On a Wave Map Equation Arising in General Relativity

HANS RINGSTRÖM
Max Planck Institut für Gravitationsphysik

Abstract

We consider a class of space-times for which the essential part of Einstein's equations can be written as a wave map equation. The domain is not the standard one, but the target is hyperbolic space. One ends up with a $1+1$ nonlinear wave equation, where the space variable belongs to the circle and the time variable belongs to the positive real numbers. The main objective of this paper is to analyze the asymptotics of solutions to these equations as $t \to \infty$. For each point in time, the solution defines a loop in hyperbolic space, and the first result is that the length of this loop tends to 0 as $t^{-1/2}$ as $t \to \infty$. In other words, the solution in some sense becomes spatially homogeneous. However, the asymptotic behavior need not be similar to that of spatially homogeneous solutions to the equations. The orbits of such solutions are either a point or a geodesic in the hyperbolic plane. In the nonhomogeneous case, one gets the following asymptotic behavior in the upper half-plane (after applying an isometry of hyperbolic space if necessary):

(1) The solution converges to a point.
(2) The solution converges to the origin on the boundary along a straight line (which need not be perpendicular to the boundary).
(3) The solution goes to infinity along a curve $y = \text{const}$.
(4) The solution oscillates around a circle inside the upper half-plane.

Thus, even though the solutions become spatially homogeneous in the sense that the spatial variations die out, the asymptotic behavior may be radically different from anything observed for spatially homogeneous solutions of the equations. This analysis can then be applied to draw conclusions concerning the associated class of space-times. For instance, one obtains the leading-order behavior of the functions appearing in the metric, and one can conclude future causal geodesic completeness. © 2004 Wiley Periodicals, Inc.

1 Introduction

Let us first give a brief background to the problem. In the study of the expanding direction of cosmological space-times, the results obtained so far can roughly be divided into small data results and results obtained for situations with symmetry. The small data results without symmetry are often very difficult to prove, but as opposed to cases when one has imposed symmetry conditions, one does get conclusions for an open set of initial data. On the other hand, these results concern initial data close to known solutions, and what one obtains is typically that the perturbed solutions decay to the known ones.

Communications on Pure and Applied Mathematics, Vol. LVII, 0657–0703 (2004)

In a way, the study of situations with symmetry is a complementary approach. In some sense, one considers an empty set of initial data, but on the other hand, one need not start with initial data close to something known. Thus, there is the possibility that one may observe some unexpected nonlinear behavior. The symmetry classes for which one can describe the asymptotics in detail consist mainly of spatially homogeneous solutions. However, even in this case, one gets quite interesting behavior, especially if one also considers the direction towards the singularity. In fact, this case is not completely understood at this time. In this paper, we consider the so-called Gowdy space-times. These admit a two-dimensional group of isometries acting on spatial Cauchy surfaces, so that the equations one ends up with are a system of nonlinear wave equations in $1 + 1$ dimensions. This class has received considerable attention, probably due to the fact that it is on the borderline; it is not trivial to analyze it, but the set of equations is manageable.

The Gowdy vacuum space-times were first introduced in [3] (see also [2]), and in [4] the fundamental questions concerning global existence were answered. The following conditions can be used to define a member of this class:

- It is a time-orientable, globally hyperbolic, vacuum Lorentz manifold.
- It has compact spatial Cauchy surfaces.
- There is a smooth effective group action of $U(1) \times U(1)$ on the Cauchy surfaces under which the metric is invariant.
- The twist constants vanish.

Let us explain the terminology. A group action of a Lie group G on a manifold M is effective if $gp = p$ for all $p \in M$ implies $g = e$. Due to the existence of the symmetries, we get two Killing fields. Let us call them X and Y. The twist constants are defined by

$$\kappa_X = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta \quad \text{and} \quad \kappa_Y = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta \,.$$

The fact that they are constants is due to the field equations. By the existence of the effective group action, one can draw the conclusion that the spatial Cauchy surfaces have topology $\mathbb{T}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, or a Lens space. In all the cases except $\mathbb{T}^3$, the twist constants have to vanish. However, in the case of $\mathbb{T}^3$ they need not vanish, and the condition that they vanish is the most unnatural of the ones on the list above. Since one only expects there to be a causally geodesically complete direction in the $\mathbb{T}^3$ case, and since the equations are much more complicated when the twist constants are not 0, we will only consider the Gowdy $\mathbb{T}^3$-case.

We refer the interested reader to [2, 3] for a proof of these statements. In [2], the symmetries are imposed on initial data, which is perhaps somewhat more natural. We will take the Gowdy vacuum space-times on $\mathbb{R}_+ \times \mathbb{T}^3$ to be metrics of the form (1.1). This is in fact not quite true (see [2, pp. 116–117]); we have set some constants to 0. However, the mentioned class is a natural subclass, and the discrepancy should not cause any major difficulties.

The subject of this paper is the asymptotic behavior of metrics of the form

$$\begin{aligned} g = {} & t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) \\ & + t\big[e^P\, d\sigma^2 + 2e^P Q\, d\sigma\, d\delta + (e^P Q^2 + e^{-P})d\delta^2\big] \quad \text{as } t \to \infty. \end{aligned} \tag{1.1}$$

Here $t \in \mathbb{R}_+ = (0, \infty)$, (θ, σ, δ) are coordinates on $\mathbb{T}^3$ and P, and Q and λ are functions of (t, θ). The evolution equations become

$$P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} - e^{2P}\big(Q_t^2 - Q_\theta^2\big) = 0\,, \tag{1.2}$$

$$Q_{tt} + \frac{1}{t}Q_t - Q_{\theta\theta} + 2(P_t Q_t - P_\theta Q_\theta) = 0\,, \tag{1.3}$$

and the constraints

$$\lambda_t = t\big[P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)\big]\,, \tag{1.4}$$

$$\lambda_\theta = 2t(P_\theta P_t + e^{2P} Q_\theta Q_t)\,. \tag{1.5}$$

Obviously, the constraints are decoupled from the evolution equations except for the condition on P and Q implied by (1.5). The procedure for constructing a metric is thus to choose initial data for P and Q and their time derivatives such that there is a λ satisfying (1.5). One then solves (1.2)–(1.3) after which (1.4) determines λ up to a constant. Finally, one can check that (1.5) holds for all time. Consequently, the equations of interest are the two nonlinear coupled wave equations (1.2)–(1.3). In this parametrization, the expanding direction corresponds to $t \to \infty$, and our main concern will be the asymptotics of solutions to (1.2)–(1.3) as $t \to \infty$.

Equations (1.2)–(1.3) can be interpreted as a wave map equation. In fact, let

$$g_0 = -dt^2 + d\theta^2 + t^2\, d\phi^2 \tag{1.6}$$

be a metric on $\mathbb{R}_+ \times \mathbb{T}^2$, and let

$$g_1 = dP^2 + e^{2P}\, dQ^2 \tag{1.7}$$

be a metric on $\mathbb{R}^2$. Then (1.2)–(1.3) are the wave map equations for a map from $(\mathbb{R}_+ \times \mathbb{T}^2, g_0)$ to $(\mathbb{R}^2, g_1)$, which is independent of the ϕ-variable on $\mathbb{T}^2$. Note that $(\mathbb{R}^2, g_1)$ is isometric to the upper half-plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric $g_H = (dx^2 + dy^2)/y^2$ under the isometry $(Q, P) \mapsto (Q, e^{-P})$. One important consequence of this is that isometries of the hyperbolic plane map solutions of (1.2)–(1.3) to solutions. Another important consequence is the existence of certain conserved quantities, which we will write down in a moment. It will be convenient to carry out the analysis in the (P, Q)–variables, but the conclusions take their most natural form in the (x, y)–variables. For this reason we will use the different variables in parallel.

The starting point of this paper was the numerical studies carried out by Beverly Berger and Vincent Moncrief; see [1]. One object they considered was

$$l(t) = \int_{\mathbb{S}^1} \sqrt{P_\theta^2 + e^{2P} Q_\theta^2}\, d\theta\,. \tag{1.8}$$

This is the length of the closed curve in hyperbolic space defined by P and Q for a fixed time t. Their studies indicated that it should decay as $t^{-1/2}$. This statement can then be interpreted as saying that the solution becomes more and more spatially homogeneous. In fact, they observed that

$$H = \frac{1}{2}\int_{\mathbb{S}^1} \left[P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)\right]d\theta \tag{1.9}$$

decays as $1/t$. Note that this implies that $l(t) \leq Kt^{-1/2}$, where l is defined by (1.8). In this paper we prove the following:

THEOREM 1.1 *Consider a solution to* (1.2)–(1.3). *Then there is a* $T \geq 1$ *and a* K *such that for all* $t \geq T$*, the energy* H *defined by* (1.9) *satisfies*

$$H(t) \leq \frac{K}{t}\,. \tag{1.10}$$

REMARK The analogous statement is true for more general classes of equations than (1.2)–(1.3); see Theorem 7.1. Below, we will use the letter K to denote some constant whose value is of no importance, and we will in general assume $t \geq 1$ in the estimates we write down. In part, the latter is due to the fact that we are only interested in the future, but if we have an estimate $H \leq K_1/t$, we in some cases also wish to be able to bound H in terms of K_1, whence the bound $t \geq 1$ is natural.

PROOF: This follows from Theorem 7.1. □

Furthermore, in most cases studied numerically, the analysis suggested that given a solution to (1.2) and (1.3), one can find a spatially homogeneous solution to the equations such that the difference between the solution one started with and the spatially homogeneous solution decays to 0 in the supremum norm. It turns out that this is not always true.

In order to discuss the asymptotics, we need to introduce some terminology. Consider a solution to (1.2)–(1.3). Then we have the following constants:

$$A = \int_{\mathbb{S}^1} \left\{2Q(tQ_t)e^{2P} - 2(tP_t)\right\}d\theta\,, \tag{1.11}$$

$$B = \int_{\mathbb{S}^1} e^{2P}(tQ_t)d\theta\,, \tag{1.12}$$

$$C = \int_{\mathbb{S}^1} \left\{(tQ_t)(1 - e^{2P}Q^2) + 2Q(tP_t)\right\}d\theta\,. \tag{1.13}$$

As has been mentioned, (1.2)–(1.3) can be given a Lagrangian formulation. Since the Lagrangian is invariant under the isometries of the hyperbolic space, we get conserved quantities due to Noether's theorem. Thus, for example, the fact that

A is constant is a consequence of the fact that dilations are isometries of the upper half-plane, and the conservation of B follows from the fact that translations in Q are isometries. Of course, one can check that A, B, and C are constants by differentiating with respect to time and using the equations. When one maps a solution to a solution by an isometry of the hyperbolic plane, the constants generally change. However, there is one combination, $A^2 + 4BC$, that is unchanged, and this object will play an important role in the analysis. We will also use the notation $\alpha = A/(2\pi)$, $\beta = B/(2\pi)$, $\gamma = C/(2\pi)$, and

$$\delta = \frac{\sqrt{|\alpha^2 + 4\beta\gamma|}}{2}. \tag{1.14}$$

In the spatially homogeneous case

$$\alpha^2 + 4\beta\gamma = 4t^2\left(P_t^2 + e^{2P} Q_t^2\right). \tag{1.15}$$

Thus, spatially homogeneous solutions to (1.2)–(1.3) satisfy $A^2 + 4BC \geq 0$ with equality if and only if the solution is trivial; that is, P and Q are constants. However, if $A_0, B_0, C_0 \in \mathbb{R}$, there are solutions whose (A, B, C) equal (A_0, B_0, C_0); see (6.9)–(6.10). Thus no matter what the value of $A^2 + 4BC$ is, one can find nontrivial data that yield this value. It turns out that the asymptotics are very different depending on whether $A^2 + 4BC$ is positive, zero, or negative. If $A^2 + 4BC > 0$, then the solution qualitatively behaves like a spatially homogeneous solution, but not if the opposite inequality holds. In fact, we have the following:

THEOREM 1.2 *Consider a solution to* (1.2)–(1.3). *Let* $\mathbf{x} = (x, y) = (Q, e^{-P})$, *and let* d_H *be the metric induced by the Riemannian metric* g_H. *Then there is a* K, *a* $T > 0$, *and a curve* Γ *such that*

$$d_H(\mathbf{x}(t,\theta), \Gamma) \leq K t^{-1/2} \quad \text{for all } t \geq T.$$

The possibilities for Γ *are as follows*:

- *If all the constants* A, B, *and* C *are* 0, Γ *is a point.*
- *If* $A^2 + 4BC = 0$ *but the constants are not all* 0, Γ *is either a horocycle (i.e., a circle touching the boundary) or a curve* $y =$ const.
- *If* $A^2 + 4BC > 0$, Γ *is either a circle intersecting the boundary transversally or a straight line intersecting the boundary transversally.*
- *If* $A^2 + 4BC < 0$, Γ *is a circle inside the upper half-plane.*

REMARK We use the convention $d_H(\mathbf{x}, \Gamma) = \inf_{\mathbf{x}_0 \in \Gamma} d_H(\mathbf{x}, \mathbf{x}_0)$. The circle one obtains in the case $A^2 + 4BC < 0$ may be degenerate, that is, a point. Note also that since $l(t)$ defined by (1.8) decays as $t^{-1/2}$, we have

$$\sup_{\theta_1, \theta_2 \in \mathbb{S}^1} d_H[\mathbf{x}(t,\theta_1), \mathbf{x}(t,\theta_2)] \leq K t^{-1/2}. \tag{1.16}$$

PROOF: If all the conserved quantities are 0, the statement follows by combining Corollary 8.10, Proposition 8.11, and the decay of the energy. If $A^2+4BC > 0$, the statement follows from the discussion following the proof of Theorem 8.12.

The case where $A^2 + 4BC = 0$ but the conserved quantities are not all 0 is discussed after the statement of Theorem 8.14. Finally, the discussion following the proof of Proposition 8.16 deals with the case $A^2 + 4BC < 0$. □

In a generalized sense, one can thus say that the solution converges to a circle. Since there are four different kinds of circles in the upper half-plane (points, nondegenerate circles inside the upper half-plane, circles touching the boundary, and circles intersecting the boundary transversally), one gets the four cases above. Concerning the case where the solution converges to a point, not that much more remains to be said, but in the other cases, it is of interest to know how the solution moves along the respective curves.

THEOREM 1.3 *Consider a solution to* (1.2)–(1.3) *with* $A^2 + 4BC > 0$. *Then there is an isometry of the hyperbolic plane, a* K, *a* $T > 0$, *and constants* c_1 *and* c_2 *such that if* $(Q_1, P_1) = (x, -\ln y)$ *is the transformed solution,*

$$\left\| \frac{x}{y} - c_1 \right\|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2} \tag{1.17}$$

and

$$\| \ln y + \delta \ln t + c_2 \|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2} \tag{1.18}$$

for all $t \geq T$, *where* δ *is given by* (1.14).

PROOF: See Theorem 8.12. □

Equation (1.17) says that the distance from the solution to the straight line $x = c_1 y$ decays to 0 as $t^{-1/2}$, and (1.18) shows that the solution is moving towards the boundary of hyperbolic space along this straight line. In the spatially homogeneous case and in the polarized case ($Q = 0$), the constant c_1 is always 0; compare the discussion following Lemma 8.2. The question is then if this is true in general. By Proposition 8.13 we conclude that given c_1', $D > 0$, and $\eta > 0$, there is a solution with $A^2 + 4BC = D$ and $|c_1 - c_1'| \leq \eta$, where c_1 is associated with the solution according to (1.17). Thus, unlike the polarized and spatially homogeneous solutions, the asymptotic curve need not be a geodesic of the hyperbolic plane. One is then led to ask, Is there a natural characterization of solutions with $c_1 = 0$? Equations (1.17) and (1.18) yield a measure of how fast the solution is moving towards the boundary. One consequence of these estimates is that if one fixes a point inside the hyperbolic plane, the distance from this point to the solution at time t is $\delta \ln t$ up to an error that is bounded, irrespective of which θ-coordinate one chooses. Figure 1.1 illustrates the behavior. The closed curve is supposed to illustrate the loop. Note that if we apply an isometry to this picture, we will typically get a circle instead of a straight line.

THEOREM 1.4 *Consider a solution to* (1.2)–(1.3) *with* $A^2 + 4BC = 0$ *but for which not all the constants are* 0. *Then there is an isometry of the hyperbolic*

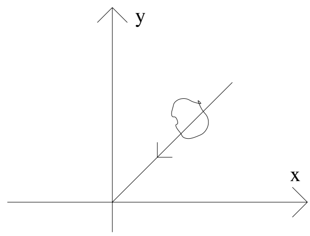

FIGURE 1.1. Asymptotics when $\alpha^2 + 4\beta\gamma > 0$.

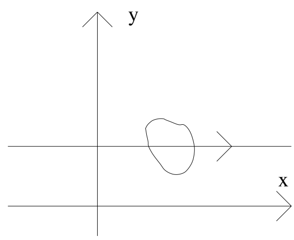

FIGURE 1.2. Asymptotics when $\alpha^2 + 4\beta\gamma = 0$ but not all conserved quantities are 0.

plane, a K, a $T > 0$, and constants c_1 and c_2 such that if $(Q_1, P_1) = (x, -\ln y)$ is the transformed solution,

$$\|y - c_1\|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2} \tag{1.19}$$

and

$$\|x - \ln t - c_2\|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2} \tag{1.20}$$

for all $t \geq T$.

PROOF: See Theorem 8.14. □

Note that the conditions of the theorem are inconsistent with spatial homogeneity. Equation (1.19) says that the distance from the solution to the curve $y = c_1$ decays to 0 as $t^{-1/2}$, and by (1.20), the solution is diverging to infinity along this curve. Furthermore, one can use (1.19) and (1.20) to conclude that if one fixes a point in the hyperbolic plane, the distance from this point to the solution at time t is $2 \ln \ln t$ up to a bounded error term. Figure 1.2 illustrates the behavior. Note that if one applies an isometry to this picture, one typically gets a horocycle instead of a straight line.

Let us now consider the case $A^2 + 4BC < 0$. The first question to ask is whether the limit circle is always a point. That this is not the case follows from Proposition 8.17. Again, it would be of interest to characterize those solutions that have the degenerate behavior. We still do not know how the solutions behave along

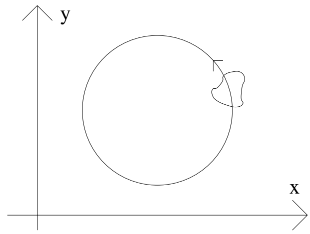

FIGURE 1.3. Asymptotics when $\alpha^2 + 4\beta\gamma < 0$.

the circle; it might be that the solutions converge to a point. However, we have the following:

THEOREM 1.5 *Consider a solution to* (1.2)–(1.3) *such that* $A^2 + 4BC < 0$. *If the circle* Γ *obtained in Theorem* 1.2 *is not a point, there is a* K *and* $T > 0$ *and for every* $t_0 \geq T$ *a curve* γ_{t_0} *with the properties*

$$\gamma_{t_0}(\mathbb{R}_+) = \Gamma\,, \quad d_H[\mathbf{x}(t,\theta), \gamma_{t_0}(t)] \leq K t_0^{-1/2}\,, \quad \textit{for all } t \geq t_0$$

and

$$\gamma_{t_0}\left[t_1 \exp\left(\frac{2\pi}{\delta}\right)\right] = \gamma_{t_0}(t_1)\,, \quad g_H(\gamma_{t_0}'(t), \gamma_{t_0}'(t)) = \frac{r^2\delta^2}{t^2}\,,$$

where $2\pi r$ *is the length of the circle* Γ *with respect to the hyperbolic metric.*

REMARK We note that one can give an explicit expression for the curve γ_{t_0}; see Proposition 8.18.

PROOF: The conclusions of the theorem can be deduced from Proposition 8.18. □

Consequently, the solution oscillates forever along the circle and is more or less periodic in a logarithmic time coordinate. Observe that the solutions in this case behave in a way unlike anything seen when studying spatially homogeneous solutions to the equations; spatially homogeneous solutions are either constant or go to the boundary along a geodesic. However, the solution becomes spatially homogeneous in the sense that (1.16) is satisfied. Thus solutions that become spatially homogeneous in the limit, in the sense (1.16), need not at all behave like spatially homogeneous solutions to the equations. Figure 1.3 illustrates the behavior.

The above information, together with some further analysis, can be used to obtain the following result:

THEOREM 1.6 *Consider a solution to* (1.2)–(1.3). *Then if* H *is given by* (1.9), *there is a* K, *a* $T > 0$, *and a constant* c_H *such that*

$$|tH(t) - c_H| \leq \frac{K}{t} \quad \textit{for all } t \geq T. \tag{1.21}$$

Furthermore, if c_H *is* 0, *the solution is independent of* θ, *and in that case,* $t^2 H(t)$ *is constant.*

PROOF: Section 9 consists of a proof of this statement. □

Note that this proves that estimate (1.10) is optimal for solutions that are not spatially homogeneous. Note furthermore that (1.21) is also optimal, in the sense that one cannot obtain a better decay estimate that holds for all solutions to (1.2)–(1.3). In fact, for a nontrivial spatially homogeneous solution, $t^2 H(t) = c_0 > 0$, which makes it impossible to have anything of the form $o(t^{-1})$ on the right-hand side of (1.21).

In the end, we are interested in the metric (1.1) and thus in the behavior of the functions P, Q, and λ. From the proofs of Theorems 1.2 through 1.5, one can deduce the behavior of P and Q; see Proposition 8.18 and the discussion in the paragraph preceding Lemma 8.15 for details. The leading-order behavior of λ can interestingly enough be deduced immediately from Theorem 1.6.

THEOREM 1.7 *Consider a solution to* (1.2)–(1.5). *Then, if the solution is not independent of* θ, *there is a* $T > 0$ *and a* K *such that*

$$\|\lambda(t,\cdot) - c_\lambda t\|_{C(\mathbb{S}^1,\mathbb{R})} \le K \ln t \quad \textit{for all } t \ge T \quad \textit{where } c_\lambda > 0.$$

PROOF: By (1.4)–(1.5), we have $|\lambda_\theta| \le \lambda_t$. Thus

$$\|\lambda - \langle\lambda\rangle\|_{C(\mathbb{S}^1,\mathbb{R})} \le \int_{\mathbb{S}^1} |\lambda_\theta| d\theta \le \int_{\mathbb{S}^1} \lambda_t \, d\theta = 2tH(t) \le K .$$

By (1.21), we also have

$$\left| \langle\lambda_t\rangle - \frac{c_H}{\pi} \right| \le \frac{K}{t} ,$$

where c_H is positive under the assumptions of the theorem. We get the conclusion of the theorem. □

The above statements can be obtained without any control of the sup norm of the derivatives of P and Q. In some situations, however, it is of interest to have such control.

PROPOSITION 1.8 *Consider a solution to* (1.2)–(1.3). *Then*

$$\|P_t\|_{C(\mathbb{S}^1,\mathbb{R})} + \|P_\theta\|_{C(\mathbb{S}^1,\mathbb{R})} + \|e^P Q_t\|_{C(\mathbb{S}^1,\mathbb{R})} + \|e^P Q_\theta\|_{C(\mathbb{S}^1,\mathbb{R})} \le K t^{-1/2} .$$

The proof is to be found in Section 10. Using the above information, one finally obtains the following theorem, whose proof is to be found in Section 11.

THEOREM 1.9 *Consider a metric given by* (1.1), *where* λ, P, *and* Q *are solutions to* (1.2)–(1.5). *Assume, furthermore, that the metric is not independent of* θ. *Let* $\gamma : (s_-, s_+) \to \mathbb{R}_+ \times \mathbb{T}^3$ *be an inextendible causal geodesic with respect to this metric and assume that* $\langle\gamma', \partial_t\rangle < 0$. *Then* γ *is future complete.*

For the case where the solution is independent of θ, we refer the reader to the literature on spatially homogeneous solutions.

2 Generalities

We will formulate some of the results in a more general setting in order to give a feeling for what the structure is that makes the argument work. We will always consider wave maps from $\mathbb{R}_+ \times \mathbb{T}^2$ with the metric (1.6) that are independent of the ϕ-variable, but let us for the moment assume a general target metric $\bar{g}$ on $\mathbb{R}^k$. The relevant Lagrangian density is

$$\mathcal{L} = \frac{t}{2}\bar{g}_{\alpha\beta}(f)\{-f_t^\alpha f_t^\beta + f_\theta^\alpha f_\theta^\beta\}. \tag{2.1}$$

Assume from now on that $f : \mathbb{R}_+ \times \mathbb{S}^1 \to \mathbb{R}^k$ satisfies the corresponding Euler-Lagrange equations. If we define H and $\hat{H}$ by

$$tH = \hat{H} = \frac{t}{2}\int_{\mathbb{S}^1} \bar{g}_{\alpha\beta}\{f_t^\alpha f_t^\beta + f_\theta^\alpha f_\theta^\beta\}d\theta, \tag{2.2}$$

we have

$$\frac{d\hat{H}}{dt} = \frac{1}{t}\int_{\mathbb{S}^1} \mathcal{L}\, d\theta \quad \text{and} \quad \frac{dH}{dt} = -\frac{1}{t}\int_{\mathbb{S}^1} \bar{g}_{\alpha\beta} f_t^\alpha f_t^\beta\, d\theta. \tag{2.3}$$

This is one important consequence of the geometric setting.

Another is the following. Let

$$\mathcal{A} = \frac{t}{2}\bar{g}_{\alpha\beta}(f_t^\alpha + f_\theta^\alpha)(f_t^\beta + f_\theta^\beta), \quad \mathcal{B} = \frac{t}{2}\bar{g}_{\alpha\beta}(f_t^\alpha - f_\theta^\alpha)(f_t^\beta - f_\theta^\beta). \tag{2.4}$$

Then

$$(\partial_t - \partial_\theta)\mathcal{A} = (\partial_t + \partial_\theta)\mathcal{B} = \frac{1}{t}\mathcal{L}. \tag{2.5}$$

Most of the arguments will require more structure than this, and we will from now on only consider metrics of the form

$$\bar{g} = \sum_{i=1}^{n} dP^i \otimes dP^i + \sum_{i,j=1}^{m} g_{ij}(P)dQ^i \otimes dQ^j. \tag{2.6}$$

If the matrix $M(P)$ is defined as having entries $g_{ij}(P)$, we will for most arguments also require

$$\sum_{k=1}^{n} \sup_{x\in\mathbb{R}^n} \left| M^{-1/2}(x)\frac{\partial M}{\partial x^k}(x)M^{-1/2}(x)\right| \le K_M < \infty. \tag{2.7}$$

By $M^{-1/2}$ we mean the unique positive definite and symmetric matrix B such that $B^2 = M^{-1}$. One example of metrics satisfying these conditions is $(m+1)$–dimensional hyperbolic space. This can be viewed as $\mathbb{R}^{m+1}$ with the metric

$$\bar{g} = dP \otimes dP + e^{2P}\sum_{i=1}^{m} dQ^i \otimes dQ^i.$$

Let us for reference write down the equations corresponding to (2.6). Due to the structure of the metric it is natural to divide the equations into two blocks. We have

$$P^i_{tt} + \frac{1}{t} P^i_t - P^i_{\theta\theta} - \frac{1}{2} \frac{\partial g_{kl}}{\partial P^i} \big(Q^k_t Q^l_t - Q^k_\theta Q^l_\theta \big) = 0 \tag{2.8}$$

and

$$\partial_t \big(t g_{ij} Q^j_t \big) - \partial_\theta \big(t g_{ij} Q^j_\theta \big) = 0 \,. \tag{2.9}$$

3 Global Existence

The arguments in this section are of course standard, but we wish to prove the following for the sake of completeness.

THEOREM 3.1 *Consider* (2.8)–(2.9). *Given smooth initial data given at some* $t_0 \in \mathbb{R}_+$*, there is a unique smooth solution to these equations on all of* $\mathbb{R}_+$.

PROOF: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (2.4) where $\bar{g}$ is given by (2.6), and let

$$F_1(u, \theta) = \mathcal{A}(u, \theta - u) \,, \quad F_2(u, \theta) = \mathcal{B}(u, \theta + u) \,, \quad E_i(u) = \sup_{\theta \in \mathbb{S}^1} F_i(u, \theta) \,,$$

and

$$E = E_1 + E_2 \,.$$

By (2.5), we have

$$\begin{aligned} |F_1(u_1, \theta) - F_1(u_0, \theta)| = \left| \int_{u_0}^{u_1} \partial_u F_1(u, \theta) du \right| &= \left| \int_{u_0}^{u_1} \frac{1}{u} \mathcal{L}(u, \theta - u) du \right| \\ &\le \left| \int_{u_0}^{u_1} \frac{1}{2u} E(u) du \right| , \end{aligned}$$

and similarly for F_2. Taking the supremum over θ and then adding, we get

$$E(u_1) \le E(u_0) + \left| \int_{u_0}^{u_1} \frac{1}{u} E(u) du \right| .$$

For $u_1 \ge u_0$, we can apply Grönwall's lemma to obtain

$$E(u_1) \le \frac{u_1}{u_0} E(u_0) \quad \text{for all } u_1 \ge u_0. \tag{3.1}$$

In order to analyze the case $u_1 \le u_0$, define

$$h(u) = E(u_0) - \int_{u_0}^{u} \frac{1}{v} E(v) dv \,.$$

Then

$$h' = -\frac{1}{u} E \ge -\frac{1}{u} h \,.$$

This implies

$$E(u_1) \le \frac{u_0}{u_1} E(u_0) \quad \text{for all } u_1 \le u_0.$$

Thus E is bounded on compact subintervals of $\mathbb{R}_+$. Consequently, P is bounded on such intervals, so that the metric $g_{ij}(P)$ is equivalent to the Euclidean metric on $\mathbb{R}^m$ on compact subintervals of $\mathbb{R}_+$. Consequently, the sup norm of P and the first derivatives of P and Q are bounded on compact subintervals of $\mathbb{R}_+$. Using this together with energy estimates, one can control the higher-order derivatives in L^2, and thus one obtains global existence. □

Note that (3.1) gives bounds on the sup norm of the derivatives, but no decay. After knowing that the energy decays and after having analyzed the behavior of P and Q, one can, however, improve the argument to obtain the decay of Proposition 1.8.

4 Method

The first step in the analysis is to prove that the energy decays as $1/t$. The method to prove this is one that in principle has a wider range of applicability. For this reason, and for reasons of exposition, we would here like to apply it to two simple examples. The examples are linear, and in general one cannot expect the arguments to carry over to the nonlinear case. However, it will turn out that similar arguments work if the initial data are small. One then has to prove that the energy converges to zero separately in order to obtain the desired decay for general initial data in the nonlinear setting.

The starting point was the following ODE example, which was brought to our attention by Vincent Moncrief in a talk given at the AEI. Consider the ODE

$$\ddot{x} + 2a\dot{x} + b^2 x = 0\,,$$

where $a > 0$ and $b^2 > a^2$. The goal is to find a decay estimate for the energy

$$H = \frac{1}{2}(\dot{x}^2 + b^2 x^2)$$

without solving the equation. Compute

$$\frac{dH}{dt} = -2a\dot{x}^2\,.$$

Thus H decreases, but we cannot even conclude that H converges to 0, even though we know that H converges to 0 exponentially. The idea is then to introduce a correction term

$$\Gamma = ax\dot{x}\,.$$

The function Γ has two important properties. The first property is

$$|\Gamma| = \left|\frac{a}{b}\right| |bx\dot{x}| \leq \left|\frac{a}{b}\right| \frac{1}{2}(\dot{x}^2 + b^2 x^2) = \left|\frac{a}{b}\right| H\,.$$

Since $|a/b| < 1$, there are constants c_1 and $c_2 > 0$, depending only on $|a/b|$, such that

$$c_1 H \leq H + \Gamma \leq c_2 H\,.$$

The second property is that

$$\frac{d(H+\Gamma)}{dt} = -2a(H+\Gamma)\,.$$

We conclude that $H \le K\exp(-2at)$, which is an optimal estimate.

Let us consider the polarized Gowdy case, i.e., (1.2)–(1.3) with $Q = 0$. The relevant equation is

$$P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} = 0\,.$$

The natural energy is

$$H = \frac{1}{2}\int_{\mathbb{S}^1} \left(P_t^2 + P_\theta^2\right) d\theta\,.$$

We have

$$\frac{dH}{dt} = -\frac{1}{t}\int_{\mathbb{S}^1} P_t^2\, d\theta\,.$$

In analogy with the previous example, it seems natural to introduce a correction

$$\tilde{\Gamma} = \frac{1}{2t}\int_{\mathbb{S}^1} P P_t\, d\theta\,.$$

However, this cannot be bounded in terms of H, so an argument similar to the one given above cannot work. A more promising candidate is

$$\Gamma = \frac{1}{2t}\int_{\mathbb{S}^1} (P - \langle P\rangle) P_t\, d\theta$$

where we have used the notation

$$\langle P\rangle = \frac{1}{2\pi}\int_{\mathbb{S}^1} P\, d\theta\,.$$

Observe that

$$\int_{\mathbb{S}^1} (P - \langle P\rangle)^2\, d\theta = 2\pi \sum_{n\in\mathbb{Z}} |a_n|^2 \le 2\pi \sum_{n\in\mathbb{Z}} n^2 |a_n|^2 = \int_{\mathbb{S}^1} P_\theta^2\, d\theta$$

since $a_0 = 0$. Thus

$$|\Gamma| \le \frac{1}{2t}H\,.$$

Furthermore,

$$\frac{d(H+\Gamma)}{dt} = -\frac{1}{t}(H+\Gamma) - \frac{1}{t}\Gamma - \frac{1}{2t}\langle P_t\rangle^2 \le -\frac{1}{t}(H+\Gamma) + \frac{1}{2t^2}H\,.$$

Thus

$$H \le \frac{K}{t}\,. \tag{4.1}$$

Since $\langle P \rangle$ is a spatially homogeneous solution to the equation, and (4.1) implies that

$$\|P - \langle P \rangle\|_{C(\mathbb{S}^1,\mathbb{R})} \leq K t^{-1/2},$$

we conclude that the distance from P to a spatially homogeneous solution of the equations decays to 0 as t tends to ∞.

Observe that similar arguments can be used to prove that

$$\sum_{|\alpha|=k} \int_{\mathbb{T}^d} \left[(\partial^\alpha \partial_t P)^2 + |\nabla \partial^\alpha P|^2\right] dx \leq \frac{C_k}{t}$$

for solutions P to

$$P_{tt} + \frac{1}{t} P_t - \Delta P = 0 \quad \text{on } (0,\infty) \times \mathbb{T}^d.$$

5 Model Metrics

Let M be a smooth map from $\mathbb{R}^n$ into the set of symmetric and positive definite $m \times m$ matrices. Assume furthermore that M satisfies (2.7). In this section we wish to write down some consequences of this condition that will be of importance later. By K we will denote any constant whose value is of no importance. Introduce the notation

$$(v, w)_{M(x)} = {}^{\mathsf{T}} v M(x) w \quad \text{and} \quad |v|_{M(x)} = [(v,v)_{M(x)}]^{1/2}.$$

By $|\cdot|$, we denote the ordinary Euclidean norm on $\mathbb{R}^k$, and by $|B|$, where B is a matrix, we mean

$$\sup_{|x|=1} |Bx|.$$

LEMMA 5.1 *Consider an M as above. Then*

$$|(v,w)_{M(x)}| \leq |v|_{M(x)} |w|_{M(x)}, \tag{5.1}$$

$$|v|_{M(x_1)} \leq \exp\{K|x_1 - x_0|\} |v|_{M(x_0)}, \tag{5.2}$$

$$\left| {}^{\mathsf{T}} v \frac{\partial M}{\partial x^k}(x) w \right| \leq K |v|_{M(x)} |w|_{M(x)}, \tag{5.3}$$

$$\left| {}^{\mathsf{T}} v [M(x_1) - M(x_0)] v \right| \leq K \exp\{K|x_1 - x_0|\} \, |x_1 - x_0| \, |v|^2_{M(x_0)}. \tag{5.4}$$

PROOF: If B is a positive definite and symmetric matric, we denote by $B^{1/2}$ the unique positive definite and symmetric matrix C such that $C^2 = B$. Since

$$\left|M^{1/2}(x) v\right|^2 = {}^{\mathsf{T}} v M(x) v$$

due to the symmetry of $M^{1/2}$, we conclude that (5.1) holds and that (5.3) holds due to (2.7). In order to prove (5.2), let

$$x(t) = (1-t) x_0 + t x_1. \tag{5.5}$$

By (5.3), we have

$$\frac{d}{dt}\left\{{}^{\mathsf{T}}v(M\circ x)v\right\} = {}^{\mathsf{T}}v\left(\frac{\partial M}{\partial x^k}\circ x\right)v\left(x_1^k - x_0^k\right) \le K|x_1 - x_0|\,{}^{\mathsf{T}}v(M\circ x)v\,.$$

We deduce that (5.2) holds. In order to prove (5.4), let x be as in (5.5). Combining (5.2) and (5.3), we get

$$\begin{aligned}\left|{}^{\mathsf{T}}v[M(x_1) - M(x_0)]v\right| &= \left|\int_0^1 {}^{\mathsf{T}}v\frac{\partial M}{\partial x^k}[x(t)]\left(x_1^k - x_0^k\right)v\,dt\right| \\ &\le K\exp\{K|x_1 - x_0|\}\,|x_1 - x_0|\,|v|^2_{M(x_0)}\,.\end{aligned}$$

Thus (5.4) holds. □

The following statements are presented here in order not to interrupt the flow of later proofs.

LEMMA 5.2 *Assume that* $P \in C^\infty(\mathbb{S}^1, \mathbb{R}^n)$, $Q \in C^\infty(\mathbb{S}^1, \mathbb{R}^m)$, *and that* M *is a smooth function from* $\mathbb{R}^n$ *to the symmetric and positive definite* $m\times m$ *matrices satisfying* (2.7). *Then*

$$\||Q - \langle Q\rangle|_{M\circ P}\|_{C(\mathbb{S}^1,\mathbb{R})} \le K_1\left(\int_{\mathbb{S}^1} |Q_\theta|^2_{M\circ P}\,d\theta\right)^{1/2} \tag{5.6}$$

and

$$\int_{\mathbb{S}^1} |\langle Q\rangle|^2_{M\circ P}\,d\theta \le K_2\int_{\mathbb{S}^1} |Q|^2_{M\circ P}\,d\theta\,. \tag{5.7}$$

The constants K_1 *and* K_2 *depend on* $\|P - \langle P\rangle\|_{C(S^1,\mathbb{R}^n)}$ *and on* K_M *appearing in* (2.7).

PROOF: Using (5.2), we have

$$\left\||Q - \langle Q\rangle|^2_{M\circ P}\right\|_{C(\mathbb{S}^1,\mathbb{R})} \le K\left\||Q - \langle Q\rangle|^2_{M(\langle P\rangle)}\right\|_{C(\mathbb{S}^1,\mathbb{R})}\,.$$

Let e_k be an orthonormal basis with respect to $(\,\cdot\,,\,\cdot\,)_{M(\langle P\rangle)}$. Then

$$\left\||Q - \langle Q\rangle|^2_{M(\langle P\rangle)}\right\|_{C(\mathbb{S}^1,\mathbb{R})} \le \sum_{k=1}^m \left\|(Q - \langle Q\rangle, e_k)^2_{M(\langle P\rangle)}\right\|_{C(\mathbb{S}^1,\mathbb{R})}\,.$$

Note that

$$\int_{\mathbb{S}^1} (Q - \langle Q\rangle, e_k)_{M(\langle P\rangle)}\,d\theta = 0\,,$$

since Q is the only object in this expression depending on θ. Consequently, there is a $\theta_k \in \mathbb{S}^1$ such that $(Q(\theta_k) - \langle Q\rangle, e_k)_{M(\langle P\rangle)} = 0$. Thus

$$
\begin{aligned}
&\left\|(Q - \langle Q\rangle, e_k)^2_{M(\langle P\rangle)}\right\|_{C(\mathbb{S}^1,\mathbb{R})} \\
&\quad \le 2\int_{\mathbb{S}^1} \left|(Q_\theta, e_k)_{M(\langle P\rangle)}(Q - \langle Q\rangle, e_k)_{M(\langle P\rangle)}\right| d\theta \\
&\quad \le 2\left[\int_{\mathbb{S}^1} (Q_\theta, e_k)^2_{M(\langle P\rangle)}\, d\theta\right]^{1/2}\left[\int_{\mathbb{S}^1} (Q - \langle Q\rangle, e_k)^2_{M(\langle P\rangle)}\, d\theta\right]^{1/2} \\
&\quad \le 2\left[\int_{\mathbb{S}^1} |Q_\theta|^2_{M(\langle P\rangle)}\, d\theta\right]^{1/2}\left[\int_{\mathbb{S}^1} |Q - \langle Q\rangle|^2_{M(\langle P\rangle)}\, d\theta\right]^{1/2} \\
&\quad \le K\left\||Q - \langle Q\rangle|^2_{M\circ P}\right\|^{1/2}_{C(\mathbb{S}^1,\mathbb{R})}\left[\int_{\mathbb{S}^1} |Q_\theta|^2_{M\circ P}\, d\theta\right]^{1/2},
\end{aligned}
$$

where we have used (5.2) again in the last step. Combining the above inequalities we obtain (5.6). Observe that the constants depend only on $\|P - \langle P\rangle\|_{C(\mathbb{S}^1,\mathbb{R}^n)}$ and K_M. Using (5.2) and Hölder's inequality, we estimate

$$
\begin{aligned}
\int_{\mathbb{S}^1} {}^{\mathsf{T}}\langle Q\rangle(M\circ P)\langle Q\rangle d\theta &\le K\,{}^{\mathsf{T}}\langle Q\rangle M(\langle P\rangle)\langle Q\rangle \\
&= K\left|(\langle M^{1/2}(\langle P\rangle)Q\rangle\right|^2 \\
&\le K\left[\frac{1}{2\pi}\int_{\mathbb{S}^1} |M^{1/2}(\langle P\rangle)Q| d\theta\right]^2 \\
&\le K\int_{\mathbb{S}^1} |M^{1/2}(\langle P\rangle)Q|^2\, d\theta \le K\int_{\mathbb{S}^1} {}^{\mathsf{T}}Q(M\circ P)Q\, d\theta\,.
\end{aligned}
$$

The lemma follows. □

6 Small Data

Consider the energy H defined by (2.2) when the metric $\bar{g}$ is given by (2.6). Then the following holds:

THEOREM 6.1 *Consider a metric of the form* (2.6) *satisfying* (2.7). *Then there is an* $\eta > 0$ *such that for functions* $P \in C^\infty(\mathbb{R}_+\times\mathbb{S}^1, \mathbb{R}^n)$ *and* $Q \in C^\infty(\mathbb{R}_+\times\mathbb{S}^1, \mathbb{R}^m)$ *solving* (2.8) *and* (2.9) *and satisfying* $H(t_0) \le \eta$ *for some* $t_0 \in \mathbb{R}_+$, *there is a* K *such that*

$$H(t) \le \frac{K}{t} \quad \textit{for all } t \ge t_0.$$

Here H is defined by (2.2).

PROOF: Note that H is decreasing due to (2.3) so that $H(t) \leq \eta$ for all $t \geq t_0$. Furthermore,

$$\|P - \langle P\rangle\|_{C(\mathbb{S}^1,\mathbb{R}^n)} \leq KH^{1/2} \leq K\eta^{1/2} \quad \text{for all } t \geq t_0. \tag{6.1}$$

Below, the constants in most estimates depend on the sup norm of $P - \langle P\rangle$, but as the estimates are only of interest to the future of a given time, this is not a problem. Note also that all the constants below that depend on this sup norm decay with the sup norm. Thus if we assume $\eta \leq 1$, we can use the same constants for all solutions in the regions where $H \leq 1$.

We use the method presented in Section 4. The main point is thus to construct suitable corrections. Due to the structure of the metric, it is natural to divide these corrections into two parts. Consider

$$\Gamma^P = \frac{1}{2t}\sum_{i=1}^{n}\int_{\mathbb{S}^1}(P^i - \langle P^i\rangle)P_t^i\, d\theta\,.$$

Since the metric is of the form (2.6), we can argue as in Section 4 in order to obtain

$$|\Gamma^P| \leq \frac{K}{t}H\,.$$

Let us compute, using (2.8),

$$\begin{aligned}\frac{d\Gamma^P}{dt} &= -\frac{1}{t}\Gamma^P + \frac{1}{2t}\int_{\mathbb{S}^1}|P_t|^2\, d\theta - \frac{\pi}{t}|\langle P_t\rangle|^2 \\ &\quad + \frac{1}{2t}\sum_{i=1}^{n}\int_{\mathbb{S}^1}(P^i - \langle P^i\rangle)\left[-\frac{1}{t}P_t^i + P_{\theta\theta}^i + \frac{1}{2}\frac{\partial g_{kl}}{\partial P^i}(Q_t^k Q_t^l - Q_\theta^k Q_\theta^l)\right]d\theta\,.\end{aligned}$$

Consider the integrand of the last term. The terms involving P_t^i/t yield $-\Gamma^P/t$, and the terms involving $P_{\theta\theta}^i$ we integrate partially. We thus get the estimate

$$\begin{aligned}\frac{d\Gamma^P}{dt} &\leq -\frac{2}{t}\Gamma^P + \frac{1}{2t}\int_{\mathbb{S}^1}\left[|P_t|^2 - |P_\theta|^2\right]d\theta \\ &\quad + \frac{1}{2t}\sum_{i=1}^{n}\int_{\mathbb{S}^1}(P^i - \langle P^i\rangle)\frac{1}{2}\frac{\partial g_{kl}}{\partial P^i}\left(Q_t^k Q_t^l - Q_\theta^k Q_\theta^l\right)d\theta\,.\end{aligned}$$

Combining (2.7), (5.3), and (6.1), we get the conclusion that the last term can be estimated by $KH^{3/2}/t$. Thus

$$\frac{d\Gamma^P}{dt} \leq -\frac{2}{t}\Gamma^P + \frac{1}{2t}\int_{\mathbb{S}^1}\left[|P_t|^2 - |P_\theta|^2\right]d\theta + \frac{K}{t}H^{3/2}\,. \tag{6.2}$$

Let

$$\Gamma^Q = \frac{1}{2t}\int_{\mathbb{S}^1} g_{ij}(\langle P\rangle)(Q^i - \langle Q^i\rangle)Q_t^j \, d\theta \, .$$

Observe that by (5.1), (5.2), (5.6), and (6.1),

$$|\Gamma^Q| \le \frac{1}{2t}\int_{\mathbb{S}^1} |(Q-\langle Q\rangle, Q_t)_{M\circ\langle P\rangle}|d\theta \le \frac{K}{t}\int_{\mathbb{S}^1} |Q-\langle Q\rangle|_{M\circ P}\,|Q_t|_{M\circ P}\, d\theta \le \frac{K}{t}H \, ,$$

where we used Hölder's inequality in the last step. Let us compute

$$\begin{aligned} \frac{d\Gamma^Q}{dt} &= -\frac{1}{t}\Gamma^Q + \frac{1}{2t}\int_{\mathbb{S}^1} \frac{\partial g_{ij}}{\partial P^k}(\langle P\rangle)\langle P_t^k\rangle(Q^i - \langle Q^i\rangle)Q_t^j \, d\theta \\ &\quad + \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)Q_t^j Q_t^k \, d\theta - \frac{\pi}{t} g_{jk}(\langle P\rangle)\langle Q_t^j\rangle\langle Q_t^k\rangle \\ &\quad + \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle)Q_{tt}^k \, d\theta \, . \end{aligned} \tag{6.3}$$

Note here that the term which only involves averages has a sign. This is the reason for using $g_{ij}(\langle P\rangle)$ in the definition of Γ^Q instead of $g_{ij}(P)$. Observe that $|\langle P_t^k\rangle| \le KH^{1/2}$ due to Hölder's inequality. Combining this with (5.2), (5.3), (5.6), and (6.1), we get

$$\left|\frac{1}{2t}\int_{\mathbb{S}^1} \frac{\partial g_{ij}}{\partial P^k}(\langle P\rangle)\langle P_t^k\rangle(Q^i - \langle Q^i\rangle)Q_t^j \, d\theta\right| \le \frac{K}{t}H^{3/2} \, . \tag{6.4}$$

Using (2.9), we get

$$\begin{aligned} &\frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle)Q_{tt}^k \, d\theta \\ &\quad = \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle) \\ &\qquad\qquad \cdot\left[-\frac{1}{t}Q_t^k + Q_{\theta\theta}^k - g^{ki}\frac{\partial g_{io}}{\partial P^l}\big(P_t^l Q_t^o - P_\theta^l Q_\theta^o\big)\right]d\theta \\ &\quad = -\frac{1}{t}\Gamma^Q - \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)Q_\theta^j Q_\theta^k \, d\theta \\ &\qquad - \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle)v^k \, d\theta \, . \end{aligned} \tag{6.5}$$

where we have introduced the notation

$$v^k = g^{ki}\frac{\partial g_{io}}{\partial P^l}\big(P^l_t Q^o_t - P^l_\theta Q^o_\theta\big)\,.$$

By (5.1), (5.2), (5.6), and (6.1), we have

$$\left|\frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle)v^k\,d\theta\right| \le \frac{K}{t}H^{1/2}\int_{\mathbb{S}^1}\big(g_{jk}(P)v^k v^j\big)^{1/2}\,d\theta\,. \tag{6.6}$$

Let us use the notation that b^{ij} are the components of $M^{-1/2}$ and b_{ij} are the components of $M^{1/2}$. Then $(g_{jk}(P)v^j v^k)^{1/2}$ coincides with the Euclidean norm of the vector with components

$$b^{il}\frac{\partial g_{ij}}{\partial P^k}\big[P^k_t Q^j_t - P^k_\theta Q^j_\theta\big] = b^{il}\frac{\partial g_{ij}}{\partial P^k}b^{jo}\big[P^k_t b_{or}Q^r_t - P^k_\theta b_{or}Q^r_\theta\big]\,.$$

Due to (2.7), we thus get

$$\big(g_{jk}v^j v^k\big)^{1/2} \le K\big[|P_t|^2 + |P_\theta|^2 + |Q_t|^2_{M\circ P} + |Q_\theta|^2_{M\circ P}\big]\,.$$

Combining this with (6.5) and (6.6), we get

$$\frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)(Q^j - \langle Q^j\rangle)Q^k_{tt}\,d\theta \le \\ -\frac{1}{t}\Gamma^Q - \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)Q^j_\theta Q^k_\theta\,d\theta + \frac{K}{t}H^{3/2}\,.$$

Combining this with (6.3) and (6.4), we get

$$\frac{d\Gamma^Q}{dt} \le -\frac{2}{t}\Gamma^Q + \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(\langle P\rangle)\big[Q^j_t Q^k_t - Q^j_\theta Q^k_\theta\big]d\theta + \frac{K}{t}H^{3/2}\,,$$

where we have discarded the term in (6.3) that involves only averages. This is not quite what we want. We would prefer to have $g_{jk}(P)$ instead of $g_{jk}(\langle P\rangle)$. Using (5.4) and (6.1), we can, however, conclude that changing $\langle P\rangle$ to P does not cost us more than $KH^{3/2}/t$. Thus

$$\frac{d\Gamma^Q}{dt} \le -\frac{2}{t}\Gamma^Q + \frac{1}{2t}\int_{\mathbb{S}^1} g_{jk}(P)\big[Q^j_t Q^k_t - Q^j_\theta Q^k_\theta\big]d\theta + \frac{K}{t}H^{3/2}\,. \tag{6.7}$$

Letting $\Gamma = \Gamma^P + \Gamma^Q$ and using (6.2), (6.7), and (2.3), we get

$$\frac{d(H+\Gamma)}{dt} \le -\frac{1}{t}(H+\Gamma) - \frac{1}{t}\Gamma + \frac{K}{t}H^{3/2}\,. \tag{6.8}$$

Assuming $\eta \le 1$ and only considering the solution for $t \ge t_0$, the constant K is independent of the sup norm of $P - \langle P\rangle$. Since $|\Gamma| \le KH/t$, we can assume $|\Gamma|/H$ to be as small as we wish by waiting long enough. With the notation $\mathcal{E} = H + \Gamma$

and assuming η to be small enough, we get the conclusion that there is a $1 > \delta > 0$ and a $T > 0$ such that

$$\frac{d\mathcal{E}}{dt} \leq -\frac{\delta}{t}\mathcal{E} \quad \text{and} \quad \mathcal{E} > 0 \quad \text{for all } t \geq T.$$

Thus

$$\mathcal{E}(t) \leq \left(\frac{T}{t}\right)^{\delta} \mathcal{E}(T) \quad \text{for all } t \geq T.$$

Since

$$|\Gamma(t)| \leq \frac{K}{t}\mathcal{E} \quad \text{and} \quad \left|\frac{H^{3/2}}{t}\right| \leq \frac{K}{t^{\delta/2}}\frac{\mathcal{E}}{t} \quad \text{for } t \geq T' \text{ big enough,}$$

we get

$$\frac{d\mathcal{E}}{dt} \leq -\frac{1}{t}\left[1 - \frac{K}{t^{\delta/2}}\right]\mathcal{E} \quad \text{for } t \geq T'.$$

The theorem follows. □

For the remainder of this section, we will only consider solutions to (1.2)–(1.3). In order to prove that the circles obtained as the limit curves in the case $A^2 + 4BC < 0$ are not all points, it is necessary to consider families of solutions to (1.2) and (1.3) and to have constants that are the same for all the members of the family. In particular, we will be interested in the following family: Let p_0, A_0, B_0, $C_0 \in \mathbb{R}$, and $t_0 > 0$. Then

$$Q(t_0, \theta) = \frac{1}{t_0^{1/2}} \cos\theta\,, \quad P(t_0, \theta) = p_0\,, \quad Q_t(t_0, \theta) = \frac{B_0 e^{-2p_0}}{2\pi t_0}\,, \tag{6.9}$$

$$P_t(t_0, \theta) = -\frac{1}{4\pi t_0} A_0 + \frac{1}{4\pi t_0^{1/2}}\left[2C_0 - 2B_0 e^{-2p_0} + \frac{B_0}{t_0}\right]\cos\theta\,, \tag{6.10}$$

yield $A = A_0$, $B = B_0$, and $C = C_0$. Furthermore, it is clear that there is a smooth function λ such that

$$\lambda_\theta = 2t_0\big(P_\theta P_t + e^{2P} Q_\theta Q_t\big) \quad \text{at } t_0.$$

Consider initial data of this form. Assume that $\mathcal{P} = (p_0, A_0, B_0, C_0)$ are fixed constants, and consider the family of solutions obtained by varying t_0. We will use the notation $P(\mathcal{P}, t_0)$ and $Q(\mathcal{P}, t_0)$ to denote the solution to (1.2) and (1.3) found by specifying initial data as in (6.9) and (6.10). Furthermore, we will use $P(\mathcal{P}, t_0; t, \theta)$ to denote the solution evaluated at (t, θ), and similarly $H(\mathcal{P}, t_0; t)$. We have

$$\lim_{t_0 \to \infty} t_0 H(\mathcal{P}, t_0; t_0) = \frac{\pi}{2} e^{2p_0} + \frac{1}{8\pi}\big(C_0 - B_0 e^{-2p_0}\big)^2 = c_H(\mathcal{P})\,.$$

LEMMA 6.2 *Let $\mathcal{P}$ be fixed. Consider the family of solutions to* (1.2)–(1.3) *obtained by varying t_0 in* (6.9)–(6.10). *Then there is a $t_{\mathcal{P}}$ such that, with notation as above,*

$$H(\mathcal{P}, t_0; t) \leq \frac{2c_H(\mathcal{P})}{t} \quad \textit{for all } t \geq t_0 \geq t_{\mathcal{P}} \geq 1.$$

PROOF: There is a $t_{\mathcal{P},1}$ such that $H(\mathcal{P}, t_0; t) \leq 1$ for all $t \geq t_0 \geq t_{\mathcal{P},1}$. Since the constants appearing in the arguments presented in this section only depend on the size of H, we thus have

$$\frac{d\mathcal{E}(\mathcal{P}, t_0; t)}{dt} \leq -\frac{1}{t}\mathcal{E}(\mathcal{P}, t_0; t) - \frac{1}{t}\Gamma(\mathcal{P}, t_0; t) + \epsilon(\mathcal{P}, t_0; t)\,,$$

where $\mathcal{E} = H + \Gamma$,

$$|\Gamma(\mathcal{P}, t_0; t)| \leq \frac{K}{t}H(\mathcal{P}, t_0; t) \quad \text{and} \quad |\epsilon(\mathcal{P}, t_0; t)| \leq \frac{K}{t}H^{3/2}(\mathcal{P}, t_0; t)\,,$$

and the inequalities hold for $t \geq t_0 \geq t_{\mathcal{P},1}$ with constants independent of $(\mathcal{P}, t_0)$. We can thus choose a $t_{\mathcal{P},2} \geq t_{\mathcal{P},1}$ such that

$$H(\mathcal{P}, t_0; t) \leq 2\mathcal{E}(\mathcal{P}, t_0; t) \quad \text{and} \quad \frac{d\mathcal{E}(\mathcal{P}, t_0; t)}{dt} \leq -\frac{1}{2t}\mathcal{E}(\mathcal{P}, t_0; t)$$

for all $t \geq t_0 \geq t_{\mathcal{P},2}$. Thus

$$\mathcal{E}(\mathcal{P}, t_0; t) \leq \left(\frac{t_0}{t}\right)^{1/2}\mathcal{E}(\mathcal{P}, t_0; t_0) \quad \text{for } t \geq t_0 \geq t_{\mathcal{P},2}.$$

We conclude that

$$\frac{d\mathcal{E}(\mathcal{P}, t_0; t)}{dt} \leq \left[-\frac{1}{t} + \frac{K}{t^2} + \frac{K}{t^{5/4}}t_0^{1/4}\mathcal{E}^{1/2}(\mathcal{P}, t_0; t_0)\right]\mathcal{E}(\mathcal{P}, t_0; t)\,,$$

whence

$$\mathcal{E}(\mathcal{P}, t_0; t) \leq \frac{t_0\mathcal{E}(\mathcal{P}, t_0; t_0)}{t}\exp\left[\frac{K}{t_0} + K\mathcal{E}^{1/2}(\mathcal{P}, t_0; t_0)\right].$$

Since t times the right-hand side tends to $c_H(\mathcal{P})$ as $t_0 \to \infty$, the lemma follows. □

7 Large Data

Let us prove that the above asymptotic behavior is true for general initial data.

THEOREM 7.1 *Consider a metric of the form* (2.6) *satisfying* (2.7). *If*

$$P \in C^\infty(\mathbb{R}_+ \times \mathbb{S}^1, \mathbb{R}^n) \quad \textit{and} \quad Q \in C^\infty(\mathbb{R}_+ \times \mathbb{S}^1, \mathbb{R}^m)$$

are solutions to (2.8) *and* (2.9), *then*

$$H(t) \leq \frac{K}{t}$$

for all $t \geq t_0$ and some $t_0 > 0$. Here H is defined by (2.2).

PROOF: Note that H is bounded to the future due to (2.3) so that even though most constants depend on the sup norm of $P - \langle P \rangle$, this will be bounded for the entire future. Due to (2.3), we conclude that

$$\frac{1}{t}\int_{\mathbb{S}^1}[|P_t|^2 + |Q_t|^2_{M\circ P}]d\theta \in L^1([t_0, \infty)) \quad \text{for any } t_0 > 0. \tag{7.1}$$

Note also that by applying (5.7) with Q replaced by Q_t, we have

$$\int_{\mathbb{S}^1} |\langle Q_t\rangle|^2_{M\circ P}\, d\theta \le K \int_{\mathbb{S}^1} |Q_t|^2_{M\circ P}\, d\theta$$

so that

$$\frac{1}{t}\int_{\mathbb{S}^1} |\langle Q_t\rangle|^2_{M\circ P}\, d\theta \quad \text{and} \quad \frac{1}{t}\int_{\mathbb{S}^1} (Q_t, \langle Q_t\rangle)_{M\circ P}\, d\theta$$

are both $L^1([t_0,\infty))$ for $t_0 > 0$. By "$\cdots$" we will below denote things that converge as $t \to \infty$. Consider

$$\begin{aligned}
&\int_{t_0}^{t} \frac{1}{s}\int_{\mathbb{S}^1} \big[-|Q_t|^2_{M\circ P} + |Q_\theta|^2_{M\circ P}\big] d\theta\, ds \\
&= \int_{t_0}^{t} \frac{1}{s}\int_{\mathbb{S}^1} \big[-(Q_t - \langle Q_t\rangle, Q_t)_{M\circ P} + |Q_\theta|^2_{M\circ P}\big] d\theta\, ds + \cdots \\
&= \left[-\frac{1}{s}\int_{\mathbb{S}^1} (Q - \langle Q\rangle, Q_t)_{M\circ P}\, d\theta\right]_{t_0}^{t} \\
&\quad + \int_{t_0}^{t}\int_{\mathbb{S}^1} \left[\partial_t\left(\frac{1}{s} g_{ij}(P) Q_t^i\right) - \partial_\theta\left(\frac{1}{s} g_{ij}(P) Q_\theta^i\right)\right](Q^j - \langle Q^j\rangle) d\theta\, ds + \cdots \\
&= -\int_{t_0}^{t} \frac{2}{s^2}\int_{\mathbb{S}^1} (Q - \langle Q\rangle, Q_t)_{M\circ P}\, d\theta\, ds + \cdots = \cdots ,
\end{aligned}$$

where we used (2.9) in the second-to-last inequality. We have also made use of estimates such as (5.6) and the fact that we know H to be bounded. We conclude that

$$\frac{1}{t}\int_{\mathbb{S}^1} |Q_\theta|^2_{M\circ P}\, d\theta \in L^1([t_0,\infty)) \quad \text{for all } t_0 > 0.$$

Due to (7.1) and Hölder's inequality, we conclude that $(1/t)|\langle P_t\rangle|^2$ is in $L^1([t_0,\infty))$ so that

$$\frac{1}{t}\int_{\mathbb{S}^1} \langle P_t\rangle \cdot P_t\, d\theta \in L^1([t_0,\infty))\,.$$

Consequently,

$$\begin{aligned}
&\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \big[-|P_t|^2 + |P_\theta|^2\big] d\theta\, ds \\
&= \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \big[-(P_t - \langle P_t\rangle)\cdot P_t + |P_\theta|^2\big] d\theta\, ds + \cdots \\
&= \left[-\frac{1}{s}\int_{\mathbb{S}^1} (P - \langle P\rangle)\cdot P_t\, d\theta\right]_{t_0}^{t} - \int_{t_0}^{t} \frac{1}{s^2}\int_{\mathbb{S}^1} P_t\cdot (P - \langle P\rangle) d\theta\, ds \\
&\quad + \int_{t_0}^{t} \frac{1}{s}\int_{\mathbb{S}^1} [P_{tt} - P_{\theta\theta}]\cdot (P - \langle P\rangle) d\theta\, ds + \cdots \\
&= -\int_{t_0}^{t} \frac{1}{s^2}\int_{\mathbb{S}^1} P_t\cdot (P - \langle P\rangle) d\theta\, ds \\
&\quad + \int_{t_0}^{t} \frac{1}{2s}\int_{\mathbb{S}^1} \frac{\partial g_{kl}}{\partial P^i}\big(Q_t^k Q_t^l - Q_\theta^k Q_\theta^l\big)(P^i - \langle P^i\rangle) d\theta\, ds + \cdots ,
\end{aligned}$$

where we have used the fact that H is bounded to the future and (2.8). Observe that

$$\left|\frac{1}{2s}\int_{\mathbb{S}^1} \frac{\partial g_{kl}}{\partial P^i}\big(Q_t^k Q_t^l - Q_\theta^k Q_\theta^l\big)(P^i - \langle P^i\rangle) d\theta\right| \le \frac{K}{s}\int_{\mathbb{S}^1} \big[|Q_t|^2_{M\circ P} + |Q_\theta|^2_{M\circ P}\big] d\theta$$

due to (5.3) and the fact that the sup norm of $P - \langle P\rangle$ is bounded to the future. We conclude that

$$\int_{t_0}^{t} \frac{1}{s}\int_{\mathbb{S}^1} \big[-|P_t|^2 + |P_\theta|^2\big] d\theta\, ds$$

converges as $t \to \infty$. Consequently,

$$\frac{1}{t}\int_{\mathbb{S}^1} |P_\theta|^2\, d\theta \in L^1([t_0, \infty))$$

so that

$$\frac{1}{t} H \in L^1([t_0, \infty))\,.$$

Since H is monotonically decaying, we conclude that it converges to 0. Combining this with Theorem 6.1, we get the conclusion of the theorem. □

8 Behavior of the Mean Values

From now on, we will only consider solutions to (1.2)–(1.3). The main point in the analysis of the behavior of the mean values is to interpret (1.11)–(1.13) as ODEs for the mean values. A first step in this direction is taken by the following lemma.

LEMMA 8.1 *Consider a solution to* (1.2)–(1.3). *Then*

$$t\langle P_t\rangle = \beta\langle Q\rangle - \frac{\alpha}{2} + \frac{1}{2\pi}\int_{\mathbb{S}^1} te^{2P}(Q - \langle Q\rangle)Q_t\,d\theta\,, \tag{8.1}$$

$$te^{\langle P\rangle}\langle Q_t\rangle = \beta e^{-\langle P\rangle} - \frac{1}{2\pi}e^{\langle P\rangle}\int_{\mathbb{S}^1}(e^{2P-2\langle P\rangle} - 1)tQ_t\,d\theta\,, \tag{8.2}$$

and

$$\begin{aligned} t\langle Q_t\rangle = {}& \gamma + \alpha\langle Q\rangle - \beta\langle Q\rangle^2 + \frac{t}{\pi}\int_{\mathbb{S}^1}(\langle Q\rangle - Q)P_t\,d\theta \\ & + \frac{t}{2\pi}\int_{\mathbb{S}^1} e^{2P}Q_t(Q - \langle Q\rangle)^2\,d\theta\,. \end{aligned} \tag{8.3}$$

PROOF: Observe that

$$\int_{\mathbb{S}^1} Q(tQ_t)e^{2P}\,d\theta = B\langle Q\rangle + \int_{\mathbb{S}^1} te^{2P}(Q - \langle Q\rangle)Q_t\,d\theta$$

so that (1.11) implies (8.1). Furthermore,

$$\begin{aligned} \int_{\mathbb{S}^1} e^{2P}Q^2(tQ_t)d\theta &= \int_{\mathbb{S}^1} e^{2P}tQ_tQ(Q - \langle Q\rangle)d\theta + \langle Q\rangle\left(\frac{A}{2} + t\int_{\mathbb{S}^1} P_t\,d\theta\right) \\ &= \langle Q\rangle\left(\frac{A}{2} + t\int_{\mathbb{S}^1} P_t\,d\theta\right) - B\langle Q\rangle^2 + \langle Q\rangle\left(\frac{A}{2} + t\int_{\mathbb{S}^1} P_t\,d\theta\right) \\ &\quad + \int_{\mathbb{S}^1} e^{2P}tQ_t(Q - \langle Q\rangle)^2\,d\theta \\ &= \langle Q\rangle(A + 2t\int_{\mathbb{S}^1} P_t\,d\theta) - B\langle Q\rangle^2 + \int_{\mathbb{S}^1} e^{2P}tQ_t(Q - \langle Q\rangle)^2\,d\theta\,. \end{aligned}$$

Combining this with (1.13) yields (8.3). Finally, (8.2) is a reformulation of (1.12) obtained by multiplying by $e^{-\langle P\rangle}/(2\pi)$ and proceeding similarly to the above. □

It will turn out to be easier to analyze these equations for certain combinations of the constants than for others. Since the constants change when one applies an isometry of the hyperbolic plane, it is natural to try to find an isometry that yields equations that are as simple as possible.

LEMMA 8.2 *Consider a solution to* (1.2)–(1.3). *If* $A^2 + 4BC > 0$, *there is an isometry such that if* A_1, B_1, *and* C_1 *are the constants of the transformed solution, then* $A_1 = -\sqrt{A^2 + 4BC}$ *and* $B_1 = C_1 = 0$. *If* $A^2 + 4BC = 0$, *there is an isometry such that the constants of the transformed solution are* $A_1 = B_1 = 0$ *and* $C_1 = 4\pi$ *or* $C_1 = 0$.

REMARK In the case $A^2 + 4BC < 0$, one cannot achieve $B = 0$, since $A^2 + 4BC$ is invariant under the isometries.

PROOF: Let us give a list of isometries and how the constants change.

Translation:

$$(Q, P) \mapsto (Q + \Delta, P) \quad \text{yields} \quad (A, B, C) \mapsto (A + 2\Delta B, B, C - \Delta A - \Delta^2 B)\,.$$

Dilation:

$$(Q, P) \mapsto (\eta Q, P - \ln \eta) \quad \text{yields} \quad (A, B, C) \mapsto \left(A, \frac{1}{\eta} B, \eta C\right).$$

Inversion:

$$(Q, e^{-P}) \mapsto \left(-\frac{Q}{Q^2 + e^{-2P}}, \frac{e^{-P}}{Q^2 + e^{-2P}}\right) \quad \text{yields} \quad (A, B, C) \mapsto -(A, C, B)\,.$$

The above isometries generate all the orientation-preserving isometries of the hyperbolic plane. To get all the isometries, one only needs to add a reflection:

$$(Q, P) \mapsto (-Q, P) \quad \text{yields} \quad (A, B, C) \mapsto (A, -B, -C)\,.$$

One can compute explicitly that all the above transformations leave the expression $A^2 + 4BC$ invariant.

Let us assume that $B \neq 0$. One obtains $B = 0$ by carrying out a translation by $\Delta = -A/(2B) + \sqrt{A^2 + 4BC}/(2B)$ and then an inversion. If $A^2 + 4BC > 0$, one can make the C-constant 0 by the translation $\Delta = B/\sqrt{A^2 + 4BC}$. This yields the form given in the statement of the lemma. If $A^2 + 4BC = 0$, we can use a reflection and dilation, if necessary, to obtain the desired form.

If $B = 0$ and $A^2 + 4BC = 0$, one applies a dilation and reflection as above, and if $B = 0$ and $A^2 + 4BC > 0$, one first makes a translation to set the C-constant to 0, and then one makes an inversion, if necessary, to get the right sign on the A-constant. □

Let us apply the above lemma to the spatially homogeneous solutions. In that case, either $A^2 + 4BC > 0$ or the solution is constant due to (1.15). Thus, in the nontrivial case, there is an isometry such that the transformed solution has $B_1 = C_1 = 0$ and $A_1 < 0$. However, this implies that $Q_1 = 0$ in the spatially homogeneous case. In other words, all spatially homogeneous solutions to (1.2)–(1.3) can be obtained by applying the isometries of the hyperbolic plane to the polarized ($Q = 0$) spatially homogeneous solutions. In particular, all nontrivial spatially homogeneous solutions follow geodesics of the hyperbolic plane.

The starting point for the analysis is the following.

LEMMA 8.3 *Consider a solution to* (1.2)–(1.3) *with* $B \neq 0$. *Assume* $H(t) \leq K_1/t$ *for* $t \geq T \geq 1$. *If* $A^2 + 4BC \geq 0$, *then* $\langle Q \rangle$ *is bounded and* $\langle P \rangle$ *is bounded from below for* $t \geq T$. *If* $A^2 + 4BC < 0$, $\langle P \rangle$ *and* $\langle Q \rangle$ *are both bounded for* $t \geq T$. *The bounds depend only on the constants* A, B, C, *and* K_1.

PROOF: Multiply (8.3) with $e^{\langle P \rangle}$ and eliminate $te^{\langle P \rangle}\langle Q_t \rangle$ by using (8.2). After some rearrangements one obtains

$$\left|\beta\left\{e^{-\langle P \rangle} + e^{\langle P \rangle}\left[\left(\langle Q \rangle - \frac{\alpha}{2\beta}\right)^2 - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2}\right]\right\}\right| \leq K\,,$$

where K depends only on K_1. In the case $A^2 + 4BC < 0$, this implies first of all that $\langle P \rangle$ is bounded and then that $\langle Q \rangle$ is bounded. The lemma follows in this particular case. Concerning the remaining case, if $|\langle Q \rangle - \alpha/(2\beta)| \geq \delta/|\beta|$, where δ is defined by (1.14), both terms within the curly brackets have the same sign, whence we get a bound on $\langle P \rangle$ from below when this inequality is satisfied. This then implies a bound on $\langle Q \rangle$. Combining the bound on $\langle Q \rangle$ with the inequality, we get a bound on $\langle P \rangle$ from below. □

LEMMA 8.4 *Fix* $\mathcal{P} = (p_0, A_0, B_0, C_0)$ *with* $B_0 \neq 0$. *Consider the family of solutions determined by* $(\mathcal{P}, t_0)$ *by giving initial data as in* (6.9) *and* (6.10). *With notation as in Lemma* 6.2, *there are constants* $c_{\langle Q \rangle}(\mathcal{P})$ *and* $c_{\langle P \rangle, l}(\mathcal{P})$ *such that*

$$|\langle Q \rangle(\mathcal{P}, t_0; t)| \leq c_{\langle Q \rangle}(\mathcal{P}) \quad \text{and} \quad \langle P \rangle(\mathcal{P}, t_0; t) \geq c_{\langle P \rangle, l}(\mathcal{P})$$

for all $t \geq t_0 \geq t_{\mathcal{P}}$. *If* $A^2 + 4BC < 0$, *one also has such a uniform bound on* $\langle P(\mathcal{P}, t_0; t) \rangle$ *from above.*

PROOF: This follows by combining Lemma 8.3 with Lemma 6.2. □

LEMMA 8.5 *Consider a solution to* (1.2)–(1.3). *Then there is a* T *such that*

$$g = \langle P_t \rangle^2 + e^{2\langle P \rangle}\langle Q_t \rangle^2 \leq \frac{K}{t^2} \quad \text{for } t \geq T,$$

where the first equality defines g.

REMARK Note that the estimate $H \leq K/t$ combined with Hölder's inequality only yields $g \leq K/t$. The added decay obtained in this lemma is crucial to everything that follows.

PROOF: If $B = 0$, then (8.1) implies that

$$\langle P_t \rangle^2 \leq \frac{K}{t^2}\,.$$

If $B \neq 0$, then the same statement holds due to the fact that $\langle Q \rangle$ is bounded in that case; compare Lemma 8.3. Consider (8.2). The second term on the right-hand side is always bounded. The first term is also bounded, regardless of whether $B = 0$, due to Lemma 8.3. The lemma follows. □

LEMMA 8.6 *Fix* $\mathcal{P} = (p_0, A_0, B_0, C_0)$. *Consider the family of solutions determined by* $(\mathcal{P}, t_0)$ *by giving initial data as in* (6.9) *and* (6.10). *With notation as in Lemma* 6.2*, there is a constant* $c_g(\mathcal{P})$ *such that*

$$g(\mathcal{P}, t_0; t) \le \frac{c_g(\mathcal{P})}{t^2} \quad \text{for all } t \ge t_0 \ge t_{\mathcal{P}}.$$

The essential technical tools in the analysis are the following two lemmas.

LEMMA 8.7 *Consider a solution to* (1.2)–(1.3) *and let* $f \in C^\infty(\mathbb{R}_+, \mathbb{R})$ *satisfy*

$$|f| \le K \quad \text{and} \quad |f_t| \le \frac{K}{t^{1/2}} \quad \text{for } t \ge T.$$

Then, if $t \ge t_0 \ge T$,

$$\int_{t_0}^{t} f\left[\langle P_t\rangle - \frac{\beta}{s}\langle Q\rangle + \frac{\alpha}{2s}\right]ds = O\big(t_0^{-1/2}\big).$$

PROOF: Consider

$$\int_{t_0}^{t} f \int_{\mathbb{S}^1} e^{2P}(Q - \langle Q\rangle)Q_t\, d\theta\, ds.$$

Observe that

$$\int_{\mathbb{S}^1} e^{2P}(Q - \langle Q\rangle)\langle Q_t\rangle d\theta = O(t^{-3/2})$$

due to Lemma 8.5 and $H \le K/t$. Consequently,

$$\int_{\mathbb{S}^1} e^{2P}(Q - \langle Q\rangle)Q_t\, d\theta = \int_{\mathbb{S}^1} e^{2P}(Q - \langle Q\rangle)(Q_t - \langle Q_t\rangle)d\theta + O(t^{-3/2}).$$

Thus

$$\begin{aligned}
&\int_{t_0}^{t} f \int_{\mathbb{S}^1} e^{2P}(Q - \langle Q\rangle)Q_t\, d\theta\, ds \\
&= \frac{1}{2}\int_{t_0}^{t} f \int_{\mathbb{S}^1} e^{2P}\partial_t(Q - \langle Q\rangle)^2\, d\theta\, ds + O(t_0^{-1/2}) \\
&= \left[\frac{1}{2}\int_{\mathbb{S}^1} f e^{2P}(Q - \langle Q\rangle)^2\, d\theta\right]_{t_0}^{t} \\
&\quad - \int_{t_0}^{t}\int_{\mathbb{S}^1}\left(f P_t + \frac{1}{2}f_t\right)e^{2P}(Q - \langle Q\rangle)^2\, d\theta\, ds + O\big(t_0^{-1/2}\big) \\
&= O\big(t_0^{-1/2}\big).
\end{aligned}$$

Inserting this in (8.1) we get the conclusion of the lemma. □

LEMMA 8.8 *Consider a solution to* (1.2)–(1.3) *and let* $f \in C^\infty(\mathbb{R}_+, \mathbb{R})$ *satisfy*

$$|e^{-\langle P\rangle} f| \leq K \quad \textit{and} \quad |e^{-\langle P\rangle} f_t| \leq \frac{K}{t^{1/2}} \quad \textit{for } t \geq T. \tag{8.4}$$

Then, if $t \geq t_0 \geq T$,

$$\int_{t_0}^t f\left[2\langle Q_t\rangle - \frac{\gamma}{s} - \frac{\alpha}{s}\langle Q\rangle + \frac{\beta}{s}\langle Q\rangle^2 - \frac{\beta}{s}e^{-2\langle P\rangle}\right]ds = O\big(t_0^{-1/2}\big). \tag{8.5}$$

PROOF: Consider (8.3). Observe that the last term on the right-hand side is $O(t^{-3/2})$ if we multiply by f/t, due to (8.4). Consider now the second-to-last term. Dividing by t and integrating in time, we obtain

$$\begin{aligned}
&\frac{1}{\pi}\int_{t_0}^t f \int_{\mathbb{S}^1} (\langle Q\rangle - Q)P_t \, d\theta\, ds \\
&\quad = \frac{1}{\pi}\int_{t_0}^t f \int_{\mathbb{S}^1} (\langle Q\rangle - Q)(P_t - \langle P_t\rangle)d\theta\, ds + O\big(t_0^{-1/2}\big) \\
&\quad = \left[\frac{1}{\pi}\int_{\mathbb{S}^1} f(\langle Q\rangle - Q)(P - \langle P\rangle)d\theta\right]_{t_0}^t \\
&\qquad - \frac{1}{\pi}\int_{t_0}^t \int_{\mathbb{S}^1} [f(\langle Q_t\rangle - Q_t) + f_t(\langle Q\rangle - Q)](P - \langle P\rangle)d\theta\, ds + O\big(t_0^{-1/2}\big) \\
&\quad = \frac{1}{\pi}\int_{t_0}^t f \int_{\mathbb{S}^1} Q_t(P - \langle P\rangle)d\theta\, ds + O\big(t_0^{-1/2}\big),
\end{aligned}$$

where we have used (8.4) and Lemma 8.5. Due to (8.2) we have, by Taylor-expanding $\exp[2P - 2\langle P\rangle] - 1$,

$$\langle Q_t\rangle = \frac{\beta}{t}e^{-2\langle P\rangle} - \frac{1}{\pi}\int_{\mathbb{S}^1} (P - \langle P\rangle)Q_t \, d\theta + O\big(e^{-\langle P\rangle}t^{-3/2}\big), \tag{8.6}$$

whence

$$\frac{1}{\pi}\int_{t_0}^t f \int_{\mathbb{S}^1} Q_t(P - \langle P\rangle)d\theta\, ds = \int_{t_0}^t f\left[-\langle Q_t\rangle + \frac{\beta}{s}e^{-2\langle P\rangle}\right]ds + O\big(t_0^{-1/2}\big).$$

Inserting this information in (8.3), we obtain (8.5). □

It will be of interest to apply the two above lemmas to families of solutions to (1.2)–(1.3). What one wants to know in such a situation is that the constant hidden in $O(t_0^{-1/2})$ can be chosen to be the same for the entire family. Assume that the family is parametrized by a parameter r. Assume furthermore that for $t \geq t_0 \geq 1$, there are functions $f(r;\cdot)$ such that the following estimates hold:

$$|f(r;t)| \leq K_1, \quad |f_t(r;t)| \leq K_2 t^{-1/2}, \quad H(r;t) \leq K_3 t^{-1}, \quad g(r;t) \leq K_4 t^{-2},$$

for all $t \geq t_0$ with constants K_1 through K_4 independent of r. Then one can convince oneself that one can use the same constant in Lemma 8.7 for the entire family. The statement concerning Lemma 8.8 is similar.

LEMMA 8.9 *Consider a solution to* (1.2)–(1.3). *Then if* $t > t_0 \geq 1$

$$\int_{t_0}^{t} \left[\langle P_t \rangle - \frac{\beta}{s} \langle Q \rangle + \frac{\alpha}{2s} \right] ds = O\big(t_0^{-1/2}\big) . \tag{8.7}$$

PROOF: Apply Lemma 8.7 with $f = 1$. □

COROLLARY 8.10 *Consider a solution to* (1.2)–(1.3). *If* $B = 0$ *there is a constant* c_P *and a* $T > 0$ *such that*

$$\langle P \rangle + \frac{\alpha}{2} \ln t - c_P = O(t^{-1/2}) \quad \textit{for all } t \geq T.$$

PROPOSITION 8.11 *Consider a solution to* (1.2)–(1.3). *If* $B = 0$, *then there is a constant* c_Q *and a* $T > 0$ *such that for* $t \geq T$,

$$\left| e^{\langle P \rangle} \left(\langle Q \rangle + \frac{\gamma}{\alpha} \right) - c_Q \right| \leq K t^{-1/2} \quad \textit{if } \alpha \neq 0 \tag{8.8}$$

and

$$\left| \langle Q \rangle - \frac{\gamma}{2} \ln t - c_Q \right| \leq K t^{-1/2} \quad \textit{if } \alpha = 0 . \tag{8.9}$$

REMARK The case $\beta = \alpha = 0$ and $\gamma \neq 0$ cannot occur in the spatially homogeneous case.

PROOF: Let $f = t^{-\alpha/2}$. Then (8.4) is satisfied due to Corollary 8.10. Lemma 8.8 then yields

$$\int_{t_0}^{t} \left[s^{-\alpha/2} \langle Q_t \rangle - \frac{\alpha}{2s} s^{-\alpha/2} \langle Q \rangle - \frac{\gamma}{2s} s^{-\alpha/2} \right] ds = O\big(t_0^{-1/2}\big) ,$$

since $\beta = 0$. Since the first two terms in the integrand can be written as the derivative of $s^{-\alpha/2}\langle Q \rangle$, we get the conclusion that there is a c such that

$$\left| t^{-\alpha/2} \langle Q \rangle + \frac{\gamma}{\alpha} t^{-\alpha/2} - c \right| \leq K t^{-1/2} \quad \text{if } \alpha \neq 0.$$

If $\alpha = 0$, we get (8.9). Combining this with Corollary 8.10, we obtain the conclusion of the proposition. □

THEOREM 8.12 *Consider a solution to* (1.2)–(1.3) *with* $A^2 + 4BC > 0$. *Then there is an isometry of the hyperbolic plane such that if* (Q_1, P_1) *is the transformed solution,*

$$\left\| P_1 - \delta \ln t - c_P \right\|_{C(\mathbb{S}^1,\mathbb{R})} + \left\| e^{P_1} Q_1 - c_1 \right\|_{C(\mathbb{S}^1,\mathbb{R})} \leq K t^{-1/2}, \tag{8.10}$$

where δ *is given by* (1.14).

PROOF: The theorem follows by combining Lemma 8.2, Corollary 8.10, Proposition 8.11, and the decay of the energy. □

Note that if one lets $\mathbf{x}_1 = (x_1, y_1) = (Q_1, e^{-P_1})$, then the distance from (x_1, y_1) to the line Γ, defined by $x = c_1 y$, with respect to the hyperbolic metric tends to 0 as $t^{-1/2}$. In fact, if $\mathbf{x} = (x, y) = (c_1 y_1, y_1)$ and $\gamma(s) = (sx + (1-s)x_1, y_1)$, then γ joins $\mathbf{x}$ and $\mathbf{x}_1$ and the length of γ with respect to the hyperbolic metric decays as $t^{-1/2}$ due to Theorem 8.12. Thus $d_H(\mathbf{x}_1, \Gamma) \leq K t^{-1/2}$. Since this estimate is invariant under isometries, we get the statement of Theorem 1.2 in the case that $A^2 + 4BC > 0$.

The question remains to what values c_1 may converge.

PROPOSITION 8.13 *Let* $A_0, c_1', \eta \in \mathbb{R}$ *with* $-A_0, \eta > 0$. *Then there is a solution to* (1.2)–(1.3) *with* $(A, B, C) = (A_0, 0, 0)$ *such that if* c_1 *is the constant appearing in* (8.10), $|c_1 - c_1'| \leq \eta$.

PROOF: We first need to modify the initial data (6.9)–(6.10) slightly. Let $A_0 < 0$ and $p_0, q_0 \in \mathbb{R}$ be arbitrary. Define $B_0 = 0$ and $C_0 = q_0 A_0$. Performing a translation of the corresponding initial data (6.9)–(6.10) by q_0, one obtains

$$Q(t_0, \theta) = q_0 + \frac{1}{t_0^{1/2}} \cos\theta, \quad P(t_0, \theta) = p_0, \quad Q_t(t_0, \theta) = 0, \tag{8.11}$$

$$P_t(t_0, \theta) = -\frac{1}{4\pi t_0} A_0 + \frac{q_0 A_0}{2\pi t_0^{1/2}} \cos\theta. \tag{8.12}$$

The corresponding family of solutions will have $(A, B, C) = (A_0, 0, 0)$. Let $\mathcal{Q} = (A_0, p_0, q_0)$, and denote a solution to (1.2)–(1.3) with the initial data (8.11)–(8.12) by $P(\mathcal{Q}, t_0; t, \theta)$, etc. Observe that the family has been obtained by a translation of the standard family we have been considering and that the energy is invariant under translations. Thus, we can apply Lemma 6.2 to obtain the existence of a $t_{\mathcal{Q}}$ such that

$$H(\mathcal{Q}, t_0; t) \leq \frac{K(\mathcal{Q})}{t}$$

for all $t \geq t_0 \geq t_{\mathcal{Q}} \geq 1$. Due to this, the fact that (8.1) and (8.2) hold and the fact that $\beta = 0$ for this family, we obtain

$$t^2 \big[\langle P_t(\mathcal{Q}, t_0; t) \rangle^2 + \exp\big(2\langle P(\mathcal{Q}, t_0; t) \rangle\big) \langle Q_t(\mathcal{Q}, t_0; t) \rangle^2 \big] \leq K(\mathcal{Q})$$

for all $t \geq t_0 \geq t_{\mathcal{Q}} \geq 1$. By the argument presented after the proof of Lemma 8.8, we can apply Lemma 8.7 for $f = 1$ and obtain an estimate which is uniform in the sense that

$$\left| \int_{t_0}^{t} \left[\langle P_t(\mathcal{Q}, t_0; s) \rangle + \frac{\alpha}{2s} \right] ds \right| \leq K(\mathcal{Q}) t_0^{-1/2}$$

for all $t \geq t_0 \geq t_{\mathcal{Q}} \geq 1$. Thus

$$\left| \langle P(\mathcal{Q}, t_0; t) \rangle - p_0 + \frac{\alpha}{2} \ln \frac{t}{t_0} \right| \leq K(\mathcal{Q}) t_0^{-1/2} . \tag{8.13}$$

Define

$$f(\mathcal{Q}, t_0; t) = \exp\left(p_0 - \frac{\alpha}{2} \ln \frac{t}{t_0} \right) .$$

Then

$$\exp\left[-\langle P(\mathcal{Q}, t_0; t) \rangle \right] |f(\mathcal{Q}, t_0; t)| \leq \exp\left[K(\mathcal{Q}) t_0^{-1/2} \right]$$

and

$$\exp\left[-\langle P(\mathcal{Q}, t_0; t) \rangle \right] |f_t(\mathcal{Q}, t_0; t)| \leq \frac{|\alpha|}{2t} \exp\left[K(\mathcal{Q}) t_0^{-1/2} \right] .$$

As was observed after its proof, one can apply Lemma 8.8 with f as above and obtain uniform estimates in the sense that

$$\left| \int_{t_0}^{t} f(\mathcal{Q}, t_0; s) \left[\langle Q_t(\mathcal{Q}, t_0; s) \rangle - \frac{\alpha}{2s} \langle Q(\mathcal{Q}, t_0; s) \rangle \right] ds \right| \leq K(\mathcal{Q}) t_0^{-1/2}$$

for $t \geq t_0 \geq t_{\mathcal{Q}} \geq 1$. Thus

$$\left| f(\mathcal{Q}, t_0; t) \langle Q(\mathcal{Q}, t_0; t) \rangle - e^{p_0} q_0 \right| \leq K(\mathcal{Q}) t_0^{-1/2} .$$

Combining this with (8.13), we get

$$\left| \exp[\langle P(\mathcal{Q}, t_0; t) \rangle] \langle Q(\mathcal{Q}, t_0; t) \rangle - e^{p_0} q_0 \right| \leq K(\mathcal{Q}) t_0^{-1/2} .$$

By choosing p_0 and q_0 so that $e^{p_0} q_0 = c_1'$ and letting t_0 be big enough, we obtain the conclusion of the proposition. □

Similarly to the proof of Theorem 8.12, we have the following:

THEOREM 8.14 *Consider a solution to* (1.2)–(1.3) *with* $A^2 + 4BC = 0$ *but for which not all the constants are* 0. *Then there is an isometry of the hyperbolic plane such that if* (Q_1, P_1) *is the transformed solution,*

$$\|P_1 - c_P\|_{C(\mathbb{S}^1, \mathbb{R})} + \|Q_1 - \ln t - c_Q\|_{C(\mathbb{S}^1, \mathbb{R})} \leq K t^{-1/2} .$$

We deduce from this theorem that the distance from the solution $(x_1, y_1) = (Q_1, e^{-P_1})$ to the curve $y = \text{const}$ decays to 0 as $t^{-1/2}$ with respect to the hyperbolic metric. The corresponding statement in Theorem 1.2 follows.

For the sake of completeness, let us say something about how P and Q behave after undoing the isometries in Theorems 8.12 and 8.14. Consider first the proof of Lemma 8.2. In order to obtain $(A, B, C) = (0, 0, 4\pi)$ in the case $A^2 + 4BC = 0$, one has to carry out the following operations (assuming B was not 0 to start with):

first a translation, then an inversion, then a dilation, and finally, if necessary, a reflection. Let us reverse this procedure. We begin by taking a solution with the asymptotics as given in Theorem 8.14 and to start with, we carry out a dilation and possibly a reflection. The asymptotics one then obtains are

$$\|P_1 - c_P\|_{C(\mathbb{S}^1,\mathbb{R})} + \|Q_1 - c_0 \ln t - c_Q\|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2}\,. \tag{8.14}$$

Here c_0 can be any nonzero number. Then one should carry out an inversion. Let $P_0 = c_P$, $Q_0 = c_0 \ln t + c_Q$, and

$$\left(Q_2, e^{-P_2}\right) = \left(-\frac{Q_1}{Q_1^2 + e^{-2P_1}}, \frac{e^{-P_1}}{Q_1^2 + e^{-2P_1}}\right). \tag{8.15}$$

Due to (8.14), we have

$$Q_2 + \frac{Q_0}{Q_0^2 + e^{-2P_0}} = O[(\ln t)^{-2} t^{-1/2}]$$

and

$$e^{-P_2} - \frac{e^{-P_0}}{Q_0^2 + e^{-2P_0}} = O[(\ln t)^{-2} t^{-1/2}]\,.$$

Thus

$$\begin{aligned} P_2 - P_0 - \ln\left[Q_0^2 + e^{-2P_0}\right] &= \ln \frac{e^{P_1}(Q_1^2 + e^{-2P_1})}{e^{P_0}(Q_0^2 + e^{-2P_0})} \\ &= \ln[1 + O(t^{-1/2})] = O(t^{-1/2})\,. \end{aligned}$$

Finally, one should carry out a translation. Let us consider the case $A^2+4BC > 0$ and $B \neq 0$. Then one has to perform a translation, an inversion, and finally a nonzero translation in order to obtain the asymptotics formulated in Theorem 8.12. In order to undo these operations, we therefore first have to carry out a nonzero translation. We then obtain

$$\left\|P_1 - \delta \ln t - c_P\right\|_{C(\mathbb{S}^1,\mathbb{R})} + \left\|e^{P_1}(Q_1 - c_Q) - c_1\right\|_{C(\mathbb{S}^1,\mathbb{R})} \leq Kt^{-1/2}\,,$$

where $c_Q \neq 0$. Introduce $P_0 = \delta \ln t + c_P$, $Q_0 = c_Q + c_1 e^{-P_0}$, and (Q_2, P_2) by (8.15). We obtain

$$Q_2 + \frac{Q_0}{Q_0^2 + e^{-2P_0}} = O(t^{-\delta-1/2})$$

due to the fact that $\delta > 0$ and Q_0 converges to a nonzero value. Furthermore,

$$e^{-P_2} - \frac{e^{-P_0}}{Q_0^2 + e^{-2P_0}} = O(t^{-\delta-1/2})$$

so that

$$\begin{aligned} P_2 - P_0 - \ln\left(Q_0^2 + e^{-2P_0}\right) &= \ln \frac{e^{P_1}(Q_1^2 + e^{-2P_1})}{e^{P_0}(Q_0^2 + e^{-2P_0})} \\ &= \ln[1 + O(t^{-1/2})] = O(t^{-1/2})\,. \end{aligned}$$

Thus we obtain the behavior in the cases $B \neq 0$ and $A^2+4BC \geq 0$. It is interesting to note that one can use (8.7) and (8.16) below to find the leading-order behavior of the solution in these cases. However, it is quite difficult, particularly in the case $A^2 + 4BC = 0$. In a sense, it is perhaps not so surprising, since, for example, if $A^2 + 4BC = 0$ and $B \neq 0$, then the asymptotics are that P and Q approach the boundary of hyperbolic space along a horocycle.

LEMMA 8.15 *Consider a solution to* (1.2)–(1.3). *If* $B \neq 0$, *then there is a* $T > 0$ *such that for* $t \geq t_0 \geq T$,

$$\int_{t_0}^{t} \left[2\langle Q_t \rangle + \frac{\beta}{s} \left\{ \left(\langle Q \rangle - \frac{\alpha}{2\beta} \right)^2 - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} \right\} - \frac{\beta}{s} e^{-2\langle P \rangle} \right] ds = O\big(t_0^{-1/2}\big) . \tag{8.16}$$

PROOF: Due to Lemma 8.3, we know that $\langle P \rangle$ is bounded from below. Thus we can apply Lemma 8.8 with $f = 1$ to obtain the conclusion of the lemma. □

Before we state the next proposition, it seems natural to give some intuition motivating the result. Consider (8.7) and (8.16). Assuming the error terms to be 0, one gets

$$\int_{t_0}^{t} \left[\langle P_t \rangle - \frac{\beta}{s} \left(\langle Q \rangle - \frac{\alpha}{2\beta} \right) \right] ds = 0$$

and similarly for $\langle Q_t \rangle$. Introducing $u = \langle Q \rangle - \alpha/(2\beta)$ and $v = \langle P \rangle$, and differentiating the above-mentioned equations, one obtains

$$\dot{v} = \frac{\beta}{t} u \quad \text{and} \quad 2\dot{u} = \frac{\beta}{t} \left\{ e^{-2v} + \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} - u^2 \right\} .$$

If we change time coordinates so that $2t\dot{u}/\beta = u'$ and assume $(\alpha^2+4\beta\gamma)/(4\beta^2) = -1$, we get the equations

$$u' = e^{-2v} - 1 - u^2 \quad \text{and} \quad v' = 2u . \tag{8.17}$$

Solutions to these equations have the property that

$$u^2 e^v + e^v + e^{-v} = \text{const} . \tag{8.18}$$

In fact, one can prove that if u and v are solutions to (8.17) that are nontrivial, there is a $v_0 > 0$ and a τ_0 such that the solution can be written

$$[u(\tau), v(\tau)] = \left(\frac{\sinh v_0 \sin 2(\tau - \tau_0)}{\cosh v_0 - \sinh v_0 \cos 2(\tau - \tau_0)}, \ln[\cosh v_0 - \sinh v_0 \cos 2(\tau - \tau_0)] \right) .$$

In reality, we do have error terms that have to be dealt with. However, it is natural to try to prove that an analogue of the conserved quantity (8.18) converges to some value as t tends to ∞.

PROPOSITION 8.16 *Consider a solution to* (1.2)–(1.3). *Assume that* $A^2 + 4BC < 0$. *Then there is a constant* c_N *and a* $T > 0$ *such that for* $t \geq T$

$$\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 e^{\langle P\rangle} - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} e^{\langle P\rangle} + e^{-\langle P\rangle} - c_N = O(t^{-1/2}) .$$

PROOF: Consider

$$\int_{t_0}^{t} \partial_s \left[\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 e^{\langle P\rangle} - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} e^{\langle P\rangle} + e^{-\langle P\rangle}\right] ds .$$

Due to Lemma 8.3 we know that $\langle P\rangle$ and $\langle Q\rangle$ are bounded. Due to Lemma 8.5, any $f = h(\langle P\rangle, \langle Q\rangle)$ for $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$ can thus be used when applying Lemma 8.7 and 8.8. We get

$$\int_{t_0}^{t} e^{\langle P\rangle}\langle P_t\rangle ds = \int_{t_0}^{t} \frac{\beta}{s} e^{\langle P\rangle}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right) ds + O\left(t_0^{-1/2}\right) ,$$

$$\int_{t_0}^{t} e^{-\langle P\rangle}\langle P_t\rangle ds = \int_{t_0}^{t} \frac{\beta}{s} e^{-\langle P\rangle}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right) ds + O\left(t_0^{-1/2}\right) ,$$

$$\int_{t_0}^{t} \left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 e^{\langle P\rangle}\langle P_t\rangle ds = \int_{t_0}^{t} \frac{\beta}{s}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^3 e^{\langle P\rangle} ds + O]\left(t_0^{-1/2}\right) ,$$

and

$$2\int_{t_0}^{t} \left(\langle Q\rangle - \frac{\alpha}{2\beta}\right) e^{\langle P\rangle}\langle Q_t\rangle ds =$$
$$-\int_{t_0}^{t} \left(\langle Q\rangle - \frac{\alpha}{2\beta}\right) e^{\langle P\rangle} \frac{\beta}{s}\left[\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} - e^{-2\langle P\rangle}\right] ds + O\left(t_0^{-1/2}\right) .$$

Combining these observations, we get the conclusion of the proposition. □

Observe that the transformation $(Q, P) \mapsto (Q, e^{-P})$ takes the curve in the statement of Proposition 8.16 to a circle. From the estimates it follows that the distance from the solution to this curve decays as $t^{-1/2}$, and the statement in Theorem 1.2 concerning the case $A^2 + 4BC < 0$ follows.

At this point it becomes clear why we have insisted on obtaining uniform estimates. The function

$$\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 e^{\langle P\rangle} - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} e^{\langle P\rangle} + e^{-\langle P\rangle} \tag{8.19}$$

has a minimum $2|\delta/\beta|$, where δ is given by (1.14), achieved for

$$\langle Q\rangle = \frac{\alpha}{2\beta} \quad \text{and} \quad \langle P\rangle = -\ln\left|\frac{\delta}{\beta}\right| .$$

If c_N equals this minimum value, $\langle P\rangle$ and $\langle Q\rangle$ have to converge to the corresponding values. Since we have no control over c_N, one can then ask the question whether c_N always corresponds to the minimum value. That this is not the case follows from the following proposition:

PROPOSITION 8.17 *Fix* $\mathcal{P} = (p_0, A_0, B_0, C_0)$ *with* $A_0^2 + 4B_0C_0 < 0$. *Consider the family of solutions determined by* $(\mathcal{P}, t_0)$ *by giving initial data as in* (6.9) *and* (6.10). *With notation as in Lemma* 6.2*, there are constants* $c_l(\mathcal{P})$ *and* $c_N(\mathcal{P}, t_0) \geq 0$ *such that*

$$\left|\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 e^{\langle P\rangle} - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} e^{\langle P\rangle} + e^{-\langle P\rangle} - c_N(\mathcal{P}, t_0)\right| \leq c_l(\mathcal{P}) t^{-1/2}$$

for $t \geq t_0 \geq t_{\mathcal{P}} \geq 1$, *where we have omitted the argument* $(\mathcal{P}, t_0; t)$ *for the sake of brevity.*

PROOF: First recall the observations following the proof of Lemma 8.8. The parameter in this case is the starting time t_0. The proposition follows by going through the proof of Proposition 8.16 and keeping these observations in mind. □

By the above proposition we can, given B_1 and C_1 with $B_1C_1 < 0$, $c \geq 2|C_1/B_1|^{1/2}$, and $\eta > 0$, find a solution with $(A, B, C) = (0, B_1, C_1)$ such that $|c - c_N| \leq \eta$ by varying p_0 and the starting time t_0. In this way, we can control the length of the circle with respect to the hyperbolic metric to which the solution asymptotes. Consider a solution with $A^2 + 4BC < 0$. By the proof of Lemma 8.2, we can carry out a translation of it so that the transformed solution has A-constant $A_1 = 0$. Thus there is no restriction in assuming $A_1 = 0$. What the above observations say is that there are basically no restrictions on what the length of the circle with respect to the hyperbolic metric might be.

Let us try to describe how the solution behaves when the circle is nontrivial. Assume that $c_N > 2|\delta/\beta|$. Introduce the variables

$$(u, v) = \left[\frac{\delta e^{\langle P\rangle}}{\beta} - \frac{\beta c_N}{2\delta}, e^{\langle P\rangle}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)\right].$$

Observe that

$$u^2 + v^2 = -1 + \frac{\beta^2 c_N^2}{4\delta^2} + O(t^{-1/2}).$$

Let

$$r_N = \left(-1 + \frac{\beta^2 c_N^2}{4\delta^2}\right)^{1/2}$$

and define

$$(\tilde{u}, \tilde{v}) = r_N[\sin(\delta \ln t + \phi_0), \cos(\delta \ln t + \phi_0)],$$

where ϕ_0 has been chosen so that $\mathbf{u} = {}^{\mathsf{T}}(u, v)$ is parallel to $\tilde{\mathbf{u}} = {}^{\mathsf{T}}(\tilde{u}, \tilde{v})$ when $t = t_0$.

PROPOSITION 8.18 *Consider a solution to* (1.2)–(1.3) *such that* $A^2 + 4BC < 0$. *Assume that the* c_N *associated with this solution satisfies* $c_N > 2|\delta/\beta|$. *Let* $T \geq 1$

be great enough that $u^2 + v^2 > 0$ for all $t \geq T$. Let $t_0 \geq T$ and assume, with notation as above, ϕ_0 to be such that $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are parallel for $t = t_0$. Then

$$\left\| P - \ln\left[\frac{\beta^2 c_N}{2\delta^2} + \frac{\beta}{\delta} r_N \sin(\delta \ln t + \phi_0)\right]\right\|_{C(\mathbb{S}^1,\mathbb{R})} \leq K t_0^{-1/2}$$

and

$$\left\| Q - \frac{\alpha}{2\beta} - \frac{r_N \cos(\delta \ln t + \phi_0)}{\frac{\beta^2 c_N}{2\delta^2} + \frac{\beta}{\delta} r_N \sin(\delta \ln t + \phi_0)}\right\|_{C(\mathbb{S}^1,\mathbb{R})} \leq K t_0^{-1/2}$$

for $t \geq t_0$, where K depends only on T and the initial data.

PROOF: Let

$$\Phi = \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}$$

where $\xi(t) = \delta \ln t + \phi_0$. If

$$A = \frac{\delta}{t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then $\Phi' = -A\Phi$. Observe also that $[A, \Phi] = 0$. For $t \geq t_0$,

$$\begin{aligned} \Phi(t)(\mathbf{u} - \tilde{\mathbf{u}})(t) - \Phi(t_0)(\mathbf{u} - \tilde{\mathbf{u}})(t_0) &= \int_{t_0}^{t} [-A\Phi(\mathbf{u} - \tilde{\mathbf{u}}) + \Phi(\mathbf{u} - \tilde{\mathbf{u}})']ds \\ &= \int_{t_0}^{t} (-A\Phi\mathbf{u} + \Phi\mathbf{u}')ds\,, \end{aligned} \tag{8.20}$$

since $\tilde{\mathbf{u}}' = A\tilde{\mathbf{u}}$. Observe that we can apply Lemma 8.7 and 8.8 if f is given by $f = h(\sin\xi, \cos\xi, \langle P\rangle, \langle Q\rangle)$ for an arbitrary $h \in C^\infty(\mathbb{R}^4, \mathbb{R})$ if $t \geq 1$. Consequently, we have

$$\int_{t_0}^{t} \Phi\mathbf{u}'\, ds = \int_{t_0}^{t} \Phi\mathbf{v}\, ds + O\big(t_0^{-1/2}\big)\,,$$

with $\mathbf{v} = {}^{\mathsf{T}}(v_1, v_2)$, where

$$v_1 = \frac{\delta}{\beta} e^{\langle P\rangle} \frac{\beta}{s}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right) = \frac{\delta}{s} v\,,$$

and

$$\begin{aligned} v_2 &= \frac{\beta}{s} e^{\langle P\rangle}(\langle Q\rangle - \frac{\alpha}{2\beta})^2 - e^{\langle P\rangle}\frac{\beta}{2s}\left\{\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 - \frac{\alpha^2 + 4\beta\gamma}{4\beta^2} - e^{-2\langle P\rangle}\right\} \\ &= \frac{\beta}{2s}\left\{e^{\langle P\rangle}\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 - \frac{\delta^2}{\beta^2} e^{\langle P\rangle} + e^{-\langle P\rangle}\right\} \\ &= \frac{\beta}{2s}\left(c_N - 2\frac{\delta^2}{\beta^2} e^{\langle P\rangle}\right) + O(s^{-3/2}) \\ &= -\frac{\delta}{s} u + O(s^{-3/2})\,. \end{aligned}$$

Consequently,

$$\int_{t_0}^{t} \Phi \mathbf{u}'\,ds = \int_{t_0}^{t} \Phi \mathbf{v}\,ds + O\big(t_0^{-1/2}\big) = \int_{t_0}^{t} \Phi[A\mathbf{u} + O(s^{-3/2})]ds + O\big(t_0^{-1/2}\big)\,.$$

Combining this with (8.20), we get the conclusion that

$$\Phi(t)(\mathbf{u} - \tilde{\mathbf{u}})(t) - \Phi(t_0)(\mathbf{u} - \tilde{\mathbf{u}})(t_0) = O\big(t_0^{-1/2}\big)\,.$$

Since ϕ_0 has been chosen so that $(\mathbf{u} - \tilde{\mathbf{u}})(t_0) = O(t_0^{-1/2})$ and since $\Phi \in \mathrm{SO}(2)$, we get the conclusion that

$$\|\mathbf{u}(t) - \tilde{\mathbf{u}}(t)\| \le K t_0^{-1/2} \quad \text{for all } t \ge t_0.$$

This can be used to derive the conclusion of the proposition. □

9 Asymptotic Behavior of the Energy

Up to this point, we have only an estimate of the energy H, but with the information obtained in the previous section, we can do better.

PROPOSITION 9.1 *Consider a solution to* (1.2)–(1.3). *Then there is a constant* $c_H \ge 0$ *and a* $T > 0$ *such that for all* $t \ge T$,

$$|tH(t) - c_H| \le \frac{K}{t}\,.$$

PROOF: Compute

$$\frac{d(tH)}{dt} = \frac{1}{2}\int_{\mathbb{S}^1} \big[P_\theta^2 - P_t^2 + e^{2P}(Q_\theta^2 - Q_t^2)\big]d\theta\,.$$

Using the fact that A is a conserved quantity, we compute

$$\begin{aligned}
&\int_{t_0}^{t}\int_{\mathbb{S}^1} e^{2P}\big(Q_\theta^2 - Q_t^2\big)d\theta\,ds \\
&\quad= \int_{t_0}^{t}\int_{\mathbb{S}^1} \bigg[-\partial_t\big(e^{2P}QQ_t\big) \\
&\qquad\qquad + e^{2P}Q\bigg(Q_{tt} + \frac{1}{s}Q_t + 2P_tQ_t - Q_{\theta\theta} - 2P_\theta Q_\theta\bigg) \\
&\qquad\qquad - \frac{1}{s}e^{2P}QQ_t\bigg]d\theta\,ds \\
&= -\bigg[\frac{A}{2s} + \int_{\mathbb{S}^1} P_t\,d\theta\bigg]_{t_0}^{t} - \int_{t_0}^{t}\bigg[\frac{A}{2s^2} + \frac{1}{s}\int_{\mathbb{S}^1} P_t\,d\theta\bigg]ds\,.
\end{aligned}$$

Thus, for $t \geq t_0 \geq T$,

$$\left| \int_{t_0}^{t} \int_{\mathbb{S}^1} e^{2P} (Q_\theta^2 - Q_t^2) d\theta\, ds \right| \leq K t_0^{-1}.$$

Note that

$$\int_{t_0}^{t} \int_{\mathbb{S}^1} P_t \langle P_t \rangle d\theta\, ds = \int_{t_0}^{t} 2\pi \langle P_t \rangle^2 ds = O\big(t_0^{-1}\big),$$

so that

$$\begin{aligned}
&\int_{t_0}^{t} \int_{\mathbb{S}^1} \big(P_\theta^2 - P_t^2\big) d\theta\, ds \\
&\quad = \int_{t_0}^{t} \int_{\mathbb{S}^1} \big[P_\theta^2 - P_t (P_t - \langle P_t \rangle)\big] d\theta\, ds + O\big(t_0^{-1}\big) \\
&\quad = \left[- \int_{\mathbb{S}^1} (P - \langle P \rangle) P_t\, d\theta \right]_{t_0}^{t} \\
&\qquad + \int_{t_0}^{t} \int_{\mathbb{S}^1} \left[-\frac{1}{s} P_t + e^{2P} \big(Q_t^2 - Q_\theta^2\big) \right] (P - \langle P \rangle) d\theta\, ds + O\big(t_0^{-1}\big) \\
&\quad = \int_{t_0}^{t} \int_{\mathbb{S}^1} e^{2P} \big(Q_t^2 - Q_\theta^2\big) (P - \langle P \rangle) d\theta\, ds + O\big(t_0^{-1}\big).
\end{aligned}$$

Let us compute, for $t \geq t_0 \geq T$,

$$\begin{aligned}
&\int_{t_0}^{t} \int_{\mathbb{S}^1} e^{2P} \big(Q_t^2 - Q_\theta^2\big) (P - \langle P \rangle) d\theta\, ds \\
&\quad = \int_{t_0}^{t} \int_{\mathbb{S}^1} e^{2P} \big[Q_t \partial_t (Q - \langle Q \rangle) - Q_\theta^2\big] (P - \langle P \rangle) d\theta\, ds + O\big(t_0^{-1}\big) \\
&\quad = \int_{t_0}^{t} \int_{\mathbb{S}^1} \big[\partial_t \big\{ e^{2P} Q_t (Q - \langle Q \rangle)(P - \langle P \rangle) \big\} \\
&\qquad\quad - \partial_t \big(e^{2P} Q_t\big) (Q - \langle Q \rangle)(P - \langle P \rangle) - e^{2P} Q_t (Q - \langle Q \rangle)(P_t - \langle P_t \rangle) \\
&\qquad\quad + \partial_\theta \big(e^{2P} Q_\theta\big) (Q - \langle Q \rangle)(P - \langle P \rangle) + e^{2P} Q_\theta (Q - \langle Q \rangle) P_\theta \big] d\theta\, ds \\
&\qquad + O\big(t_0^{-1}\big).
\end{aligned}$$

Observe that the terms involving $\partial_t (e^{2P} Q_t)$ and $\partial_\theta (e^{2P} Q_\theta)$ can be neglected due to (1.3) and the decay of the energy. The first term in the integrand can be ignored as well. The part of the third term in the integrand arising from $\langle P_t \rangle$ can also be neglected. Finally, replacing Q_t with $Q_t - \langle Q_t \rangle$ and e^{2P} with $e^{2\langle P \rangle}$ causes no harm.

Thus,

$$
\begin{aligned}
\int_{t_0}^{t} \int_{\mathbb{S}^1} & e^{2P} \big(Q_t^2 - Q_\theta^2\big)(P - \langle P \rangle) d\theta\, ds \\
&= \frac{1}{2} \int_{t_0}^{t} \int_{\mathbb{S}^1} e^{2\langle P \rangle} \big[\partial_\theta (Q - \langle Q \rangle)^2 P_\theta - \partial_t (Q - \langle Q \rangle)^2 P_t\big] d\theta\, ds + O\big(t_0^{-1}\big) \\
&= \frac{1}{2} \int_{t_0}^{t} \int_{\mathbb{S}^1} \big[e^{2\langle P \rangle} (Q - \langle Q \rangle)^2 (P_{tt} - P_{\theta\theta}) - \partial_t \big\{ e^{2\langle P \rangle} (Q - \langle Q \rangle)^2 P_t \big\} \\
&\qquad + 2\langle P_t \rangle e^{2\langle P \rangle} (Q - \langle Q \rangle)^2 P_t\big] d\theta\, ds + O\big(t_0^{-1}\big) \\
&= O\big(t_0^{-1}\big) .
\end{aligned}
$$

The proposition follows. □

In order to prove that if $t^2 H(t)$ is bounded, then the solution is independent of the θ-variable, we need to prove the following technical lemma:

LEMMA 9.2 *Consider a solution to* (1.2)–(1.3). *If* $t^2 H$ *is bounded, then*

$$
\lim_{t \to \infty} [t^2 H(t) - t\psi(t)] = 0 , \tag{9.1}
$$

where

$$
\psi = \pi t \big(\langle P_t \rangle^2 + e^{2\langle P \rangle} \langle Q_t \rangle^2\big) .
$$

PROOF: Going through the proof of Theorem 6.1, one can see that

$$
\frac{d(H + \Gamma)}{dt} = -\frac{1}{t}(H + \Gamma) - \frac{1}{t}\Gamma - \frac{1}{t^2}\psi + \epsilon ,
$$

where

$$
|\epsilon| \le \frac{K}{t} H^{3/2} .
$$

Introducing $\mathcal{E} = t^2(H + \Gamma)$ and assuming t to be big enough, we thus have

$$
\frac{d\mathcal{E}}{dt} = \frac{1}{t}\mathcal{E} - \psi + \epsilon_1 + \epsilon_2 \tag{9.2}
$$

where

$$
|\epsilon_1| \le K \frac{\mathcal{E}}{t^2} \quad \text{and} \quad |\epsilon_2| \le K \frac{\mathcal{E}^{3/2}}{t^2} .
$$

Furthermore,

$$
\psi \le \frac{t}{2} \int_{\mathbb{S}^1} \big[P_t^2 + e^{2P} Q_t^2\big] d\theta \left(1 + \frac{K}{t}\right) .
$$

Thus

$$
\frac{1}{t}\mathcal{E} - \psi \ge -\frac{K}{t^2}
$$

so that the right-hand side of (9.2) is bounded by $-Kt^{-2}$ from below. Observe that there is a $C_3 > 0$ such that

$$\left|\frac{d(\mathcal{E} - t\psi)}{dt}\right| \leq \frac{C_3}{t}$$

where we have used (1.2)–(1.3) and the fact that $t^2 H$ is bounded.

Assume there is an $\eta > 0$ and a time sequence $t_k \to \infty$ such that

$$(\mathcal{E} - t\psi)(t_k) \geq \eta\,.$$

By the above, if $t \geq t_k$,

$$|(\mathcal{E} - t\psi)(t) - (\mathcal{E} - t\psi)(t_k)| \leq C_3 \ln \frac{t}{t_k}\,.$$

Choose t_k' such that

$$C_3 \ln \frac{t_k'}{t_k} = \frac{\eta}{2}\,.$$

Then

$$(\mathcal{E} - t\psi)(t) \geq \frac{\eta}{2} \quad \text{in } [t_k, t_k']$$

and

$$\int_{t_k}^{t_k'} \frac{1}{t}(\mathcal{E} - t\psi)dt \geq \frac{\eta}{2} \ln \frac{t_k'}{t_k} = \frac{\eta^2}{4C_3}\,.$$

Since the negative contributions to the integral are negligible, we get the conclusion that $\mathcal{E} \to \infty$, a contradiction. Thus $\mathcal{E} - t\psi \to 0$ as $t \to \infty$. Since $\mathcal{E} - t^2 H \to 0$ as $t \to \infty$, the lemma follows. □

LEMMA 9.3 *Consider a solution to* (1.2) *and* (1.3). *If* $A^2 + 4BC \geq 0$ *and* $t^2 H$ *is bounded, then the solution is independent of* θ.

PROOF: Due to Lemma 8.2 and the fact that H is invariant under the isometries, we can assume that $B = 0$. Under the assumptions of the lemma, (8.2) implies

$$e^{\langle P \rangle} \langle Q_t \rangle = O(t^{-2})$$

whence Lemma 9.2 yields

$$\lim_{t \to \infty} \left[t^2 H - \pi t^2 \langle P_t \rangle^2\right] = 0\,.$$

By (8.1), we get the conclusion that $t\langle P_t \rangle \to -\alpha/2$, whence $t^2 H \to \pi\alpha^2/4$. Since the derivative of $t^2 H$ is nonnegative, we get the conclusion that

$$t^2 H \leq \frac{\pi\alpha^2}{4}\,.$$

Using (8.1) again, we obtain

$$\frac{t^2}{2}\int_{\mathbb{S}^1} P_t^2\,d\theta \ge \pi t^2\langle P_t\rangle^2 \ge \frac{\pi\alpha^2}{4} - \frac{\alpha}{2}\int_{\mathbb{S}^1} te^{2P}(Q-\langle Q\rangle)Q_t\,d\theta$$
$$\ge \frac{\pi\alpha^2}{4} - Kt\int_{\mathbb{S}^1} e^{2P}\big(Q_\theta^2+Q_t^2\big)d\theta\,.$$

Thus, if t is large enough that $Kt \le t^2/4$, we can conclude that

$$\frac{\pi\alpha^2}{4} \le \frac{t^2}{2}\int_{\mathbb{S}^1} P_t^2\,d\theta + Kt\int_{\mathbb{S}^1} e^{2P}\big(Q_\theta^2+Q_t^2\big)d\theta$$
$$\le \frac{t^2}{2}\int_{\mathbb{S}^1} P_t^2\,d\theta + \frac{t^2}{4}\int_{\mathbb{S}^1} e^{2P}\big(Q_\theta^2+Q_t^2\big)d\theta \le t^2H \le \frac{\pi\alpha^2}{4}\,.$$

Thus

$$\int_{\mathbb{S}^1}\big[P_\theta^2 + e^{2P}(Q_t^2+Q_\theta^2)\big]d\theta = 0\,,$$

whence the solution is independent of θ and Q is constant. □

LEMMA 9.4 *Consider a solution to* (1.2) *and* (1.3). *If* $A^2+4BC<0$, *then* t^2H *is unbounded.*

PROOF: Let us assume that $t^2H \le K < \infty$. We know that $\langle P\rangle$ is bounded in this case, so (8.3) implies

$$t\langle Q_t\rangle = -\beta\left[\left(\langle Q\rangle - \frac{\alpha}{2\beta}\right)^2 - \frac{\alpha^2+4\beta\gamma}{4\beta^2}\right] + O(t^{-1})\,.$$

Thus $A^2+4BC<0$ is not possible. □

Adding up the results of this section, we obtain Theorem 1.6.

10 The Sup Norm of the Derivatives

Observe that the arguments carried out so far yield no control over the behavior of the derivatives of P and Q in the sup norm. In this section we will try to remedy this.

PROOF OF PROPOSITION 1.8: The approach is the same as in Theorem 3.1. Let F_1, F_2, E_1, E_2, and E be as in the proof of that theorem, where the metric $\bar{g}$ used to define $\mathcal{A}$ and $\mathcal{B}$ is given by g_1 defined in (1.7). Observe that $F_1, F_2 \ge 0$ and that

$$s\big[P_t^2+P_\theta^2+e^{2P}(Q_t^2+Q_\theta^2)\big](s,\theta) \le E(s)\,.$$

Due to (2.5), we have

$$(10.1)\qquad 2\frac{\partial F_2}{\partial s}(s,\theta)=\big[-P_t^2-e^{2P}Q_t^2+P_\theta^2+e^{2P}Q_\theta^2\big](s,\theta+s)\,.$$

Consider, for $s\geq s_0\geq 1$,

$$\begin{aligned}
&\int_{s_0}^s\big[-P_t^2(u,\theta+u)+P_\theta^2(u,\theta+u)\big]du\\
&\quad=\int_{s_0}^s[-P_t(u,\theta+u)+P_\theta(u,\theta+u)][P_t(u,\theta+u)+P_\theta(u,\theta+u)]du\\
&\quad=\int_{s_0}^s[-P_t(u,\theta+u)+P_\theta(u,\theta+u)]\\
&\qquad\qquad\cdot[\partial_u(P(u,\theta+u)-\langle P\rangle(u))+\langle P_t\rangle(u)]du\\
&\quad=\int_{s_0}^s\langle P_t\rangle(u)[-P_t(u,\theta+u)+P_\theta(u,\theta+u)]du\\
&\qquad+\big[\{-P_t(u,\theta+u)+P_\theta(u,\theta+u)\}\{P(u,\theta+u)-\langle P\rangle(u)\}\big]_{s_0}^s\\
&\qquad-\int_{s_0}^s[-P_{tt}(u,\theta+u)+P_{\theta\theta}(u,\theta+u)][P(u,\theta+u)-\langle P\rangle(u)]du\,.
\end{aligned}$$

Since $|\langle P_t\rangle|\leq K/t$, we have

$$\begin{aligned}
\left|\int_{s_0}^s\langle P_t\rangle(u)[-P_t(u,\theta+u)+P_\theta(u,\theta+u)]du\right|&\leq K\int_{s_0}^s u^{-3/2}E^{1/2}(u)du\\
&\leq K\int_{s_0}^s u^{-3/2}[1+E(u)]du
\end{aligned}$$

and

$$\begin{aligned}
&\big|[\{-P_t(u,\theta+u)+P_\theta(u,\theta+u)\}\{P(u,\theta+u)-\langle P\rangle(u)\}]_{s_0}^s\big|\\
&\qquad\qquad\leq K(s_0)+\frac{K}{s}E^{1/2}(s)\leq K(s_0)+\frac{K}{s}[1+E(s)]\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{s_0}^s[-P_{tt}(u,\theta+u)+P_{\theta\theta}(u,\theta+u)][P(u,\theta+u)-\langle P\rangle(u)]du\\
&\quad=\int_{s_0}^s\left[\frac{1}{u}P_t(u,\theta+u)-\exp[2P(u,\theta+u)]\big(Q_t^2(u,\theta+u)-Q_\theta^2(u,\theta+u)\big)\right]\\
&\qquad\qquad\cdot[P(u,\theta+u)-\langle P\rangle(u)]du\,.
\end{aligned}$$

However,

$$\left|\int_{s_0}^{s}\exp[2P(u,\theta+u)]\big[Q_t^2(u,\theta+u)-Q_\theta^2(u,\theta+u)\big][P(u,\theta+u)-\langle P\rangle(u)]du\right|\le K\int_{s_0}^{s}\frac{1}{u^{3/2}}E(u)du$$

and

$$\left|\int_{s_0}^{s}\frac{1}{u}P_t(u,\theta+u)[P(u,\theta+u)-\langle P\rangle(u)]du\right|\le\int_{s_0}^{s}\frac{K}{u^2}[1+E(u)]du\,.$$

Adding up we get

$$\left|\int_{s_0}^{s}\big[-P_t^2(u,\theta+u)+P_\theta^2(u,\theta+u)\big]du\right|\le K(s_0)+\frac{K}{s}E(s)+K\int_{s_0}^{s}u^{-3/2}E(u)du\,. \tag{10.2}$$

Consider, for $s\ge s_0\ge 1$,

$$\begin{aligned}&\int_{s_0}^{s}\exp[2P(u,\theta+u)]\big\{Q_\theta^2(u,\theta+u)-Q_t^2(u,\theta+u)\big\}du\\&=\int_{s_0}^{s}\{\exp[2P(u,\theta+u)]-\exp[2\langle P\rangle(u)]\}\big\{Q_\theta^2(u,\theta+u)-Q_t^2(u,\theta+u)\big\}du\\&\quad+\int_{s_0}^{s}e^{2\langle P\rangle(u)}[Q_\theta(u,\theta+u)-Q_t(u,\theta+u)]\\&\qquad\cdot\{\partial_u[Q(u,\theta+u)-\langle Q\rangle(u)]+\langle Q_t\rangle(u)\}du\\&=I_1+I_2=I_1+(I_{21}+I_{22})\,,\end{aligned}$$

where I_1 is the first integral and I_2 is the second integral. I_{21} contains the part of I_2 due to the term involving the ∂_u-derivative. Similarly to the above, we then have

$$|I_1|\le K\int_{s_0}^{s}u^{-3/2}E(u)du\quad\text{and}\quad|I_{22}|\le K\int_{s_0}^{s}u^{-3/2}[1+E(u)]du\,.$$

The natural way to estimate I_{21} is to first carry out a partial integration

$$\begin{aligned}I_{21}&=\Big[e^{2\langle P\rangle(u)}\{Q_\theta(u,\theta+u)-Q_t(u,\theta+u)\}\{Q(u,\theta+u)-\langle Q\rangle(u)\}\Big]_{s_0}^{s}\\&\quad-\int_{s_0}^{s}2\langle P_t\rangle(u)e^{2\langle P\rangle(u)}[Q_\theta(u,\theta+u)-Q_t(u,\theta+u)]\\&\qquad\cdot[Q(u,\theta+u)-\langle Q\rangle(u)]du\\&\quad-\int_{s_0}^{s}e^{2\langle P\rangle(u)}[Q_{\theta\theta}(u,\theta+u)-Q_{tt}(u,\theta+u)][Q(u,\theta+u)-\langle Q\rangle(u)]du\,.\end{aligned}$$

Call the three terms on the right-hand side I_{211}, I_{212}, and I_{213}, respectively. We have

$$|I_{211}|\le K(s_0)+\frac{K}{s}E(s)\,,\qquad|I_{212}|\le K\int_{s_0}^{s}\frac{1}{u^2}[1+E(u)]du$$

and, using (1.3),

$$|I_{213}| \leq K \int_{s_0}^{s} \frac{1}{u^{3/2}} [1 + E(u)] du .$$

Adding up, we have

$$\left| \int_{s_0}^{s} \exp[2P(u, \theta + u)] \big\{ Q_\theta^2(u, \theta + u) - Q_t^2(u, \theta + u) \big\} du \right| \leq K(s_0) + \frac{K}{s} E(s) + K \int_{s_0}^{s} \frac{1}{u^{3/2}} E(u) du . \tag{10.3}$$

Combining (10.1), (10.2), and (10.3), we get the conclusion that

$$F_2(s, \theta) \leq F_2(s_0, \theta) + K(s_0) + \frac{K}{s} E(s) + K \int_{s_0}^{s} \frac{1}{u^{3/2}} E(u) du ,$$

whence

$$E_2(s) \leq E_2(s_0) + K(s_0) + \frac{K}{s} E(s) + K \int_{s_0}^{s} \frac{1}{u^{3/2}} E(u) du .$$

The argument for F_1 is similar, and we conclude that

$$E(s) \leq K(s_0) + \frac{K}{s} E(s) + K \int_{s_0}^{s} \frac{1}{u^{3/2}} E(u) du ,$$

whence

$$E(s) \leq K(s_0) + K \int_{s_0}^{s} \frac{1}{u^{3/2}} E(u) du .$$

Grönwall's lemma yields the conclusion of the proposition. □

11 Causal Geodesic Completeness

We are now in a position to prove Theorem 1.9.

PROOF: Introduce the global orthonormal frame

$$e_0 = t^{1/4} e^{-\lambda/4} \partial_t , \qquad e_1 = t^{1/4} e^{-\lambda/4} \partial_\theta ,$$
$$e_2 = t^{-1/2} e^{-P/2} \partial_\sigma , \qquad e_3 = t^{-1/2} e^{P/2} (-Q \partial_\sigma + \partial_\delta) .$$

Let

$$f_0 = -\langle \gamma', e_0 \rangle \quad \text{and} \quad f_k = \langle \gamma', e_k \rangle .$$

Since the curve is a causal geodesic,

$$-f_0^2 + \sum f_k^2 = c \quad \text{where } c \leq 0 \text{ is a constant.} \tag{11.1}$$

Let the time component of γ be denoted γ_0. We have

$$\frac{d\gamma_0}{ds} = -t^{1/2} e^{-\lambda/2} \langle \gamma', \partial_t \rangle = t^{1/4} e^{-\lambda/4} f_0 .$$

Note that $\gamma_0(s) \to \infty$ as $s \to s_+$. The reason is the following: Let $s_0 \in (s_-, s_+)$. Since $d\gamma_0/ds > 0$, the assumption that γ_0 is bounded from above leads to the conclusion that γ_0 converges to a finite value. Furthermore, the curve is contained in a compact set for $s \in [s_0, s_+)$. Due to the causality of the curve, one can conclude that $\gamma(s)$ converges as $s \to s_+$. This implies that it is extendible as a continuous curve and thus as a geodesic.

Let $k = 2, 3$ and consider

$$\begin{aligned} \frac{df_k}{ds} = \langle \gamma', \nabla_{\gamma'} e_k \rangle = \sum_{l=2}^{3} f_0 f_l [\langle e_0, \nabla_{e_l} e_k \rangle + \langle e_l, \nabla_{e_0} e_k \rangle] \\ + \sum_{l=2}^{3} f_1 f_l [\langle e_1, \nabla_{e_l} e_k \rangle + \langle e_l, \nabla_{e_1} e_k \rangle] . \end{aligned} \tag{11.2}$$

Let $\phi = t^{1/4} e^{-\lambda/4}$. Then one can estimate, using Proposition 1.8,

$$|\langle e_\mu, \nabla_{e_\nu} e_\kappa \rangle| \le K t^{-1/2} \phi$$

if two of μ, ν, and κ belong to $\{2, 3\}$. Combining this with (11.2) and the fact that the curve is causal, one concludes that

$$\frac{d}{ds}\big[1 + f_2^2 + f_3^2\big] \le K t^{-1/2} \phi f_0 \big[1 + f_2^2 + f_3^2\big] ,$$

where the 1 has been included for convenience. Since $\phi f_0 = dt/ds$, one concludes that

$$1 + f_2^2 + f_3^2 \le K \exp(K t^{1/2}) . \tag{11.3}$$

Compute

$$\frac{df_0}{ds} = -\frac{1}{4}\lambda_\theta \phi f_1 f_0 + \frac{1}{4}\big(t^{-1} - \lambda_t\big)\phi f_1^2 - \sum_{k,l=2}^{3} f_k f_l \langle e_k, \nabla_{e_l} e_0 \rangle . \tag{11.4}$$

Observe that λ_t is bounded due to Proposition 1.8. By (11.1) we thus get

$$\begin{aligned} \frac{1}{4}\big(t^{-1} - \lambda_t\big) f_1^2 &= \frac{1}{4}\big(t^{-1} - \lambda_t\big) f_0^2 + \frac{1}{4}\big(t^{-1} - \lambda_t\big)\left(\sum_{k=1}^{3} f_k^2 - f_0^2\right) \\ &\quad - \frac{1}{4}\big(t^{-1} - \lambda_t\big)\big(f_2^2 + f_3^2\big) \\ &\le \frac{1}{4}\big(t^{-1} - \lambda_t\big) f_0^2 + K\big[1 + f_2^2 + f_3^2\big] . \end{aligned}$$

Combining this with (11.4), we get the conclusion

$$\frac{df_0}{ds} \le \left[-\frac{1}{4}\lambda_\theta f_1 + \frac{1}{4}(t^{-1} - \lambda_t) f_0\right]\phi f_0 + K\phi\big[1 + f_2^2 + f_3^2\big] .$$

Note that

$$\frac{d}{ds}\ln\phi = \frac{d}{ds}\left(\frac{1}{4}\ln t - \frac{1}{4}\lambda\right) = -\frac{1}{4}\lambda_\theta \phi f_1 + \frac{1}{4}(t^{-1} - \lambda_t)\phi f_0 .$$

Thus

$$\frac{1}{f_0}\frac{df_0}{ds} \le \frac{d}{ds}\ln\phi + K\phi\frac{1+f_2^2+f_3^2}{f_0}. \tag{11.5}$$

Let us prove that ϕf_0 is bounded to the future. Assume the contrary. Then there is a $k_0 \in \mathbb{N}$ and for each $k \ge k_0$ an interval $[s_{1,k}, s_{2,k}]$ such that

$$k_0 \le \phi(s) f_0(s) \le k$$

in the interval, the lower endpoint being achieved for $s = s_{1,k}$ and the higher for $s_{2,k}$. Due to Theorem 1.7, the fact that the solution is spatially inhomogeneous implies that $\lambda(t) \ge 4at$ for t big enough, where $a > 0$. We have, using this observation together with (11.3),

$$\begin{aligned}\int_{s_{1,k}}^{s_{2,k}} \phi\frac{1+f_2^2+f_3^2}{f_0}\,ds &\le \frac{1}{k_0^2}\int_{s_{1,k}}^{s_{2,k}} \phi^2\big[1+f_2^2+f_3^2\big]\phi f_0\,ds \\ &\le \frac{K}{k_0^2}\int_{t_{1,k}}^{t_{2,k}} e^{-at}\,dt \le K,\end{aligned}$$

where K is independent of k. Let us use the notation $\lambda[\gamma(s_{i,k})] = \lambda_{i,k}$ and $\gamma_0(s_{i,k}) = t_{i,k}$ for $i = 1, 2$. Observe that $t_{1,k}$ is bounded from below by a positive constant, that $t_{2,k}$ tends to infinity, that $|\lambda_{i,k} - \langle\lambda\rangle(t_{i,k})| \le K$, and that

$$\langle\lambda\rangle(t_{2,k}) - \langle\lambda\rangle(t_{1,k}) \ge a(t_{2,k} - t_{1,k})$$

for k great enough, where $a > 0$. We conclude that $\phi_{2,k}/\phi_{1,k}$ is bounded by a constant independent of k (note that if $t_{2,k} - t_{1,k}$ tends to infinity, the quotient tends to 0). By (11.5), we conclude that $f_0(s_{2,k})/f_0(s_{1,k})$ is bounded by a constant independent of k. We have a contradiction. Thus ϕf_0 is bounded. Since $\phi f_0 = d\gamma_0/ds$ and $\gamma_0(s) \to \infty$ as $s \to s_+$, we conclude that $s_+ = \infty$; that is, the geodesic is future complete. □

12 Discussion

There are several questions concerning the problem presented in this paper that have not been answered. If one is interested in considering curvature, one would, for instance, be interested in the behavior of higher-order derivatives. Furthermore, one would like to find some geometric condition that makes the argument concerning the decay of the energy work. For example, if the target space is complex hyperbolic space, the method presented in this paper does not immediately apply, but it can be modified to fit that setting. In terms of physics, this target corresponds to Einstein's equations coupled to Maxwell's equations under the same symmetry assumptions. Furthermore, the method of obtaining the equations for the mean values is rather haphazard, which is of course not satisfactory. If the target space is a higher-dimensional hyperbolic space, one can derive equations of the same form as those written down in Lemmas 8.7 and 8.8, but again it is difficult to see a more general pattern.

Acknowledgments. The author is grateful to Vincent Moncrief and Beverly Berger for describing their work on the expanding direction of Gowdy prior to publication. Their insight was the starting point of this paper. The formulation of the results would have been less appealing without the encouragement and support of Alan Rendall. Part of this work was carried out while the author was attending the Workshop on General Relativity at the mathematics department of Stanford University and AIM. The author would like to express his gratitude to AIM for the support obtained during this visit.

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HANS RINGSTRÖM
Max Planck Institut für Gravitationsphysik
Albert Einstein Institut
Am Mühlenberg 1
D-14476 Golm
GERMANY
E-mail: hansr@aei-potsdam.mpg.de

Received June 2003.