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The Ramond-Ramond sector of string theory beyond leading order

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Abstract

Present knowledge of higher-derivative terms in string effective actions is, with a few exceptions, restricted to the NS-NS sector, a situation which prevents the development of a variety of interesting applications for which the RR terms are relevant. We here provide the formalism as well as efficient techniques to determine the latter directly from string-amplitude calculations. As an illustration of these methods, we compute the dependence of the type-IIB action on the three- and five-form RR field strengths at four-point, genus-one, order-$(\alpha')^3$ level. We explicitly verify that our results are in accord with the SL(2,$\mathbb{Z}$) S-duality invariance of type-IIB string theory. Extensions of our method to other bosonic terms in the type-II effective actions are discussed as well.

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1 Introduction

A considerable amount of information about string and M-theory can be extracted from the low-energy effective field theory actions, in particular once one includes corrections that go beyond the leading supergravity terms. A clear illustration of this fact can be found in the large body of literature in which the effects of terms of higher order in the Riemann tensor have been studied. This particular set of corrections already provides an important testing ground for candidate microscopic versions of M-theory [1]. Upon reduction to four dimensions, such higher-order terms influence the couplings of the scalar fields [2, 3]. They have also been argued to lead to induced Einstein-Hilbert terms, which are important for gravity localisation phenomena [4, 5] as they arise in brane-world scenarios. Furthermore, they may play a role in supersymmetry breaking [6, 7] by producing a potential that stabilises certain moduli for non-supersymmetric vacua. In black-hole and black-brane physics, higher-derivative terms lead to modifications of the thermodynamics, and thus implicitly to new tests of the string-theory description of the entropy of such objects [8, 9]. Another important set of applications can be found in the context of the AdS/CFT correspondence, where the effects of higher-derivative terms in the supergravity action map to $1/N$ effects on the Yang-Mills side [10, 11, 12]. This list is far from exhaustive, but it should suffice to illustrate the importance of having a good understanding of higher-derivative terms.

While higher-derivative terms are thus of considerable interest, they are very hard to obtain explicitly and this has been a serious obstacle to further development of the applications just mentioned. One particularly limiting factor is the fact that while pure graviton corrections are relatively easy to compute, this is not the case for terms involving any of the other supergravity fields. While the dependence on fermionic fields is perhaps not particularly relevant except for supersymmetry considerations, there is a clear need to determine the way in which the other bosonic fields appear in the action. With [13] we have initiated a programme to determine these terms using a combination of techniques from string theory and supersymmetry. We refer the reader to the introduction and discussion of that paper, as well as the review [14], for an overview of the existing knowledge of higher-derivative terms in string and M-theory effective actions and the methods used to obtain them.

The present paper is part of this long-term programme and aims at the construction of certain terms that contain not just the graviton but also gauge fields present in the supergravity multiplet.[1] Since we restrict ourselves to bosonic action terms, a supersymmetry approach as used in [14] would be very cumbersome, as the terms involving fermions greatly outnumber the purely bosonic ones. Because of the presence of spin-fields, a sigma-model β-function calculation does not appear to be feasible either.[2] For the ten-dimensional theories, it is more straightforward to determine the effective action directly from string-amplitude calculations, which is what we will do in the present paper. Our main aim is to present the formalism and machinery behind the calculations that lead to a determination of the terms involving gravitons and Ramond-Ramond gauge fields (a topic which, rather surprisingly, has remained practically untouched in the literature so far). In a certain sense our calculations thus complete the work of Gross and Sloan [16], which only deals with the Neveu-Schwarz three-form field strength.

In more detail, our calculations will be concerned with type-IIB four-point functions at genus one, involving external three-form or five-form Ramond-Ramond states. The restriction to four-point functions has been made because many (though not all) conceptual and technical problems are already present here; we will comment on extensions to higher-point functions in the discussion. The restriction to genus one is motivated by the fact that at this order in the string coupling, amplitudes with fewer than four external particles vanish identically. Genus-one amplitudes thus only begin to contribute to the effective action at the level of the $(\alpha')^3 R^4$ term and the terms related to it by supersymmetry. Furthermore, genus-one calculations turn out to be technically simpler than their cousins at genus-zero, and for the type-IIB string they are in any case related by the SL(2,$\mathbb{Z}$) duality symmetry of the effective field theory.

The first issue that requires attention is the calculation of the world-sheet correlators. While closed-form expressions exist for correlators of both the bosonic [17] and the fermionic world-sheet fields [18] for genus one (and in fact for higher genera as well), the actual evaluation of our amplitudes is still rather involved. As in previous calculations involving four external particles, we eventually find that intricate cancellations due to Riemann identities cause the integral over the vertex-operator insertion points to reduce to an integral over a constant. This is a particular consequence of the low number of external particles and is not expected to hold once higher orders are considered.

Subsequently, for reasons that will be explained, one needs an efficient way to identify and subtract terms from the amplitude which vanish by the the Bianchi identities, and to organise the result in terms of the tensors that appear in the effective action. The spin fields in the world-sheet correlators, coming from the vertex operators of the Ramond-Ramond states,

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[1] We should add at this point that an additional reason for being interested in gauge-field terms is the role they play in modifying the structure of supersymmetry transformations. In the context of M-theory, we have argued in [13] that modifications to the superspace torsion constraint in eleven dimensions are only visible when the four-form gauge-field strength is taken into account. A similar conclusion is expected in the ten-dimensional theories. However, progress along these lines will also require the construction of gauge-field terms involving fermions, something we will not attempt in the present paper.

[2] While the approach of Berkovits and Howe [15] allows one to deduce target-space equations of motion in RR backgrounds by solving BRST-nilpotency and holomorphicity conditions, it seems rather complicated to extend their results to three-loop level, which would be required to see higher-derivative interactions in type-II theories.

imply that the amplitude is composed of traces over (long strings of) gamma matrices, whose complexity increases rapidly with the number of gamma matrices. We will show how the amplitude and the resulting expressions for the effective action can be organised and analysed efficiently by employing group-theoretical arguments. This is illustrated most convincingly by the compact form of the final result of the five-form effective action, given in equation (2.13) below.

As we have pointed out in the previous paper in this series [19], when doing this kind of amplitude calculations one frequently uncovers various ill-understood aspects of string perturbation theory. The present paper is no exception to this empirical rule, and we will again encounter various technical issues which have received very little attention in the literature so far. As such, our paper paves the way for the complete order-$(\alpha')^3$ analysis of the effective action, to be presented elsewhere, and more generally we hope that our explicit and systematic presentation of the calculation will be of practical use to others.

As announced, our main result is the determination of the $R^2(DF_{(5)})^2$ and $R^2(DF_{(3)})^2$ terms in the effective action, which are given in equations (2.13) and (2.27) respectively. Apart from the intrinsic importance of actually being able to calculate these and similar other terms completely and efficiently, we will use the last part of our paper to emphasise that there are several reasons why these terms are particularly interesting from a physical point of view.

Firstly, we will show how the three-form terms in the RR sector are directly related, by SL(2,$\mathbb{Z}$) symmetry, to similar terms involving the NS-NS three-form which were computed by Gross and Sloan [16]. A strong check on our calculations is the fact that our three-form action indeed precisely satisfies this duality requirement, despite the fact that it arises in a completely different way from the string calculations (the world-sheet correlators in the NS-NS and RR sectors are completely different). This provides a very interesting perturbative verification of the SL(2,$\mathbb{Z}$) symmetry of type-IIB string theory.

Secondly, we will comment on the relation of our calculation to the predictions of the linearised type-IIB superfield. It has recently been shown by Berkovits and Howe that an extension of this superfield construction to the non-linear level is problematic (a detailed argument can be found in de Haro et al. [20]). This implies that our string-based methods are at present the only way in which the effective action terms at higher order in the fields can be reliably determined. However, one still expects the superfield to be able to predict four-field terms in the action. We will indeed show how the superfield predictions at this level fit in precisely with our string theory results.

We will end this paper with an outline of the computation of the $R^3(F_{(5)})^2$ terms in the effective action, which are most interesting from the point of view of physical applications and whose computation will rely heavily on the results and methods we have derived here. We will also comment on similar computations involving space-time fermions.

2 Four-boson string amplitudes

2.1 World-sheet correlators and modular integrals

When computing string amplitudes, there are in general many different ways in which the momenta and polarisation tensors can be contracted in the final result. Our approach will be to classify these Lorentz-invariant contractions using a group-theoretical method; this is the key element that allows us to present the amplitude, and later on the resulting effective action, in a compact form. However, there are many Lorentz-invariant combinations which

Y	$V_F^{(-1/2,-1/2)}$	$V_F^{(-1/2,-1/2)}$	$V_\zeta^{(0,0)}$	$V_\zeta^{(0,0)}$
$\partial X\Psi$	S_{L}	S_{L}	$\partial X + k\Psi\Psi$	$\partial X + k\Psi\Psi$
	$\Gamma^{[5]}e^{ik_1\cdot X}$	$\Gamma^{[5]}e^{ik_2\cdot X}$	$e^{ik_3\cdot X}$	$e^{ik_4\cdot X}$
$\bar\partial X\tilde\Psi$	S_{R}	S_{R}	$\bar\partial X + k\tilde\Psi\tilde\Psi$	$\bar\partial X + k\tilde\Psi\tilde\Psi$

Table 1: Schematic form of the two-graviton, two-five-form string amplitude. We have indicated the ghost numbers of the vertex operators, and split the correlator in left- and right-moving parts. Y denotes the picture changing operator.

are simply incompatible with the exchange symmetries of the external states in the amplitude. Before doing the classification, we would like to eliminate such incompatible contractions.

Since we are interested in the $R^2(DF_{(5)})^2$ terms in the effective action, we will have to compute the string amplitude with two gravitons and two five-forms. This amplitude is manifestly invariant under the exchange of e.g. the two five-form vertex operators, which involves the exchange of the polarisation tensors $F^{(1)}$ and $F^{(2)}$, the momenta $k^{(1)}$ and $k^{(2)}$, as well as the insertion point variables z_1 and z_2. In order to deduce the symmetry properties under exchange of the polarisation tensors and external momenta only, we need to analyse the z-dependence of the string integrand. Understanding the z-dependence is also important because we eventually wish to compute the full amplitude, not just the tensorial structures, and we will therefore have to perform the final integral over all the insertion point variables. In the present section we determine this z-dependence of the combined world-sheet correlators.

The correlator for the two-graviton, two-five-form amplitude can be summarised using a notation we have used also in [13]; see table 1. This is based on the explicit form of the five-form vertex operator in the $(-\frac{1}{2},-\frac{1}{2})$ ghost picture,

$$V_{F_{(5)}}^{(-1/2,-1/2)}(k) = \frac{1}{5!}\int \mathrm{d}^2 z\, F_{mnpqr}\,(\bar S_{\mathrm{L}}\Gamma^{mnpqr}S_{\mathrm{R}})\, e^{-\phi/2-\tilde\phi/2}e^{ik\cdot X}\,, \tag{2.1}$$

as well as on the standard one for the graviton with $(0,0)$ ghost charges,

$$V_\zeta^{(0,0)}(k) = \int \mathrm{d}^2 z\, \zeta_{mn}(\partial X^m - ik\cdot\Psi\Psi^m)(\bar\partial X^n - ik\cdot\tilde\Psi\tilde\Psi^n)\, e^{ik\cdot X}\,. \tag{2.2}$$

(We will comment below on the fact that these vertex operators should, in principle, still be amended with world-sheet gravitino terms.) Using the fact that fermionic correlators of the form $\langle SS\,\Psi\rangle$ vanish identically, there are thus, a priori, two different types of terms in each chiral sector of the string amplitude: those with three RNS fermions Ψ and those with five. We will argue below that the terms involving $\langle SS\,\Psi\!:\!\Psi\Psi\!:\rangle$ correlators do not contribute at eight-derivative order in the amplitude, so that the full four-particle amplitude is given by the single term

$$\begin{aligned}\mathcal{A} = k^{(3)}_{r_1}k^{(3)}_{s_1}k^{(4)}_{r_3}k^{(4)}_{s_3}\zeta^{(3)}_{r_2s_2}\zeta^{(4)}_{r_4s_4}\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\mathrm{Im}\,\tau}\prod_{i=1}^{4}\int_{\mathcal{T}}\mathrm{d}^2 z_i \\ \lim_{\substack{v\to z_1\\ w\to z_2}}\Big[\mathcal{B}_{mn}\,\mathrm{Tr}\Big(\not{F}^{(1)}\tilde{\mathcal{F}}^{ns_1s_2s_3s_4}(\not{F}^{(2)})^{\mathrm{T}}(\mathcal{F}^{mr_1r_2r_3r_4})^{\mathrm{T}}\Big)\,\mathcal{G}\tilde{\mathcal{G}}\Big]\,.\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
\mathcal{B}^{mn} = \Big[\Big(\sum_{i=1}^{4} k_i^m \big[-i\partial_v \ln\theta_1(v-z_i) + \frac{2\pi}{\mathrm{Im}\,\tau} \mathrm{Im}\,(v-z_i) \big] \Big) \\
\times \Big(\sum_{j=1}^{4} k_j^n \big[-i\bar{\partial}_w \ln\bar{\theta}_1(\bar{w}-\bar{z}_j) - \frac{2\pi}{\mathrm{Im}\,\tau} \mathrm{Im}\,(w-z_j) \big] \Big) \\
- \frac{2\pi}{\mathrm{Im}\,\tau} \eta^{mn} \Big] \times \Big\langle \prod_{m=1}^{4} e^{ik_m \cdot X(z_m)} \Big\rangle ,
\end{aligned}
$$

Table 2: Explicit expression for the bosonic correlator $\langle \partial X \bar{\partial} X \prod_{i=1}^{4} e^{ikX} \rangle$. Despite the appearance of the sum, only one term survives when this correlator is combined with the fermionic one. Note that our way of taking care of the picture changer differs from the method in Atick and Sen [18] (who take the limit $v \to z_1$ before computing the bosonic correlator); the result is, however, equivalent.

The bosonic correlator that appears here is given by

$$
\mathcal{B}^{mn} = \big\langle \partial X^m(v) \bar{\partial} X^n(w) \prod_{i=1}^{4} \exp\left[ik_i \cdot X(z_i)\right] \big\rangle , \tag{2.4}
$$

while the fermionic and ghost correlators are

$$
\mathcal{F}^{mr_1r_2r_3r_4} = \big\langle \Psi^m(v)\, S_{\mathrm{L}}(z_1) S_{\mathrm{L}}(z_2) \ {:}\Psi^{r_1}\Psi^{r_2}{:}(z_3) \ {:}\Psi^{r_3}\Psi^{r_4}{:}(z_4) \big\rangle , \tag{2.5}
$$

$$
\mathcal{G} = \big\langle \exp\left[-\phi(v)\right] \exp\left[-\phi(z_1)/2\right] \exp\left[-\phi(z_2)/2\right] \big\rangle , \tag{2.6}
$$

with similar expressions for the right-moving sector. These can all be computed explicitly: techniques for both the bosonic and the fermionic correlators have been given by Atick and Sen [18]. Details can be found in tables 2 and 3.

Since the holomorphic and anti-holomorphic fermionic correlators come with a $(v - z_1)$ and a $(\bar{w} - \bar{z}_2)$ zero, respectively, the only relevant term in the bosonic correlator given above is the momentum-dependent one containing a $(v - z_1)^{-1}(\bar{w} - \bar{z}_2)^{-1}$ pole. The terms coming from the zero-mode contraction of the bosons in the two picture changers, for instance, do not contribute to the amplitude. Note also that it is, a priori, possible to obtain second-order momentum factors from the correlator of the exponentials. However, at genus one, this correlator of exponentials leads to an expansion $1 + (\alpha')^3 k^6 + \cdots$, i.e. the order k^2 and k^4 terms are absent (this can be deduced from the explicit calculation of the correlator of exponentials by Green and Vanhove [21]; see the discussion around equation (5.2) in that paper).

This balance between zeroes from the fermionic correlators and poles from the bosonic ones is also the reason why the amplitude does not receive any contributions from $\langle SS\,\Psi : \Psi\Psi : \rangle$ correlators: these can be shown to be proportional to a double zero in each chiral half, given by $(v - z_1)^2(\bar{w} - \bar{z}_2)^2$. The corresponding bosonic correlator does contain a second order pole, but only at order k^4 in the external momenta (the resulting term in the amplitude presumably talks to field-theory subtraction terms from the effective action at lower order). For our calculation these terms are, in any case, irrelevant.

Note that we are in the fortunate situation that our amplitude does not receive any contributions from the pinch singularities discussed by, e.g., Minahan [22] and Bern et al. [23].

structure	r_1	r_2	r_3	r_4	m	α_1	α_2	Γ	$\langle\cdots\rangle/K_\nu$	K_ν	coeff
$(\Gamma^{r_1 r_2 r_3 r_4 m})_{\alpha_1\alpha_2}$	1	3	5	7	8	$(---++)$	$(-----)$	$-i$	$-i(v-z_1)$	1	1
$(\Gamma^{r_1 r_2 r_3})_{\alpha_1\alpha_2}\,\delta^{r_4 m}$	1	3	5	7	7	$(---++)$	$(-----)$	$\frac{1}{2}$	$(v-z_1)$	1	2
$(\Gamma^{r_3 r_4 r_1})_{\alpha_1\alpha_2}\,\delta^{r_2 m}$	5	7	1	3	7	$(---++)$	$(-----)$	$\frac{1}{2}$	$(v-z_1)$	1	2
$(\Gamma^{r_2 r_4 m})_{\alpha_1\alpha_2}\,\delta^{r_1 r_3}$	1	3	1	5	7	$(+---+)$	$(-----)$		0		0
$(\Gamma^{r_1})_{\alpha_1\alpha_2}\,\delta^{r_2 r_3}\delta^{r_4 m}$	1	3	3	5	5	$(-++++)$	$(-----)$	$\frac{1}{4}$	$-2(v-z_1)$	-1	8
$(\Gamma^{r_3})_{\alpha_1\alpha_2}\,\delta^{r_4 r_1}\delta^{r_2 m}$	3	5	1	3	5	$(-++++)$	$(-----)$	$\frac{1}{4}$	$-2(v-z_1)$	-1	8
$(\Gamma^{m})_{\alpha_1\alpha_2}\,\delta^{r_1 r_3}\delta^{r_2 r_4}$	1	3	1	3	5	$(++-++)$	$(-----)$	1	$(v-z_1)$	2	2

Table 3: Decomposition into irreducible tensor structures of the fermionic correlator $\langle \Psi(v)\, S(z_1) S(z_2) :\!\Psi\Psi\!:(z_3)\ :\!\Psi\Psi\!:(z_4)\,\rangle$. These expressions should still be antisymmetrised, with unit weight, in r_1, r_2 and r_3, r_4 respectively. The seven structures correspond to the number of singlets in the $\mathbf{10}\otimes\mathbf{16}\otimes\mathbf{16}\otimes\mathbf{45}\otimes\mathbf{45}$ tensor product. We are using the picture changing choice $v\to z_1$. The values of the r-indices refer to real world-sheet fermions, which are related to the complex SO(2) fermions by $\Psi^1 = \psi+\bar\psi$ and $\Psi^2 = -i(\psi-\bar\psi)$ and similarly for the other eight. The values of the spinor indices are chosen such that a non-zero entry of the Γ matrix product is selected; this number is listed in the "Γ" column (the explicit expressions for these matrices in the particular basis that we are using can be found in appendix A.2 of [19]). The last three columns give the result of the "unnormalised" Atick&Sen expressions presented in appendix A.2, the normalisation factor and the final result for the coefficient of each tensorial structure in the amplitude (obtained when the pole of the bosonic correlator is taken into account).

That is, our amplitude does not contain integrals of the form $\int \mathrm{d}^2\nu_i\, k_i\cdot k_j |\nu_i-\nu_j|^{-2-k_i\cdot k_j} = 1$. This is simply because our bosonic poles do not survive but instead get cancelled by the zeroes from the fermionic correlator.

We should at this stage also comment on the presence of terms in the vertex operators which involve the world-sheet gravitinos. As we have pointed out in [19], these terms sometimes result in additional contributions to the amplitude. In the present situation, however, world-sheet gravitino terms are harmless. This is essentially because the replacement of a picture changer with a term involving a world-sheet gravitino results in a bosonic correlator in which both a ∂X and a $\bar\partial X$ factor have been removed (this can be deduced simply from the BRST transformation rules as spelled out in equation (2.7) of [19]). In the present situation this means that the bosonic correlator reduces to the correlator of plane waves, while the fermionic correlator remains unchanged. The zero of the latter is then no longer balanced by a pole from the bosonic correlator, and as a result the contribution of these world-sheet gravitino terms to the amplitude vanishes.[3]

Just as for the four-graviton amplitude, we observe that after the use of the Riemann identity, and after the picture-changing limits have been taken, the integral over the insertion points and the modular parameter becomes trivial, as the integrand reduces to a constant.[4]

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[3] This point can be summarised by the general rule of thumb that the world-sheet gravitino terms in vertex operators are *only* relevant for amplitudes in which, before the replacement of picture changers with world-sheet gravitinos, there are non-trivial contributions arising from the zero-mode part of a $\langle\partial X\bar\partial X\rangle$ contraction.

[4] The Riemann identity (A.25) can be used by virtue of the fact that one of the $\theta_\nu(\frac{z_1-z_2}{2})$ factors of the

This implies that symmetry properties of the amplitude under exchange of vertex operators reduce simply to symmetry properties under exchange of momenta and polarisation tensors (the exchange of the insertion points becomes a trivial operation). As we will see in the next section, this fact significantly reduces the number of Lorentz-invariant structures that need to be taken into account. With the basis of Lorentz invariants that we will construct there, we can then go back to the string amplitude and compute the final step, namely the trace of the gamma matrices in eq. (2.3). This calculation will be tackled in section 2.3.

2.2 Group-theoretical classification of terms in the effective action

Having determined the world-sheet correlators which form the building blocks of the string amplitude, we are now in a position to determine a general basis of four-field terms, compatible with that string amplitude, in which we will express the effective action. For certain applications it may be useful to have access to the explicit index contractions of the two Weyl tensors and two five-form fields (together with their derivatives). However, here we will present our results using a more compact notation, based on a group-theoretical classification of the various factors that make up a term. More details of such group-theoretical decompositions can be found in Fulling et al. [24].

There are several reasons for adopting this approach. One is that the it allows for a nice and compact way of expressing the effective action which is completely general and does not rely on any a priori knowledge about the origin of the various terms.[5] At the practical level, this way of organising the result has also provided very strong checks on signs and factors in the calculation. Finally, and most importantly, it allows for a systematic identification of the terms proportional to Bianchi identities. This is crucial when analysing the amplitudes directly in terms of the RR field strengths that appear in the vertex operators, since the physical state conditions for these fields correspond to the requirements $*d{*}F^{+}_{(5)} = 0 = dF^{+}_{(5)}$ (with analogous expressions for the RR three-form). In coordinate notation, the former condition just translates to $D^k F^{+}_{km_2m_3m_4m_5} = 0$ and is therefore trivial to handle. Imposing the latter, however, requires that the fully antisymmetrised tensors $D_{[m}F^{+}_{m_1m_2m_3m_4m_5]}$ be identifiable in general $W^2(DF^{+}_{(5)})^2$ contractions, something which the group-theoretical decompositions allow us to achieve.

For the $W^2(DF^{+}_{(5)})^2$ terms in the action, it is useful to first group together terms according to the rank of the W^2 and $(DF^{+}_{(5)})^2$ blocks, i.e. by the number of indices which contract between the two blocks. These quadratic blocks can then be decomposed further into distinct

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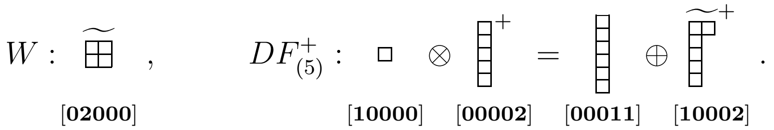
fermionic correlator cancels against a part of the ghost correlator,

$$\mathcal{G}(v,z_1,z_2) = \frac{\theta_1\big(v-z_1\big)^{\frac{1}{2}}\theta_1\big(v-z_2\big)^{\frac{1}{2}}}{\theta_1\big(z_2-z_1\big)^{\frac{1}{4}}\theta_\nu\big(v-\frac{z_1+z_2}{2}\big)} \tag{2.7}$$

after the limit $v \to z_1$ has been taken.

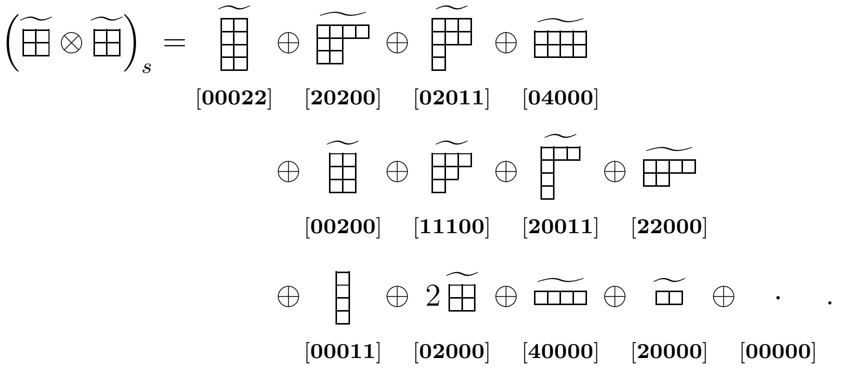
[5] For the four-graviton terms, a compact notation is achieved by using the t_8 and ϵ_{10} symbols, by means of which the type-IIB four-graviton interaction term can be written as $(t_8t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10})W^4$. However, these tensors do not suffice for the terms under consideration in the present paper. (See [13] for another set of terms, namely fermi bilinears, where more complicated tensor structures arise.)

irreducible representations. The two basic tensors sit in the representations



(2.8)

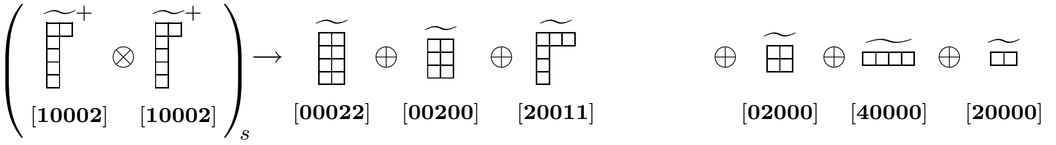
The symmetric product of two Weyl tensors yields the following decomposition in terms of Young tableaux:



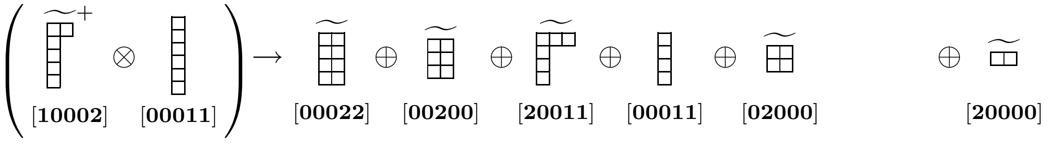
(2.9)

However, not all of these representations are compatible with the symmetries of the amplitude. For instance, the rank-eight contractions are fully antisymmetric in the four indices of the "*r*" and "*s*" sets respectively, as can be deduced from the form of the fermionic correlator given in table 3. As a consequence, of the four a priori possible rank-eight tableaux, only the [00022] can appear in the amplitude. Similar arguments can be used to discard the [11100] and [22000], leaving [00200] and [20011] as the only rank-six tableaux. All tableaux of rank four and lower remain, however.

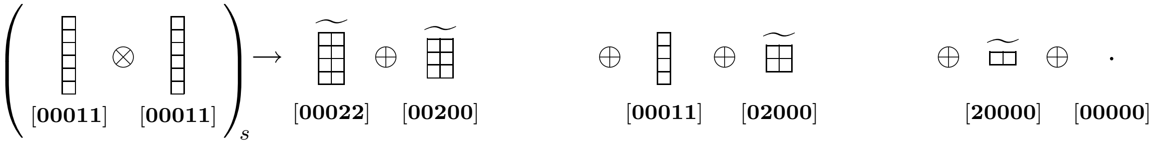
The product of the two $DF^+_{(5)}$ tensors can be decomposed similarly. Keeping only the tableaux that are compatible with the symmetries of the string amplitude, one finds



(2.10a)

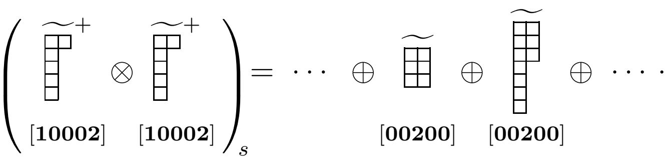


(2.10b)



(2.10c)

When deriving these decompositions, one has to pay attention to the fact that representations sometimes can be realised in terms of two different Young tableaux. This happens because of the presence of the invariant epsilon tensor. The main example that appears in the decomposition above is



(2.11)

Only the first tableau is compatible with the structure of the string amplitude, which explains why the [00200] only occurs with unit multiplicity on the first line of (2.10).

The invariant action should now be expressible in terms of a basis of scalars constructed from tensor products of the W^2 and $(DF^+_{(5)})^2$ representations. Inspecting (2.9) and (2.10), this gives a total of twenty-one terms, of which seven remain when the Bianchi identities are imposed. A list of these seven on-shell invariants, i.e. the tensor contractions corresponding to the Young tableaux, is constructed in appendix A.1 (in our calculations we have, however, kept track of all twenty-one invariants as a check on signs and factors, and to have an efficient way to eliminate terms proportional to Bianchi identities). In order to write down the full on-shell superinvariant with two five-form tensors that appears in string theory, we then only have to give the numerical values of the *seven* coefficients of these building blocks. These coefficients are computed in the following section, and the final result can be found in equation (2.13).

2.3 Amplitude results and the effective action

All ingredients that are necessary to extract an effective action from the string amplitude (2.3) are now available. Inserting the correlators computed in section 2.1, doing the (trivial) integral over the modular parameter and performing the spinorial traces, we end up with a result which is to be reproduced by a yet-to-be-determined effective action. The latter can be expressed in the group theory basis discussed in the previous section.

In general, constructing an effective action from a set of string amplitudes is a non-trivial procedure. One starts at the lowest order in the external fields at which string theory produces a non-zero amplitude, and constructs a corresponding term in the effective action which reproduces this result. Iterating this procedure with more and more external states in the string amplitude, one generically finds that the corresponding field theory amplitude receives contributions from all lower order terms in the action. This so-called field-theory subtraction problem is, fortunately, absent for our four-point calculation. This is because of the fact that there are no non-vanishing three-point amplitudes at genus one. The entire four-point amplitude must therefore be generated by a new four-field term in the effective action. A related problem concerns the contribution to the field theory amplitude of vertices in the effective action which appear at lower order in the derivatives. Again we find that, due to the low number of external states, this issue does not cause any problems in the present situation.

All this means that we can transcribe the amplitude in a relatively straightforward way to an effective action. Just before doing the traces, this leads to

$$\begin{aligned}\mathcal{L} &= W_{r_1r_2s_1s_2}W_{r_3r_4s_3s_4}\\ &\quad\times \mathrm{Tr}\,\Big\{D_m\not{F}^+\Big(\Gamma^{r_1\ldots r_4m}+4\,\Gamma^{[r_1r_2r_3}\eta^{r_4]m}+8\,\Gamma^{r_1}\eta^{r_2r_3}\eta^{r_4m}+8\,\Gamma^{r_3}\eta^{r_4r_1}\eta^{r_2m}+2\,\Gamma^m\eta^{r_1r_3}\eta^{r_2r_4}\Big)\\ &\qquad\quad\times D_n\not{F}^+\Big(\Gamma^{s_1\ldots s_4n}+4\,\Gamma^{[s_1s_2s_3}\eta^{s_4]n}+8\,\Gamma^{s_1}\eta^{s_2s_3}\eta^{s_4n}+8\,\Gamma^{s_3}\eta^{s_4s_1}\eta^{s_2n}+2\,\Gamma^n\eta^{s_1s_3}\eta^{s_2s_4}\Big)\Big\}\end{aligned} \tag{2.12}$$

Here we have used the fact that the two $\Gamma^{[3]}$ terms on the second and third line of Table 2 combine to form a tensor fully antisymmetric in $r_1,\ldots,r_4$ (and similarly in $s_1,\ldots,s_4$ for the right-moving sector). Observe that all terms come with a plus sign, despite the appearance of the gamma-matrix transpose in (2.3) for the anti-holomorphic sector. Rather than explaining from first principles how these signs arise, we will instead argue for their correctness in section 2.4 by observing that this is the only combination which leads to a result consistent with the type-IIB SL(2,$\mathbb{Z}$) duality symmetry. Alternatively, observe that these signs are the only ones leading to an amplitude which is symmetric under exchange of the external five-form states.

Before we proceed to rewrite this trace in a more usable form, we should make a few comments. Firstly, one may wonder how self-duality of the five-form polarisation tensors—a direct consequence of the Weyl-spinor contraction of $\Gamma^{[5]}$ in the vertex operator (2.1)—is to be dealt with in the actual calculations. Our approach has been to work with manifestly self-dual five-forms, i.e., to let the projection operator $\mathcal{P}_+ = \frac{1}{2}(\mathbb{1} + *)$ act on each five-form field strength. For the amplitude trace in (2.3) (or (2.12)), it is straightforward to show that this has the effect of producing an overall factor $\text{Tr}(\Pi_+)$, where $\Pi_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma^\#)$ denotes the Weyl projection operators. Since the trace is over Weyl spinors with a U(1) R-charge, this results in an overall multiplicative factor of 32. In addition, there is another overall multiplicative factor of 2, resulting from adding the parity-even and -odd parts of the trace (which become identical after imposing self-duality).

Secondly, we should comment on the dependence of the effective action on Ricci tensor and scalar factors. When expanded to linear order in the fluctuations these factors vanish by virtue of the on-shell conditions on the vertex operator polarisation tensors. Therefore, one would in principle have to analyse higher-point amplitudes to determine the dependence on these tensors. For applications in which the background is taken to be Ricci-flat at lowest order in α', the fact that these terms are not known may, however, not necessarily pose a problem.[6]

The spinorial traces that are left in (2.12) are rather complicated and lead to a plethora of terms without any obvious structure. As announced, the most systematic way to organise the result is to use the group-theory basis constructed in section 2.2. Expressed in this way we find the following expression for the four-field effective action at order $(\alpha')^3$, at up to two powers of the five-form field strength and up to terms appearing in the lowest-order equations of motion:

$$\begin{aligned} S_{\text{IIB}}^{W^2(DF^+_{(5)})^2} = \int \mathrm{d}^{10}x\,\sqrt{-g}\,\Big(DF^+_{(5)}\Big|_{\widetilde{\tiny\yng(2,1,1,1,1)}^{\,+}}\Big)^2 \\ \times\Big(\quad (16+\lambda)\,W^2\Big|_{\widetilde{\tiny\yng(2,2,2,2)}} - 4(16-\lambda)\,W^2\Big|_{\widetilde{\tiny\yng(2,2,2)}} + 192\,W^2\Big|_{\widetilde{\tiny\yng(3,1,1)}} \\ +\tfrac{16}{15}(16+\lambda)\,W^2\Big|_{\widetilde{\tiny\yng(2,2)}_A} + \tfrac{32}{3}\,W^2\Big|_{\widetilde{\tiny\yng(4)}} + \tfrac{1}{21}(16+\lambda)\,W^2\Big|_{\widetilde{\tiny\yng(2)}}\Big)\,, \end{aligned} \tag{2.13}$$

where we have ignored an overall normalisation constant. The coefficient of the remaining basis element $\widetilde{\tiny\yng(2,2)}_S$ vanishes identically. The explicit index contractions of the various terms can, if necessary, be obtained from the expressions given in appendix A.1.

Moreover, we have here introduced a constant λ to parameterise an ambiguity in the four-field action as determined purely from a four-point string-amplitude calculation. This

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[6] We adopt here the point of view that we first determine the effective action in a fixed basis of the fields, corresponding to the string theory vertex operators, and only afterwards analyse how field redefinitions might simplify this action. It is impossible to analyse the latter issue when the dependence on Ricci tensor and scalar terms is unknown, as these are related to $(F^+_{(5)})^2$ type terms of rank two by the lowest-order equations of motion.

ambiguity is directly analogous to the one at four-point level in the coefficient of the $\epsilon_{10}\epsilon_{10}W^4$ term: both the latter and the part of the action proportional to λ are total derivatives at linearised level, and their respective coefficients can therefore not be reliably determined by four-point calculations.[7] While the coefficient of the $\epsilon_{10}\epsilon_{10}W^4$ action was fixed by Grisaru et al. [25] by means of a four-loop sigma-model β-function calculation, it could alternatively have been determined by the calculation of suitable higher-point string amplitudes. We will not attempt to determine λ by calculating a five-point string amplitude here. In the discussion, we shall however comment on a very suggestive way to fix its value based on the linear scalar superfield.

We should stress that the fact that we have restricted the higher-derivative action to depend on only the self-dual part of the five-form field strength causes no problems. Rather, it is the correct thing to do, and leads, upon varying the action (2.13) with respect to the four-form gauge potential C_4, to an α'-corrected self-duality relation for the composite field strength

$$\tilde{F}_5 := \mathrm{d}C_4 + \tfrac{1}{2}B_2 \wedge F_3 - \tfrac{1}{2}H_3 \wedge C_2 \tag{2.14}$$

(here $H_3 = \mathrm{d}B_2$ and $F_3 = \mathrm{d}C_2$). To see how this comes about, first recall that the equation of motion for C_4 to lowest order, i.e. the self-duality condition $*\tilde{F}_5 = \tilde{F}_5$, cannot be obtained from a covariant action functional [26]. The best we can do is to write down an action $S^{(0)}$ which is *compatible* with self-duality, in the sense that the associated Euler–Lagrange equation for C_4 combined with the Bianchi identity for $\tilde{F}_5$ requires the latter to be self-dual up to an exact form. Explicitly, the two equations read, respectively,

$$\mathrm{d}{*\tilde{F}_5} = H_3 \wedge F_3 \quad \text{and} \quad \mathrm{d}\tilde{F}_5 = H_3 \wedge F_3\,. \tag{2.15}$$

This logic repeats itself at higher order in α'. The field equation for C_4 derived from the action including $S^{(0)}$ and our four-field result (2.13)—or, more generally, the full order-$(\alpha')^3$ action $S^{(3)}$, which is assumed to depend on C_4 only through $\tilde{F}_5^+$ and $D\tilde{F}_5^+$—takes the form

$$\mathrm{d}\left[*\tilde{F}_5 - 2\,\mathcal{P}_-\left(\frac{\delta S^{(3)}}{\delta \tilde{F}_5^+}\right)\right] = H_3 \wedge F_3\,, \tag{2.16}$$

where $\mathcal{P}_- = \frac{1}{2}(1 - *)$ is the projection operator onto antiself-dual five-forms. Combined with the Bianchi identity for $\tilde{F}_{(5)}$, this equation is compatible (in the sense discussed above) with the α'-corrected self-duality relation

$$\tilde{F}_5 + \frac{\delta S^{(3)}}{\delta \tilde{F}_5^+} = *\left[\tilde{F}_5 + \frac{\delta S^{(3)}}{\delta \tilde{F}_5^+}\right]\,. \tag{2.17}$$

The presence of $\mathcal{P}_-$ in (2.16), which is crucial for consistency, is a direct consequence of the fact that $\tilde{F}_5$ is projected onto its self-dual part in $S^{(3)}$. (In the vertex operators that appear in our four-point calculations one only needs the linearised version of this constraint, which is of course simply stating that the polarisation tensor is self-dual.)

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[7] The ambiguous part of the action was determined by linearising a general linear combination of the seven basis invariants and subsequently imposing on-shell identities and momentum conservation, and finally requiring the resulting expression to vanish.

2.4 A cross-check using SL(2,$\mathbb{Z}$) invariance

The calculation for the five-form terms which have discussed so far shares many aspects with the calculation one has to do in order to determine the dependence of the effective action on the Ramond-Ramond three-form field strength. The resulting $R^2(DF_{(3)})^2$ terms in the effective action should, as we explain below, be related by SL(2,$\mathbb{Z}$) symmetry to the $R^2(DH_{(3)})^2$ terms involving the Neveu-Schwarz three-form field strength, which have been computed by Gross and Sloan [16].[8] Matching their calculation with ours leads to an interesting check of the duality symmetry of the type-IIB theory, while at the same time providing a very strong check on all signs and factors in our string calculation.

To explain the logic, recall that SL(2,$\mathbb{Z}$) symmetry dictates that the three-form field strengths should appear in the effective action only through the SL(2,$\mathbb{Z}$)-invariant combination

$$G_{(3)} = \frac{1}{\sqrt{\operatorname{Im}\tau}}\big(F_{(3)} + \tau H_{(3)}\big)\,, \tag{2.18}$$

with U(1) charge one, as well as its complex conjugate. The complex coupling τ is formed from the Ramond-Ramond scalar and the dilaton,

$$\tau = \chi + ie^{-\phi}\,. \tag{2.19}$$

Terms in the effective action are then built from (2.18) as well as other SL(2,$\mathbb{Z}$) singlets, whose transformation behaviour under the local U(1) symmetry is compensated for by prefactor modular functions $f^{(w,-w)}$ of the complex coupling τ (see e.g. Green and Sethi [27] for more details).

The terms computed by Gross and Sloan [16] are those independent of χ, i.e. for which $\tau = i\operatorname{Im}\tau$. We will also restrict ourselves to this case, because we did not compute string amplitudes with an external Ramond-Ramond scalar. In the Einstein frame, the NS-NS field-strength part of the genus-zero effective action is then of the form[9]

$$e^{-5\phi/2}\sqrt{-g}R^2(DH_{(3)})^2\Big|_{\text{Einstein}} \rightarrow e^{-2\phi}\sqrt{-g}R^2(DH_{(3)})^2\Big|_{\text{string}}\,. \tag{2.21}$$

with the specific tensorial structure given by Gross and Sloan [16]. Their result can be made SL(2,$\mathbb{Z}$) invariant in two ways,

$$\begin{aligned} e^{-3\phi/2}R^2 DG_{(3)}DG^*_{(3)}\Big|_{\text{Einstein}} &= e^{-5\phi/2}R^2(DH_{(3)})^2 + e^{-\phi/2}R^2(DF_{(3)})^2\Big|_{\text{Einstein}}\,,\\ -e^{-3\phi/2}R^2\,\tfrac{1}{2}\big(DG_{(3)}DG_{(3)} + \text{c.c.}\big)\Big|_{\text{Einstein}} &= e^{-5\phi/2}R^2(DH_{(3)})^2 - e^{-\phi/2}R^2(DF_{(3)})^2\Big|_{\text{Einstein}}\,. \end{aligned} \tag{2.22}$$

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[8] At the level of the effective supergravity theories we are discussing the symmetry group is SL(2,$\mathbb{R}$), but we will refer to the smaller, discrete subgroup that is the S-duality symmetry of the full type-IIB string theory.

[9] We remind the reader that under a super-Weyl rescaling from the Einstein frame to the string frame the metric transforms as $g^E_{\mu\nu} = e^{-\phi/2}g^S_{\mu\nu}$ and that this induces the transformation

$$\begin{aligned} &\int \mathrm{d}^{10}x\,\sqrt{-g}\,\Big[R + (\alpha')^3 e^{-3\phi/2}t_8t_8R^4 - \tfrac{1}{2}(\partial_\mu\phi)^2\Big]\Big|_{\text{Einstein}}\\ &\qquad\qquad = \int \mathrm{d}^{10}x\,\sqrt{-g}\,e^{-2\phi}\Big[R + (\alpha')^3 t_8t_8R^4 + 4(\partial_\mu\phi)^2\Big]\Big|_{\text{string}} \end{aligned} \tag{2.20}$$

on the genus-zero string effective action.

We thus find that the string effective action should contain terms of the form

$$R^2(DF_{(3)})^2\Big|_{\text{string}}\,, \tag{2.23}$$

with tensorial structures identical to those of the Neveu-Schwarz dependent terms. Moreover, by the general logic sketched above, terms with this structure appear both at tree level and at genus one. The relative coefficient with respect to the Neveu-Schwarz sector depends on the unknown relative normalisation between the two type of terms given above (this coefficient has been conjectured by Kehagias and Partouche [28] but an explicit derivation of it is lacking).

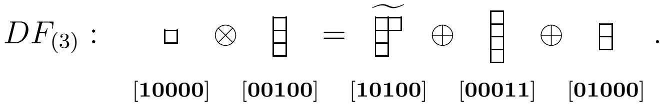
The most basic way in which the above prediction of the duality symmetry can be verified is by comparing the amplitude with two gravitons and two external NS-NS three-form states to the one with two gravitons and two external RR states. The most interesting aspect of this comparison is that the fermionic world-sheet correlators that appear in these two calculations are completely different. In the NS-NS sector (for full details we refer to Gross and Sloan [16]) the relevant correlator is formed from RNS fermions only, taking the form

$$\langle :\!\Psi\Psi\!:(z_1)\ :\!\Psi\Psi\!:(z_2)\ :\!\Psi\Psi\!:(z_3)\ :\!\Psi\Psi\!:(z_4)\,\rangle \tag{2.24}$$

in each of the chiral sectors. The correlator in the RR sector, on the other hand, contains spin fields which lead to traces over gamma matrices, as we have explained in detail in section 2.1. We have compared the long expressions for the $ggH_{(3)}H_{(3)}$ and $ggF_{(3)}F_{(3)}$ amplitudes at eighth order in the momenta. Remarkably, after taking into account the on-shell conditions on the polarisation tensors and imposing momentum conservation, the two amplitudes (each with several hundred different terms) agree perfectly!

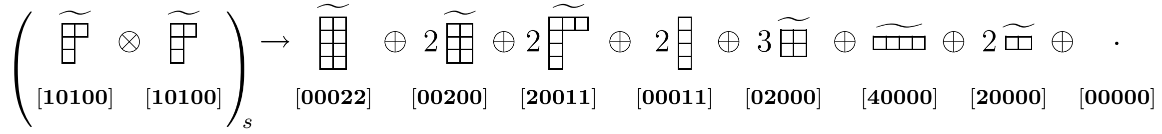
In order to establish the SL(2,$\mathbb{Z}$) duality at the level of the full covariant effective action, one would need to have precise control over the terms in the action that lead to a vanishing on-shell four-point function. Since the five-point $gggH_{(3)}H_{(3)}$ amplitude has never been fully computed in the literature, these terms are ambiguous already in the NS-NS sector.[10] In order to enable a systematic future comparison, we have therefore again decomposed the effective action in a basis of irreducible invariants and parameterised the ambiguity by determining those linear combinations of the basis invariants that lead to vanishing four-point functions.

This decomposition is similar in spirit to the one given in the previous section for the five-form terms. The three-form derivatives come in three different irreducible representations:



(2.25)

We can immediately discard the [01000] as it corresponds to the lowest-order equation-of-motion term $D^m F_{mnp}$, which we simply set to zero throughout the calculations. The products of the remaining two $DF_{(3)}$ (or $DH_{(3)}$) representations have the decompositions



(2.26a)

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[10] There is a widely expressed belief that the $W^2DH^2_{(3)}$ terms can simply be obtained from the W^4 terms by using a generalised spin connection modified by the addition of a torsion term in the form of the NS-NS three-form field strength. There is, however, no proof that this mechanism works in general (see, e.g., Metsaev and Tseytlin [29] for a discussion) and we will therefore not rely on it here.

$$\left(\underset{[\mathbf{10100}]}{\widetilde{\ydiagram{2,1,1}}}\otimes\underset{[\mathbf{00011}]}{\ydiagram{1,1,1,1}}\right)\to\underset{[\mathbf{00200}]}{\widetilde{\ydiagram{2,2,2}}}\oplus\underset{[\mathbf{20011}]}{\widetilde{\ydiagram{3,1,1,1}}}\oplus\underset{[\mathbf{00011}]}{\ydiagram{1,1,1,1}}\oplus\underset{[\mathbf{02000}]}{\widetilde{\ydiagram{2,2}}}\oplus\underset{[\mathbf{20000}]}{\widetilde{\ydiagram{2}}}\tag{2.26b}$$

$$\left(\underset{[\mathbf{00011}]}{\ydiagram{1,1,1,1}}\otimes\underset{[\mathbf{00011}]}{\ydiagram{1,1,1,1}}\right)_s\to\underset{[\mathbf{00022}]}{\widetilde{\ydiagram{2,2,2,2}}}\oplus\underset{[\mathbf{00200}]}{\widetilde{\ydiagram{2,2,2}}}\oplus\underset{[\mathbf{00011}]}{\ydiagram{1,1,1,1}}\oplus\underset{[\mathbf{02000}]}{\widetilde{\ydiagram{2,2}}}\oplus\underset{[\mathbf{20000}]}{\widetilde{\ydiagram{2}}}\oplus\underset{[\mathbf{00000}]}{\cdot}\tag{2.26c}$$

Here we have again left out representations not present in the expansion of two Weyl tensors (given in (2.9)) as well as terms which are not compatible with the symmetries of the tensorial structures listed in table 3.[11]

Verifying SL(2,$\mathbb{Z}$) symmetry at the level of the effective actions amounts to establishing that the respective expansions in basis invariants for $W^2(DH_{(3)})^2$ and $W^2(DF_{(3)})^2$ agree when all terms proportional to $DH_{(3)}$ or $DF_{(3)}$ in either the [01000] or the [00011] representation have been set to zero. Combining the W^2 expansion (2.9) with the first line of (2.26), one finds that this corresponds to a matching of 17 coefficients.

The decomposition of the effective action on the group-theory basis is thus more complicated than in the case of the five-form action. The three invariants based on the $\widetilde{\ydiagram{2,2}}_S$ representation again do not appear, while the terms that lead to a vanishing four-point amplitude form a four-parameter family of ambiguous terms parameterised by $\lambda_1 \ldots \lambda_4$. Including these ambiguous terms, the expression for the action at four-point level reads

$$\begin{aligned}
S_{\rm IIB}^{W^2DF^2_{(3)}} = \int \mathrm{d}^{10}x\,\sqrt{-g}\,\Big(DF_{(3)}\Big|_{\widetilde{\ydiagram{2,1,1}}}\Big)^2 \\
\times\Bigg((-240+\lambda_1)\,W^2\Big|_{\widetilde{\ydiagram{2,2,2,2}}} + (-64+\lambda_2)\,W^2\Big|_{\widetilde{\ydiagram{2,2,2}}_a} \\
+ 12\,(32+\tfrac{2}{5}\lambda_1-\lambda_2)\,W^2\Big|_{\widetilde{\ydiagram{2,2,2}}_b} - 96\,W^2\Big|_{\widetilde{\ydiagram{3,1,1,1}}_a} + 96\,W^2\Big|_{\widetilde{\ydiagram{3,1,1,1}}_b} \\
- 64\,W^2\Big|_{\widetilde{\ydiagram{4}}} + (32+\lambda_4)\,W^2\Big|_{\ydiagram{1,1,1,1}_a} - 32\,W^2\Big|_{\ydiagram{1,1,1,1}_b} + (\tfrac{96}{5}+\lambda_3)\,W^2\Big|_{\widetilde{\ydiagram{2,2}}_{Aa}} \\
+ (\tfrac{256}{5}+\tfrac{32}{25}\lambda_1-2\lambda_2-\tfrac{4}{3}\lambda_3)\,W^2\Big|_{\widetilde{\ydiagram{2,2}}_{Ab}} + \tfrac{8}{5}\,(-32-\tfrac{1}{5}\lambda_1)\,W^2\Big|_{\widetilde{\ydiagram{2,2}}_{Ac}} \\
+ (-\tfrac{16}{105}\lambda_1+\tfrac{25}{84}\lambda_2+\tfrac{5}{18}\lambda_3)\,W^2\Big|_{\widetilde{\ydiagram{2}}_a} + (\tfrac{8}{7}+\tfrac{1}{35}\lambda_1-\tfrac{5}{84}\lambda_2-\tfrac{5}{36}\lambda_3)\,W^2\Big|_{\widetilde{\ydiagram{2}}_b} \\
+ (\tfrac{68}{315}-\tfrac{11}{3150}\lambda_1+\tfrac{1}{144}\lambda_2+\tfrac{1}{108}\lambda_3)\,W^2\Big|_{\cdot}\Bigg)\,.
\end{aligned}\tag{2.27}$$

We have again discarded an overall normalisation constant. The indices on the tableaux serve to distinguish inequivalent ways of constructing terms with the same Young symmetries. We refrain from giving the explicit expressions for these 14 basis invariants here, as the SL(2,$\mathbb{Z}$)

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[11] Once more, one has to be careful to recognise that one representation can appear as two different Young tableaux; see the discussion around (2.11). In the present case, this happens for the [20011] representation in the second line of (2.26).

symmetry in this case provides us with a more compact way of writing the action: by a suitable choice of the parameters λ_i, the action given above can be reduced to the form

$$(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10})\, W^2 (DF_{(3)})^2 \,. \tag{2.28}$$

The parameter values corresponding to this action are given by $\lambda_1 = 240$, $\lambda_2 = 192$, $\lambda_3 = -576/5$ and $\lambda_4 = 0$. In particular, they have been adjusted so that the rank-eight contractions between the W^2 and $DF^2_{(3)}$ blocks cancel between the two terms in (2.28). This gives precisely the RR three-form version of the action obtained by inserting a shifted spin connection $\tilde{\omega} = \omega + H_{(3)}$ in the familiar W^4 action of the type-IIB theory, showing that our result (2.27) is compatible also with a stronger check of the SL(2,$\mathbb{Z}$) duality symmetry.[12]

3 Discussion and comments on applications

3.1 Comparison with predictions of the linearised superfield

Berkovits and Howe have recently shown that the linearised scalar superfield of the type-IIB theory [30, 31] does not admit a non-linear extension, essentially because no chiral measure can be constructed (a detailed argument can be found in de Haro et al. [20]). In the present section we would like to make a few concluding remarks relating our four-point amplitudes to the predictions of the *linearised* superfield. We would, however, like to stress that, despite the comments we will make in this section, the direct string amplitude computations of the type presented in our paper are currently the only known way of constructing the full dependence of the type-IIB invariant on the Ramond-Ramond gauge fields.

In the linearised superfield approach, the action arises as a sixteen-dimensional fermionic integral over four powers of the scalar superfield Φ,

$$S_{\text{IIB}} = \int \mathrm{d}^{10}x\, \mathrm{d}^{16}\theta\, e\, \Phi^4 \,. \tag{3.1}$$

This integral is very hard to do explicitly except when one restricts to pure graviton terms, but we can again, as in [19], use arguments based on representation theory to compare with our string-based result. The linearised scalar superfield contains, at fourth order in θ, the terms

$$\Phi = \cdots + (\theta \mathcal{C} \Gamma^{r_1 r_2 r_3}\theta)\,(\theta \mathcal{C} \Gamma^{s_1 s_2 s_3}\theta)\,\Big(R_{r_1 r_2 s_1 s_2}\eta_{r_3 s_3} + D_{r_1} F_{r_2 r_3 s_1 s_2 s_3}\Big) + \cdots \,. \tag{3.2}$$

The four powers of the anti-commuting θ restrict the fields to the representations

$$(\otimes_{i=1}^{4}\mathbf{16})_a = \underset{[\mathbf{02000}]}{\widetilde{\ydiagram{2,2}}} \;\oplus\; \underset{[\mathbf{10002}]}{\widetilde{\ydiagram{2,1,1,1,1}}^{\,+}} \,, \tag{3.3}$$

where the tensors on the right-hand side correspond to the Weyl tensor and the non-trivial representation of $DF^+_{(5)}$, respectively. A similar argument restricts the way in which W^2 and

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[12] As a curiosity, let us mention that the above values of the λ parameters correspond to the $\epsilon_{10}\epsilon_{10}$ part of (2.28) multiplied by a factor of two. Hence, the action (2.27) with all λ_i set to zero also takes the form (2.28) but with the opposite sign for the second term.

$(DF^{+}_{(5)})^2$ tensors can appear in the action. Namely, at level θ^8 in Φ^2 one finds the decomposition

$$(\otimes_{i=1}^{8}\mathbf{16})_a = \underset{[\mathbf{00200}]}{\widetilde{\ydiagram{2,2,2}}} \oplus \underset{[\mathbf{20011}]}{\widetilde{\ydiagram{3,1,1,1}}} \oplus \underset{[\mathbf{40000}]}{\widetilde{\ydiagram{4}}}\ . \tag{3.4}$$

We observe that this is, in general, more restrictive than the action given in (2.13).

However, as we have noted in section 2.3, there is a one-parameter ambiguity in our effective action (2.13) which arises because the four-point amplitude does not suffice to fix the coefficients of terms in the action that lead to vanishing on-shell four-point functions. This ambiguity, parameterised by λ in (2.13), is precisely such that the action can be reduced to the three representations given in (3.4) above: for the special value $\lambda = -16$ only these representations survive. Assuming that the superfield integral (3.1) produces the string superinvariant—and the fact that there exists a value for λ for which the action satisfies the restriction imposed by (3.4) indeed lends a certain credence to this assumption—we can thus use this argument to fix the four-field $R^2(DF^{+}_{(5)})^2$ action completely.

As a side note, the above decomposition also explains the precise relative coefficient between the $t_8t_8W^4$ and $\epsilon_{10}\epsilon_{10}W^4$ parts of the superfield W^4 invariant (which happens to agree with the corresponding part of the type-IIB four-point effective action): there is a unique linear combination of the two that, like (3.4), contains no rank-eight contractions between two W^2 factors.[13]

3.2 Applications, conclusions and outlook

We have shown how systematic string-amplitude calculations and a group-theoretical approach to the construction of field-theory invariants can be used to successfully compute the string effective action including Ramond-Ramond fields. This is a technically rather demanding project, but we have shown that the results can be cast in a very compact form. We have illustrated our methods by giving an explicit expression for the four-field terms in the type-IIB effective action involving two powers of the three-form or five-form field strength. Clearly, our work is a necessary prerequisite for any future calculations of this type that extend to higher order in the fields. For this reason, we consider it very important that our calculations passed the rather spectacular SL(2,$\mathbb{Z}$) match.

One particularly interesting term in the effective action that can be determined using our setup is the $W^3(F^{+}_{(5)})^2$ term; in fact, the results obtained in the present paper are crucial in order to determine these terms. We have done some preliminary work to investigate the complexity of this problem. Firstly, while the string amplitudes now contain considerably more terms, the correlators are again all "topological" in the sense that they reduce to constants by virtue of the Riemann identity. Secondly, one now encounters field-theory subtraction issues: the five-point field theory amplitude receives contributions from tree-level graphs formed from one lowest-order supergravity vertex and one four-field vertex coming from the $W^2(DF^{+}_{(5)})^2$ terms computed in the present paper. Despite these complications, the effective action still has a manageable expansion in group theory invariants. This is determined by the two tensor

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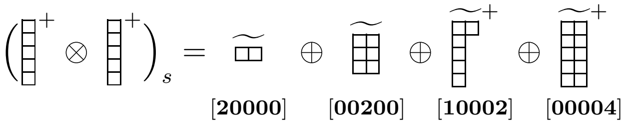
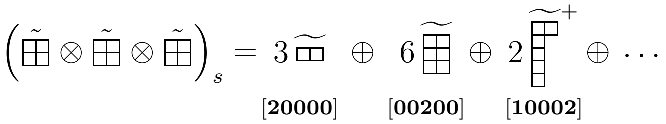
[13] Here one should again take note of the discussion around (2.11). For the case at hand, the irreducible representation [20011] has a dual realisation in terms of a rank-eight tensor. However, this tensor is antisymmetric in six indices and can therefore not be be formed out of two Weyl tensors.

products



(3.5a)



(3.5b)

where the ellipses denote representations that do not appear in the expansion (3.5a). From the overlap between the two expansions we see that there are, at most, eleven independent $W^3\,(F^+_{(5)})^2$ invariants. Computing this part of the type-IIB action will be an interesting application of the procedure given in the present paper.

Another set of terms which is related to the ones computed here are those involving spacetime fermions. Part of our motivation for the paper [13] was to deduce the modifications to the supersymmetry transformation rules and the resulting superalgebra in order to determine the superspace torsion constraints. For this purpose it is presumably sufficient to determine the fermion bilinear terms in the effective action.

Finally, let us comment on a more conceptual lesson which can be learned from our analysis. Despite the very low number of basis invariants from which our effective action is constructed, the corresponding expressions as given in the appendix are quite lengthy and the resulting action, when written out in this brute-force way, would be rather intractable. It thus seems clear that any actual applications along the lines of Frolov et al. [12] or de Haro et al. [9] would benefit from keeping the Young symmetrisers implicit as long as possible.

A similar systematic approach is also highly desirable for the determination of corrections to supersymmetry transformation rules. As we have already shown in [13], supersymmetry mixes the tensorial structures of the various terms in the effective action in a very complicated way. By writing not just the action but also the supersymmetry transformation rules using a group-theory basis, it is likely that one can cast the $(\alpha')^3$ corrections to these rules in a manageable form, despite the underlying complexity of the calculations. We will leave these issues for future work.

Acknowledgements

We thank Stefan Theisen and Pierre Vanhove for discussions, and Paul Howe for valuable remarks regarding the geometry of type-IIB superspace. For computation of the gamma-matrix traces, the Mathematica package `GAMMA` by Gran [32] has been helpful. Similarly, the programme `LiE` [33] has been most useful for calculating SO(1,9) expansions.

A Appendix

A.1 Explicit expressions for tensor polynomials

In this section we list the explicit expressions for the tensor polynomials that appear in the effective action at order $(\alpha')^3$.

The first step in their construction is to decompose the W^2 and $(DF^+_{(5)})^2$ products in irreducible representations, as given in (2.9) and (2.10) of the main text. The explicit forms of the Weyl tensor building blocks are given by

$$\begin{aligned} W^2\Big|_{\widetilde{\ydiagram{2,2,2,2}}} = {} & W_{r_1r_2s_1s_2}W_{r_3r_4s_3s_4} - 2\,\delta^{r_1}_{s_1}W_{d_1r_2s_2s_3}W_{r_3r_4d_1s_4} - \tfrac{4}{5}\delta^{r_1r_2}_{s_1s_2}W_{d_1r_3d_2s_3}W_{d_1s_4d_2r_4} \\ & + \tfrac{1}{5}\delta^{r_1r_2}_{s_1s_2}W_{d_1d_2r_3r_4}W_{d_1d_2s_3s_4} - \tfrac{2}{15}\delta^{r_1r_2r_3}_{s_1s_2s_3}W_{d_1d_2d_3r_4}W_{d_1d_2d_3s_4} \\ & + \tfrac{1}{210}\delta^{r_1r_2r_3r_4}_{s_1s_2s_3s_4}W^2\,, \end{aligned} \tag{A.1}$$

$$\begin{aligned} W^2\Big|_{\widetilde{\ydiagram{2,2,2}}} = {} & \tfrac{1}{2}W_{d_1r_1s_1s_2}W_{d_1s_3r_2r_3} + \tfrac{1}{3}\delta^{r_1}_{s_1}W_{d_1r_2d_2s_2}W_{d_1s_3d_2r_3} \\ & + \tfrac{1}{14}\delta^{r_1r_2}_{s_1s_2}W_{d_1d_2d_3r_3}W_{d_1d_2d_3s_3} - \tfrac{1}{12}\delta^{r_1}_{s_1}W_{d_1d_2r_2r_3}W_{d_1d_2s_2s_3} \\ & - \tfrac{1}{336}\delta^{r_1r_2r_3}_{s_1s_2s_3}W^2\,, \end{aligned} \tag{A.2}$$

$$\begin{aligned} W^2\Big|_{\widetilde{\ydiagram{3,1,1,1}}} = {} & W_{d_1s_1[r_1r_2}W_{|d_1s_2|r_3r_4]} - \tfrac{1}{12}\delta^{s_1s_2}W_{d_1d_2[r_1r_2}W_{|d_1d_2|r_3r_4]} \\ & + \tfrac{1}{18}\delta^{(s_1}{}_{[r_1}W_{|d_1d_2|r_2r_3}W_{|d_1d_2|r_4]}{}^{s_2)} \\ & + \tfrac{1}{9}\delta^{(s_1}{}_{[r_1}W_{|d_1d_2|r_2}{}^{s_2)}W_{d_1d_2|r_3r_4]}\,, \end{aligned} \tag{A.3}$$

$$\begin{aligned} W^2\Big|_{\widetilde{\ydiagram{2,2}}_A} = {} & \tfrac{2}{3}W_{d_1r_1d_2r_2}W_{d_1s_1d_2s_2} - \tfrac{2}{3}W_{d_1r_1d_2s_1}W_{d_1s_2d_2r_2} \\ & - \tfrac{1}{4}\delta^{r_1}_{s_1}W_{d_1d_2d_3r_2}W_{d_1d_2d_3s_2} + \tfrac{1}{72}\delta^{r_1r_2}_{s_1s_2}W^2\,, \end{aligned} \tag{A.4}$$

$$\begin{aligned} W^2\Big|_{\widetilde{\ydiagram{2,2}}_S} = {} & \tfrac{2}{3}W_{d_1r_1d_2s_1}W_{d_1r_2d_2s_2} + \tfrac{2}{3}W_{d_1r_1d_2s_1}W_{d_1s_2d_2r_2} \\ & + \tfrac{1}{4}\delta^{r_1}_{s_1}W^{d_1d_2d_3r_2}W_{d_1d_2d_3s_2} - \tfrac{1}{72}\delta^{r_1r_2}_{s_1s_2}W^2\,, \end{aligned} \tag{A.5}$$

$$W^2\Big|_{\widetilde{\ydiagram{4}}} = W_{d_1(r_1|d_2|r_2}W_{|d_1|s_1|d_2|s_2)} - \tfrac{3}{14}\eta_{(r_1r_2}W_{s_1|d_1d_2d_3|}W_{s_2)d_1d_2d_3} + \tfrac{1}{112}\eta_{(r_1r_2}\eta_{s_1s_2)}W^2\,, \tag{A.6}$$

$$W^2\Big|_{\ydiagram{1,1,1,1}} = W_{d_1[r_1|d_2|r_2}W_{|d_1|s_1|d_2|s_2]}\,, \tag{A.7}$$

$$W^2\Big|_{\widetilde{\ydiagram{2}}} = W_{d_1d_2d_3r_1}W_{d_1d_2d_3s_1} - \tfrac{1}{10}\eta_{r_1s_1}W^2\,. \tag{A.8}$$

These expressions are obtained by applying the Young-tableau symmetrisers and subsequently subtracting traces in order to reduce to an irreducible representation. Antisymmetry is assumed on the index sets r_i and s_i except where otherwise explicitly indicated. The non-trivial representation contained in $DF^{+}_{(5)}$ is given by

$$\begin{aligned} DF^{+}_{(5)}\Big|_{\widetilde{\yng(2,1,1,1,1)}^{+}} = {}& \frac{5}{6}D_r F^{+}_{r_1r_2r_3r_4r_5} + \frac{1}{6}\Big(D_{r_1}F^{+}_{rr_2r_3r_4r_5} - D_{r_2}F^{+}_{rr_1r_3r_4r_5} \\ & \qquad + D_{r_3}F^{+}_{rr_1r_2r_4r_5} - D_{r_4}F^{+}_{rr_1r_2r_3r_5} + D_{r_5}F^{+}_{rr_1r_2r_3r_4}\Big) \\ & - \frac{1}{10}\Big(\eta_{rr_1}D_{d_1}F^{+}_{d_1r_2r_3r_4r_5} - \eta_{rr_2}D_{d_1}F^{+}_{d_1r_1r_3r_4r_5} \\ & \qquad - \eta_{rr_3}D_{d_1}F^{+}_{d_1r_1r_2r_4r_5} - \eta_{rr_4}D_{d_1}F^{+}_{d_1r_1r_2r_3r_5} - \eta_{rr_5}D_{d_1}F^{+}_{d_1r_1r_2r_3r_4}\Big)\,. \end{aligned} \tag{A.9}$$

There are *a priori* two ways to contract two of these $DF^{+}_{(5)}$ factors in such a way as to obtain a tensor of rank eight. However, the classification in section 2.2 has shown that there can be only one independent object of this type. Indeed, explicit calculation shows that one of these contractions vanishes:

$$\bigg(\Big(D_m F^{+}_{m_1m_2m_3m_4m_5}\, D_n F^{+}_{n_1n_2n_3n_4n_5}\Big)\Big|_{\widetilde{\yng(2,1,1,1,1)}^{+}\otimes\widetilde{\yng(2,1,1,1,1)}^{+}} \times \eta^{mn}\eta^{m_5n_5}\bigg)\bigg|_{\widetilde{\yng(2,2,2,2)}} = 0\,. \tag{A.10}$$

The other contraction does lead to a non-vanishing result:

$$\Big(D_m F^{+}_{m_1m_2m_3m_4m_5}\, D_n F^{+}_{n_1n_2n_3n_4n_5}\Big)\Big|_{\widetilde{\yng(2,1,1,1,1)}^{+}\otimes\widetilde{\yng(2,1,1,1,1)}^{+}} \times \eta^{mn_5}\eta^{m_5n} \neq 0\,. \tag{A.11}$$

The explicit expression can easily be obtained from (A.9).

The invariants used in the action (2.13) can be constructed from the ingredients given above. In the following we will suppress all terms proportional to $D^m F^{+}_{mr_2\cdots r_5}$ as they are lowest-order equations of motion. Within each invariant, the terms can be classified according to the number of indices that are contracted between the W^2 and the $(DF^{+}_{(5)})^2$ blocks. For space reasons, we list here only the terms with the maximal number of contractions between the two blocks.[14] These are given by

$$\begin{aligned} W^2\Big|_{\widetilde{\yng(2,2,2,2)}}\big(DF^{+}_{(5)}\Big|_{\widetilde{\yng(2,1,1,1,1)}^{+}}\big)^2 = {}& -\tfrac{5}{9}D_kF_{d_1d_2d_3d_4}\,D_lF_{d_5d_6d_7k}\,W_{d_1d_2d_7l}W_{d_3d_4d_5d_6} \\ & -\tfrac{5}{18}D_lF_{d_1d_2d_3d_4}\,D_lF_{d_5d_6d_7d_8}\,W_{d_1d_2d_5d_6}W_{d_3d_4d_7d_8} \\ & +\tfrac{5}{18}D_kF_{d_1d_2d_3d_4l}\,D_lF_{d_5d_6d_7d_8k}\,W_{d_1d_2d_5d_6}W_{d_3d_4d_7d_8} \\ & -\tfrac{5}{9}D_kF_{d_1d_2d_3l}\,D_lF_{d_4d_5d_6d_7}\,W_{d_1d_2d_6d_7}W_{d_3kd_4d_5} \\ & +\tfrac{2}{9}D_kF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_6l}W_{d_3kd_4d_5} \\ & +\tfrac{2}{9}D_kF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_4d_5}W_{d_3kd_6l} \\ & -2D_kF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_6k}W_{d_3ld_4d_5} \\ & -2D_kF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_4d_5}W_{d_3ld_6k} \\ & +\text{lower-rank contractions}\,, \end{aligned} \tag{A.12}$$

---

[14] Note, however, that the lower-rank contractions arising from the trace-subtraction terms in (A.1)–(A.8) (or, alternatively, those in the corresponding $(DF^{+}_{(5)})^2$ expressions) are crucial to obtain a correct decomposition of the action in irreducible parts.

$$\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{2,2,2}}}(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{\,+}})^2 = {} & \tfrac{1}{6}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_5d_3k}W_{d_2d_4d_5l} \\
& - \tfrac{1}{36}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1kd_5l}W_{d_2d_5d_3d_4} \\
& + \tfrac{1}{12}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1ld_5k}W_{d_2d_5d_3d_4} \\
& - \tfrac{1}{18}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_3d_5k}W_{d_2d_5d_4l} \\
& - \tfrac{1}{9}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_5d_3k}W_{d_2d_5d_4l} \\
& - \tfrac{1}{72}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_5l}W_{d_3d_4d_5k} \\
& + \tfrac{1}{24}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_5k}W_{d_3d_4d_5l} \\
& + \tfrac{1}{6}D_lF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_6d_7}W_{d_3d_7d_4d_5} \\
& + \tfrac{1}{6}D_kF_{d_1d_2d_3l}\,D_lF_{d_4d_5d_6k}\,W_{d_1d_2d_6d_7}W_{d_3d_7d_4d_5} \\
& + \tfrac{1}{12}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_4d_5}W_{d_3kd_5l} \\
& - \tfrac{1}{36}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_4d_5}W_{d_3ld_5k} \\
& + \text{lower-rank contractions}\,,
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{3,1,1,1}}}(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{\,+}})^2 = {} & \tfrac{1}{6}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_5d_3k}W_{d_2d_4d_5l} \\
& - \tfrac{1}{36}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1kd_5l}W_{d_2d_5d_3d_4} \\
& + \tfrac{1}{12}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1ld_5k}W_{d_2d_5d_3d_4} \\
& - \tfrac{1}{18}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_3d_5k}W_{d_2d_5d_4l} \\
& - \tfrac{1}{9}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_5d_3k}W_{d_2d_5d_4l} \\
& - \tfrac{1}{72}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_5l}W_{d_3d_4d_5k} \\
& + \tfrac{1}{24}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_5k}W_{d_3d_4d_5l} \\
& + \tfrac{1}{6}D_lF_{d_1d_2d_3}\,D_lF_{d_4d_5d_6}\,W_{d_1d_2d_6d_7}W_{d_3d_7d_4d_5} \\
& + \tfrac{1}{6}D_kF_{d_1d_2d_3l}\,D_lF_{d_4d_5d_6k}\,W_{d_1d_2d_6d_7}W_{d_3d_7d_4d_5} \\
& + \tfrac{1}{12}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_4d_5}W_{d_3kd_5l} \\
& - \tfrac{1}{36}D_kF_{d_1d_2}\,D_lF_{d_3d_4}\,W_{d_1d_2d_4d_5}W_{d_3ld_5k} \\
& + \text{lower-rank contractions}\,,
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{2,2}}_A}(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{\,+}})^2 = {} & -\tfrac{1}{72}D_kF_{d_1l}\,D_lF_{d_2d_3}\,W_{d_1kd_4d_5}W_{d_2d_3d_4d_5} \\
& + \tfrac{1}{144}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_4d_3k}W_{d_2d_3d_4l} \\
& + \tfrac{1}{18}D_kF_{d_1d_2}\,D_lF_{d_3k}\,W_{d_1d_5d_4l}W_{d_2d_4d_3d_5} \\
& + \tfrac{1}{24}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_3d_4l}W_{d_2d_4d_3k} \\
& - \tfrac{7}{144}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_4d_3l}W_{d_2d_4d_3k} \\
& + \text{p.t.o.}
\end{aligned} \tag{A.15}$$

$$
\begin{aligned}
& -\tfrac{7}{144} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_4 d_3 k} W_{d_2 d_4 d_3 l} \\
& -\tfrac{1}{144} D_l F_{d_1 d_2}\, D_l F_{d_3 d_4}\, W_{d_1 d_3 d_5 d_6} W_{d_2 d_4 d_5 d_6} \\
& +\tfrac{1}{48} D_k F_{d_1 d_2 l}\, D_l F_{d_3 d_4 k}\, W_{d_1 d_3 d_5 d_6} W_{d_2 d_4 d_5 d_6} \\
& +\tfrac{1}{72} D_l F_{d_1 d_2}\, D_l F_{d_3 d_4}\, W_{d_1 d_5 d_3 d_6} W_{d_2 d_5 d_4 d_6} \\
& -\tfrac{1}{24} D_k F_{d_1 d_2 l}\, D_l F_{d_3 d_4 k}\, W_{d_1 d_5 d_3 d_6} W_{d_2 d_5 d_4 d_6} \\
& +\tfrac{1}{18} D_k F_{d_1 l}\, D_l F_{d_2 d_3}\, W_{d_1 d_4 d_3 d_5} W_{d_2 d_5 d_4 k} \\
& -\tfrac{1}{36} D_k F_{d_1 l}\, D_l F_{d_2 d_3}\, W_{d_1 d_3 d_4 d_5} W_{d_2 k d_4 d_5} \\
& -\tfrac{1}{288} D_l F_{d_1 d_2}\, D_l F_{d_3 d_4}\, W_{d_1 d_2 d_5 d_6} W_{d_3 d_4 d_5 d_6} \\
& +\tfrac{1}{96} D_k F_{d_1 d_2 l}\, D_l F_{d_3 d_4 k}\, W_{d_1 d_2 d_5 d_6} W_{d_3 d_4 d_5 d_6} \\
& -\tfrac{1}{288} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_2 d_3 d_4} W_{d_3 d_4 k l} \\
& +\tfrac{7}{144} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_3 d_2 d_4} W_{d_3 k d_4 l} \\
& -\tfrac{1}{72} D_k F_{d_1 d_2}\, D_l F_{d_3 k}\, W_{d_1 d_2 d_4 d_5} W_{d_3 l d_4 d_5} \\
& + \text{lower-rank contractions}\,,
\end{aligned}
$$

$$
\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{2,2}}_S} \big(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{\,+}}\big)^2 = {} & \tfrac{1}{36} D_k F_{d_1 d_2}\, D_l F_{d_3 k}\, W_{d_1 l d_4 d_5} W_{d_2 d_3 d_4 d_5} \\
& +\tfrac{1}{24} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_4 d_3 k} W_{d_2 d_3 d_4 l} \\
& -\tfrac{1}{9} D_k F_{d_1 d_2}\, D_l F_{d_3 k}\, W_{d_1 d_5 d_4 l} W_{d_2 d_4 d_3 d_5} \\
& +\tfrac{1}{144} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_3 d_4 l} W_{d_2 d_4 d_3 k} \\
& +\tfrac{1}{144} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_4 d_3 l} W_{d_2 d_4 d_3 k} \\
& +\tfrac{1}{24} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_4 d_3 k} W_{d_2 d_4 d_3 l} \\
& +\tfrac{1}{144} D_l F_{d_1 d_2}\, D_l F_{d_3 d_4}\, W_{d_1 d_3 d_5 d_6} W_{d_2 d_4 d_5 d_6} \\
& -\tfrac{1}{48} D_k F_{d_1 d_2 l}\, D_l F_{d_3 d_4 k}\, W_{d_1 d_3 d_5 d_6} W_{d_2 d_4 d_5 d_6} \\
& -\tfrac{1}{36} D_l F_{d_1 d_2}\, D_l F_{d_3 d_4}\, W_{d_1 d_5 d_3 d_6} W_{d_2 d_5 d_4 d_6} \\
& +\tfrac{1}{12} D_k F_{d_1 d_2 l}\, D_l F_{d_3 d_4 k}\, W_{d_1 d_5 d_3 d_6} W_{d_2 d_5 d_4 d_6} \\
& -\tfrac{1}{9} D_k F_{d_1 l}\, D_l F_{d_2 d_3}\, W_{d_1 d_4 d_3 d_5} W_{d_2 d_5 d_4 k} \\
& +\tfrac{1}{36} D_k F_{d_1 l}\, D_l F_{d_2 d_3}\, W_{d_1 d_3 d_4 d_5} W_{d_2 k d_4 d_5} \\
& +\tfrac{7}{288} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_2 d_3 d_4} W_{d_3 d_4 k l} \\
& -\tfrac{7}{72} D_k F_{d_1}\, D_l F_{d_2}\, W_{d_1 d_3 d_2 d_4} W_{d_3 k d_4 l} \\
& + \text{lower-rank contractions}\,,
\end{aligned}
\tag{A.16}
$$

$$
\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{4}}}(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{+}})^2 = {}& \tfrac{1}{6}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_4d_3k}W_{d_2d_3d_4l} \\
&+\tfrac{1}{6}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_3d_4l}W_{d_2d_4d_3k} \\
&+\tfrac{1}{6}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_4d_3l}W_{d_2d_4d_3k} \\
&+\tfrac{1}{6}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_4d_3k}W_{d_2d_4d_3l} \\
&-\tfrac{1}{12}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_2d_3d_4}W_{d_3d_4kl} \\
&+\tfrac{1}{3}D_kF_{d_1}\,D_lF_{d_2}\,W_{d_1d_3d_2d_4}W_{d_3kd_4l} \\
&+\text{lower-rank contractions}\,,
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
W^2\Big|_{\widetilde{\ydiagram{2}}}(DF^+_{(5)}\Big|_{\widetilde{\ydiagram{2,1,1,1,1}}^{+}})^2 = {}& \tfrac{1}{144}D_kF\,D_lF\,W_{d_1kd_2d_3}W_{d_1ld_2d_3} \\
&+\tfrac{9}{144}D_lF_{d_1}\,D_lF_{d_2}\,W_{d_1d_3d_4d_5}W_{d_2d_3d_4d_5} \\
&+\tfrac{7}{36}D_kF_{d_1l}\,D_lF_{d_2k}\,W_{d_1d_3d_4d_5}W_{d_2d_3d_4d_5} \\
&-\tfrac{1}{144}D_kF_l\,D_lF_{d_1}\,W_{d_1d_2d_3d_4}W_{d_2kd_3d_4} \\
&-\tfrac{1}{144}D_kF_{d_1}\,D_lF_k\,W_{d_1d_2d_3d_4}W_{d_2ld_3d_4} \\
&+\text{lower-rank contractions}\,.
\end{aligned}
\tag{A.18}
$$

In the above expressions we have dropped all index pairs contracted between the two field strengths, so $F_{d_1d_2d_3d_4}F_{d_5d_6d_7d_8} \equiv F_{d_1d_2d_3d_4}{}^{m}F_{d_5d_6d_7d_8m}$, and so on. For ease of notation, we have also left out the labels indicating that $F_{(5)}$ is self-dual. As was the case for the amplitude trace, the $(DF^+_{(5)})^2$ tensors were obtained as the sum of parity-even and -odd parts after the extraction of a self-duality projector $\mathcal{P}_+ = \frac{1}{2}(\mathbb{1} + *)$ from each of the $F^+_{(5)}$ tensors.

In the actual computation, the invariants constructed from $DF^+_{(5)}$ in the fully antisymmetric representation are also required in order to complete the basis. This has enabled us to make a strong consistency check on the calculation, as it allowed us to verify that the amplitude can indeed be expanded in the basis of invariants predicted by group theory. Because such terms vanish identically on shell and therefore can be left out of the effective action, and because they are rather lengthy expressions, we do not list them here.

A.2 Fermionic world-sheet correlators

The general world-sheet correlator for fermions was derived by Atick and Sen [18]; it takes the form

$$
\begin{aligned}
&\Big\langle \prod_{i=1}^{N_1} S^+(\widetilde{y}_i) \prod_{i=1}^{N_2} S^-(y_i) \prod_{i=1}^{N_3} \bar{\Psi}(\widetilde{z}_i) \prod_{i=1}^{N_4} \Psi(z_i) \Big\rangle_\nu \\
&\qquad = K_\nu \frac{\prod\limits_{i<j} \theta_1(\widetilde{y}_i - \widetilde{y}_j)^{\frac{1}{4}} \prod\limits_{i<j} \theta_1(y_i - y_j)^{\frac{1}{4}} \prod\limits_{i<j} \theta_1(\widetilde{z}_i - \widetilde{z}_j) \prod\limits_{i<j} \theta_1(z_i - z_j)}{\prod\limits_{i,j} \theta_1(y_i - \widetilde{y}_j)^{\frac{1}{4}} \prod\limits_{i,j} \theta_1(z_i - \widetilde{z}_i)} \\
&\qquad\quad \times \frac{\prod\limits_{i,j} \theta_1(z_i - \widetilde{y}_j)^{\frac{1}{2}} \prod\limits_{i,j} \theta_1(\widetilde{z}_i - y_j)^{\frac{1}{2}}}{\prod\limits_{i,j} \theta_1(\widetilde{z}_i - \widetilde{y}_j)^{\frac{1}{2}} \prod\limits_{i,j} \theta_1(z_i - y_j)^{\frac{1}{2}}} \\
&\qquad\quad \times \theta_\nu \left(\frac{1}{2} \sum_i \widetilde{y}_i - \frac{1}{2} \sum_i y_i + \sum_i z_i - \sum_i \widetilde{z}_i \right) .
\end{aligned}
\tag{A.19}
$$

The normalisation constant K_ν is determined by taking limits of the insertion points such that all θ_1 functions reduce to poles or zeroes, for which one uses

$$
\lim_{z \to 0} \theta_1(z) = \theta_1'(0) \cdot z \,. \tag{A.20}
$$

Because of the operator product expansion, the total correlator in this limit should be equal to the product of these poles and zeroes, times the expectation value of the identity in each spin structure,

$$
\langle 1 \rangle_\nu = \left(\frac{\theta_\nu(0)}{\theta_1'(0)} \right)^4 . \tag{A.21}
$$

In the situations we analyse, the θ_ν factors of the fermionic correlator and the one from the ghost correlator conspire to give the $\theta_\nu(0)$. Therefore, the procedure described here fixes K_ν in terms of a power of $\theta_1'(0)$. In order to determine the overall sign, one has keep track of the ordering of the fermions, especially when converting the covariant expression to one written in terms of the five helicity basis fermions.

As an example, consider the fifth line of table 3. The relevant fermion correlator (multiplied with the one for the ghosts) is

$$
\begin{aligned}
&\sum_\nu (-)^{\nu-1} \Big\langle S_{\alpha_1}(z_1) S_{\alpha_2}(z_2) \Psi^1(z_3) \Psi^3(z_3) \Psi^3(z_4) \Psi^5(z_4) \Psi^5(v) \Big\rangle_\nu \, \mathcal{G}(z_1, z_2, v)_\nu \\
&\qquad = \sum_\nu (-)^{\nu-1} \Big\langle S_-(z_1) S_-(z_2) \big(\psi(z_3) + \bar{\psi}(z_3)\big) \Big\rangle_\nu \\
&\qquad\qquad \Big\langle S_+(z_1) S_-(z_2) \big(\psi(z_3) + \bar{\psi}(z_3)\big) \big(\psi(z_4) + \bar{\psi}(z_4)\big) \Big\rangle_\nu \\
&\qquad\qquad \Big\langle S_+(z_1) S_-(z_2) \big(\psi(z_4) + \bar{\psi}(z_4)\big) \big(\psi(v) + \bar{\psi}(v)\big) \Big\rangle_\nu \\
&\qquad\qquad \Big\langle S_+(z_1) S_-(z_2) \Big\rangle_\nu^2 \mathcal{G}(z_1, z_2, v)_\nu \,.
\end{aligned}
\tag{A.22}
$$

In the limit $z_1 \to z_2$ and $z_3 \to z_4$ one can use (A.20) to simplify the result obtained using (A.19). One can also analyse the expected behaviour of the amplitude by looking at the operator product expansion. Using the ghost correlator

$$\mathcal{G}(z_1, z_2, v)_\nu = \theta_1(v - z_1)^{\frac{1}{2}}\, \theta_1(v - z_2)^{\frac{1}{2}}\, \theta_1(z_2 - z_1)^{-\frac{1}{4}}\, \theta_\nu \left(\frac{z_1 - z_2}{2} \right)^{-1} , \tag{A.23}$$

one finds the limiting expression

$$\sum_\nu (-)^{\nu-1} \left(\frac{\theta_\nu(0)}{\theta_1'(0)} \right)^4 \frac{4\,(v - z_2)(z_2 - z_3)^2}{(z_3 - v)(z_1 - z_3)(z_3 - z_4)(z_2 - z_3)(z_1 - z_3)(z_1 - z_2)} \times \begin{cases} -K_\nu\, \theta_1'(0) & \text{from (A.19)}\,, \\ 1 & \text{from the OPEs}\,. \end{cases} \tag{A.24}$$

Comparing the two, one thus deduces that the normalisation constant is $K_\nu = -1/\theta_1'(0)$ (independent of the spin structure sector).

Having determined the normalisation, the sum over spin structures is then evaluated using the Riemann identity

$$\begin{aligned} \sum_{\nu=1,2,3,4} (-)^{\nu-1}\, \theta_\nu(z_1|\tau)\theta_\nu(z_2|\tau)\theta_\nu(z_3|\tau)\theta_\nu(z_4|\tau) &= 2\,\theta_1\left(\frac{z_1 + z_2 + z_3 + z_4}{2} \Big| \tau \right) \theta_1\left(\frac{z_1 + z_2 - z_3 - z_4}{2} \Big| \tau \right) \\ &\quad \times \theta_1\left(\frac{z_1 - z_2 - z_3 + z_4}{2} \Big| \tau \right) \theta_1\left(\frac{z_1 - z_2 + z_3 - z_4}{2} \Big| \tau \right) . \end{aligned} \tag{A.25}$$

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