + (\alpha_1^4 - \alpha_1\alpha_3) \Big(\left\{ {ab\gamma \atop ba\gamma} \right\} + \left\{ {\alpha bc \atop \alpha cb} \right\} \Big)\\
&\quad + (\tfrac{5}{4}\alpha_1^4 - \alpha_1\alpha_3 + i\alpha_1^2\gamma_3 + i\delta_1) \Big(\left\{ {\alpha bc \atop b\alpha c} \right\} + \left\{ {ab\gamma \atop a\gamma b} \right\} \Big)\\
&\quad + (\tfrac{5}{4}\alpha_1^4 - \alpha_1\alpha_3 - i\alpha_1^2\gamma_3 - i\delta_1) \Big(\left\{ {a\beta c \atop \beta a c} \right\} + \left\{ {a\beta c \atop ac\beta} \right\} \Big)\\
&\quad + (\alpha_1^4 - \alpha_1\alpha_3 + i\delta_1) \Big(\left\{ {\alpha b\gamma \atop b\alpha\gamma} \right\} + \left\{ {\alpha b\gamma \atop \alpha\gamma b} \right\} \Big)\\
&\quad + (\alpha_1^4 - \alpha_1\alpha_3 - i\delta_1) \Big(\left\{ {a\beta\gamma \atop \beta a\gamma} \right\} + \left\{ {\alpha\beta c \atop \alpha c\beta} \right\} \Big)\\
&\quad + (-\tfrac{7}{4}\alpha_1^4 + \alpha_1\alpha_3) \Big(\left\{ {\alpha\beta c \atop \beta\alpha c} \right\} + \left\{ {a\beta\gamma \atop a\gamma\beta} \right\} \Big) + (-\tfrac{7}{4}\alpha_1^4 + \alpha_1\alpha_3 - \delta_3) \Big(\left\{ {\alpha\beta\gamma \atop \beta\alpha\gamma} \right\} + \left\{ {\alpha\beta\gamma \atop \alpha\gamma\beta} \right\} \Big)\\
&\quad - \tfrac{1}{2}\alpha_1^4 \Big(\left\{ {abc \atop cab} \right\} + \left\{ {abc \atop bca} \right\} \Big) + \delta_3 \Big(\left\{ {\alpha\beta\gamma \atop \gamma\alpha\beta} \right\} + \left\{ {\alpha\beta\gamma \atop \beta\gamma\alpha} \right\} \Big)\\
&\quad + (-\tfrac{1}{4}\alpha_1^4 + i\alpha_1^2\gamma_1) \Big(\left\{ {\alpha bc \atop c\alpha b} \right\} + \left\{ {ab\gamma \atop b\gamma a} \right\} \Big) + (\tfrac{1}{4}\alpha_1^4 + i\alpha_1^2\gamma_2) \Big(\left\{ {a\beta\gamma \atop \gamma a\beta} \right\} + \left\{ {\alpha\beta c \atop \beta c\alpha} \right\} \Big)\\
&\quad + (-\tfrac{1}{4}\alpha_1^4 - i\alpha_1^2\gamma_1) \Big(\left\{ {a\beta c \atop ca\beta} \right\} + \left\{ {a\beta c \atop \beta ca} \right\} \Big) + (\tfrac{1}{4}\alpha_1^4 - i\alpha_1^2\gamma_2) \Big(\left\{ {\alpha b\gamma \atop \gamma\alpha b} \right\} + \left\{ {\alpha b\gamma \atop b\gamma\alpha} \right\} \Big)\\
&\quad - \tfrac{1}{2}\alpha_1^4 \Big(\left\{ {ab\gamma \atop \gamma ba} \right\} + \left\{ {\alpha bc \atop cb\alpha} \right\} \Big) + \tfrac{1}{2}\alpha_1^4 \left\{ {\alpha b\gamma \atop \gamma b\alpha} \right\} - \delta_3 \left\{ {\alpha\beta\gamma \atop \gamma\beta\alpha} \right\}\\
&\quad + \tfrac{3}{2}\alpha_1^4 \Big(\left[{\alpha\beta \atop \alpha\beta} \right] - \left[{\alpha\beta \atop \beta\alpha} \right] \Big),\\
(\mathfrak{Q}_0)^a{}_\alpha &= e^{i\beta_1} \left\{ {a \atop \alpha} \right\},\\
(\mathfrak{Q}_1)^a{}_\alpha &= \tfrac{1}{\sqrt{2}}\, \alpha_1\, e^{i\beta_1 + i\beta_2} \varepsilon_{\alpha\beta}\varepsilon^{abc} \left\{ {\beta \atop bc} \right\},\\
(\mathfrak{Q}_2)^a{}_\alpha &= e^{i\beta_1} (\tfrac{1}{4}\alpha_1^2 - i\tfrac{1}{2}\gamma_3 + i\tfrac{1}{2}\gamma_4) \Big(\left\{ {ab \atop \alpha b} \right\} + \left\{ {ba \atop b\alpha} \right\} \Big) + e^{i\beta_1} (\tfrac{1}{2} i\gamma_3 + \tfrac{1}{2} i\gamma_4) \Big(\left\{ {a\beta \atop \alpha\beta} \right\} - \left\{ {\beta a \atop \beta\alpha} \right\} \Big)\\
&\quad + e^{i\beta_1} (-\tfrac{1}{4}\alpha_1^2 - i\gamma_1) \Big(\left\{ {ab \atop b\alpha} \right\} + \left\{ {ba \atop \alpha b} \right\} \Big) + e^{i\beta_1} (\tfrac{1}{4}\alpha_1^2 + i\gamma_2) \Big(\left\{ {a\beta \atop \beta\alpha} \right\} - \left\{ {\beta a \atop \alpha\beta} \right\} \Big),\\
(\mathfrak{S}_0)^\alpha{}_a &= e^{-i\beta_1} \left\{ {\alpha \atop a} \right\},\\
(\mathfrak{S}_1)^\alpha{}_a &= \tfrac{1}{\sqrt{2}}\, \alpha_1\, e^{-i\beta_1 - i\beta_2} \varepsilon_{abc}\varepsilon^{\alpha\beta} \left\{ {bc \atop \beta} \right\},\\
(\mathfrak{S}_2)^\alpha{}_a &= e^{-i\beta_1} (\tfrac{1}{4}\alpha_1^2 + \tfrac{1}{2} i\gamma_3 - \tfrac{1}{2} i\gamma_4) \Big(\left\{ {ab \atop ab} \right\} + \left\{ {b\alpha \atop ba} \right\} \Big) + e^{-i\beta_1} (-\tfrac{1}{2} i\gamma_3 - \tfrac{1}{2} i\gamma_4) \Big(\left\{ {\alpha\beta \atop a\beta} \right\} - \left\{ {\beta\alpha \atop \beta a} \right\} \Big)\\
&\quad + e^{-i\beta_1} (-\tfrac{1}{4}\alpha_1^2 + i\gamma_1) \Big(\left\{ {ab \atop ba} \right\} + \left\{ {b\alpha \atop ab} \right\} \Big) + e^{-i\beta_1} (\tfrac{1}{4}\alpha_1^2 - i\gamma_2) \Big(- \left\{ {\alpha\beta \atop \beta a} \right\} + \left\{ {\beta\alpha \atop a\beta} \right\} \Big).
\end{aligned}
$$

Table 5.2: Two-loop deformations of the generators

up to gauge transformations. The similarity transformation amounts to adding commutators with $\mathcal{H}_0, \mathfrak{Q}_0, \mathfrak{S}_0$

$$\mathcal{H}_2 \mapsto \mathcal{H}_2 + [T_2, \mathcal{H}_0], \quad \mathfrak{J}_2 \mapsto \mathfrak{J}_2 + [T_2, \mathfrak{J}_0]. \tag{5.67}$$

These commutators vanish for $\mathcal{H}_0$ and $\mathcal{L}$

$$[\mathcal{H}_0, \mathfrak{J}_0] = [\mathcal{L}, \mathfrak{J}_0] = [\mathcal{H}_0, \mathcal{H}_0] = [\mathcal{L}, \mathcal{H}_0] = 0. \tag{5.68}$$

In other words, conjugation with $g^2\mathcal{H}_0$ and $g^2\mathcal{L}$ will have no effect on $\mathcal{H}_2, \mathfrak{Q}_2, \mathfrak{S}_2$. The remaining four structures in (5.44,5.45) do not commute with $\mathcal{H}_0, \mathfrak{Q}_0, \mathfrak{S}_0$ and amount to the constants $\gamma_{1,2,3,4}$. Note that γ_4 is related to the structure $\mathfrak{D}_0$ and does not appear in $\mathcal{H}_2$ because of $[\mathfrak{D}_0, \mathcal{H}_0] = 0$.

- Finally, we are allowed to perform a transformation of the coupling constant

$$\mathfrak{J}(g) \mapsto \mathfrak{J}(f(g)). \tag{5.69}$$

If we use the function $f(g) = \alpha_1 g + \alpha_3 g^3$ we find that

$$\mathcal{H}_2 \mapsto \alpha_1^4 \mathcal{H}_2 + 2\alpha_1\alpha_3\mathcal{H}_0, \tag{5.70}$$

which explains the degree of freedom α_3.

5.4.3 Short States and Wrapping Interactions

The second order interactions $\mathcal{H}_2$ act on three fields. We should also determine its action on the states of length two[8]

$$\mathcal{O}_{(ab)} = |ab\rangle = \operatorname{Tr}\phi_a\phi_b, \quad \mathcal{O}_{a\beta} = |a\beta\rangle = \operatorname{Tr}\phi_a\psi_\beta, \quad \mathcal{O}_1 = \varepsilon^{\alpha\beta}|\alpha\beta\rangle = \varepsilon^{\alpha\beta}\operatorname{Tr}\psi_\alpha\psi_\beta. \tag{5.71}$$

Together, these form the protected half-BPS multiplet $[0;0;0,2]$. It is therefore reassuring to see that $\mathcal{O}_{(ab)}$ and $\mathcal{O}_{a\beta}$ are annihilated by $\mathcal{H}_0, \mathcal{H}_1$; just as well they should be annihilated by $\mathcal{H}_2$. For $\mathcal{O}_1$ the situation is different: It is annihilated by $\mathcal{H}_0$, but $\mathcal{H}_1$ produces the operator

$$\mathcal{O}_2 = \varepsilon^{abc}|abc\rangle = \varepsilon^{abc}\operatorname{Tr}\phi_a\phi_b\phi_c. \tag{5.72}$$

The action of $\mathcal{H}(g)$ on these two operators up to second order is given by

$$\mathcal{H}(g)\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix} = \begin{pmatrix}\epsilon g^2 & -2\sqrt{2}\,e^{i\beta_2}\alpha_1^3 g\\ -9\sqrt{2}\,e^{i\beta_2}e^{-i\beta_2}\alpha_1^3 g & 6\alpha_1^2 - 18\alpha_1^2 g^2 + 12\alpha_1\alpha_3 g^2\end{pmatrix}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix}, \tag{5.73}$$

where we have assumed that $\mathcal{H}_2\mathcal{O}_1 = \epsilon\mathcal{O}_1$. The eigenvalues of this matrix at fourth order are given by

$$E_1 = \epsilon g^2 - 6\alpha_1^4 g^2, \quad E_2 = 6\alpha_1^2 - 12\alpha_1^4 g^2 + 12\alpha_1\alpha_3 g^2. \tag{5.74}$$

Due to its half-BPS nature, the energy of the diagonalised $\mathcal{O}_1$ must be exactly zero, $E_1 = 0$, and we set

$$\epsilon = 6\alpha_1^4. \tag{5.75}$$

The second order Hamiltonian for states of length two should thus annihilate the states $\mathcal{O}_{(ab)}, \mathcal{O}_{a\beta}$ and yield $6\alpha_1^4\mathcal{O}_1$ when acting on $\mathcal{O}_1$. This is achieved by a wrapping interaction, c.f. Sec. 2.6.3,

$$\mathcal{H}_2 = \ldots + \tfrac{3}{2}\alpha_1^4\big(\left[\begin{smallmatrix}\alpha\beta\\ \alpha\beta\end{smallmatrix}\right] - \left[\begin{smallmatrix}\alpha\beta\\ \beta\alpha\end{smallmatrix}\right]\big). \tag{5.76}$$

[8] Length-one states are U(1) fields and do not interact at all.

5.4.4 Conclusions

We see that for all free parameters in Tab. 5.2 there is an associated symmetry of the algebra relations and we can say that the two-loop contribution is uniquely fixed. The only parameter that influences energies is α_3; we cannot remove it by algebraic considerations. The parameters $\gamma_{1,2,3,4}$ rotate only the eigenstates. Finally, the parameters $\delta_{1,2,3}$ are there only because we were not careful enough in finding *independent* structures (for $\mathcal{H}_2$ there are only $25 = 28 - 3$ independent structures). They have no effect at all.

5.5 Three-Loops

For the fourth order contributions $\mathcal{H}_4, \mathfrak{Q}_4, \mathfrak{S}_4$ we find in total $208 + 56 + 56$ parity conserving structures; they all conserve the number of fields[9]. Of these only $173 + 32 + 32$ are independent due to identities as discussed above. We impose the constraint (5.39) at fourth order

$$[\mathfrak{Q}, \mathcal{H}] = [\mathfrak{S}, \mathcal{H}] = \mathcal{O}(g^5) \tag{5.77}$$

and find that the algebra relations fix 202 coefficients (plus one coefficient at third order). This leaves 35 free coefficients. The anticommutator of supercharges (5.9) at fourth order

$$\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta (\tfrac{1}{3}\mathfrak{D}_0 + \tfrac{1}{2} g^2 \mathcal{H}) + \mathcal{O}(g^5) \tag{5.78}$$

is satisfied automatically.

As we have learned above, the commutators at fourth order are not sufficient to ensure consistency for splitting multiplets at the unitarity bound, we should also consider fifth order. To perform those commutators would be even harder. We therefore consider a set of probe multiplets at the unitarity bound. By requiring that the three-loop energy shifts coincide within submultiplets we are able to fix another 8 coefficients.

Still this leaves 27 coefficients to be fixed, however, almost all of them rotate the space of states. Experimentally, we found that only 4 coefficients affect the energies. The remaining 23 coefficients can be attributed to similarity transformations. As before, the number of similarity transformations equals the number of structures for $\mathcal{H}_2$, i.e. 25. This means that there must be 2 commuting generators which are readily found to be $g^4\mathcal{H}_0$ and $g^4 L$. We summarise our findings concerning the number of coefficients in Tab. 5.4. The symmetries indicated in the table refer to $\mathcal{L}$ (which is conserved at leading order but broken at first order, hence the -1 at $k = 1$), $g^2\mathcal{L}$ (broken at third order), $g^2\mathcal{H}$, $g^4\mathcal{L}$ (will break at fifth order) and $g^4\mathcal{H}$. This sequence of symmetries will continue at higher orders, but there will be additional ones due to integrability, see Sec. 6.1.4.

Let us now discuss the relevant coefficients. One coefficient is due to a redefinition of the coupling constant and cannot be fixed algebraically. To constrain the other three we will need further input. Unfortunately, the resulting generators are too lengthy to be displayed here. Instead, let us have a look at the set of totally bosonic states. In this subsector (which is closed only when further restricted to two flavours, i.e. the $\mathfrak{su}(2)$ subsector) $\mathcal{H}_4$ is presented in Tab. 5.6.

The coefficients in Tab. 5.6 can be understood as follows. The coefficients $\sigma_{1,2,3,4}$ are relevant. One of them, σ_1, multiplies the structure $\mathcal{H}_0$ (up to spectator legs) and therefore

[9] At sixth order the number of fields can be changed by two using four antisymmetric tensors.

k	0	1	2	3	4
$\mathcal{H}_k$	6	2	25	18	173
$\mathfrak{Q}_k$	1	1	4	6	32
$\mathfrak{S}_k$	1	1	4	6	32
total	8	4	33	30	237
fixed at $\mathcal{O}(g^k)$	5	2	25	26	202
fixed at $\mathcal{O}(g^{k+1})$	1	1	3	1	8
relevant	1	0	1	0	4
irrelevant	1	1	4	3	23
symmetries	1	−1	2	−1	2
$\mathcal{H}_{k-2}$	2	0	6	2	25

Table 5.4: Number of coefficients. $\mathcal{H}_k, \mathfrak{Q}_k, \mathfrak{S}_k$ give the number of independent structures that can be used for the construction of generators. The algebra relations fix a certain number of coefficients. Of the remaining coefficients, some are relevant for energies and some correspond to similarity transformations generated by the structures in $\mathcal{H}_{k-2}$. Some of the similarity transformations are symmetries.

corresponds to a redefinition of the coupling constant as in (5.69). The coefficients $\zeta_{1,2,3}$ multiply structures which are actually zero: More explicitly, ζ_1 multiplies $\left\{ \begin{smallmatrix} abcd \\ [abcd] \end{smallmatrix} \right\}$ and $\zeta_{2,3}$ can be gauged away by removing spectator legs. Finally, the coefficients $\xi_{1,2,3}$ are related to similarity transformations and have no effect on the energy shifts.

The crucial point is that we want $\mathcal{H}_4$ to be generated by Feynman diagrams. Here we can make use of a special property of the scalar sector, c.f. Sec. 2.5.3. The Feynman diagrams with the maximum number of eight legs do not have internal index loops. In the planar case, such diagrams must be iterated one-loop diagrams. This implies that we can only have three permutations of *adjacent* fields. The structures

$$\left\{ \begin{smallmatrix} abcd \\ cdab \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ bdca \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ dbac \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ cbda \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ dacb \end{smallmatrix} \right\}, \quad \left\{ \begin{smallmatrix} abcd \\ cdba \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ dcab \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} abcd \\ dbca \end{smallmatrix} \right\}, \quad \left\{ \begin{smallmatrix} abcd \\ dcba \end{smallmatrix} \right\} \tag{5.79}$$

consist of four, five or six crossings of adjacent fields and are therefore excluded. We must set their coefficients to zero

$$\sigma_3 = \sigma_4 = 0, \qquad \zeta_1 = \xi_1 = 0. \tag{5.80}$$

The final relevant coefficient σ_2 multiplies a structure $\mathcal{Q}_{4,0}$ which commutes with $\mathcal{H}_0$. This issue is related to integrability, see Sec. 6.1.4. At this point we cannot determine σ_2, but believe that it will be fixed due to the anticommutator $\{\mathfrak{Q}, \mathfrak{S}\}$ at $\mathcal{O}(g^6)$ (four-loops).

5.6 Spectrum

In this section we fix the remaining degrees of freedom within the Hamiltonian and apply it to a number of states to find their energies.

5.6.1 The Remaining Coefficients

First of all, we would like to fix the remaining relevant coefficients. We cannot do this algebraically because most of them correspond to symmetries of the commutation

$$
\begin{aligned}
\mathcal{H}_4 = {} & \left(\tfrac{15}{2}\alpha_1^6 - 8\alpha_1^3\alpha_3 + \sigma_1 - \tfrac{1}{3}\sigma_2 + 12\sigma_3 - 2\sigma_4 + \xi_2 + \zeta_1\right)\left\{{abcd \atop abcd}\right\} \\
& + \left(-\tfrac{13}{4}\alpha_1^6 + 3\alpha_1^3\alpha_3 - \tfrac{1}{4}\sigma_1 + \tfrac{1}{6}\sigma_2 - 3\sigma_3 + \sigma_4 - \tfrac{1}{2}\xi_2 - \zeta_1 + \zeta_3\right)\left(\left\{{abcd \atop abdc}\right\} + \left\{{abcd \atop bacd}\right\}\right) \\
& + \left(-\tfrac{13}{2}\alpha_1^6 + 6\alpha_1^3\alpha_3 - \tfrac{1}{2}\sigma_1 + \tfrac{1}{3}\sigma_2 - 6\sigma_3 + 2\sigma_4 - \xi_2 - \zeta_1 - 2\zeta_3\right)\left\{{abcd \atop acbd}\right\} \\
& + \left(\tfrac{3}{2}\alpha_1^6 - \alpha_1^3\alpha_3 - \sigma_4 + \tfrac{1}{2}\xi_2 + \zeta_1 - i\zeta_2\right)\left(\left\{{abcd \atop acdb}\right\} + \left\{{abcd \atop cabd}\right\}\right) \\
& + \left(\tfrac{3}{2}\alpha_1^6 - \alpha_1^3\alpha_3 - \sigma_4 + \tfrac{1}{2}\xi_2 + \zeta_1 + i\zeta_2\right)\left(\left\{{abcd \atop adbc}\right\} + \left\{{abcd \atop bcad}\right\}\right) \\
& + \left(-\tfrac{1}{6}\sigma_2 - \tfrac{1}{2}\xi_2 - \zeta_1\right)\left(\left\{{abcd \atop adcb}\right\} + \left\{{abcd \atop cbad}\right\}\right) + \left(\tfrac{1}{2}\alpha_1^6 - 2\sigma_3 + \zeta_1\right)\left\{{abcd \atop badc}\right\} \\
& + \left(-\tfrac{1}{2}\alpha_1^6 + \tfrac{1}{3}\sigma_2 - 4\sigma_3 + 2\sigma_4 - \zeta_1\right)\left(\left\{{abcd \atop bcda}\right\} + \left\{{abcd \atop dabc}\right\}\right) \\
& + \left(-\tfrac{1}{3}\sigma_2 + 4\sigma_3 + \sigma_4 - i\xi_3 - \zeta_1\right)\left\{{abcd \atop bdac}\right\} + \left(-\tfrac{1}{3}\sigma_2 + 4\sigma_3 + \sigma_4 + i\xi_3 - \zeta_1\right)\left\{{abcd \atop cadb}\right\} \\
& + \left(\sigma_3 - \sigma_4 + i\xi_1 + \zeta_1\right)\left(\left\{{abcd \atop bdca}\right\} + \left\{{abcd \atop dbac}\right\}\right) + \left(\sigma_3 - \sigma_4 - i\xi_1 + \zeta_1\right)\left(\left\{{abcd \atop cbda}\right\} + \left\{{abcd \atop dacb}\right\}\right) \\
& + \left(-2\sigma_3 - 2\sigma_4 + \zeta_1\right)\left\{{abcd \atop cdab}\right\} + \left(-\zeta_1\right)\left(\left\{{abcd \atop cdba}\right\} + \left\{{abcd \atop dcab}\right\}\right) \\
& + \left(2\sigma_4 - \zeta_1\right)\left\{{abcd \atop dbca}\right\} + \left(\zeta_1\right)\left\{{abcd \atop dcba}\right\}.
\end{aligned}
$$

Table 5.6: $\mathcal{H}_4$ acting on bosonic states.

relations. One of them is a redefinition of the coupling constant

$$g \mapsto f(g). \tag{5.81}$$

Unlike the other symmetries, this transformation has relevant consequences, it implies that energies are changed according to

$$E(g) \mapsto E(f(g)). \tag{5.82}$$

In order to match these degrees of freedom to $\mathcal{N} = 4$ SYM we should use some scaling dimension that is known to all orders in perturbation theory. In fact it is sufficient to use the scaling behaviour in the BMN limit, c.f. Sec. 3.6, i.e. for large J, all properly rescaled quantities should depend only on

$$\hat{g} = \frac{g}{J} \qquad \text{or} \qquad \lambda' = \frac{\lambda}{J^2} = \frac{8\pi^2 g^2}{J^2} = 8\pi^2\hat{g}^2. \tag{5.83}$$

Let us assume this to be the case. If we redefine the coupling constant g we obtain for the rescaled coupling constant

$$\hat{g} \mapsto \frac{f(g)}{J} = \frac{f_1 g + f_3 g^3 + \dots}{J} = f_1\hat{g} + f_3\hat{g}^3 J^2 + \dots \ . \tag{5.84}$$

The problem is that all the higher expansion coefficients of f yield divergent contributions in the BMN limit $J \to \infty$. Thus all $\hat{g}$ dependent quantities will also become divergent. The only degree of freedom compatible with BMN scaling behaviour is to change g by a constant factor f_1.

In our model we would like to define the coupling constant by fixing α_3, σ_1 in such a way as to obtain a good scaling behaviour of energies in the BMN limit. It is not possible to achieve proper scaling by adjusting α_3, σ_1 alone; also σ_2 multiplies a structure with a

wrong scaling. This is fortunate, because it allows us to determine σ_2 as well, in total we find

$$\alpha_3 = 0, \quad \sigma_1 = 0, \quad \sigma_2 = 0. \tag{5.85}$$

Afterwards we can only change α_1. This final degree of freedom is eliminated by a single known scaling dimension, e.g. the one of the Konishi multiplet (2.20) $E_0 = 6$, or using the quantitative BMN energy formula (3.99). It fixes α_1 to unity

$$\alpha_1 = 1. \tag{5.86}$$

We conclude that *the planar three-loop Hamiltonian is uniquely fixed* by the symmetry algebra, field theory and the BMN scaling behaviour.[10] Together with the fact that this model is a closed subsector of $\mathcal{N} = 4$ SYM we have derived the planar dilatation generator in the $\mathfrak{su}(2|3)$ subsector at three-loops. Similarly, this model is a closed subsector of the BMN matrix model and the two Hamiltonians must agree up to three loops (after a redefinition of the coupling constant and provided that the BMN matrix model has a BMN limit). This is indeed the case as shown in [79].

5.6.2 Lowest-Lying States

We are now ready to compute numerical values for some energies. For this we should consider the charges D_0, s, p, q, L of a state and compute the number of constituent fields according to (5.15). These are arranged within a trace in all possible ways

$$\mathcal{E}_n = (\mathrm{Tr}\, \phi_1^{n_1} \phi_2^{n_2} \phi_3^{n_3} \psi_1^{n_4} \psi_2^{n_5}, \ldots). \tag{5.87}$$

Note that the length L is not a good quantum number at $\mathcal{O}(g)$, so we must include states of all admissible lengths in (5.87). In practice this means that we may replace a complete set of bosons $\phi_1\phi_2\phi_3$ by a complete set of fermions $\psi_1\psi_2$, (5.23). Due to conservation of charges, the Hamiltonian closes on this set of states and we can evaluate its matrix elements[11]

$$\mathcal{H}(g)\, \mathcal{E}_i = \mathcal{E}_j\, H^j{}_i(g). \tag{5.88}$$

It is a straightforward task to find the eigenvalues and their perturbations

$$\mathcal{H}(g) = \mathcal{H}_0 + V(g), \qquad \text{with } V = \mathcal{O}(g). \tag{5.89}$$

Diagonalising the leading order matrix $\mathcal{H}_0$ is a non-linear problem. The resulting eigenvalues represent the one-loop energies E_0. Now we pick an eigenvalue $e = E_0$ of $\mathcal{H}_0$ and consider the subspace of states with energy e. The higher-order energy shifts are given by (in contrast to the formula in [79] V' was constructed such that conjugation symmetry is

[10] Without BMN scaling the constants $\alpha_1, \alpha_3, \sigma_1, \sigma_2$ remain unknown. However, $\alpha_1, \alpha_3, \sigma_1$ are related to a redefinition of the coupling constant and, as we shall see in Sec. 6.1.4, there is a natural explanation for σ_2 in terms of integrability. It therefore makes sense to say that the Hamiltonian is uniquely fixed (up to symmetries) even without making use of BMN scaling.

[11] Although the Hilbert space is infinite-dimensional, the Hamiltonian acts on a space of fixed dimension D_0. Therefore the matrix $H^j{}_i(g)$ has a finite size.

preserved)

$$\begin{aligned} V' = \sum\nolimits_e \Pi_e \Big[& V + V\Delta_e V + \big(V\Delta_e V\Delta_e V - \tfrac{1}{2}V\Delta_e^2 V\Pi_e V - \tfrac{1}{2}V\Pi_e V\Delta_e^2 V\big) \\ & + V\Delta_e V\Delta_e V\Delta_e V - \tfrac{1}{2}V\Delta_e^2 V\Pi_e V\Delta_e V - \tfrac{1}{2}V\Delta_e V\Pi_e V\Delta_e^2 V \\ & - \tfrac{1}{2}V\Delta_e^2 V\Delta_e V\Pi_e V - \tfrac{1}{2}V\Pi_e V\Delta_e V\Delta_e^2 V \\ & - \tfrac{1}{2}V\Delta_e V\Delta_e^2 V\Pi_e V - \tfrac{1}{2}V\Pi_e V\Delta_e^2 V\Delta_e V \\ & + \tfrac{1}{3}V\Delta_e^3 V\Pi_e V\Pi_e V + \tfrac{1}{3}V\Pi_e V\Delta_e^3 V\Pi_e V + \tfrac{1}{3}V\Pi_e V\Pi_e V\Delta_e^3 V + \dots \Big] \Pi_e. \end{aligned} \tag{5.90}$$

The propagator Δ_e is given by

$$\Delta_e = \frac{1-\Pi_e}{e-\mathcal{H}_0} \tag{5.91}$$

and Π_e projects to the subspace with leading correction e. If there is only a single state with one-loop energy e, (5.90) gives its higher order corrections. For degenerate states at one-loop, (5.89,5.90) must be applied iteratively until the resulting matrix V' becomes diagonal[12].

Next, it is important to know the multiplets of states. In the interacting theory there are two types of single-trace multiplets, half-BPS and long ones. The half-BPS multiplets are easily identified, there is one multiplet with labels

$$D_0 = L = p, \quad E = 0, \quad [0;0;0,p], \quad P = (-1)^p \tag{5.92}$$

for each p, they receive no corrections to their energy. Long multiplets are not so easy to find. By means of a `C++` computer programme we have constructed the spectrum of all states explicitly (up to some energy bound) and iteratively removed the multiplets corresponding to the leftover highest weight state (this 'sieve' algorithm, also reminiscent of the standard algorithm for division, is described in more detail in [105, 106]). For a set of states with given charges as in (5.87) this also tells us how many representatives there are from each of the multiplets and allows us to identify the energy we are interested in.

Finally, to obtain the energy shift of a given multiplet, a lot of work can be saved by choosing a suitable representative. Resolving the mixing problem for the highest weight state is usually more involved than for a descendant. For instance, highest weight states involve all three flavours of bosons, $n_1, n_2, n_3 \geq 1$. This increases the number of permutations in (5.87) and also gives rise to mixing between states of different lengths. The matrix $H^m{}_n$ will be unnecessarily large. If, instead, one applies three supergenerators $\mathfrak{Q}^1{}_4\mathfrak{Q}^2{}_4\mathfrak{Q}^3{}_4$, i.e.

$$n_1 \mapsto n_1 - 1, \quad n_2 \mapsto n_2 - 1, \quad n_3 \mapsto n_3 - 1, \quad n_4 \mapsto n_4 + 3, \tag{5.93}$$

the state becomes more uniform. This decreases the number of permutations and, in the case of multiplets at the unitarity bound (5.16), mixing between states of different lengths is prevented due to $n_3 = n_5 = 0$.

We summarise our findings for states of dimension $D_0 \leq 8.5$ in Tab. 5.8. We have labelled the states by their dimension D_0, classical $\mathfrak{su}(2|3)$ Dynkin labels, and classical length L. For each multiplet we have given its energy $E = E_0 + g^2 E_2 + g^4 E_4 + \mathcal{O}(g^6)$ up

[12] In principle it could happen that states with equal leading order energy e have matrix elements at $\mathcal{O}(g)$. In this case the energy would have an expansion in terms of $g \sim \sqrt{\lambda}$ instead of $g^2 \sim \lambda$ similar to the peculiarities noticed in [38]. It would be interesting to see if this does indeed happen or, if not, why?

D_0	$\mathfrak{su}(2\|3)$	L	$\big(E_0, E_2, E_4\big)^P$
2	$[0;0;0,2]^{\dagger\bullet}$	2	$(0,0,0)^+$
3	$[0;0;0,3]^{\dagger\bullet}$	3	$(0,0,0)^-$
3	$[0;1;0,0]^{*\bullet}$	3	$\big(6,-12,42\big)^+$
4	$[0;0;0,4]^{\dagger\bullet}$	4	$(0,0,0)^+$
4	$[0;1;0,1]^{*\bullet}$	4	$\big(4,-6,17\big)^-$
5	$[0;0;0,5]^{\dagger\bullet}$	5	$(0,0,0)^-$
5	$[0;1;0,2]^{*\bullet}$	5	$\big(10E-20,-17E+60,\frac{117}{2}E-230\big)^+$
5	$[0;1;1,0]^{*\bullet}$	5	$\big(6,-9,\frac{63}{2}\big)^-$
6	$[0;0;0,6]^{\dagger\bullet}$	6	$(0,0,0)^+$
6	$[0;1;0,3]^{*\bullet}$	6	$\big(2,-\frac{3}{2},\frac{37}{16}\big)^-, \big(6,-\frac{21}{2},\frac{555}{16}\big)^-$
6	$[0;1;1,1]^{*\bullet}$	6	$\big(5,-\frac{15}{2},25\big)^\pm$
6	$[0;2;0,0]$	6	$\big(14E-36,-24E+90,\frac{173}{2}E-315\big)^+$
6	$[2;3;0,0]^*$	5	$\big(10,-20,\frac{145}{2}\big)^-$
6.5	$[1;2;0,2]^*$	6	$\big(8,-14,49\big)^\pm$
7	$[0;0;0,7]^{\dagger\bullet}$	7	$\big(0,0,0\big)^-$
7	$[0;1;0,4]^{*\bullet}$	7	$\big(14E^2-56E+56,-23E^2+172E-224,79E^2-695E+966\big)^+$
7	$[0;1;1,2]^{*\bullet}$	7	$\big(4,-5,14\big)^\pm, \big(6,-9,33\big)^-$
7	$[0;1;2,0]^{*\bullet}$	7	$\big(20E^2-116E+200,-32E^2+340E-800,112E^2-1400E+3600\big)^+$
7	$[0;2;0,1]$	7	$\big(22E^2-144E+248,-37E^2+460E-1016,125E^2-1893E+4438\big)^-$
7	$[2;3;0,1]^*$	6	$\big(8,-14,46\big)^+$
7.5	$[1;2;0,3]^*$	7	$\big(7,-12,\frac{83}{2}\big)^\pm$
7.5	$[1;2;1,1]^*$	7	$\big(6,-\frac{33}{4},\frac{1557}{64}\big)^\pm, \big(10,-\frac{75}{4},\frac{4315}{64}\big)^\pm$
7.5	$[1;3;0,0]$	7	$\big(9,-15,51\big)^\pm$
8	$[0;0;0,8]^{\dagger\bullet}$	8	$\big(0,0,0\big)^+$
8	$[0;1;0,5]^{*\bullet}$	8	$\big(4,-5,\frac{49}{4}\big)^-, \big(8E-8,-13E+18,\frac{179}{4}E-61\big)^-$
8	$[0;1;1,3]^{*\bullet}$	8	$\big(17E^2-90E+147,-\frac{51}{2}E^2+\frac{525}{2}E-\frac{1239}{2},\frac{169}{2}E^2-\frac{2091}{2}E+\frac{5649}{2}\big)^\pm$
8	$[0;1;2,1]^{*\bullet}$	8	$\big(5,-\frac{15}{2},\frac{55}{2}\big)^\pm, \big(12E-24,-18E+54,57E-171\big)^-$
8	$[0;2;0,2]$	8	$\big(7,-\frac{19}{2},\frac{59}{2}\big)^\pm, \big(44E^5-768E^4+6752E^3-31168E^2+70528E-60224,A,B\big)^+$
8	$[0;2;1,0]$	8	$\big(9,-\frac{31}{2},\frac{103}{2}\big)^\pm, \big(24E^2-172E+344,-39E^2+524E-1372,138E^2-2209E+6198\big)^-$
8	$[2;3;0,2]^*$	7	$\big(28E^2-252E+728,-51E^2+906E-3864,179E^2-3965E+20090\big)^-$
8	$[2;3;1,0]^*$	7	$\big(8,-\frac{25}{2},\frac{687}{16}\big)^+, \big(12,-\frac{45}{2},\frac{1281}{16}\big)^+$
8.5	$[1;2;0,4]^*$	8	$\big(6,-\frac{19}{2},\frac{247}{8}\big)^\pm, \big(8,-\frac{29}{2},\frac{427}{8}\big)^\pm$
8.5	$[1;2;1,2]^*$	8	$\big(31E^3-350E^2+1704E-3016,-50E^3+1111E^2-7971E+18452,C\big)^\pm$
8.5	$[1;2;2,0]^*$	8	$\big(8,-13,\frac{343}{8}\big)^\pm, \big(15E-48,-23E+135,\frac{595}{8}E-\frac{4023}{8}\big)^\pm$
8.5	$[1;3;0,1]$	8	$\big(8,-13,\frac{173}{4}\big)^\pm, \big(10,-\frac{67}{4},\frac{3725}{64}\big)^\pm, \big(19E-86,-\frac{133}{4}E+\frac{1169}{4},\frac{7395}{64}E-\frac{79503}{64}\big)^\pm$

$A = -73E^5 + 2486E^4 - 31804E^3 + 188280E^2 - 506048E + 487104$
$B = 251E^5 - 10452E^4 + 156202E^3 - 1041992E^2 + 3055168E - 3125328$
$C = \frac{337}{2}E^3 - \frac{18363}{4}E^2 + 38740E - 102390$

Table 5.8: Spectrum of highest weight states with $D_0 \leq 8.5$ in the dynamic $\mathfrak{su}(2|3)$ spin chain. Please refer to the end of Sec. 5.6.2 for explanations.

to three-loops and parity P. A pair of degenerate states with opposite parity is labelled by $P = \pm$. For convenience we have indicated the shortening conditions relevant for the $\mathfrak{su}(2|3)$ representations: Half-BPS multiplets and multiplets at the unitarity bound (which split at $g = 0$) are labelled by † and *, respectively. For $s = 0$ some of the components are in the $\mathfrak{su}(2)$ subsector, such multiplets are indicated by $^\bullet$.

Generically, the one-loop energies are not fractional numbers, but solutions to some algebraic equations. We refrain from solving them (numerically), but instead give the equations. In the table such states are indicated as polynomials $X_{0,2,4}(E)$ of degree $k-1$. The energies are obtained as solutions to the equation

$$E^k = X_0(E) + g^2 X_2(E) + g^4 X_4(E) + \mathcal{O}(g^6), \tag{5.94}$$

see also Sec. 3.5.1. The scaling dimension of the corresponding gauge theory states are given by $D(g) = D_0 + g^2 E(g)$.

For example, the three-loop planar scaling dimension of the Konishi operator $\mathcal{K} = \eta^{mn} \operatorname{Tr} \Phi_m \Phi_n$ introduced in Sec. 2.1.1 is given by (see Fig. 5.8, line 3 corresponds to a descendant of $\mathcal{K}$)

$$D = 2 + 6g^2 - 12g^4 + 42g^6 + \mathcal{O}(g^8) = 2 + \frac{3g^2_{\text{YM}}N}{4\pi^2} - \frac{3g^2_{\text{YM}}N}{16\pi^2} + \frac{21g^2_{\text{YM}}N}{256\pi^2} + \dots . \tag{5.95}$$

The two-loop result was computed in [42] and the three-loop coefficient was first conjectured in [38]. It was later derived in [51] using the methods described in the current chapter. This result was recently confirmed by independent arguments based on extracting the $\mathcal{N} = 4$ SYM anomalous dimensions of twist-two operators (c.f. Sec. 3.5.2) from the exact QCD result. The three-loop QCD result became available after an impressive, full-fledged and rigorous field theoretic computation by Moch, Vermaseren and Vogt [127]. To generalise to $\mathcal{N} = 4$ SYM, it was observed that in maximally supersymmetric gauge theory only terms of 'highest transcendentality' seem to arise. Here, terms similar to $\zeta(k)$ have transcendentality k and an ℓ-loop anomalous dimension in $\mathcal{N} = 4$ SYM should have transcendentality $2\ell-1$. Even more remarkably, for purely gluonic amplitudes, the contributions of highest transcendentality appear to independent of the matter content. If true, one can truncate to highest transcendentality [128] to obtain the anomalous dimensions of twist-two operators from QCD. The conjecture of [128] for the lowest twist-two operator, which is part of the Konishi multiplet, *agrees* with the result (5.95) in a spectacular fashion.

5.6.3 Two Excitations

We can use our above results to find the energy of two-excitation states up to three-loops. In this subsector they are represented by the highest weight modules with Dynkin labels $[0; 1; 0, J-2]$ and length $L = J+1$. All 'flavours' of two-excitation states are part of the same multiplet, c.f. Sec. 3.5.4, and it is convenient to use a descendant in the $\mathfrak{su}(2)$ subsector as in Sec. 3.6

$$\mathcal{E}^J_p = \operatorname{Tr} \phi \, \mathcal{Z}^p \, \phi \, \mathcal{Z}^{J-p}. \tag{5.96}$$

The action of the one-loop Hamiltonian was found in (3.105)

$$\mathcal{H}_0 \, \mathcal{E}^J_p = -2\delta_{p\neq J} \, \mathcal{E}^J_{p+1} + 2(\delta_{p\neq J} + \delta_{p\neq 0}) \, \mathcal{E}^J_p - 2\delta_{p\neq 0} \, \mathcal{E}^J_{p-1} \tag{5.97}$$

and the exact eigenstates are given in (3.87)

$$\mathcal{O}^J_{0,n} = \frac{1}{J+1}\sum_{p=0}^{J} \cos\frac{\pi n(2p+1)}{J+1}\,\mathcal{E}^J_p \tag{5.98}$$

with the exact one-loop energy (3.86)

$$E^J_{0,n} = 8\sin^2\frac{\pi n}{J+1}\,. \tag{5.99}$$

Let us state the inverse transformation of the discrete cosine transform (5.98)

$$\mathcal{E}^J_p = \mathcal{O}^J_{0,0} + 2\sum_{n=1}^{[J/2]} \cos\left(\frac{\pi n(2p+1)}{J+1}\right)\,\mathcal{O}^J_{0,n}. \tag{5.100}$$

We act with the two-loop Hamiltonian and find that it mostly equals the square of the one-loop Hamiltonian

$$\mathcal{H}_2\,\mathcal{E}^J_p = -\tfrac{1}{4}\mathcal{H}_0^2\,\mathcal{E}^J_p + V_2\,\mathcal{E}^J_p, \tag{5.101}$$

up to a contact-interaction of the two excitations

$$V_2\,\mathcal{E}^J_p = (\delta_{p,0}+\delta_{p,J}-\delta_{p,1}-\delta_{p,J-1})\big(\mathcal{E}^J_1 - \mathcal{E}^J_0\big). \tag{5.102}$$

We now face the problem that the states $\mathcal{O}^J_{0,n}$ are no longer eigenstates of $\mathcal{H}_2$, since $\mathcal{H}_0$ and V_2 do not commute. We find

$$V_2\mathcal{O}^J_{0,n} = -\frac{64}{J+1}\sin^2\frac{\pi n}{J+1}\cos\frac{\pi n}{J+1}\sum_{n'=1}^{[J/2]}\sin^2\frac{\pi n'}{J+1}\cos\frac{\pi n'}{J+1}\,\mathcal{O}^J_{0,n'}. \tag{5.103}$$

However, we can treat $\mathcal{H}_2$ as a perturbation and thus find that the two-loop part of the planar anomalous dimension is the diagonal ($m = m'$) piece of $\mathcal{H}_2$. We obtain the following two-loop energy shift

$$E^J_{2,n} = 64\,\sin^4\frac{\pi n}{J+1}\left(-\frac{1}{4} - \frac{\cos^2\frac{\pi n}{J+1}}{J+1}\right). \tag{5.104}$$

Furthermore, using standard perturbation theory, we can also find the perturbative correction to the eigenstates: They involve the coupling constant dependent redefinition

$$\mathcal{O}^J_n = \mathcal{O}^J_{0,n} + g^2\mathcal{O}^J_{2,n} + \dots \tag{5.105}$$

with

$$\mathcal{O}^J_{2,n} = -\frac{64}{J+1}\sum_{\substack{n'=1\\ n'\neq n}}^{[J/2]} \frac{\sin^2\frac{\pi n}{J+1}\cos\frac{\pi n}{J+1}\sin^2\frac{\pi n'}{J+1}\cos\frac{\pi n'}{J+1}}{\sin^2\frac{\pi n}{J+1}-\sin^2\frac{\pi n'}{J+1}}\,\mathcal{O}^J_{0,n'}. \tag{5.106}$$

This mixing of modes is a complicating feature that we can expect at each further quantum loop order; remarkably, it is absent in the large J (BMN) limit.

We move on to three-loops and find that the result agrees with a general formula

$$D_n^J = D_0 + g^2 E = J + 2 + \sum_{\ell=1}^{\infty} \left(8g^2 \sin^2 \frac{\pi n}{J+1}\right)^{\ell} \left(c_\ell + \sum_{k,l=1}^{\ell-1} c_{\ell,k,l} \frac{\cos^{2l} \frac{\pi n}{J+1}}{(J+1)^k}\right) \tag{5.107}$$

with the coefficients up to three-loops given by [38]

$$c_1 = 1, \qquad c_2 = -\tfrac{1}{4}, \quad c_{2,1,1} = -1, \qquad c_3 = \tfrac{1}{8}, \quad c_{3,k,l} = \begin{pmatrix} +\frac{3}{4} & +\frac{1}{2} \\ -\frac{3}{4} & +\frac{5}{2} \end{pmatrix}. \tag{5.108}$$

As expected, the formula reproduces the energies of all states $[0;1;0,J-2]$ in Tab. 5.8.

The coefficients c_ℓ agree with the prediction of string theory on plane-waves (3.99,3.98)

$$D_n^J = J + 2\sqrt{1+\lambda' n^2} + \mathcal{O}(J^{-1}), \qquad \lambda' = 8\pi^2 g^2 / J^2. \tag{5.109}$$

This is a non-trivial result: Although we have only made use of the qualitative BMN limit, the quantitative BMN energy formula seems to be the outcome.

We can also compare our result to string theory on a near plane-wave background [26,88]. This corresponds to an expansion of the results in powers of $1/J$. Let us expand our result (5.107,5.108) to first order

$$\begin{aligned} D_n^J &= J + 2 + \sum_{\ell=1}^{\infty} \left(\lambda' n^2\right)^{\ell} \left(c_\ell + J^{-1}\left(-2\ell c_\ell + \textstyle\sum_{l=1}^{\ell-1} c_{\ell,1,l}\right) + \mathcal{O}(J^{-2})\right) \\ &= J + 2 + \left(\lambda' n^2\right)\left(1 - 2J^{-1}\right) + \left(\lambda' n^2\right)^2 \left(-\tfrac{1}{4} + 0J^{-1}\right) + \left(\lambda' n^2\right)^3 \left(\tfrac{1}{8} + \tfrac{1}{2} J^{-1}\right) + \dots . \end{aligned} \tag{5.110}$$

This is to be compared to the near plane-wave string theory result [88] (the comparison takes place at level 4 of the multiplet)

$$\begin{aligned} D_n^J &= J + 2\sqrt{1+\lambda' n^2} - 2\lambda' n^2\, J^{-1} + \mathcal{O}(J^{-2}) \\ &= J + 2 + \left(\lambda' n^2\right)\left(1 - 2J^{-1}\right) + \left(\lambda' n^2\right)^2 \left(-\tfrac{1}{4} + 0J^{-1}\right) + \left(\lambda' n^2\right)^3 \left(\tfrac{1}{8} + 0J^{-1}\right) + \dots . \end{aligned} \tag{5.111}$$

Structurally, both expression are equivalent and all coefficients agree except single one at $\mathcal{O}(\lambda'^3 J^{-1})$. The same kind of disagreement was also observed for three excitations [129,102] and arbitrarily many of scalar type [130,131]. We will see further evidence of a disagreement between string theory and gauge theory starting at three loops in Sec. 6.5; we will discuss this issue there.

5.6.4 An Eighth-BPS state

Let us take a peek at non-planar physics within this sector: The lowest-dimensional eighth-BPS state is expected to be a triple-trace state with weight $w = (6;0,0;0,0,4;0,6)$. Using the non-planar, one-loop Hamiltonian we find this protected state

$$\begin{aligned} \mathcal{O}_{1/8-\mathrm{BPS}} = \varepsilon^{abc}\varepsilon^{def} \big[& N(N^2-3)\,\mathrm{Tr}\,\phi_a\phi_d\,\mathrm{Tr}\,\phi_b\phi_e\,\mathrm{Tr}\,\phi_c\phi_f \\ & + 6(N^2-1)\,\mathrm{Tr}\,\phi_a\phi_d\,\mathrm{Tr}\,\phi_b\phi_c\phi_e\phi_f - 12N\,\mathrm{Tr}\,\phi_a\phi_b\phi_c\phi_d\phi_e\phi_f \\ & + 8N\,\mathrm{Tr}\,\phi_a\phi_d\phi_b\phi_e\phi_c\phi_f + 4\,\mathrm{Tr}\,\phi_a\phi_b\phi_c\,\mathrm{Tr}\,\phi_d\phi_e\phi_f \big]. \end{aligned} \tag{5.112}$$

It is annihilated by the operators

$$\varepsilon_{abc}[\check{\phi}^b, \check{\phi}^c], \qquad \varepsilon_{abc}\varepsilon^{\alpha\beta}\,\mathrm{Tr}\,\psi_\alpha[\check{\phi}^a, [\check{\phi}^b, [\check{\phi}^c, \psi_\beta]]], \tag{5.113}$$

which are part of the non-planar generalisations of $\mathfrak{S}_1, \mathcal{H}_0$ and $\mathcal{H}_1$. This implies that the state is protected (at least at one-loop).

It would be interesting to generalise some of the results of this chapter to include non-planar corrections. At two-loops this might indeed be feasible as there are only few non-planar graphs.

Chapter 6

Higher-Loop Integrability

In this final chapter we would like to put together the results of the previous two chapters. In Ch. 5 we have seen how to make use of the interacting algebra to find higher-loop corrections. In Ch. 4 we have investigated the integrability of planar $\mathcal{N} = 4$ SYM at the one-loop level and demonstrated its usefulness. An obvious question is whether the integrable structures persist even at higher-loops. This will be the subject of the current chapter.

To start off, we shall introduce higher-loop integrability and argue that $\mathcal{N} = 4$ gauge theory (or, more precisely, the subsector discussed in Ch. 5) is indeed integrable at higher-loops. The main part of the chapter is devoted to the investigation of an integrable model in the $\mathfrak{su}(2)$ subsector. By making some assumptions on the form of interactions, we will find that this model is *uniquely* determined at five-loops and, excitingly, reproduces the plane wave energy formula. What is more, we find the corresponding Bethe ansatz to compute the spectrum at an arbitrary order in perturbation theory!

6.1 Higher-Loop Spin Chains

First of all, we would like to describe the notion of integrability for spin chains at higher-loops. We then go on by explaining why we believe that these structures should apply to higher-loop $\mathcal{N} = 4$ SYM. Finally we will investigate the scaling behaviour of charges in the thermodynamic limit and describe how they can be defined canonically.

6.1.1 Aspects of Higher-Loop Integrability

To describe higher-loop corrections to scaling dimensions we have promoted the Hamiltonian $\mathcal{H} = \mathcal{H}_0$ of Ch. 3 to a function of the coupling constant $\mathcal{H}(g)$ in Ch. 5. At zero coupling one recovers the one-loop Hamiltonian $\mathcal{H}(0) = \mathcal{H}_0$. For higher-loop integrability we do the same and promote the charges $\mathcal{Q}_r = \mathcal{Q}_{r,0}$ to functions $\mathcal{Q}_r(g)$ with $\mathcal{Q}_r(0) = \mathcal{Q}_{r,0}$. A Hamiltonian $\mathcal{H}(g) = \mathcal{Q}_2(g)$ is considered to be higher-loop integrable if there exist conserved charges $\mathcal{Q}_r(g)$ with

$$\text{`}\textit{higher-loop integrability}\text{'}: \quad [\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = [\mathfrak{J}(g), \mathcal{Q}_r(g)] = 0. \tag{6.1}$$

In the case of $\mathcal{N} = 4$ SYM, the symmetry algebra is $\mathfrak{psu}(2,2|4) \times \mathfrak{u}(1)$. The $\mathfrak{u}(1)$ factor corresponds to the anomalous piece of the dilatation operator $\delta\mathfrak{D}(g)$ which is conserved

in two-point functions. We will argue that $\mathcal{N} = 4$ SYM in the planar limit might be integrable for arbitrary values of the coupling constant g. The higher charges $\mathcal{Q}_r(g)$ form an abelian algebra which enlarges the symmetry in the planar limit

$$\mathfrak{psu}(2,2|4) \times \mathfrak{u}(1) \longrightarrow \mathfrak{psu}(2,2|4) \times \mathfrak{u}(1)^\infty. \tag{6.2}$$

Here, the $\mathfrak{u}(1)$ anomalous dimension is absorbed into the set of integrable charges by the identification $\delta\mathfrak{D}(g) = g^2\mathcal{H}(g) = g^2\mathcal{Q}_2(g)$.

An obstacle to the investigation of higher-loop integrability is that it apparently cannot be described with the formalism introduced in Sec. 4.1. The reason is that higher-loop interactions are between several nearby fields, whereas an ordinary integrable spin chain involves nearest-neighbour interactions only.[1] What is more, higher-loop interactions can change the length of the spin chain giving rise to completely new structures. In order to construct a higher-loop integrable spin chain, the R-matrix of Sec. 4.1 appears to be not suitable. For instance, it describes the scattering of two elementary spins and it is not yet understood how to generalise it to the interactions that occur at higher-loops. Consequently, we cannot even attempt to prove the associated Yang-Baxter equation, which would make higher-loop integrability manifest. Finally, there is not yet a constructive means to obtain higher charges.

In Sec. 4.1.3 we have shown how to extract the charges $\mathcal{Q}_{r,0}$ from the transfer matrix $\mathcal{T}(u,0)$. Here, we can take the opposite direction and package all charges into a transfer matrix

$$\mathcal{T}(u,g) = \exp i \sum_{r=2}^{\infty} u^{r-1}\, \mathcal{Q}_r(g), \tag{6.3}$$

which should satisfy the equivalent of (4.11)

$$[\mathfrak{J}(g), \mathcal{T}(u,g)] = [\mathcal{T}(u,g), \mathcal{T}(v,g)] = 0. \tag{6.4}$$

An interpretation of the transfer matrix might be as follows: An ordinary spin chain can be considered to be a composite object assembled from individual fields/spins/particles. The transfer matrix is derived from scattering processes of the individual particles. At higher-loops it is not known how to make sense of parton scattering processes. Nevertheless, it might still be useful to consider the transfer matrix as a Wilson loop around the composite object. The identification of particles with fields/spins, however, would be lost for two reasons: Interactions take place between more than two fields and the number of fields is not even preserved. Instead, the Bethe ansatz (c.f. Sec. 4.3) offers an alternative notion of particles: The composite object has some vacuum state and its excitations correspond to particles. The interactions of such excitation particles are pairwise (right-hand side of the Bethe equations) and can therefore be integrable. This picture may be generalised to higher-loops without complications.

6.1.2 The Local Charges

We do not know how to obtain the higher charges $\mathcal{Q}_r$ explicitly and are therefore forced to construct them by hand such that they satisfy (6.1). We cannot expect this

[1] Although non-nearest neighbour interactions of several spins are included in the tower of higher charges $\mathcal{Q}_r$, these cannot be related to the higher orders of the Hamiltonian because the charges commute among themselves, whereas the $\mathcal{H}_k$'s in general do not.

to be feasible for finite values of g and restrict ourselves to a perturbative treatment. In fact, we know (parts of) $\mathcal{H}(g)$ only up to order g^4 (three-loops) and we may construct the charges $\mathcal{Q}_r(g)$ only up to the same order. The algebra need not be satisfied exactly, but only up to terms of higher order in perturbation theory

$$[\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = [\mathfrak{J}(g), \mathcal{Q}_r(g)] = \mathcal{O}(g^{2\ell}). \tag{6.5}$$

For all-loop integrability the remaining higher-order terms would have to be cancelled by higher-order corrections to the charges. We now expand the charges in powers of the coupling constant g

$$\mathcal{Q}_r(g) = \sum_{k=0}^{\infty} g^k \, \mathcal{Q}_{r,k}. \tag{6.6}$$

The integrability condition (6.1) translates to the statement of *perturbative integrability*

$$\text{'}\textit{perturbative integrability}\text{'}: \quad \sum_{k=0}^{l} [\mathcal{Q}_{r,k}, \mathcal{Q}_{s,l-k}] = \sum_{k=0}^{l} [\mathfrak{J}_k, \mathcal{Q}_{r,l-k}] = 0. \tag{6.7}$$

One important consideration for the construction of charges is their representation as an interaction acting on the spin chain, see Sec. 2.6.2. In Sec. 4.1.3 we have learned that the local charges $\mathcal{Q}_{r,0}$ act on r adjacent spins. In other words, the charge $\mathcal{Q}_{r,0}$ has $2r$ legs, r incoming and r outgoing ones. Although the higher-loop form of interactions described in Sec. 2.6.2 applies to quantities that appear in correlators, it seems natural to generalise it to the charges. An order g^k correction to a generator $\mathfrak{J}$ involves $k+2$ legs and we conclude that for each power of g we should have one leg. In total, a charge $\mathcal{Q}_{r,k}$ should have $2r+k$ legs

$$\mathcal{Q}_{r,k} \sim \left\{ {\mathcal{A}_1 \ldots \mathcal{A}_{E_\mathrm{i}} \atop \mathcal{B}_1 \ldots \mathcal{B}_{E_\mathrm{o}}} \right\}, \quad \text{with} \quad E_\mathrm{i} + E_\mathrm{o} = 2r + k. \tag{6.8}$$

It is also natural to assume that the charges have a definite parity, the same as at leading order

$$\mathfrak{p}\, \mathcal{Q}_r(g)\, \mathfrak{p}^{-1} = (-1)^r \mathcal{Q}_r(g). \tag{6.9}$$

Finally, the charges preserve the classical dimension

$$[\mathfrak{D}_0, \mathcal{Q}_r(g)] = 0 \tag{6.10}$$

when we identify the second charge with the anomalous dimension $\delta D(g) = g^2 \mathcal{Q}_2(g)$, because $\mathfrak{D}(g)$ and $\mathcal{Q}_2(g)$ commute with all charges and so does their difference.

Let us now comment on the role of the Hamiltonian $\mathcal{H}(g)$. On the one hand, it belongs to the symmetry algebra $\mathfrak{psu}(2,2|4)$ when combined with $\mathfrak{D}_0$

$$\mathfrak{D}_0 + g^2 \mathcal{H}(g) = \mathfrak{D}_0 + \delta\mathfrak{D}(g) = \mathfrak{D}(g) \in \mathfrak{psu}(2,2|4). \tag{6.11}$$

On the other hand, $\mathcal{H}(g)$ is also one of the integrable charges. As such it is a generator of the abelian algebra $\mathfrak{u}(1)^\infty$ defined by (6.1)

$$\mathcal{H}(g) = \mathcal{Q}_2(g) \in \mathfrak{u}(1)^\infty. \tag{6.12}$$

This is somewhat different from the situation at one-loop where the symmetry algebra is taken strictly at $g = 0$. The one-loop anomalous dilatation generator is an independent object and belongs only to the abelian algebra $\mathfrak{u}(1)^\infty$.

6.1.3 Parity Pairs

As we have seen in Sec. 4.1.4, the integrable structure gives rise to pairs of states with degenerate energies and opposite parities. We have proved integrability only at one-loop, but a closer look at Tab. 5.8 reveals that the degeneracy of all one-loop pairs ($^{\pm}$) is preserved even at three-loops![2] This is so for the pairs of the $\mathfrak{su}(2)$ sector ($^{\bullet}$) [38], for pairs at the unitarity bound (*), but also, and most importantly, for pairs away from the unitarity bound (unmarked). As discussed at the end of Sec. 5.1.4 all states of such a multiplet are superpositions of states of different lengths. This is interesting because it shows that also for truly *dynamic spin chains* with a fluctuating number of sites, integrability is an option.

We do not know how to use the R-matrix formalism beyond one-loop, if this is possible at all. Therefore we cannot rigorously prove higher-loop integrability by means of a Yang-Baxter equation. One might construct several of the higher charges explicitly at higher-loops and thus make integrability very plausible, but this would not constitute a proof. Instead, we shall be satisfied by demonstrating the preservation of degenerate pairs at three-loops. This is certainly a necessary condition for integrability, but at first sight it appears not to imply the existence of commuting charges. Nevertheless, there are some indications that pairing is indeed sufficient to guarantee integrability. First of all, a systematic pairing is most naturally explained by the following set of identities

$$[\mathcal{H}(g), \mathcal{Q}(g)] = [\mathfrak{p}, \mathcal{H}(g)] = \{\mathfrak{p}, \mathcal{Q}(g)\} = 0 \tag{6.13}$$

among the Hamiltonian $\mathcal{H}(g)$, parity $\mathfrak{p}$ and some charge $\mathcal{Q}(g)$. The investigations for the model in Sec. 6.2 have shown that indeed we can construct a charge $\mathcal{Q}(g) = \mathcal{Q}_3(g)$ whenever the spectrum has sufficiently many pairs without imposing further constraints on the Hamiltonian $\mathcal{H}(g)$. More remarkably, it always turned out to be possible to construct conserved charges $\mathcal{Q}_r(g)$ as soon as the Hamiltonian pairs up states. The charges do not only commute with the Hamiltonian, but also among themselves, i.e. they automatically satisfy (6.1). This parallels earlier observations [80] that it appears close to impossible to construct systems with $[\mathcal{Q}_2, \mathcal{Q}_3] = 0$ which do not have arbitrarily many other commuting charges $\mathcal{Q}_r$, i.e. which are not integrable.

It is therefore likely that planar $\mathcal{N} = 4$ SYM (at least) in the $\mathfrak{su}(2|3)$ subsector and (at least) at three-loops is integrable. In agreement with the findings of [79] we conclude that integrability appears to be a consequence of field theory combined with symmetry and does not depend on the specific model very much. It strongly supports the idea of all-loop integrability in planar $\mathcal{N} = 4$ SYM. What is more, the dynamic aspects of higher-loop spin chains appear to be no obstruction. Let us emphasise, though, that a rigorous proof of higher-loop integrability remains a challenge.

6.1.4 The $\mathfrak{su}(2|3)$ Sector Revisited

At this point we can reinvestigate the undetermined coefficients of the $\mathfrak{su}(2|3)$ spin chain at three-loops, c.f. Sec. 5.5. By imposing

$$[\mathfrak{Q}(g), \mathcal{H}(g)] = [\mathfrak{S}(g), \mathcal{H}(g)] = \mathcal{O}(g^6) \tag{6.14}$$

[2]Let us emphasise that, even if the table was computed assuming BMN scaling to fix the values of $\alpha_1, \alpha_3, \sigma_1, \sigma_2$, the pairing holds for generic values.

we found that $\mathcal{H}_4$ depends on four relevant coefficients $\sigma_{1,2,3,4}$. The coefficients $\sigma_{3,4}$ multiply invalid structures, whereas σ_1 corresponds to a redefinition of the coupling constant. The remaining coefficient σ_2 multiplies $\mathcal{Q}_{4,0}$ which is structurally equivalent to $\mathcal{H}_4$ and satisfies $[\mathfrak{J}(g), \mathcal{Q}_4(g)] = 0$ as well. In fact, by imposing $[\mathfrak{Q}(g), X(g)] = [\mathfrak{S}(g), X(g)] = 0$ we do not only find $\mathcal{H}(g)$, but also all the other even generators $\mathcal{Q}_r(g)$ of the abelian algebra of integrable charges $\mathfrak{u}(1)^\infty$. Thus σ_2 corresponds to the transformation

$$\mathcal{H}(g) \mapsto \mathcal{H}(g) + \sigma_2\, g^4\, \mathcal{Q}_4(g), \tag{6.15}$$

which has no influence on (6.14) due to $[\mathfrak{J}(g), \mathcal{Q}_r(g)] = 0$. This degree of freedom may be fixed by considering the anticommutator of supercharges (5.9) at order g^6

$$\sum_{k=0}^{6} \left\{ (\mathfrak{S}_k)^\alpha{}_a, (\mathfrak{Q}_{6-k})^b{}_\beta \right\} = \tfrac{1}{2}\delta^b_a \delta^\alpha_\beta \mathcal{H}_4. \tag{6.16}$$

Unfortunately, this equation involves $\mathfrak{Q}_6, \mathfrak{S}_6$ which are part of a four-loop calculation and out of reach here. We believe that (6.16) will force the coefficient σ_2 to vanish, and in conclusion all corrections up to three loops would be determined uniquely (up to a redefinition of the coupling constant). At higher loops this picture is expected to continue: While $[\mathfrak{Q}(g), X(g)] = [\mathfrak{S}(g), X(g)] = 0$ determines the even elements $\mathcal{Q}_r$ of $\mathfrak{u}(1)^\infty$, the anticommutator $\{\mathfrak{S}(g), \mathfrak{Q}(g)\}$ yields the one element $\mathcal{Q}_2(g) = \mathcal{H}(g)$ which is also associated to $\mathfrak{su}(2|3)$ as the generator $\mathfrak{D}_0 + \frac{3}{2}g^2\mathcal{H}(g)$. It is reasonable to assume that σ_2 may alternatively be fixed by the non-planar algebra where conservation of the charge $\mathcal{Q}_4$ is lost.

Let us comment on the effect of integrability on the degrees of freedom for similarity transformations. Similarity transformations are symmetries of the algebra relations and thus give rise to undetermined coefficients in the construction of the most general deformation of generators. In Sec. 5.5 we argued that the coefficients which arise for the generators $\mathfrak{J}_k$ are in one-to-one correspondence with the structures that can be used for the construction of $\mathcal{H}_{k-2}$, see also Tab. 5.4. However there are also some similarity transformations which do not change the generators. These are generated by invariant operators such as $\mathcal{H}(g)$ and, to some extent, the length $\mathcal{L}$. In an integrable system there are more invariant operators: The charges $\mathcal{Q}_r(g)$. Only the even charges $\mathcal{Q}_r$ are compatible with the structure of $\mathcal{H}_{k-2}$. For example, the fourth charge will appear as a symmetry in Tab. 5.4 starting at sixth order (four-loops).

6.1.5 The Thermodynamic Limit

The thermodynamic limit is the limit in which the length of the spin chain L as well as the number of excitations is taken to infinity while focusing on the the low-energy spectrum, c.f. Sec. 4.5. In this limit it was observed that the r-th charge $Q_{r,0}$ at one-loop scales as L^{1-r} [68]. Here, we would like to generalise the thermodynamic limit to higher-loops. From the investigation of the closely related BMN limit (c.f. Sec. 3.6) as well as from classical spinning strings (c.f. Sec. 4.6), we infer that each power of the coupling constant g should be accompanied by one power of $1/L$. It is common belief that this scaling behaviour holds for perturbative gauge theory, but it is clearly not a firm fact. We shall assume its validity for several reasons: Firstly, it was not only confirmed at

one-loop, but also at two-loops [25, 38]. It is a nice structure and conceptually it would be somewhat disappointing if broken at some higher loop order. Secondly, the AdS/CFT correspondence seems to suggest it. Thirdly, it will allow us to define charges uniquely, see below, and arrive at a *unique* result in Sec. 6.2.

In conclusion, the scaling of charges in the thermodynamic limit is given by[3]

$$\tilde{\mathcal{Q}}_{r,k} = \lim_{L\to\infty} L^{k+r-1}\mathcal{Q}_{r,k}, \qquad \tilde{\mathcal{T}}(\tilde{u},\tilde{g}) = \lim_{L\to\infty} \mathcal{T}(L\tilde{u}, L\tilde{g}). \tag{6.17}$$

6.1.6 Canonical Charges

As the charges form an abelian algebra, one can replace $\mathcal{Q}_r(g)$ by some polynomial $\mathcal{Q}'_r(g)$ in the charges without changing the algebra. We will now prove the uniqueness of the 'canonical' set of charges with good structural (c.f. Sec. 6.1.2) and scaling properties (c.f. Sec. 6.1.5). We will start by assuming that the charges $\mathcal{Q}_r(g)$ are canonical and show that we cannot change them without spoiling the one of the properties.

The charges can be written as a local interaction $\mathcal{Q}_{r,k} = \sum_{p=1}^{L} \mathcal{Q}_{r,k,p\ldots}$ with the local charge density $\mathcal{Q}_{r,k,p\ldots}$. A generic polynomial transformation would therefore make $\mathcal{Q}'_r$ multi-local in general. To preserve locality we are restricted to linear transformations which are generated by

$$\mathcal{Q}'_r(g) = \mathcal{Q}_r(g) + \alpha_{r,s,l}\, g^{2l}\, \mathcal{Q}_{r+2s}(g). \tag{6.18}$$

We now find two constraints on l and s: On the one hand, there is the structural constraint (6.8) which tells that $\mathcal{Q}'_{r,k}$ can only have $2r+k$ legs. This requires

$$l \geq 2s. \tag{6.19}$$

On the other hand, a correct scaling in the thermodynamic limit (6.17) requires that $\mathcal{Q}'_{r,k}$ scales as $\mathcal{O}(L^{1-k-r})$. In order not to spoil scaling, we need

$$l \leq s. \tag{6.20}$$

Together, these two constraints imply $l \leq 0$, but we do not allow negative powers of g. In total we get $l = s = 0$ or, in other words, $\mathcal{Q}_r$ can only be rescaled by a constant $\alpha_{r,0,0}$. Finally, this constant can be fixed by using the canonical transfer matrix $\mathcal{T}(u,0)$ of the one-loop spin-chain, c.f. Sec. 4.1.3.

In conclusion, the canonical definition for $\mathcal{Q}_r(g)$ is unique (if it exists). As $\mathcal{H}(g)$ is subject to the same constraints as $\mathcal{Q}_2(g)$, both of them must be equal $\mathcal{H}(g) = \mathcal{Q}_2(g)$.

6.2 The $\mathfrak{su}(2)$ Sector at Higher-Loops

In this section we will construct a model for higher-loop anomalous dimensions in the $\mathfrak{su}(2)$ subsector of $\mathcal{N} = 4$ SYM. We will rely on three assumptions on the form of interactions: (*i*) Integrability, (*ii*) the thermodynamic limit and (*iii*) some constraints inspired by Feynman diagrammatics. Note that none of these assumptions should be taken

[3] In the BMN we would account for the finite number of excitations by the slightly modified definitions $\hat{\mathcal{Q}}_{r,k} = L^{k+r}\mathcal{Q}_{r,k}$ and $\hat{\mathcal{T}}(\tilde{u},\tilde{g}) = \mathcal{T}(L\tilde{u}, L\tilde{g})^L$

as a firm fact. Whether or not they are fully justified in (perturbative) $\mathcal{N}=4$ SYM will not be the subject of this chapter, but we believe that the model shares several features with higher-loop gauge theory and therefore deserves an investigation. Intriguingly, it will turn out to be *unique* up to (at least) five-loops and agree with the excitation energy formula in the BMN limit! At any rate, this makes it a very interesting model to consider in its own right. With some luck, the assumptions (*i-iii*) will turn out to be valid for $\mathcal{N}=4$ SYM and we have constructed the planar, five-loop[4] dilatation generator in the $\mathfrak{su}(2)$ subsector.

6.2.1 Interactions

In the $\mathfrak{su}(2)$ sector, the number of field sites is conserved. In particular, this implies that odd powers of g are not allowed. Furthermore, rotations are manifestly realised, the $\mathfrak{su}(2)$ generators do not receive radiative corrections. Therefore the interactions can only be of the form (see Sec. 2.6.2)

$$\left\{ \begin{smallmatrix} a_1 \dots a_E \\ a_{\pi(1)} \dots a_{\pi(E)} \end{smallmatrix} \right\} \tag{6.21}$$

with π some permutation of E elements.[5] Any permutation can be represented in terms of elementary permutations $\mathcal{P}_{p,p+1}$ of adjacent fields. A generic term will be written as

$$\{p_1, p_2, \dots\} = \sum_{p=1}^{L} \mathcal{P}_{p+p_1,p+p_1+1} \mathcal{P}_{p+p_2,p+p_2+1} \dots . \tag{6.22}$$

For example, in this notation the one-loop dilatation generator (3.75) is given by

$$\mathcal{H}_0 = \mathcal{Q}_{2,0} = \big(\{\} - \{1\}\big). \tag{6.23}$$

This notation is useful due to the nature of maximal scalar diagrams as discussed at the end of Sec. 2.5.3: An interaction of scalars at ℓ loops with the maximal number of $2+2\ell$ legs can be composed from ℓ crossings of scalar lines. In the planar limit, the crossings correspond to elementary permutations and at ℓ-loops there should be no more than ℓ permutations. In field theory this is a feature of maximal diagrams, but here we will assume the pattern to hold in general. Furthermore, in Sec. 4.1.3 we have learned that the r-th charge at leading (one-loop) order can be constructed from $r-1$ copies of the Hamiltonian density which, in this case (6.23), is essentially an elementary permutation. We will therefore assume the contributions to the charges to be of the form

$$\mathcal{Q}_{r,2\ell-2} \sim \{p_1, \dots, p_m\} \quad \text{with } m \leq r+\ell-2 \text{ and } 1 \leq p_i \leq r+\ell-2. \tag{6.24}$$

Finally, the even (odd) charges should be parity even (odd) and (anti)symmetric.[6] Parity acts on the interactions as

$$\mathfrak{p}\,\{p_1, \dots, p_m\}\,\mathfrak{p}^{-1} = \{-p_1, \dots, -p_m\}, \tag{6.25}$$

[4] We disregard wrapping interactions, see Sec. 2.6.3, i.e. this applies only to states of length $L \geq 6$.

[5] Note that we will consider the states to be sufficiently long and drop wrapping interactions (c.f. Sec. 2.6.3). We will comment on the role of wrappings in Sec. 6.5.4.

[6] In fact, the Hamiltonian $\mathcal{H}(g)$ and charges $\mathcal{Q}_r(g)$ should be hermitian. The coefficients of the interaction structures should therefore be real (imaginary) for even (odd) r. Reality of the Hamiltonian follows from the equivalence of the Hamiltonian for the $\mathfrak{su}(2)$ sector and its conjugate.

whereas symmetry acts as

$$\{p_1,\dots,p_m\}^{\intercal} = \{p_m,\dots,p_1\}. \tag{6.26}$$

Symmetry will ensure that the eigenvalues of the charges are real.

Note that the interaction symbols are subject to several identities which can be used to bring them into some normal form. One identity involves a repeated elementary permutation

$$\{\dots,p,p,\dots\} = \{\dots,\dots\}. \tag{6.27}$$

Another obvious identity

$$\{\dots,p,p',\dots\} = \{\dots,p',p\dots\} \quad \text{if} \quad |p-p'|\geq 2 \tag{6.28}$$

allows to commute two non-overlapping elementary permutations. A third identity is due to gauge invariance or cyclic invariance of interactions

$$\{p_1+p',\dots,p_m+p'\} = \{p_1,\dots,p_m\}. \tag{6.29}$$

Finally, the spin at each site can take two different values and we cannot antisymmetrise more than two spins. This leads to the the identity

$$\begin{aligned}\{\dots,p,p\pm 1,p,\dots\} = \{\dots,\dots\} - \{\dots,p,\dots\} - \{\dots,p\pm 1,\dots\}\\ + \{\dots,p,p\pm 1,\dots\} + \{\dots,p\pm 1,p,\dots\}.\end{aligned} \tag{6.30}$$

6.2.2 The Higher Charges

We would now like to construct some of the higher charges for the model in Ch. 5 explicitly. Let us start by writing down the Hamiltonian in Tab. 5.2,5.6, restricted to the $\mathfrak{su}(2)$ subsector in the notation introduced above

$$\begin{aligned}\mathcal{H}_0 &= \{\} - \{1\},\\ \mathcal{H}_2 &= (-2+2\alpha_3)\{\} + (3-2\alpha_3)\{1\} - \tfrac{1}{2}\big(\{1,2\}+\{2,1\}\big),\\ \mathcal{H}_4 &= \big(\tfrac{15}{2} - 8\alpha_3 + \sigma_1 - \tfrac{2}{3}\sigma_2\big)\{\} + \big(-13+12\alpha_3-\sigma_1+\tfrac{4}{3}\sigma_2\big)\{1\} + \tfrac{1}{2}\{1,3\}\\ &\quad + \big(3-2\alpha_3-\tfrac{1}{3}\sigma_2\big)\big(\{1,2\}+\{2,1\}\big) + \big(-\tfrac{1}{2}+\tfrac{1}{3}\sigma_2\big)\big(\{1,2,3\}+\{3,2,1\}\big)\\ &\quad + \big(-\tfrac{1}{3}\sigma_2 - i\xi_3\big)\{2,1,3\} + \big(-\tfrac{1}{3}\sigma_2+i\xi_3\big)\{1,3,2\}.\end{aligned} \tag{6.31}$$

Here, we should set $\sigma_1=\sigma_2=\alpha_3=0$ to obtain the correct scaling behaviour in the thermodynamic limit, see Sec. 5.6.1. Furthermore, the coefficient ξ_3 is related to a similarity transformation; it consequently does not affect scaling dimensions and we set it to zero.[7] According to Sec. 4.1.3, the leading order third charge is given by

$$\mathcal{Q}_{3,0} = \tfrac{i}{2}\big(\{1,2\}-\{2,1\}\big). \tag{6.32}$$

As discussed in Sec. 6.1.4, the leading fourth charge can be read off from $\mathcal{H}_4$ as the structure multiplied by σ_2

$$\begin{aligned}\mathcal{Q}_{4,0} = -\tfrac{2}{3}\{\} + \tfrac{4}{3}\{1\} - \tfrac{1}{3}(\{1,2\}+\{2,1\})\\ - \tfrac{1}{3}\big(\{1,3,2\}+\{2,1,3\}\big) + \tfrac{1}{3}\big(\{1,2,3\}+\{3,2,1\}\big).\end{aligned} \tag{6.33}$$

[7] Furthermore, the coefficients of interaction structures can be assumed to be real for the Hamiltonian.

Both of them commute with $\mathcal{H}_0$ and among themselves.

Let us now go ahead and compute the first correction to a higher charge. For $\mathcal{Q}_{3,2}$ the only suitable structures are $(\{1,2\}-\{2,1\})$ and $(\{1,2,3\}-\{3,2,1\})$. We demand that $\mathcal{Q}_3(g)$ commutes with $\mathcal{H}(g)$ in perturbation theory

$$[\mathcal{H}_0, \mathcal{Q}_{3,2}] + [\mathcal{H}_2, \mathcal{Q}_{3,0}] = 0 \tag{6.34}$$

and find the coefficient of $(\{1,2,3\}-\{3,2,1\})$ to be fixed to $i/2$. The coefficient of $(\{1,2\}-\{2,1\})$ cannot be determined because this structure is proportional to $\mathcal{Q}_{3,0}$ and commutes with $\mathcal{H}_0$ by construction. We can only fix it by demanding a correct scaling behaviour in the thermodynamic limit, c.f. Sec. 6.1.5,6.1.6 and obtain

$$\mathcal{Q}_{3,2} = -2i\big(\{1,2\}-\{2,1\}\big) + \tfrac{i}{2}\big(\{1,2,3\}-\{3,2,1\}\big). \tag{6.35}$$

We proceed in the same way to determine the integrable charges $\mathcal{Q}_3, \mathcal{Q}_4$ up to $\mathcal{O}(g^4)$ (three-loops). The unique solution with correct scaling in the thermodynamic limit is presented in Tab. 6.2.

6.2.3 Higher-Loop Construction

Here we would like to construct the most general (i) integrable higher-loop spin chain with (ii) the proposed scaling in the thermodynamic limit. For that purpose, we make the most general ansatz for the charges in terms of (iii) permutation symbols $\{\ldots\}$ multiplied by undetermined coefficients. We then demand that the charges mutually commute and have the right scaling behaviour in the thermodynamic limit. We will use the computer algebra system `Mathematica` to preform all necessary commutators and solve the arising sets of linear equations to determine the coefficients. Some of the methods used in the construction are given in App. E. Let us describe the details of our construction of the integrable model:[8]

- We make the ansatz that the charges $\mathcal{Q}_{r,2\ell-2}$ with r even (odd), have even (odd) parity and are (anti)symmetric. They consist of no more than $r+\ell-2$ elementary permutations ranging over $r+\ell-1$ adjacent sites, see Sec. 6.2.1.
- We compute the commutator of $\mathcal{H}$ and $\mathcal{Q}_3$ (up to five-loops). By demanding that it should vanish, we obtain a set of linear equations among the coefficients. We solve it for coefficients of $\mathcal{Q}_3$ as far as possible, but some equations relate coefficients of $\mathcal{Q}_2$ only among themselves.
- Alternatively, we may ignore the charge $\mathcal{Q}_3$ and only demand that all pairs in the spectrum of $\mathcal{H}$ remain degenerate at higher-loops. This yields the same set of constraints for the coefficients of $\mathcal{H}$ (up to four-loops).
- We then compute the commutator of $\mathcal{H}$ and $\mathcal{Q}_r$ (for $r=3,4$ up to $\ell=4$ and for $r=5,6$ up to $\ell=2$). This constrains most coefficients of $\mathcal{Q}_r$ and remarkably yields no new constraints for $\mathcal{H}$.

[8] We keep track of the number of free coefficients in Tab. 6.4. We shall also indicate in the text at what loop order ℓ the individual calculations have been performed.

$$
\begin{aligned}
\mathcal{H}_0 &= +\{\} - \{1\},\\
\mathcal{H}_2 &= -2\{\} + 3\{1\} - \tfrac{1}{2}\big(\{1,2\} + \{2,1\}\big),\\
\mathcal{H}_4 &= +\tfrac{15}{2}\{\} - 13\{1\} + \tfrac{1}{2}\{1,3\} + 3(\{1,2\} + \{2,1\}) - \tfrac{1}{2}\big(\{1,2,3\} + \{3,2,1\}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{3,0} &= +\tfrac{i}{2}\big(\{1,2\} - \{2,1\}\big),\\
\mathcal{Q}_{3,2} &= -2i\big(\{1,2\} - \{2,1\}\big) + \tfrac{i}{2}\big(\{1,2,3\} - \{3,2,1\}\big),\\
\mathcal{Q}_{3,4} &= +\tfrac{73i}{8}\big(\{1,2\} - \{2,1\}\big) - \tfrac{i}{4}\big(\{1,2,4\} + \{1,3,4\} - \{1,4,3\} - \{2,1,4\}\big)\\
&\quad - \tfrac{7i}{2}\big(\{1,2,3\} - \{3,2,1\}\big) - \tfrac{i}{8}\big(\{1,3,2,4\} - \{2,1,4,3\}\big)\\
&\quad - \tfrac{i}{8}\big(\{1,2,4,3\} - \{1,4,3,2\} + \{2,1,3,4\} - \{3,2,1,4\}\big)\\
&\quad + \tfrac{5i}{8}\big(\{1,2,3,4\} - \{4,3,2,1\}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{4,0} &= -\tfrac{2}{3}\{\} + \tfrac{4}{3}\{1\} - \tfrac{1}{3}(\{1,2\} + \{2,1\})\\
&\quad - \tfrac{1}{3}\big(\{1,3,2\} + \{2,1,3\}\big) + \tfrac{1}{3}\big(\{1,2,3\} + \{3,2,1\}\big),\\
\mathcal{Q}_{4,2} &= +5\{\} - \tfrac{31}{3}\{1\} + \tfrac{2}{3}\{1,3\} + \tfrac{17}{6}\big(\{1,2\} + \{2,1\}\big) + \tfrac{11}{6}\big(\{1,3,2\} + \{2,1,3\}\big)\\
&\quad - \tfrac{13}{6}\big(\{1,2,3\} + \{3,2,1\}\big) - \tfrac{1}{3}\{2,1,3,2\} - \tfrac{1}{6}\big(\{1,3,2,4\} + \{2,1,4,3\}\big)\\
&\quad - \tfrac{1}{6}\big(\{1,2,4,3\} + \{1,4,3,2\} + \{2,1,3,4\} + \{3,2,1,4\}\big)\\
&\quad + \tfrac{1}{2}\big(\{1,2,3,4\} + \{4,3,2,1\}\big),\\
\mathcal{Q}_{4,4} &= -\tfrac{63}{2}\{\} + \tfrac{401}{6}\{1\} - \tfrac{20}{3}\{1,3\} - \tfrac{5}{6}\{1,4\} - \tfrac{77}{4}\big(\{1,2\} + \{2,1\}\big)\\
&\quad - \tfrac{61}{6}\big(\{1,3,2\} + \{2,1,3\}\big) + \tfrac{1}{2}\big(\{1,2,4\} + \{1,3,4\} + \{1,4,3\} + \{2,1,4\}\big)\\
&\quad + \tfrac{83}{6}\big(\{1,2,3\} + \{3,2,1\}\big) + \tfrac{8}{3}\{2,1,3,2\} - \tfrac{1}{6}\big(\{1,2,4,5\} + \{2,1,5,4\}\big)\\
&\quad + \tfrac{1}{6}\big(\{1,3,2,5\} + \{1,3,5,4\} + \{1,4,3,5\} + \{2,1,3,5\}\big)\\
&\quad + \tfrac{19}{12}\big(\{1,3,2,4\} + \{2,1,4,3\}\big) + \tfrac{1}{6}\big(\{1,2,5,4\} + \{2,1,4,5\}\big)\\
&\quad + \tfrac{17}{12}\big(\{1,2,4,3\} + \{1,4,3,2\} + \{2,1,3,4\} + \{3,2,1,4\}\big)\\
&\quad - \tfrac{1}{6}\big(\{1,2,3,5\} + \{1,3,4,5\} + \{1,5,4,3\} + \{3,2,1,5\}\big)\\
&\quad - \tfrac{19}{4}\big(\{1,2,3,4\} + \{4,3,2,1\}\big) + \tfrac{1}{12}\big(\{1,4,3,2,5\} + \{2,1,3,5,4\}\big)\\
&\quad + \tfrac{1}{12}\big(\{1,3,2,5,4\} + \{2,1,4,3,5\}\big) + \tfrac{1}{12}\big(\{1,2,5,4,3\} + \{3,2,1,4,5\}\big)\\
&\quad - \tfrac{1}{6}\big(\{1,3,2,4,3\} + \{2,1,3,2,4\} + \{2,1,4,3,2\} + \{3,2,1,4,3\}\big)\\
&\quad - \tfrac{1}{4}\big(\{1,2,4,3,5\} + \{1,3,2,4,5\} + \{2,1,5,4,3\} + \{3,2,1,5,4\}\big)\\
&\quad - \tfrac{1}{4}\big(\{1,2,3,5,4\} + \{1,5,4,3,2\} + \{2,1,3,4,5\} + \{4,3,2,1,5\}\big)\\
&\quad + \tfrac{3}{4}\big(\{1,2,3,4,5\} + \{5,4,3,2,1\}\big).
\end{aligned}
$$

Table 6.2: The Hamiltonian $\mathcal{H} = \mathcal{Q}_2$ and the charges $\mathcal{Q}_3, \mathcal{Q}_4$ at three-loops.

ℓ	1	2	3	4	5
structures for $\mathcal{H}_{2\ell-2}$	2	3	6	12	27
integrability	0	0	2	5	17
integrable $\mathcal{H}_{2\ell-2}$	2	3	4	7	10
structures for $T_{2\ell-2}$	0	0	0	1	3
$\mathcal{H}_{2\ell-2}$ relevant	2	3	4	6	7
propagation	1	2	3	4	5
two-spin interaction	0	0	1	2	2(+1)
remaining d.o.f.	1	1	0	0	0

ℓ	1	2	3	4	5
structures for $\mathcal{Q}_{3,2\ell-2}$	1	2	6	15	46
integrability	0	1	4	13	43
$\mathcal{Q}_{3,2\ell-2}$	1	1	2	2	3
structures for $\mathcal{Q}_{4,2\ell-2}$	6	12	27	63	
integrability	3	9	23	59	
$\mathcal{Q}_{4,2\ell-2}$	3	3	4	4	

Table 6.4: Number of coefficients for the higher-loop integrable $\mathfrak{su}(2)$ spin chain

- All the obtained charges $\mathcal{Q}_r$ commute among themselves without further constraints. We notice that the remaining degrees of freedom correspond precisely to linear redefinitions of charges, c.f. the right hand side of Tab. 6.4: We can rescale the charge $\mathcal{Q}_3$ by a function of the coupling constant, this yields one degree of freedom at each loop order. We may also add $g^4 f(g^2)\mathcal{Q}_5(g)$ which is structurally equivalent to $\mathcal{Q}_3(g)$, this yields one degree of freedom starting from $\ell = 3$. For $\mathcal{Q}_4$ the story is equivalent. Here, we can always add the length operator $\mathcal{L}$ as well as the the Hamiltonian $\mathcal{H}$ or rescale by a function. This gives three degrees of freedom for all loop orders. Starting from $\ell = 3$ there are further degrees of freedom due to $\mathcal{Q}_6$. All in all, this is just the expected number of coefficients. In the thermodynamic limit (should it exist at all), all coefficients would be fixed for the canonical charges, see Sec. 6.1.6.

In this way we have established the most general integrable system for the assumed set of interactions. We conclude that, indeed, the pairing of states appears to be a *sufficient* condition for integrability (in this model). Next we would like to impose the thermodynamic limit. For the thermodynamic limit we make use of two processes: Propagation of a single excitation and interaction of two excitations. These should yield the correct scaling behaviour at each loop order.

- In Sec. 3.6 we have seen that the planar one-loop dilatation operator acts on the position of a single excitation as a lattice Laplacian $\Box$. The resulting eigenstates are Fourier modes. The lowest, non-zero eigenvalue of the Laplacian is proportional to $1/L^2$, exactly the right behaviour for $\mathcal{H}_0$. Due to the form of the interaction, $\mathcal{H}_{2\ell-2}$ must act as a polynomial in $\Box$ of degree ℓ. For the correct scaling behaviour, $1/L^{2\ell}$, all terms $\Box^k$ with $k < \ell$ should be absent. In general, this determines ℓ coefficients, see Tab. 6.4. We shall not fix the coefficient of $\Box^\ell$ although the quantitative BMN excitation energy formula $\mathcal{H}(g) = (\sqrt{1-2g^2\Box}-1)/g^2$ predicts it

$$\mathcal{H}_{2\ell-2} \sim -2^{-\ell-1} C_{\ell-1} \Box^\ell \tag{6.36}$$

  with $C_k = (2k)!/k!(k+1)!$ the Catalan numbers governing the expansion of the square root.

- The interaction of two excitations is a more delicate issue, it is obtained by acting with $\mathcal{H}_{2\ell-2}$ on two excitations and subtracting the contribution from the propagation of the individual excitations. The remainder can only be non-zero if both excitations

are close (the distance depends on the loop order), in other words the remainder is a contact interaction. This interaction must also be suppressed by sufficiently high powers of $1/L$.

Let us investigate the first order effect of the contact term for states with only two excitations, see Sec. 5.6.3. The first order is determined by diagonal scattering, we therefore compute the matrix element of $\mathcal{O}_{0,n}$ going into itself. First of all, this is suppressed by $1/L$ due to phase space considerations. Furthermore, there is a suppression of $n^2 n'^2/L^4$ for the process $\mathcal{O}_{0,n} \to \mathcal{O}_{0,n'}$. This is due to the zero mode state $\mathcal{O}_{0,0}$ which must be annihilated and can never be produced. In total there is a suppression of $1/L^5$ for the contact term, this is sufficient for $\ell = 2$. Starting from three-loops, the contact term may violate the scaling behaviour and there will be additional constraints. At three-loops a single constraint is enough to remove terms of order $1/L^5$. At four-loops we need to remove terms of orders $1/L^5$ and $1/L^7$ yielding two constraints. At $\ell = 5$ there are three constraints, but only two independent coefficients which may influence the scaling behaviour. Miraculously, the three constraints seem to be degenerate such that the scaling in the thermodynamic limit appears to be fine at five-loops.

To impose the constraints is not a trivial task.[9] The problem is that also iterated contact terms of perturbation theory may violate the scaling in the thermodynamic limit. These must be cancelled by higher order contact terms. We will therefore consider only states with exactly two excitations for which closed expression can be found. We will assume that the conjectured energy formula (5.107) holds to all orders. We will then compute the energies of several two-excitation states at higher-loops and match them with the formula. This can only be possible if a qualitative BMN limit exists and the coefficients are adjusted such that Hamiltonian provides the correct scaling.

- At this point nearly all relevant coefficients are fixed. However, starting at four-loops, there are some free coefficients left which have no influence on the scaling dimensions. These are due to the freedom to rotate the space of states with an orthogonal transformation generated by an antisymmetric operator A. For the four-loop interactions there is precisely one antisymmetric operator A_6, it happens to be proportional to $[\mathcal{Q}_{2,0}, \mathcal{Q}_{2,2}]$. It gives rise to the following similarity transformation

$$\mathcal{Q}_r(g) \mapsto \exp(\alpha g^6 A_6)\, \mathcal{Q}_r(g)\, \exp(-\alpha g^6 A_6). \tag{6.37}$$

The leading term in $\mathcal{H} = \mathcal{Q}_2$ due to the transformation is

$$\mathcal{H}_6 \mapsto \mathcal{H}_6 + \alpha[A_6, \mathcal{H}_0]. \tag{6.38}$$

Consequently, the eigenvalues of $\mathcal{H}(g)$ are not changed and α only affects the eigenvectors. Similarly, at five-loops there are three even antisymmetric operators for the construction of A_8.

There are some interesting points to be mentioned regarding this solution. First of all, integrability and the thermodynamic limit fix exactly the right number of coefficients for

[9] It would be interesting to find a general criterion which determines whether an interaction of two or more excitations has the correct scaling or not.

a unique solution. For this solution, the contribution $\delta\hat{E}$ of one excitation to the energy in the BMN limit is given by (recall that $\hat{g}^2 = g^2/J^2 = \lambda'/8\pi^2$ and $D = D_0 + \hat{g}^2\hat{E}$)

$$\delta\hat{E}_n = \frac{c_1}{\hat{g}^2}\Big(\sqrt{1 + c_2\, 8\pi^2 n^2 \hat{g}^2} - 1\Big) + \mathcal{O}(g^{10}) \tag{6.39}$$

The constants c_1, c_2 correspond to symmetries of the equations, they can therefore not be fixed by algebraic arguments. We will set them to their physical values, $c_1 = c_2 = 1$. It is interesting to observe that the BMN quantitative square-root formula for the energy of one excitation is predicted correctly; we have only demanded that the thermodynamic (i.e. qualitative BMN limit) limit exists. Finally, let us mention that the three-loop contribution agrees precisely with the calculation of Ch. 5. For the physical choice of c_1, c_2 we present the four-loop and five-loop contribution to the Hamiltonian in Tab. 6.6.

6.3 Spectrum

We can now apply the higher-loop integrable Hamiltonian to obtain some energies. In addition we can evaluate the integrable charges on the eigenstates of the Hamiltonian.

6.3.1 Lowest-Lying States

In preparation for the next section, it will be helpful to know the spectrum of lowest-lying modes for our spin chain. To obtain a matrix representation for the operators, we have used standard higher order quantum mechanical perturbation theory. We have applied the Hamiltonian $\mathcal{H}$ and charges $\mathcal{Q}_3, \mathcal{Q}_4$ up to four-loops to all states with a given length L and number of excitations K. The computations were performed using the routines in App. E. Then, the leading order energy matrix was diagonalised to obtain the leading order energy eigenvalues. Next, the off-diagonal terms at higher-loops were removed iteratively by a sequence of similarity transformations, see also Sec. 5.6.2. Afterwards, the Hamiltonian is diagonal and we can read off the energy eigenvalues. The same similarity transformations which were used to make $\mathcal{H}$ diagonal also diagonalise $\mathcal{Q}_3, \mathcal{Q}_4$ and we may read off their eigenvalues.

We present our findings up to $L = 8$ in Tab. 6.8 (we omit the protected states with $K = 0$) which is read as follows: For each state there is a polynomial and we write down the coefficients up to $\mathcal{O}(g^8)$ and $\mathcal{O}(x^2)$. For single states the polynomial $X(x, g)$ equals simply

$$X(x, g) = E(g) + x^2 Q_4(g). \tag{6.40}$$

If there is more than one state transforming in the same representation, the eigenvalues are solutions to algebraic equations. These could be solved numerically, here we prefer to state the exact algebraic equation in terms of a polynomial $X(\omega, x, g)$ of degree $k - 1$ in ω (k is also the number of lines in one block, one for each coefficient of the polynomial). The energy and charge eigenvalues are determined through the formula

$$\omega = E(g) + xQ_3(g) + x^2 Q_4(g), \qquad \omega^k = X(\omega, x, g). \tag{6.41}$$

At first sight the terms linear in x may appear spurious and the corresponding charge $Q_3(g)$ would have to be zero. For unpaired states with non-degenerate $Q_2(g)$ this is true,

$$
\begin{aligned}
\mathcal{H}_6 = {} & -35\{\} + \big(67 + \tfrac{1}{2}\alpha\big)\{1\} + \big(-\tfrac{21}{4} - \tfrac{1}{4}\alpha\big)\{1,3\} - \tfrac{1}{4}\{1,4\} \\
& + \big(-\tfrac{151}{8} - \tfrac{1}{2}\alpha\big)\big(\{1,2\} + \{2,1\}\big) + \tfrac{1}{4}\alpha\big(\{1,3,2\} + \{2,1,3\}\big) \\
& + \tfrac{1}{4}\big(\{1,2,4\} + \{1,3,4\} + \{1,4,3\} + \{2,1,4\}\big) + \big(6 + \tfrac{1}{4}\alpha\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\
& + \big(-\tfrac{3}{4} - \tfrac{1}{4}\alpha\big)\{2,1,3,2\} + \big(\tfrac{9}{8} + \tfrac{1}{4}\alpha\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
& + \big(-\tfrac{1}{2} - \tfrac{1}{8}\alpha\big)\big(\{1,2,4,3\} + \{1,4,3,2\} + \{2,1,3,4\} + \{3,2,1,4\}\big) \\
& - \tfrac{5}{8}\big(\{1,2,3,4\} + \{4,3,2,1\}\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_8 = {} & +\big(\tfrac{1479}{8} + \tfrac{1}{4}\alpha\big)\{\} + \big(-\tfrac{1043}{4} - \tfrac{3}{4}\alpha - 2\beta_2 - \beta_3\big)\{1\} \\
& + \big(-19 + \tfrac{5}{8}\alpha + 4\beta_1 + \beta_2 + \tfrac{1}{2}\beta_3\big)\{1,3\} + \big(5 - 4\beta_1 - 4\beta_2\big)\{1,4\} + \tfrac{1}{8}\{1,5\} \\
& + \beta_3\big(\{1,2\} + \{2,1\}\big) - \tfrac{1}{4}\{1,3,5\} + \big(\tfrac{251}{4} + \beta_2 - \tfrac{1}{2}\beta_3\big)\big(\{1,3,2\} + \{2,1,3\}\big) \\
& + \big(-3 + 2\beta_2\big)\big(\{1,2,4\} + \{1,3,4\} + \{1,4,3\} + \{2,1,4\}\big) \\
& - \tfrac{1}{8}\big(\{1,2,5\} + \{1,4,5\} + \{1,5,4\} + \{2,1,5\}\big) \\
& + \big(\tfrac{41}{4} + \tfrac{1}{4}\alpha + 3\beta_2 - \tfrac{1}{2}\beta_3\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\
& + \big(-\tfrac{107}{2} - \tfrac{1}{8}\alpha + \beta_2 + \tfrac{1}{2}\beta_3\big)\{2,1,3,2\} \\
& + \big(\tfrac{1}{4} - \beta_1\big)\big(\{1,3,2,5\} + \{1,3,5,4\} + \{1,4,3,5\} + \{2,1,3,5\}\big) \\
& + \big(\tfrac{183}{4} + 2\beta_1 - \beta_2 - \tfrac{1}{2}\beta_3\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
& + \big(-\tfrac{3}{4} + 2\beta_1\big)\big(\{1,2,5,4\} + \{2,1,4,5\}\big) + \big(1 - 2\beta_1\big)\big(\{1,2,4,5\} + \{2,1,5,4\}\big) \\
& + \big(-\tfrac{51}{2} - \tfrac{1}{4}\alpha - \beta_1 - \tfrac{5}{2}\beta_2 + \tfrac{1}{4}\beta_3\big)\big(\{1,2,4,3\} + \{1,4,3,2\} + \{2,1,3,4\} + \{3,2,1,4\}\big) \\
& + \beta_1\big(\{1,2,3,5\} + \{1,3,4,5\} + \{1,5,4,3\} + \{3,2,1,5\}\big) \\
& + \big(\tfrac{35}{4} - 2\beta_2\big)\big(\{1,2,3,4\} + \{4,3,2,1\}\big) \\
& + \big(-\tfrac{7}{8} - \tfrac{1}{4}\alpha - 2\beta_2\big)\big(\{1,4,3,2,5\} + \{2,1,3,5,4\}\big) \\
& + \big(\tfrac{1}{2} + \tfrac{1}{8}\alpha\big)\big(\{1,3,2,5,4\} + \{2,1,4,3,5\}\big) \\
& + \big(\tfrac{5}{8} + \tfrac{1}{8}\alpha + \beta_2\big)\big(\{1,3,2,4,3\} + \{2,1,3,2,4\} + \{2,1,4,3,2\} + \{3,2,1,4,3\}\big) \\
& + \big(\tfrac{1}{4} + \tfrac{1}{8}\alpha + 2\beta_2\big)\big(\{1,2,5,4,3\} + \{3,2,1,4,5\}\big) \\
& + \big(\tfrac{1}{4} - \beta_2\big)\big(\{1,2,4,3,5\} + \{1,3,2,4,5\} + \{2,1,5,4,3\} + \{3,2,1,5,4\}\big) \\
& + \beta_2\big(\{1,2,3,5,4\} + \{1,5,4,3,2\} + \{2,1,3,4,5\} + \{4,3,2,1,5\}\big) \\
& - \tfrac{7}{8}\big(\{1,2,3,4,5\} + \{5,4,3,2,1\}\big)
\end{aligned}
$$

Table 6.6: Four-loop and five-loop contributions to the Hamiltonian.

L	K	P	g^0x^0	g^2x^0	g^4x^0	g^6x^0	g^0x^2	g^2x^2	g^4x^2	g^6x^2
4	2	+	$+6$	-12	$+42$	$*$	$+0$	$*$	$*$	$*$
5	2	−	$+4$	-6	$+17$	$-\frac{115}{2}$	$+\frac{8}{3}$	-8	$*$	$*$
6	2	+	$+10\omega$	-17ω	$+\frac{117}{2}\omega$	$-\frac{1037}{4}\omega$	$-\frac{10}{3}\omega$	$+30\omega$	$-\frac{381}{2}\omega$	$*$
			-20	$+60$	-230	$+1025$	$+0$	$-\frac{140}{3}$	$+420$	$*$
6	3	−	$+6$	-9	$+\frac{63}{2}$	$-\frac{621}{4}$	-6	$+36$	$+0$	$*$
7	2	−	$+2$	$-\frac{3}{2}$	$+\frac{37}{16}$	$-\frac{283}{64}$	$+\frac{4}{3}$	$-\frac{5}{2}$	$+\frac{81}{16}$	$-\frac{707}{64}$
7	2	−	$+6$	$-\frac{21}{2}$	$+\frac{555}{16}$	$-\frac{8997}{64}$	$+0$	$+\frac{9}{2}$	$-\frac{513}{16}$	$+\frac{11907}{64}$
7	3	±	$+10\omega$	-15ω	$+50\omega$	$-\frac{875}{4}\omega$	$-\frac{10}{3}\omega$	$+25\omega$	$-\frac{285}{2}\omega$	$+\frac{1615}{2}\omega$
			-25	$+75$	$-\frac{1225}{4}$	$+\frac{5875}{4}$	$+\frac{245}{12}$	-180	$+\frac{28145}{24}$	$-\frac{86875}{12}$
8	2	+	$+14\omega^2$	$-23\omega^2$	$+79\omega^2$	$-349\omega^2$	$-\frac{14}{3}\omega^2$	$+39\omega^2$	$-250\omega^2$	$+\frac{4691}{3}\omega^2$
			-56ω	$+172\omega$	-695ω	$+3254\omega$	$+\frac{56}{3}\omega$	$-\frac{700}{3}\omega$	$+\frac{5258}{3}\omega$	-11822ω
			$+56$	-224	$+966$	-4585	-0	$+168$	$-\frac{5054}{3}$	$+\frac{38269}{3}$
8	3	−	$+6$	-9	$+33$	-162	-6	$+33$	-192	$+1191$
8	3	±	$+8\omega$	-10ω	$+28\omega$	-102ω	$+\frac{4}{3}\omega$	-2ω	$+4\omega$	$-\frac{26}{3}\omega$
			-16	$+40$	-137	$+548$	$-\frac{4}{3}$	$-\frac{40}{3}$	$+\frac{328}{3}$	$-\frac{1948}{3}$
8	4	+	$+20\omega^2$	$-32\omega^2$	$+112\omega^2$	$-511\omega^2$	$-\frac{32}{3}\omega^2$	$+72\omega^2$	$-442\omega^2$	$+\frac{8264}{3}\omega^2$
			-116ω	$+340\omega$	-1400ω	$+6938\omega$	$+\frac{392}{3}\omega$	$-\frac{3100}{3}\omega$	$+\frac{20708}{3}\omega$	-45348ω
			$+200$	-800	$+3600$	-18400	-320	$+2800$	$-\frac{58400}{3}$	$+\frac{389680}{3}$

Table 6.8: Four-loop energies and charges $Q_{3,4}$. See Sec. 6.3.1 for an explanation.

but not so for pairs of degenerate states. Then the solution of the algebraic equation leads to terms of the sort $\sqrt{0+x^2} = \pm x$, where the 0 is meant to represent the degeneracy. Note that for some states the interaction is longer than the state. In such a case, indicated by $*$ in the table, we do not know the energy/charge eigenvalue, see also Sec. 6.5.4.

6.3.2 Two Excitations

Now that the Hamiltonian is known up to five loops, we may continue the analysis of the two-excitation states in Sec. 5.6.3. In principle, we should diagonalise the energy in perturbation theory, however, this is very labourious. Instead we will assume the all-loop formula (5.107) to be correct and match the coefficients to sufficiently many two-excitation states.[10] When the coefficients have been determined, we may compare the formula to further states and find agreement. We take this as compelling evidence that the obtained formula and coefficients are indeed correct. We present a summary of findings in Tab. 6.10.

An application of the exact energies of two-excitation operators is the near BMN limit of $\mathcal{O}(1/J)$ corrections. Some inspired guessing yields an all-loop expression for the near BMN limit which agrees with Tab. 6.10 at five-loops

$$D_n^J = J + 2\sqrt{1+\lambda' n^2} - \frac{4\lambda' n^2}{J\sqrt{1+\lambda' n^2}} + \frac{2\lambda' n^2}{J(1+\lambda' n^2)} + \mathcal{O}(1/J^2). \tag{6.42}$$

[10] In fact, the five-loop coefficients have been obtained in a more convenient way, see Sec. 6.4.2.

$$D_n^J = J + 2 + \sum_{\ell=1}^{\infty} \left(\frac{g_{\rm YM}^2 N}{\pi^2} \sin^2 \frac{\pi n}{J+1} \right)^{\ell} \left(c_\ell + \sum_{k,l=1}^{\ell-1} c_{\ell,k,l} \frac{\cos^{2l} \frac{\pi n}{J+1}}{(J+1)^k} \right),$$

$$\begin{aligned}
c_1 &= +1, \\
c_2 &= -\frac{1}{4}, & c_{2,1,1} &= -1, \\
c_3 &= +\frac{1}{8}, & c_{3,k,l} &= \begin{pmatrix} +\frac{3}{4} & +\frac{1}{2} \\ -\frac{3}{4} & +\frac{5}{2} \end{pmatrix}, \\
c_4 &= -\frac{5}{64}, & c_{4,k,l} &= \begin{pmatrix} -\frac{5}{8} & -\frac{5}{12} & -\frac{1}{3} \\ +\frac{3}{4} & -\frac{7}{4} & -\frac{7}{2} \\ -\frac{1}{2} & +\frac{59}{12} & -\frac{49}{6} \end{pmatrix}, \\
c_5 &= +\frac{7}{128}, & c_{5,k,l} &= \begin{pmatrix} +\frac{35}{64} & +\frac{35}{96} & +\frac{7}{24} & +\frac{1}{4} \\ -\frac{45}{64} & +\frac{185}{96} & +\frac{131}{48} & +\frac{33}{8} \\ +\frac{5}{8} & -\frac{125}{24} & -\frac{13}{24} & +\frac{81}{4} \\ -\frac{5}{16} & +\frac{305}{48} & -\frac{1319}{48} & +\frac{243}{8} \end{pmatrix}.
\end{aligned}$$

Table 6.10: Planar scaling dimension of two-excitation states.

The first $1/J$ term can be regarded as a renormalisation of the term $\lambda' n^2$ in the first square root. For instance, we might replace J in the definition of λ' by $L = J + 2$ to absorb the second term into the leading order energy. Incidentally, this yields precisely the coupling constant $\tilde{g} = g/L$ for the thermodynamic limit (see Sec. 6.4.3). Unfortunately, as we have seen in Sec. 5.6.3, this formula does not agree with the expression for the near plane-wave limit (5.111) derived in [88]

$$D_n^J = J + 2\sqrt{1 + \lambda' n^2} - \frac{2\lambda' n^2}{J} + \mathcal{O}(1/J^2). \tag{6.43}$$

A curious observation is that the coefficient $c_{\ell,1,1}$ equals $2\ell c_\ell$. At order $1/J$, it cancels the effect of the expansion of the leading order sine. Only at one-loop there is no $c_{\ell,1,1}$ to cancel $2c_1$. We find exactly the string theory prediction when we set $c_{\ell,1,l} = 0$ for $l > 1$.

6.3.3 Three Excitations

Let us continue the analysis of unpaired three-excitation states at higher loops. We find for the scaling dimensions

$$\begin{aligned}
D &= 2, \\
D &= 4 + 6g^2 - 12g^4 + \tfrac{84}{2} g^6 + \dots, \\
D &= 6 + 6g^2 - 9g^4 + \tfrac{63}{2} g^6 - \tfrac{621}{4} g^8 + \tfrac{7047}{8} g^{10} + \dots, \\
D &= 8 + 6g^2 - 9g^4 + \tfrac{66}{2} g^6 - \tfrac{648}{4} g^8 + \tfrac{7212}{8} g^{10} + \dots, \\
D &= 10 + 6g^2 - 9g^4 + \tfrac{66}{2} g^6 - \tfrac{645}{4} g^8 + \tfrac{7179}{8} g^{10} + \dots, \\
D &= 12 + 6g^2 - 9g^4 + \tfrac{66}{2} g^6 - \tfrac{645}{4} g^8 + \tfrac{7182}{8} g^{10} + \dots, \\
D &= 14 + 6g^2 - 9g^4 + \tfrac{66}{2} g^6 - \tfrac{645}{4} g^8 + \tfrac{7182}{8} g^{10} + \dots, \\
&\dots,
\end{aligned} \tag{6.44}$$

where we have added the dimension-two half-BPS state and a Konishi descendant which appear to be the natural first two elements of this sequence. Note that the exact one-loop form of the eigenstates is corrected at higher-loops.

We observe that all corrections D_k to the scaling dimensions below the 'diagonal' $k \leq L-2$, are equal. Incidentally, the coefficients agree with the formula

$$D(g) = L + \left(\sqrt{1+8g^2}-1\right) + \left(\sqrt{1+2g^2}-1\right) + \left(\sqrt{1+2g^2}-1\right). \tag{6.45}$$

We may interpret the three terms in parenthesis as the energies of the three excitations. Then this form can be taken as a clear confirmation of an integrable system with elastic scattering of excitations.

Only if the loop order is at least half the classical dimension at $\mathcal{O}(g^L)$ the pattern breaks down. Interestingly, if the loop order is exactly half the classical dimension, the coefficient is decreased by $3 \cdot 2^{2-\ell}$. It would be of great importance to understand the changes further away from the diagonal. This might provide us with clues about wrapping interactions, which, in the above example, obscure the scaling dimension of the Konishi state beyond three-loops.

For completeness, we state a similar all-loop conjecture for the higher charges [89] to generalise (4.98)

$$Q_r(g) = \frac{i}{r-1}\left(\frac{1+(-1)^r}{\left(\frac{i}{4}+\frac{i}{4}\sqrt{1+8g^2}\right)^{r-1}} + \frac{1+(-1)^r}{\left(\frac{i}{2}+\frac{i}{2}\sqrt{1+2g^2}\right)^{r-1}}\right). \tag{6.46}$$

Note that Q_r is accurate only up order g^{L-2-r}. The corresponding transfer matrix, to be compared to the one-loop counterpart (4.97), is

$$T(x,g) = \frac{x-\left(\frac{i}{4}+\frac{i}{4}\sqrt{1+8g^2}\right)}{x+\left(\frac{i}{4}+\frac{i}{4}\sqrt{1+8g^2}\right)}\,\frac{x-\left(\frac{i}{2}+\frac{i}{2}\sqrt{1+2g^2}\right)}{x+\left(\frac{i}{2}+\frac{i}{2}\sqrt{1+2g^2}\right)} + \dots . \tag{6.47}$$

Here we have used the symbol x instead of u for the spectral parameter; the reason will become more apparent in the next section.

6.4 Long-Range Bethe Ansatz

In Sec. 6.2 we have investigated the $\mathfrak{su}(2)$ subsector up to five-loops assuming that higher-loop integrability holds and that the thermodynamic limits exists. Remarkably, these requirements were sufficient to obtain a *unique* system! For an integrable system we might hope for a Bethe ansatz to describe the energy eigenvalues. Serban and Staudacher have shown that the Inozemtsev long-range spin chain and associated *asymptotic* Bethe ansatz [78] can be used to reproduce this model up to three-loops [87]. At four-loops there is, however, a fundamental difference and the scaling in the thermodynamic limit breaks down in the Inozemtsev chain. In this context, asymptotic refers to the fact that the Bethe ansatz is only reliable up to L loops, where L is the length of the chain.

6.4.1 Ansatz

Without further ado, let us write down an ansatz [89] to reproduce the results of the previous section. The universal Bethe equations are the same as for the Inozemtsev

chain [78] proposed in [87]

$$\frac{P_L(u_k-\frac{i}{2})}{P_L(u_k+\frac{i}{2})}=\prod_{\substack{l=1\\l\neq k}}^{K}\frac{u_k-u_l-i}{u_k-u_l+i}\,. \tag{6.48}$$

Here we use the Bethe roots u_k instead of momenta p_k as the fundamental variables, see [89] for a description of the transformation. The precise model is specified by the function $P_L(u)$. For the model derived in Sec. 6.2 we suggest[11]

$$P_L(u)=x(u)^L+\left(\frac{g^2}{2x(u)}\right)^L \tag{6.49}$$

with the function $x(u)$ defined as[12]

$$x(u)=\tfrac{1}{2}u+\tfrac{1}{2}u\sqrt{1-2g^2/u^2}\,. \tag{6.50}$$

This relation is the main difference to the Inozemtsev chain. Inspired by the findings of Sec. 6.3.3 we propose the energy to be given by

$$E=\sum_{k=1}^{K}\left(\frac{i}{x(u_k+\frac{i}{2})}-\frac{i}{x(u_k-\frac{i}{2})}\right)+\mathcal{O}(g^{2L-2}),\qquad D=L+g^2E. \tag{6.51}$$

The unknown terms of order g^{2L-2} are related to the asymptotic nature of our Bethe ansatz.

Furthermore, the charges are apparently given by

$$Q_r=\sum_{k=1}^{K}\frac{i}{r-1}\left(\frac{1}{x(u_k+\frac{i}{2})^{r-1}}-\frac{1}{x(u_k-\frac{i}{2})^{r-1}}\right)+\mathcal{O}(g^{2L-2r+2}). \tag{6.52}$$

They can be summed up into a transfer matrix

$$T(x)=U\exp\sum_{r=2}^{\infty}iu^{r-1}\mathcal{Q}_r+\ldots=\prod_{k=1}^{K}\frac{x-x(u_k+\frac{i}{2})}{x-x(u_k-\frac{i}{2})}+\ldots\,, \tag{6.53}$$

where the dots indicate further possible terms like x^L or g^{2L} which cannot be seen for the lower charges or at lower loop orders. The transfer matrix at $u=0$ gives the shift eigenvalue

$$1=U=T(0)=\prod_{k=1}^{K}\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})} \tag{6.54}$$

which should equal $U=1$ for gauge theory states with cyclic symmetry.

[11] The alternative model specified by $P'_L(u)=x(u)^L$ is equivalent to our model at the desired accuracy. Beyond that order, one of the two functions might be preferred, but probably the model changes substantially. For the singular solutions in Sec. 4.4.3, the simplified, non-polynomial function $P'_L(u)$ leads to problems due to overlapping divergencies.

[12] At $g=0$ we reproduce the one-loop Bethe ansatz.

The function $P_L(u)$ is indeed a polynomial of degree L in u, which can be derived from the inverse of (6.50)

$$u(x) = x + \frac{g^2}{2x}\,. \tag{6.55}$$

Therefore, the equation (6.48) is the Bethe equation of an inhomogeneous spin chain, see Sec. 4.1.2 and [59]. The polynomial can be factorised and we obtain for the inhomogeneities v_p

$$P_L(u) = \prod_{p=1}^{L}(u - v_p) \quad \text{with} \quad v_p = \sqrt{2}\, g \cos\frac{\pi(2p-1)}{2L}\,. \tag{6.56}$$

Now it can be noticed that the physical transfer matrix $T(x)$ is not the natural transfer matrix $T'(u)$ associated to the inhomogeneous spin chain

$$T'(u) = \prod_{k=1}^{K}\frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \frac{P_L(u)}{P_L(u+i)}\prod_{k=1}^{K}\frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{i}{2}}\,. \tag{6.57}$$

The Bethe equations follow from this transfer matrix by cancellation of poles at $u_k - \frac{i}{2}$. The charges Q'_r derived from $T'(u)$ are given as in (4.54). In perturbation theory we can relate these charges to the physical charges Q_r by

$$Q_r = Q'_r + \tfrac{1}{2}(r+1)g^2 Q'_{r+2} + \tfrac{1}{8}(r+2)(r+3)g^4 Q'_{r+4} + \dots\,. \tag{6.58}$$

Let us first of all comment on the inhomogeneity. Our spin chain is homogeneous, how can the Bethe ansatz of an inhomogeneous spin chain describe our model? First of all, the equation (6.58) is merely an eigenvalue equation, it does not directly relate the homogeneous and inhomogeneous charges, $\mathcal{Q}_r$ and $\mathcal{Q}'_s$; it merely tells us that there is a similarity transformation between the two. Similar maps are encountered within the inhomogeneous spin chain itself: On the one hand, the order of the inhomogeneities v_p does not matter for the Bethe ansatz and thus for the eigenvalues Q'_r of the charges. On the other hand, it should certainly influence the charge operators $\mathcal{Q}'_r$. Consequently, the eigenstates should be related by a similarity transformation.[13] To understand our model better, it would be essential to investigate this point further and find the map between our homogeneous spin chain model and the common inhomogeneous spin chain.

In our equations, the map between x and u (6.50,6.55) plays a major role. It is a double covering map, for every value of u there are two corresponding values of x, namely

$$u \longleftrightarrow \left\{x, \frac{g^2}{2x}\right\}. \tag{6.59}$$

For small values of g, where the asymptotic Bethe ansatz describes the long-range spin chain, we will always assume that $x \approx u$. When g is taken to be large (if this makes sense at all is a different question), however, special care would be needed in selecting the appropriate branch. The double covering map for x and u has an analog for the transfer matrices $T(x)$ and $T'(u)$. We find the relation

$$\frac{T(x)\,T(g^2/2x)}{T(0,g)} \approx T'(u). \tag{6.60}$$

[13] The inhomogeneities v_p and v_{p+1} can be interchanged by conjugation with $\mathcal{R}_{p,p+1}(v_p - v_{p+1})$.

which holds if the second term in (6.57) is dropped. It can be proved by using the double covering relation

$$(x-x')\left(1-\frac{g^2}{2xx'}\right)=\left(x+\frac{g^2}{2x}\right)-\left(x'+\frac{g^2}{2x'}\right)=u-u'. \tag{6.61}$$

We believe it is important to further study the implications of the double covering maps. This might lead to insight into the definition of our model, possibly even beyond wrapping order.

Now we have totally self-consistent Bethe equations with associated transfer matrix elements $T'(u)$. Unfortunately, $T'(u)$ does not directly describe physical quantities, such as the energy E. These are encoded in the physical transfer matrix elements $T(x)$ which involve the function $x(u)$ and are ambiguous due to the two branches of the square root. This is not a problem in perturbation theory, however, even there inconsistencies are observed at higher order in g [89]. Remarkably, these appear precisely at the order where wrapping interactions start to contribute and our asymptotic Bethe ansatz is fully consistent to the desired accuracy. Conversely, there are signs of the missing of wrapping terms. We hope that finding a cure for the problems beyond wrapping order might help to find a generalization of the Bethe equations which include wrapping interactions. Presumably these equations will have a substantially different form, see Sec. 6.4.4.

6.4.2 Results

Here we summarise the results of a comparison of the above Bethe ansatz with the spectrum of the spin chain model. For the details of the comparison we refer the reader to the article [89].

- The energy of states with two excitations agrees with the formula given in Tab. 6.10. In fact, the five-loop result was obtained using the Bethe ansatz, where this is a straightforward task. It was shown to agree with the model in a number of cases. Furthermore, it is possible to derive all-loop results in the near BMN limit. The energy as an analytic function in λ' is indeed given by the conjectured formula (6.42).
- The general BMN energy formula (3.99,3.98) is easily confirmed.
- The unpaired three-excitation states are singular. We can treat these solutions by demanding cancellation of singularities in the transfer matrix $T'(u)$. Remarkably, the results agree with their respective mirror solutions, see Sec. 4.3.1. These have $L-2$ excitations instead of 3 and are regular. Up to $L=10$ their energies do agree with (6.44).
- All states with $L\leq 8$ and all unpaired ones with $L\leq 10$ have been computed in the Bethe ansatz. Their energies agree with Tab. 6.8.
- We have also compared some higher charges of the Bethe ansatz with the corresponding explicit computations. They agree.

In conclusion, we can say that for all considered examples, the Bethe ansatz yields precisely the same spectrum as the integrable spin chain model constructed in Sec. 6.2. It shows that an integrable spin chain with a well-defined thermodynamic limit (see

Sec. 6.1.5) is very likely to exist in contrast to the doubts raised in [87]. In terms of the long-range Bethe ansatz there may seem to be many such models. These would be obtained by replacing the phase relation (6.49) and energy formula (6.51) by some other function that is well-behaved in the limit. If we however demand that the model is related to an inhomogeneous spin chain, we find a unique model with thermodynamic scaling behaviour, see [89] for details.

The upshot for the integrable spin chain model is similar: In its construction we have assumed a very specific form of interactions and the obtained Hamiltonian has turned out to be unique (at five loops). In other words, the very relations (6.49,6.51) are special and correspond to the assumed form of interactions (*iii*).[14] At any rate, the relations (6.49,6.51) are very suggestive in view of a correspondence to string theory on plane waves, see Sec. 3.6. It is therefore not inconceivable, that the Bethe ansatz indeed describes planar $\mathcal{N}=4$ gauge theory in the $\mathfrak{su}(2)$ subsector at higher-loops.

6.4.3 The Thermodynamic Limit

Here we shall only present the thermodynamic limit of the equations in (6.4.1). The overall structure is the same as described in Sec. 4.5 only that we now introduce dependence on the effective coupling constant

$$\tilde{g} = g/L. \tag{6.62}$$

An application and sample calculation is found in [89].

As before, we shall assume that the roots $\tilde{u}_k = u_k/L$ condense on a disconnected contour $\mathcal{C}$ in the complex plane with density function $\rho(\tilde{u})$. The density is again normalised by the total filling fraction $\tilde{K} = K/L$

$$\int_{\mathcal{C}} d\tilde{u}\,\rho(\tilde{u}) = \tilde{K}. \tag{6.63}$$

We find for the energy and the momentum constraint

$$\tilde{E} = \int_{\mathcal{C}} \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}}\,\frac{1}{\tilde{x}(\tilde{u})}\,, \qquad 2\pi n = \int_{\mathcal{C}} \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}}\,, \tag{6.64}$$

with the map between $\tilde{x}$ and $\tilde{u}$ is given by[15]

$$\tilde{x}(\tilde{u}) = \tfrac{1}{2}\tilde{u} + \tfrac{1}{2}\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}\,, \qquad \tilde{u}(\tilde{x}) = \tilde{x} + \frac{g^2}{2\tilde{x}}\,, \tag{6.65}$$

whereas the higher-loop generalisation of the Bethe equation reads

$$2\pi n_{\tilde{u}} - \frac{1}{\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}} = 2\fint_{\mathcal{C}} \frac{d\tilde{v}\,\rho(\tilde{v})}{\tilde{v}-\tilde{u}}\,. \tag{6.66}$$

In the thermodynamic limit, the physical charges and resolvent are given by

$$\tilde{Q}_r = \int_{\mathcal{C}} \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}}\,\frac{1}{\tilde{x}(\tilde{u})^{r-1}}\,, \qquad G(\tilde{x}) = \int_{\mathcal{C}} \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}}\,\frac{\tilde{x}(\tilde{u})}{\tilde{x}(\tilde{u})-\tilde{x}}\,. \tag{6.67}$$

[14] This picture is rather similar to the Inozemtsev spin chain where the requirement of pairwise interactions of spins at a distance was shown to lead to the phase relation of the Inozemtsev-Bethe ansatz.

[15] Note the useful identity $\tilde{x} - g^2/2\tilde{x} = \tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}$.

As in the one-loop case, the Bethe equation (6.66) can alternatively be written as a consistency condition on the singular transfer matrix $\tilde{T}'(\tilde{u}) \sim 2\cos G'_{\text{sing}}(\tilde{u})$ with

$$G'_{\text{sing}}(\tilde{u}) = G'(\tilde{u}) + \frac{1}{2\tilde{u}\sqrt{1-2\tilde{g}^2/\tilde{u}^2}} \qquad \text{and} \qquad G'(\tilde{u}) = \int_{\mathcal{C}} \frac{d\tilde{v}\,\rho(\tilde{v})}{\tilde{v}-\tilde{u}}\,. \tag{6.68}$$

The function $2\cos G'_{\text{sing}}(\tilde{u})$ is single-valued if it obeys (4.113)

$$G'_{\text{sing}}(\tilde{u}+i\epsilon) + G'_{\text{sing}}(\tilde{u}-i\epsilon) = 2\pi n_{\tilde{u}} \tag{6.69}$$

across a cut of G' at $\tilde{u}$. At this point, it is however not clear how the physical transfer matrix $\tilde{T}(x)$ is related to the physical resolvent $G(x)$ and if there is also a consistency requirement which leads to the Bethe equations. This is largely related to mirror cuts in $\tilde{T}(\tilde{g}^2/2\tilde{x})$ which are due to the double covering map (6.61).

6.4.4 Bethe Ansätze for Bigger Subsectors

The Bethe ansatz has proved to be a very powerful tool in obtaining the spectrum at high loop orders. It would therefore be extremely interesting and important to generalise it to bigger subsectors than the $\mathfrak{su}(2)$ subsector, preferably to the complete $\mathfrak{psu}(2,2|4)$ spin chain. Despite some attempts we have not succeeded in finding suitable equations beyond the $\mathfrak{su}(2)$ sector.[16] We would thus like to present a number of considerations for the construction of a complete all-loop Bethe ansatz.

- The most important issue seems to be multiplet shortening, see also Sec. 1.8,4.3.5. The spectrum contains a number of multiplets which are short in the free theory. A short multiplet cannot acquire an anomalous dimension (energy) unless it combines with other short multiplets to form a long one (in analogy to the Higgs mechanism).

  The one-loop Bethe ansatz was not constructed to respect multiplet joining, nevertheless it does display this feature. For all short, non-protected multiplets compatible short multiplets can be found. All of these have not only equal anomalous dimension but also equal higher charges so that they can join in the interacting theory. The one-loop Bethe ansatz has a solution for all highest-weight states of the multiplets. In particular, there are multiple solutions for splitting multiplets. At higher-loops these multiplets join, consequently the complete Bethe ansatz should only find the highest weight of the long multiplet. The highest weights of the submultiplets should not generalise or display some other kind of inconsistency beyond one-loop.

- There is another issue related to multiplet shortening. The Bethe ansatz not only yields solutions corresponding to gauge theory states, but also solutions with non-zero momentum, $U \neq 1$. A naive generalisation to higher-loops would also produce these states. This, however, would be inconsistent for a simple reason: Short multiplets in the non-zero momentum sector generically do not have suitable partners to join up. Let us consider the state

  $$|X\rangle = |\mathcal{Z}\phi\rangle - |\phi\mathcal{Z}\rangle. \tag{6.70}$$

  This state does not obey the momentum constraint, i.e. $\operatorname{Tr}|X\rangle = 0$ or $\mathcal{U}|X\rangle = -|X\rangle$. It is not physical, but nevertheless reproduced by the Bethe ansatz. Its Dynkin labels

[16] In the thermodynamic limit there are simplifications which allow to guess the Bethe equations [132].

are $[0;0;1,0,1;0;0]$, it is the primary of a BPS multiplet. In principle this multiplet could join with another short multiplet of lower dimension to form a long one. Here this is not possible, there are no potential partners. Consequently, this multiplet would have to be protected, but a direct computation using the one-loop Hamiltonian or Bethe ansatz yields $E = 4$ which is inconsistent. A prospective Bethe ansatz should take this into account and exclusively yield solutions with zero momentum.

- There is a complication which applies to non-compact and supersymmetric representations: The Dynkin labels r, r_1, r_2 contain the anomalous dimension δD and thus change with g. For the Bethe ansatz, the labels also specify the number of excitations for any given state (c.f. Sec. 4.3.4) which certainly must be positive integers. It is not clear if and how these two points can be combined. If possible, it is reasonable to believe that the Bethe equations will be a self-consistency equation on the energy (this is somewhat similar to the integral equations which appear in string theory [75]). In perturbation theory at each order, the corrected energy would have to be used as input for the next order.

- The length L and hypercharge B are not good quantum numbers in dynamic spin chains. However, they are also used as input for the Bethe equations, see Sec. 4.3.4. It is not clear how to identify states when L and B cannot be fixed; ideally the Bethe ansatz for a dynamic spin chain should not distinguish between states with different L, B.

All in all this suggests that the prospective all-loop $\mathfrak{psu}(2,2|4)$ Bethe ansatz, if it exists,[17] has a rather different structure than the one-loop approximation. Even more, the one-loop ansatz might turn out not to be a smooth limit of it. This could indeed be a blessing in disguise because a substantially different Bethe ansatz might allow for the inclusion of wrapping interactions, see Sec. 6.5.4 or even incorporate them naturally.

The arguments presented above do not apply to the $\mathfrak{su}(2)$ subsector because there is no multiplet shortening and the $\mathfrak{su}(2)$ labels are not affected by the anomalous dimension. This explains why it was relatively easy to find our all-loop generalisation.

For a generalisation one might take a more pragmatic point of view and merely consider the classical algebra as the symmetry algebra. All classical representation labels would be perfectly well-defined and integer. The Bethe ansatz should yield some energies which we interpret as the corrections to the scaling dimensions. It would then be essential that all submultiplets have precisely the same energy and charges. Solutions with non-zero momentum would have to be ignored. The only shortcoming of such an ansatz would be that it does not explain the truly interacting structure of the algebra.

6.5 Stringing Spins at Higher Loops

Now that we have a Bethe ansatz for higher-loop scaling dimensions we may continue the comparison of spinning strings and gauge theory started in Sec. 4.6.

[17] Given that $\mathcal{N} = 4$ SYM is likely to be an integrable model, even beyond one-loop, we can hope for some associated Bethe ansatz.

6.5.1 Spinning out of Control

The first steps in this direction have been performed in [87] using the Inozemtsev spin chain, which is consistent with the results of the previous and current chapter up to three-loops. Further progress was made in [75, 70, 89]. Here we shall only summarise the results.

In the thermodynamic limit the perturbative Bethe equations reduce to expressions similar to the ones given in Sec. 4.5, but with a few additional g-dependent terms, c.f. Sec. 6.4.3. These terms modify the solution in two ways. On the one hand, the contour will experience a perturbative deformation and, on the other hand, the energy formula receives radiative corrections. Together, these determine the higher-loop contributions to the energy, either implicitly or explicitly. It was then found that the two-loop correction indeed coincides with the prediction from string theory [87]. Moreover the higher charges do agree [70]. This result was subsequently generalized to all solutions within the $\mathfrak{su}(2)$ subsector by comparing their Bethe ansätze [75, 89].

Interestingly, *the agreement does not persist at the three-loop level* [87]. One might argue that this due to a flawed gauge theory Hamiltonian. Although this is a possibility, it would not explain the discrepancy: The authors of [87] investigated whether agreement can be achieved by modifying the phase relation and expression for the energy in the most general way compatible with the scaling behaviour of string theory: The outcome was negative. Therefore it may seem impossible to construct a weak coupling integrable spin chain to reproduce string theory at 'three-loops'. Giving up on integrability is not an option either, because (classical) string theory on $AdS_5 \times S^5$ is integrable [83] and the spectra could not possibly agree.

The problem parallels the earlier three-loop disagreement with near plane-wave string theory [88] discussed in Sec. 5.6.3. In fact, it appears that the mismatch in these two examples is related: Both of them constitute a deviation from the BMN limit, either by considering many excitations or a state where the length is not strictly infinite.[18]

6.5.2 Order of Limits

The above problems suggest that either the correspondence between string theory and gauge theory breaks down at three-loops or some subtlety has not been taken into account properly [87, 75, 89]. Indeed, there may be a fundamental problem in the comparison:

The comparison takes place in the thermodynamic limit $L \to \infty$ and in an expansion around $\tilde{g} = g/L = 0$. However starting with an exact function $F(g, L)$, we must decide which limit is taken first. It turns out that for classical string theory, the thermodynamic limit $L \to \infty$ is a basic assumption. The resulting energy may then be expanded in powers of $\tilde{g}$. In contrast, gauge theory takes the other path. The computations are based on perturbation theory around $g = 0$. This expansion coincides with the expansion in $\tilde{g}$ because the coefficients turn out to be suppressed by sufficiently many powers of $1/L$. Now the order of limits does potentially matter. This is best illustrated in the non-commutative diagram Fig. 6.2 and the example in Sec. 6.5.3. Semi-classical string

[18] One could extract the expansion of the function $\tilde{E}(\alpha)$ for small α, (c.f. Sec. 4.6) from a large L expansion. For that one would consider an arbitrary number of excitations, K. For the $1/L^n$ correction to the energy one should find no more than n powers of K. The term K^n/L^n is to be interpreted as α^n, whereas all lower powers of K would have to be dropped.

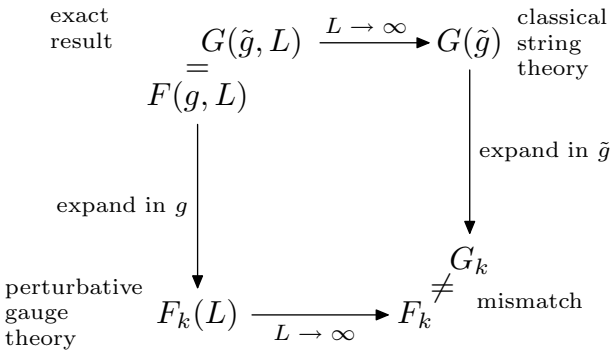

Figure 6.2: A possible explanation for both the near BMN and the FT spinning strings disagreement. F_ℓ excludes gauge theory wrapping effects, while G_ℓ is expected to include them.

theory corresponds to the upper right corner of the diagram, i.e. it requires the large spin limit. Conversely, perturbative gauge theory is situated at the lower left corner, where the length L is finite, but only the first few orders in g are known.[19]

The BMN and FT proposals are both based on the assumption that the diagram in Fig. 6.2 does commute. In other words one should be able to compare, order by order, the gauge theory loop expansion with the string theory expansion in $\tilde{g}$. That this might in fact not be true was first hinted at in another context in [133]. Another, more closely related, instance where the different limiting procedures lead to different results can be found in [87]. For the hyperbolic Inozemtsev spin chain it was shown that the order of limits does matter. In the 'gauge theory' order, this spin chain appears to have no proper thermodynamic limit. For the 'string theory' order, i.e. when the thermodynamic limit is taken right from the start, it is meaningful!

In order to make contact with string theory we propose to sum up the perturbation series in λ before taking the thermodynamic limit. In this case, the comparison would take place at the upper right corner of Fig. 6.2.[20] With the all-loop spin chain at hand this may indeed be feasible. In contrast to the Inozemtsev chain, there appears to be no difference between the two orders of limits because the thermodynamic limit is well-behaved in perturbation theory. However one has to take into account wrapping interactions which could violate thermodynamic scaling behaviour. These arise at higher loop orders ℓ when the interaction stretches all around the state, i.e. when $\ell \geq L$. We will discuss them in Sec. 6.5.4 after an example, which illustrates the potential importance of these interactions.

[19] We recall that the number of known terms grows with L, if our spin chain ansatz is correct.

[20] If we wish to compare at the lower left corner we should sum up all quantum corrections in string theory before we compare to perturbative gauge theory. There the $1/L$ alias $1/g$ suppression of quantum effects was derived assuming a large g. For small g this simplification is not justified and additional effects may contribute.

6.5.3 Example

Here we present an example where one can see the importance of the order of limits. We choose a function

$$F(g,L) = \frac{g^{2L}}{(c+g^2)^L} = \left(1+\frac{c}{\tilde{g}^2 L^2}\right)^{-L} = G(\tilde{g},L). \tag{6.71}$$

In perturbation theory around $g=0$ we find that the function vanishes at L leading loop orders

$$F(g,L) = \sum_{k=0}^{\infty} F_k\, g^k = \frac{g^{2L}}{c^L} - \frac{g^{2L+2}}{c^{L+1}} + \frac{g^{2L+4}}{c^{L+2}} + \ldots, \qquad \text{i.e.} \quad F_k(L) = 0 \quad \text{for} \quad k < 2L. \tag{6.72}$$

The leading factor g^{2L} mimics the effect of wrapping interactions in gauge theory as explained below. When we now go to the thermodynamic limit, $L \to \infty$, we see that all coefficients F_k are zero.

Now let us take the thermodynamic limit first. The large L limit of $G(\tilde{g},L) = \sum_k G_k\, \tilde{g}^k$ yields $G(\tilde{g}) = 1$ in a straightforward fashion. This result depends crucially on the function $F(g,L)$. Currently, we do not know how to incorporate wrapping interactions, but g^{2L} alone would not have a sensible thermodynamic limit. To compensate this, we have introduced some function $1/(c+g^2)^L$. Clearly we cannot currently prove that gauge theory produces a function like this, but it appears to be a definite possibility. In our toy example, the expansion in $\tilde{g}$ gives $G_0 = 1$ and $G_k = 0$ otherwise.

In conclusion we find $G_0 = 1$ while $F_0 = 0$ which demonstrates the non-commutativity of the diagram in Fig. 6.2 in an example potentially relevant to our context. It is not hard to construct a function $F(g,L)$ which yields arbitrary coefficients G_k while all F_k remain zero.

Note however that there is a sign of the non-commutativity in (6.72): A correct scaling behaviour would require the coefficient F_k to scale as L^{-k}. In particular for $k = 2L$, the coefficient should scale as L^{-2L} instead of c^{-L}. Therefore one can say that the function F violates the scaling law even at weak coupling, but in a mild way that is easily overlooked. This parallels the above observations for the Inozemtsev spin chain that a proper scaling behaviour might be obscured in perturbation theory.

6.5.4 Wrapping Interactions

One very important aspect of the spin chain are wrapping interactions (c.f. Sec. 2.6.3). These interactions appear when the expected length of the interactions, $\ell+1$, exceeds the length of the state L. Let us note that the *asymptotic* Bethe ansatz for the Inozemtsev spin chain apparently does not incorporate wrapping interactions correctly. As our ansatz is very similar, we expect the same to apply here.

For a fixed length L, the wrapping interactions are irrelevant at lower loop orders, but at higher-loops they are the only contribution. Therefore, at finite values of the coupling constant, wrapping interactions dominate and the dependence on the coupling constant may change completely. This is very appealing from the point of view of the AdS/CFT correspondence, which predicts the scaling dimension of a generic, unprotected state to grow like $\sqrt{g} \sim \lambda^{1/4}$ for large g [5].Instead, the energy formula (6.51) seems to suggest

a linear growth in g, but it is valid only for sufficiently low loop orders. Therefore we suspect that wrapping interactions may be responsible for the conjectured $\sqrt{g}$ behaviour at large g. Note that a generalisation of the Bethe equations for classical string theory [75] to towards the quantum regime has been conjectured in [130]. These equations reproduce the $\lambda^{1/4}$ behaviour as well as the near plane wave results of [88, 129, 102, 131].

In the same spirit, wrappings may be important for the comparison between gauge theory and string theory in the case of spinning strings and the near BMN limit [89] where they become an order-of-limits issue: When we assume the length of the state to be sufficiently large, wrapping interactions are suppressed. Then we find that g and $1/L$ combine and the energy is a function of $\tilde{g}$. However, for a comparison to string theory, we might wish to take g finite for a fixed length. Again we should find that g and $1/L$ combine (possibly for a completely different reason) and obtain a function of $\tilde{g}$. Here we may expect a qualitative difference because wrappings dominate the complete tail of the perturbative expansion for any fixed length. Summing up the perturbative series we might get a totally different function, as demonstrated by the example in Sec. 6.5.3.

Unfortunately, we currently have no handle on wrapping interactions. In the algebraic analysis of Ch. 5, there seem to be few constraints on their form. In fact, virtually any higher-loop anomalous dimensions can be assigned to multiplets with a small length by means of wrapping interactions. For the Bethe ansatz the inclusion of wrapping interactions will probably require a substantially different form. For instance, an exact, *non-asymptotic* Bethe ansatz for the Inozemtsev chain is not known [78].

All in all, a better understanding of wrapping interactions might be of great importance to the AdS/CFT correspondence and our understanding of higher-loop conformal gauge theory.

Conclusions

In this dissertation we have developed means to efficiently compute and investigate scaling dimensions of local operators in a perturbative four-dimensional conformal field theory. The central object is the *dilatation operator*; it is one of the generators of the (super)conformal algebra and it measures scaling dimensions. In the example of $\mathcal{N} = 4$ supersymmetric gauge theory, we have made use of the symmetry algebra and structural properties of Feynman graphs to derive this generator up to a few quantum loops; the analysis is purely algebraic, no actual (higher-loop) computations were required. The obtained results have allowed us to prove that the planar dilatation operator is completely *integrable*, not only at one-loop, but possibly even at higher-loops and for finite values of the coupling constant.

Apart from generic interest in the structure of field theories at higher-loops, our investigations are motivated by the AdS/CFT correspondence. One prediction of this conjecture is the agreement of the spectrum of scaling dimensions of local operators in $\mathcal{N} = 4$ SYM with the spectrum of energies of IIB string theory on $AdS_5 \times S^5$. The dilatation operator has become a versatile tool for testing and even proving parts of the AdS/CFT correspondence.

There are various ways to deduce scaling dimensions of local operators and their quantum corrections (i.e. the anomalous dimensions) from field theory correlators. They are convenient when interested in particular and rather simple local operators at low loop orders. Unfortunately, the AdS/CFT correspondence relates this regime of operators to an essentially inaccessible strong-coupling regime in string theory. Conversely, the perturbative regime of string theory is usually mapped to an inaccessible regime in gauge theory. In recent years, it has emerged that this incompatibility can be overcome when focusing on states with a large spin quantum number on S^5, or equivalently of $\mathfrak{so}(6)$. In gauge theory this requires operators with a large number of constituents in which case the direct approach requires too much work. It is therefore desirable to have some technology to deal with such states in an efficient way.

The dilatation generator is perfectly suited to investigate scaling dimensions. As compared to conventional methods, it offers several advantages: Once it is obtained from field theory, the computation of scaling dimensions is turned into a combinatorial exercise. Furthermore, the dilatation operator is an algebraic object and one can save a significant amount of work by simplifying it, before it is applied to specific states. Moreover, one can try to obtain the dilatation operator without performing labourious field theory computations; this constitutes one of the key results of the current work. Finally, whereas scaling dimensions are just a set of data, the dilatation generator comprises the *whole spectrum*; it allows to compare on an abstract level and thus prove the equivalence of certain spectra instead of performing tests for an (inevitably small) set of states. For instance, when

non-planar corrections are taken into account, the dilatation generator can split and join traces (alias strings) making it very reminiscent of a stringy Hamiltonian. What is more, in $\mathcal{N} = 4$ SYM in the BMN limit, it was shown to coincide also *quantitatively* with the plane-wave string field theory Hamiltonian. This proves the agreement of arbitrary-genus contributions to scaling dimensions/energy and constitutes a strong test of the AdS/CFT or plane-wave/BMN correspondence.

Here, we have chosen $\mathcal{N} = 4$ supersymmetric gauge theory as a model quantum conformal field theory in four spacetime dimensions. For this model we have first investigated the dilatation operator in the one-loop approximation. We have started by making the most general ansatz compatible with field theory, which involves infinitely many undetermined coefficients. Subsequently, we have used conformal invariance to reduce the independent coefficients, first to one infinite sequence, then to just a single one. Being related to a rescaling of the coupling constant, it is not possible to assign a value to the leftover coefficient except by actually computing it in field theory. Therefore, the complete one-loop dilatation operator of $\mathcal{N} = 4$ SYM is entirely fixed by symmetry (up to obvious transformations).

Similar conclusions hold also at higher-loops: A general treatment would have required very much work, therefore we have first investigated subsectors on which the dilatation operator closes. In an interesting one, the $\mathfrak{su}(2|3)$ subsector, we were able to obtain planar three-loop corrections by algebraic means. Again, the result has turned out to be unique up to symmetries of the defining relations. This enables us to perform a very exact test of AdS/CFT correspondence and the near plane-wave/BMN limit thereof. Remarkably, this test has revealed a discrepancy starting only at three-loops. We have presented a possible explanation in terms of an order of limits problem, but more importantly, it demonstrates that we can find relevant and unexpected physics even in a higher-loop computation!

The dilatation operator is not only useful to obtain scaling dimensions, but it is interesting in itself. One exciting feature of the planar dilatation operator, with very important consequences, is its apparent integrability. As far as the spectrum is concerned, integrability merely leads to a curiosity: For generic multiplets of local operators there is a partner with exactly degenerate anomalous dimension. Below the surface, however, the existence of arbitrarily many conserved commuting charges strongly constrains the dilatation operator. This leads to a major simplification in computing scaling dimensions due to the algebraic Bethe ansatz.

The Bethe ansatz is especially powerful in the thermodynamic limit, i.e. for local operators with a large number of constituent fields. This limit is practically inaccessible by conventional methods, however, here the Bethe equations turn into integral equations which can still be solved in practice. The AdS/CFT correspondence relates the thermodynamic limit of $\mathcal{N} = 4$ SYM to classical spinning string configurations on $AdS_5 \times S^5$. The energy is usually given by intricate algebraic, elliptic or hyperelliptic functions of the ratios of the involved spins. In a number of cases, agreement between gauge theory and string theory could be shown, which confirms the correspondence with unprecedented accuracy. Even more, the Bethe equations in the thermodynamic limit could be shown to coincide with integral equations derived from string theory!

Integrability of the planar dilatation operator was first found at one-loop and for certain subsectors of states. More accurately, it was shown that the dilatation operator is isomorphic to the Hamiltonian of an integrable quantum spin chain with $\mathfrak{so}(6)$ or

$\mathfrak{sl}(2)$ symmetry. In this work we have extended these one-loop results to the complete spectrum of states and full $\mathfrak{psu}(2,2|4)$ superconformal symmetry. Even more exciting is a generalisation of integrability to higher-loops, an entirely new topic. We have found several indications for integrability beyond one-loop although a framework to investigate, prove and exploit it, is yet to be established. Most importantly, the three-loop corrections to the dilatation operator within the $\mathfrak{su}(2|3)$ subsector preserve the above-mentioned degeneracy of pairs. Furthermore, the integrable spin chain due to Inozemtsev reproduces three-loop planar scaling dimensions correctly. Finally, the sigma model of string theory on $AdS_5 \times S^5$ is integrable and, via the AdS/CFT correspondence, one expects the same feature for the corresponding gauge theory.

In order to investigate higher-loop integrability, we have constructed a deformation of the Heisenberg integrable spin chain model. The assumed form of interactions is inspired by field theory and conjectures about $\mathcal{N} = 4$ SYM. If all our conjectures are justified, the model will describe planar anomalous dimensions. Independently of that question, the model displays some remarkable features: Our assumptions have turned out to be sufficiently constraining such that we obtain a unique result up to at least five-loops. Intriguingly, it reproduces the BMN energy formula correctly. Moreover, we have found a long-range Bethe ansatz, very similar to the one describing the Inozemtsev spin chain, which reproduces the spectrum of the spin chain exactly. But this is not all: It is valid for *all loop orders*, at least if the spin chain is sufficiently long! Have we hereby opened up a window for finite coupling constant?

In conclusion, we have presented a versatile technology to investigate scaling dimensions in a conformal field theory. We have completed the one-loop calculation and even had a glimpse of higher-loop physics. Hopefully, making full use of integrability will finally allow us to leave the weak coupling regime.

Outlook

There is a wide range of open questions and problems which can be addressed with the ideas and methods presented in this work; we will group them according to the topics presented in the individual chapters. Let us start with *the dilatation operator in general* which has been considered in Ch. 2:

- It would be very helpful to develop techniques, similar to the ones presented here, for the efficient evaluation of structure constants of the operator product expansion, see Sec. 1.10. Apart from the scaling dimensions, they are the other central quantity in a conformal field theory of local operators. The structure constants may be obtained from three-point functions, but a direct computation is 'contaminated' by useless finite and divergent contributions from the perturbative expansion of the scaling dimensions. One therefore wonders whether one may generalise our methodology and develop purely algebraic techniques for directly finding the structure constants. See e.g. [134] for some work in that direction. Especially within the BMN three-point functions are important for the comparison to the plane-wave string theory [135], c.f. [108, 21, 22, 136, 49] for some results.
- It might be interesting to extend the current analysis to non-perturbative effects like instantons. Possibly the symmetry algebra also puts constraints on these and a direct computation as in [137] might be simplified or even bypassed.

In Ch. 3 we have investigated the *dilatation operator at one-loop*:

- We have focussed on $\mathcal{N} = 4$ SYM in this work, but there are a few further four-dimensional conformal quantum field theories with $\mathcal{N} = 2$ supersymmetry. For these, the determination of the dilatation generator might shed some light on holographic dualities away from the well-studied case of $AdS_5 \times S^5$. Even if the superconformal algebra is smaller, we expect that it is similarly constraining and our results should generalise in a rather straightforward fashion. Some advances in this direction have been made in e.g. [138, 139].
- Even in a QFT with broken conformal invariance [56], the techniques developed in this work can be used to investigate logarithmic corrections to two-point functions and scattering amplitudes in a systematic way. In particular in QCD at large N_c and deep inelastic scattering, similar techniques are at use (see e.g. [52–55]).

Questions related to the *dilatation operator at higher-loops* in Ch. 5:

- The complete one-loop contribution to the dilatation operator has turned out to be totally fixed by superconformal symmetry; the same might apply at two or even higher-

loops. This conjecture is not unreasonable, as the action is unique and entirely determined by $\mathfrak{psu}(2,2|4)$. Whether or not the conjecture is true, it would be great to derive the complete two-loop deformation.

- In Ch. 3,5 we have considered the algebras $\mathfrak{psu}(2,2|4)$ and $\mathfrak{su}(2|3)$ with the fundamental fields transforming in certain representations. An interesting mathematical question is, which algebras and representations are suited for consistent deformations of the algebra generators? Are they all related to (conformal) field theories?
- A very important issue is the form of wrapping interactions. For those, our methods appear to be not very constraining at higher loop orders. A non-planar treatment might solve the problem, but the complexity will increase drastically due to the large amount of such digrams. For operators of a finite length, wrappings dominate the tail of the perturbation series. They are therefore of tremendous importance for the finite coupling regime.
- It would be nice to confirm some of the higher-loop results of this work explicitly in field theory. Although we believe our computations are rigorous, we had to rely on some basic assumptions (e.g. the BMN-limit) which are not firm facts of gauge theory yet.
- A curious observation is that all the anomalous dimensions we have found are solutions to algebraic equations. In contrast, higher-loop amplitudes usually involve transcendental numbers such as $\zeta(3), \zeta(5), \ldots$. Are these merely a renormalisation artefact or do they appear at some higher loop order?

  A related question is whether the coupling constant is renormalised by a finite amount. For instance, this happens in the BMN matrix model (c.f. Sec. 2.1.6), where a redefinition of the coupling constant is required to achieve a proper scaling in the BMN limit [79] (it is a non-trivial result that this is possible at all).

One-loop integrability was the subject of Ch. 4:

- Clearly, the deep question is, *why* does integrability emerge from the planar $\mathcal{N} = 4$ gauge theory? Of course, via the the AdS/CFT correspondence, one could take integrability of the classical string sigma model [82–84] as compelling evidence, nevertheless we believe there should also be an intrinsically field theoretical reason.
- We know that the spectrum of $\mathcal{Q}_2$ is related to the spectrum of anomalous dimensions. A natural question to ask is whether the spectra of the higher charges $\mathcal{Q}_r$ have a physical meaning in the gauge theory.
- Integrability implies the appearance of degenerate pairs in the planar spectrum due to an interplay of two charges and parity. However, there appears to be no deeper reason for the pairs in terms of representation theory. It would be very desirable to understand the degeneracy of pairs better, in terms of $\mathcal{N} = 4$ SYM as well as in terms of the AdS/CFT correspondence. A possible explanation would be that symmetry extends by some compact SO(2) group whose representations are either uncharged singlets or charged doublets. The integrable charges $\mathcal{Q}_r$ are not suitable SO(2) generators because their spectrum is non-integer. The SO(2) symmetry naturally combines with the parity $\mathbb{Z}_2$ to O(2) and $1/N$ corrections break it to $\mathbb{Z}_2$.

Could the conjectured SO(2) symmetry be related to the conjectured modular invariance of $\mathcal{N}=4$ SYM? In the unbroken form, $\mathrm{SL}(2,\mathbb{R})$, the SO(2) subgroup of modular invariance would pair up states. When broken to $\mathrm{SL}(2,\mathbb{Z})$ by higher-genus corrections, there is no SO(2) group to protect the pairing and the degeneracy is lifted.

- Integrability seems to apply to a wider range of field theories. The obvious candidates are conformal field theories, see e.g. [140]. However, in theories where conformal invariance is broken by quantum effects, such as massless QCD, one may still investigate the one-loop dilatation operator, for which conformal symmetry applies, see e.g. [139,141]. In QCD, following pioneering work of Lipatov [52], methods of integrability have also had much impact, see e.g. [53–55].
- On the one hand, we have demonstrated in Sec. 3.3 that the superconformal algebra leads to a unique one-loop anomalous dilatation operator. On the other hand, there is a unique standard spin chain Hamiltonian with $\mathfrak{psu}(2,2|4)$ symmetry. In fact, both operators turn out to be the same, which shows that conformal symmetry and integrability go hand in hand. This is remarkable because we consider a four-dimensional field theory here. For a two-dimensional theory this relationship is well-understood. For more details, see the end of Sec. 4.3.5.

Questions related to *higher-loop integrability* in Ch. 6:

- An improved notion of higher-loop integrability and, even more urgently, a better understanding of the long-range spin chain in Ch. 6 or the dynamic spin chain in Ch. 5 is required.
- Can we find a Bethe ansatz for the dynamic spin chain in Ch. 5? If so, can we generalise it to the complete spin chain for $\mathcal{N}=4$ SYM? This presumably would be a non-compact, dynamic $\mathfrak{psu}(2,2|4)$ super spin chain with long-range interactions. See also Sec. 6.4.4 for further comments.
- The Bethe ansätze for the Inozemtsev spin chain and our long-range spin chain investigated in Sec. 6.4 apparently do not incorporate wrapping interactions (see Sec. 5.4.3,6.5.4). A key to unravel planar $\mathcal{N}=4$ gauge theory at all loops would be to modify the equations to account for wrappings. Unfortunately, it is hard to find the correct wrapping interactions in field theory. Here, the investigation of the unknown terms in the physical transfer matrix (6.53) (from which the Bethe equations should follow as a consistency condition) or a better understanding of the double covering map (6.59,6.60) might help.
- Is integrability related to the closure of the interacting algebra or to some property of field theory renormalisation? Can we *prove* higher-loop integrability in some way?
- An interesting model is the BMN matrix model. It behaves quite similarly to $\mathcal{N}=4$ SYM, but consists of only a finite number of fields. Explicit higher-loop computations are feasible [101, 79], and show partial agreement with $\mathcal{N}=4$ SYM. Here we might learn about wrapping-interactions and dynamic aspects explicitly.

Questions regarding *the AdS/CFT correspondence* in the context of the plane-wave/BMN correspondence and spinning strings:

- The plane-wave/BMN correspondence and the topic of spinning strings can be investigated for theories with less supersymmetry. They are especially interesting because they involve also open strings/traces as opposed to the maximally supersymmetric case with only closed strings. Many investigations are devoted to this topic, e.g. [142, 138, 143, 140], but so far there are no higher-loop results for gauge theory.
- The equivalence of the dilatation operator in the BMN limit and the plane-wave string field theory Hamiltonian has been shown at one-loop in the case of two [47] and three [115] excitations (impurities). One could try to prove the equivalence for an arbitrary number of excitations. A generalisation to higher-loops would be interesting: At two-loops one might have to consider a g-dependent change of basis to avoid the mismatch reported in [47]. At three-loops it would be exciting to see if problems of the kind encountered for the near plane-wave background [88] or for spinning strings [87] also exist for non-planar corrections.
- In Sec. 6.5 we have offered a possible explanation for the apparent disagreement of gauge theory and string theory in the case of near plane-waves [88] or spinning strings [87]. The discrepancy, however, turns out to starts merely at three loops; why do we find agreement at one loop and two loops? At lower loop orders, the structure might allow only for a unique answer on either side. Together with the structural equivalence of both models, one being a lattice discretisation of the other, the agreement may be explained.

Appendix A

An Example

Here we would like to demonstrate how to apply the dilatation operator step-by-step in order to introduce our notation. In the following two sections we will present some essential matrix model and spin chain technology.

A.1 Non-Planar Application

Consider two $N \times N$ matrices $\mathcal{Z}$ and ϕ. Their elements are given by the variables $\mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}$ and $\phi^{\mathfrak{a}}{}_{\mathfrak{b}}$ with indices $\mathfrak{a}, \mathfrak{b}, \ldots$ ranging from 1 to N. We would now like to write down a polynomial $\mathcal{O} = \mathcal{O}(\mathcal{Z}, \phi)$ in the elements of $\mathcal{Z}$ and ϕ that is invariant under similarity transformations $\mathcal{Z} \mapsto T\mathcal{Z}T^{-1}, \phi \mapsto T\phi T^{-1}$. This is achieved conveniently by taking traces of matrices such as

$$\mathcal{O}(\mathcal{Z}, \phi) = \operatorname{Tr} \mathcal{Z}\phi\mathcal{Z}\phi - \operatorname{Tr} \mathcal{Z}\mathcal{Z}\phi\phi = \sum_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}=1}^{N} \left(\mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{c}}\, \mathcal{Z}^{\mathfrak{c}}{}_{\mathfrak{d}}\, \phi^{\mathfrak{d}}{}_{\mathfrak{a}} - \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \mathcal{Z}^{\mathfrak{b}}{}_{\mathfrak{c}}\, \phi^{\mathfrak{c}}{}_{\mathfrak{d}}\, \phi^{\mathfrak{d}}{}_{\mathfrak{a}} \right). \tag{A.1}$$

We now introduce a differential operator $\mathcal{H}$ on polynomials of the matrix elements

$$\mathcal{H} = -N^{-1} \operatorname{Tr} [\mathcal{Z}, \phi][\check{\mathcal{Z}}, \check{\phi}], \tag{A.2}$$

where the derivatives $\check{\mathcal{Z}}$ and $\check{\phi}$ are defined as follows[1]

$$\check{\mathcal{Z}}^{\mathfrak{a}}{}_{\mathfrak{b}} = \frac{\partial}{\partial \mathcal{Z}^{\mathfrak{b}}{}_{\mathfrak{a}}}, \qquad \check{\phi}^{\mathfrak{a}}{}_{\mathfrak{b}} = \frac{\partial}{\partial \phi^{\mathfrak{b}}{}_{\mathfrak{a}}}. \tag{A.3}$$

Let us act with $\mathcal{H}$ on $\mathcal{O}$. In this elementary form it is quite tedious, so let us restrict to the first terms

$$\begin{aligned} \operatorname{Tr} \mathcal{Z}\phi\check{\mathcal{Z}}\check{\phi} \operatorname{Tr} \mathcal{Z}\phi\mathcal{Z}\phi = \sum_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f},\mathfrak{g},\mathfrak{h}=1}^{N} \Big[& \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{c}}\, \delta^{\mathfrak{c}}{}_{\mathfrak{f}}\, \delta^{\mathfrak{d}}{}_{\mathfrak{g}} \quad \delta^{\mathfrak{e}}{}_{\mathfrak{d}}\, \delta^{\mathfrak{f}}{}_{\mathfrak{a}}\, \mathcal{Z}^{\mathfrak{g}}{}_{\mathfrak{h}}\, \phi^{\mathfrak{h}}{}_{\mathfrak{e}} \\ & + \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{c}}\, \delta^{\mathfrak{c}}{}_{\mathfrak{f}}\, \delta^{\mathfrak{d}}{}_{\mathfrak{e}} \quad \delta^{\mathfrak{e}}{}_{\mathfrak{d}}\, \phi^{\mathfrak{f}}{}_{\mathfrak{g}}\, \mathcal{Z}^{\mathfrak{g}}{}_{\mathfrak{h}}\, \delta^{\mathfrak{h}}{}_{\mathfrak{a}} \\ & + \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{c}}\, \delta^{\mathfrak{c}}{}_{\mathfrak{h}}\, \delta^{\mathfrak{d}}{}_{\mathfrak{g}} \quad \mathcal{Z}^{\mathfrak{e}}{}_{\mathfrak{f}}\, \delta^{\mathfrak{f}}{}_{\mathfrak{a}}\, \delta^{\mathfrak{g}}{}_{\mathfrak{d}}\, \phi^{\mathfrak{h}}{}_{\mathfrak{e}} \\ & + \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{c}}\, \delta^{\mathfrak{c}}{}_{\mathfrak{h}}\, \delta^{\mathfrak{d}}{}_{\mathfrak{e}} \quad \mathcal{Z}^{\mathfrak{e}}{}_{\mathfrak{f}}\, \phi^{\mathfrak{f}}{}_{\mathfrak{g}}\, \delta^{\mathfrak{g}}{}_{\mathfrak{d}}\, \delta^{\mathfrak{h}}{}_{\mathfrak{a}} \Big] \end{aligned}$$

[1] In the language of canonical quantisation, the fields $\mathcal{Z}, \phi$ and the variations $\check{\mathcal{Z}}, \check{\phi}$ correspond to creation and annihilation operators, respectively.

$$= \sum_{\mathfrak{a},\mathfrak{b},\mathfrak{e},\mathfrak{f}=1}^{N} \left(2\mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{a}}\, \mathcal{Z}^{\mathfrak{e}}{}_{\mathfrak{f}}\, \phi^{\mathfrak{f}}{}_{\mathfrak{e}} + 2N \mathcal{Z}^{\mathfrak{a}}{}_{\mathfrak{b}}\, \phi^{\mathfrak{b}}{}_{\mathfrak{f}}\, \phi^{\mathfrak{f}}{}_{\mathfrak{e}}\, \mathcal{Z}^{\mathfrak{e}}{}_{\mathfrak{a}} \right)$$
$$= 2 \operatorname{Tr} \mathcal{Z}\phi \ \operatorname{Tr} \mathcal{Z}\phi + 2N \operatorname{Tr} \mathcal{Z}\phi\phi\mathcal{Z}. \tag{A.4}$$

This calculation can be significantly abbreviated by parameterising the matrices by $\mathrm{U}(N)$ generators $\mathfrak{t}_{\mathfrak{m}}$ and using the fusion and fission rules

$$\mathfrak{g}^{\mathfrak{m}\mathfrak{n}} \operatorname{Tr} X\mathfrak{t}_{\mathfrak{m}} \operatorname{Tr} Y\mathfrak{t}_{\mathfrak{n}} = \operatorname{Tr} XY, \qquad \mathfrak{g}^{\mathfrak{m}\mathfrak{n}} \operatorname{Tr} X\mathfrak{t}_{\mathfrak{m}} Y\mathfrak{t}_{\mathfrak{n}} = \operatorname{Tr} X \operatorname{Tr} Y. \tag{A.5}$$

The action (A.4) is now

$$\begin{aligned} \operatorname{Tr} \mathcal{Z}\phi\check{\mathcal{Z}}\check{\phi} \operatorname{Tr} \mathcal{Z}\phi\mathcal{Z}\phi &= 2\mathfrak{g}^{\mathfrak{m}\mathfrak{p}}\mathfrak{g}^{\mathfrak{n}\mathfrak{q}} \operatorname{Tr} \mathcal{Z}\phi\,\mathfrak{t}_{\mathfrak{m}}\mathfrak{t}_{\mathfrak{n}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{p}}\mathfrak{t}_{\mathfrak{q}}\mathcal{Z}\phi + 2\mathfrak{g}^{\mathfrak{m}\mathfrak{p}}\mathfrak{g}^{\mathfrak{n}\mathfrak{q}} \operatorname{Tr} \mathcal{Z}\phi\,\mathfrak{t}_{\mathfrak{m}}\mathfrak{t}_{\mathfrak{n}} \operatorname{Tr} \mathfrak{t}_{\mathfrak{p}}\phi\mathcal{Z}\,\mathfrak{t}_{\mathfrak{q}} \\ &= 2\mathfrak{g}^{\mathfrak{n}\mathfrak{q}} \operatorname{Tr} \mathcal{Z}\phi\,\mathfrak{t}_{\mathfrak{q}}\mathcal{Z}\phi\,\mathfrak{t}_{\mathfrak{n}} + 2\mathfrak{g}^{\mathfrak{n}\mathfrak{q}} \operatorname{Tr} \mathcal{Z}\phi\phi\mathcal{Z}\,\mathfrak{t}_{\mathfrak{q}}\mathfrak{t}_{\mathfrak{n}} \\ &= 2 \operatorname{Tr} \mathcal{Z}\phi \operatorname{Tr} \mathcal{Z}\phi + 2 \operatorname{Tr} \mathcal{Z}\phi\phi\mathcal{Z} \operatorname{Tr} 1. \end{aligned} \tag{A.6}$$

Summing up all contributions in $\mathcal{H}\,\mathcal{O}$ we get

$$\mathcal{H}\,\mathcal{O} = 6 \operatorname{Tr} \mathcal{Z}\phi\mathcal{Z}\phi - 6 \operatorname{Tr} \mathcal{Z}\mathcal{Z}\phi\phi = E\,\mathcal{O}, \qquad E = 6. \tag{A.7}$$

Now it is time to interpret our calculations in terms of physics. The polynomial $\mathcal{O}$ is a gauge invariant local operator (state) and $\mathcal{H} = \mathfrak{D}_2$ is the one-loop dilatation operator. We have thus found that $\mathcal{O}$ is an eigenstate of $\mathcal{H}$ with energy is 6; its anomalous dimension therefore $6g^2$. We note our definition of coupling constant g in terms of the ordinary Yang-Mills coupling constant g_{YM} and rank N of the $\mathrm{U}(N)$ gauge group

$$g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} = \frac{\lambda}{8\pi^2}\,. \tag{A.8}$$

The classical dimension of $\mathcal{O}$ is computed using the operator $\mathfrak{D}_0$

$$\mathfrak{D}_0 = \operatorname{Tr} \mathcal{Z}\check{\mathcal{Z}} + \operatorname{Tr} \phi\check{\phi}, \qquad \mathfrak{D}_0\,\mathcal{O} = 4\,\mathcal{O}, \tag{A.9}$$

here it just counts the number of constituent fields. In conclusion, the scaling dimension of $\mathcal{O}$ up to one-loop is given by

$$D = 4 + 6g^2 + \ldots = 4 + \frac{3g_{\mathrm{YM}}^2 N}{4\pi^2}\,. \tag{A.10}$$

The state $\mathcal{O}$ is a descendant of the Konishi operator $\operatorname{Tr} \Phi_m\Phi_m$, see e.g. Sec. 2.4.4.

A.2 Planar Application

Let us repeat the example in the planar limit. Consider basis states of the type $|0010110\ldots10\rangle$. The labels are identified cyclically, e.g. $|00010110\ldots1\rangle$ represents the same state. A generic state $\mathcal{O}$ is a linear combination of these, for instance

$$\mathcal{O} = |0101\rangle - |0011\rangle. \tag{A.11}$$

Now consider a linear operator $\mathcal{H}$ on the space of states

$$\mathcal{H} = \sum_{p=1}^{L} \mathcal{H}_{p,p+1}, \qquad \mathcal{H}_{12} = 1 - \mathcal{P}_{12}. \tag{A.12}$$

The operator acts on all pairs of adjacent labels (enumerated by p) within a state. For each pair, it returns the same state, 1, minus the state with both labels interchanged, $\mathcal{P}_{p,p+1}$. Note that $\mathcal{H}_{L,L+1}$ is to be interpreted as $\mathcal{H}_{L,1}$ due to the cyclic nature of states. Furthermore, it suffices to give $\mathcal{H}_{12}$ acting on the first two labels. The action of $\mathcal{H}_{p,p+1}$ on the other labels is equivalent. For example

$$\mathcal{H}_{12}|0101\rangle = |0101\rangle - |1001\rangle. \tag{A.13}$$

In total we get for $\mathcal{H}\,\mathcal{O}$

$$\begin{aligned}
\mathcal{H}\mathcal{O} &= \big(\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{34} + \mathcal{H}_{41}\big)|0101\rangle - \big(\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{34} + \mathcal{H}_{41}\big)|0011\rangle \\
&= +\,|0101\rangle - |1001\rangle + |0101\rangle - |0011\rangle + |0101\rangle - |0110\rangle + |0101\rangle - |1100\rangle \\
&\quad - |0011\rangle + |0011\rangle - |0011\rangle + |0101\rangle - |0011\rangle + |0011\rangle - |0011\rangle + |1010\rangle \\
&= +6|0101\rangle - 6|0011\rangle = 6\,\mathcal{O}.
\end{aligned} \tag{A.14}$$

The physical interpretation is as in the previous section. The major difference is that double-trace terms which arise in a non-planar computation, c.f. (A.4), are discarded right away in the planar limit, c.f. (A.13). In this particular example, the non-planar terms cancel in the end and therefore the planar approximation happens to be exact.

Appendix B

Spinors in Various Dimensions

In this appendix we present a selection of useful identities when dealing with chiral spinors in four, six and ten dimensions.

B.1 Four Dimensions

There are two types of spinor indices, $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ belonging to the two $\mathfrak{su}(2)$ factors of $\mathfrak{so}(4)$. There are two types of invariant objects, ε and σ. There are four types of totally antisymmetric tensors $\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}, \varepsilon^{\dot{\alpha}\dot{\beta}}$ and it is convenient to use four types of sigma symbols (Pauli matrices) $\sigma^{\mu}_{\dot{\alpha}\beta}, \sigma^{\mu}_{\beta\dot{\alpha}}, \sigma_{\mu}^{\dot{\alpha}\beta}, \sigma_{\mu}^{\beta\dot{\alpha}}$. We can now suppress spinor indices and use a matrix notation, in all cases it should be clear which symbol to use. The σ's are defined by the relation

$$\sigma_{\{\mu}\sigma_{\nu\}} = \eta_{\mu\nu}. \tag{B.1}$$

The different ordering of spinor indices was introduced artificially, we remove it by the identification

$$\sigma^{\mu}_{\dot{\alpha}\beta} = \sigma^{\mu}_{\beta\dot{\alpha}}, \qquad \sigma_{\mu}^{\dot{\alpha}\beta} = \sigma_{\mu}^{\beta\dot{\alpha}}. \tag{B.2}$$

Here are some identities involving ε's in matrix notation

$$\varepsilon^{\mathsf{T}} = -\varepsilon, \qquad \varepsilon\varepsilon = -1, \qquad \varepsilon\sigma_{\mu} = \sigma_{\mu}\varepsilon. \tag{B.3}$$

The Fierz identities for the σ's read

$$\sigma_{\mu}^{\dot{\alpha}\beta}\sigma^{\mu}_{\dot{\gamma}\delta} = 2\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\beta}_{\delta}, \qquad \sigma_{\mu}^{\dot{\alpha}\beta}\sigma^{\mu,\dot{\gamma}\delta} = 2\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}, \qquad \sigma_{\mu,\dot{\alpha}\beta}\sigma^{\mu}_{\dot{\gamma}\delta} = 2\varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon_{\beta\delta} \tag{B.4}$$

and the completeness relation for antisymmetric tensors is

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} - \delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} = 2\delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]}. \tag{B.5}$$

B.2 Six Dimensions

In six dimensions there is one type of spinor index $a = 1, 2, 3, 4$, two totally antisymmetric tensors $\varepsilon_{abcd}, \varepsilon^{abcd}$ and two sigma symbols σ^{m}_{ab} and σ_{m}^{ab}. Again we can suppress spinor indices in a matrix notation. The sigma symbols are antisymmetric

$$\sigma_{m}^{\mathsf{T}} = -\sigma_{m} \tag{B.6}$$

and its indices can be raised or lowered by the totally antisymmetric tensor

$$\sigma^{m,ab} = \tfrac{1}{2}\varepsilon^{abcd}\sigma^{m}_{cd}, \qquad \sigma_{m,ab} = \tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd}_{m}. \tag{B.7}$$

They satisfy the Clifford algebra

$$\sigma_{\{m}\sigma_{n\}} = \eta_{mn}. \tag{B.8}$$

Finally, we note the Fierz identities for the $\sigma's$

$$\sigma^{ab}_{m}\sigma^{m}_{cd} = 2\delta^{a}_{d}\delta^{b}_{c} - 2\delta^{a}_{c}\delta^{b}_{d}, \qquad \sigma^{ab}_{m}\sigma^{m,cd} = -2\varepsilon^{abcd}, \qquad \sigma_{m,ab}\sigma^{m}_{cd} = -2\varepsilon_{abcd}. \tag{B.9}$$

B.3 Ten Dimensions

We will denote spinor indices in ten dimensions by $A, B, \ldots = 1, \ldots, 16$. There are two sigma symbols Σ^{M}_{AB} and Σ^{AB}_{M} and we can suppress spinor indices. The sigma symbols are symmetric

$$\Sigma^{\mathsf{T}}_{M} = \Sigma_{M} \tag{B.10}$$

and satisfy

$$\Sigma_{\{M}\Sigma_{N\}} = \eta_{MN}. \tag{B.11}$$

For the construction of supersymmetric gauge theory, there is one particularly useful identity

$$\Sigma_{M,AB}\Sigma^{M}_{CD} + \Sigma_{M,AC}\Sigma^{M}_{DB} + \Sigma_{M,AD}\Sigma^{M}_{BC} = 0. \tag{B.12}$$

In order to obtain $\mathcal{N} = 4$ SYM from the ten-dimensional supersymmetric gauge theory we reduce the ten-dimensional spacetime to four spacetime and six internal dimensions. We will assume that a spinor Ψ^{A} in ten dimensions decomposes into $\Psi_{\alpha a} + \Psi^{a}_{\dot\alpha}$ in $4+6$ dimensions. Accordingly, the sigma symbols in ten dimensions split as follows

$$\begin{aligned}
\Sigma^{AB}_{\mu} &= \sigma_{\mu,\alpha\dot\beta}\delta^{b}_{a} + \sigma_{\mu,\dot\alpha\beta}\delta^{b}_{a},\\
\Sigma_{\mu,AB} &= \sigma^{\alpha\dot\beta}_{\mu}\delta^{a}_{b} + \sigma^{\dot\alpha\beta}_{\mu}\delta^{b}_{a},\\
\Sigma^{AB}_{m} &= -\sigma_{m,ab}\varepsilon_{\alpha\beta} - \sigma^{ab}_{m}\varepsilon_{\dot\alpha\dot\beta},\\
\Sigma_{m,AB} &= \sigma^{ab}_{m}\varepsilon^{\alpha\beta} + \sigma_{m,ab}\varepsilon^{\dot\alpha\dot\beta}.
\end{aligned} \tag{B.13}$$

Appendix C

SYM in Ten Dimensions

Four-dimensional maximally supersymmetric Yang-Mills theory is most conveniently derived from its ten-dimensional supersymmetric counterpart. We will therefore present the ten-dimensional theory, in superspace or in components, in the following two sections. In this work we have not made use of these theories, except maybe for the dimensional reduction scheme, which relies on the component theory.

C.1 Ten-Dimensional Gauge Theory in Superspace

Let us first consider $\mathcal{N} = 1$ gauge theory in ten-dimensional superspace [144]. Superspace is parameterised by ten bosonic coordinates X^M and sixteen fermionic coordinates Θ^A. Here, indices $M, N, \ldots$ refer to ten-component vectors and indices $A, B, \ldots$ to sixteen-component spinors of $\mathfrak{so}(10)$. The covariant derivatives on this space are defined as

$$D_M = \partial_M, \qquad D_A = \partial_A + \Sigma^M_{AB}\Theta^B\partial_M. \tag{C.1}$$

The fermionic derivatives satisfy the anticommutation relation

$$\{D_A, D_B\} = 2\Sigma^M_{AB}D_M, \tag{C.2}$$

while commutators with a bosonic derivative D_M vanish.[1] In other words, superspace is a space with torsion given by Σ^M_{AB}. The matrices Σ^M are the chiral projections of the gamma matrices of $\mathfrak{so}(10)$. They are symmetric $\Sigma^M_{AB} = \Sigma^M_{BA}$ and obey

$$\Sigma^{\{M}_{AB}\Sigma^{N\},BC} = \delta^C_A\delta^{MN}. \tag{C.3}$$

In App. B.3 we present our notation and some useful identities.

To have real fermionic coordinates Θ^A, the signature of spacetime must be either $(9,1)$ or $(5,5)$. Whereas Minkowski space $(9,1)$ is certainly the correct choice in terms of physics, it may be more useful to work in Euclidean space $(10,0)$ when computing Feynman diagrams. In fact, the actual signature of spacetime does not matter for all the algebraic considerations in this work as we can do Wick rotations at any point. We will therefore not distinguish between different signatures of spacetime and algebras. Alternatively, one may consider a complexified space/algebra where the signature is irrelevant.

[1] This space may also be considered as the quotient space G/H of the super Poincaré group G as defined by (C.1) and the Lorentz group H.

On this space we define a gauge theory with the supercovariant derivatives

$$\mathcal{D}_M = D_M - ig\mathcal{A}_M, \qquad \mathcal{D}_A = D_A - ig\mathcal{A}_A. \tag{C.4}$$

We will assume the gauge group to be SU(N) or U(N) and all adjoint fields $\mathcal{A}$ to be (traceless) Hermitian $N \times N$ matrices. Under a gauge transformation $U(X, \Theta) \in \mathrm{U}(N)$ the gauge fields transform canonically according to

$$\mathcal{A}_M \mapsto U\mathcal{A}_M U^{-1} - ig^{-1}\, D_M U\, U^{-1}, \qquad \mathcal{A}_A \mapsto U\mathcal{A}_A U^{-1} - ig^{-1}\, D_A U\, U^{-1}. \tag{C.5}$$

The covariant field strengths of the gauge field are

$$\begin{aligned} \{\mathcal{D}_A, \mathcal{D}_B\} &= 2\Sigma^M_{AB}\mathcal{D}_M - ig\mathcal{F}_{AB}, \\ [\mathcal{D}_A, \mathcal{D}_M] &= -ig\mathcal{F}_{AM}, \\ [\mathcal{D}_M, \mathcal{D}_N] &= -ig\mathcal{F}_{MN}. \end{aligned} \tag{C.6}$$

We can now impose a constraint on the gauge field, namely that the field strength $\mathcal{F}_{AB}$ vanishes

$$\mathcal{F}_{AB} = 0. \tag{C.7}$$

This field strength can be decomposed into two $\mathfrak{so}(10)$ irreducible modules, **10** and **126**. The vanishing of the **10** part determines the bosonic gauge field $\mathcal{A}_M$ in terms of the fermionic one. The **126** part has much more drastic consequences as it not only reduces the number of independent components, but also implies equations of motion for the gauge field. Before stating these, we present two important consequences of the constraint and the Jacobi identities

$$\begin{aligned} [\mathcal{D}_A, \mathcal{D}_M] &= -ig\mathcal{F}_{AM} = ig\Sigma_{M,AB}\Psi^B, \\ \{\mathcal{D}_A, \Psi^B\} &= \tfrac{i}{2}g^{-1}\Sigma^{M,BC}\Sigma^N_{CA}[\mathcal{D}_M, \mathcal{D}_N] = \tfrac{1}{2}\Sigma^{M,BC}\Sigma^N_{CA}\mathcal{F}_{MN}. \end{aligned} \tag{C.8}$$

The first shows that the **144** part of the field strength $\mathcal{F}_{AM}$ is zero, it can be proved by using the Jacobi identity and inserting the constraint. The second one can be proved by projecting on the **1**, **45**, **210** parts and using the Jacobi identity and constraint. The equations of motion which follow from (C.7) in much the same way as (C.8) are

$$\begin{aligned} [\mathcal{D}_N, \mathcal{F}^{MN}] &= -\tfrac{i}{2}g\Sigma^M_{AB}\{\Psi^A, \Psi^B\}, \\ \Sigma^M_{AB}[\mathcal{D}_M, \Psi^B] &= 0. \end{aligned} \tag{C.9}$$

C.2 Ten-Dimensional Gauge Theory in Components

Let us now leave superspace and represent superfields by a collection of ordinary fields. It has been shown that the only independent components of the gauge field are the $\Theta = 0$ components of $\mathcal{A}_M$ and Ψ^B once we impose the constraint (C.7). All other components can be gauged away or are bosonic derivatives of the fundamental fields. By fixing the gauge along the Θ coordinates we can restrict to the $\Theta = 0$ components of $\mathcal{A}_M$ and Ψ^B. These will become the fundamental fields of the field theory

$$\mathcal{W} = (\mathcal{D}_M, \Psi^A), \tag{C.10}$$

which we will collectively refer to as $\mathcal{W}$.

The equations of motion (C.9) force the fundamental fields $\mathcal{A}_M$ and Ψ^B on shell. These can be encoded into the Lagrangian

$$\mathcal{L}_{10}[\mathcal{W}] = \tfrac{1}{4}\operatorname{Tr}\mathcal{F}^{MN}\mathcal{F}_{MN} + \tfrac{1}{2}\operatorname{Tr}\Psi^A\Sigma^M_{AB}\mathcal{D}_M\Psi^B. \tag{C.11}$$

This is the Lagrangian of a ten-dimensional gauge field $\mathcal{A}_M$ coupled to a (real) chiral spinor Ψ^A in the adjoint representation of the gauge group. The covariant derivative $\mathcal{D}_M\mathcal{W}$ of an adjoint field $\mathcal{W}$ is defined as the commutator

$$\mathcal{D}_M\mathcal{W} := [\mathcal{D}_M, \mathcal{W}] = \partial_M\mathcal{W} - ig\mathcal{A}_M\mathcal{W} + ig\mathcal{W}\mathcal{A}_M. \tag{C.12}$$

Although we have dropped fermionic coordinates Θ, translations along them are still possible. For that purpose we introduce fermionic translation generators $\mathfrak{Q}_A$ which act on fields as though they were derivatives along Θ. Equivalently we introduce bosonic translation generators $\mathfrak{P}_M$ which act on fields rather than coordinates

$$\mathfrak{Q}_A \mathrel{\widehat{=}} D_A, \qquad \mathfrak{P}_M \mathrel{\widehat{=}} D_M. \tag{C.13}$$

The derivatives are taken to be covariant when acting on gauge invariant states. Written in terms of variations $\delta_{\epsilon,e} = \epsilon^A\,\mathfrak{Q}_A + e^M\,\mathfrak{P}_M$ supertranslations of the fundamental fields are given by

$$\begin{aligned}\delta_{\epsilon,e}\,\mathcal{D}_M &= ig\epsilon^A\Sigma_{M,AB}\Psi^B + ige^N\mathcal{F}_{MN},\\ \delta_{\epsilon,e}\,\Psi^A &= \tfrac{1}{2}\Sigma^{M,AB}\Sigma^N_{BC}\epsilon^C\mathcal{F}_{MN} + e^M\mathcal{D}_M\Psi^A.\end{aligned} \tag{C.14}$$

These satisfy the usual supersymmetry relations (C.2) or (C.6). Note, however, that the algebra closes only on-shell, i.e. up to terms proportional to the equations of motion. The Lagrangian (C.11) is invariant under fermionic translations up to a total derivative.

C.3 $\mathcal{N} = 4$ SYM from Ten Dimensions

To obtain $\mathcal{N} = 4$ SYM from the ten-dimensional theories, we split up ten-dimensional spacetime into a four-dimensional spacetime and a six-dimensional internal space. The fields split up according to $\mathcal{D}_M = \mathcal{D}_\mu - ig\Phi_m$, where Φ is a multiplet of six scalars, and $\Psi^A = \Psi_{\alpha a} + \dot\Psi^a_{\dot\alpha}$. The decomposition of the sigma symbols is presented in App. B.3.

Appendix D

The Algebra $\mathfrak{u}(2,2|4)$

In this appendix we shall present the algebra $\mathfrak{u}(2,2|4)$, a slightly enlarged version of the superconformal algebra, decomposed in terms of spacetime and internal $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ symmetry.

D.1 Commutation Relations

The (complex) algebra $\mathfrak{u}(2,2|4) = \mathfrak{gl}(4|4)$ is the algebra of $(4|4) \times (4|4)$ (complex) supermatrices. Using the generators $\mathfrak{J}$ we can parameterise an element $j{\cdot}\mathfrak{J}$ of the algebra by the adjoint vector j. For our purposes it is useful to break up the matrix in $2|4|2$ (even|odd|even) rows and columns, the supermatrices will split up according to

$$j{\cdot}\mathfrak{J} = \left(\begin{array}{c|c|c} l^{\beta}{}_{\alpha} + \frac{1}{2}\delta^{\beta}_{\alpha}(d+b-c) & q^{\beta}{}_{a} & p^{\beta\dot{\alpha}} \\ \hline s^{b}{}_{\alpha} & r^{b}{}_{a} - \frac{1}{2}\delta^{b}_{a}c & \dot{q}^{b\dot{\alpha}} \\ \hline -k_{\dot{\beta}\alpha} & -\dot{s}_{\dot{\beta}a} & -\dot{l}_{\dot{\beta}}{}^{\dot{\alpha}} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}(-d+b-c) \end{array}\right). \tag{D.1}$$

The commutation relations of the generators can be read off from the matrix representation of $[j{\cdot}\mathfrak{J}, j'{\cdot}\mathfrak{J}]$. Let us now discuss the generators independently. First of all, there are the $\mathfrak{su}(2), \mathfrak{su}(4), \mathfrak{su}(2)$ rotation generators $\mathfrak{L}^{\alpha}{}_{\beta}, \mathfrak{R}^{a}{}_{b}, \dot{\mathfrak{L}}^{\dot{\beta}}{}_{\dot{\alpha}}$. The indices of any generator $\mathfrak{J}$ transform canonically according to

$$\begin{aligned} [\mathfrak{L}^{\alpha}{}_{\beta}, \mathfrak{J}_{\gamma}] &= \delta^{\alpha}_{\gamma}\mathfrak{J}_{\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}\mathfrak{J}_{\gamma}, \quad [\mathfrak{L}^{\alpha}{}_{\beta}, \mathfrak{J}^{\gamma}] = -\delta^{\gamma}_{\beta}\mathfrak{J}^{\alpha} + \tfrac{1}{2}\delta^{\alpha}_{\beta}\mathfrak{J}^{\gamma}, \\ [\mathfrak{R}^{a}{}_{b}, \mathfrak{J}_{c}] &= \delta^{a}_{c}\mathfrak{J}_{b} - \tfrac{1}{4}\delta^{a}_{b}\mathfrak{J}_{c}, \quad [\mathfrak{R}^{a}{}_{b}, \mathfrak{J}^{c}] = -\delta^{c}_{b}\mathfrak{J}^{a} + \tfrac{1}{4}\delta^{a}_{b}\mathfrak{J}^{c}, \\ [\dot{\mathfrak{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathfrak{J}_{\dot{\gamma}}] &= \delta^{\dot{\alpha}}_{\dot{\gamma}}\mathfrak{J}_{\dot{\beta}} - \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\mathfrak{J}_{\dot{\gamma}}, \quad [\dot{\mathfrak{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathfrak{J}^{\dot{\gamma}}] = -\delta^{\dot{\gamma}}_{\dot{\beta}}\mathfrak{J}^{\dot{\alpha}} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\mathfrak{J}^{\dot{\gamma}}. \end{aligned} \tag{D.2}$$

The charges $\mathfrak{D}, \mathfrak{B}, \mathfrak{C}$ (dilatation generator, hypercharge, central charge) of the generators are given by

$$[\mathfrak{D}, \mathfrak{J}] = \dim(\mathfrak{J})\,\mathfrak{J}, \qquad [\mathfrak{B}, \mathfrak{J}] = \mathrm{hyp}(\mathfrak{J})\,\mathfrak{J}, \qquad [\mathfrak{C}, \mathfrak{J}] = 0 \tag{D.3}$$

with non-vanishing dimensions

$$\dim(\mathfrak{P}) = -\dim(\mathfrak{K}) = 1, \quad \dim(\mathfrak{Q}) = \dim(\dot{\mathfrak{Q}}) = -\dim(\mathfrak{S}) = -\dim(\dot{\mathfrak{S}}) = \tfrac{1}{2} \tag{D.4}$$

and non-vanishing hypercharges

$$\mathrm{hyp}(\mathfrak{Q}) = -\mathrm{hyp}(\dot{\mathfrak{Q}}) = -\mathrm{hyp}(\mathfrak{S}) = \mathrm{hyp}(\dot{\mathfrak{S}}) = \tfrac{1}{2}. \tag{D.5}$$

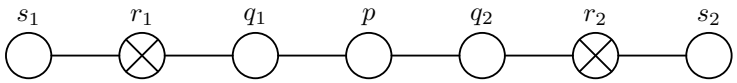

Figure D.2: Dynkin labels $[s_1; r_1; q_1, r, q_2; r_2; s_2]$ of $\mathfrak{su}(2,2|4)$. Note that in our convention the sign of the odd labels r_1, r_2 to be negative for the antisymmetric product of two fundamental representations.

Finally there are the translations $\mathfrak{P}_{\dot\alpha\beta}$, boosts $\mathfrak{K}_{\alpha\dot\beta}$ as well as their fermionic partners, the supertranslations $\mathfrak{Q}^a{}_\beta, \dot{\mathfrak{Q}}_{\dot\alpha b}$ and superboosts $\mathfrak{S}^\alpha{}_b, \dot{\mathfrak{S}}^{a\dot\beta}$. The translations and boosts commuting into themselves are given by

$$\begin{aligned}
[\mathfrak{S}^\alpha{}_a, \mathfrak{P}_{\dot\beta\gamma}] &= \delta^\alpha_\gamma \dot{\mathfrak{Q}}_{\dot\beta a}, & [\mathfrak{K}^{\alpha\dot\beta}, \dot{\mathfrak{Q}}_{\dot\gamma c}] &= \delta^{\dot\beta}_{\dot\gamma} \mathfrak{S}^\alpha{}_c, \\
[\dot{\mathfrak{S}}^{a\dot\alpha}, \mathfrak{P}_{\dot\beta\gamma}] &= \delta^{\dot\alpha}_{\dot\beta} \mathfrak{Q}^a{}_\gamma, & [\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{Q}^c{}_\gamma] &= \delta^\alpha_\gamma \dot{\mathfrak{S}}^{c\dot\beta}, \\
\{\dot{\mathfrak{Q}}_{\dot\alpha a}, \mathfrak{Q}^b{}_\beta\} &= \delta^b_a \mathfrak{P}_{\dot\alpha\beta}, & \{\dot{\mathfrak{S}}^{a\dot\alpha}, \mathfrak{S}^\beta{}_b\} &= \delta^a_b \mathfrak{K}^{\beta\dot\alpha},
\end{aligned} \tag{D.6}$$

while the translations and boosts commuting into rotations are given by

$$\begin{aligned}
[\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{P}_{\dot\gamma\delta}] &= \delta^{\dot\beta}_{\dot\gamma} \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\gamma \dot{\mathfrak{L}}^{\dot\beta}{}_{\dot\delta} + \delta^\alpha_\gamma \delta^{\dot\beta}_{\dot\delta} \mathfrak{D}, \\
\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\} &= \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \tfrac{1}{2}\delta^b_a \delta^\alpha_\beta (\mathfrak{D} - \mathfrak{C}), \\
\{\dot{\mathfrak{S}}^{a\dot\alpha}, \dot{\mathfrak{Q}}_{\dot\beta b}\} &= \delta^a_b \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta} \mathfrak{R}^a{}_b + \tfrac{1}{2}\delta^a_b \delta^{\dot\alpha}_{\dot\beta} (\mathfrak{D} + \mathfrak{C}).
\end{aligned} \tag{D.7}$$

As we see, the hypercharge $\mathfrak{B}$ never appears on the right hand side, it can be dropped, leading to $\mathfrak{su}(2,2|4)$. Conversely, when restricting to representations with zero central charge $\mathfrak{C}$, the resulting algebra is $\mathfrak{pu}(2,2|4)$, which becomes $\mathfrak{psu}(2,2|4)$ after removing $\mathfrak{B}$ as well.

In this work we deal with two further operators, $\mathcal{L}$ and $\delta\mathfrak{D} = g^2\mathcal{H}$, which are not part of $\mathfrak{u}(2,2|4)$. The operator $\mathcal{L}$ measures the length, i.e. the number of components fields, of a state. The anomalous dilatation operator $\delta\mathfrak{D}(g) = \mathfrak{D}(g) - \mathfrak{D}(0)$, or equivalently the Hamiltonian $\mathcal{H}$, commutes with $\mathfrak{u}(2,2|4)$.

D.2 Labels

Let us collect some of our notation concerning labels of states and multiplets, see also Sec. 1.5. The Dynkin labels of $\mathfrak{su}(2,2|4)$ are, c.f. Fig. D.2,

$$[s_1; r_1; q_1, p, q_2; r_2; s_2], \tag{D.8}$$

where $[q_1, p, q_2]$ and $[s_1, r, s_2]$ with $r = -r_1 - q_1 - p - q_2 - r_2$ are the Dynkin labels of $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$, respectively. Note that s_1, s_2 are *twice* the spins of the Lorentz algebra. These labels are given as the eigenvalues $L^\alpha{}_\beta, \dot L^{\dot\alpha}{}_{\dot\beta}, R^a{}_b$ of Cartan generators $\mathfrak{L}^\alpha{}_\beta, \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}, \mathfrak{R}^a{}_b$ $(\alpha = \beta, \dot\alpha = \dot\beta, a = b)$

$$\begin{aligned}
s_1 &= L^2{}_2 - L^1{}_1, & s_2 &= \dot L^2{}_2 - \dot L^1{}_1, \\
r_1 &= \tfrac{1}{2}D - \tfrac{1}{2}C - L^1{}_1 + R^1{}_1, & r_2 &= \tfrac{1}{2}D + \tfrac{1}{2}C - \dot L^1{}_1 - R^4{}_4, \\
q_1 &= R^2{}_2 - R^1{}_1, & q_2 &= R^4{}_4 - R^3{}_3, \\
p &= R^3{}_3 - R^2{}_2, & r &= -D + L^1{}_1 + \dot L^1{}_1.
\end{aligned} \tag{D.9}$$

We also use the notation

$$(D_0, s_1, s_2; q_1, p, q_2; B, L) \tag{D.10}$$

to describe states of the classical theory. The label B is the hypercharge measured by $\mathfrak{B}$. The length L corresponds to the number of fields within a state, it is measured by the operator $\mathcal{L}$ which is not part of $\mathfrak{u}(2,2|4)$. D_0 is the classical dimension and we introduce E as the 'energy' or anomalous dimension measured by $\mathcal{H} = g^{-2}\delta\mathfrak{D}$. Together they form the scaling dimension $D = D_0 + g^2E$ which is measured by $\mathfrak{D} = \mathfrak{D}_0 + g^2\mathcal{H}$. Usually we will state only the classical dimension D_0 to specify a state; the corresponding energy E will be the main result of a our computations. It is useful to know how to translate between the dimension D and Dynkin labels r_1, r_2, r, see (D.9),

$$\begin{aligned}
r_1 &= \tfrac{1}{2}D - \tfrac{1}{2}C - \tfrac{1}{2}p - \tfrac{3}{4}q_1 - \tfrac{1}{4}q_2 + \tfrac{1}{2}s_1,\\
r_2 &= \tfrac{1}{2}D + \tfrac{1}{2}C - \tfrac{1}{2}p - \tfrac{1}{4}q_1 - \tfrac{3}{4}q_2 + \tfrac{1}{2}s_2,\\
r &= -D - \tfrac{1}{2}s_1 - \tfrac{1}{2}s_2,\\
r &= -r_1 - q_1 - p - q_2 - r_2\\
D &= -\tfrac{1}{2}s_1 + r_1 + q_1 + p + q_2 + r_2 - \tfrac{1}{2}s_2.
\end{aligned} \tag{D.11}$$

D.3 The Quadratic Casimir

The quadratic Casimir of $\mathfrak{u}(2,2|4)$ is

$$\begin{aligned}
\mathfrak{J}^2 = {}& \tfrac{1}{2}\mathfrak{D}^2 + \tfrac{1}{2}\mathfrak{L}^{\gamma}{}_{\delta}\mathfrak{L}^{\delta}{}_{\gamma} + \tfrac{1}{2}\dot{\mathfrak{L}}^{\dot\gamma}{}_{\dot\delta}\dot{\mathfrak{L}}^{\dot\delta}{}_{\dot\gamma} - \tfrac{1}{2}\mathfrak{K}^{c}{}_{d}\mathfrak{K}^{d}{}_{c}\\
& - \tfrac{1}{2}[\mathfrak{Q}^{c}{}_{\gamma}, \mathfrak{S}^{\gamma}{}_{c}] - \tfrac{1}{2}[\dot{\mathfrak{Q}}_{\dot\gamma c}, \dot{\mathfrak{S}}^{\dot\gamma c}] - \tfrac{1}{2}\{\mathfrak{P}_{\dot\gamma\delta}, \mathfrak{K}^{\delta\dot\gamma}\} - \mathfrak{B}\mathfrak{C}.
\end{aligned} \tag{D.12}$$

In $\mathfrak{psu}(2,2|4)$ the last term $\mathfrak{B}\mathfrak{C}$ is absent. For highest weight states, which are annihilated by raising operators $\mathfrak{J}^+$ (1.53), we can conveniently evaluate $\mathfrak{J}^2$ by using the standard trick of turning the anticommutators into commutators. We find

$$\begin{aligned}
J^2 = {}& \tfrac{1}{4}s_1(s_1+2) + \tfrac{1}{4}s_2(s_2+2) + \tfrac{1}{2}D^2 + 2D - BC\\
& - \tfrac{1}{4}q_1(q_1+2) - \tfrac{1}{4}q_2(q_2+2) - \tfrac{1}{8}(2p+q_1+q_2)^2 - (2p+q_1+q_2).
\end{aligned} \tag{D.13}$$

D.4 The Oscillator Representation

Let us explain the use of oscillators for fields and generators in terms of the algebra $\mathfrak{gl}(M)$: We write[1]

$$\mathfrak{J}^{A}{}_{B} = \mathbf{A}^{\dagger}_{B}\mathbf{A}^{A}, \qquad \text{with } A, B = 1, \ldots, M. \tag{D.14}$$

Using the commutators

$$[\mathbf{A}^{A}, \mathbf{A}^{\dagger}_{B}] = \delta^{A}_{B}, \qquad [\mathbf{A}^{A}, \mathbf{A}^{B}] = [\mathbf{A}^{\dagger}_{A}, \mathbf{A}^{\dagger}_{B}] = 0 \tag{D.15}$$

it is a straightforward exercise to show that the generators $\mathfrak{J}$ satisfy the $\mathfrak{gl}(M)$ algebra.

[1]Strictly speaking, the oscillators $\mathbf{A}$ and $\mathbf{A}^{\dagger}$ are independent. Only in one of the real forms of $\mathfrak{gl}(M)$ they would be related by conjugation.

To construct an oscillator representation for $\mathfrak{u}(2,2|4)$, c.f. [93], we will consider two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ with $\alpha, \dot\alpha = 1,2$ and one set of fermionic oscillator $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ with $a = 1,2,3,4$. The non-vanishing commutators of oscillators are taken to be

$$[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta, \qquad [\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \qquad \{\mathbf{c}^a, \mathbf{c}^\dagger_b\} = \delta^a_b. \tag{D.16}$$

We assume that the oscillators $\mathbf{A}^A = (\mathbf{a}, \mathbf{b}^\dagger, \mathbf{c})$ and $\mathbf{A}^\dagger_A = (\mathbf{a}^\dagger, -\mathbf{b}, \mathbf{c}^\dagger)$ form a fundamental and conjugate fundamental multiplet of $\mathfrak{u}(2,2|4)$. Then, the bilinears $\mathbf{A}^\dagger_A \mathbf{A}^B$ generate the algebra $\mathfrak{u}(2,2|4)$ as described above. By comparing to the matrix form (D.1) we can read off the generators in $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ notation. The canonical forms for rotation generators of $\mathfrak{su}(2)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$ are

$$\begin{aligned}
\mathfrak{L}^\alpha{}_\beta &= \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \tfrac{1}{2}\delta^\alpha_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma, \\
\dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} &= \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^{\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}, \\
\mathfrak{R}^a{}_b &= \mathbf{c}^\dagger_b \mathbf{c}^a - \tfrac{1}{4}\delta^a_b \mathbf{c}^\dagger_c \mathbf{c}^c.
\end{aligned} \tag{D.17}$$

Under these the fields (1.78) transform canonically. We write the corresponding three $\mathfrak{u}(1)$ charges as

$$\begin{aligned}
\mathfrak{D} &= 1 + \tfrac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \tfrac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}, \\
\mathfrak{C} &= 1 - \tfrac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \tfrac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} - \tfrac{1}{2}\mathbf{c}^\dagger_c \mathbf{c}^c, \\
\mathfrak{B} &= \qquad \tfrac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma - \tfrac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}.
\end{aligned} \tag{D.18}$$

The remaining off-diagonal generators are

$$\begin{aligned}
\mathfrak{Q}^a{}_\alpha &= \mathbf{a}^\dagger_\alpha \mathbf{c}^a, \qquad & \mathfrak{S}^\alpha{}_a &= \mathbf{c}^\dagger_a \mathbf{a}^\alpha, \\
\dot{\mathfrak{Q}}_{\dot\alpha a} &= \mathbf{b}^\dagger_{\dot\alpha} \mathbf{c}^\dagger_a, \qquad & \dot{\mathfrak{S}}^{\dot\alpha a} &= \mathbf{b}^{\dot\alpha} \mathbf{c}^a, \\
\mathfrak{P}_{\alpha\dot\beta} &= \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta}, \qquad & \mathfrak{K}^{\alpha\dot\beta} &= \mathbf{a}^\alpha \mathbf{b}^{\dot\beta}.
\end{aligned} \tag{D.19}$$

Quite naturally the algebra $\mathfrak{u}(2,2|4)$ is realised by the generators (D.17,D.18,D.19).[2] We have written this in a $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(4)$ covariant way. In fact one can combine the indices a and α into a superindex and obtain a manifest $\mathfrak{su}(2) \times \mathfrak{su}(2|4)$ notation. The generators with two lower or two upper indices, $\mathfrak{P}, \dot{\mathfrak{Q}}, \mathfrak{K}, \dot{\mathfrak{S}}$, together with the remaining charges complete the $\mathfrak{u}(2,2|4)$ algebra.

Instead of one fermionic oscillator $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ with $a = 1,2,3,4$, we can also introduce two pairs of oscillators $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ and $(\mathbf{d}^{\dot a}, \mathbf{d}^\dagger_{\dot a})$ with $a, \dot a = 1,2$. These should be grouped as $\mathbf{A}^A = (\mathbf{a}, \mathbf{b}^\dagger, \mathbf{c}, \mathbf{d}^\dagger)$ and $\mathbf{A}^\dagger_A = (\mathbf{a}^\dagger, -\mathbf{b}, \mathbf{c}^\dagger, \mathbf{d})$ into fundamental representations of $\mathfrak{u}(2,2|4)$. Despite the fact that only $\mathfrak{su}(2)^4$ (or $\mathfrak{psu}(2|2)^2$ when using superindices) is manifest in this notation, it has the added benefit that we can define a physical vacuum state $|\mathcal{Z}\rangle$ by

$$\mathbf{a}^\alpha |\mathcal{Z}\rangle = \mathbf{b}^{\dot\alpha} |\mathcal{Z}\rangle = \mathbf{c}^a |\mathcal{Z}\rangle = \mathbf{d}^{\dot a} |\mathcal{Z}\rangle = 0. \tag{D.20}$$

This is the highest weight state of the field multiplet, see Sec. 1.9.

[2] Note that a shift of $\mathfrak{B}$ by a constant (-1) does not modify the algebra. Then the 1 in $\mathfrak{D}, \mathfrak{C}, \mathfrak{B}$ can be absorbed into $1 + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma}\mathbf{b}^{\dot\gamma} = \frac{1}{2}\mathbf{b}^{\dot\gamma}\mathbf{b}^\dagger_{\dot\gamma}$ to yield a canonical form.

$$n_{\mathbf{a}} = \begin{pmatrix} \frac{1}{2}D_0 + \frac{1}{2}B - \frac{1}{2}L - \frac{1}{2}s_1 \\ \frac{1}{2}D_0 + \frac{1}{2}B - \frac{1}{2}L + \frac{1}{2}s_1 \end{pmatrix}, \quad n_{\mathbf{b}} = \begin{pmatrix} \frac{1}{2}D_0 - \frac{1}{2}B - \frac{1}{2}L - \frac{1}{2}s_2 \\ \frac{1}{2}D_0 - \frac{1}{2}B - \frac{1}{2}L + \frac{1}{2}s_2, \end{pmatrix},$$

$$n_{\mathbf{c}} = \begin{pmatrix} \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \end{pmatrix}, \quad n_{\mathbf{d}} = \begin{pmatrix} \frac{1}{2}L + \frac{1}{2}B - \frac{1}{2}p - \frac{1}{4}q_1 - \frac{3}{4}q_2 \\ \frac{1}{2}L + \frac{1}{2}B - \frac{1}{2}p - \frac{1}{4}q_1 + \frac{1}{4}q_2 \end{pmatrix},$$

$$n_{\mathbf{c}} = \begin{pmatrix} \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B + \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B + \frac{1}{2}p + \frac{1}{4}q_1 + \frac{3}{4}q_2 \end{pmatrix}.$$

Table D.2: Oscillator excitation numbers for a state with given charges.

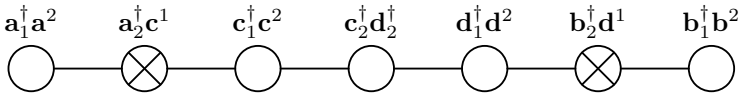

Figure D.4: Oscillator representation of simple roots.

In this context it is useful to know how to represent a state with a given weight

$$w = (D_0; s_1, s_2; q_1, p, q_2; B, L) \tag{D.21}$$

in terms of excitations of the oscillators. We introduce a multi-particle vacuum operator $|\mathcal{Z}, L\rangle$ which is the tensor product of L vacua $|\mathcal{Z}\rangle$. The oscillators $\mathbf{a}^\dagger_{p,\alpha}, \mathbf{b}^\dagger_{p,\dot\alpha}, \mathbf{c}^\dagger_{p,a}, \mathbf{d}^\dagger_{p,\dot a}$. now act on site p, where commutators of two oscillators vanish unless the sites agree. Equivalently, we define the unphysical multi-particle vacuum state $|0, L\rangle$. A generic state is written as

$$(\mathbf{a}^\dagger)^{n_{\mathbf{a}}}(\mathbf{b}^\dagger)^{n_{\mathbf{b}}}(\mathbf{c}^\dagger)^{n_{\mathbf{c}}}(\mathbf{d}^\dagger)^{n_{\mathbf{d}}}|\mathcal{Z}, L\rangle \qquad \text{or} \qquad (\mathbf{a}^\dagger)^{n_{\mathbf{a}}}(\mathbf{b}^\dagger)^{n_{\mathbf{b}}}(\mathbf{c}^\dagger)^{n_{\mathbf{c}}}|0, L\rangle. \tag{D.22}$$

By considering the weights of the oscillators as well as the central charge constraint, we find the number of excitations as given in Tab. D.2.[3] It is also useful to know how to represent the generators corresponding to the simple roots in terms of creation and annihilation operators, c.f. Fig. D.4.

[3]The components of the vectors correspond to the numbers of each component of the oscillators.

Appendix E

Tools for the $\mathfrak{su}(2)$ Sector

In this appendix we present a basic set of `Mathematica` routines to deal with $\mathfrak{su}(2)$ spin chains at higher-loops.

E.1 States

One of the basic objects is a *state* $|\ldots\rangle$, in `Mathematica` it will be represented by the function `Chain[...]`, e.g.

$$\texttt{Chain[0,0,1,0,1,1,0]} \mathrel{\widehat{=}} |0,0,1,0,1,1,0\rangle = \mathrm{Tr}\, \mathcal{Z}\mathcal{Z}\phi\mathcal{Z}\phi\phi\mathcal{Z}. \tag{E.1}$$

This function is undefined and `Mathematica` will leave it as it stands and not try to evaluate it. For example one can construct linear combinations of states, e.g.

$$2|0,0,1,1\rangle + |0,1,0,1\rangle \mathrel{\widehat{=}} \texttt{2 Chain[0,0,1,1] + Chain[0,1,0,1]}. \tag{E.2}$$

We consider spin chain states only modulo cyclic permutations. The order has to be implemented manually and we need a routine to shift states into some canonical order

```
SortChain[X_] := X /. C_Chain :>
  Module[{k}, Sort[Table[RotateLeft[C, k], {k, Length[C]}]][[1]]];
```

This function returns the argument `X` with all chains ordered. It works as follows: First of all, all states `C=Chain[...]` within `X` are found. For each `C` a list of all possible cyclic permutations is generated and sorted. The first element is taken as the canonically ordered state and returned.

We can now define a simple operation on states, the parity $\mathfrak{p}$ which reverses the spin chain and multiplies by $(-1)^L$

```
ChainParity[X_] := X /. C_Chain :> (-1)^Length[C] SortChain[Reverse[C]];
```

Conveniently, it shifts the states into a canonical order.

E.2 Interactions

The other basic object is an *interaction* $\{\ldots\}$ which will be represented by `Perm[...]`

$$\texttt{Perm[1,3,2]} \mathrel{\widehat{=}} \{1,3,2\}. \tag{E.3}$$

We need a representation for the action of permutation symbols $\{\ldots\}|\ldots\rangle$, this is done by

```
ApplyPerm[P_, C_] := P /. P0_Perm :> (C /. C0_Chain -> ApplyPermPC[P0, C0]);
ApplyPermPC[P_Perm, C_Chain] :=
  Module[{s}, Sum[PermuteList[C, P, s], {s, Length[C]}]];
```

The function `ApplyPerm` assumes `P` and `C` are linear combinations of interactions and states. It distributes the elementary interactions and states and passes on to `ApplyPermPC` for an elementary pair. This uses another function `PermuteList` to apply the permutation to each site `s` of the spin chain

```
PermuteList[C_Chain, Perm[], s_] := C;
PermuteList[C_Chain, P_Perm, s_] :=
  PermuteList[PermuteElements[C, Last[P] + s], Drop[P, -1], s];
PermuteElements[C_Chain, p_] :=
  Module[{p0 = Mod[p, Length[C], 1], p1 = Mod[p + 1, Length[C], 1]},
    ReplacePart[ReplacePart[C, C[[p0]], p1], C[[p1]], p0]];
```

The routine `PermuteList` recursively works on the permutation symbol `P` from the right and uses `PermuteElements` to perform the pairwise permutations.

E.3 Spectrum

To find the spectrum of an operator, we need to find a complete basis of states on which the operator closes. This basis is generated by

```
GenerateChains[L_, K_] :=
  (Chain @@ Join[Array[1 &, K], Array[0 &, L - K]])
    // Permutations // SortChain // Union;
```

The function returns a basis of states of `L` sites with `n` excitations. It is then convenient to have a method to evaluate the action of an operator on a basis of states

```
ActionMatrix[P_, C_] := CoeffList[ApplyPerm[P, C] // SortChain, C];
```

It returns a matrix that is equivalent to the action of `P` in the basis `C`. It requires the multi-purpose function

```
CoeffList[X_, L_] := Map[Coefficient[X, #] &, L];
```

which expands a linear expression `X` over a basis of atoms `L`.

E.4 An Example

We can now find the energy of states with length `L=4` and `K=2` excitations. Let us first construct a basis of states

```
Ops = GenerateChains[4, 2]
> {Chain[0, 0, 1, 1], Chain[0, 1, 0, 1]}
```

The one-loop Hamiltonian is given by $H_0 = \{\} - \{1\}$ so let us define

```
Ham = Perm[] - Perm[1];
```

and act on the above basis

```
MHam = ActionMatrix[Ham, Ops]
> {{2, -4}, {-2, 4}}
```

The eigenvalues are

```
Eigenvalues[MHam]
> {0, 6}
Eigenvectors[MHam]
> {{2, 1}, {-1, 1}}
```

where the energy $E = 0$ belongs to the state $2|0,0,1,1\rangle + |0,1,0,1\rangle$ and $E = 6$ to the Konishi state $-|0,0,1,1\rangle + |0,1,0,1\rangle$.

E.5 Commutators

For investigations of integrability we require methods to compute commutators of interactions abstractly. This is a straightforward implementation of the commutator of permutation symbols

```
CommutePerm[X_, Y_] :=
  X /. P1_Perm :> (Y /. P2_Perm -> CommutePerm12[P1, P2]) // SimplifyPerm;
CommutePerm12[Perm[X___], Perm[Y___]] :=
  Module[{MX = Max[X, 0] + 1, MY = Max[Y, 0] + 1, k},
    Sum[Perm @@ Join[{X} + k - 1, {Y} + MX - 1] -
        Perm @@ Join[{Y} + MX - 1, {X} + k - 1], {k, MX + MY - 1}]];
```

As above, `CommutePerm` distributes linear combinations and calls `CommutePerm12` for elementary commutators. Finally, we should simplify the permutation symbols using the rules in Sec. 6.2.1

```
SimplifyPerm[YY_] := (YY //.
  {Perm[X__ /; Min[X] != 1] :> Perm @@ ({X} - Min[X] + 1),
   Perm[X___, y_, y_, Z___] -> Perm[X, Z],
   Perm[X___, y_, z_, y_, W___]
     /; (z == y + 1) || (z == y - 1)
     -> Perm[X, W] - Perm[X, y, W] - Perm[X, z, W]
        + Perm[X, y, z, W] + Perm[X, z, y, W],
   Perm[X___, y_, z_, W___]
     /; z < y - 1
     -> Perm[X, z, y, W],
   Perm[X___, y_, z_, W__, y_, V___]
     /; (z == y - 1) && (! MemberQ[{W}, k_ /; k > y - 2])
     -> Perm[X, y, z, y, W, V],
   Perm[X___, y_, W__, z_, y_, V___]
     /; (z == y + 1) && (! MemberQ[{W}, k_ /; k < y + 2])
     -> Perm[X, W, y, z, y, V]}) // Expand;
```

For the construction of higher-loop interactions, one can write functions to construct a basis of interaction symbols for a given loop order and to solve commutators for unknown coefficients. As the construction still requires a sufficient amount of manual work, we will not present these here.

Appendix F

The Harmonic Action

The Hamiltonian density $\mathcal{H}_{12}$ is given by an $\mathfrak{psu}(2,2|4)$ invariant function acting on two sites,

$$\mathcal{H}_{12} = 2h(\mathcal{J}_{12}). \tag{F.1}$$

We will now describe explicitly how $\mathcal{H}_{12}$ acts on a state of two fields in the oscillator representation, see Sec. 1.9.

F.1 Generic Invariant Action

We will investigate the action of a generic function $f(\mathcal{J}_{12})$ on two oscillator sites. Let us introduce a collective oscillator $\mathbf{A}_A^\dagger = (\mathbf{a}_\alpha^\dagger, \mathbf{b}_{\dot\alpha}^\dagger, \mathbf{c}_a^\dagger, \mathbf{d}_{\dot a}^\dagger)$. A general state in $\mathcal{V}_{\rm F} \times \mathcal{V}_{\rm F}$ can be written as

$$|p_1, \dots, p_n; A\rangle = \mathbf{A}^\dagger_{p_1,A_1} \dots \mathbf{A}^\dagger_{p_n,A_n} |\mathcal{Z}\mathcal{Z}\rangle, \tag{F.2}$$

subject to the central charge constraints $\mathfrak{C}_1|X\rangle = \mathfrak{C}_2|X\rangle = 0$. The label $p_k = 1, 2$ determines the site on which the k-th oscillator acts. The action of $\mathfrak{psu}(2,2|4)$ conserves the number of each type of oscillator; it can however move oscillators between both sites. Therefore the action of $f(\mathcal{J}_{12})$ is

$$f(\mathcal{J}_{12})\,|p_1, \dots, p_n; A\rangle = \sum_{p'_1,\dots p'_n} c_{p,p',A}\,\delta_{C_1,0}\delta_{C_2,0}\,|p'_1, \dots, p'_n; A\rangle \tag{F.3}$$

with some coefficients $c_{s,s',A}$. The sums go over the sites $1, 2$ and $\delta_{C_1,0}$, $\delta_{C_2,0}$ project to states where the central charge at each site is zero. In view of the fact that oscillators represent indices of fields, see (1.79), a generic invariant operator $f(\mathcal{J}_{12})$ acts on two fields by moving indices between them.

F.2 The Harmonic Action

The action of the harmonic numbers within the Hamiltonian density $\mathcal{H}_{12} = 2h(\mathcal{J}_{12})$ turns out to be particularly simple. It does not depend on the types of oscillators A_k, but only on the number of oscillators which change the site

$$\mathcal{H}_{12}|s_1, \dots, s_n; A\rangle = \sum_{s'_1,\dots s'_n} c_{n,n_{12},n_{21}}\,\delta_{C_1,0}\delta_{C_2,0}\,|s'_1, \dots, s'_n; A\rangle. \tag{F.4}$$

Here n_{12}, n_{21} count the number of oscillators hopping from site 1 to 2 or vice versa. The coefficients $c_{n,n_{12},n_{21}}$ are given by

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}} \frac{\Gamma\big(\frac{1}{2}(n_{12}+n_{21})\big)\Gamma\big(1+\frac{1}{2}(n-n_{12}-n_{21})\big)}{\Gamma\big(1+\frac{1}{2}n\big)}\,. \tag{F.5}$$

In the special case of no oscillator hopping we find

$$c_{n,0,0} = h(\tfrac{1}{2}n), \tag{F.6}$$

which can be regarded as a regularisation of (F.5). We will refer to this action given by (F.4,F.5,F.6) as the '*harmonic action*'. Interestingly, we find that the action of the Hamiltonian density using the $\mathfrak{su}(4|2) \times \mathfrak{su}(2)$ invariant vacuum (c.f. Sec. 1.9) is given by exactly the same expressions.

F.3 Proof

To prove that $\mathcal{H}_{12}$ is given by (F.4,F.5,F.6) it suffices to show

$$[\mathfrak{J}_{12}, \mathcal{H}_{12}] = 0, \qquad \mathcal{H}_{12}\,\mathcal{V}_j = 2h(j)\,\mathcal{V}_j. \tag{F.7}$$

The invariance of $\mathcal{H}_{12}$ under the subalgebra $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$ is straightforward: These generators only change the types of oscillators, whereas the harmonic action does not depend on that. In contrast, the remaining generators change the number of oscillators by two.

Consider a generator which increases the number of oscillators by two, e.g. $\mathfrak{P}_{12,\alpha\dot\beta}$, and act with $\mathcal{H}_{12}\mathfrak{P}_{12,\alpha\dot\beta}$ on a generic state. First we apply $\mathfrak{P}$

$$\mathfrak{P}_{12,\alpha\dot\beta}|p_1,\ldots,p_n;A\rangle = |1,1,p_1,\ldots,p_n;A'\rangle + |2,2,p_1,\ldots,p_n;A'\rangle, \tag{F.8}$$

and get a state with two new oscillators, $A' = (\alpha,\dot\beta,A)$. We apply the Hamiltonian density (F.4) to this state and get eight terms (to be summed over all p'_k). In two of these terms, both new oscillators end up at site 1

$$c_{n+2,n_{12},n_{21}}|1,1,p'_1,\ldots,p'_n;A'\rangle + c_{n+2,n_{12},n_{21}+2}|1,1,p'_1,\ldots,p'_n;A'\rangle. \tag{F.9}$$

Here, n_{12}, n_{21} refer only to the hopping of the old oscillators. Both coefficients can be combined using (F.5)

$$c_{n+2,n_{12},n_{21}} + c_{n+2,n_{12},n_{21}+2} = c_{n,n_{12},n_{21}}. \tag{F.10}$$

We pull the additional two oscillators out of the state and get

$$\big(c_{n+2,n_{12},n_{21}} + c_{n+2,n_{12},n_{21}+2}\big)|1,1,p'_1,\ldots,p'_n;A'\rangle = \mathfrak{P}_{1,\alpha\dot\beta}\,c_{n,n_{12},n_{21}}|p'_1,\ldots,p'_n;A\rangle. \tag{F.11}$$

Summing over all p'_k therefore yields $\mathfrak{P}_{1,\alpha\dot\beta}\mathcal{H}_{12}|p_1,\ldots,p_n;A\rangle$. If both new oscillators end up at site 2 we get an equivalent result. It remains to be shown that the other four terms cancel. Two of these are

$$c_{n+2,n_{12},n_{21}+1}|1,2,p'_1,\ldots,p'_n;A'\rangle + c_{n+2,n_{12}+1,n_{21}}|1,2,p'_1,\ldots,p'_n;A'\rangle. \tag{F.12}$$

The absolute values in (F.5) match for $c_{n+2,n_{12},n_{21}+1}$ and $c_{n+2,n_{12}+1,n_{21}}$ and we sum up the signs

$$(-1)^{1+n_{12}n_{21}+n_{12}} + (-1)^{1+n_{12}n_{21}+n_{21}} = (-1)^{1+n_{12}n_{21}}\big((-1)^{n_{12}} + (-1)^{n_{21}}\big). \tag{F.13}$$

Now, oscillators always hop in pairs due to the central charge constraint. One of the new oscillators has changed the site, so the number of old oscillators changing site must be odd. The above two signs must be opposite and cancel in the sum. The same is true for the remaining two terms. This concludes the proof for $[\mathfrak{P}_{12,\alpha\dot\beta}, \mathcal{H}_{12}] = 0$. The proof for the other generators which increase the number of oscillators is equivalent. To prove invariance under the remaining generators, we note that these remove two oscillators from one of the two sites. Assume it will remove the first two oscillators from a state (for each two oscillators that are removed, the argument will be the same). Now, the argument is essentially the same as the proof for $\mathfrak{P}_{12,\alpha\dot\beta}$ read in the opposite direction.

To prove that the eigenvalues of $\mathcal{H}_{12}$ are given by $2h(j)$, we act on a state $|j\rangle$ of $\mathcal{V}_j$ within the bosonic $\mathfrak{su}(1,1)$ subsector, c.f. Sec. 3.4

$$|j\rangle = \sum_{k=0}^{j} \frac{(-1)^k j!}{k!(j-k)!}\,|k, j-k\rangle \tag{F.14}$$

with a single spin given by

$$|k\rangle = \frac{1}{k!}\,(\mathbf{a}_2^\dagger)^k(\mathbf{b}_2^\dagger)^k\,|\mathcal{Z}\rangle. \tag{F.15}$$

The state $|j\rangle$ has a definite spin and is therefore an eigenstate of $\mathcal{H}_{12}$. We can choose to calculate only the coefficient of $|j,0\rangle$ in $\mathcal{H}_{12}|j\rangle$. It is given by (see Sec. 3.2.6)

$$h(j) + \sum_{k=1}^{j} \frac{(-1)^{1+k} j!}{k\,k!(j-k)!} = 2h(j), \tag{F.16}$$

which proves that $\mathcal{H}_{12} = 2h(\mathcal{J}_{12})$. This concludes the proof of (F.7).

F.4 An Example

We will now determine the planar anomalous dimensions of some states with weight $(2;0,0;0,0,0;0,2)$ to demonstrate how to apply the above Hamiltonian. Using Tab. D.2 we find that we have to excite each of the four oscillators $\mathbf{c},\mathbf{d}$ once. There must be an equal number of $\mathbf{c}$ and $\mathbf{d}$ oscillators on each site due to the central charge constraint and the three distinct configurations are

$$\begin{aligned}
|1212\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{2,2}^\dagger \mathbf{d}_{1,1}^\dagger \mathbf{d}_{2,2}^\dagger |\mathcal{Z}\mathcal{Z}\rangle,\\
|1221\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{2,2}^\dagger \mathbf{d}_{2,1}^\dagger \mathbf{d}_{1,2}^\dagger |\mathcal{Z}\mathcal{Z}\rangle,\\
|1111\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{1,2}^\dagger \mathbf{d}_{1,1}^\dagger \mathbf{d}_{1,2}^\dagger |\mathcal{Z}\mathcal{Z}\rangle.
\end{aligned} \tag{F.17}$$

Let us now act with $\mathcal{H}_{12}$ on these states, we find

$$\begin{aligned}
\mathcal{H}_{12}|1212\rangle &= c_{4,0,0}|1212\rangle + c_{4,0,2}|1111\rangle + c_{4,2,0}|2222\rangle\\
&\quad + c_{4,1,1}|2112\rangle + c_{4,1,1}|1221\rangle + c_{4,2,2}|2121\rangle\\
&= \tfrac{3}{2}|1212\rangle - \tfrac{1}{2}|1111\rangle - \tfrac{1}{2}|2222\rangle + \tfrac{1}{2}|2112\rangle + \tfrac{1}{2}|1221\rangle - \tfrac{1}{2}|2121\rangle\\
&= |1212\rangle + |1221\rangle - |1111\rangle
\end{aligned} \tag{F.18}$$

using (F.4,F.5,F.6) and cyclicity of the trace. Evaluating the Hamiltonian for the remaining two states $|1212\rangle$ and $|1221\rangle$ we find the energy matrix

$$H = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \tag{F.19}$$

the factor of 2 is due to $\mathcal{H} = \mathcal{H}_{12} + \mathcal{H}_{21}$. One eigenstate is

$$|\mathcal{K}\rangle = |1212\rangle + |1221\rangle - |1111\rangle. \tag{F.20}$$

with energy $E = 6$; it is clearly part of the Konishi multiplet. The other two, $|1212\rangle + |1111\rangle$ and $|1221\rangle + |1111\rangle$, have vanishing energy and belong to the half-BPS current multiplet.

Acknowledgements

Here, I would like to express my gratitude to my advisor, Matthias Staudacher, for all the support, encouragement, advice, time, stimulating discussions, disputes, chocolates and fantastic collaboration throughout the last two years. Without his keen sense of interesting physical problems, this work would not be the same. Also, I thank him for being the sociable person he is, this made my time in Potsdam as well as on joint travels both exciting and pleasant.

I have also benefitted very much from everyday discussions with Gleb Arutyunov, Charlotte Kristjansen and Jan Plefka and thank them for sharing their knowledge on various aspects of physics. Especially, I will never forget the afternoons spent in Matthias's office drinking tea and discussing the latest discoveries with each other.

I am grateful to Massimo Bianchi, Virginia Dippel, Sergey Frolov, Charlotte Kristjansen, Joe Minahan, Francisco Morales, Jan Plefka, Henning Samtleben, Gordon Semenoff, Matthias Staudacher, Arkady Tseytlin and Kostya Zarembo for the fruitful collaboration on the projects on which this thesis is based in part.

I thank Virginia Dippel, Markus Pössel, Matthias Staudacher and Arkady Tseytlin for proof-reading the manuscript and suggesting valuable improvements to make your reading of this text more enjoyable. I also thank the referees of the dissertation, Curtis Callan, Hermann Nicolai and Jan Plefka, for their time. Again, thanks are due to Matthias for all the help while preparing earlier articles.

For interesting discussions of physics and otherwise, it is a pleasure to thank Vladimir Bazhanov, Andrei Belitsky, David Berenstein, Vladimir Braun, Chong-Sun Chu, Neil Constable, Luise Dolan, Thomas Fischbacher, Dan Freedman, Jaume Gomis, Alexander Gorsky, Christine Gottschalkson, Petra Gutjahr, Paul Howe, Hikaru Kawai, Volodya Kazakov, Nakwoo Kim, Axel Kleinschmidt, Thomas Klose, Gregory Korchemsky, David Kosower, Stefano Kovacs, Juan Maldacena, Shiraz Minwalla, Chiara Nappi, Ari Pankiewicz, Christina Pappa, Kasper Peeters, Markus Pössel, Thomas Quella, Rodolfo Russo, Anton Ryzhov, Kazuhiro Sakai, Dierk Schleicher, John Schwarz, Didina Serban, Corneliu Sochichiu, Emery Sokatchev, Marcus Spradlin, Bogdan Stefański, Ian Swanson, Diana Vaman, Herman Verlinde, Edward Witten, Maria Zamaklar and fellow physicists I have met at seminars, schools and conferences as well as everybody I have unfortunately missed.

I would like to thank my office-mates, colleagues and members of staff at the Institute for the pleasant, inspiring atmosphere and for making my work so enjoyable. In particular, I acknowledge the prompt help by the computer support when my laptop went to the happy hunting grounds after four and a half years of loyal service around the world.

Ich möchte mich ganz herzlich bei Anna, Thomas, Maissi, Miriam und meinen Freunden für ihre fortwährende Unterstützung bedanken, ohne die diese Arbeit in der jetzigen Form nicht möglich gewesen wäre.

Schließlich bedanke ich mich bei der *Studienstiftung des deutschen Volkes* für die Unterstützung durch ein Promotionsförderungsstipendium, sowie auch bei meinem Vertrauensdozenten Prof. Dr. Harald Uhlig und Mit-Stiftis für ihr Engagement. Die Sommerakademie 2003 in St. Johann hat sehr zur Erweiterung meiner Horizonte beigetragen. Ebenso hoffe ich auf dem nun folgenden Sprachkurs in Amboise nicht nur meine Französischkenntnisse verbessern zu können, sondern mich auch von den Strapazen des Anfertigens dieser Arbeit zu erholen.

Thank You!

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