Global Small Solutions of the Vlasov-Maxwell System in the Absence of Incoming Radiation

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Abstract

We consider a modified version of the Vlasov-Maxwell system in which the usual Maxwell fields are replaced by their retarded parts. We show that solutions of this modified system exist globally for a small initial density of particles and that they describe a system without incoming radiation.

1 Introduction and main result

The relativistic Vlasov-Maxwell system (RVM) models the dynamics of a plasma consisting of a large number of charged particles under the assumption that the particles interact only by the electrodynamic forces that the fields generate collectively. In particular, collisions between particles and external forces are assumed to be negligible.

Examples of physical systems which are thought to be well-modelled by RVM are the solar wind and the ionosphere.

Given the huge number of particles which form the plasma it should be hopeless to attempt to describe the state of the plasma by looking at the position and the velocity of each individual particle. Therefore a statistical description of the matter is needed. In the framework of kinetic theory the microscopic state of the plasma is described by specifying a distribution function in the phase space for each species of particle. Let us assume for simplicity that the plasma consists of a single species of particle with unit mass and charge and set also the speed of light equal to one (i.e. $c=1$). We denote by $f(t,x,p)$ the probability density to find a particle at time t at position x with momentum p, where $(t,x,p)\in\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_p$. Clearly, $f\geq 0$. The charge density and the current density of the plasma are given respectively by

$$\rho(t,x)=\int_{\mathbb{R}^3} dp\, f(t,x,p),\quad j(t,x)=\int_{\mathbb{R}^3} dp\,\widehat{p}f(t,x,p), \tag{1.1}$$

where we denoted by $\widehat{p}$ the relativistic velocity of a particle with momentum p, that is

$$\widehat{p} = \frac{p}{\sqrt{1+|p|^2}}. \tag{1.2}$$

The electromagnetic field (E,B) generated by the plasma solves the Maxwell equations

$$\begin{cases} \partial_t E = \partial_x \wedge B - 4\pi j, & \partial_x \cdot E = 4\pi\rho, \\ \partial_t B = -\partial_x \wedge E, & \partial_x \cdot B = 0. \end{cases} \tag{1.3}$$

The system is closed by requiring that f be a solution of the Vlasov continuity equation

$$\partial_t f + \widehat{p} \cdot \partial_x f + (E + \widehat{p} \wedge B) \cdot \partial_p f = 0. \tag{1.4}$$

The RVM system consists of the set of equations (1.1)–(1.4). A short survey on the initial value problem for this system will be given at the end of this introduction. For later convenience we recall here the definition of the total energy of a solution of RVM, which is $\mathcal{E}_{\mathrm{tot}}(t) = \mathcal{E}_{\mathrm{kin}}(t) + \mathcal{E}_{\mathrm{field}}(t)$, where $\mathcal{E}_{\mathrm{kin}}(t)$ is the kinetic energy of the particles,

$$\mathcal{E}_{\mathrm{kin}}(t) = \int dx \int dp \sqrt{1+|p|^2} f(t,x,p)$$

and $\mathcal{E}_{\mathrm{field}}(t)$ is the field energy,

$$\mathcal{E}_{\mathrm{field}}(t) = \frac{1}{2} \int dx \,(|E(t,x)|^2 + |B(t,x)|^2).$$

(In the previous definitions it is understood that the integrals are extended over $\mathbb{R}^3$). For smooth solutions of RVM the total energy is finite and conserved for all times provided it is finite at the time $t=0$ (cf. [2]).

In this paper we are interested in those solutions of RVM which are characterized by the property of being isolated from *incoming radiation*. Let us first discuss these solutions heuristically and then we will give their precise definition.

The radiation is defined as the part of the electromagnetic field which carries energy to *null infinity*, that is to that part of the infinity of the Minkowski space which is reached along the null and asymptotically null curves. The null infinity is distinguished in *future* null infinity, which is reached in the limit $t \to +\infty$, $|x| \to +\infty$, at constant *retarded* time $u = t - |x|$, and *past* null infinity, which is reached in the limit $t \to -\infty$, $|x| \to +\infty$, now at constant *advanced* time, $v = t + |x|$. Correspondingly one defines *outgoing* and *incoming* radiation to be the part of the electromagnetic field which propagates energy to future and past null infinity respectively.

Since RVM is symmetric with respect to the transformation $t \to -t$ (time reflection[1]), this system will contain in general outgoing as well as incoming radiation. In order to give a precise definition of solutions of RVM which do not contain incoming radiation, let us consider the energy $\mathcal{E}_{\mathrm{in}}(v_1, v_2)$ carried by the field to past null

[1] Namely, if $(f(t,x,p), E(t,x), B(t,x))$ is a solution, then $(f(-t,x,-p), E(-t,x), -B(-t,x))$ gives a new solution of RVM.

infinity in the interval $[v_1,v_2]$ of the advanced time. This quantity can be formally calculated by the limit

$$\mathcal{E}_{\mathrm{in}}(v_1,v_2)=-\lim_{r\to+\infty}\int_{v_1}^{v_2}dv\int_{|x|=r}dx\,(E\wedge B)\cdot\omega_{|t=v-r},$$

where $\omega=x/|x|$ and $E\wedge B$ is the Poynting vector (the minus sign comes from the convention to consider positive the flux of energy flowing in onto the system). We will say that a solution of RVM is isolated from incoming radiation if $\mathcal{E}_{\mathrm{in}}(v_1,v_2)=0$, for all $v_1,v_2\in\mathbb{R}$.

In this paper we are mainly concerned with the question whether the solutions of RVM isolated from incoming radiation are represented by the retarded solution of the equations. For this purpose we restrict ourselves to consider the system

$$\partial_t f+\widehat{p}\cdot\partial_x f+(E_{\mathrm{ret}}+\widehat{p}\wedge B_{\mathrm{ret}})\cdot\partial_p f=0, \tag{1.5}$$

$$E_{\mathrm{ret}}(t,x)=-\int\frac{dy}{|x-y|}(\partial_x\rho+\partial_t j)(t-|x-y|,y), \tag{1.6}$$

$$B_{\mathrm{ret}}(t,x)=\int\frac{dy}{|x-y|}(\partial_x\wedge j)(t-|x-y|,y), \tag{1.7}$$

where ρ and j are defined by (1.1). We will refer to the system (1.5)–(1.7) as the *retarded* relativistic Vlasov-Maxwell system, or $\mathrm{RVM}_{\mathrm{ret}}$ for short.

Let us briefly comment in which sense the solutions of $\mathrm{RVM}_{\mathrm{ret}}$ have to be considered as solutions of RVM. Assume first that f_{ret} is a C^1 global solution of $\mathrm{RVM}_{\mathrm{ret}}$ and that also $(E_{\mathrm{ret}},B_{\mathrm{ret}})$ is C^1. By means of (1.5), ρ and j satisfy the continuity equation, $\partial_t\rho+\partial_x\cdot j=0$, and therefore the retarded field *is* a solution of the Maxwell equations. Thus, $(f_{\mathrm{ret}},E_{\mathrm{ret}},B_{\mathrm{ret}})$ is a solution of RVM. The same is true if f_{ret} is a *semiglobal* solution of $\mathrm{RVM}_{\mathrm{ret}}$, i.e. a solution defined for $t\in(-\infty,T]$, where $T\in\mathbb{R}$. However it is clear that there is no meaningful notion of local solutions of $\mathrm{RVM}_{\mathrm{ret}}$. For the retarded field at a point (t,x) is obtained by integration over the *whole* past light cone with vertex in (t,x) (no initial data for the field are imposed) and so the field at time t is determined if and only if a solution has been constructed in the interval $(-\infty,t]$.

We can now state the main result of this paper. This is a global existence and uniqueness theorem for small data of solutions of $\mathrm{RVM}_{\mathrm{ret}}$ which we also show to be isolated from incoming radiation in the sense specified above.

Theorem 1 *Let $f^{\mathrm{in}}(x,p)\geq 0$ be given in $C^2_0(\mathbb{R}^3_x\times\mathbb{R}^3_p)$ and $R>0$ such that $f^{\mathrm{in}}(x,p)=0$ for $|x|^2+|p|^2\geq R^2$. Define*

$$\Delta=\sum_{|\mu|=0}^{2}\|\nabla^\mu f^{\mathrm{in}}\|_\infty,$$

where $\mu\in\mathbb{N}^6$ is a multi-index. Then there exists a constant $\varepsilon>0$ depending only on R such that for $\Delta\leq\varepsilon$, $\mathrm{RVM}_{\mathrm{ret}}$ has a unique C^1 global solution f_{ret} satisfying $f_{\mathrm{ret}}(0,x,p)=f^{\mathrm{in}}(x,p)$. Moreover $(E_{\mathrm{ret}},B_{\mathrm{ret}})\in C^1(\mathbb{R}_t\times\mathbb{R}^3_x)$ and there exists a positive *constant $C=C(R)$ such that the field satisfies the following estimates for all $(t,x)\in\mathbb{R}_t\times\mathbb{R}^3_x$:*

$$|E_{\rm ret}(t,x)|+|B_{\rm ret}(t,x)| \leq C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}, \qquad (1.8)$$
$$|DE_{\rm ret}(t,x)|+|DB_{\rm ret}(t,x)| \leq C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-7/4}, \qquad (1.9)$$

where D denotes any first order derivative. Moreover $(f_{\rm ret},E_{\rm ret},B_{\rm ret})$ is the unique solution of RVM which satisfies (1.8), (1.9) and $f_{\rm ret}(0,x,p)=f^{\rm in}(x,p)$.

The uniqueness assertion of theorem 1 will be made more precise in proposition 3 below. The fact that the solution of theorem 1 is isolated from incoming radiation is a consequence of the estimate (1.8).

This paper is organized as follows. In section 2 we recall a few facts on RVM which will be needed in the sequel. In section 3 we prove the main estimates on the retarded field and the uniqueness part of theorem 1. The existence part is proved in section 4. An appendix is devoted to the proof of two technical lemmas.

To conclude this introduction we mention some important results on the initial value problem for RVM. Existence of a unique solution for a short time has been proved in [17]. A unique global solution is shown to exist in [3] under the *a priori* assumption that there exists a function $\beta\in C^0(\mathbb{R})$ such that $\mathcal{P}(t)\leq\beta(t), \forall t\in\mathbb{R}$, where $\mathcal{P}(t)$ denotes the maximum momentum of the particles up to the time t, i.e.:

$$\mathcal{P}(t)=\sup_{0\leq s\leq t}\{|p|:f(s,x,p)\neq 0, \text{ for some } x\in\mathbb{R}^3\}.$$

A different proof of this result based on the Fourier transform was given recently in [9].

The result in [3] was applied to prove global existence and uniqueness under different smallness assumptions on the initial data (cf. [5, 6, 13]) and for arbitrarily large data in two space dimensions (i.e. $x\in\mathbb{R}^2$) in [7]. Existence, but not uniqueness, of global weak solutions is proved in [1].

The non-relativistic limit of RVM is the Vlasov-Poisson system (VP). For a single species of particles with unit positive charge and mass the VP system is given by

$$\partial_t f+v\cdot\partial_x f+\partial_x U\cdot\partial_v f=0,$$

$$\Delta U=4\pi\rho,$$

where U is the electrostatic potential, v the classical velocity of the particles, $f=f(t,x,v)$ and $\rho=\int dv\, f$. The initial value problem for VP has been proved to be correctly set for general initial data in [11, 12] (cf. also [14, 16]) and the convergence of solutions of RVM to solutions of VP, when the speed of light tends to infinity, has been established rigorously in [15]. The *a priori* estimates proved in [8] show that the solutions of VP do not contain radiation. In order to measure an energy lost to infinity for VP (in a non-relativistic sense, i.e. at *spacelike* infinity) an extra dipole term has to be added into the equations (cf. [10]).

2 Preliminary results

In this section we recall some well-known results on RVM which will be used later on. We start by fixing a bit of notation. The symbol T will denote the *free transport* operator, that is

$$T=\partial_t+\widehat{p}\cdot\partial_x.$$

We denote by C a generic constant which may change from line to line but which depends only on R. If a constant depends on R and on other parameters, it will be denoted by C_*. The partial derivative with respect to x_i $(i=1,2,3)$ will be denoted by ∂_{x_i}, while any derivative of order k with respect to t and/or x will be denoted by D^k (namely, $Dg=\partial_t g$ or $\partial_{x_i}g$, $D^2g=\partial_t^a\partial_{x_i}^b\partial_{x_j}^c g$, $a+b+c=2$ and so on, with the convention $D^0g=g$). The L^∞ norm of a function $g(x_1,...,x_n)$ with respect to the variables $x_{k+1},...,x_n$ will be denoted by $\|g(x_1,...,x_k)\|_\infty$. The L^p norm is denoted by $\|\cdot\|_{L^p}$. The notation $\|\cdot\|_w$ is used for the norm defined in section 4 below (cf. (3.4)). We also set $F=(E,B)$.

The Vlasov equation can be reduced to a system of ordinary differential equations by using the method of characteristics. Consider the following "initial" value problem for the function $(X,P):\mathbb{R}_s\to\mathbb{R}^6$:

$$\frac{dX}{ds}=\widehat{P}, \tag{2.1}$$

$$\frac{dP}{ds}=E(s,X)+\widehat{P}\wedge B(s,X), \tag{2.2}$$

$$(X(t),P(t))=(x,p). \tag{2.3}$$

Let $(X(s,t,x,p),P(s,t,x,p))$ denote the solution of the previous problem (sometimes it will be denoted by $(X(s),P(s))$ for short). Then the solution of the Vlasov equation is given by

$$f(t,x,p)=f^{\rm in}(X(0,t,x,p),P(0,t,x,p)). \tag{2.4}$$

By (2.4), $\mathrm{supp}[f(t)]=\{(x,p):f(t,x,p)\neq 0\}\subseteq\Xi(t)$ where

$$\Xi(t)=\{(x,p)\in\mathbb{R}^3_x\times\mathbb{R}^3_p \text{ s.t. } |X(0,t,x,p)|^2+|P(0,t,x,p)|^2\leq R^2\}.$$

Moreover, since the characteristics flow preserves the Lebesgue measure, the L^p norm in phase space of the particle density is conserved:

$$\|f(t)\|_{L^p}=\|f^{\rm in}\|_{L^p},\quad \forall 1\leq p\leq\infty, t\in\mathbb{R}. \tag{2.5}$$

We also recall the following

Definition 1 *A solution (f,F) of RVM is said to satisfy the "Free Streaming Condition" (FSC) with respect to the constant $\eta>0$ if there exists $\alpha>\frac{1}{2}$ such that*

$$|F(t,x)|\leq\eta(1+|t|+|x|)^{-\alpha}(1+R+|t|-|x|)^{-\alpha},$$
$$|\partial_x F(t,x)|\leq\eta(1+|t|+|x|)^{-\alpha}(1+R+|t|-|x|)^{-\alpha-1},$$

for $t\in\mathbb{R}$ and $|x|\leq R+|t|$.

The following lemma contains some estimates on the characteristics which are due to FSC.

Lemma 1 *There exists a constant $\eta_0 > 0$ such that if (f,F) is a C^1 solution of RVM which satisfies FSC with respect to $\eta \le \eta_0$, then for all $(x,p) \in \Xi(t)$ and $t \in \mathbb{R}$:*

$$\mathcal{P}(t) \le 2R, \tag{2.6}$$

$$|\partial_x (X,P)(0,t,x,p)| \le C, \tag{2.7}$$

$$|\partial_p (X,P)(0,t,x,p)| \le C(1+|t|). \tag{2.8}$$

Moreover for all $(x,p_i) \in \Xi(t)$ (i=1,2) and $t \in \mathbb{R}$:

$$|X(0,t,x,p_1) - X(0,t,x,p_2)| \ge C|p_1 - p_2|\,|t|. \tag{2.9}$$

Proof: The estimates (2.6) and (2.9) are proved for example in [6], lemmas 1 and 2. The proof of (2.8) is identical to the one of (2.7) and the latter is given in lemma 5.6 of [13]. □

We will use repeatedly the following consequence of (2.6). Assume that $\mathcal{P}(t) \le \beta, \forall t \in \mathbb{R}$, for some positive constant β. Then

$$|X(s,t,x,p)| \le R + a(\beta)|s|, \quad \forall (x,p) \in \Xi(t), \forall (s,t) \in \mathbb{R}^2, \tag{2.10}$$

where

$$a(\beta) = \beta / \sqrt{1+\beta^2} < 1. \tag{2.11}$$

In fact by (2.1), say for $s \ge 0$, $|X(s)| \le R + \sup_{0 \le \tau \le s} |\widehat{P}(\tau,t,x,p)|s$. Moreover

$$|\widehat{P}|^2 = 1 - \frac{1}{1+|P|^2} \le 1 - \frac{1}{1+\beta^2} = \frac{\beta^2}{1+\beta^2} = a(\beta)^2$$

and therefore $\sup_{0 \le \tau \le s} |\widehat{P}(\tau,t,x,p)| \le a(\beta)$. In particular, setting $s = t$ in (2.10),

$$f(t,x,p) = 0, \text{ for } |x| \ge R + a(\beta)|t|. \tag{2.12}$$

Corollary 1 *Under the assumptions of lemma 1,*

$$\|\partial_x^j \partial_p^k f(t)\|_\infty \le C \|\nabla f^{\mathrm{in}}\|_\infty (1+|t|)^k, \quad j+k = 1, \tag{2.13}$$

$$\mathrm{Vol}[\mathrm{supp}\, f(t,x,\cdot)] \le C(1+|t|+|x|)^{-3}. \tag{2.14}$$

Proof: The estimates (2.13) follow directly from (2.4), (2.7), (2.8). For (2.14) define $\mathcal{A}(t,x) = \mathrm{supp} f(t,x,\cdot) = \{p \in \mathbb{R}^3 : f(t,x,p) \ne 0\}$. By (2.4) and (2.6) we have

$$\mathcal{A}(t,x) \subseteq \{p : |p| \le 2R\} \cap \{p : |X(0,t,x,p)| \le R\} = \mathcal{U} \cap \mathcal{V}.$$

For $|t| \le 1$ we use that $\mathrm{Vol}[\mathcal{A}(t,x)] \le \mathrm{Vol}[\mathcal{U}] \le C \le C(1+|t|)^{-3}$. For $|t| > 1$ we use that $\mathrm{Vol}[\mathcal{A}(t,x)] \le \mathrm{Vol}[\mathcal{V}]$. If $p_1, p_2 \in \mathcal{V}$ and $\eta \le \eta_0$ then, by inequality (2.9),

$$C|p_1 - p_2|\,|t| \le |X(0,t,x,p_1) - X(0,t,x,p_2)| \le 2R.$$

This means that the set $\mathcal{V}$ is contained in a ball with radius $C|t|^{-1}$, whose volume is then bounded by $C(1+|t|)^{-3}$. Moreover, for $|x|\le R+|t|$ we have also $C(1+|t|)^{-3}\le C(1+|t|+|x|)^{-3}$ and so (2.14) is proved. □

A key ingredient in the proof of theorem 1 is the analogue for the retarded solution of the integral representation formulae for the field and the gradient of the field which have been proved in [3]. We denote by $K_{\rm ret}$ the Lorentz force, $K_{\rm ret}=E_{\rm ret}+\widehat{p}\wedge B_{\rm ret}$.

Lemma 2 *Assume $\mathcal{P}(t)\le\beta$ for some positive constant β. Then there exist two smooth functions $\mathfrak{a}_1$, $\mathfrak{a}_2$ uniformly bounded in the support of f such that*

$$\begin{aligned}E_{\rm ret}(t,x)= &-\int_{\Omega_a}\frac{dy}{|x-y|^2}\int_{|p|\le\beta}dp\,\mathfrak{a}_1(\omega,p)f(t-|x-y|,y,p)\\ &-\int_{\Omega_a}\frac{dy}{|x-y|}\int_{|p|\le\beta}dp\,\mathfrak{a}_2(\omega,p)K_{\rm ret}f(t-|x-y|,y,p),\end{aligned}\tag{2.15}$$

where $\omega=(y-x)/|y-x|$, $a=a(\beta)$ is given by (2.11) and Ω_a denotes the set

$$\Omega_a(t,x)=\{y\in\mathbb{R}^3:|y|\le R+a|t-|x-y||\}.$$

An analogous representation formula with two slightly different bounded kernels holds also for $B_{\rm ret}$.

Sketch of the proof: The proof of lemma 2 is identical to the one of theorem 3 in [3], the kernels of the integral representations being also the same. We give here the idea of the proof for sake of completeness. By (1.6) and (1.1) we have

$$E^i_{\rm ret}(t,x)=-\int_{\Omega_a}\frac{dy}{|x-y|}\int_{|p|\le\beta}dp\,(\partial_{y_i}f+\widehat{p}_i\partial_t f)(t-|x-y|,y,p).\tag{2.16}$$

To justify that the integral w.r.t. y in (2.16) is extended over the set Ω_a we notice that, by (2.12), $f(t-|x-y|,y,p)=0$ for $|y|\ge R+a|t-|x-y||$. In particular, since $a<1$, then $\Omega_a(t,x)$ is bounded for any fixed $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. Now we express $\partial_{y_i}f(t-|x-y|,y,p)$ and $\partial_t f(t-|x-y|,y,p)$ in terms of the perfect derivatives of $f(t-|x-y|,y,p)$ via the identities

$$\begin{aligned}\partial_{y_i}f(t-|x-y|,y,p) &= \omega_i(1+\omega\cdot\widehat{p})^{-1}Tf(t-|x-y|,y,p)\\ &\quad+\left(\delta_{ik}-\frac{\omega_i\widehat{p}_k}{1+\omega\cdot\widehat{p}}\right)\partial_{y_k}[f(t-|x-y|,y,p)],\end{aligned}\tag{2.17}$$

$$\begin{aligned}\partial_t f(t-|x-y|,y,p) &= (1+\omega\cdot\widehat{p})^{-1}Tf(t-|x-y|,y,p)\\ &\quad-\frac{\widehat{p}_k}{1+\omega\cdot\widehat{p}}\partial_{y_k}[f(t-|x-y|,y,p)].\end{aligned}\tag{2.18}$$

We substitute (2.17) and (2.18) into (2.16) and integrate by parts. Since f vanishes on the boundary of Ω_a, we obtain

$$E_{\rm ret}(t,x)=-\int_{\Omega_a}\frac{dy}{|x-y|^2}\int_{|p|\le\beta}dp\,\mathfrak{a}_1(\omega,p)f(t-|x-y|,y,p)$$

$$- \int_{\Omega_a} \frac{dy}{|x-y|} \int_{|p|\le\beta} dp\, \mathfrak{b}(\omega,p) Tf(t-|x-y|,y,p), \tag{2.19}$$

where

$$\mathfrak{a}_1(\omega,p) = \frac{\omega+\widehat{p}}{(1+p^2)(1+\omega\cdot\widehat{p})^2}, \tag{2.20}$$

$$\mathfrak{b}(\omega,p) = \frac{\omega+\widehat{p}}{1+\omega\cdot\widehat{p}}. \tag{2.21}$$

By (1.5), $Tf = -K_{\mathrm{ret}}\cdot\nabla_p f$. Substituting into (2.19) and integrating by parts in p, we get (2.15) with $\mathfrak{a}_2 = \partial_p \mathfrak{b}$ (again, since f vanishes for $|p|=\beta$ there are no boundary terms). The kernels $\mathfrak{a}_1$, $\mathfrak{a}_2$ are bounded by $C\sqrt{1+p^2}$ (see [4]). Thus in the present case, because of our assumption $\mathcal{P}(t)\le\beta$, they are uniformly bounded. □

The following lemma contains the analogous representation for the derivatives of the retarded field and corresponds to theorem 4 of [3]:

Lemma 3 *Assume $\mathcal{P}(t)\le\beta$ for some positive constant β. Then there exist smooth functions $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ uniformly bounded in the support of f such that*

$$\begin{aligned} DE_{\mathrm{ret}}(t,x) = &- \int_{\Omega_a} \frac{dy}{|x-y|^3} \int_{|p|\le\beta} dp\, \mathfrak{b}_1(\omega,p) f(t-|x-y|,y,p) \\ &- \int_{\Omega_a} \frac{dy}{|x-y|^2} \int_{|p|\le\beta} dp\, \mathfrak{b}_2(\omega,p) K_{\mathrm{ret}} f(t-|x-y|,y,p) \\ &- \int_{\Omega_a} \frac{dy}{|x-y|} \int_{|p|\le\beta} dp\, \mathfrak{b}_3(\omega,p) D(K_{\mathrm{ret}} f)(t-|x-y|,y,p). \end{aligned} \tag{2.22}$$

Moreover the kernel $\mathfrak{b}_1(\omega,p)$ satisfies

$$\int_{S^2} \mathfrak{b}_1(\omega,p)\, d\omega = 0. \tag{2.23}$$

The derivatives of B_{ret} admit a similar representation with three different bounded kernels $\mathfrak{b}_1'$, $\mathfrak{b}_2'$, $\mathfrak{b}_3'$ and $\mathfrak{b}_1'$ also satisfies (2.23).

Sketch of the proof: Let I_1, I_2 denote the two integrals in (2.15). By differentiating I_2 we obtain the third term in (2.22) with $\mathfrak{b}_3 = \mathfrak{a}_2$. Differentiating I_1 we get

$$DI_1(t,x) = -\int_{\Omega_a} \frac{dy}{|x-y|^2} \int_{|p|\le\beta} dp\, \mathfrak{a}_1(\omega,p) Df(t-|x-y|,y,p).$$

The absence of boundary terms is again due to the fact that f vanishes on the boundary of Ω_a. In the previous expression we use again (2.17), (2.18) and then integrate by parts. We end up with (2.22) after defining properly the various kernels. The exact form of the latter quantities is given in [3] but here it is not important; the crucial point is that the kernels are uniformly bounded for $|p|\le\beta$. The identity (2.23) is proved in [3]. □

3 Estimates on the retarded field and uniqueness

All the estimates in this paper will be based on the following two lemmas:

Lemma 4 *Let $I_n^q(t,x)$ ($n=1,2$; $q\in\mathbb{R}$) denote the integral*

$$I_n^q(t,x)=\int \frac{dy}{|x-y|^n}(1+|t-|x-y||+|y|)^{-q}.$$

Then for all $(t,x)\in\mathbb{R}_t\times\mathbb{R}_x^3$ the following estimates hold:

$$I_1^q \le C(1+|t|+|x|)^{-1}(1+|t-|x||)^{-q+3}, \quad q>3,$$
$$I_2^q \le C(1+|t|+|x|)^{-1}(1+|t-|x||)^{-q+2}, \quad q>2.$$

Lemma 5 *Assume $g\in C^1\cap L^\infty(\mathbb{R}_t\times\mathbb{R}_x^3\times\mathbb{R}_p^3)$ and vanishes for $|p|\ge\beta$. Assume also that*

$$\mathrm{Vol}[\mathrm{supp}\, g(t,x,\cdot)]\le C(1+|t|+|x|)^{-3}, \tag{3.1}$$

$$Dg\in L^\infty(\mathbb{R}_t\times\mathbb{R}_x^3\times\mathbb{R}_p^3). \tag{3.2}$$

Let $\mathfrak{b}_1(\omega,p)$ be smooth and satisfy (2.23). Then the integral

$$I(t,x)=\int\frac{dy}{|x-y|^3}\int_{|p|\le\beta} dp\,\mathfrak{b}_1(\omega,p)g(t-|x-y|,y,p), \tag{3.3}$$

satisfies the estimate

$$|I(t,x)|\le C_*(\|g\|_\infty+\|Dg\|_\infty)(1+|t|+|x|)^{-1}(1+|t-|x||)^{-7/4},$$

for all $(t,x)\in\mathbb{R}_t\times\mathbb{R}_x^3$, where $C_*=C_*(\beta)$.

The quite long and technical proofs of lemmas 4 and 5 are postponed in appendix.

We denote by $\|F\|_w$ the weighted norm:

$$\|F\|_w=\sup_{t,x}[(1+|t|+|x|)(1+|t-|x||)^w|F(t,x)|], \tag{3.4}$$

where $w>0$ and set $F_{\mathrm{ret}}=(E_{\mathrm{ret}},B_{\mathrm{ret}})$. In the following two propositions we estimate the retarded field generated by a solution f_{ret} of $\mathrm{RVM}_{\mathrm{ret}}$ with initial data and regularity as stated in theorem 1.

Proposition 1 *Assume $\mathcal{P}(t)\le\beta$ and (3.1) holds for $g\equiv f_{\mathrm{ret}}$. Then there exists a constant $\varepsilon>0$ which depends on R and β such that for $\|f^{\mathrm{in}}\|_\infty\le\varepsilon$ the retarded field satisfies the estimate*

$$\|F_{\mathrm{ret}}\|_1\le C_*\|f^{\mathrm{in}}\|_\infty, \tag{3.5}$$

where $C_*=C_*(R,\beta)$.

Proof: Using (3.1) to estimate (2.15) we get, with the notation of lemma 4,

$$|E_{\rm ret}(t,x)| \le C_* \|f^{\rm in}\|_\infty I_2^3(t,x) + C_* \|f^{\rm in}\|_\infty \int_{\Omega_a} \frac{dy}{|x-y|} \frac{|F_{\rm ret}(t-|x-y|,y)|}{(1+|t-|x-y||+|y|)^3}.$$

An analogous estimate holds for $B_{\rm ret}$ and so we have

$$\begin{aligned} |F_{\rm ret}(t,x)| &\le C_* \|f^{\rm in}\|_\infty I_2^3(t,x) + C_* \|f^{\rm in}\|_\infty \int_{\Omega_a} \frac{dy}{|x-y|} \frac{|F_{\rm ret}(t-|x-y|,y)|}{(1+|t-|x-y||+|y|)^3} \\ &\le C_* \|f^{\rm in}\|_\infty I_2^3(t,x) + C_* \|f^{\rm in}\|_\infty \|F_{\rm ret}\|_1 \\ &\quad \times \int_{\Omega_a} \frac{dy}{|x-y|} (1+|t-|x-y||+|y|)^{-4} (1+|t-|x-y|-|y||)^{-1} \\ &\le C_* \|f^{\rm in}\|_\infty I_2^3(t,x) + C_* \|f^{\rm in}\|_\infty \|F_{\rm ret}\|_1 I_1^5(t,x). \end{aligned}$$

Here we used that

$$(1+|t-|x-y|-|y||)^{-1} \le C_* (1+|t-|x-y||+|y|)^{-1} \tag{3.6}$$

holds for $y \in \Omega_a$. In fact

$$\begin{aligned} \frac{1+|t-|x-y||+|y|}{1+|t-|x-y|-|y||} &\le (1+2R) \frac{1+|t-|x-y||+|y|}{1+2R+|t-|x-y|-|y||} \\ &\le (1+2R) \frac{1+R+(1+a)|t-|x-y||}{1+R+(1-a)|t-|x-y||} \\ &\le 2\left(\frac{1+2R}{1-a}\right) = C_*. \end{aligned}$$

Hence, by lemma 4

$$(1+|t|+|x|)(1+|t-|x||)|F_{\rm ret}(t,x)| \le C_* \|f^{\rm in}\|_\infty + C_* \|f^{\rm in}\|_\infty \|F_{\rm ret}\|_1$$

and so $(1-C_*\|f^{\rm in}\|_\infty)\|F_{\rm ret}\|_1 \le C_* \|f^{\rm in}\|_\infty$, which implies (3.5) for $\|f^{\rm in}\|_\infty \le 1/2C_*$. □

By the same argument we can prove the following *a priori* estimate on the derivatives of the field.

Proposition 2 *Assume $\mathcal{P}(t) \le \beta$ and (3.1), (3.2) hold for $g \equiv f_{\rm ret}$. Then for a proper small $\|f^{\rm in}\|_\infty$, the retarded field satisfies*

$$\|DF_{\rm ret}\|_{7/4} \le C_* z, \tag{3.7}$$

where $z = (1+\|f^{\rm in}\|_\infty)(\|f^{\rm in}\|_\infty + \|Df\|_\infty)$ and $C_* = C_*(R,\beta)$.

Proof: By (2.22) we have,

$$\begin{aligned} |DE_{\rm ret}(t,x)| \le\ & |I| + |II| + |III| \\ & + C_* \|f\|_\infty \int_{\Omega_a} \frac{dy}{|x-y|} \frac{|DF_{\rm ret}(t-|x-y|,y)|}{(1+|t-|x-y||+|y|)^3}, \end{aligned}$$

where I is the integral (3.3) with $g \equiv f_{\rm ret}$ and

$$II(t,x) = \int_{\Omega_a} \frac{dy}{|x-y|^2} \int_{|p|\le\beta} dp\, \mathfrak{b}_2(\omega,p) K_{\rm ret} f(t-|x-y|,y,p),$$
$$III(t,x) = \int_{\Omega_a} \frac{dy}{|x-y|} \int_{|p|\le\beta} dp\, \mathfrak{b}_3(\omega,p) K_{\rm ret} Df(t-|x-y|,y,p).$$

To estimate II and III we use (3.5) and (3.6). So doing we get

$$II(t,x) \le C_* \|f^{\rm in}\|_\infty^2 I_2^5(t,x), \quad III(t,x) \le C_* \|f^{\rm in}\|_\infty \|Df\|_\infty I_1^5(t,x)$$

and therefore, using lemmas 4 and 5,

$$\begin{aligned} |DF_{\rm ret}(t,x)| &\le C_* z (1+|t|+|x|)^{-1} (1+|t-|x||)^{-7/4} \\ &\quad + C_* \|f^{\rm in}\|_\infty \int_{\Omega_a} \frac{dy}{|x-y|} \frac{|DF_{\rm ret}(t-|x-y|,y)|}{(1+|t-|x-y||+|y|)^3} \\ &\le \frac{C_* z}{(1+|t|+|x|)(1+|t-|x||)^{7/4}} + C_* \|f^{\rm in}\|_\infty \|DF_{\rm ret}\|_{7/4} I_1^{23/4}. \end{aligned}$$

Hence, by lemma 4,

$$\|DF_{\rm ret}\|_{7/4} (1 - C_* \|f^{\rm in}\|_\infty) \le C_* z,$$

which concludes the proof. □

We also notice that (3.5), (3.7) implies FSC w.r.t. $\eta = C_* z$. In particular for the approximation sequence defined in section 4 below we will have $\eta = C\Delta$ for a proper small Δ.

To conclude this section we prove the uniqueness part of theorem 1:

Proposition 3 *Let $f^{\rm in}(x,p) \ge 0$ be given in $C_0^1(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ and consider the following class of solutions of RVM:*

$$\begin{aligned} \mathfrak{D}(f^{\rm in},\eta) = \big\{ (f,F) \in C^1 : \; & f(0,x,p) = f^{\rm in}(x,p), \\ & \|f^{\rm in}\|_\infty + \|\nabla f^{\rm in}\|_\infty \le \eta, \\ & (f,F) \text{ satisfies FSC w.r.t } \eta, \\ & F(t,\cdot) \in L^2(\mathbb{R}^3), \|F(t,\cdot)\|_{L^2} \to 0 \text{ as } t \to -\infty \big\}. \end{aligned}$$

Then there exists a positive constant η_0 such that for $\eta \le \eta_0$ either $\mathfrak{D}(f^{\rm in},\eta)$ is empty or it contains only one element.

Proof: Let (f_1,E_1,B_1) and (f_2,E_2,B_2) be two solutions of RVM in $\mathfrak{D}(f^{\rm in},\eta)$ and put $\delta f = (f_1 - f_2)$, $\delta E = (E_1 - E_2)$, $\delta B = (B_1 - B_2)$. Then $(\delta f, \delta E, \delta B)$ satisfies the system

$$\partial_t \delta f + \widehat{p} \cdot \partial_x \delta f + (E_1 + \widehat{p} \wedge B_1) \cdot \partial_p \delta f = -(\delta E + \widehat{p} \wedge \delta B) \cdot \partial_p f_2, \tag{3.8}$$

$$\begin{cases} \partial_t \delta E = \partial_x \wedge \delta B - 4\pi \delta j, & \partial_x \cdot \delta E = 4\pi \delta\rho, \\ \partial_t \delta B = -\partial_x \wedge \delta E, & \partial_x \cdot \delta B = 0, \end{cases} \tag{3.9}$$

with initial data $\delta f(0,x,p)=0$ and where $\delta\rho=\int dp\,\delta f$, $\delta j=\int dp\,\widehat{p}\,\delta f$. Our aim is to show that $\delta f=\delta E=\delta B\equiv 0$. However we remark at this point that it is sufficient to prove this for $t\le 0$. For, if the uniqueness holds in the past, then (f_i,E_i,B_i), $i=1,2$, will be solutions of RVM with the same initial data and then, since for a proper small η the estimate $\mathcal{P}(t)\le 2R$ is satisfied for all $t\ge 0$ (see lemma 1), the uniqueness in the future follows by [3]. Hence we assume $t\le 0$ in the rest of the proof. The L^2 solution $\delta F=(\delta E,\delta B)$ of (3.9) which satisfies $\|\delta F(t,\cdot)\|_{L^2}\to 0$ for $t\to-\infty$ is unique, because the L^2 norm of a solution of the *homogeneous* Maxwell equations is constant. We claim that, for a proper small η, this solution is represented by

$$\delta E(t,x) = -\int \frac{dy}{|x-y|}(\partial_x\delta\rho+\partial_t\delta j)(t-|x-y|,y), \tag{3.10}$$

$$\delta B(t,x) = \int \frac{dy}{|x-y|}(\partial_x\wedge\delta j)(t-|x-y|,y). \tag{3.11}$$

(Note that (3.10), (3.11) define a solution of (3.9) because $\delta\rho,\delta j$ satisfy the continuity equation $\partial_t\delta\rho+\partial_x\cdot\delta j=0$ as a consequence of (3.8)). To this purpose we first note that $\|\delta f(t)\|_\infty\le 2\|f^{\rm in}\|_\infty$ and that, for a proper small η,

$$\delta f(t,x,p)=0, \quad \text{for } |x|\ge R-a(2R)t, \tag{3.12}$$

$$\mathrm{Vol}[\mathrm{supp}\,\delta f(t,x,\cdot)]\le C(1-t+|x|)^{-3}, \tag{3.13}$$

$$\|\partial_x^j\partial_p^k\delta f(t)\|_\infty\le C\|\nabla f^{\rm in}\|_\infty(1-t)^k, \quad j+k=1, \tag{3.14}$$

cf. (2.12) and corollary 1. Moreover, the function (3.10) admits an integral representation formula similar to that one given in lemma 2:

$$\begin{aligned}\delta E(t,x) = &-\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)\delta f(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{b}(\omega,p)T\delta f(t-|x-y|,y,p),\end{aligned} \tag{3.15}$$

cf. (2.19) (it is understood that the integrals in y are over Ω_a and the ones in p are over $\{|p|\le 2R\}$). By (3.8) we have

$$T\delta f=-(E_1+\widehat{p}\wedge B_1)\cdot\partial_p\delta f-(\delta E+\widehat{p}\wedge\delta B)\cdot\partial_p f_2.$$

Substituting into (3.15) and integrating by parts in p we get

$$\begin{aligned}\delta E(t,x) = &-\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)\delta f(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)(E_1+\widehat{p}\wedge B_1)\delta f(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)(\delta E+\widehat{p}\wedge\delta B)f_2(t-|x-y|,y,p)\\ =\,& I+II+III,\end{aligned} \tag{3.16}$$

where $\mathfrak{a}_2=\partial_p\mathfrak{b}$ (cf. (2.21)). By (3.13), the integral I is bounded by

$$|I(t,x)|\le C\|\delta f\|_\infty I_2^3(t,x)\le C\|f^{\rm in}\|_\infty(1-t+|x|)^{-2}.$$

To estimate $II(t,x)$ we use that for $y\in\Omega_a$ the free streaming condition in the past gives

$$\begin{aligned}|(E_1+\widehat{p}\wedge B_1)| &\le \eta(1-t+|x-y|+|y|)^{-\alpha}(1+R-t+|x-y|-|y|)^{-\alpha}\\ &\le C(1-t+|x-y|+|y|)^{-2\alpha}\le C(1-t+|x-y|+|y|)^{-1},\end{aligned}$$

since $\alpha>\frac{1}{2}$. The same applies for $\delta E+\widehat{p}\wedge\delta B$ in $III(t,x)$ and so we get

$$|II(t,x)|+|III(t,x)|\le C(\|\delta f\|_\infty+\|f^{\rm in}\|_\infty)I_1^4(t,x)\le C\|f^{\rm in}\|_\infty(1-t+|x|)^{-2}.$$

Substituting these estimates into (3.16) and using the same argument for δB we get

$$|\delta F|\le C\|f^{\rm in}\|_\infty(1-t+|x|)^{-2}$$

and so $\|\delta F(t,\cdot)\|_{L^2}\to 0$ as $t\to-\infty$, as we claimed. We are able now to complete the proof of proposition 3. Let us introduce

$$\|\delta F\|'=\sup_{x;t\le 0}[(1-t+|x|)|\delta F(t,x)|].$$

By (3.16) we have

$$|\delta E(t,x)|\le C(\|\delta f\|_\infty+\|f^{\rm in}\|_\infty\|\delta F\|')(1-t+|x|)^{-2}. \tag{3.17}$$

On the other hand, integrating (3.8) along the characteristics of the Vlasov equation and using (3.14) we get

$$\|\delta f\|_\infty\le C\|\nabla f^{\rm in}\|_\infty\|\delta F\|'(1-t). \tag{3.18}$$

Combining (3.17) and (3.18) we get

$$|\delta E(t,x)|\le C[\|\nabla f^{\rm in}\|_\infty+\|f^{\rm in}\|_\infty]\|\delta F\|'(1-t+|x|)^{-1}.$$

Hence, from the analogous estimate on δB we find

$$\|\delta F\|'\le C\eta\|\delta F\|_0'$$

which entails $\|\delta F\|'=0$ for $\eta<C^{-1}$ and thus $\delta F=\delta f=0$. □

We remark that the meaning of the last condition in the definition of $\mathfrak{D}(f^{\rm in},\eta)$ is that all the energy is contained in the particles in the limit $t\to-\infty$. However this energy is not carried to past null infinity since the particles always move, even asymptotically, at velocities strictly smaller than the speed of light. The solution of theorem 1 belongs to the class $\mathfrak{D}(f^{\rm in},C\Delta)$ and therefore, for a proper small Δ, it is unique in this class.

4 Proof of existence

The existence part of theorem 1 is proved by a standard recursive argument which we split in three steps:

Step 1: The approximation sequence

We define: $f_1(t,x,p) = f^{\text{in}}(x - \widehat{p}t, p)$,

$$E_1(t,x) = \int \frac{dy}{|x-y|}[-\partial_x \rho_1 - \partial_t j_1](t - |x-y|, y),$$
$$B_1(t,x) = \int \frac{dy}{|x-y|}[\partial_x \wedge j_1](t - |x-y|, y),$$

where $\rho_1 = \int dp f_1$, $j_1 = \int dp \widehat{p} f_1$ and set $F_1 = (E_1, B_1)$. This solution corresponds to the case in which the particles do not interact with the field, i.e. the force term in (1.5) is omitted. Now, supposing that f_n is been defined, we build ρ_n, j_n, E_n, B_n via the formulae $\rho_n = \int dp f_n$, $j_n = \int dp \widehat{p} f_n$,

$$E_n(t,x) = \int \frac{dy}{|x-y|}[-\partial_x \rho_n - \partial_t j_n](t - |x-y|, y),$$
$$B_n(t,x) = \int \frac{dy}{|x-y|}[\partial_x \wedge j_n](t - |x-y|, y)$$

and put $F_n = (E_n, B_n)$. Now consider the following initial value problem for the function $(X,P): \mathbb{R}_s \to \mathbb{R}^6$:

$$\frac{dX}{ds} = \widehat{P}, \quad \frac{dP}{ds} = E_n + \widehat{P} \wedge B_n, \tag{4.1}$$
$$(X(t), P(t)) = (x,p). \tag{4.2}$$

Let $(X_{n+1}(s,t,x,p), P_{n+1}(s,t,x,p))$ denote the classical solution of the previous problem (sometimes it will be denoted by $(X_{n+1}(s), P_{n+1}(s))$ for short) and define f_{n+1} as

$$f_{n+1}(t,x,p) = f^{\text{in}}(X_{n+1}(0,t,x,p), P_{n+1}(0,t,x,p)). \tag{4.3}$$

f_{n+1} solves the following *linear* equation:

$$\partial_t f_{n+1} + \widehat{p} \cdot \partial_x f_{n+1} + (E_n + \widehat{p} \wedge B_n) \cdot \partial_p f_{n+1} = 0, \tag{4.4}$$

with initial datum $f_{n+1}(0,x,p) = f^{\text{in}}(x,p)$. The following lemma is easily proved by induction:

Lemma 6 *For a proper small Δ, the sequence (f_n, F_n) is constituted by C^2 functions and the following estimates hold $\forall n \in \mathbb{N}$: For $j,k \in \{0,1,2\}$, $0 \le j+k \le 2$,*

$$\|D^j \partial_p^k f_n(t)\|_\infty \le C\Delta(1+|t|)^k, \quad \forall t \in \mathbb{R}; \tag{4.5}$$

for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^3_x$:

$$\mathcal{P}_n(t) = \sup_{0 \le s \le t} \{|p| : f_n(s,x,p) \ne 0, \text{ for some } x\} \le 2R, \tag{4.6}$$

$$\text{Vol}[\text{supp} f_n(t,x,\cdot)] \le C(1+|t|+|x|)^{-3}, \tag{4.7}$$
$$|F_n(t,x)| \le C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}, \tag{4.8}$$
$$|D^k F_n(t,x)| \le C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-7/4}, \quad k = 1,2. \tag{4.9}$$

Proof: The estimates (4.5), (4.6) and (4.7) in the case $n=1$ follow directly from the definition of f_1. For (4.8) in the case $n=1$, note that the integral representation formula for E_1 reduces to the first integral of (2.15), namely

$$E_1(t,x)=-\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)f_1(t-|x-y|,y,p).$$

(From now on it will be understood that the integrals in p are over the set $\{|p|\leq 2R\}$ and the ones in y over $\Omega_a(t,x)$, with $a=a(2R)$). The previous integral is bounded by $C\Delta I_2^3(t,x)$, i.e., using lemma 4,

$$|E_1(t,x)|\leq C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}.$$

The same is true for B_1 and therefore (4.8) in the case $n=1$ is proved. Analogously, the representation formula for DE_1 reduces to the integral (3.3), with $g\equiv f_1$ and therefore lemma 5 gives $(4.9)_{k=1}$ in the case $n=1$. In a similar way, for D^2E_1 one has

$$D^2E_1(t,x)=\int\frac{dy}{|x-y|^3}\int dp\,\mathfrak{b}_1(\omega,p)Df_1(t-|x-y|,y,p),$$

which, applying lemma 5 to $g\equiv Df_1$, is estimated by

$$\begin{aligned}|D^2E_1(t,x)| &\leq C\frac{(\|Df_1\|_\infty+\|D^2f_1\|_\infty)}{(1+|t|+|x|)(1+|t-|x||)^{7/4}}\\ &\leq C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-7/4}.\end{aligned}$$

Now assume that (4.5)–(4.9) hold for (f_n,F_n). Since FSC is satisfied for $\eta=C\Delta$, then for a proper small Δ the estimates (4.6), (4.7) follow by lemma 1 and corollary 1. The same is true for (4.5) in the case $j+k\leq 1$, while the case $j+k=2$ follows by using the estimates on the second derivatives of F_n. Precisely, the argument for the proof of lemma 1 shows that the characteristics satisfy the estimate

$$|\partial_x^j\partial_p^k(X,P)(0,t,x,p)|\leq C(1+|t|)^k,\quad \text{for } j+k=2,$$

provided that the field satisfies FSC (for a proper small η) and

$$|\partial_x^2F(t,x)|\leq C(1+|t|+|x|)^{-\alpha}(1+|t|-|x|)^{-\alpha-1}.$$

The details are omitted because they are the same as for the proof of lemma 1. To complete the proof of (4.8) we apply lemma 4 to the representation formula

$$\begin{aligned}E_{n+1}(t,x) &= -\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)f_{n+1}(t-|x-y|,y,p)\\ &\quad -\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)(E_n+\widehat{p}\wedge B_n)f_{n+1}(t-|x-y|,y,p),\end{aligned}$$

which is proved as lemma 2. Similar equations can be written for the first and the second derivatives of E_{n+1}. By applying lemmas 4 and 5 to these equations, the estimate (4.9) follows after a straightforward argument. □

Step 2: Convergence in the C^0 norm

In this step we will prove the convergence of the sequence F_n with respect to the norm $\|\cdot\|_{3/4}$. Indeed the convergence holds in the norm (3.4) for all $0<w<1$. The choice $w=3/4$ suffices for our purpose and it is made only for sake of simplicity.

Proposition 4 *For properly small initial data, the sequence F_n converges in the norm $\|\cdot\|_{3/4}$.*

Proof: Put $\delta f_{n,m}=f_n-f_m$ and $\delta F_{n,m}=F_n-F_m$. The analogue of (2.15) for the approximation sequence is

$$\begin{aligned} E_n(t,x) = &-\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)f_n(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)K_{n-1}f_n(t-|x-y|,y,p), \end{aligned}\tag{4.10}$$

where $K_{n-1}=E_{n-1}+\widehat{p}\wedge B_{n-1}$. Thus

$$\begin{aligned} \delta E_{n,m} = &-\int\frac{dy}{|x-y|^2}\int dp\,\mathfrak{a}_1(\omega,p)\delta f_{n,m}(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)K_{n-1}\delta f_{n,m}(t-|x-y|,y,p)\\ &-\int\frac{dy}{|x-y|}\int dp\,\mathfrak{a}_2(\omega,p)\delta K_{n-1,m-1}f_m(t-|x-y|,y,p). \end{aligned}$$

Estimating:

$$\begin{aligned} |\delta E_{n,m}| \le C\bigg(&\int\frac{dy}{|x-y|^2}\int dp|\delta f_{n,m}|(t-|x-y|,y,p)\\ &+\int\frac{dy}{|x-y|}\int dp|F_{n-1}||\delta f_{n,m}|(t-|x-y|,y,p)\\ &+\int\frac{dy}{|x-y|}\int dp|\delta F_{n-1,m-1}|f_m(t-|x-y|,y,p)\bigg)=C\Big(I_1+I_2+I_3\Big). \end{aligned}$$

For I_3 we use that

$$\begin{aligned} I_3 &\le C\Delta\int\frac{dy}{|x-y|}\frac{|\delta F_{n-1,m-1}|(t-|x-y|,y)}{(1+|t-|x-y||+|y|)^3}\\ &\le C\Delta\|\delta F_{n-1,m-1}\|_{3/4}\int\frac{dy}{|x-y|}(1+|t-|x-y||+|y|)^{-19/4}\\ &\le C\Delta\|\delta F_{n-1,m-1}\|_{3/4}(1+|t|+|x|)^{-1}(1+|t-|x||)^{-7/4}. \end{aligned}\tag{4.11}$$

Here we used (3.6) with $\beta\equiv 2R$. To estimate I_1 and I_2 in a proper way we need to carry out a factor $\|\delta F_{n-1,m-1}\|_{3/4}$. To this purpose we notice that, by (4.4),

$$\partial_t\delta f_{n,m}+\widehat{p}\cdot\partial_x\delta f_{n,m}+K_{n-1}\cdot\partial_p\delta f_{n,m}=-\delta K_{n-1,m-1}\cdot\partial_p f_m.$$

Integrating along the characteristics of the Vlasov equation we get

$$\delta f_{n,m}(t,x,p) = -\int_0^t (\delta E_{n-1,m-1} + \widehat{P}_n \wedge \delta B_{n-1,m-1}) \cdot \partial_p f_m(\tau, X_n(\tau), P_n(\tau)) d\tau.$$

From the previous equation, inequality (4.5) and the estimate $|X_n(\tau)| \le R + a|\tau|$ we deduce

$$|\delta f_{n,m}(t,x,p)| \le C\Delta \|\delta F_{n-1,m-1}\|_{3/4} (1+|t|)^{1/4}. \tag{4.12}$$

Hence

$$\begin{aligned} I_1 &\le C\Delta \|\delta F_{n-1,m-1}\|_{3/4} \int \frac{dy}{|x-y|^2} (1+|t-|x-y||+|y|)^{-11/4} \\ &\le C\Delta \|\delta F_{n-1,m-1}\|_{3/4} (1+|t|+|x|)^{-1} (1+|t-|x||)^{-3/4}, \end{aligned}$$

$$\begin{aligned} I_2 &\le C\Delta \|\delta F_{n-1,m-1}\|_{3/4} \int \frac{dy}{|x-y|} (1+|t-|x-y||+|y|)^{-19/4} \\ &\le C\Delta \|\delta F_{n-1,m-1}\|_{3/4} (1+|t|+|x|)^{-1} (1+|t-|x||)^{-7/4}. \end{aligned}$$

Adding the various estimates we get

$$(1+|t|+|x|)(1+|t-|x||)^{-3/4} |\delta E_{n,m}(t,x)| \le C\Delta \|\delta F_{n-1,m-1}\|_{3/4}.$$

An identical estimate holds for $\delta B_{n,m}$ and therefore we finally get

$$\|\delta F_{n,m}\|_{3/4} \le C\Delta \|\delta F_{n-1,m-1}\|_{3/4}. \tag{4.13}$$

If the initial data are small enough in order that $C\Delta < 1$, then F_n is a Cauchy sequence in the norm $\|\cdot\|_{3/4}$ and so it converges uniformly and the limit function $F = (E,B)$ satisfies

$$|F(t,x)| \le C\Delta (1+|t|+|x|)^{-1} (1+|t-|x||)^{-3/4}. \tag{4.14}$$

□

By (4.12), the sequence $f_n(t,x,p)$ converges uniformly with respect to $(t,x,p) \in [-T,T] \times \mathbb{R}^3_x \times \mathbb{R}^3_p$, for all $T > 0$. The limit function (f,F) of the sequence (f_n, F_n) is a continuous solution of $\mathrm{RVM}_{\mathrm{ret}}$. Moreover, substituting (4.14) into the second integral in the right hand side of (2.15), we find that $E(t,x)$ satisfies the estimate

$$|E(t,x)| \le C\Delta (I_2^3 + I_1^{19/4}) \le C\Delta (1+|t|+|x|)^{-1} (1+|t-|x||)^{-1}.$$

The same is true for the magnetic field and so (1.8) is proved.

Corollary 2 *The following inequalities hold for all $t \in \mathbb{R}$ and $(x,p) \in \Xi(t)$:*

$$|\delta f_{n,m}(t,x,p)| \le (1+|t|)^{1/4} q_{n,m}, \tag{4.15}$$

$$|\delta X_{n,m}(0)| \le (1+|t|) q_{n,m}, \tag{4.16}$$

$$|\delta P_{n,m}(0)| \le q_{n,m}, \tag{4.17}$$

where $q_{n,m} \to 0$, as $n,m \to \infty$.

Proof: (4.15) follows by (4.12). To prove (4.16), (4.17) we use that, by means of (4.1), say for $0 \le s \le t$,

$$|\delta X_{n,m}(s)| \le \int_s^t d\tau\, |\delta \widehat{P}_{n,m}(\tau)| d\tau \le C \int_s^t d\tau\, |\delta P_{n,m}(\tau)|.$$

Moreover by the known C^1 bounds and the Cauchy property of F_n in the norm $\|\cdot\|_{3/4}$,

$$\begin{aligned}
|\delta P_{n,m}(s)| &\le \int_s^t d\tau\, |K_{n-1}(\tau, X_n(\tau), P_n(\tau)) - K_{m-1}(\tau, X_m(\tau), P_m(\tau))| \\
&\le \int_s^t d\tau\, |K_{n-1}(\tau, X_n(\tau), P_n(\tau)) - K_{n-1}(\tau, X_m(\tau), P_n(\tau))| \\
&\quad + \int_s^t d\tau\, |K_{n-1}(\tau, X_m(\tau), P_n(\tau)) - K_{n-1}(\tau, X_m(\tau), P_m(\tau))| \\
&\quad + \int_s^t d\tau\, |K_{n-1}(\tau, X_m(\tau), P_m(\tau)) - K_{m-1}(\tau, X_m(\tau), P_m(\tau))| \\
&\le q_{n,m} + C \int_s^t d\tau\, (1+\tau)^{-11/4} |\delta X_{n,m}(\tau)| + C \int_s^t d\tau\, (1+\tau)^{-2} |\delta P_{n,m}(\tau)|.
\end{aligned}$$

Combining the last two inequalities we get

$$|\delta P_{n,m}(s)| \le q_{n,m} + C \int_s^t d\tau [(\tau - s)(1+\tau)^{-11/4} + (1+\tau)^{-2}] |\delta P_{n,m}(\tau)|.$$

Hence, by the Gronwall lemma:

$$|\delta P_{n,m}(s)| \le q_{n,m}, \quad |\delta X_{n,m}(s)| \le (1+t-s) q_{n,m}, \tag{4.18}$$

which concludes the proof. □

Step 3: Convergence in the C^1 norm

In this step we will prove the convergence of the sequence DF_n with respect to the norm $\|\cdot\|_1$ (which again is not optimal but sufficient for our purpose).

Proposition 5 *For properly small initial data the sequence DF_n converges in the norm $\|\cdot\|_1$.*

Proof: Put $\delta Df_{n,m} = Df_n - Df_m$ and $\delta DE_{n,m} = DE_n - DE_m$. By (2.22) we have

$$\begin{aligned}
\delta DE_{n,m} &= \int dp \int dy \frac{\mathfrak{b}_1(\omega, p)}{|x-y|^3} \delta f_{n,m}(t - |x-y|, y, p) \\
&\quad + \int dp \int dy \frac{\mathfrak{b}_2(\omega, p)}{|x-y|^2} (f_n K_{n-1} - f_m K_{m-1})(t - |x-y|, y, p) \\
&\quad + \int dp \int dy \frac{\mathfrak{b}_3(\omega, p)}{|x-y|} [D(K_{n-1} f_n) - D(K_{m-1} f_m)](t - |x-y|, y, p) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_2 we write

$$\begin{aligned}|I_2(t,x)| &\le C\int dp \int \frac{dy}{|x-y|^2}|F_{n-1}||\delta f_{n,m}|(t-|x-y|,y,p)\\ &+C\int dp\int \frac{dy}{|x-y|^2}|\delta F_{n-1,m-1}|f_m(t-|x-y|,y,p)\\ &\le I_2^{19/4} q_{n,m} \le q_{n,m}(1+|t|+|x|)^{-1}(1+|t-|x||)^{-11/4},\end{aligned}$$

where we used the estimate (4.15) and the Cauchy property of F_n in the norm $\|\cdot\|_{3/4}$. The integral I_1 is further split as follows:

$$I_1 = \int_{|x-y|\le 1} dy \cdots + \int_{|x-y|>1} dy \cdots.$$

For the second integral we have, by (4.15),

$$\int_{|x-y|>1} dy \cdots \le q_{n,m}\int_{|x-y|>1}\frac{dy}{|x-y|^3}(1+|t-|x-y||+|y|)^{-11/4}.$$

The integral in the right hand side of the previous expression corresponds to the integral $II^{11/4}$ which has been estimated in the proof of lemma 5 (cf. (A.3) in appendix). The result is (see (A.4))

$$\int_{|x-y|>1} dy \cdots \le q_{n,m}(1+|t|+|x|)^{-1}(1+|t-|x||)^{-3/2}. \tag{4.19}$$

For the first part of the integral I_1, we have, by the same argument following eq. (A.2) in appendix,

$$\int_{|x-y|\le 1} dy \cdots \le C\int_{t-1}^{t}\frac{\|\delta D f_{n,m}(\tau)\|_\infty}{(1+|\tau|+|x|)^3}. \tag{4.20}$$

We will prove afterwards that

$$|\delta D f_{n,m}(t,x,p)| \le (1+|t|)[q_{n,m}+C\Delta\|\delta DF_{n-1,m-1}\|_1]. \tag{4.21}$$

Hence substituting into (4.20) and adding to (4.19) we get

$$|I_1| \le (1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}(q_{n,m}+C\Delta\|\delta DF_{n-1,m-1}\|_1).$$

For I_3 we expand the integrand function as

$$\begin{aligned}D(K_{n-1}f_n)-D(K_{m-1}f_m) &= (DK_{n-1})\delta f_{n,m}+(Df_m)\delta K_{n-1,m-1}\\ &+f_m\delta DK_{n-1,m-1}+K_{n-1}\delta Df_{n,m}\end{aligned}$$

and therefore, after some straightforward estimates,

$$|I_3| \le (1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}(q_{n,m}+C\Delta\|\delta DF_{n-1,m-1}\|_1).$$

Summing up the various estimates we get

$$(1+|t|+|x|)(1+|t-|x||)|\delta DE_{n,m}| \leq q_{n,m} + C\Delta\|\delta DF_{n-1,m-1}\|_1$$

and so, by the analogous estimate for any other first derivative of F_n, we conclude

$$\|\delta DF_{n,m}\|_1 \leq q_{n,m} + C\Delta\|\delta DF_{n-1,m-1}\|_1.$$

For properly small initial data, the previous inequality implies that DF_n is a Cauchy sequence in the norm $\|\cdot\|_1$ and so that it converges uniformly. Therefore, F is a C^1 function and satisfies:

$$|DF(t,x)| \leq C\Delta(1+|t|+|x|)^{-1}(1+|t-|x||)^{-1}. \tag{4.22}$$

Let us prove now the inequality (4.21), say for $t>0$. By (4.3) we have

$$\begin{aligned}|\delta Df_{n,m}(t,x,p)| &\leq |\partial_x f^{\text{in}}(X_n,P_n)||\delta DX_{n,m}| + |\partial_p f^{\text{in}}(X_n,P_n)||\delta DP_{n,m}| \\ &\quad + |DX_m||\partial_x f^{\text{in}}(X_n,P_n) - \partial_x f^{\text{in}}(X_m,P_m)| \\ &\quad + |DP_m||\partial_p f^{\text{in}}(X_n,P_n) - \partial_p f^{\text{in}}(X_m,P_m)| \\ &\leq C\Delta(|\delta DX_{n,m}| + |\delta DP_{n,m}| + |\delta X_{n,m}| + |\delta P_{n,m}|),\end{aligned} \tag{4.23}$$

evaluation of the characteristics at $s=0$ being understood. By (4.1):

$$|\delta DX_{n,m}(s)| \leq \int_s^t d\tau \left(|\delta DP_{n,m}(\tau)| + |\delta P_{n,m}(\tau)|\right), \tag{4.24}$$

$$|\delta DP_{n,m}(s)| \leq \int_s^t d\tau\, |D[K_{n-1}(\tau,X_n(\tau),\widehat{P}_n(\tau))] - D[K_{m-1}(\tau,X_m(\tau),\widehat{P}_m(\tau))]|. \tag{4.25}$$

The integrand function in (4.25) is expanded as follows:

$$\begin{aligned}&D[K_{n-1}(\tau,X_n(\tau),\widehat{P}_n(\tau))] - D[K_{m-1}(\tau,X_m(\tau),\widehat{P}_m(\tau))] \\ &= \partial_x E_{n-1}(\tau,X_n)\delta DX_{n,m} + DX_m[\partial_x E_{n-1}(\tau,X_n) - \partial_x E_{m-1}(\tau,X_m)] \\ &\quad + B_{n-1}(\tau,X_n)\wedge \delta D\widehat{P}_{n,m} + D\widehat{P}_m \wedge [B_{n-1}(\tau,X_n) - B_{m-1}(\tau,X_m)] \\ &\quad + \widehat{P}_n \wedge \partial_x B_{n-1}(\tau,X_n)\delta DX_{n,m} + \delta\widehat{P}_{x,m}\wedge \partial_x B_{n-1}(\tau,X_n)DX_m \\ &\quad + \widehat{P}_m \wedge DX_m[\partial_x B_{n-1}(\tau,X_n) - \partial_x B_{m-1}(\tau,X_m)].\end{aligned}$$

Using the known bounds on F_n and DF_n we get

$$\begin{aligned}|\delta DP_{n,m}(s)| &\leq q_{n,m} + C\int_s^t d\tau\,(1+\tau)^{-11/4}|\delta DX_{n,m}(\tau)| + C\int_s^t d\tau\,(1+\tau)^{-2}|\delta DP_{n,m}| \\ &\quad + C\int_t^s d\tau\,|B_{n-1}(\tau,X_n(\tau)) - B_{m-1}(\tau,X_m(\tau))| \\ &\quad + C\int_t^s d\tau\,|DF_{n-1}(\tau,X_n(\tau)) - DF_{m-1}(\tau,X_m(\tau))|.\end{aligned} \tag{4.26}$$

Now we substitute (4.24) into (4.26) and use

$$\begin{aligned}
&|B_{n-1}(\tau,X_n(\tau))-B_{m-1}(\tau,X_m(\tau))| \\
&\le |B_{n-1}(\tau,X_n(\tau))-B_{m-1}(\tau,X_n(\tau))|+|B_{m-1}(\tau,X_n(\tau))-B_{m-1}(\tau,X_m(\tau))| \\
&\le [(1+\tau)^{-7/4}+(1+t-\tau)(1+\tau)^{-11/4}]\,q_{n,m},
\end{aligned}$$

$$\begin{aligned}
&|DF_{n-1}(\tau,X_n(\tau))-DF_{m-1}(\tau,X_m(\tau))| \\
&\le |DF_{n-1}(\tau,X_n(\tau))-DF_{m-1}(\tau,X_n(\tau))|+|DF_{m-1}(\tau,X_n(\tau))-DF_{m-1}(\tau,X_m(\tau))| \\
&\le C(1+\tau)^{-2}\|\delta F_{n-1,m-1}\|_1+[(1+t-\tau)(1+\tau)^{-11/4}]q_{n,m},
\end{aligned}$$

which follow by the known bounds on the first and second order derivatives, the second of (4.18) and the Cauchy property of F_n in the norm $\|\cdot\|_{3/4}$. In this way we get

$$\begin{aligned}
|\delta DP_{n,m}(s)| \le\; & q_{n,m}+C\|\delta DF_{n-1,m-1}\|_1 \\
& +\int_s^t d\tau\,[(1+\tau)^{-2}+(1+t-\tau)(1+\tau)^{-11/4}]|\delta DP_{n,m}(\tau)|
\end{aligned}$$

and so, by the Gronwall lemma,

$$|\delta DP_{n,m}|\le (q_{n,m}+C\|\delta DF_{n-1,m-1}\|_1).$$

Thus by (4.24),

$$|\delta DX_{n,m}|\le (q_{n,m}+C\|\delta DF_{n-1,m-1}\|_1)(1+t-s). \tag{4.27}$$

Taking $s=0$ and substituting into (4.23), the estimate (4.21) follows after using (4.16) and (4.17). □

By means of (4.21), Df_n converges uniformly in x,p and pointwise in t. The same argument permits to prove that even the p-derivatives of f_n satisfy this property and therefore the limit function (f,F) is C^1. Substituting (4.22) into the last integral of (2.22), the estimate (1.9) is proved by using again lemmas 4 and 5. This concludes the proof of theorem 1.

Acknowledgments: The author acknowledges many useful discussions with his PhD advisor, Dr. Alan D. Rendall. Support by the European HYKE network (contract HPRN-CT-2002-00282) is also acknowledged.

Appendix

Proof of lemma 4

We will use repeatedly lemma 7 of [5], which we rewrite below in a form more suitable to our case.

Lemma A *For any function $g\in C^0(\mathbb{R}^2)$, $a>0$, $b\in(a,+\infty]$ and $n\in\mathbb{N}$:*

$$\int_{a\le|x-y|\le b}\frac{dy}{|x-y|^n}g(t-|x-y|,|y|)=\frac{2\pi}{|x|}\int_{t-b}^{t-a}\frac{d\tau}{(t-\tau)^{n-1}}\int_{||x|-t+\tau|}^{|x|+t-\tau}d\lambda\, g(\tau,\lambda)\lambda.$$

Estimate on I_1^q and I_2^q for $t \leq 0$

For $t \leq 0$ we have $|t-|x-y|| = -t+|x-y|$ and by lemma A we have:

$$\begin{aligned} I_1^q(t,x) &= \int \frac{dy}{|x-y|}(1-t+|x-y|+|y|)^{-q} \\ &= \frac{2\pi}{|x|}\int_{-\infty}^{t} d\tau \int_{||x|-t+\tau|}^{|x|+t-\tau} \frac{\lambda}{(1+\lambda-\tau)^q} d\lambda \\ &= \frac{2\pi}{|x|}\int_{-\infty}^{t-|x|} \cdots + \frac{2\pi}{|x|}\int_{t-|x|}^{t} \cdots = A+B. \end{aligned}$$

For A we use

$$\begin{aligned} A &= \frac{2\pi}{|x|}\int_{-\infty}^{t-|x|} d\tau \int_{t-|x|-\tau}^{t+|x|-\tau} \frac{\lambda}{(1+\lambda-\tau)^q} d\lambda \\ &\leq \frac{C}{|x|}\int_{-\infty}^{t-|x|} d\tau \int_{t-|x|-\tau}^{t+|x|-\tau} \frac{d\lambda}{(1+\lambda-\tau)^{q-1}} \\ &\leq C\int_{-\infty}^{t-|x|} \frac{d\tau}{(1-\tau)^{q-1}} \leq \frac{C}{(1-t+|x|)^{q-2}}. \end{aligned}$$

For B we use

$$\begin{aligned} B &= \frac{2\pi}{|x|}\int_{t-|x|}^{t} d\tau \int_{|x|-t+\tau}^{|x|+t-\tau} \frac{\lambda}{(1+\lambda-\tau)^q} d\lambda \\ &\leq \frac{C}{|x|(1-t+|x|)^{q-1}}\int_{t-|x|}^{t} (t-\tau)d\tau \\ &\leq \frac{C}{(1-t+|x|)^{q-2}}. \end{aligned}$$

For $n=2$, $t \leq 0$, we write, again using lemma A,

$$\begin{aligned} I_2^q(t,x) &= \int \frac{dy}{|x-y|^2}(1-t+|x-y|+|y|)^{-q} \\ &= \frac{2\pi}{|x|}\int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{||x|-t+\tau|}^{|x|+t-\tau} \frac{\lambda}{(1+\lambda-\tau)^q} d\lambda \\ &= \frac{2\pi}{|x|}\int_{-\infty}^{t-|x|} \cdots + \frac{2\pi}{|x|}\int_{t-|x|}^{t} \cdots = A+B. \end{aligned}$$

For A we use

$$\begin{aligned} A &= \frac{2\pi}{|x|}\int_{-\infty}^{t-|x|} \frac{d\tau}{t-\tau} \int_{t-|x|-\tau}^{t+|x|-\tau} \frac{\lambda}{(1+\lambda-\tau)^q} d\lambda \\ &\leq \frac{C}{|x|}\int_{-\infty}^{t-|x|} d\tau \left(\frac{t+|x|-\tau}{t-\tau}\right) \int_{t-|x|-\tau}^{t+|x|-\tau} \frac{d\lambda}{(1+\lambda-\tau)^q} \\ &\leq \frac{C}{|x|}\int_{-\infty}^{t-|x|} d\tau \frac{|x|}{(1-\tau)^q} = \frac{C}{(1-t+|x|)^{q-1}}. \end{aligned}$$

For B we use

$$
\begin{aligned}
B &= \frac{2\pi}{|x|}\int_{t-|x|}^{t}\frac{d\tau}{t-\tau}\int_{|x|-t+\tau}^{|x|+t-\tau}\frac{\lambda}{(1+\lambda-\tau)^q}d\lambda \\
&\le \frac{C}{|x|}\int_{t-|x|}^{t}\frac{d\tau}{t-\tau}\int_{|x|-t+\tau}^{|x|+t-\tau}\frac{1}{(1+\lambda-\tau)^{q-1}}d\lambda \\
&\le \frac{C}{|x|(1-t+|x|)^{q-1}}\int_{t-|x|}^{t}\frac{d\tau}{t-\tau}\int_{|x|-t+\tau}^{|x|+t-\tau}d\lambda \\
&\le \frac{C}{(1-t+|x|)^{q-1}}.
\end{aligned}
$$

Estimate on I_1^q for $t>0$

We split I_1^q as follows:

$$
\begin{aligned}
I_1^q(t,x) &= \int_{|x-y|\le t}\frac{dy}{|x-y|}(1+t-|x-y|+|y|)^{-q} \\
&+\int_{|x-y|\ge t}\frac{dy}{|x-y|}(1-t+|x-y|+|y|)^{-q} = I_{1A}^q+I_{1B}^q.
\end{aligned}
$$

By using lemma A we have

$$
\begin{aligned}
I_{1A}^q &= \frac{2\pi}{|x|}\int_0^t d\tau\int_{||x|-t+\tau|}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1+\tau+\lambda)^q}, \\
I_{1B}^q &= \frac{2\pi}{|x|}\int_{-\infty}^0 d\tau\int_{||x|-t+\tau|}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q}.
\end{aligned}
$$

Now define

$$
(t-|x|)_+ = \begin{cases} t-|x| & \text{if } t-|x|>0 \\ 0 & \text{if } t-|x|\le 0 \end{cases}
$$

$$
(t-|x|)_- = \begin{cases} t-|x| & \text{if } t-|x|<0 \\ 0 & \text{if } t-|x|\ge 0 \end{cases}
$$

and split the preceding integrals as follows:

$$
\begin{aligned}
I_{1A}^q &= \frac{2\pi}{|x|}\int_0^{(t-|x|)_+}d\tau\int_{t-|x|-\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1+\tau+\lambda)^q} \\
&+\frac{2\pi}{|x|}\int_{(t-|x|)_+}^{t}d\tau\int_{|x|-t+\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1+\tau+\lambda)^q} = I_{1A\alpha}^q+I_{1A\beta}^q,
\end{aligned}
$$

$$
\begin{aligned}
I_{1B}^q &= \frac{2\pi}{|x|}\int_{-\infty}^{(t-|x|)_-}d\tau\int_{t-|x|-\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q} \\
&+\frac{2\pi}{|x|}\int_{(t-|x|)_-}^{0}d\tau\int_{|x|-t+\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q} = I_{1B\alpha}^q+I_{1B\beta}^q.
\end{aligned}
$$

Thus, finally

$$I_1^q(t,x) = I_{1A\alpha}^q + I_{1A\beta}^q + I_{1B\alpha}^q + I_{1B\beta}^q. \tag{A.1}$$

Estimate for $I_{1A\alpha}^q$

$$\begin{aligned}
I_{1A\alpha}^q &\le \frac{C}{|x|}\int_0^{(t-|x|)_+} d\tau \int_{t-|x|-\tau}^{|x|+t-\tau} \frac{d\lambda}{(1+\tau+\lambda)^{q-1}} \\
&\le \frac{C}{|x|}\int_0^{(t-|x|)_+} \frac{d\tau}{(1+t-|x|)^{q-3}} \int_{t-|x|-\tau}^{|x|+t-\tau} \frac{d\lambda}{(1+\tau+\lambda)^2} \\
&\le \frac{C(t-|x|)_+}{(1+t-|x|)^{q-2}(1+t+|x|)} \le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+3}.
\end{aligned}$$

Estimate for $I_{1A\beta}^q$

$$\begin{aligned}
I_{1A\beta}^q &\le \frac{C}{|x|}\int_{(t-|x|)_+}^{t} d\tau \int_{|x|-t+\tau}^{|x|+t-\tau} \frac{d\lambda}{(1+\tau+\lambda)^{q-1}} \\
&\le \frac{C}{|x|}\int_{(t-|x|)_+}^{t} d\tau \frac{t-\tau}{(1-t+|x|+2\tau)^{q-2}(1+t+|x|)} \\
&\le \frac{C(t-(t-|x|)_+)}{|x|(1+|t|+|x|)} \int_{(t-|x|)_+}^{t} d\tau\,(1-t+|x|+2\tau)^{2-q} \\
&\le \frac{C}{1+|t|+|x|}\left(\frac{(1+|x|+(t-|x|)_+)^{q-3}}{(1+t+|x|)^{q-3}(1-t+|x|+2(t-|x|)_+)^{q-3}}\right) \\
&\le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+3}.
\end{aligned}$$

Estimate for $I_{1B\alpha}^q$

$$\begin{aligned}
I_{1B\alpha}^q &\le \frac{C}{|x|}\int_{-\infty}^{(t-|x|)_-} d\tau \int_{t-|x|-\tau}^{|x|+t-\tau} \frac{d\lambda}{(1-\tau+\lambda)^{q-1}} \\
&\le C\int_{-\infty}^{(t-|x|)_-} \frac{d\tau}{(1+t-|x|-2\tau)^{q-2}(1+t+|x|-2\tau)} \\
&\le C(1+t+|x|)^{-1}(1+t-|x|-2(t-|x|)_-)^{-q+3} \\
&\le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+3}.
\end{aligned}$$

Estimate for $I_{1B\beta}^q$

$$\begin{aligned}
I_{1B\beta}^q &\le \frac{C}{|x|}\int_{(t-|x|)_-}^{0} \frac{(t-\tau)d\tau}{(1-t+|x|)^{q-2}(1+t+|x|-2\tau)} \\
&\le \frac{C}{|x|}\frac{(t-(t-|x|)_-)|(t-|x|)_-|}{(1-t+|x|)^{q-2}(1+t+|x|)} \\
&\le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+3}.
\end{aligned}$$

Estimate on I_2^q for $t>0$

The integral I_2^q is split as I_1^q in (A.1), namely

$$I_2^q(t,x) = I_{2A\alpha}^q + I_{2A\beta}^q + I_{2B\alpha}^q + I_{2B\beta}^q,$$

where, using lemma A,

$$I_{2A\alpha}^q = \frac{2\pi}{|x|}\int_0^{(t-|x|)_+} \frac{d\tau}{t-\tau}\int_{t-|x|-\tau}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q},$$

$$I_{2A\alpha}^q = \frac{2\pi}{|x|}\int_{(t-|x|)_+}^{t} \frac{d\tau}{t-\tau}\int_{|x|-t+\tau}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q},$$

$$I_{2B\alpha}^q = \frac{2\pi}{|x|}\int_{-\infty}^{(t-|x|)_-} \frac{d\tau}{t-\tau}\int_{t-|x|-\tau}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1-\tau+\lambda)^q},$$

$$I_{2B\beta}^q = \frac{2\pi}{|x|}\int_{(t-|x|)_-}^{0} \frac{d\tau}{t-\tau}\int_{|x|-t+\tau}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1-\tau+\lambda)^q}.$$

Since the argument to estimate the preceding integrals is very similar to the one used for I_1^q, we just show how to estimate $I_{2A\alpha}^q$.

$$\begin{aligned}
I_{2A\alpha}^q &\le \frac{C}{|x|}\int_0^{(t-|x|)_+} d\tau \frac{|x|+t-\tau}{t-\tau}\int_{t-|x|-\tau}^{|x|+t-\tau} \frac{d\lambda}{(1+\lambda+\tau)^q} \\
&\le \frac{C}{|x|}\left(1+\frac{|x|}{t-(t-|x|)_+}\right)\int_0^{(t-|x|)_+} d\tau \frac{|x|}{(1+t-|x|)^{q-1}(1+t+|x|)} \\
&\le \frac{C(t-|x|)_+}{(1+t+|x|)(1+t-|x|)^{q-1}} \le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+2}.
\end{aligned}$$

Proof of lemma 5

We divide the integral $I(t,x)$ in two parts as follows:

$$I = \int_{|x-y|\le 1} dy \cdots + \int_{|x-y|>1} dy \cdots. \tag{A.2}$$

Following [3], we rewrite the first integral as

$$\begin{aligned}
\int_{|x-y|\le 1} dy \cdots &= \int_{t-1}^{t} d\tau \int dp \int_{|\omega|=1} \mathfrak{b}_1(\omega,p)\frac{g(\tau,x+(t-\tau)\omega,p)}{t-\tau} d\omega \\
&= \int_{t-1}^{t} d\tau \int dp \int_{|\omega|=1} \mathfrak{b}_1(\omega,p)\frac{g(\tau,x+(t-\tau)\omega,p)-g(\tau,x,p)}{t-\tau} d\omega,
\end{aligned}$$

where the property (2.23) has been used. Then

$$\begin{aligned}
\int_{|x-y|\le 1} dy \cdots &\le C_* \sup_{t-1\le\tau\le t} \|Dg(\tau)\|_\infty \int_{t-1}^{t} \frac{d\tau}{(1+|\tau|+|x|)^3} \\
&\le C_* \|Dg\|_\infty (1+|t|+|x|)^{-3} \\
&\le C_* \|Dg\|_\infty (1+|t|+|x|)^{-1}(1+|t-|x||)^{-2}.
\end{aligned}$$

For the second part of I we write

$$\int_{|x-y|>1} dy \cdots \le C_* \|g\|_\infty \int_{|x-y|>1} \frac{dy}{|x-y|^3}(1+|t-|x-y||+|y|)^{-3} = C_* \|g\|_\infty II(t,x).$$

Since an integral similar to $II(t,x)$ needs to be estimated to prove proposition 5, we will treat the more general case

$$II^q(t,x) = \int_{|x-y|>1} \frac{dy}{|x-y|^3}(1+|t-|x-y||+|y|)^{-q}, \quad q>2. \tag{A.3}$$

We will prove that

$$II^q(t,x) \le C(1+|t|+|x|)^{-1}(1+||t-|x||)^{-q+5/4}. \tag{A.4}$$

We start by splitting $II^q(t,x)$ as follows:

$$II^q(t,x) = \int_{1<|x-y|\le 1+|t-|x||} \cdots + \int_{|x-y|>1+|t-|x||} \cdots = II^q_A + II^q_B.$$

For II^q_B we use

$$II^q_B \le \frac{I^q_2(t,x)}{(1+|t-|x||)} \le C(1+|t|+|x|)^{-1}(1+|t-|x||)^{-q+1}.$$

The estimate on II^q_A for $t\le 0$ is

$$\begin{aligned} II^q_A &\le \int_{1<|x-y|\le 1-t+|x|} \frac{dy}{|x-y|^3}(1-t+|x-y|+|y|)^{-q} \\ &\le \frac{C}{(1-t+|x|)^q} \int_{1\le|x-y|\le 1-t+|x|} dy|x-y|^{-3} \\ &\le \frac{C\log(1-t+|x|)}{(1-t+|x|)^q} \le C(1+|t|+|x|)^{-1}(1+|t-|x||)^{-q+\frac{5}{4}}. \end{aligned}$$

The estimate on II^q_A for $t>0$ requires a more careful analysis.

Estimate on II^q_A for $t>0, |x|\le 1$

For $t\le 1$, II^q_A is dominated by the same integral extended over $\{1\le|x-y|\le 3\}$ and so the estimate is straightforward. For $t\ge 1$ we have $t-|x|\ge 0$ and so we may split II^q_A as follows:

$$\begin{aligned} II^q_A &= \int_{1\le|x-y|\le t} \frac{dy}{|x-y|^3}(1+t-|x-y|+|y|)^{-q} \\ &\quad + \int_{t\le|x-y|\le 1+t-|x|} \frac{dy}{|x-y|^3}(1-t+|x-y|+|y|)^{-q} \\ &= II^q_{A1} + II^q_{A2}. \end{aligned}$$

Using lemma A we have

$$I_{A1}^q = \frac{2\pi}{|x|}\int_0^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{||x|+\tau-t|}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1+\tau+\lambda)^q}.$$

Since $t-1\le t-|x|$, then $|x|+\tau-t\le 0$ and we have

$$\begin{aligned}
II_{A1}^q &\le \frac{C}{|x|}\int_0^{t-1}\frac{d\tau}{t-\tau}\frac{|x|+t-\tau}{t-\tau}\frac{1}{(1+t-|x|)^{q-2}}\int_{t-|x|-\tau}^{|x|+t-\tau}\frac{d\lambda}{(1+\tau+\lambda)^2}\\
&\le C(1+t-|x|)^{-q+1}(1+t+|x|)^{-1}\int_0^{t-1}d\tau(t-\tau)^{-1}\\
&\le C\frac{\log t}{(1+t-|x|)^{q-1}}(1+t+|x|)^{-1}\\
&\le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+\frac{5}{4}},
\end{aligned}$$

since $t\ge 1$ and $t-|x|\ge 0$.

For II_{A2}^q we write

$$\begin{aligned}
II_{A2}^q &= \frac{2\pi}{|x|}\int_{|x|-1}^0\frac{d\tau}{(t-\tau)^2}\int_{t-|x|-\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q}\\
&\le \frac{C}{|x|}\int_{|x|-1}^0\frac{d\tau}{t-\tau}\frac{|x|}{(1+t-|x|-2\tau)^{q-1}(1+t+|x|-2\tau)}\\
&\le C\frac{\log t+\log(1+t-|x|)}{(1+t+|x|)(1+t-|x|)^{q-1}}\\
&\le C(1+t+|x|)^{-1}(1+|t-|x||)^{-q+\frac{5}{4}}.
\end{aligned}$$

Since in the following the details are very similar, they will be omitted.

Estimate on II_A^q for $|x|>1, 0<t\le 1$

In this case we have $t-|x|\le 0$ and $|x-y|>1\ge t$ and so

$$\begin{aligned}
II_A^q &= \int_{1<|x-y|\le 1-t+|x|}\frac{dy}{|x-y|^3}(1-t+|x-y|+|y|)^{-q}\\
&= \frac{2\pi}{|x|}\int_{2t-1-|x|}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{||x|+\tau-t|}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q}.
\end{aligned}$$

Since $2t-1-|x|\le t-|x|\le t-1$, we split the last integral as follows:

$$\begin{aligned}
II_A^q &= \frac{2\pi}{|x|}\int_{2t-1-|x|}^{t-|x|}\frac{d\tau}{(t-\tau)^2}\int_{t-|x|-\tau}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q}\\
&+\frac{2\pi}{|x|}\int_{t-|x|}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau}d\lambda\frac{\lambda}{(1-\tau+\lambda)^q} = II_{A1}^q + II_{A2}^q,
\end{aligned}$$

and each component is estimated as before.

Estimate on II_A^q for $|x|>1, t>1$

Case $t-|x|\geq 0$
Since $1+t-|x|<t$, we have

$$
\begin{aligned}
II_A^q &= \int_{1<|x-y|\leq 1+t-|x|} \frac{dy}{|x-y|^3}(1+t-|x-y|+|y|)^{-q} \\
&= \frac{2\pi}{|x|}\int_{|x|-1}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{||x|+\tau-t|}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q}.
\end{aligned}
$$

We further consider separately the regions $\frac{1}{2}(t+1)<|x|\leq t$ and $1<|x|\leq\frac{1}{2}(t+1)$. In the first case one has $|x|-1>t-|x|$ and therefore II_A^q reduces to

$$
II_A^q = \frac{2\pi}{|x|}\int_{|x|-1}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q},
$$

which is estimated as before. For $1<|x|\leq\frac{1}{2}(t+1)$ we write

$$
\begin{aligned}
II_A^q = & \frac{2\pi}{|x|}\int_{|x|-1}^{t-|x|}\frac{d\tau}{(t-\tau)^2}\int_{t-|x|-\tau}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q} \\
&+\frac{2\pi}{|x|}\int_{t-|x|}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q} = II_{A1}^q + II_{A2}^q
\end{aligned}
$$

and each component is estimated as before.

Case $t-|x|<0$
For $|x|\leq 2t-1$ we write

$$
\begin{aligned}
II_A^q &= \int_{1\leq|x-y|\leq 1-t+|x|} \frac{dy}{|x-y|^3}(1+t-|x-y|+|y|)^{-q} \\
&= \frac{2\pi}{|x|}\int_{2t-|x|-1}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q},
\end{aligned}
$$

where we used that $t-|x|<2t-|x|-1$. For $|x|\geq 2t-1$ we write

$$
\begin{aligned}
II_A^q &= \int_{1\leq|x-y|\leq t} \frac{dy}{|x-y|^3}(1+t-|x-y|+|y|)^{-q} \\
&\quad+\int_{t\leq|x-y|\leq 1-t+|x|} \frac{dy}{|x-y|^3}(1-t+|x-y|+|y|)^{-q} \\
&= \frac{2\pi}{|x|}\int_{0}^{t-1}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1+\tau+\lambda)^q} \\
&\quad+\frac{2\pi}{|x|}\int_{2t-|x|-1}^{0}\frac{d\tau}{(t-\tau)^2}\int_{|x|+\tau-t}^{|x|+t-\tau} d\lambda \frac{\lambda}{(1-\tau+\lambda)^q}.
\end{aligned}
$$

The usual argument applies to estimate all the above integrals and concludes the proof of lemma 5.

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