ASYMPTOTIC EXPANSIONS CLOSE TO THE SINGULARITY IN GOWDY SPACETIMES

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Abstract. We consider Gowdy spacetimes under the assumption that the spatial hypersurfaces are diffeomorphic to the torus. The relevant equations are then wave map equations with the hyperbolic space as a target. In an article by Grubišić and Moncrief, a formal expansion of solutions in the direction toward the singularity was proposed. Later, Kichenassamy and Rendall constructed a family of real analytic solutions with the maximum number of free functions and the desired asymptotics at the singularity. The condition of real analyticity was subsequently removed by Rendall. In an article by the author, it was shown that one can put a condition on initial data that leads to asymptotic expansions. In this article, we show the following. By fixing a point in hyperbolic space, we can consider the hyperbolic distance from this point to the solution at a given spacetime point. If we fix a spatial point for the solution, it is enough to put conditions on the rate at which the hyperbolic distance tends to infinity as time tends to the singularity in order to conclude that there are smooth expansions in a neighbourhood of the given spatial point.

1. Introduction

The motivation for studying the problem discussed in this article is the desire to understand the structure of singularities in cosmological spacetimes. By the singularity theorems, cosmological spacetimes typically have a singularity in the sense of causal geodesic incompleteness. However, it seems that the methods used to obtain this result are not so well suited to answering related questions concerning e.g. curvature blow up. To proceed, it seems difficult to avoid analyzing the equations in detail. After some appropriate choice of gauge, one is then confronted with the task of analyzing the asymptotics of a non-linear hyperbolic equation. This is in general not so easy. Consequently, one often imposes some symmetry condition, and we will here consider a class of spacetimes with a two dimensional group of symmetries. The problem one ends up with is then a system of non-linear wave equations in $1+1$ dimensions.

The Gowdy spacetimes were first introduced in [4] (see also [2]), and in [7] the fundamental questions concerning global existence were answered. We will take the Gowdy vacuum spacetimes on $\mathbb{R}\times T^3$ to be metrics of the form (1). We do not wish to motivate this choice at length here, but refer the reader to the above mentioned references for further details. A brief justification is given in the introduction of [11]. Let

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2] \quad (1)$$

Here, $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on T^3. The evolution equations become

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau} Q_\theta^2) = 0 \tag{2}$$

$$Q_{\tau\tau} - e^{-2\tau} Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau} P_\theta Q_\theta) = 0, \tag{3}$$

and the constraints

$$\lambda_\tau = P_\tau^2 + e^{-2\tau} P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau} Q_\theta^2) \tag{4}$$

$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P} Q_\theta Q_\tau). \tag{5}$$

Obviously, the constraints are decoupled from the evolution equations, excepting the condition on P and Q implied by (5). Thus the equations of interest are the two non-linear coupled wave equations (2)-(3). In the above parametrization, the singularity corresponds to $\tau \to \infty$, and the subject of this article is the asymptotics of solutions to (2)-(3) as $\tau \to \infty$. The equations (2)-(3) are wave map equations. In fact, let

$$g_0 = -e^{-2\tau} d\tau^2 + d\theta^2 + e^{-2\tau} d\chi^2$$

be a metric on $\mathbb{R} \times T^2$ and let

$$g_R = dP^2 + e^{2P} dQ^2 \tag{6}$$

be a metric on $\mathbb{R}^2$. Then (2)-(3) are the wave map equations for a map from $(\mathbb{R} \times T^2, g_0)$ to $(\mathbb{R}^2, g_R)$ which is independent of the χ-coordinate. Note that $(\mathbb{R}^2, g_R)$ is isometric to the upper half plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric

$$g_H = \frac{dx^2 + dy^2}{y^2} \tag{7}$$

under the map

$$\phi_{RH}(Q, P) = (Q, e^{-P}). \tag{8}$$

Thus the target space is hyperbolic space. For the rest of the article, we will be concerned with the above mentioned wave map equations and not consider the consequences for the resulting spacetimes. The implications for the spacetime geometry can be found elsewhere, see e.g. [11].

The idea of finding expansions for solutions close to the singularity started with the article [5] by Grubišić and Moncrief. In our setting, the natural expansions are

$$P(\tau, \theta) = v(\theta)\tau + \phi(\theta) + e^{-\epsilon\tau} u(\theta, \tau) \tag{9}$$

$$Q(\tau, \theta) = q(\theta) + e^{-2v(\theta)\tau}[\psi(\theta) + w(\tau, \theta)] \tag{10}$$

where $\epsilon > 0$ and $w, u \to 0$ as $\tau \to \infty$ and $0 < v(\theta) < 1$. This should be compared with (5) and (6) of [8], where $-Z = P$, $X = Q$ and $t = e^{-\tau}$. A heuristic argument motivating the condition on the velocity can be found in [1]. In the non-generic case $Q = 0$, one can prove that (9) holds without the condition on v. This special case is called polarized Gowdy and has been studied in [6], which also considers the other topologies for Gowdy spacetimes. The numerical simulations indicate that there are exceptions to (9)-(10), called spikes, at which the kinetic energy of the wave map makes a jump. We refer the reader to [1] for numerical and to [10] for analytical results concerning spikes.

By the so called Fuchsian techniques one can construct a large family of solutions with the asymptotic behaviour (9)-(10). In fact, given functions v, ϕ, q and ψ from S^1 to $\mathbb{R}$ of a suitable degree of smoothness and subject to the condition $0 < v < 1$, one can construct solutions to (2)-(3) with asymptotics of the form (9)-(10). The proof of this in the real analytic case can be found in [8] and [9] covers the smooth case. One nice feature of this construction is the fact that one gets to specify four functions freely, just as as if though one were specifying initial data for (2)-(3).

Finally, in [11] a condition on the initial data leading to the desired asymptotics is given. Note also the related work in [3]. In [11], conditions on the H^2 norm of the first derivatives are imposed. The reason is the following. It turns out that the most important quantity to control is the velocity v appearing in (9). In other words, one wants to control what P_τ converges to. In [11], this control was achieved by integrating (2). In order to reach any conclusion, one then has to control $e^{-2\tau}P_{\theta\theta}$ in the sup norm, which is the reason for the condition on the H^2 norm of the first derivatives. Looking at the expansion (9) naively, another possibility presents itself. One expects P_τ to converge to v rapidly, so that $P_\tau - P/\tau$ should be of the order of magnitude $O(\tau^{-1})$. Assume for a moment that we can prove that we have this estimate in the sup norm. Then one easily sees that $\partial_\tau(P/\tau) = O(\tau^{-2})$. Consequently, P/τ converges to a continuous function. Since $P_\tau - P/\tau$ converges to zero, we then get the convergence of P_τ to a continuous function. What we need is consequently to prove that $P_\tau - P/\tau$ tends to zero as τ^{-1} under suitable circumstances. One quantity suited to achieve this task is

$$F(\tau) = \frac{1}{2}\sup_{\theta\in S^1}[(P_\tau - \frac{1}{\tau}P + e^{-\tau}P_\theta)^2 + e^{2P}(Q_\tau + e^{-\tau}Q_\theta)^2] + \frac{1}{2}\sup_{\theta\in S^1}[(P_\tau - \frac{1}{\tau}P - e^{-\tau}P_\theta)^2 + e^{2P}(Q_\tau - e^{-\tau}Q_\theta)^2]. \tag{11}$$

This object may seem unnatural at first, however it appears naturally in the arguments. It turns out that if $1 \le P \le \tau - 1$ in an interval $[\tau_1, \tau_2]$, then $F(\tau) \le F(\tau_1)(\tau_1/\tau)^2$ for $\tau \in [\tau_1, \tau_2]$, see Lemma 1. This estimate is optimal for solutions with asymptotics of the form (9)-(10) if we assume that $0 < v < 1$ and that ϕ is not identically zero.

Theorem 1. *Consider a solution to (2)-(3). Assume that $\tau_0 \ge 2$,*

$$1 \le P(\tau_0, \theta) \le \tau_0 - 1, \quad \gamma \le \frac{1}{\tau_0}P(\tau_0,\theta) \le 1 - \gamma \tag{12}$$

for $\theta \in S^1$ and some $\gamma > 0$, and that $F(\tau_0) \le (\gamma - \alpha)^2$ for some $\alpha > 0$, $\alpha < \gamma$. Then there are $v, \phi, q, r \in C^\infty(S^1, \mathbb{R})$, and for all non-negative integers k, polynomials $\Xi_{i,k}$, $i = 1, ..., 4$ in τ, where

$$0 < \alpha \le v \le 1 - \alpha < 1 \tag{13}$$

on S^1 such that

$$\|P_\tau - v\|_{C^k(S^1,\mathbb{R})} \le \Xi_{1,k}\exp[-\alpha\tau], \tag{14}$$

$$\|P - p\|_{C^k(S^1,\mathbb{R})} \le \Xi_{2,k}\exp[-\alpha\tau], \tag{15}$$

$$\|e^{2p}Q_\tau - r\|_{C^k(S^1,\mathbb{R})} \le \Xi_{3,k}\exp[-\alpha\tau], \tag{16}$$

and

$$\|e^{2p}(Q - q) + \frac{r}{2v}\|_{C^k(S^1,\mathbb{R})} \le \Xi_{4,k}\exp[-\alpha\tau] \tag{17}$$

for all $\tau \in [\tau_0, \infty)$*, where* $p = v \cdot \tau + \phi$*.*

Remark. The result obtained is the same as in [11], but the conditions are weaker. We wish to emphasize that the proof does not rely on [11] in any essential way. We only need two pages of estimates from that paper in order to complete the argument. Thus, the rather intricate arguments presented in [11] are avoided.

The proof is to be found at the end of Section 2. Under the conditions of the theorem, we thus obtain the expansions (9)-(10). Note the condition (12) which requires that $P \geq 1$. This is a rather unnatural condition which one would like to get rid of. It turns out that the reason one has to impose this condition is due to a bad choice of representative of hyperbolic space. Consider the asymptotic behaviour (9)-(10). Under the map (8), one gets the following picture in the upper half plane. The x-coordinate converges for every fixed θ and the y-coordinate tends to zero. In other words, for a fixed θ, the solution goes to the boundary along a geodesic, at least approximately. Let us consider the same picture in the disc model. Let D be the open unit disc in the complex plane and let

$$g_D = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{18}$$

be a metric on it. In complex notation, we have the isometry

$$\phi_{HD}(z) = \frac{z - i}{z + i} \tag{19}$$

from the upper half plane to the disc model. Note that the inverse is given by $\phi_{HD}^{-1}(w) = i(1 + w)/(1 - w)$. Say now that we have a solution tending to the boundary in the disc model. For some spatial point, the solution tends to 1, and for neighbouring points, the solution goes to the boundary a little bit below and a little bit above in the complex plane. If one translates this to the PQ-variables, one gets a very violent behaviour which is completely unrelated to the geometry of the wave map problem. For this reason, we consider the equations in the disc model. We have

$$\partial_\tau \left(\frac{z_\tau}{(1 - |z|^2)^2} \right) - e^{-2\tau} \partial_\theta \left(\frac{z_\theta}{(1 - |z|^2)^2} \right) = \frac{2z}{(1 - |z|^2)^3} [|z_\tau|^2 - e^{-2\tau} |z_\theta|^2]. \tag{20}$$

We will use the notation

$$|z_\tau|_D^2 = \frac{4}{(1 - |z|^2)^2} |z_\tau|^2 \tag{21}$$

and similarly for z_θ, $z_\tau + e^{-\tau} z_\theta$ etc. Let

$$\rho = \ln \frac{1 + |z|}{1 - |z|}. \tag{22}$$

This is the distance from the origin to the point z with respect to the hyperbolic metric. Note that ρ^2 and $\rho/|z|$ are smooth functions on the open unit disc.

Theorem 2. *Consider a solution* z *to (20). Assume that there is an* $0 < \alpha < 1$ *such that at* τ_0*,*

$$\frac{1}{2} \sup_\theta |z_\tau + e^{-\tau} z_\theta|_D^2 + \frac{1}{2} \sup_\theta |z_\tau - e^{-\tau} z_\theta|_D^2 \leq (1 - \alpha)^2. \tag{23}$$

Then there is a $v \in C^0(S^1, \mathbb{R}^2)$ *with* $|v| \leq 1 - \alpha$ *and a* C *and a* T *such that for* $\tau \geq T$,

$$\|\frac{1}{\tau}\frac{z(\tau,\cdot)}{|z(\tau,\cdot)|}\rho(\tau,\cdot) - v\|_{C^0(S^1,\mathbb{R}^2)} + \|\frac{2z_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} - v\|_{C^0(S^1,\mathbb{R}^2)}$$
$$+e^{-\tau}\|\frac{2z_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\|_{C^0(S^1,\mathbb{R}^2)} \leq C\tau^{-1}.$$

Remark. Note that if $v(\theta_0) = 0$, then the above estimate implies that $z(\tau, \theta_0)$ remains bounded as $\tau \to \infty$. In the proof, we show that the left hand side of (23) decreases as τ increases.

The proof is to be found at the end of Section 3. As noted in Lemma 5, one gets the same conclusions if one replaces the condition (23) with the condition that there is a T such that $\rho(\tau, \theta) \leq \tau - 2$ for $\tau \geq T$. Of course, one would like to have smooth expansions. Note that this can be done in all generality in the polarized case, see [6]. We are not able to prove that there are smooth expansions if $|v| = 0$, but all other cases can be dealt with in the following sense.

Theorem 3. *Consider a solution to (20). Let* $\theta_0 \in S^1$ *be a fixed angle, and assume that there is a* T *and a* $0 < \gamma < 1$ *such that*

$$\frac{\rho(\tau,\theta_0)}{\tau} \leq 1 - \gamma$$

for all $\tau \geq T$. *Assume furthermore that there is a sequence* $\tau_k \to \infty$ *such that*

$$\frac{\rho(\tau_k,\theta_0)}{\tau_k} \geq \gamma$$

Then there is an $\eta > 0$ *and an isometry* ϕ *from* (D, g_D) *to* $(\mathbb{R}^2, g_R)$ *such that if* $(Q, P) = \phi \circ z$, *then the conclusions of Theorem 1 hold if we replace* S^1 *with* I_η, *where* $I_\eta = (\theta_0 - \eta, \theta_0 + \eta)$.

The theorem follows from Lemma 6 and Lemma 7. If we skip the condition that $\rho(\tau_k, \theta_0)/\tau_k \geq \gamma$, we still get conclusions similar to those of Theorem 2, cf. Lemma 7.

When working on this article, the author was made aware of the fact that similar results had been obtained by Chae and Chruściel. Since their methods are quite different from ours, it is however our hope that the reader will find both articles interesting.

2. Estimates

The purpose of this section is to prove Theorem 1. We begin by describing the decay properties of the function F defined in (11).

Lemma 1. *Consider a solution to (2)-(3). Assume that* $\tau_2 \geq \tau_1 \geq 2$ *and that*

$$1 \leq P(\tau,\theta) \leq \tau - 1$$

for $(\tau, \theta) \in [\tau_1, \tau_2] \times S^1$. *Then if* F *is defined in (11),*

$$F(\tau) \leq F(\tau_1)\left(\frac{\tau_1}{\tau}\right)^2$$

for all $\tau \in [\tau_1, \tau_2]$.

Proof. Let

$$\mathcal{A}_1 = \frac{1}{2}e^{\tau}(P_\tau - \frac{1}{\tau}P + e^{-\tau}P_\theta)^2, \quad \mathcal{A}_2 = \frac{1}{2}e^{\tau}e^{2P}(Q_\tau + e^{-\tau}Q_\theta)^2,$$

$$\mathcal{B}_1 = \frac{1}{2}e^{\tau}(P_\tau - \frac{1}{\tau}P - e^{-\tau}P_\theta)^2, \quad \mathcal{B}_2 = \frac{1}{2}e^{\tau}e^{2P}(Q_\tau - e^{-\tau}Q_\theta)^2$$

and

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \quad \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2. \tag{24}$$

One can compute that

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A}_1 = (\frac{1}{2} - \frac{1}{\tau})e^{\tau}[(P_\tau - \frac{1}{\tau}P)^2 + e^{-2\tau}P_\theta^2] - (1 - \frac{2}{\tau})e^{-\tau}P_\theta^2$$
$$+e^{2P+\tau}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)(P_\tau - \frac{1}{\tau}P + e^{-\tau}P_\theta)$$

and that

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A}_2 = \frac{1}{2}e^{2P+\tau}[Q_\tau^2 + e^{-2\tau}Q_\theta^2] \tag{25}$$
$$-e^{2P+\tau}(P_\tau + e^{-\tau}P_\theta)(Q_\tau^2 - e^{-2\tau}Q_\theta^2) - e^{2P-\tau}Q_\theta^2.$$

Thus

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A} = (\frac{1}{2} - \frac{1}{\tau})e^{\tau}[(P_\tau - \frac{1}{\tau}P)^2 + e^{-2\tau}P_\theta^2] - (1 - \frac{2}{\tau})e^{-\tau}P_\theta^2$$
$$-\frac{P}{\tau}e^{2P+\tau}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) + \frac{1}{2}e^{2P+\tau}[Q_\tau^2 + e^{-2\tau}Q_\theta^2] - e^{2P-\tau}Q_\theta^2.$$

Assuming $1 \leq P \leq \tau - 1$ and $\tau \geq 2$, we thus get

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A} \leq (\frac{1}{2} - \frac{1}{\tau})(\mathcal{A} + \mathcal{B}).$$

Similarly, if $1 \leq P \leq \tau - 1$ and $\tau \geq 2$,

$$(\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{B} \leq (\frac{1}{2} - \frac{1}{\tau})(\mathcal{A} + \mathcal{B}).$$

Let us write down the analogue of (25) for future reference.

$$(\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{B}_2 = \frac{1}{2}e^{2P+\tau}[Q_\tau^2 + e^{-2\tau}Q_\theta^2] \tag{26}$$
$$-e^{2P+\tau}(P_\tau - e^{-\tau}P_\theta)(Q_\tau^2 - e^{-2\tau}Q_\theta^2) - e^{2P-\tau}Q_\theta^2.$$

Let

$$F_1(u,\theta) = \mathcal{A}(u, \theta + e^{-u}), \quad F_2(u,\theta) = \mathcal{B}(u, \theta - e^{-u}),$$

$$\hat{F}_i(u) = \sup_{\theta \in S^1} F_i(u,\theta), \quad \hat{F}(u) = \hat{F}_1(u) + \hat{F}_2(u). \tag{27}$$

We get

$$F_1(u_2,\theta) = F_1(u_1,\theta) + \int_{u_1}^{u_2} [(\partial_u - e^{-u}\partial_\theta)\mathcal{A}](u, \theta + e^{-u})du$$
$$\leq \hat{F}_1(u_1) + \int_{u_1}^{u_2} (\frac{1}{2} - \frac{1}{u})\hat{F}(u)du,$$

assuming $1 \leq P(\tau,\theta) \leq \tau - 1$ and $\tau \geq 2$ for $(\tau,\theta) \in [u_1, u_2] \times S^1$. Take the sup norm of this and add it to a similar estimate for $\hat{F}_2$ in order to obtain

$$\hat{F}(u_2) \leq \hat{F}(u_1) + \int_{u_1}^{u_2} (1 - \frac{2}{u})\hat{F}(u)du.$$

Grönwall's lemma implies

$$e^{-u_2}\hat{F}(u_2) \leq e^{-u_1}\hat{F}(u_1)\left(\frac{u_1}{u_2}\right)^2.$$

The lemma follows. □

Corollary 1. *Consider a solution to (2)-(3). Assume that* $\tau_0 \geq 2$,

$$1 \leq P(\tau_0,\theta) \leq \tau_0 - 1, \quad \gamma \leq \frac{1}{\tau_0}P(\tau_0,\theta) \leq 1 - \gamma$$

for $\theta \in S^1$ *and some* $\gamma > 0$, *and that* $F(\tau_0) \leq (\gamma - \alpha)^2$ *for some* $\alpha > 0$, $\alpha < \gamma$. *Then*

$$F(\tau) \leq F(\tau_0)\left(\frac{\tau_0}{\tau}\right)^2$$

for all $\tau \geq \tau_0$. *Furthermore, there is a function* $v \in C^0(S^1,\mathbb{R})$ *and a constant* $0 < C < \infty$ *such that*

$$\|P_\tau(\tau,\cdot) - v\|_{C^0(S^1,\mathbb{R})} \leq \frac{C}{\tau}$$

for $\tau \geq \tau_0$ *and* $\alpha \leq v(\theta) \leq 1 - \alpha$ *for all* $\theta \in S^1$.

Proof. Consider the set

$$\mathcal{S} = \{\tau \geq \tau_0 : s \in [\tau_0,\tau] \Rightarrow 1 \leq P(s,\theta) \leq s - 1 \ \ \forall\theta \in S^1\}.$$

This set is closed and connected by definition. From the conditions of the lemma, it is clear that it is non-empty. We wish to prove that it is open (in the subspace topology on $[\tau_0,\infty)$). Assume that $\tau_1 \in \mathcal{S}$. Let us estimate, using Lemma 1,

$$|\frac{P(\tau_1,\theta)}{\tau_1} - \frac{P(\tau_0,\theta)}{\tau_0}| = |\int_{\tau_0}^{\tau_1} \frac{1}{\tau}[P_\tau(\tau,\theta) - \frac{1}{\tau}P(\tau,\theta)]d\tau| \leq \int_{\tau_0}^{\tau_1} \frac{F^{1/2}(\tau)}{\tau}d\tau \tag{28}$$

$$\leq (\gamma - \alpha)\int_{\tau_0}^{\tau_1} \frac{\tau_0}{\tau^2}d\tau \leq (\gamma - \alpha)(1 - \frac{\tau_0}{\tau_1}).$$

Using Lemma 1 once again we conclude that

$$|P_\tau(\tau_1,\theta) - \frac{P(\tau_1,\theta)}{\tau_1}| \leq (\gamma - \alpha)\frac{\tau_0}{\tau_1}.$$

Combining these two estimates, we get

$$|P_\tau(\tau_1,\theta) - \frac{P(\tau_0,\theta)}{\tau_0}| \leq \gamma - \alpha.$$

Since $\gamma \leq P(\tau_0,\theta)/\tau_0 \leq 1 - \gamma$ for all $\theta \in S^1$, $\alpha \leq P_\tau(\tau_1,\theta) \leq 1 - \alpha$ for all $\theta \in S^1$. This implies that there is a set of the form $[\tau_1, \tau_1 + \epsilon)$ with $\epsilon > 0$ such that $1 \leq P(s,\theta) \leq s - 1$ for all s in this set and $\theta \in S^1$. Consequently, $\mathcal{S}$ is open, so that $\mathcal{S} = [\tau_0,\infty)$. An estimate similar to that of (28) yields the conclusion that for $\tau_2 \geq \tau_1 \geq \tau_0$

$$\|\frac{P(\tau_2,\cdot)}{\tau_2} - \frac{P(\tau_1,\cdot)}{\tau_1}\|_{C^0(S^1,\mathbb{R})} \leq \frac{C}{\tau_1}.$$

Thus there is a $v \in C^0(S^1, \mathbb{R})$ such that

$$\|\frac{P(\tau,\cdot)}{\tau} - v\|_{C^0(S^1,\mathbb{R})} \leq \frac{C}{\tau}.$$

Letting τ_1 tend to infinity in (28), we get the conclusion that $\alpha \leq v(\theta) \leq 1-\alpha$ for all $\theta \in S^1$. Combining this with the fact that F decays like τ^{-2}, we get the conclusions of the corollary. □

Lemma 2. *Consider a solution to (2)-(3) with initial data satisfying the conditions in the statement of Corollary 1. Then*

$$\|e^{2P}[Q_\tau^2(\tau,\cdot) + e^{-2\tau}Q_\theta^2(\tau,\cdot)]\|_{C^0(S^1,\mathbb{R})} \leq Ce^{-\alpha\tau}.$$

Proof. Consider (25) and (26). By Corollary 1 we conclude that $e^{-\tau}P_\theta$ converges to zero in the sup norm. Since we also know that P_τ converges to v, we conclude that for τ great enough,

$$\frac{\alpha}{2} \leq P_\tau \pm e^{-\tau}P_\theta \leq 1 - \frac{\alpha}{2}.$$

Combining this with (25) and (26) we obtain

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A}_2 \leq (\frac{1}{2} - \frac{\alpha}{2})(\mathcal{A}_2 + \mathcal{B}_2), \quad (\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{B}_2 \leq (\frac{1}{2} - \frac{\alpha}{2})(\mathcal{A}_2 + \mathcal{B}_2).$$

The remaining part of the argument is similar to the end of the proof of Lemma 1. □

Consider the energies

$$\mathcal{E}_k = \frac{1}{2}\int_{S^1}[(\partial_\theta^k\partial_\tau P)^2 + e^{-2\tau}(\partial_\theta^{k+1}P)^2]d\theta$$

and

$$E_k = \frac{1}{2}\int_{S^1} e^{2P}[(\partial_\theta^k\partial_\tau Q)^2 + e^{-2\tau}(\partial_\theta^{k+1}Q)^2]d\theta.$$

We will also be interested in the energies

$$H_k = 1 + \mathcal{E}_k + E_k + \frac{1}{2}e^{-\alpha\tau}\int_{S^1}(\partial_\theta^k P)^2 d\theta. \tag{29}$$

Note that

$$\frac{d\mathcal{E}_k}{d\tau} \leq \int_{S^1}\partial_\theta^k[e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)]\partial_\theta^k\partial_\tau P d\theta. \tag{30}$$

Furthermore, since we can assume $\alpha/2 \leq P_\tau \leq 1-\alpha/2$ for τ large enough, one can estimate

$$\frac{dE_k}{d\tau} \leq -\alpha E_k + \int_{S^1} f_k\partial_\theta^k\partial_\tau Q d\theta \tag{31}$$

for τ large enough, where

$$f_k = \partial_\tau(e^{2P}\partial_\theta^k\partial_\tau Q) - e^{-2\tau}\partial_\theta(e^{2P}\partial_\theta^{k+1}Q). \tag{32}$$

Note that $f_0 = 0$ and that

$$f_{k+1} = \partial_\theta f_k - 2P_\theta f_k - 2P_{\theta\tau}e^{2P}\partial_\theta^k\partial_\tau Q + 2P_{\theta\theta}e^{2P-2\tau}\partial_\theta^{k+1}Q. \tag{33}$$

The details of the derivations of the last three equations can be found in the proof of Lemma 5.2 of [11].

Lemma 3. *Consider a solution to (2)-(3) with initial data satisfying the conditions in the statement of Corollary 1. Then H_1 defined by (29) is bounded to the future.*

Proof. We wish to prove this by proving an estimate of the form

$$\frac{dH_1}{d\tau} \leq Ce^{-\beta\tau}H_1,$$

where $\beta > 0$. Let us prove this with H_1 on the left hand side replaced by the constituents of H_1. By (30) we have

$$\frac{d\mathcal{E}_1}{d\tau} \leq \int_{S^1} \partial_\theta[e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)]P_{\tau\theta}d\theta \leq \sqrt{2}\|\partial_\theta[e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)]\|_{L^2(S^1,\mathbb{R})}\mathcal{E}_1^{1/2}.$$

Let us estimate the L^2-norm appearing on the right hand side. There are two cases to deal with. Estimate, using Lemma 2,

$$\|P_\theta e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)\|_{L^2(S^1,\mathbb{R})} \leq Ce^{-\alpha\tau}e^{\alpha\tau/2}H_1^{1/2},$$

which is of the desired form. The other case can be estimated

$$\|e^{2P}(Q_\tau Q_{\tau\theta} - e^{-2\tau}Q_\theta Q_{\theta\theta})\|_{L^2(S^1,\mathbb{R})} \leq Ce^{-\alpha\tau/2}E_1^{1/2},$$

using Lemma 2. By (31)-(33) and the fact that $f_0 = 0$, we get

$$\frac{dE_1}{d\tau} \leq \int_{S^1}[-2P_{\tau\theta}e^{2P}Q_\tau Q_{\tau\theta} + 2e^{2P-2\tau}P_{\theta\theta}Q_\theta Q_{\tau\theta}]d\theta.$$

Thus we can use Lemma 2 in order to obtain

$$\frac{dE_1}{d\tau} \leq Ce^{-\alpha\tau/2}(E_1 + \mathcal{E}_1).$$

Finally

$$\frac{d}{d\tau}[\frac{1}{2}e^{-\alpha\tau}\int_{S^1} P_\theta^2 d\theta] = -\frac{\alpha}{2}e^{-\alpha\tau}\int_{S^1} P_\theta^2 d\theta + e^{-\alpha\tau}\int_{S^1} P_\theta P_{\tau\theta}d\theta \leq 2e^{-\alpha\tau/2}H_1^{1/2}\mathcal{E}_1^{1/2}.$$

The lemma follows. □

Corollary 2. *Consider a solution to (2)-(3) with initial data satisfying the conditions in the statement of Corollary 1. Then*

$$E_1(\tau) \leq C(1 + \tau^2)e^{-\alpha\tau} \tag{34}$$

for all $\tau \geq \tau_0$.

Proof. By (31)-(33) and the fact that $f_0 = 0$, we get

$$\frac{dE_1}{d\tau} \leq -\alpha E_1 + C\exp[-\alpha\tau/2]E_1^{1/2},$$

where we have also used Lemma 2 and the fact that H_1 is bounded. This yields the conclusion of the corollary. □

Lemma 4. *Consider a solution to (2)-(3) with initial data satisfying the conditions in the statement of Corollary 1. Then for every k, there is a constant C_k and a polynomial $\mathcal{P}_k$ in τ such that for $\tau \geq 0$*

$$H_k \leq C_k, \quad E_k \leq \mathcal{P}_k(\tau)e^{-\alpha\tau}.$$

Proof. Let us make the inductive assumption that

$$H_i \leq C_i, \quad E_i \leq \mathcal{P}_i(\tau)e^{-\alpha\tau}$$

for $i = 1, ..., k$, where the C_i are constants and the $\mathcal{P}_i$ are polynomials. By Lemma 3 and (34), we know this to be true for $k = 1$. Note that if we let

$$\mathcal{F}_l = \frac{1}{2}\int_{S^1}(\partial_\theta^{l+1}P)^2 d\theta,$$

we have

$$\frac{d\mathcal{F}_l}{d\tau} = \int_{S^1}\partial_\theta^{l+1}P\partial_\theta^{l+1}\partial_\tau P d\theta \leq 2\mathcal{F}_l^{1/2}\mathcal{H}_{l+1}^{1/2},$$

whence

$$\mathcal{F}_l \leq C_l(1+\tau^2)$$

for $l = 0, ..., k-1$. Consequently, we have, using the definition of H_k,

$$\|\partial_\theta^l P\|_{C^0(S^1,\mathbb{R})} \leq C_l(1+\tau^2)^{1/2}, \quad \|\partial_\theta^k P\|_{C^0(S^1,\mathbb{R})} \leq C_k e^{\alpha\tau/2}H_{k+1}^{1/2}, \tag{35}$$

$$\|\partial_\theta^k P\|_{L^2(S^1,\mathbb{R})} \leq C_k(1+\tau^2)^{1/2}, \quad \|\partial_\theta^{k+1} P\|_{L^2(S^1,\mathbb{R})} \leq C_k e^{\alpha\tau/2}H_{k+1}^{1/2}, \tag{36}$$

where $l = 1, ..., k-1$. Note that if $l \geq 1$, $\partial_\theta^l\partial_\tau Q$ has to have a zero on the circle since its average over the circle is zero. Thus

$$\begin{aligned}\|e^P\partial_\theta^l\partial_\tau Q\|_{C^0(S^1,\mathbb{R})} &\leq \int_{S^1}|P_\theta e^P\partial_\theta^l\partial_\tau Q + e^P\partial_\theta^{l+1}\partial_\tau Q|d\theta \\ &\leq \mathcal{Q}_l e^{-\alpha\tau/2} + 2\pi^{1/2}E_{l+1}^{1/2},\end{aligned} \tag{37}$$

for $l = 1, ..., k$, where $\mathcal{Q}_l$ is a polynomial in τ and we have used the induction hypothesis and the first inequality in (36) with $k = 1$, which holds due to Lemma 3. If $l \leq k-1$, we thus get exponential decay of the left hand side, due to the induction hypothesis. A similar estimate holds if we replace Q_τ with $e^{-\tau}Q_\theta$.

The idea of proof is the same as in Lemma 3. Consider

$$\frac{d\mathcal{E}_{k+1}}{d\tau} \leq \int_{S^1}\partial_\theta^{k+1}[e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2)]\partial_\theta^{k+1}\partial_\tau P d\theta.$$

By Hölder's inequality, we need to estimate expressions of the form

$$\|\partial_\theta^{l_1}P\cdots\partial_\theta^{l_o}Pe^{2P}\partial_\theta^m\partial_\tau Q\partial_\theta^n\partial_\tau Q\|_{L^2(S^1,\mathbb{R})} \tag{38}$$

where $l_1 + ... + l_o + m + n = k+1$, and the analogous expressions with Q_τ replaced by $e^{-\tau}Q_\theta$. We wish to estimate this by $H_{k+1}^{1/2}$ times something that decays exponentially. If the biggest l_i is $k+1$, we can use the second inequality in (36) and Lemma 2. If there are two l_i which equal k, then $k = 1$ and we can use Lemma 2, the second inequality in (35) and the first inequality in (36) to obtain the desired estimate. If there is only one l_i equal to k and the rest are smaller, we can take out all factors $\partial_\theta^{l_j}P$ using the first or the second of (35). The remaining terms are then estimated using Lemma 2 or (34). If all the l_i are less than or equal to $k-1$, we can take out the derivatives of P in the sup norm with a polynomial bound in τ, cf. the first inequality in (35). Thus, we need not concern ourselves with derivatives hitting P in what follows. If one of m and n are less than or equal to $k-1$, we can take out the corresponding factor in the sup norm, using (37) and the fact that E_{l+1} decays exponentially for $l \leq k-1$. The problem which remains is the case where $m = n = k$. Then $k = 1$ and we can use (34) and (37) to take out one factor in the sup norm and estimate the remaining L^2 norm by $E_1^{1/2}$. The argument for expressions of the form (38) where one has replaced Q_τ with $e^{-\tau}Q_\theta$ is similar.

Consider (31)-(33). Since $f_0 = 0$, we inductively get the conclusion that f_{k+1} is a sum of terms of the form

$$\partial_\theta^{m+1}\partial_\tau P e^{2P}\partial_\theta^l \partial_\tau Q, \tag{39}$$

where $m+l=k$ and terms where one replaces P_τ with $e^{-\tau}P_\theta$ and Q_τ with $e^{-\tau}Q_\theta$. Note that the effect of the operator $\partial_\theta - 2P_\theta$ on an expression of the form $e^{2P}g$ is to only differentiate g. This is the reason for (39). Discarding the first term in (31), we see that the estimate we wish to achieve is

$$\|e^{-P}f_{k+1}\|_{L^2(S^1,\mathbb{R})} \leq Ce^{-\beta\tau}H_{k+1}^{1/2}.$$

Due to the form (39), the relevant things to estimate are thus

$$\|\partial_\theta^{m+1}\partial_\tau P e^{P}\partial_\theta^l \partial_\tau Q\|_{L^2(S^1,\mathbb{R})}, \tag{40}$$

where $l+m=k$. If $l \leq k-1$, we use (37) to estimate $e^P\partial_\theta^l\partial_\tau Q$ in the sup norm. If $l=k$, we take out $P_{\tau\theta}$ in the sup norm and use the induction hypothesis to conclude that what remains decays exponentially. If $k=1$, we can estimate $P_{\tau\theta}$ by $H_{k+1}^{1/2}$. Otherwise, it is bounded. The arguments for the terms where one replaces P_τ with $e^{-\tau}P_\theta$ and Q_τ with $e^{-\tau}Q_\theta$ are similar.

Finally

$$\frac{d}{d\tau}[\frac{1}{2}e^{-\alpha\tau}\int_{S^1}(\partial_\theta^{k+1}P)^2 d\theta] \leq e^{-\alpha\tau}\int_{S^1}\partial_\theta^{k+1}P\partial_\theta^{k+1}\partial_\tau P d\theta \leq 2e^{-\alpha\tau/2}H_{k+1}.$$

In consequence, we have the desired inequality for H_{k+1}' and we conclude that H_{k+1} is bounded. In order to complete the induction, we need to prove the decay of E_{k+1}. Due to (31), we only need to prove that expressions of the form (40) can be estimated by a polynomial times $e^{-\alpha\tau/2}$ (and the analogous expressions where ∂_τ is replaced with $e^{-\tau}\partial_\theta$). If $l \leq k-1$, we can use (37) and the boundedness of H_{k+1}. Since $P_{\tau\theta}$ is bounded in the sup norm due to the boundedness of H_2, and since E_k has the desired decay, we can also deal with the case $l=k$. The lemma follows. □

Proof of Theorem 1. The conditions of Corollary 1 are satisfied. By Lemma 4, we conclude that $\mathcal{E}_k$ is bounded and that E_k decays as $e^{-\alpha\tau}$ times a polynomial for all k. Furthermore, for all k, there is a polynomial $\mathcal{Q}_k$ in τ such that for all k

$$\|e^P\partial_\theta^k\partial_\tau Q\|_{C^0(S^1,\mathbb{R})} + \|e^{P-\tau}\partial_\theta^{k+1}Q\|_{C^0(S^1,\mathbb{R})} \leq \mathcal{Q}_k(\tau)e^{-\alpha\tau/2}.$$

This follows from (37) and Lemma 4. When one has these estimates, it is not so difficult to combine them with the equations in order to get the conclusions of the theorem. However, the arguments require almost two pages, so we refer the reader to pp. 23-25 of [11] for the details. □

3. The disc model

We begin by proving a result analogous to Lemma 1 in the disc model.

Lemma 5. *Let z be a solution to (20) and assume that there is a T such that $\rho(\tau,\theta) \leq \tau - 2$ for $(\tau,\theta) \in [T,\infty) \times S^1$, where ρ is defined in (22). Then there is a $v \in C^0(S^1,\mathbb{R}^2)$ with $|v| \leq 1$ and a C such that for $\tau \geq T$,*

$$\|\frac{1}{\tau}\frac{z(\tau,\cdot)}{|z(\tau,\cdot)|}\rho(\tau,\cdot) - v\|_{C^0(S^1,\mathbb{R}^2)} + \|\frac{2z_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} - v\|_{C^0(S^1,\mathbb{R}^2)}$$
$$+e^{-\tau}\|\frac{2z_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\|_{C^0(S^1,\mathbb{R}^2)} \leq C\tau^{-1}.$$

Proof. If z is a solution of (20), let

(41) $$\mathcal{A}_1 = \frac{1}{2}e^\tau|z_\tau + e^{-\tau}z_\theta|_D^2, \quad \mathcal{A}_2 = \frac{1}{2}e^\tau|z_\tau - e^{-\tau}z_\theta|_D^2,$$

with notation as in (21), and

$$\mathcal{C}_1 = -\frac{1}{2\tau}e^\tau(\partial_\tau + e^{-\tau}\partial_\theta)(\rho)^2 + \frac{1}{2}e^\tau\frac{\rho^2}{\tau^2}, \quad \mathcal{C}_2 = -\frac{1}{2\tau}e^\tau(\partial_\tau - e^{-\tau}\partial_\theta)(\rho)^2 + \frac{1}{2}e^\tau\frac{\rho^2}{\tau^2}.$$

Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are smooth functions. If we let

$$f = \frac{1}{2\tau}(1-|z|^2)\frac{z}{|z|}\rho,$$

which is a smooth function, then

(42) $$\mathcal{B}_1 = \mathcal{A}_1 + \mathcal{C}_1 = \frac{1}{2}e^\tau|z_\tau - f + e^{-\tau}z_\theta|_D^2$$

and similarly for $\mathcal{B}_2 = \mathcal{A}_2 + \mathcal{C}_2$. Consequently $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-negative. We have

(43) $$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A}_1 = (\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{A}_2 = \frac{1}{2}e^\tau[|z_\tau|_D^2 - e^{-2\tau}|z_\theta|_D^2].$$

Let us compute

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{C}_1 = \frac{1}{\tau^2}e^\tau\partial_\tau(\rho^2) - \frac{1}{2\tau}e^\tau\partial_\tau(\rho^2) + \frac{1}{2}e^\tau\frac{\rho^2}{\tau^2} - \frac{1}{\tau}e^\tau\frac{\rho^2}{\tau^2}$$
$$-\frac{1}{2\tau}e^\tau(\partial_\tau^2 - e^{-2\tau}\partial_\theta^2)(\rho^2).$$

If $|z| > 0$, we can compute

(44) $$\frac{1}{2}(\partial_\tau^2 - e^{-2\tau}\partial_\theta^2)(\rho^2) = (\rho_\tau^2 - e^{-2\tau}\rho_\theta^2) + \rho(\rho_{\tau\tau} - e^{-2\tau}\rho_{\theta\theta}),$$

and that

$$\sinh\rho(\rho_{\tau\tau} - e^{-2\tau}\rho_{\theta\theta}) = \cosh\rho[|z_\tau|_D^2 - e^{-2\tau}|z_\theta|_D^2 - \rho_\tau^2 + e^{-2\tau}\rho_\theta^2].$$

Observe that

(45) $$\rho_\tau^2 + \sinh^2\rho|\partial_\tau(\frac{z}{|z|})|^2 = |z_\tau|_D^2$$

and similarly for the θ derivatives. Consequently

$$|z_\tau|_D^2 - \rho_\tau^2 \geq 0.$$

Since $\rho\cosh\rho \geq \sinh\rho$, we conclude that

$$-\rho(\rho_{\tau\tau} - e^{-2\tau}\rho_{\theta\theta}) \leq -|z_\tau|_D^2 + \rho_\tau^2 - e^{-2\tau}\rho_\theta^2 + \frac{\rho\cosh\rho}{\sinh\rho}e^{-2\tau}|z_\theta|_D^2$$

Combining this with (44), we get

$$-\frac{1}{2}(\partial_\tau^2 - e^{-2\tau}\partial_\theta^2)(\rho^2) \le -|z_\tau|_D^2 + \frac{\rho\cosh\rho}{\sinh\rho}e^{-2\tau}|z_\theta|_D^2.$$

Since $\rho\cosh\rho/\sinh\rho \le \tau - 1$ if $\rho \le \tau - 2$, we get

$$-\frac{1}{2\tau}e^\tau(\partial_\tau^2 - e^{-2\tau}\partial_\theta^2)(\rho^2) \le e^{-\tau}|z_\theta|_D^2 - \frac{1}{\tau}e^\tau[|z_\tau|_D^2 + e^{-2\tau}|z_\theta|_D^2].$$

We conclude that if $\rho \le \tau - 2$, then

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{B}_1 \le (\frac{1}{2} - \frac{1}{\tau})e^\tau[|z_\tau|_D^2 + e^{-2\tau}|z_\theta|_D^2] + \frac{1}{\tau^2}e^\tau\partial_\tau(\rho^2) \tag{46}$$

$$-\frac{1}{2\tau}e^\tau\partial_\tau(\rho^2) + \frac{1}{2}e^\tau\frac{\rho^2}{\tau^2} - \frac{1}{\tau}e^\tau\frac{\rho^2}{\tau^2} = (\frac{1}{2} - \frac{1}{\tau})(\mathcal{B}_1 + \mathcal{B}_2).$$

Similarly,

$$(\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{B}_2 \le (\frac{1}{2} - \frac{1}{\tau})(\mathcal{B}_1 + \mathcal{B}_2). \tag{47}$$

The above derivation was carried out under the assumption that $|z| > 0$. However, if $z(\tau,\theta) = 0$, then ρ^2 can be replaced with $4|z|^2$ when calculating second derivatives. Thus, at the point (τ,θ), we have

$$(\partial_\tau^2 - e^{-2\tau}\partial_\theta^2)(\rho^2) = 8(|z_\tau|^2 - e^{-2\tau}|z_\theta|^2).$$

Using this observation, one can see that (46) and (47) hold regardless of whether $|z| = 0$ or not. Let

$$G(\tau) = e^{-\tau}\sup_\theta \mathcal{B}_1(\tau,\theta) + e^{-\tau}\sup_\theta \mathcal{B}_2(\tau,\theta),$$

and assume that there is a T such that $\rho(\tau,\theta) \le \tau - 2$ for all $(\tau,\theta) \in [T,\infty) \times S^1$. Then an argument similar to the proof of Lemma 1 shows that

$$G(\tau) \le G(\tau_0)\left(\frac{\tau_0}{\tau}\right)^2$$

if $\tau \ge \tau_0 \ge T$. Assuming $|z| > 0$, we can use (45) in order to obtain

$$e^{-\tau}\mathcal{B}_1 + e^{-\tau}\mathcal{B}_2 = (\rho_\tau - \frac{1}{\tau}\rho)^2 + \sinh^2\rho|\partial_\tau(\frac{z}{|z|})|^2 + e^{-2\tau}|z_\theta|_D^2.$$

Consequently, there is a constant C such that for $\tau \ge T$

$$(\rho_\tau - \frac{1}{\tau}\rho)^2 + \sinh^2\rho|\partial_\tau(\frac{z}{|z|})|^2 \le C\tau^{-2}$$

assuming $|z| > 0$. Define g by

$$g = \frac{1}{\tau}\frac{z}{|z|}\rho.$$

Let us compute, assuming $|z| > 0$,

$$\partial_\tau g = \frac{z}{|z|}\frac{1}{\tau}(\rho_\tau - \frac{1}{\tau}\rho) + \frac{1}{\tau}\partial_\tau(\frac{z}{|z|})\rho.$$

Since $\rho/\sinh\rho$ is bounded, we get the conclusion that

$$|\partial_\tau g| \le C\tau^{-2} \tag{48}$$

for $\tau \ge T$, assuming $|z| > 0$. If $z = 0$, we get

$$\partial_\tau g = \frac{2}{\tau}z_\tau.$$

However, in this case

$$e^{-\tau}\mathcal{B}_1 + e^{-\tau}\mathcal{B}_2 = 4|z_\tau|^2 + 4e^{-2\tau}|z_\theta|^2.$$

Consequently (48) holds regardless of whether z is zero or not. If $\tau_2 \geq \tau_1 \geq T$, we thus get the conclusion that

$$\|g(\tau_2,\cdot) - g(\tau_1,\cdot)\|_{C^0(S^1,\mathbb{R})} \leq C\tau_1^{-1}.$$

In other words, there is a $v \in C^0(S^1,\mathbb{R}^2)$ such that

$$\|g(\tau,\cdot) - v\|_{C^0(S^1,\mathbb{R}^2)} \leq C\tau^{-1}.$$

Note that $|v| \leq 1$, since $|g| \leq 1$ for $\tau \geq T$. Since

$$e^{-\tau}\mathcal{B}_1 + e^{-\tau}\mathcal{B}_2 = |\frac{2z_\tau}{1-|z|^2} - g|^2 + e^{-2\tau}|z_\theta|_D^2,$$

the lemma follows. □

Proof of Theorem 2. Consider $\mathcal{A}_1$ and $\mathcal{A}_2$ defined in (41). Due to (43), we have

$$(\partial_\tau - e^{-\tau}\partial_\theta)\mathcal{A}_1 = (\partial_\tau + e^{-\tau}\partial_\theta)\mathcal{A}_2 \leq \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2).$$

This can be used to conclude that the object appearing on the left hand side of (23) is monotonically decaying to the future. The argument is similar to the proof of Lemma 1. Consequently, $|z_\tau|_D \leq 1-\alpha$ to the future. Due to (45), this means that $\rho(\tau,\theta) \leq (1-\alpha)\tau + C$ for $\tau \geq \tau_0$. At late enough times, the conditions of Lemma 5 are thus fulfilled. The theorem follows. □

4. Local results

Here we wish to state some results that are local in the spatial coordinate. We also wish to relate the pictures in the different models of hyperbolic space. Note that we have the three models of hyperbolic space (6), (7) and (18) which are related by the isometries (8) and (19).

Lemma 6. *Consider a solution to (20). Let $\theta_0 \in S^1$ and assume that there are $\gamma, \epsilon > 0$, T and $v \in C^0(I_\epsilon,\mathbb{R}^2)$, where $I_\epsilon = (\theta_0 - \epsilon, \theta_0 + \epsilon)$, $2\gamma \leq |v| \leq 1 - 2\gamma$ in I_ϵ and*

$$\|\frac{1}{\tau}\frac{z(\tau,\cdot)}{|z(\tau,\cdot)|}\rho(\tau,\cdot) - v\|_{C^0(I_\epsilon,\mathbb{R}^2)} \leq C\tau^{-1} \tag{49}$$

for $\tau \geq T$. Then there is an $\eta > 0$ and an isometry ϕ from (D, g_D) to $(\mathbb{R}^2, g_R)$ such that if $(Q,P) = \phi \circ z$, then the conclusions of Theorem 1 hold if we replace S^1 with I_η.

Proof. Using the isometries (8) and (19), we conclude that the relation between variables in the disc z and the PQ-variables given by

$$(Q,P) = [-\frac{2\mathrm{Im}z}{1+|z|^2 - 2\mathrm{Re}z}, -\ln(1-|z|^2) + \ln(1+|z|^2 - 2\mathrm{Re}z)] \tag{50}$$

is an isometry. Using (22) and (50), we have

$$P = \rho - 2\ln(1+|z|) + \ln(1+|z|^2 - 2\mathrm{Re}z). \tag{51}$$

By the assumptions of the lemma, there is a positive γ such that

$$2\gamma \leq |v(\theta)| \leq 1 - 2\gamma \tag{52}$$

for $\theta \in I_\epsilon$ and by making ϵ smaller, if necessary, we can assume that the image of I_ϵ under $v/|v|$ is contained in an angle interval of length less that π. We can also assume that $\rho \geq 1$ in $[\tau_0, \infty) \times I_\epsilon$. By (49) we have

$$\left\| \frac{z(\tau, \cdot)}{|z(\tau, \cdot)|} - \frac{v}{|v|} \right\|_{C^0(I_\epsilon, \mathbb{R}^2)} \leq C\tau^{-1}$$

for $\tau \geq \tau_0$. Consequently, we can assume that $z/|z|$ is contained in an angle interval of $3\pi/2$. Thus we can perform a rotation to the solution in order to ensure that

$$|z/|z| - 1| \geq c \tag{53}$$

for some positive constant c and $(\tau, \theta) \in [\tau_0, \infty) \times I_\epsilon$. From now on, we will assume that such a rotation has been carried out. By (51) and (53), we conclude that $P = \rho + O(1)$. Combining this with (49) and (52), we get the conclusion that there is a T such that for $(\tau, \theta) \in [T, \infty) \times I_\epsilon$,

$$1 \leq P(\tau, \theta) \leq \tau - 1, \quad \gamma \leq \frac{P(\tau, \theta)}{\tau} \leq 1 - \gamma.$$

We can now carry out a localized version of the argument presented in the proof of Lemma 1. Instead of considering the set $[T, \infty) \times S^1$ we consider (τ, θ) such that $\tau \geq T$ and

$$\theta \in I_{\epsilon,T,\tau} = (\theta_0 - \epsilon - e^{-\tau} + e^{-T}, \theta_0 + \epsilon + e^{-\tau} - e^{-T}),$$

where we assume that T is big enough that $e^{-T} \leq \epsilon/2$. We conclude that if

$$F_{\epsilon,T}(\tau) = \frac{1}{2} e^{-\tau} \sup_{\theta \in I_{\epsilon,T,\tau}} \mathcal{A}(\tau, \theta) + \frac{1}{2} e^{-\tau} \sup_{\theta \in I_{\epsilon,T,\tau}} \mathcal{B}(\tau, \theta),$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in (24), then

$$F_{\epsilon,T}(\tau) \leq C\tau^{-2}.$$

For any $\xi > 0$ there is a T_2 such that if $\tau \geq T_2$, then we can change the initial data outside of $(\theta_0 - \epsilon/4, \theta_0 + \epsilon/4)$ so that $F(\tau) \leq \xi$, where F is defined in (11) and $\gamma \leq P(\tau, \cdot)/\tau \leq 1 - \gamma$. Let us be explicit concerning the change of the initial data. Let $\chi \in C^\infty(S^1, \mathbb{R})$ be such that $\chi(\theta) = 1$ for θ outside of $(\theta_0 - \epsilon/2, \theta_0 + \epsilon/2)$ and $\chi(\theta) = 0$ for $\theta \in (\theta_0 - \epsilon/4, \theta_0 + \epsilon/4)$. Since Q is bounded in the relevant set, due to (50) and (53), the modifications

$$\tilde{P} = (1 - \chi)P + \chi\gamma\tau, \quad \tilde{P}_\tau = (1 - \chi)P_\tau + \chi\gamma, \quad \tilde{Q} = (1 - \chi)Q, \quad \tilde{Q}_\tau = (1 - \chi)Q_\tau$$

yield the desired conclusion for large enough τ. By modifying the initial data as above at a late enough time T_3, the conditions of Theorem 1 will be fulfilled. Furthermore, we can assume that we have the original solution in the strip $[T_3, \infty) \times (\theta_0 - \epsilon/8, \theta_0 + \epsilon/8)$. Consequently, we get the conclusions of Theorem 1 localized to the above mentioned strip. □

Lemma 7. *Consider a solution to (20). Let $\theta_0 \in S^1$ be a fixed angle, and assume that there is a T and a $0 < \gamma < 1$ such that*

$$\frac{\rho(\tau, \theta_0)}{\tau} \leq 1 - \gamma$$

for all $\tau \geq T$. Then there is an $\eta > 0$ and a $v \in C^0(I_\eta, \mathbb{R}^2)$ such that for $\tau \geq T$,

$$\left\| \frac{1}{\tau} \frac{z(\tau, \cdot)}{|z(\tau, \cdot)|} \rho(\tau, \cdot) - v \right\|_{C^0(I_\eta, \mathbb{R}^2)} + \left\| \frac{2z_\tau(\tau, \cdot)}{1 - |z(\tau, \cdot)|^2} - v \right\|_{C^0(I_\eta, \mathbb{R}^2)}$$

$$+e^{-\tau}\|\frac{2z_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\|_{C^0(I_\eta,\mathbb{R}^2)} \leq C\tau^{-1}.$$

Proof. Let

$$\mathcal{S}_\tau = (\theta_0 - 2e^{-\tau}, \theta_0 + 2e^{-\tau}),$$

where we assume that τ is big enough that $2e^{-\tau} \leq \pi/2$. We know that

$$|z_\tau|_D^2 + e^{-2\tau}|z_\theta|_D^2 \leq C$$

for $(\tau,\theta) \in [T,\infty) \times S^1$ and that (45) holds, and a similar division for the spatial derivatives, assuming $|z| > 0$. Consequently,

$$|\frac{\rho(\tau,\theta_1)}{\tau} - \frac{\rho(\tau,\theta_2)}{\tau}| \leq \frac{C}{\tau},$$

assuming $\theta_i \in \mathcal{S}_\tau$. Thus there is a $T_1 \geq T$ such that (assuming $3\beta = \gamma$)

$$\rho(\tau,\theta)/\tau \leq 1 - 2\beta \quad \text{and} \quad \rho(\tau,\theta) \leq \tau - 2 \tag{54}$$

for $\tau \in [T_1,\infty)$ and $\theta \in \mathcal{S}_\tau$. The idea of the argument is as follows. First we consider a quantity similar to the G introduced in the proof of Lemma 5, the only difference being that we take supremum over $\mathcal{S}_\tau$ instead of S^1. At a late enough time, say τ_1, the quantity analogous to G has become small enough that we can close the argument and make statements concerning the domain determined by $\mathcal{S}_{\tau_1}$. This domain contains an open subset of the singularity.

Let $\mathcal{B}_1$ be defined as in (42) and similarly for $\mathcal{B}_2$. Let $\tau \geq \tau_0 \geq T_1$. Define

$$L_1(u,\theta) = \mathcal{B}_1(u,\theta + e^{-u} - e^{-\tau}) \quad \text{and} \quad L_2(u,\theta) = \mathcal{B}_2(u,\theta - e^{-u} + e^{-\tau}),$$

where $\theta \in \mathcal{S}_\tau$. Note that

$$\theta + e^{-u} - e^{-\tau}, \theta - e^{-u} + e^{-\tau} \in \mathcal{S}_u$$

for $u \leq \tau$ and $\theta \in \mathcal{S}_\tau$. Let

$$\hat{L}_i(u) = \sup_{\theta\in\mathcal{S}_u} \mathcal{B}_i(u,\theta) \quad \text{and} \quad \hat{L} = \hat{L}_1 + \hat{L}_2.$$

Due to (46) we have, for $\theta \in \mathcal{S}_\tau$,

$$\begin{aligned}\mathcal{B}_1(\tau,\theta) &= L_1(\tau_0,\theta) + \int_{\tau_0}^{\tau} [(\partial_u - e^{-u}\partial_\theta)\mathcal{B}_1](u,\theta + e^{-u} - e^{-\tau})du \\ &\leq \hat{L}_1(\tau_0) + \int_{\tau_0}^{\tau} (\frac{1}{2} - \frac{1}{u})(\hat{L}_1 + \hat{L}_2)du.\end{aligned}$$

Taking the supremum over $\theta \in \mathcal{S}_\tau$ and adding a similar estimate for $\mathcal{B}_2$, we get the conclusion that

$$\hat{L}(\tau) \leq \hat{L}(\tau_0) + \int_{\tau_0}^{\tau} (1 - \frac{2}{u})\hat{L}(u)du.$$

Consequently,

$$e^{-\tau}\hat{L}(\tau) \leq e^{-\tau_0}\hat{L}(\tau_0)\frac{\tau_0^2}{\tau^2}.$$

Let us consider a similar argument on a different set. Let

$$\mathcal{S}_{\tau_1,\tau} = (\theta_0 - e^{-\tau} - e^{-\tau_1}, \theta_0 + e^{-\tau} + e^{-\tau_1}).$$

Note that $\mathcal{S}_{\tau_1,\tau_1} = \mathcal{S}_{\tau_1}$. For $\theta \in \mathcal{S}_{\tau_1,\tau}$, let

$$K_1(u,\theta) = \mathcal{B}_1(u,\theta + e^{-u} - e^{-\tau}) \quad \text{and} \quad K_2(u,\theta) = \mathcal{B}_2(u,\theta - e^{-u} + e^{-\tau}).$$

For $u \leq \tau$, we have

$$\theta + e^{-u} - e^{-\tau}, \theta - e^{-u} + e^{-\tau} \in \mathcal{S}_{\tau_1,u}.$$

Letting

$$\hat{K}_i(u) = \sup_{\theta \in \mathcal{S}_{\tau_1,u}} \mathcal{B}_i(u,\theta) \quad \text{and} \quad \hat{K} = \hat{K}_1 + \hat{K}_2,$$

we can argue similarly to the above in order to obtain

$$e^{-\tau}\hat{K}(\tau) \leq e^{-\tau_1}\hat{K}(\tau_1)\frac{\tau_1^2}{\tau^2}, \tag{55}$$

assuming $\rho(u,\theta) \leq u-2$ for $u \in [\tau_1,\tau]$ and $\theta \in \mathcal{S}_{\tau_1,u}$. Note that $\hat{K}(\tau_1) = \hat{F}(\tau_1)$. The problem is of course that we cannot assume that $\rho \leq \tau - 2$ in the relevant set. Let us assume that τ_1 is big enough so that

$$e^{-\tau_1}\hat{K}(\tau_1) \leq \beta^2, \quad \tau_1\beta \geq 2 \tag{56}$$

and $\tau_1 \geq T_1$. Due to (54), we know that $\rho(\tau_1,\theta)/\tau_1 \leq 1 - 2\beta$ for $\theta \in \mathcal{S}_{\tau_1}$. Let

$$\mathcal{S} = \{\tau \in [\tau_1,\infty) : s \in [\tau_1,\tau], \theta \in \mathcal{S}_{\tau_1,s} \Rightarrow \frac{\rho(s,\theta)}{s} \leq 1-\beta\}.$$

Note that $\mathcal{S}$ is closed, connected and non-empty. Let us prove that it is open. Let $\tau \in \mathcal{S}$. Then (55) is applicable in $[\tau_1,\tau]$ due to (56). Consequently, if $\theta \in \mathcal{S}_{\tau_1,\tau}$, then

$$|\frac{\rho(\tau,\theta)}{\tau} - \frac{\rho(\tau_1,\theta)}{\tau_1}| = |\int_{\tau_1}^{\tau} \frac{1}{s}(\rho_s - \frac{\rho}{s})ds| \leq \int_{\tau_1}^{\tau} \beta\frac{\tau_1}{s^2}ds = \beta(1-\frac{\tau_1}{\tau}).$$

This implies, using (54), that

$$\frac{\rho(\tau,\theta)}{\tau} \leq 1 - \beta(1+\frac{\tau_1}{\tau}).$$

Consequently, $\mathcal{S}$ is open. The conclusions of the lemma follow by an argument similar to the proof of Lemma 5. □

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