Smoothness at Null Infinity and the Structure of Initial Data

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Abstract. We describe our present understanding of the relations between the behavior of asymptotically flat Cauchy data for Einstein's vacuum field equations near space-like infinity and the asymptotic behavior of their evolution in time at null infinity.

1. Introduction

There are no doubts any longer that the idea of *gravitational radiation* refers to a real physical phenomenon. Framing, however, a precise underlying mathematical concept still poses problems. The work on gravitational radiation by Pirani [62], Trautman [69], Sachs [63], [64], Bondi [10], Newman and Penrose [58] and others, which was brought in a sense to a conclusion by Penrose [59], [60], is based on the idealization of an *isolated self-gravitating system*. It requires information on the long time evolution of gravitational fields which at the time could only be guessed. Ten years before these developments Y. Choquet-Bruhat had achieved a breakthrough in the mathematical analysis of the local Cauchy problem for Einstein's field equations [11]. However, the technical means to derive the fall-off behavior of gravitational fields at far distances and late times from 'basic principles' were not available in the 1960's. In the meantime there has been a considerable progress in controlling the asymptotic structure of solutions to Einstein's field equations but it is still not quite clear which 'basic principles' to assume here.

In the following we shall report on work which aims at closing various gaps in the study of gravitational radiation, the analysis of the Einstein equations, and the calculation of wave forms. Sections 2–5 present a fairly detailed discussion of the underlying analytical structures and of the recent results which led to the author's present understanding of the situation. To maintain the flow of the arguments, the reader is referred for derivations to the original literature. In Sections 6 and 7 will be given new results and detailed arguments.

Penrose's proposal to characterize far fields of *isolated systems* in terms of their conformal structure ([59], [60]) has been criticized over the years on several grounds: various variations, alternatives, etc. have been proposed (cf. [5], [13], [14], [17], [26], [66], [70], [73], and references given therein). Some authors consider the smoothness requirements on the conformal boundary as too restrictive and suggest generalizations (cf. [17], [70], [73]). Doubts have been raised as to whether non-trivial asymptotically simple solutions to the vacuum field equations exist at all ([14]) and it has been argued that the smoothness of the conformal boundary required in [59] excludes interesting physics ([13]). The wide range of opinions on the subject is illustrated by the curious contrast between this emphasis on subtleties of the asymptotic smoothness and claims that 'null infinity is too far away for modelling real physics' (cf. [26], [66]).

In [26] even the *asymptotically flat model* is abandoned and replaced by a *time-like cut model*. The latter introduces a spatially compact time-like hypersurface $\mathcal{T}$ which is chosen in an ad hoc fashion to cut off 'the system of interest' from the rest of the ambient universe. The idea then is to study the system which has thus been 'isolated' as an object of its own.

The usefulness of any such suggestion can only be demonstrated by analyzing its mathematical feasibility. This becomes clear when one tries to calculate wave forms numerically. Such calculations cannot be based on hand waving or physical intuition. The design of an effective numerical computer code requires a precise mathematical formulation.

The analysis of the time-like cut model reduces to a study of the initial boundary value problem for Einstein's field equations in which boundary data are prescribed on $\mathcal{T}$ and Cauchy data are given on a space-like hypersurface $\mathcal{S}$ which intersects $\mathcal{T}$ in the space-like surface $\partial\mathcal{S}$. In [42] has been given a fairly complete analysis of this problem for Einstein's vacuum field equations. This study is only local in time, but it provides insights into the basic problem. So far, the time-like cut model raises many more questions than it appears able to answer.

How is $\mathcal{T}$ to be chosen? Physical considerations may lead to suggestions when the system of interest is 'sufficiently far' away from other systems. However, there is in general no preferred physical or geometrical choice for $\mathcal{T}$. (It is instructive to compare this with the anti-de Sitter-type solutions, where the time-like boundary $\mathcal{J}$ at space-like and null infinity is determined geometrically and the boundary data can be prescribed in covariant form (cf. [36]).)

The boundary must be characterized by some implicit or explicit geometrical condition. A natural choice is to prescribe its mean extrinsic curvature. Its evolution in time is then defined implicitly by a quasi-linear wave equation which itself depends in a non-local way on the data given on $\mathcal{S}$ and $\mathcal{T}$ (cf. [42]). Long time calculations thus require an extra effort to control the regularity of the boundary.

The gauge is related on the time-like boundary $\mathcal{T}$ directly to the evolution process. It depends on the (implicit) choice of a time-like unit vector field tangent to $\mathcal{T}$. While the data which are prescribed on the space-like hypersurface $\mathcal{S}$ allow one to analyze the local geometry near $\mathcal{S}$ at any desired order, the data which can be prescribed on the boundary $\mathcal{T}$ provide very little information on the local geometry near $\mathcal{T}$. All this makes it particularly difficult to show that the gauge and the constraints are preserved under the evolution in time.

These properties imply in general a non-covariance of the boundary conditions and data. Moreover, due to the fact that no causal direction is distinguished on $\mathcal{T}$ there does not seem to exist a natural 'no incoming radiation condition' and, in particular, no natural concept of 'outgoing radiation'. In fact, it appears difficult to associate with the initial boundary value problem any 'simple' quantities which characterize the system and its dynamics and which can be related to observational data.

While the discussion in [42] singles out data which are mathematically admissible, it is far from clear what should be prescribed on $\mathcal{T}$ from the physical point of view. The 'correct' data induced by the ambient universe will never be known. The information fed into 'the system' by the data prescribed on $\mathcal{T}$ can hardly be assessed. In long time calculations it may alter the character of the system drastically.

Because of these difficulties the time-like cut model appears not very promising. Nevertheless, it is of interest because of its similarity to the standard approach to numerical relativity, where an artificial time-like boundary is introduced to render the computational grid finite. It is expected here that the assumption of asymptotic flatness together with a judicious choice of the boundary will alleviate some of the difficulties pointed out above.

At present the only satisfactory solution to the gravitational radiation problem is based on the assumption of asymptotical flatness and the most elegant and geometrically natural definition of the latter is provided by the idea of the *conformal boundary at null infinity* introduced in [59]. While useful physical concepts can be associated with a conformal boundary which is sufficiently smooth (cf. [4], [46], [61] and the references given there), the possible degree of differentiability, which encodes the fall-off behavior of the gravitational field, still poses questions. This article deals with this particular issue and tries to disentangle its various aspects and difficulties.

Einstein's field equations admit certain conformal representations which in the following will be referred to as *conformal field equations*. These equations are 'regular' in the sense that they imply in a suitable gauge equations which are hyperbolic even at points of null infinity ([28], [29]). This fact has been used to show that the smoothness of the conformal boundary is preserved if it is guaranteed on the initial slice $\mathcal{S}$ of an hyperboloidal initial value problem ([31], [33], cf. also [38]). The subsequent analysis of hyperboloidal initial data ([3], [1], [2]) showed the existence of a large class of smooth hyperboloidal data for the conformal field equations. The construction of such data requires the 'free data' to satisfy a finite number of conditions at the space-like boundary $\partial\mathcal{S}$ at which the hyperboloidal slice $\mathcal{S}$ intersects future null infinity $\mathcal{J}^+$.

However, the work referred to above also shows the existence of a large class of hyperboloidal data which are smooth on $\mathcal{S} \setminus \partial\mathcal{S}$ but possess a non-trivial *polyhomogeneous expansion* at $\partial\mathcal{S}$, i.e., an asymptotic expansion in terms of $x^k \log^j x$ where x is a defining function of the boundary $\partial\mathcal{S}$, which vanishes on $\partial\mathcal{S}$. Logarithmic terms can occur as a consequence of the constraint equations even if the free data extend smoothly to $\partial\mathcal{S}$. Recently, it has been shown that certain hyperboloidal data which are polyhomogeneous at $\partial\mathcal{S}$ evolve into solutions to the conformal field equations which possess *generalized conformal boundaries* near the initial slice ([18], [57]). While the precise behavior of these solutions near that boundary still needs to be analyzed, the result shows that the use of the conformal field equations and the characterization of the edge of space-time in terms of its conformal structure are not restricted to asymptotically regular situations.

We conclude from these results that in the standard Cauchy problem the field equations decide on the degree of smoothness of the conformal boundary at null infinity in arbitrarily small neighborhoods of space-like infinity.

There are other reasons to study the region near space-like infinity. The hyperboloidal initial value problem is intrinsically time-asymmetric. To analyze in the same picture incoming radiation, a non-linear scattering process, and outgoing radiation, one needs to include space-like infinity (as pointed out already in [60]). Also, if the hyperboloidal data are not distinguished by special features as, for instance, the presence of a trapped surface, it is not clear which part of the imagined space-time is covered by their evolution. They could represent a hypersurface close to time-like infinity or close to a Cauchy hypersurface (a difficulty shared with the characteristic initial value problem and the initial boundary value problem).

This should not obscure the fact that numerical calculations of space-times from hyperboloidal data allow one to determine wave forms for many 'realistic' physical processes. So far the only semi-global calculations of space-times, including their radiation fields at null infinity, are based on hyperboloidal and characteristic initial value problems (cf. [27], [52], [53], and the article by L. Lehner and O. Reula, this volume).

We are thus left with the following task: (i) characterize the data which evolve near space-like infinity into solutions of prescribed smoothness at null infinity, (ii) analyze for which of these data physical concepts and requirements (linear and angular momentum at space-like and null infinity, reduction of the asymptotic gauge (BMS) group to a Poincaré group, ...) can be meaningfully introduced and a satisfactory physical picture can be established.

The first step is technically the most difficult one. It requires us to control under fairly general assumptions the effect of the quasi-linear, gauge hyperbolic field equations over infinite regions of space-time. Moreover, the asymptotic behavior of the solutions has to be determined with a precision which excludes any further refinement.

Once this step has been taken, many considerations of the second step will reduce to straightforward, though possibly quite lengthy, calculations. Nevertheless, the second step is of crucial importance. At this stage one has to observe that the notion of asymptotic flatness is not part of the general theory; it is an idealization which is chosen to serve a purpose. While it is suggested to us by important solutions such as those of the Kerr family, it is far from being determined by the equations alone. There remains a large freedom to decide on the asymptotic behavior of the fields.

To make one's choice, one needs to know the mathematical options and has to decide on the physical questions to be answered. A theorem which characterizes the most general Cauchy data on $\tilde{S} = \mathbb{R}^3$ for which the maximal globally hyperbolic Einstein development is null geodesically complete and for which the Riemann tensor goes to zero at (null) infinity would be mathematically quite an achievement but, by itself, insufficient from the point of view of physics.

We are not interested here in discussing 'observations' in asymptotically flat solutions which refer to the roughness of the asymptotic structure as, for instance, in [70]. We rather wish to understand whether (i) the solution models a 'system of physical interest' and (ii) its far field and asymptotic structure allow one to extract information on the system which characterizes its physical nature and can be related to observational data.

This task is neither easy nor well defined. The studies of the last 40 years provide some understanding of the situations one may expect to observe (collapse to a black hole, mergers of black holes, ...). By exploring, however, the questions above in a general setting, new phenomena may be encountered (cf. [12], [49] for an example). But given that the interior is understood to some degree, what do we do about (ii)?

Recent results on the constraint equations exhibit possibilities to modify asymptotically flat vacuum data 'far out' without affecting the interior. The data can be made to agree near space-like infinity with exact Schwarzschild or Kerr data ([20], [21]), with even more general static resp. stationary data, or with data which are only *asymptotically static resp. stationary* ([16]) (cf. also the discussion in the article by R. Bartnik and J. Isenberg, this volume, for other techniques of modifying or extending solutions to the constraints).

These results have been used to settle a question which has been open for a long time. Since data which are static or stationary near space-like infinity evolve into solutions which possess a smooth conformal boundary at null infinity (cf. [23]), these solutions contain smooth hyperboloidal hypersurfaces. Recently P. Chruściel and E. Delay have shown the existence of families of Cauchy data on $\mathbb{R}^3$ which are static outside a fixed radius and have members of arbitrarily small ADM-mass. The corresponding solutions contain hyperboloidal hypersurfaces to which the results of [33] apply. This demonstrates the existence of non-trivial asymptotically simple solutions to Einstein's vacuum field equations with prescribed smoothness of the asymptotic structure ([15]).

More recently S. Klainerman and F. Nicolò revisited their work in [55] and showed ([56]) that their solutions will have the *Sachs peeling property* ([63], [64]) if the data are subject to certain asymptotic conditions. However, the class of data which meet these requirements still needs further analysis.

The new flexibility in constructing asymptotically flat initial data also allows one to illustrate some difficulties of the asymptotically flat space-time model. Let $(\mathcal{S}, d)$ denote the initial data where $\mathcal{S}$ is the hypersurface considered in our discussion of the time-like cut model and d indicates the fields induced on $\mathcal{S}$ by the cosmological model. Suppose that $(\mathcal{S}', d')$ is an asymptotically flat initial data set for which there exists an embedding $\phi : \mathcal{S} \rightarrow \mathcal{S}'$ such that the push forward of d by ϕ is in a suitable sense 'close' to d' on $\phi(\mathcal{S})$. The evolution in time of the data $(\mathcal{S}', d')$ can then be considered in some neighborhood of $\phi(\mathcal{S})$ as a good approximation of the evolution of d in the cosmological model.

If the set $\mathcal{S}$ is chosen large enough and close to the region where the system is undergoing a wave generation process, the main part of the wave signal will reach null infinity at a finite retarded time. The fact that for very late times the data on $\mathcal{S}' \setminus \phi(\mathcal{S})$ will create a deviation of our solution from the cosmological one is likely to be irrelevant in many interesting situations. From a pragmatical point of view it may be considered the main purpose of the asymptotically flat space-time extension beyond the domain of dependence of $\phi(\mathcal{S})$ to allow perturbations of the gravitational field generated near $\phi(\mathcal{S})$ to unfold into a clean wave signal which can be read off at null infinity.

Since changes near space-like infinity affect the field, however weakly, at all later times, they may have an important effect in the case of black hole solutions. One may envisage the collapse of pure gravitational radiation to a black hole as being modelled by vacuum solutions which arise from smooth asymptotically flat data on $\mathbb{R}^3$, admit smooth, complete (cf. [47]) conformal boundaries $\mathcal{J}^{\pm}$, and possess future event horizons while all past directed null geodesics require endpoints on $\mathcal{J}^-$. At present nothing is known about such solutions and they may not exist. Is it conceivable then that there exist solutions which show all the (suitably generalized) features listed above but have a rough conformal boundary? Could such boundaries allow for a 'higher radiation content'? If that were the case the restriction to smooth conformal boundaries might preclude the discussion of certain interesting physical phenomena.

Clearly, the large freedom in constructing asymptotically flat extensions should neither be used to import irrelevant information into the system nor to suppress important features. The replacement of an extension by one which is strictly Kerr (say) near space-like infinity introduces a transition zone on the initial hypersurface which mediates between the given and the Kerr data. The resulting 'wrinkles' in the solution are recorded in the radiation field at null infinity. Is this information physically insignificant or does it indicate that something important has been ironed out by forming the new extension?

This question points again to the need of understanding the detailed behavior of the fields near space-like infinity. In the standard conformal representation space-like infinity with respect to the solution space-time is represented by a point, usually denoted by i^0. With respect to an asymptotically flat Cauchy hypersurface $\tilde{\mathcal{S}}$ space-like infinity is then also represented by a point, denoted by i, which becomes under conformal compactification a point in the extended 3-manifold $\mathcal{S} = \tilde{\mathcal{S}} \cup \{i\}$. Unfortunately, if $m_{ADM} \neq 0$ the conformal initial data are strongly singular at i. This is the basic stumbling block for analyzing the field near space-like infinity in terms of the standard conformal rescaling.

The constraint equations on the Cauchy hypersurface $\tilde{\mathcal{S}}$ impose only weak restrictions on the asymptotic behavior of the data near space-like infinity. It is easy to construct data which at higher orders will become quite 'rough' near i and which can be expected to affect the smoothness of the fields near null infinity in a physically meaningless way. Thus one will have to make a reasonable choice and find a class of data which allows one to perform a sufficiently detailed analysis of their evolution in time while still being sufficiently general to recognize basic features of the asymptotic behavior at null infinity.

In the following it is assumed that the data on $\tilde{\mathcal{S}}$ represent a space-like slice of time reflection symmetry, so that the second fundamental form vanishes, and that their conformal structure, represented by a conformal 3-metric h on $\mathcal{S}$, extends smoothly to the point i. Only these conditions will be used in the following discussion, no a priori assumptions on the evolution in time will be made. We note that the assumptions are made to simplify the calculations, there exists a large space for generalizations.

Somewhat unexpectedly, the attempt to analyze for data as described above the evolution near space-like infinity i in the context of the conformal field equations led to a *finite regularization of the singularity at space-like infinity* ([37]).

In a certain conformal scaling of the conformal initial data the choice of the frame and the coordinates is combined with a blow-up of the point i to a sphere $\mathcal{I}^0$ such that the initial data and the gauge of the evolution system become smoothly extendable to $\mathcal{I}^0$ *in a different conformal scaling.* Moreover, the *general conformal field equations* imply in that scaling a system of hyperbolic reduced equations which also extends smoothly to $\mathcal{I}^0$ (Section 5.5). This allows one to define a *regular finite initial value problem at space-like infinity.*

Under the evolution defined by the extended reduced system the set $\mathcal{I}^0$ evolves into a set $\mathcal{I} =]-1,1[\times\mathcal{I}^0$, which represents a boundary of the physical space-time. This *cylinder at space-like infinity* is defined solely in terms of conformal geometry and the general conformal field equations.

In the given coordinates, the sets $\mathcal{J}^{\pm}$ which represent near space-like infinity the conformal boundary at null infinity are at a finite location. They 'touch' the set $\mathcal{I}$ at certain *critical sets* $\mathcal{I}^{\pm} = \{\pm 1\} \times \mathcal{I}^0$, which can be regarded as the two components of the boundary of $\mathcal{I}$ and, simultaneously, as boundaries of $\mathcal{J}^{\pm}$. Due to the peculiarities of the construction the solution is determined on the closure $\bar{\mathcal{I}} \equiv \mathcal{I} \cup \mathcal{I}^- \cup \mathcal{I}^+ \simeq [-1,1] \times \mathcal{I}^0$ of $\mathcal{I}$ uniquely by the data on $\tilde{\mathcal{S}}$ and there is no freedom to prescribe boundary data on $\bar{\mathcal{I}}$.

This setting discloses the structure which decides on the asymptotic smoothness of the fields. At the critical sets $\mathcal{I}^{\pm}$ occurs a *break-down of the hyperbolicity of* *the reduced equations.* As explained in Section 5.3, a subtle interplay of this degeneracy with the structure of the initial data near $\mathcal{I}^0$, which is mediated by certain *transport equations* on $\mathcal{I}$, turns out critical for the smoothness of the conformal structure at null infinity. This peculiar situation is not suggested by general PDE theory, it is a specific feature of Einstein's theory, the geometric nature of the field equations, and general properties of conformal structures.

The transport equations, which are linear hyperbolic equations interior to $\mathcal{I}$, allow one to calculate the coefficients u^p of the Taylor series of the solution at $\mathcal{I}$ from data implied on $\mathcal{I}^0$ by the Cauchy data on $\mathcal{S}$ (cf. 5.82). Near $\mathcal{I}^{\pm}$ this series can be interpreted as an asymptotic expansion. The coefficients u^p are smooth functions on $\mathcal{I}$ which can be calculated order by order by following an algorithmic procedure.

It turns out that the coefficients u^p develop in general logarithmic singularities at the critical sets $\mathcal{I}^{\pm}$. This behavior foreshadows a possible non-smoothness of the conformal structure at null infinity. In the linearized setting it follows that the logarithmic singularities are transported along the generators of null infinity ([39]). In the non-linear case their effect on the conformal structure at null infinity is not under control yet, however, the solutions are unlikely to be better behaved than in the linear case.

The evidence obtained so far suggests cases which range from conformal structures of high differentiability to ones with low smoothness at null infinity. The non-smoothness may take the form of polylogarithmic expansions in terms of expressions $c\,(1-\tau)^k \log^j (1-\tau)$. Here τ is a coordinate with $\tau \to 1$ from below on $\mathcal{J}^+$, the coefficients $c = c(\rho, t)$ are smooth functions of a coordinate ρ along the null generators and suitable angular coordinates t on the set of null generators of $\mathcal{J}^+$, and k, j are non-negative integers. If k is small enough Sachs peeling fails and Penrose compactification results in weak differentiability.

Can one 'loose physics' if one insists on extensions which are smooth at null infinity? This certainly would be true if the coefficients c would encode important physical information. The discussion of the regularity conditions in Section 5.5 suggests that at low orders the coefficients are determined by the data in an arbitrarily small neighborhood of space-like infinity. By the results on the constraints referred to above these data seem to be rather arbitrary, only weakly related to 'the system' characterized by the data on $\mathcal{S}$, and thus of little relevance. As described in the following the situation is more complicated at higher orders, depends then in a more subtle way on the non-linearity of the equations, and requires further study.

Since the coefficients u^p can be calculated at arbitrary orders, we expect that this analysis will also allow us to describe in detail the relations between physical concepts defined at space-like infinity and concepts defined on null infinity (ADM resp. Bondi linear and angular momentum, etc.). The behavior of the fields at the sets $\mathcal{I}^{\pm}$ is also critical for the possibility to reduce the BMS group to the Poincaré group (cf. [41]). Thus, the precise understanding of the behavior of the fields near the critical sets should provide us with a rather complete physical picture.

Eventually one would like to make statements on the smoothness of the solution space-time at null infinity in terms of properties of the initial data. Thus one needs to control how the behavior of the asymptotic expansion at the critical sets depends on the structure of the initial data and to derive *regularity conditions* on the initial data which are necessary and sufficient for the smoothness of the coefficients u^p at the critical sets.

The information on the coefficients which is available so far has been used to derive conditions which are *necessary* for the regularity ([37]). The derivation of the complete condition is still difficult because of the algebraic complexities of the calculations involved.

Recently J.A. Valiente Kroon obtained with the help of an algebraic computer program complete and explicit expressions at higher order which are pointing at the possibility that *asymptotic staticity* (or, if the time reflection symmetry is dropped, *asymptotic stationarity*) may play a decisive role in deriving sufficient regularity conditions ([71], [72]).

We are thus led to revisit the static vacuum solutions (Sections 4.2, 6 and 7). Because of the loss of hyperbolicity of the conformal field equations at $\mathcal{I}^{\pm}$, it is not clear whether the smoothness of the conformal structure at null infinity observed for static and stationary vacuum solution can be understood as resulting from the possible regularity of the extended solutions at the critical sets. We show that for static solutions our setting is smooth, in fact real analytic, in a neighborhood of the set $\mathcal{I}^- \cup \mathcal{I} \cup \mathcal{I}^+$.

This narrows down the range in which the final regularity condition is to be found. We know that asymptotic staticity is sufficient and that the conditions found in [37], which are implied by asymptotic staticity, are necessary for the regularity of the asymptotic expansion on $\bar{\mathcal{I}}$. There is still the possibility that the final condition ' fizzles out' and depends on the specific data but we expect to arrive at the end at a definite, geometric condition.

To assess how restrictive such conditions would be, it is instructive to consider the results by Chruściel and Delay in [16]. They allow us to conclude that there exist large classes of solutions to the constraints, which are essentially arbitrary on given compact subsets of the initial hypersurface $\tilde{S}$, whose evolutions in time admit asymptotic expansions at $\mathcal{I}^{\pm}$ with coefficients which extend smoothly to the critical sets $\mathcal{I}^{\pm}$ up to a given or at all orders.

So far we ignored a question which is of central importance for gravitational wave astronomy: can the replacement of an asymptotically flat extension by another one result in a drastic change of the wave signal at null infinity? If that were the case, it would be hard to see how specific physical processes could be identified in the wave forms calculated at $\mathcal{J}^+$.

There should be characteristics of wave signals which are specific to 'the system' and which are stable under changes of the asymptotically flat extension if these extensions are restricted to 'reasonable' classes. This problem should be amenable to analytical and numerical investigations and we expect our analysis to contribute to its solution. In fact, it appears that with the regularity conditions mentioned above the field equations themselves hint at a first 'reasonable' class of asymptotically flat extensions.

2. Conformal field equations

Our analysis of the gravitational field near space-like and null infinity relies on a certain conformal representation of Einstein's vacuum field equations referred to as the *general conformal field equations*. We give a short introduction to these equations and point out various facts about the equations and the underlying mathematical structures which will be important in the following. For derivations, detailed arguments, and further background material such as the theory of *normal conformal Cartan connections*, which is the natural home of many of the concepts used in the following, we refer the reader to the original article [36] and the survey article [38].

The aim is to discuss a solution $(\tilde{M}, \tilde{g})$ to Einsteins vacuum field equation

$$R_{\mu\nu}[\tilde{g}] = 0, \tag{2.1}$$

in terms of a suitably chosen *conformal factor* Θ and the *conformal metric* $g = \Theta^2\,\tilde{g}$. Denoting by ∇ the Levi-Civita connection of g, the transformation law of the Ricci tensor under the conformal rescaling above takes in four dimensions the form

$$R_{\nu\rho}[g] = R_{\nu\rho}[\tilde{g}] - \frac{2}{\Theta}\,\nabla_\nu\,\nabla_\rho\,\Theta - g_{\nu\rho}\,g^{\lambda\delta}\left(\frac{1}{\Theta}\,\nabla_\lambda\,\nabla_\delta\,\Theta - \frac{3}{\Theta^2}\,\nabla_\lambda\,\Theta\,\nabla_\delta\,\Theta\right). \tag{2.2}$$

If Θ is considered here as being given, equation (2.1) implies with (2.2) an equation for g with a similar principal part as (2.1).

As explained in the next section, we will mainly be interested in the behavior of the field in space-time domains where $\Theta \to 0$. Because the right-hand side of (2.2) is formally singular in this limit, an abstract discussion of the solutions near the sets $\{\Theta = 0\}$ becomes very delicate. It will be seen, however, that under suitable assumptions on the initial data for the field and with an appropriate behavior of the conformal factor the right-hand side of (2.2) can attain smooth limits. This result is obtained by a more sophisticated use of the behavior of the fields and the equations under transformations which preserve the conformal structure.

2.1. The general conformal field equations

In [28], [29] has been obtained a system of equations which is regular in the sense that there occur no factors Θ^{-1} on the right-hand sides or factors Θ in the principal part of the equations. Its unknowns are Θ, g, and fields derived from them such as the *rescaled conformal Weyl tensor* $W^i{}_{jkl} \equiv \Theta^{-1}\,C^i{}_{jkl}$. These have been used to derive various results about the asymptotic behavior of solutions to the Einstein equations. The specific behavior of the conformal fields near space-like infinity discussed in the next sections asks, however, for a particularly careful analysis of the equations and the gauge conditions. It turns out that this is considerably simplified by making use of the full freedom offered by conformal structures.

A *Weyl connection* for the conformal structure defined by g is a torsion free connection $\hat{\nabla}$ which satisfies

$$\hat{\nabla}_\rho\, g_{\mu\nu} = -2\, f_\rho\, g_{\mu\nu}. \tag{2.3}$$

with some 1-form f_ρ. It is distinguished by the fact that it preserves the conformal structure of g (and thus of $\tilde{g}$). If a frame $\{e_k\}_{k=0,1,2,3}$ is conformal at a given point p in the sense that it satisfies there $g(e_j, e_k) = \Lambda^2\, \eta_{jk}$ with some $\Lambda > 0$ and $\eta_{jk} = diag\,(1, -1, -1, -1)$, then it satisfies such a relation with a point dependent function Λ along a given curve γ through p if it is parallely transported along γ with respect to the connection $\hat{\nabla}$. In particular, if the 1-form f_ρ is closed the connection $\hat{\nabla}$ is locally the Levi-Civita connection of a metric in the conformal class.

Assuming under $\tilde{g} \to g = \Theta^2\, \tilde{g}$ the transformation law

$$\tilde{f}_\rho \to f_\rho = \tilde{f}_\rho - \Theta^{-1}\, \tilde{\nabla}_\rho \Theta,$$

the defining property (2.3) is expressed in terms of the metric $\tilde{g}$ equivalently by $\hat{\nabla}_\rho\, \tilde{g}_{\mu\nu} = -2\, \tilde{f}_\rho\, \tilde{g}_{\mu\nu}$ where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{g}$. It follows from (2.3) that the connection $\hat{\nabla}$ defines with the connection ∇ the difference tensor $\hat{\nabla} - \nabla = S(f)$ given by the specific expression

$$S(f)_\mu{}^\rho{}_\nu \equiv \delta^\rho{}_\mu\, f_\nu + \delta^\rho{}_\nu\, f_\mu - g_{\mu\nu}\, g^{\rho\lambda}\, f_\lambda. \tag{2.4}$$

This, in turn, can be used to specify $\hat{\nabla}$ in terms of ∇ and the 1-form f_ρ. The three connections are related by

$$\hat{\nabla} - \tilde{\nabla} = S(\tilde{f}), \quad \nabla - \tilde{\nabla} = S(\Theta^{-1}\, d\Theta), \quad \hat{\nabla} - \nabla = S(f). \tag{2.5}$$

Important for us will also be the 1-form

$$d_\mu \equiv \Theta\, \tilde{f}_\mu = \Theta\, f_\mu + \nabla_\mu \Theta. \tag{2.6}$$

The decomposition

$$R^\mu{}_{\nu\lambda\rho} = 2\,\{g^\mu{}_{[\lambda}\, L_{\rho]\nu} - g_{\nu[\lambda}\, L_{\rho]}{}^\mu\} + C^\mu{}_{\nu\lambda\rho}, \tag{2.7}$$

of the curvature tensor of ∇ in terms of the trace free conformal Weyl tensor $C^\mu{}_{\nu\lambda\rho}$ and the Schouten tensor

$$L_{\mu\nu} = \frac{1}{2}\, R_{\mu\nu} - \frac{1}{12}\, R\, g_{\mu\nu}, \tag{2.8}$$

which carries the information about the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, has an analogue for $\hat{\nabla}$ which takes the form

$$\hat{R}^\mu{}_{\nu\lambda\rho} = 2\,\{g^\mu{}_{[\lambda}\, \hat{L}_{\rho]\nu} - g^\mu{}_\nu\, \hat{L}_{[\lambda\rho]} - g_{\nu[\lambda}\, \hat{L}_{\rho]}{}^\mu\} + C^\mu{}_{\nu\lambda\rho}. \tag{2.9}$$

where

$$\hat{L}_{\mu\nu} = \frac{1}{2}\,\hat{R}_{(\mu\nu)} - \frac{1}{4}\,\hat{R}_{[\mu\nu]} - \frac{1}{12}\,\hat{R}\,g_{\mu\nu}, \tag{2.10}$$

contains the information about the Ricci tensor $\hat{R}_{\mu\nu} = \hat{R}^{\rho}{}_{\mu\rho\nu}$ and the Ricci scalar $\hat{R} = g^{\mu\nu}\,\hat{R}_{\mu\nu}$. The tensors (2.8) and (2.10) are related by

$$\nabla_{\mu}\,f_{\nu} - f_{\mu}\,f_{\nu} + \frac{1}{2}\,g_{\mu\nu}\,f_{\lambda}\,f^{\lambda} = L_{\mu\nu} - \hat{L}_{\mu\nu}. \tag{2.11}$$

To take care of the specific direction dependence of the various fields near space-like infinity it is convenient to express the conformal field equations in terms of a suitably chosen orthonormal frame field. Let $\{e_k\}_{k=0,1,2,3}$ be a frame field satisfying $g_{ik} \equiv g(e_i, e_k) = \eta_{ik}$, denote by ∇_k and $\hat{\nabla}_k$ the covariant derivative with respect to ∇ and $\hat{\nabla}$ in the direction of e_k, and define the connection coefficients $\Gamma_i{}^j{}_k$ and $\hat{\Gamma}_i{}^j{}_k$ of ∇ and $\hat{\nabla}$ in this frame field by $\nabla_i e_k = \Gamma_i{}^j{}_k\,e_k$ and $\hat{\nabla}_i e_k = \hat{\Gamma}_i{}^j{}_k\,e_k$ respectively. Then $\hat{\Gamma}_i{}^j{}_k = \Gamma_i{}^j{}_k + \delta^j{}_i\,f_k + \delta^j{}_k\,f_i - g_{ik}\,g^{jl}\,f_l$ with $f_k = f_{\mu}\,e^{\mu}{}_k$, where $e^{\mu}{}_k = < d\,x^{\mu},\, e_k >$ denote the frame coefficients with respect to an as yet unspecified coordinate system x^{μ}, $\mu = 0, 1, 2, 3$. We note that $f_i = \frac{1}{4}\,\hat{\Gamma}_i{}^k{}_k$ because $\Gamma_i{}^j{}_k\,g_{jl} + \Gamma_i{}^j{}_l\,g_{jk} = 0$.

If all tensor fields (except the e_k) are expressed in terms of the frame field and the corresponding connection coefficients, the *conformal field equations* used in the following are written as equations for the unknown

$$u = (e^{\mu}{}_k,\quad \hat{\Gamma}_i{}^j{}_k,\quad \hat{L}_{jk},\quad W^i{}_{jkl}), \tag{2.12}$$

and are given by

$$[e_p, e_q] = (\hat{\Gamma}_p{}^l{}_q - \hat{\Gamma}_q{}^l{}_p)\,e_l, \tag{2.13}$$

$$e_p(\hat{\Gamma}_q{}^i{}_j) - e_q(\hat{\Gamma}_p{}^i{}_j) - \hat{\Gamma}_k{}^i{}_j(\hat{\Gamma}_p{}^k{}_q - \hat{\Gamma}_q{}^k{}_p) + \hat{\Gamma}_p{}^i{}_k\hat{\Gamma}_q{}^k{}_j - \hat{\Gamma}_q{}^i{}_k\hat{\Gamma}_p{}^k{}_j \tag{2.14}$$
$$= 2\,\{g^i{}_{[p}\,\hat{L}_{q]j} - g^i{}_j\,\hat{L}_{[pq]} - g_{j[p}\,\hat{L}_{q]}{}^i\} + \Theta\,W^i{}_{jpq},$$

$$\hat{\nabla}_p\,\hat{L}_{qj} - \hat{\nabla}_q\,\hat{L}_{pj} = d_i\,W^i{}_{jpq}, \tag{2.15}$$

$$\nabla_i\,W^i{}_{jkl} = 0. \tag{2.16}$$

The square brackets in the first equation denote the commutator of vector fields. The connection ∇, which appears in the last equation, can be expressed by the relations given above in terms of $\hat{\nabla}$ and f_k. The last equation, referred to in the following as the *Bianchi equation*, is in a sense the core of the system. It is obtained from the contracted vacuum Bianchi identity $\tilde{\nabla}_{\mu}\,C^{\mu}{}_{\nu\lambda\rho} = 0$ by using the specific conformal identity $\Omega^{-1}\,\tilde{\nabla}_{\mu}\,C^{\mu}{}_{\nu\lambda\rho} = \nabla_{\mu}\,(\Omega^{-1}\,C^{\mu}{}_{\nu\lambda\rho})$. The first three equations are then essentially the *structural equations* of the theory of normal conformal Cartan connections.

No equations are given so far for the fields Θ and $d_k = \Theta\,f_k + \nabla_k\Theta$. They reflect the *conformal gauge freedom* artificially introduced here into Einstein's field equations. These fields cannot be prescribed quite arbitrarily. For solution for which the limit $\Theta \to 0$ is meaningful the latter should imply $d_k \to \nabla_k\Theta$.

The theory of normal conformal Cartan connections associates with each conformal structure a distinguished class of curves which provides a useful way of dealing with the gauge freedom. A *conformal geodesic* for $(\tilde{M}, \tilde{g})$ is a curve $x(\tau)$ in $\tilde{M}$ which solves, together with a 1-form $\tilde{f} = \tilde{f}(\tau)$ along it, the system of ODE's

$$(\tilde{\nabla}_{\dot{x}}\dot{x})^{\mu} + S(\tilde{f})_{\lambda}{}^{\mu}{}_{\rho}\,\dot{x}^{\lambda}\,\dot{x}^{\rho} = 0, \tag{2.17}$$

$$(\tilde{\nabla}_{\dot{x}}\tilde{f})_{\nu} - \frac{1}{2}\,\tilde{f}_{\mu}\,S(\tilde{f})_{\lambda}{}^{\mu}{}_{\nu}\,\dot{x}^{\lambda} = \tilde{L}_{\lambda\nu}\,\dot{x}^{\lambda}, \tag{2.18}$$

where $S(\tilde{f})$ and $\tilde{L}$ are given by (2.4) and (2.8) with g replaced by $\tilde{g}$. For any given metric in the conformal class there are more conformal geodesics than metric geodesics because for given initial data $x_* \in M$, $\dot{x}_* \in T_{x_*}M$, $\tilde{f}_* \in T^*_{x_*}M$ there exists a unique solution $x(\tau)$, $\tilde{f}(\tau)$ to (2.17), (2.18) near x_* satisfying for given $\tau_* \in \mathbb{R}$

$$x(\tau_*) = x_*, \quad \dot{x}(\tau_*) = \dot{x}_*, \quad \tilde{f}(\tau_*) = \tilde{f}_*. \tag{2.19}$$

The sign of $\tilde{g}(\dot{x}, \dot{x})$ is preserved near x_* but not its modulus.

Conformal geodesics admit, unlike metric geodesics, general fractional linear maps as parameter transformations. They are *conformal invariants*. Denote by b a smooth 1-form field. Then, if $x(\tau)$, $\tilde{f}(\tau)$ solve the conformal geodesics equations (2.17), (2.18), the pair $x(\tau)$, $\tilde{f}(\tau) - b|_{x(\tau)}$ solves the same equations with $\tilde{\nabla}$ replaced by the connection $\hat{\nabla} = \tilde{\nabla} + S(b)$ and L by $\hat{L}$, i.e., the curve $x(\tau)$, and in particular its parameter τ, are independent of the Weyl connection in the conformal class which is used to write the equations (cf. [40]).

Let there be given a smooth congruence of conformal geodesics which covers an open set U of $\tilde{M}$ such that the associated 1-forms $\tilde{f}$ define a smooth field on U. Denote by $\hat{\nabla}$ the torsion free connection on U which has with the connection $\tilde{\nabla}$ difference tensor $\hat{\nabla} - \tilde{\nabla} = S(\tilde{f})$ and denote by $\hat{L}$ the tensor (2.10) derived from $\hat{\nabla}$. Comparing with (2.11), we find that equations (2.17), (2.18) can be written

$$\hat{\nabla}_{\dot{x}}\,\dot{x} = 0, \quad \hat{L}_{\lambda\nu}\,\dot{x}^{\lambda} = 0. \tag{2.20}$$

Let e_k be a frame field satisfying along the congruence

$$\hat{\nabla}_{\dot{x}}\,e_k = \tilde{\nabla}_{\dot{x}}\,e_k + \langle\tilde{f}, e_k\rangle\dot{x} + \langle\tilde{f}, \dot{x}\rangle e_k - \tilde{g}(\dot{x}, e_k)\tilde{g}^{\sharp}(\tilde{f}, .) = 0. \tag{2.21}$$

Suppose that $\tilde{\mathcal{S}}$ is a hypersurface which is transverse to the congruence, meets each of the curves exactly once, and on which $\tilde{g}(e_i, e_k) = \Theta_*^2\eta_{ik}$ with some function $\Theta_* > 0$. It follows that $\tilde{g}(e_i, e_k) = \Theta^2\,\eta_{ik}$ on U with a function Θ which satisfies

$$\hat{\nabla}_{\dot{x}}\,\Theta = \Theta\langle\dot{x}, \tilde{f}\rangle, \quad \Theta|_{\tilde{\mathcal{S}}} = \Theta_*. \tag{2.22}$$

The observations above allow us to construct a special gauge for the conformal equations. Let $\tilde{\mathcal{S}}$ be a space-like hypersurface in the given vacuum solution $(\tilde{M}, \tilde{g})$. We choose on $\tilde{\mathcal{S}}$ a positive 'conformal factor' Θ_*, a frame field e_{k*}, and a 1-form $\tilde{f}_*$ such that $\Theta_*^2\,\tilde{g}(e_{i*}, e_{k*}) = \eta_{ik}$ and e_{0*} is orthogonal to $\tilde{\mathcal{S}}$. Then there exists through each point $x_* \in \tilde{\mathcal{S}}$ a unique conformal geodesic $(x(\tau), \tilde{f}(\tau))$ with $\tau = 0$ on $\tilde{\mathcal{S}}$ which satisfies there the initial conditions $\dot{x} = e_{0*}$, $\tilde{f} = \tilde{f}_*$.

If all data are smooth these curves define in some neighborhood U of $\tilde{S}$ a smooth caustic free congruence which covers U. Furthermore, $\tilde{f}$ defines a smooth 1-form on U which supplies a Weyl connection $\hat{\nabla}$ as described above. A smooth frame field e_k and the related conformal factor Θ are then obtained on U by solving (2.21), (2.22) for given initial data $e_k = e_{k*}$, $\Theta = \Theta_*$ on $\tilde{S}$. The frame field is orthonormal for the metric $g = \Theta^2\,\tilde{g}$. Dragging along local coordinates x^α, $\alpha = 1, 2, 3$, on $\tilde{S}$ with the congruence and setting $x^0 = \tau$ we obtain a coordinate system. This gauge is characterized on U by the explicit gauge conditions

$$\dot{x} = e_0 = \partial_\tau, \quad \hat{\Gamma}_0{}^j{}_k = 0, \quad \hat{L}_{0k} = 0. \tag{2.23}$$

Coordinates, a frame field, and a conformal factor as above are said to define a *conformal Gauss gauge.* Since metric Gauss systems are well known to quickly develop caustics, it may be mentioned that conformal Gauss systems can cover large space-time domains in a regular fashion (cf. [40]).

To obtain a closed system for all the fields entering equations (2.13)–(2.16), we could now supplement the latter by equations which are implied for the fields Θ and d_k in a conformal Gauss gauge. It turns out that such a gauge implies quite simple ordinary differential equations along the conformal geodesics defining the gauge, it holds in fact

$$\ddot{d}_0 = 0, \qquad \dot{\Theta} = d_0, \qquad \dot{d}_a = 0, \quad a = 1, 2, 3,$$

where the dot denotes differentiation with respect to the parameter τ.

Thus, the fields Θ and d_k given by a conformal Gauss can be determined in our situation explicitly ([36]): *If $\tilde{g}$ is a solution to Einstein's vacuum equations* (2.1), *the fields Θ and d_k are given by the explicit expressions*

$$\Theta = \Theta_* \left(1 + \tau\,\langle \tilde{f}, \dot{x}\rangle_* + \frac{\tau^2}{4}\,\Theta_*^2\,\left((\tilde{g}(\dot{x},\dot{x}))^2\,\tilde{g}^\sharp(\tilde{f},\tilde{f})\right)_*\right) \tag{2.24}$$

$$= \Theta_* \left(1 + \tau\,\langle \tilde{f}, \dot{x}\rangle_* + \frac{\tau^2}{4}\,\left(g^\sharp(\tilde{f},\tilde{f})\right)_*\right),$$

$$d_0 = \dot{\Theta}, \quad d_a = \Theta_* \langle \tilde{f}_*, e_{a*}\rangle, \;\; a = 1, 2, 3, \tag{2.25}$$

where $g^\sharp$ denotes the contravariant version of g and the quantities with a subscript star are considered as constant along the conformal geodesics and given by their values on $\tilde{S}$.

Assuming for Θ and d_k the expressions (2.24) and (2.25), equations (2.13)–(2.16) provide a complete system for u. In spite of the fact that we use a special gauge, we refer to this system as the *general conformal field equations* to indicate that they employ the full gauge freedom preserving conformal structures.

Equally important for us are the facts that the expression (2.24) offers the possibility to control in a conformal Gauss gauge the location of the set where $\Theta \to 0$ and that (2.24), (2.25) imply with the relation $d_k = \Theta\, f_k + \nabla_k \Theta$ in sufficiently regular situations that

$$\nabla_k\,\Theta\,\nabla^k\,\Theta \to 0 \quad \text{as} \quad \Theta \to 0. \tag{2.26}$$

2.2. Spinor version of the general conformal field equations

Writing the conformal equations in the spin frame formalism leads to various algebraic simplifications. We introduce here only the basic notions of this formalism and refer the reader to [61] for a comprehensive introduction. It should be noted, however, that our notation does not completely agree with that of [61]. In particular, if a specific frame field is used it will always be pointed out in the text but not be indicated by gothic indices.

Starting with the orthonormal frame introduced above we define null frame vector fields $e_{AA'} = \sigma^k{}_{AA'}\, e_k$ with constant van der Waerden symbols $\sigma^k{}_{AA'}$ (here and in the following indices $A, B, \ldots, A', B', \ldots$ take values 0 and 1 and the summation rule is assumed) such that

$$e_{00'} = \frac{1}{\sqrt{2}}\,(e_0 + e_3), \quad e_{11'} = \frac{1}{\sqrt{2}}\,(e_0 - e_3),$$

$$e_{01'} = \frac{1}{\sqrt{2}}\,(e_1 - i\, e_2), \quad e_{10'} = \frac{1}{\sqrt{2}}\,(e_1 + i\, e_2).$$

Then $e_{00'}$, $e_{11'}$ are real and $e_{01'}$, $e_{10'}$ are complex (conjugate) null vector fields and their scalar products are given by $g(e_{AA'}, e_{CC'}) = \epsilon_{AC}\,\epsilon_{A'C'}$ where ϵ_{AC}, $\epsilon_{A'C'}$, ϵ^{AC}, $\epsilon^{A'C'}$ denote the anti-symmetric spinor fields with $\epsilon_{01} = \epsilon_{0'1'} = \epsilon^{01} = \epsilon^{0'1'} = 1$. The latter are also used to raise and lower spinor indices according to the rules $X^A = \epsilon^{AB}\, X_B$ and $X_B = X^A\, \epsilon_{AB}$ so that $\epsilon_A{}^B = \epsilon_{AC}\, \epsilon^{BC}$ denotes the Kronecker spinor (similar rules hold for primed indices).

If connection coefficients $\Gamma_{AA'}{}^{BB'}{}_{CC'}$ are defined by writing $\nabla_{e_{AA'}} e_{CC'} = \Gamma_{AA'}{}^{BB'}{}_{CC'}\, e_{BB'}$, the spinor connection coefficients are given by $\Gamma_{AA'}{}^B{}_C = \frac{1}{2}\,\Gamma_{AA'}{}^{BE'}{}_{CE'}$ and one has

$$\Gamma_{AA'}{}^{BB'}{}_{CC'} = \Gamma_{AA'}{}^B{}_C\, \epsilon_{C'}{}^{B'} + \bar{\Gamma}_{AA'}{}^{B'}{}_{C'}\, \epsilon_C{}^B.$$

Here it is observed, as usual, that the relative order of primed and unprimed indices is irrelevant and that under complex conjugation primed indices are converted into unprimed indices and vice versa. Covariant derivatives of spinor fields are now given by similar rules as in the case of the standard frame formalism. Writing $\nabla_{AA'}$ for $\nabla_{e_{AA'}}$ we have, e.g., for a spinor field $X^A{}_B{}^{C'}$

$$\nabla_{DD'}\, X^A{}_B{}^{C'} = e_{DD'}(X^A{}_B{}^{C'}) + \Gamma_{DD'}{}^A{}_F\, X^F{}_B{}^{C'}$$
$$-\Gamma_{DD'}{}^F{}_B\, X^A{}_F{}^{C'} + \bar{\Gamma}_{DD'}{}^{C'}{}_{F'}\, X^A{}_B{}^{F'}.$$

We have similar rules for the connection $\hat{\nabla}$ and its associated connection coefficients $\hat{\Gamma}_{AA'BC}$ and it holds

$$\Gamma_{CC'AB} = \Gamma_{CC'BA}, \quad \hat{\Gamma}_{CC'AB} = \Gamma_{CC'AB} - \epsilon_{AC}\, f_{BC'}, \tag{2.27}$$

so that $\hat{\Gamma}_{CC'}{}^F{}_F = f_{CC'}$ gives the 1-form relating the connection $\hat{\nabla}$ to ∇.

The general conformal field equations are now written as equations for the unknowns

$$e_{AA'}, \quad \hat{\Gamma}_{AA'BC}, \quad \Theta_{AA'BB'}, \quad \phi_{ABCD}. \tag{2.28}$$

Here $\Theta_{AA'BB'}$ is the spinor representation of the tensor field $\hat{L}_{kj}$. It admits a decomposition of the form

$$\Theta_{AA'BB'} = \Phi_{AA'BB'} - \frac{1}{24} R\, \epsilon_{AB}\, \epsilon_{A'B'} + \Phi_{AB}\, \epsilon_{A'B'} + \bar{\Phi}_{A'B'}\, \epsilon_{AB},$$

where $\Phi_{AA'BB'} = \Phi_{BB'AA'} = \bar{\Phi}_{AA'BB'}$ represents the trace-free part of the tensor $\frac{1}{2}\hat{R}_{(jk)}$ provided by $\hat{\nabla}$ while $R = g^{jk}\hat{R}_{jk}$ is the Ricci scalar and the last two terms, with $\Phi_{ab} = \Phi_{(ab)}$, represent the anti-symmetric tensor $\frac{1}{4}\hat{R}_{[jk]}$. The symmetric spinor field $\phi_{ABCD} = \phi_{(ABCD)}$ represents the rescaled conformal Weyl tensor and is related to the latter by

$$W_{AA'BB'CC'DD'} = -\phi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} - \bar{\phi}_{A'B'C'D'}\, \epsilon_{AB}\, \epsilon_{CD}.$$

The general conformal field equations in the order (2.13), (2.14), (2.15), (2.16) now take the form

$$[e_{BB'}, e_{CC'}] = (\Gamma_{BB'}{}^{AA'}{}_{CC'} - \Gamma_{CC'}{}^{AA'}{}_{BB'})\, e_{AA'}, \tag{2.29}$$

$$\begin{aligned} & e_{CC'}(\hat{\Gamma}_{DD'}{}^{A}{}_{B}) - e_{DD'}(\hat{\Gamma}_{CC'}{}^{A}{}_{B}) \\ & -\hat{\Gamma}_{CC'}{}^{F}{}_{D}\,\hat{\Gamma}_{FD'}{}^{A}{}_{B} + \hat{\Gamma}_{DD'}{}^{F}{}_{C}\,\hat{\Gamma}_{FC'}{}^{A}{}_{B} - \bar{\hat{\Gamma}}_{CC'}{}^{F'}{}_{D'}\,\hat{\Gamma}_{DF'}{}^{A}{}_{B} \\ & +\bar{\hat{\Gamma}}_{DD'}{}^{F'}{}_{C'}\,\hat{\Gamma}_{CF'}{}^{A}{}_{B} + \hat{\Gamma}_{CC'}{}^{A}{}_{F}\,\hat{\Gamma}_{DD'}{}^{F}{}_{B} - \hat{\Gamma}_{DD'}{}^{A}{}_{F}\,\hat{\Gamma}_{CC'}{}^{F}{}_{B} \\ & = -\Theta_{BD'CC'}\,\epsilon_{D}{}^{A} + \Theta_{BC'DD'}\,\epsilon_{C}{}^{A} + \Theta\,\phi^{A}{}_{BCD}\,\epsilon_{C'D'}, \end{aligned} \tag{2.30}$$

$$\hat{\nabla}_{BB'}\Theta_{CC'AA'} - \hat{\nabla}_{AA'}\Theta_{CC'BB'} = d^{EE'}(\phi_{EABC}\epsilon_{E'C'}\epsilon_{A'B'} + \bar{\phi}_{E'A'B'C'}\epsilon_{EC}\epsilon_{AB}), \tag{2.31}$$

$$\nabla^{F}{}_{A'}\phi_{ABCF} = 0, \tag{2.32}$$

with the fields Θ, $d_{AA'}$ as given above. The simple form (2.32) of the spinor version of the Bianchi equation will be useful for us.

2.3. The reduced conformal field equations

The conformal Gauss gauge is not only distinguished by the fact that it is provided by the conformal structure itself and supplies explicit information on Θ and d_k, but also by a remarkable simplicity of the resulting evolution equations. Setting $p = 0$ in (2.13)–(2.16) and observing the gauge conditions (2.23) we obtain

$$\partial_\tau\, e^{\mu}{}_{q} = -\hat{\Gamma}_{q}{}^{l}{}_{0}\, e^{\mu}{}_{l}, \tag{2.33}$$

$$\partial_\tau\, \hat{\Gamma}_{q}{}^{i}{}_{j} = -\hat{\Gamma}_{k}{}^{i}{}_{j}\,\hat{\Gamma}_{q}{}^{k}{}_{0} + g^{i}{}_{0}\,\hat{L}_{qj} + g^{i}{}_{j}\,\hat{L}_{q0} - g_{j0}\,\hat{L}_{q}{}^{i} + \Theta\, W^{i}{}_{j0q}, \tag{2.34}$$

$$\partial_\tau\, \hat{L}_{qj} + \hat{\Gamma}_{q}{}^{k}{}_{0}\,\hat{L}_{kj} = d_i\, W^{i}{}_{j0q}, \tag{2.35}$$

$$\nabla_i W^{i}{}_{jkl} = 0.$$

While the first three equations are then ordinary differential equations along the conformal geodesics, we still have to deduce a suitable evolution system from the last equation. The Bianchi equation represents an overdetermined system of 16 equations for the 10 independent components of the rescaled conformal Weyl tensor. It implies a system of wave equations for $W^{i}{}_{jkl}$ which could be used as the evolution system. For the application studied in this article it turns out important, however, to use the first order system.

There are various ways of extracting from the Bianchi equations symmetric hyperbolic evolution systems but these are most easily found in the spin frame formalism. With the spinor field $\tau^{AA'} = \epsilon_0{}^A \epsilon_{0'}{}^{A'} + \epsilon_1{}^A \epsilon_{1'}{}^{A'}$ the gauge conditions (2.23) can be written

$$\tau^{AA'} e_{AA'} = \sqrt{2}\,\partial_\tau, \quad \tau^{AA'} \hat{\Gamma}_{AA'}{}^B{}_C = 0, \quad \tau^{BB'} \Theta_{AA'BB'} = 0. \tag{2.36}$$

Observing $\tau_{AA'} \tau^{BA'} = \epsilon_A{}^B$ and its complex conjugate version, one obtains from (2.27) and (2.36) the relation $\tau^{CC'} \Gamma_{CC'AB} = -\tau_A{}^{C'} f_{BC'}$ and thus

$$\hat{\Gamma}_{CC'AB} = \Gamma_{CC'AB} - \epsilon_{AC}\, \tau^{DD'} \Gamma_{DD'EB}\, \tau^E{}_{C'}, \tag{2.37}$$

which shows with (2.27) that $\hat{\Gamma}_{CC'AB}$ can be expressed in our gauge in terms of $\Gamma_{CC'AB}$ and vice versa.

Transvecting equations (2.29), (2.30), (2.31) suitably with $\tau^{EE'}$ thus gives the system of ODE's

$$\sqrt{2}\,\partial_\tau e^\mu{}_{CC'} = -\Gamma_{CC'}{}^{AA'}{}_{BB'}\, \tau^{BB'} e^\mu{}_{AA'}, \tag{2.38}$$

$$\sqrt{2}\,\partial_\tau \hat{\Gamma}_{DD'}{}^A{}_B + (\hat{\Gamma}_{DD'}{}^F{}_C \hat{\Gamma}_{FC'}{}^A{}_B + \bar{\hat{\Gamma}}_{DD'}{}^{F'}{}_{C'} \hat{\Gamma}_{CF'}{}^A{}_B)\, \tau^{CC'} \tag{2.39}$$
$$= \Theta_{BC'DD'}\, \tau^{AC'} + \Theta\, \phi^A{}_{BCD}\, \tau^C{}_{D'},$$

$$\sqrt{2}\,\partial_\tau \Theta_{CC'AA'} + (\hat{\Gamma}_{AA'}{}^F{}_B\, \Theta_{CC'FB'} + \bar{\hat{\Gamma}}_{AA'}{}^{F'}{}_{B'}\, \Theta_{CC'BF'})\tau^{BB'} \tag{2.40}$$
$$= -d^{EE'}(\phi_{EABC}\epsilon_{E'C'}\tau^B{}_{A'} + \bar{\phi}_{E'A'B'C'}\epsilon_{EC}\tau_A{}^{B'}).$$

We set now $\Lambda_{ABCA'} \equiv \nabla^F{}_{A'}\phi_{ABCF}$. Equation (2.32) is then equivalent to $0 = \Lambda_{ABCD} \equiv \Lambda_{ABCA'}\tau_D{}^{A'}$. On the other hand we have the decomposition $\Lambda_{ABCD} = \Lambda_{(ABCD)} - \frac{3}{4}\epsilon_{D(C}\Lambda_{AB)F}{}^F$ with irreducible parts

$$\Lambda_{(ABCD)} = -\frac{1}{2}\left(P\,\phi_{ABCD} - 2\,\mathcal{D}_{(D}{}^F\, \phi_{ABC)F}\right), \qquad \Lambda_{ABF}{}^F = \mathcal{D}^{EF}\, \phi_{ABEF}, \tag{2.41}$$

where $P = \tau^{AA'}\nabla_{AA'} = \sqrt{2}\,\nabla_{e_0}$ and $\mathcal{D}_{AB} = \tau_{(A}{}^{A'}\nabla_{B)A'}$ denote covariant directional derivative operators such that $\mathcal{D}_{00} = -\nabla_{01'}$, $\mathcal{D}_{11} = \nabla_{10'}$, and $\mathcal{D}_{01} = \mathcal{D}_{10} = \frac{1}{\sqrt{2}}\nabla_{e_3}$, (cf. [35], [36] for more details of the underlying space-spinor formalism).

In a Cauchy problem one will in general assume e_0 to be the future directed normal to the initial hypersurface $\tilde{\mathcal{S}}$. The operators $\mathcal{D}_{AB}$ then involve only differentiation in directions tangent to $\tilde{\mathcal{S}}$ and the equations $\Lambda_{ABF}{}^F = 0$ are interior equations on $\tilde{\mathcal{S}}$. They represent the six real constraint equations implied on $\tilde{\mathcal{S}}$ by the Bianchi equation.

For a symmetric spinor field $\psi_{A_1 \dots A_k}$ we define its (independent) *essential components* by $\psi_j = \psi_{(A_1 \dots A_k)_j}$, where $0 \le j \le k$ and the brackets with subscript j indicate that j of the indices in the brackets are set equal to 1 while the others are set equal to 0.

The five equations $\Lambda_{(ABCD)} = 0$ for the components of ϕ_{ABCD} contain the operator P. Multiplying by suitable binomial coefficients (and considering the frame and connection coefficients as given), we find that the system

$$-\left(\begin{matrix} 4 \\ A+B+C+D \end{matrix}\right) \Lambda_{(ABCD)} = 0, \tag{2.42}$$

has the following properties. If ϕ denotes the transpose of the $\mathbb{C}^5$-valued 'vector' $(\phi_0, \phi_1, \phi_2, \phi_3, \phi_4)$, it takes the form

$$A^\mu \, \partial_\mu \, \phi = H(x, \phi),$$

with a $\mathbb{C}^5$-valued function $H(x, \phi)$ and 5×5-matrix-valued functions A^μ which are hermitian, i.e., ${}^T\bar{A}^\mu = A^\mu$, and for which there exists at each point a covector ξ_μ such that $A^\mu \xi_\mu$ is positive definite. The system (2.42) is thus *symmetric hyperbolic* ([44], see also [43] and the references given there).

While the constraints implied on a given space-like hypersurface are determined uniquely, there is a large freedom to select useful evolution systems. In fact, any system of the form

$$\begin{aligned} 0 &= 2\,a\,\Lambda_{0001'}, \\ 0 &= (c-d)\,\Lambda_{0011'} - 2\,a\,\Lambda_{0000'}, \\ 0 &= (c+d)\,\Lambda_{0111'} - (c-d)\,\Lambda_{0010'}, \\ 0 &= 2\,e\,\Lambda_{1111'} - (c+d)\,\Lambda_{0110'}, \\ 0 &= -2\,e\,\Lambda_{1110'}, \end{aligned} \tag{2.43}$$

with $a, c, e > 0$ and $-(2\,e + c) < d < 2\,a + c$, is symmetric hyperbolic (the system (2.42) occurs here as a special case). We note that only the characteristics of these systems which are null hypersurfaces are of physical significance.

Equations (2.33), (2.34), (2.35), respectively equations (2.38), (2.39), (2.40), combined with a choice of (2.43), will be referred to in the following as the (general) *reduced conformal field equations*. Solutions to these equations solve in fact the complete system (2.12), (2.13), (2.15), (2.16) if the solution admits a Cauchy hypersurface on which the latter equations hold ([36]). In other words, propagation by the reduced field equations preserves the constraints.

2.4. The conformal constraints

To analyze solutions to the conformal field equations in the context of a Cauchy problem one needs to study the conformal constraints implied on a space-like initial hypersurface $\tilde{S}$. It will be convenient to discuss the evolution equations in terms of a conformal factor Θ which differs on $\tilde{S}$ from the one used to analyze the constraints. We thus assume Einstein's equations (2.1), a conformal rescaling

$$g = \Omega^2 \tilde{g}, \tag{2.44}$$

with a positive conformal factor Ω, and denote again the Levi-Civita connection of g by ∇. It is also convenient to derive the conformal constraints from the *metric conformal field equations*. The latter are written in terms of the unknown fields

g, Ω, $S \equiv \frac{1}{4}\nabla_\mu \nabla^\mu \Omega + \frac{1}{24} R\,\Omega$, $L_{\mu\nu}$ as in (2.8), and $W^\mu{}_{\nu\lambda\rho} = \Omega^{-1} C^\mu{}_{\nu\lambda\rho}$ and are given by equation (2.7), with $C^\mu{}_{\nu\lambda\rho} = \Omega\, W^\mu{}_{\nu\lambda\rho}$, and the equations

$$2\,\Omega\, S - \nabla_\mu \Omega\, \nabla^\mu \Omega = 0, \tag{2.45}$$

$$\nabla_\mu\, \nabla_\nu \Omega = -\Omega\, L_{\mu\nu} + S\, g_{\mu\nu}, \tag{2.46}$$

which are obtained by rewriting the trace and the trace free part of (2.2),

$$\nabla_\mu\, S = -L_{\mu\nu} \nabla^\nu \Omega, \tag{2.47}$$

$$\nabla_\lambda L_{\rho\nu} - \nabla_\rho L_{\lambda\nu} = \nabla_\mu \Omega\, W^\mu{}_{\nu\lambda\rho}, \tag{2.48}$$

which both can be obtained as integrability conditions of (2.46), and

$$\nabla_\mu W^\mu{}_{\nu\lambda\rho} = 0. \tag{2.49}$$

In these equations the Ricci scalar R is considered as the *conformal gauge source function*. Its choice, which is completely arbitrary in local studies, controls together with the initial data Ω and $d\Omega$ on $\tilde{\mathcal{S}}$ the evolution of the conformal scaling.

To derive the constraints induced by these equations on $\tilde{\mathcal{S}}$ we choose a g-orthonormal frame field $\{e_k\}_{k=0,1,2,3}$ near $\tilde{\mathcal{S}}$ such that $n \equiv e_0$ is g-normal to $\tilde{\mathcal{S}}$, write $\nabla_k \equiv \nabla_{e_k}$, $\nabla_k\, e_j = \Gamma_k{}^l{}_j\, e_l$, and express all fields (except the e_k) and equations in terms of this frame. We assume that indices $a, b, c, \ldots$ from the beginning of the alphabet take values $1, 2, 3$ and that the summation convention also holds for these indices. The inner metric h induced by g on $\tilde{\mathcal{S}}$ is then given by $h_{ab} = g(e_a, e_b) = -\delta_{ab}$ and the second fundamental form by

$$\chi_{ab} = g(\nabla_{e_a} n, e_b) = \Gamma_a{}^j{}_0\, g_{jb} = -\Gamma_a{}^0{}_b.$$

We set $\Sigma = \nabla_0\, \Omega$ and $W^*_{\mu\nu\lambda\rho} = \frac{1}{2}\, W_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta}{}_{\lambda\rho}$. If the tensor fields

$$L_{\mu\nu}, \quad L_{\mu\nu}n^\nu, \quad W_{\mu\nu\lambda\rho}, \quad W_{\mu\nu\lambda\rho}n^\nu n^\rho, \quad W^*_{\mu\nu\lambda\rho}n^\nu n^\rho, \quad W_{\mu\nu\lambda\rho}n^\nu,$$

are projected orthogonally into $\tilde{\mathcal{S}}$ and expressed in terms of the frame $\{e_a\}_{a=1,2,3}$ on $\tilde{\mathcal{S}}$, they are given by (the left-hand sides of)

$$L_{ab}, \quad L_a \equiv L_{a0},$$

$$w_{abcd} \equiv W_{abcd}, \quad w_{ab} \equiv W_{a0b0}, \quad w^*_{ab} \equiv W^*_{a0b0}, \quad w_{abc} \equiv W_{a0bc},$$

respectively and satisfy the relations

$$R = 6\, L_\mu{}^\mu = 6\,(L_{00} + L_a{}^a), \tag{2.50}$$

$$w_{abcd} = -2\{h_{a[c}w_{d]b} + h_{b[d}w_{c]a}\}, \quad w^*_{ad}\,\epsilon^d{}_{bc} = w_{abc}, \quad w^*_{ad} = -\frac{1}{2}\, w_{abc}\,\epsilon_d{}^{bc},$$

$$w_{ab} = w_{ba}, \quad w_a{}^a = 0, \quad w^*_{ab} = w^*_{ba}, \quad w^*_a{}^a = 0,$$

$$w_{abc} = -w_{acb}, \quad w^a{}_{ac} = 0, \quad w_{[abc]} = 0,$$

where indices are moved with h_{ab} and ϵ_{abc} is totally antisymmetric with $\epsilon_{123} = 1$. The tensors w_{ab} and w^*_{ab} represent the n-electric and the n-magnetic part of $W^i{}_{jkl}$ on $\tilde{\mathcal{S}}$ respectively.

Equation (2.7) implies Gauss' and Codazzi's equation on $\tilde{\mathcal{S}}$

$$r_{ab} = -\Omega\, w_{ab} + L_{ab} + L_c{}^c\, h_{ab} + \chi_c{}^c\, \chi_{ab} - \chi_{ac}\, \chi_b{}^c, \tag{2.51}$$

$$D_b\, \chi_{ca} - D_c\, \chi_{ba} = \Omega\, w_{abc} + h_{ab}\, L_c - h_{ac}\, L_b, \tag{2.52}$$

while equations (2.45)–(2.49) imply the following interior equations which only involve derivatives in the directions of e_a, $a = 1, 2, 3$, tangent to $\tilde{\mathcal{S}}$

$$2\,\Omega\, S - \Sigma^2 - D_a\Omega\, D^a\Omega = 0, \tag{2.53}$$

$$D_a\, D_b\, \Omega = -\Sigma\, \chi_{ab} - \Omega\, L_{ab} + S\, h_{ab}, \tag{2.54}$$

$$D_a\, \Sigma = \chi_a{}^c D_c\Omega - \Omega\, L_a, \tag{2.55}$$

$$D_a\, S = -D^b\, \Omega\, L_{ba} - \Sigma\, L_a, \tag{2.56}$$

$$D_a\, L_{bc} - D_b\, L_{ac} = D^e\Omega\, w_{ecab} - \Sigma\, w_{cab} - \chi_{ac}\, L_b + \chi_{bc}\, L_a, \tag{2.57}$$

$$D_a\, L_b - D_b\, L_a = D^e\Omega\, w_{eab} + \chi_a{}^c L_{bc} - \chi_b{}^c L_{ac}, \tag{2.58}$$

$$D^c\, w_{cab} = \chi^c{}_a\, w_{bc} - \chi^c{}_b\, w_{ac}, \tag{2.59}$$

$$D^a\, w_{ab} = \chi^{ac}\, w_{abc}, \tag{2.60}$$

where r_{ab} denotes the Ricci tensor of h_{ab}. These equations can be read as *conformal constraints* for the fields

$$\Omega,\ \ \Sigma,\ \ S,\ \ h_{ab},\ \ \chi_{ab},\ \ L_a,\ \ L_{ab},\ \ w_{ab},\ \ w^*_{ab}.$$

Alternatively, if a 'physical' solution $\tilde{h}_{ab}$, $\tilde{\chi}_{ab}$ to the vacuum constraints is given and a conformal factor Ω and functions Σ, R have been chosen, which are gauge dependent functions at our disposal, the equations above can be used to calculate

$$S,\ \ L_{\mu\nu},\ \ W^\mu{}_{\nu\lambda\rho},$$

from the conformal first and second fundamental forms h_{ab}, χ_{ab} of $\tilde{\mathcal{S}}$, which are related to the physical data by

$$h_{ab} = \Omega^2\, \tilde{h}_{ab}, \quad \chi_{ab} = \Omega\,(\tilde{\chi}_{ab} + \Sigma\, \tilde{h}_{ab}). \tag{2.61}$$

The equations above will be discussed in more detail in Section 4.

3. Asymptotic simplicity

To characterize the fall-off behavior of asymptotically flat solutions at null infinity in terms of geometric concepts Penrose introduced the notion of *asymptotic simplicity* ([59], [60], cf. also [61] for further discussions and references).

Definition 3.1. *A smooth space-time $(\tilde{\mathcal{M}}, \tilde{g})$ is called asymptotically simple if there exists a smooth, oriented, time-oriented, causal space-time $(\mathcal{M}, g)$ and a smooth function Ω on $\mathcal{M}$ such that:*

(i) *$\mathcal{M}$ is a manifold with boundary $\mathcal{J}$,*
(ii) *$\Omega > 0$ on $\mathcal{M} \setminus \mathcal{J}$ and $\Omega = 0$, $d\Omega \neq 0$ on $\mathcal{J}$,*
(iii) *there exists an embedding Φ of $\tilde{\mathcal{M}}$ onto $\Phi(\tilde{\mathcal{M}}) = \mathcal{M} \setminus \mathcal{J}$ which is conformal such that $\Omega^2\, \Phi^{-1*}\tilde{g} = g$,*
(iv) *each null geodesic of $(\tilde{\mathcal{M}}, \tilde{g})$ acquires two distinct endpoints on $\mathcal{J}$.*

We note that only the conformal class of $(\tilde{\mathcal{M}}, \tilde{g})$ enters the definition and it is only the conformal structure of $(\mathcal{M}, g)$ which is determined here. The set $\mathcal{J}$ is referred to as the *conformal boundary of* $(\tilde{\mathcal{M}}, \tilde{g})$ *at null infinity.* This definition is the mathematical basis for the

Penrose proposal: *Far fields of isolated gravitating systems behave like that of asymptotically simple space-times in the sense that they can be smoothly extended to null infinity, as indicated above, after suitable conformal rescalings.*

Since gravitational fields are governed by Einstein's equations, the proposal suggests a sharp characterization of the fall-off behavior implied by the field equations in terms of the purely geometrical definition (3.1).

We will be interested in the following in solutions to Einstein's vacuum field equations (2.1) which satisfy the conditions of definition (3.1) (or suitable generalizations). The two assumptions have important consequences for the structure of $(\mathcal{M}, g)$. We shall only quote those which will be used in the following discussion and refer the reader for further information to the references given above.

If the vacuum field equations hold near $\mathcal{J}$, the latter defines a smooth null hypersurface of $\mathcal{M}$ (cf. equation (2.45)). It splits into two components, $\mathcal{J}^+$ and $\mathcal{J}^-$, which are generated by the past and future endpoints of null geodesics in $\mathcal{M}$ and are thus called *future and past null infinity* (or *scri* $\pm$) respectively. Each of $\mathcal{J}^\pm$ is ruled by null generators, each set of null generators has topology S^2, and $\mathcal{J}^\pm$ have the topology of $\mathbb{R} \times S^2$. For the applications one will have to relax the conditions of definition (3.1). In particular condition (*iv*), which is important to obtain the result about the topology of $\mathcal{J}^\pm$, must be replaced by a different completeness condition if one wants to discuss space-times with black holes.

One of the main difficulties in developing a well-defined concept of *outgoing radiation* in the time-like cut model is related to the fact that there exists in general no distinguished null direction field along the time-like boundary $\mathcal{T}$. In contrast, the null generators of $\mathcal{J}^+$ define a unique causal direction field on $\mathcal{J}^+$, which is represented by $\nabla^{AA'}\Omega$. It turns out that the field $\phi_{ABCD}\, o^A\, o^B\, o^C\, o^D$ on $\mathcal{J}^+$, where o^A denotes a spinor field satisfying $o^A\, \bar{o}^{A'} = -\nabla^{AA'}\,\Omega$ on $\mathcal{J}^+$ and ϕ_{ABCD} the rescaled conformal Weyl spinor field, has a natural interpretation as the *outgoing radiation field.* Further important physical concepts can be associated with the hypersurface $\mathcal{J}^+$ or subsets of it and questions of interpretation have been extensively analyzed (cf. [4], [19], [46] and the references given there).

Critical however, and in fact a matter of controversy, have been the smoothness assumptions in the definition, which encode the fall-off behavior imposed on the physical fields. It is far from immediate that they are in harmony with the fall-off behavior imposed by the field equations. No problem arises if the proposal can be justified with C^∞ replaced by C^k with sufficiently large integer k. But there is a lower threshold for the differentiability, which is not easily specified, at which the definition will loose much of its elegance and simplicity.

In [60] it is assumed that

$$\mathcal{M} \text{ is of class } C^{k+1} \text{ and } g, \Omega \in C^k(\mathcal{M}), \;\; k \geq 3. \tag{3.1}$$

The conformal Weyl spinor $\Psi_{ABCD} = \Omega\, \phi_{ABCD}$ is then in $C^{k-2}(\mathcal{M})$. Under the further assumption

$$\Omega\, \nabla_{EE'}\, \nabla^A{}_{A'}\, \Psi_{ABCD} \rightarrow 0 \quad \text{at} \quad \mathcal{J}^+, \tag{3.2}$$

which will certainly be satisfied if $k \geq 4$ in (3.1), and the natural assumption

$$\text{the set of null generators of } \mathcal{J}^+ \text{ has topology } S^2, \tag{3.3}$$

it is then shown that Ψ_{ABCD} vanishes on $\mathcal{J}^+$. The solution is thus asymptotically flat in the most immediate sense and the rescaled conformal Weyl spinor ϕ_{ABCD} extends in a continuous fashion to $\mathcal{J}^+$. As a consequence, it follows that the space-time satisfies the *Sachs peeling property* ([60], [63], [64]) which says that in a suitably chosen spin frame the components of the conformal Weyl spinor fall-off as $\Psi_{ABCD} = O(\tilde{r}^{A+B+C+D-5})$ (where A, B, C, D take values $0, 1$) along an outgoing null geodesic when its (physical) affine parameter $\tilde{r} \rightarrow \infty$ at $\mathcal{J}^+$.

Remarkable as it is that such a conclusion can be drawn for the spin-2 nature of the field Ψ_{ABCD} and its governing field equation $\tilde{\nabla}^F{}_{A'}\, \Psi_{ABCF} = 0$, there remains the question whether the long time evolution by the field equations is such that assumptions (3.1), (3.2) or the conclusion drawn from them can be justified.

As discussed in the introduction, we know by now that these conditions can be satisfied by non-trivial solutions to the vacuum field equations. What is not known, however, is how the solutions satisfying these conditions are to be characterized in terms of their Cauchy data, whether these conditions exclude solutions modelling important physical phenomena, and if they do, what exactly goes wrong. Obviously, these questions can only be answered by analyzing the Cauchy problem for Einstein' field equations with asymptotically flat Cauchy data in the large with the goal to derive sharp results on the behavior of the field at null infinity.

The results obtained so far on the existence of solutions which admit (partial) smooth boundaries at null infinity make it clear that the key problem here is the behavior of the fields near space-like infinity. We shall not consider any further the results which lead to this conclusion (cf. [38] for a discussion and the relevant references) but concentrate on this particular problem.

To begin with we have a look at the asymptotic region of interest here in the case of Minkowski space. If the latter is given in the form $(\tilde{\mathcal{M}} \simeq \mathbb{R}^4, \tilde{g} = \eta_{\mu\nu}\, d\,y^\mu\, d\,y^\nu)$, the coordinate transformation $\Phi : y^\mu \rightarrow x^\mu = (-y_\lambda\, y^\lambda)^{-1}\, y^\mu$ renders the metric in the domain $\mathcal{D} \equiv \{y_\lambda\, y^\lambda < 0\} = \{x_\lambda\, x^\lambda < 0\}$ in the form $\tilde{g} = \Omega^{-2}\, \eta_{\mu\nu}\, d\,x^\mu\, d\,x^\nu$ with $\Omega = -x_\lambda\, x^\lambda$. The metric

$$g = \Omega^2\, \tilde{g} = \eta_{\mu\nu}\, d\,x^\mu\, d\,x^\nu, \tag{3.4}$$

thus extends smoothly to the domain $\bar{\mathcal{D}}$ of points in $\{x_\lambda\, x^\lambda \leq 0\}$ which are obtained as limits of sequences in $\mathcal{D}$. The point $x^\mu = 0$ in this set then represents space-like infinity for Minkowski space. With this understanding it is denoted by i^0. The hypersurfaces $\mathcal{J}'^{\pm} = \{x_\lambda\, x^\lambda = 0, \; \pm x^0 > 0\} \subset \bar{\mathcal{D}}$ represent parts of future and past null infinity of Minkowski space close to space-like infinity and are generated by the future and past directed null geodesics of g through i^0.

Consider the Cauchy hypersurface $\tilde{\mathcal{S}} = \{y^0 = 0\}$ of Minkowski space. The subset $\tilde{\mathcal{S}} \cap \mathcal{D}$ is mapped by Φ onto $\{x^0 = 0,\ x^\mu \neq 0\}$. Extending the latter to include the point $x^\mu = 0$ amounts to a smooth compactification $\tilde{\mathcal{S}} \to \mathcal{S} = \tilde{\mathcal{S}} \cup \{i\} \sim S^3$ such that the point i with coordinates $x^\mu = 0$ represents space-like infinity with respect the metric induced on $\tilde{\mathcal{S}}$ by $\tilde{g}$. The distinction of space-like infinity i with respect to a Cauchy data set and space-like infinity i^0 with respect to the solution space-time will become important and much clearer later on.

Denote by $\tilde{h}_{\alpha\beta}$ and $\tilde{\chi}_{\alpha\beta}$ the metric and the extrinsic curvature induced by $\tilde{g}$ on $\tilde{\mathcal{S}}$. A global representation of the conformal structure induced on $\mathcal{S}$ by $\tilde{h}_{\alpha\beta}$ is obtained by using a slightly different conformal rescaling than before. Set $h' = \Omega'^2\, \tilde{h}$ with $\Omega' = 2\,(1 + |y|^2)^{-1}$ where $|y| = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$. In terms of standard spherical coordinates θ, ϕ on $\tilde{\mathcal{S}}$ and the coordinate χ defined by $\cot \frac{\chi}{2} = |y|,\ 0 \leq \chi \leq \pi$, the conformal metric takes the form $h' = -(d\,\chi^2 + \sin^2\chi\, d\,\sigma^2)$ of the standard metric on the 3-sphere and extends smoothly to the point i, which is given by the coordinate value $\chi = 0$ and distinguished by the property that $\Omega = 0$, $d\,\Omega = 0$, $Hess\,\Omega = c\,h'$, with $c \neq 0$ at i. Here $d\,\sigma^2 \equiv d\,\theta^2 + \sin^2\theta\, d\,\phi^2$ denotes the standard line element on the 2-sphere S^2.

Since $\tilde{\chi}_{\alpha\beta} = 0$ and we are free to choose $\Sigma = 0$ in (2.61), we get $\chi'_{\alpha\beta} = 0$. By the formulas given in the previous section one can derive from the conformal Minkowski data $(\mathcal{S}, h', \chi')$ and a suitable choice of initial data for the gauge defining fields (2.24), (2.25) a conformal initial data set for the reduced conformal field equations. These allow us then to recover the well-known conformal embedding of Minkowski space into the Einstein cosmos ([60]) as a smooth solution to the regular conformal field equations ([38], [40]).

We would like to control what happens if the conformal Minkowski data are subject to finite perturbations. Under which assumptions will the corresponding solutions preserve asymptotic simplicity? This or the apparently simpler question under which conditions the solutions will preserve near space-like infinity a reasonable amount of smoothness of the conformal boundary cannot be answered by immediate applications of the conformal field equations. The reason is that the conformal data will not be smooth at the point i. The structure of the conformal initial data as well as the initial value problem for the conformal field equations near space-like infinity thus require a careful and detailed analysis. This will be carried out to some extent in the next section.

4. Asymptotically flat Cauchy data

As indicated in the case of Minkowski space above we will assume that the data for the conformal field equations are given on a 3-dimensional manifold $\mathcal{S} = \tilde{\mathcal{S}} \cup \{i\}$ which is obtained from a 'physical' 3-manifold $\tilde{\mathcal{S}}$ with an asymptotically flat end by adjoining a point i which represents space-like infinity. The data h_{ab}, χ_{ab} on $\mathcal{S}$

are thought as being obtained from the physical data $\tilde{h}_{ab}$, $\tilde{\chi}_{ab}$ by equations (2.61) with suitable choices of Ω and Σ such that all fields extend with an appropriate behavior (to be specified more precisely below) to i and $\Sigma(i) = 0$, $\Omega > 0$ on $\tilde{S}$, $\Omega(i) = 0$, $D_a \Omega(i) = 0$, $D_a D_b \Omega(i) = -2 h_{ab}$, where we assume the notation of Subsection 2.4.

The constraint equations (2.51)–(2.60) contain analogues of the vacuum constraints. The form of these equations suggests a solution procedure which does not require us to go back to the physical data. By taking the trace of equation (2.51), using (2.53) and the trace of (2.54), and writing $\Delta_h \equiv D_a D^a$, one gets

$$\Omega^2 r = -4 \Omega \Delta_h \Omega + 6 D_a\Omega D^a\Omega - 4 \Omega \Sigma \chi_c{}^c + \Omega^2((\chi_c{}^c)^2 - \chi_{ac} \chi^{ac}), \tag{4.1}$$

where r denotes the Ricci scalar of h. With $\theta = \Omega^{-\frac{1}{2}}$ this equation takes the form of Lichnerowicz' equation

$$(\Delta_h - \frac{1}{8} r)\theta = -\frac{1}{8} \theta \left((\chi_c{}^c)^2 - \chi_{ab} \chi^{ab}\right) + \frac{1}{2} \theta^3 \Sigma \chi_c{}^c. \tag{4.2}$$

By taking the trace of (2.52) and using (2.55) one gets

$$D^b(\Omega^{-2} \chi_{bc}) = \Omega^{-2} D_c \chi_b{}^b - 2 \Omega^{-3} D_c\Sigma. \tag{4.3}$$

Equations (4.2) and (4.3) correspond to the Hamiltonian and the momentum constraint respectively. Assuming now

$$\chi_a{}^a = 0 \text{ and (the choice of gauge) } \Sigma = 0 \text{ on } \mathcal{S}, \tag{4.4}$$

which imply $\tilde{\chi}_a{}^a = 0$, equations (4.2) and (4.3) suggest to proceed as follows: (i) prescribe h on $\mathcal{S}$ and solve the equation $D^a \psi_{ab} = 0$ for a symmetric h-trace free tensor field ψ_{ab} on $\mathcal{S}$, (ii) solve equation (4.2) with $\chi_{ab} = \theta^{-4}\psi_{ab}$ for a positive function θ, i.e., solve

$$(\Delta_h - \frac{1}{8} r)\theta = \frac{1}{8} \theta^{-7} \chi_{ab} \chi^{ab}, \quad \theta > 0. \tag{4.5}$$

The fields $\Omega = \theta^{-2}$, h_{ab}, and $\chi_{ab} = \Omega^2\psi_{ab}$ then solve (4.1) and (4.3). If it is required that

$$\rho \Theta \to 1, \quad \psi_{ab} = O(\frac{1}{\rho^4}) \quad \text{as } \rho \to 0, \tag{4.6}$$

where $\rho(p)$ denotes near i the h-distance of a point p from i, the fields $\tilde{h}_{ab}$ and $\tilde{\chi}_{ab}$ related by (2.61) to h_{ab} and χ_{ab} satisfy the vacuum constraints and are asymptotically flat ([34]).

Using the conformal constraints to determine the remaining conformal fields one gets

$$S = \frac{1}{3} \Delta_h \Omega + \frac{1}{12} \Omega \left(r + \chi_{ab} \chi^{ab}\right) = \frac{1}{2\Omega} D_c\Omega D^c\Omega, \tag{4.7}$$

$$L_a = \frac{1}{\Omega} D^c\Omega \chi_{ca}, \tag{4.8}$$

$$L_{ab} - \frac{1}{3} L_c{}^c h_{ab} = -\frac{1}{\Omega} \left(D_a D_b \Omega - \frac{1}{3} \Delta_h\Omega h_{ab}\right), \tag{4.9}$$

$$L_{00} = \frac{1}{6}\,R - L_c{}^c = \frac{1}{6}\,R - \frac{1}{4}\left(r + \chi_{ab}\,\chi^{ab}\right), \tag{4.10}$$

$$w_{ab} = -\frac{1}{\Omega^2}\left(D_a\,D_b\Omega - \frac{1}{3}\,\Delta_h\Omega\,h_{ab}\right) - \frac{1}{\Omega}\left(\chi_{ac}\,\chi_b{}^c - \frac{1}{3}\,\chi_{ce}\,\chi^{ce}\,h_{ab} + s_{ab}\right), \tag{4.11}$$

$$w^*_{ab} = -\frac{1}{\Omega}\,D_c\,\chi_{e(a}\,\epsilon_{b)}{}^{ce}, \tag{4.12}$$

where we set $s_{ab} = r_{ab} - \frac{1}{3}\,r\,h_{ab}$. The differential identities (2.56)–(2.60), which are not needed to get these expressions, will be then also be satisfied (cf. [31]).

In view of conditions (4.6) most of these fields will in general be singular at i. One will have $w_{ab} = O(r^{-3})$ near i whenever the ADM mass m of the initial data set $\tilde{h}_{ab}$, $\tilde{\chi}_{ab}$ does not vanish. Controlling the time evolution of these data requires a careful analysis of these singularities. As a simplifying hypothesis we assume, as in [37], that the data are time reflection symmetric and define a smooth conformal structure, i.e.,

$$h_{ab} \in C^\infty(\mathcal{S}), \qquad \chi_{ab} = 0. \tag{4.13}$$

We note that much of the following discussion can be extended to more general data such as those considered in [25] and the more general class of data discussed in [24], which includes the stationary data.

The Ricci scalar R is at our disposal. With $R = \frac{3}{2}\,r$ one gets on $\mathcal{S}$

$$L_{ab} = -\frac{1}{\Omega}\left(D_a\,D_b\,\Omega - \frac{1}{3}\,\Delta_h\Omega\,h_{ab}\right) + \frac{1}{12}\,r\,h_{ab}, \tag{4.14}$$

$$L_{00} = 0, \quad L_{0a} = 0, \quad w^*_{ab} = 0, \tag{4.15}$$

$$w_{ab} = -\frac{1}{\Omega^2}\left(D_a\,D_b\Omega - \frac{1}{3}\,\Delta_h\Omega\,h_{ab} + \Omega\,s_{ab}\right). \tag{4.16}$$

In spite of this simplification the crucial problem is still present; one finds that $w_{ab} = O(\rho^{-3})$ near i if $m \neq 0$ (cf. (5.28)).

4.1. Time reflection symmetric asymptotically flat Cauchy data

To allow for more flexibility in the following analysis, we also want to admit non-trivial cases with vanishing or negative mass. The positive mass theorem ([65]) then tells us that we must allow for non-compact $\mathcal{S}$. This will create no problems because we are interested only in the behavior of the fields near space-like infinity.

Let x^a, $a = 1, 2, 3$, denote h-normal coordinates defined on some convex open normal neighborhood $\mathcal{U}$ of i so that with $h = h_{ab}(x^c)\,d\,x^a\,d\,x^b$

$$x^a(i) = 0, \qquad x^a\,h_{ab}(x^c) = -x^a\,\delta_{ab} \quad \text{on} \quad \mathcal{U}. \tag{4.17}$$

All equations of this subsection will be written in these coordinates. We set $|x| = \sqrt{\delta_{ab}\,x^a\,x^b}$ and $\Upsilon = |x|^2 = \delta_{ab}\,x^a\,x^b$ so that

$$h^{ab}\,D_a\,\Upsilon\,D_b\,\Upsilon = -4\,\Upsilon, \tag{4.18}$$

and

$$\Upsilon(i) = 0, \quad D_a\,\Upsilon(i) = 0, \quad D_a\,D_c\,\Upsilon(i) = -2\,h_{ac}. \tag{4.19}$$

By taking derivatives of (4.18) and using (4.19) one obtains

$$D_a\, D_b\, D_c\, \Upsilon(i) = 0, \qquad D_a\, D_b\, D_c\, D_d\, \Upsilon(i) = -\frac{4}{3}\, r_{a(cd)b}[h](i), \tag{4.20}$$

where the curvature tensor of h is given by

$$r_{abcd}[h] = 2\{h_{a[c}l_{d]b} + h_{b[d}l_{c]a}\}$$

with $l_{ab}[h] = r_{ab}[h] - \frac{1}{4}\, r[h]\, h_{ab}$ because $\dim(\mathcal{S}) = 3$. Proceeding further in this way on can determine an expansion of Υ in terms curvature terms at i. The relations above imply in particular

$$(\Delta_h \Upsilon + 6)(i) = 0, \qquad D_a(\Delta_h \Upsilon + 6)(i) = 0, \qquad D_a\, D_b(\Delta_h \Upsilon + 6)(i) = \frac{4}{3}\, r_{ab}(i). \tag{4.21}$$

Equation (4.5) and the first of equations (4.6) can be combined under our assumptions into

$$(\Delta_h - \frac{1}{8}\, r)\, \theta = 4\, \pi\, \delta_i,$$

where in the coordinates x^a the symbol δ_i denotes the Dirac-measure with weight 1 at $x^a = 0$. In a neighborhood of i there exists then a representation $\theta = \frac{U}{|x|} + W$ with functions U, W which satisfy

$$(\Delta_h - \frac{1}{8}\, r) \left(\frac{U}{|x|}\right) = 4\, \pi\, \delta_i, \qquad (\Delta_h - \frac{1}{8}\, r)\, W = 0 \quad \text{near } i, \tag{4.22}$$

and

$$U(i) = 1, \quad W(i) = \frac{m}{2}, \tag{4.23}$$

where m denotes the ADM-mass of the solution. The functions U, W are analytic on $\mathcal{U}$ if h is analytic ([45]) and smooth if h is C^∞ ([25]).

The function $\sigma \equiv \frac{\Upsilon}{U^2}$ is characterized uniquely by the conditions that it is smooth, satisfies the equation $(\Delta_h - \frac{1}{8}\, r)\, \sigma^{-1/2} = 4\, \pi\, \delta_i$, and the relations

$$\sigma(i) = 0, \quad D_a \sigma(i) = 0, \quad D_a\, D_b\, \sigma(i) = -2\, h_{ab}, \tag{4.24}$$

hold, which follow from (4.19) and (4.23). If σ' is another function satisfying these conditions, then $\sigma' = \Upsilon\, U'^{-2}$ with $U' = 1 + O(|x|)$ by (4.24) and $U' \in C^\infty(\mathcal{U})$ by the results of [25]. The function $f = \sigma^{-1/2} - \sigma'^{-1/2}$ then solves $(\Delta_h - \frac{1}{8}\, r)\, f = 0$ and it follows that $f \in C^\infty(\mathcal{U})$ and $|x|\, f = U - U' \in C^\infty(\mathcal{U})$. Expanding f and $U - U'$ in terms of spherical harmonics it follows from the last equation that f vanishes at i at any order. Since f satisfies the conformal Laplace equation it follows that $f = 0$ on $\mathcal{U}$ by Theorem 17.2.6 of [51]. This implies that $\sigma' = \sigma$ on $\mathcal{U}$.

The first of equations (4.22) can be rewritten in the form

$$2\, D^a\, \Upsilon\, D_a U + (\Delta_h\, \Upsilon + 6)\, U - 2\, \Upsilon\, (\Delta_h - \frac{1}{8}\, r)\, U = 0. \tag{4.25}$$

This allows us to determine from (4.24) recursively an asymptotic expansion of U, which is convergent if h is real analytic. The Hadamard parametrix construction is based on an ansatz

$$U = \sum_{p=0}^{\infty} U_p \, \Upsilon^p, \tag{4.26}$$

by which the calculation of U is reduced to an ODE problem. The functions U_p are obtained recursively by

$$U_0 = \exp\left\{\frac{1}{4}\int_0^{\Upsilon^{\frac{1}{2}}} (\Delta_h \Upsilon + 6)\,\frac{d\rho}{\rho}\right\},$$

$$U_{p+1} = -\frac{U_0}{(4p-2)\,\Upsilon^{\frac{p+1}{2}}}\int_0^{\Upsilon^{\frac{1}{2}}} \frac{\Delta_h[U_p]\,\rho^p}{U_0}\,d\rho, \quad p = 0, 1, 2, \ldots,$$

where the integration is performed in terms of the affine parameter $\rho = \Upsilon^{\frac{1}{2}} = |x|$ along the geodesics emanating from i. It follows that

$$U(i) = 1, \quad D_a U(i) = 0, \quad D_a D_b U(i) = \frac{1}{6}\, l_{ab}[h], \tag{4.27}$$

which implies

$$D_a D_b D_c \sigma(i) = 0, \quad D_a D_b D_c D_d \sigma(i) = 2\,(h_{cd}\, l_{ab} + h_{ab}\, l_{cd}). \tag{4.28}$$

Given h and the solution W of the conformal Laplace equation in (4.22), the considerations above show us how to determine an expansion of the function

$$\Omega = \theta^{-2} = \left(\frac{1}{\sqrt{\sigma}} + W\right)^{-2} = \frac{\Upsilon}{(U + \rho W)^2}, \tag{4.29}$$

in terms of ρ at all orders. Corresponding expansions can be obtained for the conformal data (4.7), (4.14), (4.15), (4.16).

While U is thus seen to be determined locally by the metric h, the function W carries non-local information. Cases where $\partial^{\alpha}_{x^a} W(i) = 0$ for all multiindices $\alpha = (\alpha^1, \alpha^2, \alpha^3) \in \mathbb{N}^3$ with $|\alpha| \equiv \alpha^1 + \alpha^2 + \alpha^3 \le N$ for some non-negative integer N or for $N = \infty$ will also be of interest in the following. In the latter case we have in fact ([51])

$$W = 0 \quad \text{near} \quad i. \tag{4.30}$$

For convenience this case will be referred to as the *massless case.*

A rescaling

$$h \to h' = \vartheta^4\, h, \quad \Omega \to \Omega' = \vartheta^2\, \Omega,$$

with a smooth positive factor ϑ satisfying $\vartheta(i) = 1$, leaves $\tilde{h} = \Omega^{-2}\, h$ unchanged but implies changes

$$\theta \to \theta' = \vartheta^{-1}\, \theta, \quad U \to U' = \frac{|x'|}{|x|}\, \vartheta^{-1}\, U, \quad W \to W' = \vartheta^{-1}\, W,$$

where $|x'|$ is defined in terms of h'-normal coordinates $x^{a'}$ as described above. Due to the conformal covariance of the operator on the left-hand sides of equations (4.22), relations (4.22), (4.23) will then also hold with all fields replaced by the primed fields.

To reduce this freedom it has been assumed in [37] that the metric h is given near i on $\mathcal{S}$ in the *cn-gauge*. By definition, this conformal gauge is satisfied by h if there exists a 1-form l_* at i such that the following holds. If $x(\tau)$, $l(\tau)$ solve the conformal geodesic equations (with respect to h) with $x(0) = i$, $l(0) = l_*$, and $h(\dot{x}, \dot{x}) = 1$ at i, then a frame e_a which is h-orthonormal at i and satisfies $\hat{D}_{\dot{x}}\, e_a = 0$ (with $\hat{D} - D = S(l)$), stays h-orthonormal near i. This gauge can be achieved without restrictions on the mass and fixes the scaling uniquely up to a positive real number and a 1-form given at i. It admits an easy discussion of limits where $m \to 0$.

If $m > 0$, it is convenient to set above $\vartheta = \frac{2}{m}\, W$. It follows then that $W' = \frac{m}{2}$, whence $0 = (\Delta_{h'} - \frac{1}{8}\, r[h'])\, W' = -\frac{m}{16}\, r[h']$. Thus, if $m > 0$, we can always assume h to be given such that

$$r[h] = 0, \quad \Omega = \frac{\sigma}{(1 + \sqrt{\mu\, \sigma})^2} \quad \text{with} \quad \sqrt{\mu} = \frac{m}{2}. \tag{4.31}$$

In this gauge the function σ satisfies near i the equation $\Delta_h\, (\sigma^{-1/2}) = 4\, \pi\, \delta_i$, which implies by (4.24)

$$2\, \sigma\, s = D_a \sigma\, D^a \sigma \quad \text{with} \quad s \equiv \frac{1}{3}\, \Delta_h\, \sigma, \tag{4.32}$$

(note that an analogous equation holds with σ replaced by Ω). Equation (4.32) implies in turn together with (4.24) the Poisson equation above.

For later reference we note the form of the conformal Schwarzschild data in this gauge. In isotropic coordinates the Schwarzschild line element is given by

$$d\, \tilde{s}^2 = \left(\frac{1 - m/2\, \tilde{r}}{1 + m/2\, \tilde{r}}\right)^2 d\, t^2 - (1 + m/2\, \tilde{r})^4\, (d\, \tilde{r}^2 + \tilde{r}^2\, d\, \sigma^2).$$

Expressing the initial data $\tilde{h}$, $\tilde{\chi}$ induced on $\{t = 0\}$ in terms of the coordinate $\rho = 1/\tilde{r}$, one finds that $\tilde{\chi} = 0$ and $\tilde{h} = \Omega^{-2}\, h$ with

$$h = -\, (d\, \rho^2 + \rho^2\, d\, \sigma^2), \qquad \Omega = \frac{\rho^2}{(1 + \frac{m}{2}\, \rho)^2}, \tag{4.33}$$

so that $\sigma = \Upsilon = \rho^2$ resp. $U = 1$. The metric h also satisfies a cn-gauge.

4.2. Static asymptotically flat Cauchy data

Static solutions to the vacuum field equations can be written in the form

$$\tilde{g} = v^2\, dt^2 + \tilde{h},$$

with t-independent negative definite metric $\tilde{h}$ and t-independent norm

$$v = \sqrt{\tilde{g}(K, K)} > 0$$

of the time-like Killing field $K = \partial_t$. With the $\tilde{g}$-unit normal of a slice $\{t = \text{const.}\}$ being given by $\tilde{n} = v^{-1} K$ and the associated orthogonal projector by $\tilde{h}_\mu{}^\nu = \tilde{g}_\mu{}^\nu - \tilde{n}_\mu \tilde{n}^\nu$, one gets for the second fundamental form on this slice

$$\tilde{\chi}_{\mu\nu} = v^{-1} \tilde{h}_\mu{}^\rho \tilde{h}_\nu{}^\delta \tilde{\nabla}_\rho K_\delta = 0,$$

because it is symmetric by $\tilde{n}$ being hypersurface orthogonal while the second term is anti-symmetric by the Killing equation. The solutions are thus time reflection symmetric.

For these solutions the vacuum field equations are equivalent to the requirement that the *static vacuum field equations*

$$r_{ab}[\tilde{h}] = \frac{1}{v} \tilde{D}_a \tilde{D}_b v, \qquad \Delta_{\tilde{h}} v = 0, \tag{4.34}$$

hold on one and thus on any slice $\{t = \text{const.}\}$. In harmonic coordinates these equations become elliptic and $\tilde{h}$ and v thus real analytic.

We consider solutions $\tilde{h}$, v to equations (4.34) with non-vanishing ADM-mass which are given on a 3-manifold $\tilde{\mathcal{S}}$ which is mapped by suitable coordinates $\tilde{x}^a$ diffeomorphically to $\mathbb{R}^3 \setminus \bar{\mathcal{B}}$, where $\bar{\mathcal{B}}$ is a closed ball in $\mathbb{R}^3$. We assume that $\tilde{h}$ satisfies in these coordinates the usual condition of asymptotic flatness and $v \to 1$ as $|\tilde{x}| \to \infty$. The work in [6] (cf. also [54] for a strengthening of this result) then implies that the conformal structure defined by $\tilde{h}$ can be extended analytically to space-like infinity. The physical 3-metric $\tilde{h}$ therefore belongs to the class of data considered above.

For such solutions it follows from the discussion in [7] that the gauge (4.31) is achieved if any of the equivalent equations

$$v = 1 - m\sqrt{\Omega} = \frac{1 - \sqrt{\mu\,\sigma}}{1 + \sqrt{\mu\,\sigma}}, \qquad \sigma = \left(\frac{2}{m}\,\frac{1-v}{1+v}\right)^2, \tag{4.35}$$

holds. The set $\mathcal{S} = \tilde{\mathcal{S}} \cup \{i\}$ can then be endowed with a differential structure such that the metric $h = \Omega^2 \tilde{h}$ extends as a real analytic metric to i. We shall consider in the following h-normal coordinates as in (4.17) such that the functions $\sigma(x^c)$, $h_{ab}(x^c)$ are then real analytic on $\mathcal{U}$. The first of the static vacuum field equations (4.34) then implies

$$0 = \Sigma_{ab} \equiv D_a D_b \sigma - s\, h_{ab} + \sigma\,(1 - \mu\,\sigma)\, r_{ab}[h], \tag{4.36}$$

where s is defined as in (4.32). The second of equations (4.34) implies $r[\tilde{h}] = 0$ and can thus be read as a conformally covariant Laplace equation for v. Using the transformation rule for this equation and observing (4.35), we find that it transforms into

$$0 = (\Delta_h - \frac{1}{8}\, r[h])(\theta\, v) = (\Delta_h - \frac{1}{8}\, r[h])(\theta - m) \quad \text{on} \;\; \tilde{\mathcal{S}},$$

and is thus satisfied by our assumption $r[h] = 0$.

We shall repeat some of the considerations of [34] in the present conformal gauge. The fact that solutions to the conformal static field equations are real analytic and can be extended by analyticity into the complex domain allows us to use some very concise arguments. We note that the statements obtained here can also be obtained by recursive arguments. This will become important if some of the following considerations are to be transferred to C^∞ or C^k situations.

From (4.36) one gets

$$D_c \Sigma_{ab} = D_c D_a D_b \sigma - D_c s h_{ab} + \sigma (1 - \mu \sigma) D_c r_{ab} + (1 - 2 \mu \sigma) D_c \sigma r_{ab}.$$

With the Bianchi identity, which takes in the present gauge the form $D^a r_{ab} = 0$, follow the integrability conditions

$$0 = \frac{1}{2} D^c \Sigma_{ca} = D_a s + (1 - \mu \sigma) r_{ab} D^b \sigma, \tag{4.37}$$

and

$$0 = D_{[c} \Sigma_{a]b} + \frac{1}{2} D^d \Sigma_{d[c} h_{a]b} \tag{4.38}$$
$$= \sigma \left\{ (1 - \mu \sigma) D_{[c} r_{a]b} - \mu (2 D_{[c} \sigma r_{a]b} + D^d \sigma r_{d[c} h_{a]b}) \right\}.$$

Equation (4.36) thus implies an expression for the Cotton tensor, which is given in the present gauge by $b_{bca} = D_{[c} r_{a]b}$, and for its dualized version, which is given by

$$b_{ab} = \frac{1}{2} b_{acd} \epsilon_b{}^{cd} = \frac{\mu}{1 - \mu \sigma} (D_c \sigma r_{da} \epsilon_b{}^{cd} - \frac{1}{2} r_{de} D^e \sigma \epsilon_{ba}{}^d). \tag{4.39}$$

It follows that

$$D_a (2 \sigma s - D_c \sigma D^c \sigma) = \sigma D^c \Sigma_{ca} - 2 D^c \sigma \Sigma_{ca},$$

which shows that equation (4.32) is a consequence of equations (4.24) and (4.36) and that the latter contain the complete information of the conformal static field equations.

Let $e_{\mathbf{a}} = e^c{}_{\mathbf{a}} \partial_{x^c}$, $\mathbf{a} = 1, 2, 3$, now denote the h-orthonormal frame field on $\mathcal{U}$ which is parallely transported along the h-geodesics through i and satisfies $e^c{}_{\mathbf{a}} = \delta^c{}_{\mathbf{a}}$ at i. In the following we assume all tensor fields, except the frame field $e_{\mathbf{a}}$ and the coframe field $\sigma^{\mathbf{c}}$ dual to it, to be expressed in term of this frame field and set $D_{\mathbf{a}} \equiv D_{e_{\mathbf{a}}}$. The coefficients of h are then given by $h_{\mathbf{ab}} = -\delta_{\mathbf{ab}}$. Any analytic tensor field $T_{\mathbf{a}_1 \ldots \mathbf{a}_p}^{\mathbf{b}_1 \ldots \mathbf{b}_q}$ on V has an expansion of the form (cf. [37])

$$T_{\mathbf{a}_1 \ldots \mathbf{a}_p}^{\mathbf{b}_1 \ldots \mathbf{b}_q}(x) = \sum_{k \geq 0} \frac{1}{k!} x^{c_k} \ldots x^{c_1} (D_{\mathbf{c}_k} \ldots D_{\mathbf{c}_1} T_{\mathbf{a}_1 \ldots \mathbf{a}_p}^{\mathbf{b}_1 \ldots \mathbf{b}_q})(i),$$

(where the summation rule ignores whether indices are bold face or not).

We want to discuss how expansions of this form can be obtained for the fields

$$\sigma, \; s, \; r_{\mathbf{ab}},$$

which are provided by the solutions to the conformal static field equations. Once these fields are known, the coefficients of the 1-forms $\sigma^{\mathbf{a}} = \sigma^{\mathbf{a}}{}_b \, dx^b$, which provide the coordinate expression of the metric by the relation $h = -\delta_{\mathbf{ac}} \sigma^{\mathbf{a}}{}_b \sigma^{\mathbf{c}}{}_d \, dx^b \, dx^d$, and the connection coefficients $\Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}$ with respect to $e_{\mathbf{a}}$ can be obtained from the structural equations in polar coordinates (cf. [50])

$$\frac{d}{d\rho}\left(\rho\,\sigma^{\mathbf{a}}{}_{b}(\rho\,x)\right) = \delta^{\mathbf{a}}{}_{b} + \rho\,\Gamma_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{d}}(\rho\,x)\,x^{d}\,\sigma^{\mathbf{c}}{}_{b}(\rho\,x),$$

$$\frac{d}{d\rho}\left(\rho\,\Gamma_{\mathbf{a}}{}^{\mathbf{c}}{}_{\mathbf{e}}(\rho\,x)\,\sigma^{\mathbf{a}}{}_{b}(\rho\,x)\right) = \rho\,r^{\mathbf{c}}{}_{\mathbf{eda}}(\rho\,x)\,x^{d}\,\sigma^{\mathbf{a}}{}_{b}(\rho\,x).$$

For this purpose we consider the data

$$c_{\mathbf{a}_p\ldots\mathbf{a}_1\mathbf{bc}} = \mathcal{R}(D_{\mathbf{a}_p}\ldots D_{\mathbf{a}_1}\,r_{\mathbf{bc}})(i), \tag{4.40}$$

where $\mathcal{R}$ means 'trace free symmetric part of'. These data have the following interpretation. Since solutions to the conformal static field equations are real analytic in the given coordinates x^a, all the fields considered above can be extended into a complex domain $\mathcal{U}' \subset \mathbb{C}^3$ which comprises $\mathcal{U}$ as the subset of real points. The subset $\mathcal{N} = \{\Upsilon = 0\}$ of $\mathcal{U}'$, where we denote by Υ again the analytic extension of the real function denoted before by the same symbol, then defines the cone which is generated by the complex null geodesics $\mathbb{C} \supset \mathcal{O} \ni \zeta \to x^a(\zeta) = \zeta\,x^a_* \in \mathcal{U}'$ through i, where $x^a_* \neq 0$ is constant with $h_{ab}\,x^a_*\,x^b_* = 0$ at i. On $\mathcal{N}$ the field $D^a\Upsilon\,\partial_{x^a} = -2\,x^a\,\partial_{x^a}$ is tangent to the null generators of $\mathcal{N}$. The derivatives of $r_{\mathbf{ab}}\,\dot{x}^a\,\dot{x}^b$ with respect to ζ at i are given by the complex numbers

$$x^{a_p}_*\ldots x^{a_1}_*\,x^b_* x^c_* D_{\mathbf{a}_p}\ldots D_{\mathbf{a}_1} r_{\mathbf{bc}}(i)$$
$$= \iota^{A_p}\iota^{B_p}\ldots\iota^{A_1}\iota^{B_1}\iota^{C}\ldots\iota^{F} D_{A_p\,B_p}\ldots D_{A_1B_1}\,r_{CDEF}(i)$$

where the term on the left-hand side is rewritten on the right-hand side in space spinor notation and it is used that $x^{AB}_* \equiv \sigma^{AB}{}_a\,x^a_* = \iota^A\,\iota^B$ with some spinor ι^A because x^a_* is a null vector. Allowing x^a_* to vary over the null cone at i, i.e., allowing ι^A to vary over $P^1(\mathbb{C})$, we can extract from the numbers above the real quantities

$$c_{A_p\,B_p\ldots A_1B_1CDEF} = D_{(A_p\,B_p}\cdots D_{A_1B_1}\,r_{CDEF)}(i), \tag{4.41}$$

which are equivalent to (4.40). Giving the data (4.40) is thus equivalent to giving $r_{\mathbf{ab}}(\zeta\,x^a_*)\,\dot{x}^a\,\dot{x}^b$ where x^a_* varies over a cut of the complex null cone at i or to giving, up to a scaling, the restriction of $r_{ab}\,D^a\Upsilon\,D^b\Upsilon$ to $\mathcal{N}$. The data (4.40) are in one-to-one correspondence to the multipole moments considered in [6].

We consider now the Bianchi identity $D^{\mathbf{a}}\,r_{\mathbf{ab}} = 0$ and equation (4.38). In space spinor notation they combine into the concise form

$$(1-\mu\,\sigma)\,D_A{}^E r_{BCDE} = 2\,\mu\,r_{E(BCD}\,D_{A)}{}^E\sigma. \tag{4.42}$$

Note that the contraction and symmetrization on the right-hand side project out precisely the information contained in $r_{\mathbf{ab}}\,D^{\mathbf{a}}\Gamma\,D^{\mathbf{b}}\Gamma$ while the contraction which occurs on the left-hand side prevents us from using the equation to calculate any of the information in (4.41). We use equations (4.32), (4.37) in frame notation. By taking formal derivatives of these equations one can determine from (4.24) and the data (4.40) all derivatives of σ, s, and $r_{\mathbf{ab}}$ at i. The complete set of data (4.40) resp. (4.41) is required for this and these data determine the expansion uniquely.

This procedure has been formalized in the theory of 'exact sets of fields' discussed in [61], where equations of the type (4.42) are considered.

The formulation given above suggests proving a Cauchy-Kowalevska type results for equations (4.32), (4.37), (4.42) with data prescribed on $\mathcal{N}$. Although the existence of the vertex at i may create some difficulties in the present case, this problem has much in common with the characteristic initial value problem for Einstein's field equations for which the existence of analytic solutions has been shown ([30]). At present, no decay estimates for the $c_{\mathbf{a}_p \ldots \mathbf{a}_1 \mathbf{b} \mathbf{c}}$ as $p \to \infty$ are available which would ensure the convergence of these series. To simplify the following discussion we shall assume that the series considered above do converge.

We return to the coordinate formalism and show that this procedure provides a solution to the original equation (4.36), i.e., the quantity Σ_{ab} defined from the fields σ, s, and r_{ab} by the procedure above does vanish. We show first that $\Sigma_{ab} = 0$ on $\mathcal{N}$. Since $\sigma = 0$ on $\mathcal{N}$, this amounts to showing that $m_{ab} \equiv D_a\, D_b\, \sigma - s\, h_{ab}$ vanishes on $\mathcal{N}$. Differentiating twice the equation $D_a \sigma\, D^a \sigma - 2\, \sigma\, s = 0$, which has been solved as part of the procedure above, observing that $D^c\, \Sigma_{cd} = 0$ and restricting the resulting equation to $\mathcal{N}$ gives the linear ODE

$$D^c \sigma\, D_c\, m_{ab} = -D_a D^c \sigma\; m_{cb},$$

along the null generators of $\mathcal{N}$. Observing that $D_a\, D_b\, \sigma = -2\, h_{ab} + O(\Upsilon)$, this ODE can be written along the null geodesics $x^a(\zeta) = \zeta\, x^a_*$ considered above in the form

$$\frac{d}{d\,\zeta}(\zeta\, m_{ab}) = A_a^{\ c}\, \zeta\, m_{cb},$$

with a smooth function $A_a^{\ c} = A_a^{\ c}(\zeta)$. This implies the desired result. In view of (4.37), (4.38) it shows that we solved the problem

$$D^c\, \Sigma_{ca} = 0, \quad D_{[c}\, \Sigma_{a]b} = 0 \quad \text{near} \;\; i, \quad \Sigma_{ab} = 0 \quad \text{on} \;\; \mathcal{N}.$$

The first two equations combine in space spinor notation into $D_A{}^E\, \Sigma_{BCDE} = 0$ with symmetric spinor field Σ_{ABCD}. Following again the arguments of [61], we conclude that $\Sigma_{ab} = 0$.

Equation (4.39) implies

$$D^a \sigma\, D^b \sigma\, b_{ab} = 0 \quad \text{on} \quad V. \tag{4.43}$$

A rescaling $h \to h' = \vartheta^4\, h$ with a positive (analytic) conformal factor gives $\sigma \to \sigma' = \vartheta^2 \sigma$ and $b_{ab} \to b'_{ab} = \vartheta^{-2} b_{ab}$, whence

$$D^a \sigma\, D^b \sigma\, b_{ab} \to (D^a \sigma\, D^b \sigma\, b_{ab})' =$$

$$\vartheta^{-6} D^a \sigma\, D^b \sigma\, b_{ab} + 4\, \sigma\, \vartheta^{-7} D^a \sigma\, D^b \vartheta\, b_{ab} + 4\, \sigma^2\, \vartheta^{-8} D^a \vartheta\, D^b \vartheta\, b_{ab}.$$

This shows that (4.43) is not conformally invariant, but it also shows that the relation

$$D^a \Upsilon\, D^b \Upsilon\, b_{ab}|_{\mathcal{N}} = 0, \tag{4.44}$$

implied by (4.43), is conformally invariant. Using again the argument which allowed us to get the quantities (4.41), we can translate this onto the equivalent relations

$$\mathcal{R}(D_{a_p}\cdots D_{a_1}\,b_{bc}(i)) = 0, \quad p = 0,1,2,\ldots, \tag{4.45}$$

which take in space spinor notation the form (cf. (5.89))

$$D_{(A_p\,B_p}\cdots D_{A_1B_1}\,b_{CDEF)}(i) = 0, \quad p = 0,1,2,\ldots. \tag{4.46}$$

We note that for given integer $p_* > 0$ the string of such conditions with $0 \leq p \leq p_*$ is conformally invariant.

Since these condition have a particular bearing on the smoothness of gravitational fields at null infinity ([34], [37]) and it is not clear whether static equations are of a greater significance in this context than expected so far, we take a closer look at (4.43). If we apply the operators D_aD_b and $D_aD_bD_c$ to (4.43) and restrict the resulting equation to i, we get the relations $b_{ab}(i) = 0$ and $D_{(a}b_{bc)}(i) = 0$ respectively, which agree with (4.45) at the corresponding orders because $D^a b_{ab} = 0$. However, if we proceed similarly with $D_aD_bD_cD_d$, we get

$$D_{(a}D_b\,b_{cd)}(i) = 0. \tag{4.47}$$

Since (4.45) with $p = 2$ can be written in the form

$$D_{(a}D_b\,b_{cd)}(i) = \frac{1}{7}\,h_{(ab}\,\Delta_h\,b_{cd)}(i),$$

the relation (4.47) implies in particular that $\Delta_h\,b_{cd}(i) = 0$. It appears that in general this equation cannot be deduced in the present gauge from known general identities and (4.44) alone. There will be similar such conditions at higher orders. While the particular form of them may depend on the conformal gauge, the existence of properties which go beyond (4.45) does not. In any case, these observations show that there is a gap between h satisfying the regularity conditions (4.45) and h being conformally static.

This situation is also illustrated by the following observation. If the data provided by h are conformally flat in a neighborhood of i they trivially satisfy conditions (4.45). Without further assumptions the solution θ to the Lichnerowicz equation which relates h to the induced vacuum data $\tilde{h} = \theta^4\,h$ can still be quite general. However, if $\tilde{h}$ is static the function θ must be very special.

Lemma 4.1. *An asymptotically flat, static initial data set for the vacuum field equations with conformal metric h and positive ADM mass m is locally conformally flat if and only if it satisfies near i in the gauge* (4.31) *the equation* $r_{ab}[h] = 0$ *and thus in the normal coordinates* (4.17)

$$h = -\delta_{ab}\,dx^a\,dx^b, \quad U = 1, \quad \theta = \frac{1}{|x|} + \frac{m}{2}.$$

Remark: This tells us that *the only asymptotically flat, static vacuum data which are locally conformally flat near space-like infinity are the Schwarzschild data* (4.33). The result of ([71]), which suggests that conformal flatness of the data

h near i and the smoothness requirement on the functions u^p at $I^{\pm}$ imply that the solution be asymptotically Schwarzschild, can thus be reformulated as saying that for the given data the smoothness requirement implies the solution to be asymptotically static at space-like infinity.

Proof. By (4.38) the solution is locally conformally flat if and only if $2\,D_{[c}\sigma\, r_{a]b} = h_{b[c}\, r_{a]d} D^d\sigma$. Applying D_e to this equation and observing (4.36) and again $D_{[c}\, r_{a]b} = 0$, one gets after a contraction

$$D^c\sigma\, D_c\, r_{ab} = -3\, s\, r_{ab} + \sigma\,(1-\mu\,\sigma)\,(h_{ab}\, r_{cd}\, r^{cd} - 3\, r_{ac}\, r_b{}^c). \tag{4.48}$$

This equation can be read as an ODE along the integral curves of the vector field $D^c\,\sigma$. It follows from (4.32) that $u^a = (2\,\sigma\,|s|)^{-\frac{1}{2}}\, D^a\,\sigma$ is a unit vector field (with direction dependent limits at i). Because of

$$u^a\, D_a\, \Upsilon = -\left(\frac{2\,\Upsilon}{|s|}\right)^{1/2} (4\,U^{-1} + 2\,U^{-2}\, D^a\, U\, D_a\, \Upsilon) < 0,$$

its integral curves run into i and cover in fact a (possibly small) neighborhood $\mathcal{U}'$ of i. Equation (4.48) can be rewritten in the form

$$u^a\, D_a(\Upsilon^{3/2}\, r_{bc}) = A_{bc}^{de}\,(\Upsilon^{3/2}\, r_{de}),$$

with the matrix-valued function

$$A_{bc}^{de} = -\frac{3}{\sqrt{2\,\sigma\,|s|}}\,(s + 2\,U^{-1} + U^{-2}\, D^a U\, D_a\, \Upsilon)\, h^d{}_b\, h^e{}_c$$
$$+\frac{\sigma\,(1-\mu\,\sigma)}{\sqrt{2\,\sigma\,|s|}}\,(h_{bc}\, r^{de} - 3\, r^d{}_b\, h^e{}_c),$$

which is continuous on $\mathcal{U}'$. This implies that $r_{ab} = 0$ on $\mathcal{U}'$. The remaining statements follow immediately from (4.17) and (4.31).

Remark: We note that these data may be obtained in a different form if locally conformally flat data are given in the cn-gauge and one asks under which conditions they are conformally static. The data are then of the form

$$h_{ab} = -\delta_{ab}, \quad \Omega^{-\frac{1}{2}} = \theta = \frac{1}{|x|} + W, \quad \Delta_h\, W = 0, \quad W(i) = \frac{m}{2} > 0.$$

By a rescaling $h \to \vartheta^4\, h$, $\theta \to \vartheta^{-1}\,\theta = \frac{1}{\vartheta\,|x|} + \frac{m}{2}$ with $\vartheta = \frac{2}{m}\, W$ they are transformed into the present gauge. Assuming that these data satisfy the conformal static field equations and expressing the resulting equation again in terms of $h_{ab} = -\delta_{ab}$ one finds that the solution is static if and only if $2\,W\, D_a\, D_b\, W - 6\; D_a\, W\, D_b\, W + 2\, h_{ab}\, D_c\, W\, D^c\, W = 0$. Since $W > 0$ the equation can be rewritten in terms of $w = W^{-2}$, which gives

$$2\, w\, D_a\, D_b\, w = h_{ab}\, D_c\, w\, D^c\, w. \tag{4.49}$$

Applying D_c, multiplying with w, and using twice (4.49) again, we conclude that $D_a\, D_b\, D_c w = 0$, whence $w = k + k_a\, x^a + k_{ab}\, x^a\, x^b$ with some coefficients $k > 0$,

k_a, k_{ab}. This function satisfies (4.49) if $k_{ab} = h_{ab}\, k_c\, k^c/4\, k$. With $j_a = k_a/2\, k$ and $m = 2/\sqrt{k}$ this gives

$$W = \frac{m}{2}\,\frac{1}{\sqrt{1 + 2\, j_a\, x^a + j_a\, j^a\, x_b\, x^b}},$$

with constant j^a.

That these data are equivalent to the ones considered above is seen by rescaling with $\vartheta = \frac{2}{m}\, W$. By this one achieves $W = \frac{m}{2}$. For the metric $\vartheta^4\, h$ to acquire the flat standard form one needs to perform a coordinate transformation which is given by a special conformal transformation $x \to (I \circ T_c \circ I)(x)$ where I denotes the inversion $x^a \to x^a\,(\delta_{bc}\, x^b\, x^c)^{-1}$ and T_c a translation $x^a \to x^a + c^a$ with suitably chosen constant c^a.

5. A regular finite initial value problem at space-like infinity

In the conformal extension of Minkowski space described in Section 3 neighborhoods of space-like infinity, which are swept out by future complete outgoing and past complete incoming null geodesics, are squeezed into arbitrarily small neighborhoods of the point i^0. From the point of view of the causal structure it is natural to indicate space-like infinity by a point. The discussion in Section 4 shows, however, that in general i^0 cannot be a regular point of any smooth conformal extension. The condition for an extension to i^0 to be C^∞ (under our assumption (4.13)) is that the data are massless in the sense of (4.30) and that the free datum h satisfies the conditions (5.89) with $p_* = \infty$ ([34], [37]). Thus, smoothness at i^0 excludes the physically interesting cases.

A direct discussion of the initial value problem for the conformal field equations with initial data on an initial hypersurface $\mathcal{S} = \tilde{\mathcal{S}} \cup \{i\}$ such that $W^i{}_{jkl} = O(\rho^{-3})$ at i as discussed in Section 4 faces considerable technical problems. Not only the functional analytical treatment of a corresponding PDE problem poses enormous difficulties but already the choice of gauge becomes very subtle.

The setting described below has been arrived at by attempts to describe the structure of the singularity as clearly as possible and to deduce from the conformal field equations a formulation of the PDE problem which still preserves 'some sort of hyperbolicity' at space-like infinity. It is based on conformally invariant concepts so that possible singularities should be identifiable as defects of the conformal structure.

In a conformal Gauss gauge based on a Cauchy hypersurface $\tilde{\mathcal{S}}$ it turns out that after blowing up the point i into a sphere $\mathcal{I}^0$ and choosing the gauge suitably, one arrives at a formulation of the initial value problem near space-like infinity in which the data can be smoothly extended to and across $\mathcal{I}^0$. In that gauge also the evolution equations admit a smooth extension to space-like infinity. The evolution and extension process then generates from the set $\mathcal{I}^0$ a cylindrical piece of space-time boundary diffeomorphic to $]-1, 1[\times \mathcal{I}^0$, which is denoted by $\mathcal{I}$. It represents space-like infinity and can be considered as a blow-up of the point i^0. This boundary is neither postulated nor attached 'by hand'.

In this gauge the hypersurfaces $\mathcal{J}^{\pm}$ representing null infinity near space-like infinity are given by finite values of the coordinates which are explicitly known (it has to be shown, of course, that the evolution extends far enough). These hypersurfaces touch the cylinder $\mathcal{I}$ at sets $\mathcal{I}^{\pm}$ diffeomorphic to $\mathcal{I}^0$, which can be thought of as boundaries of $\mathcal{I}$ and of $\mathcal{J}^{\pm}$ respectively. The structure of the conformal field equations near the *critical sets* $\mathcal{I}^{\pm}$ appears to be the key to the question of asymptotic smoothness.

It may appear odd to squeeze space-time regions of infinite extend into arbitrarily small neighborhoods of a point i^0 and then perform a complicated blow-up to resolve the singularity on the initial hypersurface which has been created by the first step. The point of the construction is that the finiteness of the sets $\mathcal{I}^{\pm}$ allow us to disclose, to an extent that we can put our hands on it, a subtle feature of the field equations which otherwise would be hidden at infinity (in the standard vacuum representation) or in the singularity at i^0 (in the standard conformal rescaling).

In the following the setting indicated above and its various implications will be discussed in detail. While we shall add more recent results we shall follow to a large extent the original article [37]. For derivations and details we refer the reader to this or the articles quoted below.

5.1. The gauge on the initial slice and the blow-up at i

The non-smoothness of the conformal data (4.14), (4.7), (4.15), (4.16), (4.29) at i arises from the presence of various factors ρ in the explicit expressions. To properly take care of the specific radial and angular behavior of the various fields it is natural to choose the frame field in the general conformal field equations such that the spatial vector fields e_a, $a = 1, 2, 3$, are tangent to the initial hypersurface $\mathcal{S}$ and one of them, e_3 say, is radial. Since there is no preferred direction at i, this only makes sense if the frame is chosen on $\tilde{\mathcal{S}}$ such that it has direction dependent limits at i. This singular situation finds a well-organized description in terms of a smooth submanifold of the bundle of frames. To discuss the field equations in the spin frame formalism, we will consider in fact a submanifold $\mathcal{C}_e$ of the bundle of normalized spin frames over $\mathcal{S}$ near i. While the use of spinors leads to various simplifications, it should be mentioned that the construction could be carried out similarly in the standard frame formalism (cf. [38]).

5.1.1. The construction of $\mathcal{C}_e$. Consider now $\mathcal{S}$ as a space-like Cauchy hypersurface of a 4-dimensional solution space-time $(\mathcal{M}, g)$ with induced metric h on $\mathcal{S}$. Denote by $SL(\mathcal{S})$ the set of spin frames $\delta = \{\delta_A\}_{A=0,1}$ on $\mathcal{S}$ which are normalized with respect to the alternating form ϵ, such that

$$\epsilon(\delta_A, \delta_B) = \epsilon_{AB}, \quad \epsilon_{01} = 1. \tag{5.1}$$

The group

$$SL(2,\mathbb{C}) = \{t^A{}_B \in GL(2,\mathbb{C}) \mid \epsilon_{AC} t^A{}_B t^C{}_D = \epsilon_{BD}\},$$

acts on $SL(\mathcal{S})$ by $\delta \to \delta \cdot t = \{\delta_A t^A{}_B\}_{B=0,1}$. The vector field $\tau = \sqrt{2}\, e_0$, with e_0 the future directed unit normal of $\mathcal{S}$, defines a subbundle $SU(\mathcal{S})$ of $SL(\mathcal{S})$ which is given by the spin frames in $SL(\mathcal{S})$ with

$$g(\tau, \delta_A \bar{\delta}_{A'}) = \epsilon_A{}^0 \epsilon_{A'}{}^{0'} + \epsilon_A{}^1 \epsilon_{A'}{}^{1'} \equiv \tau_{AA'}. \tag{5.2}$$

It has structure group

$$SU(2) = \{t^A{}_B \in SL(2,C) \mid \tau_{AA'} t^A{}_B \bar{t}^{A'}{}_{B'} = \tau_{BB'}\}.$$

In any frame in $SU(\mathcal{S})$ the vector τ is given by $\tau^{AA'}$. In the following we use the space spinor formalism in the notation of [36]. Using the van der Waerden symbols for space spinors

$$\sigma_a{}^{AB} = \sigma_a{}^{(A}{}_{A'} \tau^{B)A'}, \quad \sigma^c{}_{AB} = \tau_{(B}{}^{A'} \sigma^c{}_{A)A'}, \quad c = 1,2,3,$$

which satisfy

$$h_{ab} = \sigma_{a\,AB} \sigma_b{}^{AB}, \qquad \epsilon_A{}^B \epsilon_{A'}{}^{B'} = \frac{1}{2} \tau_{AA'} \tau^{BB'} + \sigma^a{}_{AF} \tau^F{}_{A'} \tau^{EB'} \sigma_{a\,E}{}^B,$$

where

$$h_{ab}\, \sigma^a{}_{AB} \sigma^b{}_{CD} = -\epsilon_{A(C}\epsilon_{D)B} \equiv h_{ABCD} \quad \text{with} \quad h_{ab} = -\delta_{ab},$$

the covering map onto the connected component $SO(3)$ of the rotation group is given by

$$SU(2) \ni t^A{}_B \xrightarrow{\Psi} t^a{}_b = \sigma^a{}_{AB}\, t^A{}_C\, t^B{}_D\, \sigma_b{}^{CD} \in SO(3).$$

The induced isomorphism of Lie algebras will be denoted by Ψ_*.

The covering morphism of $SU(\mathcal{S})$ onto the bundle $O_+(\mathcal{S})$ of positively oriented orthonormal frames on $\mathcal{S}$ maps the frame $\delta \in SU(\mathcal{S})$ onto the frame with vectors $e_a = e_a(\delta) = \sigma_a{}^{AB}\, \delta_A\, \tau_B{}^{B'}\, \bar{\delta}_{B'}$ such that $h(e_a, e_b) = h_{ab}$. We use this map to pull back to $SU(\mathcal{S})$ the h-Levi-Civita connection form on $O_+(\mathcal{S})$. Combining this with the map Ψ_*^{-1}, the connection is represented by an $su(2)$-valued connection form $\check{\omega}^A{}_B$ on $SU(\mathcal{S})$. Similarly, pulling back the $\mathbb{R}^3$-valued solder form on $O_+(\mathcal{S})$ and contracting with the van der Waerden symbols results in a 1-form σ^{AB} on $SU(\mathcal{S})$ which is referred to as solder form on $SU(\mathcal{S})$.

Let $\check{H}$ denote the real horizontal vector field on $SU(\mathcal{S})$ satisfying $\langle \sigma^{AB}, \check{H} \rangle = \epsilon_0{}^{(A}\, \epsilon_1{}^{B)}$ or, equivalently,

$$T_\delta(\pi)\, \check{H}(\delta) = \delta_{(0}\, \tau_{1)}{}^{B'} \bar{\delta}_{B'} = \frac{1}{2}(\delta_0\, \bar{\delta}_{0'} - \delta_1\, \bar{\delta}_{1'}), \quad \delta \in SU(\mathcal{S}). \tag{5.3}$$

It follows that $T_{\delta\, t}(\pi)\, \check{H}(\delta\, t) = T_\delta(\pi)\, \check{H}(\delta)$ if and only if

$$t \in U(1) \equiv \{t \in SU(2) \,|\, t = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \phi \in \mathbb{R}\}.$$

The field $\check{H}$ will essentially correspond to the 'radial' vector field mentioned above.

We consider again the normal coordinates satisfying (4.17) near i, set $\mathcal{B}_e = \{p \in \mathcal{U} \mid |x(p)| < e\}$ with $e > 0$ chosen such that the closure of $\mathcal{B}_e$ in $\mathcal{S}$ is contained in $\mathcal{U}$, and denote by $(SU(\mathcal{B}_e), \pi)$ the restriction of $(SU(\mathcal{S}), \pi)$ to $\mathcal{B}_e$. Let δ^* be in the fiber $\pi^{-1}(i) \subset SU(\mathcal{B}_e)$ over i. The map $SU(2) \ni t \to \delta(t) \equiv \delta^* \cdot t \in \pi^{-1}(i)$ defines a smooth parametrization of $\pi^{-1}(i)$. We denote by $]-e, e[\ni \rho \to \delta(\rho, t) \in SU(\mathcal{B}_e)$ the integral curve of the vector field $\sqrt{2}\,\check{H}$ satisfying $\delta(0,t) = \delta(t)$ and set $\mathcal{C}_e = \{\delta(\rho, t) \in SU(\mathcal{B}_e) \mid |\rho| < e,\, t \in SU(2)\}$. This set defines a smooth submanifold of $SU(\mathcal{B}_e)$ which is diffeomorphic to $]-e, e[\times SU(2)$. The restriction of π to this set will be denoted by π'.

The symbol ρ, which has been introduced already in Section 4.1, is used here for the following reason. The integral curves of $\sqrt{2}\,\check{H}$ through $\pi^{-1}(i)$ project onto geodesics through i with h-unit tangent vector. Thus, the projection π' maps $\mathcal{C}_e$ onto $\mathcal{B}_e$. The action of $U(1)$ on $SU(\mathcal{B}_e)$ induces an action on $\mathcal{C}_e$. While $\mathcal{I}^0 \equiv \pi^{-1}(i) = \{\rho = 0\}$ is diffeomorphic to $SU(2)$, the fiber $\pi'^{-1}(p) \subset \mathcal{C}_e$ over a point p in the punctured disk $\tilde{\mathcal{B}}_e \equiv \mathcal{B}_e \setminus \{i\}$ coincides with an orbit of $U(1)$ in $SU(\mathcal{B}_e)$ on which $\rho = |x(p)|$ and another one on which $\rho = -|x(p)|$.

The map π' factorizes as $\mathcal{C}_e \xrightarrow{\pi_1} \mathcal{C}'_e \xrightarrow{\pi_2} \mathcal{B}_e$ with $\mathcal{C}'_e \equiv \mathcal{C}_e / U(1)$ diffeomorphic to $]-e, e[\times S^2$. For ρ_* with $0 < |\rho_*| < e$ the subsets $\{\rho = \rho_*\}$ of $\mathcal{C}_e$ are diffeomorphic to $SU(2)$ and the restrictions of the map π_1 to these sets define Hopf fibrations of the form

$$SU(2) \ni t \to \sqrt{2}\sigma^a{}_{AB} t^A{}_0 t^B{}_1 \in S^2 \subset \mathbb{R}^3. \tag{5.4}$$

The set $\pi_2^{-1}(\tilde{\mathcal{B}}_e)$ (resp. $\pi'^{-1}(\tilde{\mathcal{B}}_e)$) consists of two components $\mathcal{C}'^{\pm}_e$ (resp. $\mathcal{C}^{\pm}_e$) on which $\pm\rho > 0$ respectively. Each of the sets $\mathcal{C}'^{\pm}_e$ is mapped by π_2 diffeomorphically onto the punctured disk. If $\tilde{\mathcal{B}}_e$ is now identified via π_2 with $\mathcal{C}'^{+}_e$ the manifold $\tilde{\mathcal{B}}_e$ is embedded into $\mathcal{C}'_e$ such that it acquires the set $\pi_1(\mathcal{I}^0) = \pi_2^{-1}(i)$ as a boundary. The set $\bar{\mathcal{B}}_e \equiv \tilde{\mathcal{B}}_e \cup \pi_2^{-1}(i) \simeq [0, e[\times S^2$ is a smooth manifold with boundary. Viewing $\tilde{\mathcal{B}}_e$ again as the subset of $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{i\}$, we get an extension $\bar{\mathcal{S}}$ of $\tilde{\mathcal{S}}$ which can be thought of as being obtained from $\mathcal{S}$ by blowing up the point i into a sphere. This is our desired extension of the physical initial manifold and the following discussion could be carried out in terms of the 3-dimensional manifold $\bar{\mathcal{B}}_e$.

It turns out more convenient, however, to use the 4-dimensional $U(1)$ bundle $\bar{\mathcal{C}}^{+}_e = \mathcal{C}^{+}_e \cup \mathcal{I}^0 = \{\delta \in \mathcal{C}_e \mid \rho(\delta) \geq 0\} \simeq [0, e[\times SU(2)$. It is a manifold with boundary smoothly embedded into $SU(\mathcal{B}_e)$, from which it inherits various structures. The set $\mathcal{C}_e$ is conveniently parametrized by ρ and the parallelizable group $SU(2)$. The solder and the connection form on $SU(\mathcal{B}_e)$ pull back to smooth 1-forms on $\mathcal{C}_e$. We denote the latter again by σ^{ab} and $\check{\omega}^a{}_b$ respectively. Any smooth spinor field ξ on $\mathcal{B}_e$ defines on $\mathcal{C}_e$ a smooth 'spinor-valued function' which is given at $\delta \in \mathcal{C}_e$ by the components of ξ in the frame defined by δ and denoted (in the case of a covariant field) by $\xi_{A_1 \ldots A_k, A'_1 \ldots A'_j}$. We shall refer to this function as to the 'lift' of ξ.

The structure equations induce on $\mathcal{C}_e$ the equations

$$d\sigma^{AB} = -\check{\omega}^A{}_E \wedge \sigma^{EB} - \check{\omega}^B{}_E \wedge \sigma^{AE}, \tag{5.5}$$

$$d\check{\omega}^A{}_B = -\check{\omega}^A{}_E \wedge \check{\omega}^E{}_B + \check{\Omega}^A{}_B, \tag{5.6}$$

where

$$\check{\Omega}^A{}_B = \frac{1}{2} r^A{}_{BCDF}\, \sigma^{CD} \wedge \sigma^{EF}$$

denotes the curvature form determined by the curvature spinor r_{ABCDEF}. It holds

$$r_{ABCDEF} = \left(\frac{1}{2}\, s_{ABCE} - \frac{r}{12}\, h_{ABCE}\right) \epsilon_{DF} + \left(\frac{1}{2}\, s_{ABDF} - \frac{r}{12}\, h_{ABDF}\right) \epsilon_{CE} \tag{5.7}$$

where $s_{ABCD} = s_{(ABCD)}$ is the trace free part of the Ricci tensor of h and r its Ricci scalar. The curvature tensor of h is given by

$$r_{AGBHCDEF} = -r_{ABCDEF}\epsilon_{GH} - r_{GHCDEF}\epsilon_{AB}.$$

and the Bianchi identity reads $6\, D^{AB}\, s_{ABCD} = D_{CD} r$.

We use $t \in SU(2,C)$ and $x^1 \equiv \rho$ as 'coordinates' on $\mathcal{C}_e$. The vector field $\check{H}$ tangent to $\mathcal{C}_e$ then takes the form $\sqrt{2}\, \check{H} = \partial_\rho$. Consider now the basis

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{5.8}$$

of the Lie algebra $su(2)$. Here u_3 is the generator of the group $U(1)$. We denote by Z_{u_i}, $i = 0,1,2$, the Killing vector fields generated on $SU(\mathcal{B}_e)$ by u_i and the action of $SU(2)$. These fields are tangent to $\mathcal{I}^0$. We set there

$$X_+ \equiv -(Z_{u_2} + iZ_{u_1}), \; X_- \equiv -(Z_{u_2} - iZ_{u_1}), \; X \equiv -2\, i\, Z_{u_3},$$

and extend these fields smoothly to $\mathcal{C}_e$ by requiring

$$[\check{H}, X] = 0, \quad [\check{H}, X_\pm] = 0. \tag{5.9}$$

The vector fields $\check{H}$, X, X_+, X_- constitute a frame field on $\mathcal{C}_e$ which satisfies besides (5.9) the commutation relations

$$[X,\, X_+] = 2\, X_+, \; [X,\, X_-] = -2X_-, \; [X_+,\, X_-] = -X. \tag{5.10}$$

The vector field iX is tangent to the fibers defined by π_1. The complex vector fields X_+, X_- are complex conjugates of each other such that $\overline{X_- f} = X_+ f$ for any real-valued function f.

These vector fields are related to the 1-forms above by

$$\langle \sigma^{AB}, \check{H}\rangle = \epsilon_0{}^{(A}\, \epsilon_1{}^{B)}, \quad \langle \sigma^{AB}, X\rangle = 0 \quad \text{on} \quad \mathcal{C}_e, \tag{5.11}$$

$$\langle \sigma^{AB}, X_+\rangle = \rho\, \epsilon_0{}^A\, \epsilon_0{}^B + O(\rho^2), \; \langle \sigma^{AB}, X_-\rangle = -\rho\, \epsilon_1{}^A\, \epsilon_1{}^B + O(\rho^2), \tag{5.12}$$

$$\langle \check{\omega}^A{}_B, \check{H}\rangle = 0, \quad \langle \check{\omega}^A{}_B, X\rangle = \epsilon_0{}^A\, \epsilon_B{}^0 - \epsilon_1{}^A\, \epsilon_B{}^1 \quad \text{on} \quad \mathcal{C}_e, \tag{5.13}$$

$$\langle \check{\omega}^A{}_B, X_+\rangle = \epsilon_0{}^A\, \epsilon_B{}^1 + O(\rho^2), \; \langle \check{\omega}^A{}_B, X_-\rangle = -\epsilon_1{}^A\, \epsilon_B{}^0 + O(\rho^2), \tag{5.14}$$

as $\rho \to 0$.

To transfer the tensor calculus on $\mathcal{B}_e$ to $\mathcal{C}_e$ we define vector fields $c_{AB} = c_{(AB)}$ on $\mathcal{C}_e \setminus \mathcal{I}^0$ by requiring

$$\langle \sigma^{AB}, c_{CD}\rangle = \epsilon_{(C}{}^A\, \epsilon_{D)}{}^B, \quad c_{CD} = c^1{}_{CD}\partial_\rho + c^+{}_{CD}\, X_+ + c^-{}_{CD}\, X_-. \tag{5.15}$$

The first condition implies that $T_\delta(\pi')\, c_{AB} = \delta_{(A}\tau_{B)}{}^{B'}\bar{\delta}_{B'}$ for $\delta \in \mathcal{C}_e \setminus \mathcal{I}^0$, while the second removes the freedom for the vector fields to pick up an arbitrary component in the direction of X. It follows that

$$c^1{}_{AB} = x_{AB}, \quad c^+{}_{AB} = \frac{1}{\rho}\, z_{AB} + \check{c}^+{}_{AB}, \quad c^-{}_{AB} = \frac{1}{\rho}\, y_{AB} + \check{c}^-{}_{AB}, \tag{5.16}$$

with smooth functions which satisfy

$$\check{c}^\alpha{}_{AB} = O(\rho), \quad \check{c}^\alpha{}_{01} = 0, \quad \alpha = 1, +, -, \tag{5.17}$$

and

$$x_{AB} \equiv \sqrt{2}\epsilon_{(A}{}^0 \epsilon_{B)}{}^1, \quad y_{AB} \equiv -\frac{1}{\sqrt{2}}\epsilon_A{}^1 \epsilon_B{}^1, \quad z_{AB} \equiv \frac{1}{\sqrt{2}}\epsilon_A{}^0 \epsilon_B{}^0. \tag{5.18}$$

The connection coefficients with respect to c_{AB} satisfy

$$\gamma_{CD}{}^A{}_B \equiv \langle \check{\omega}^A{}_B, c_{CD}\rangle = \frac{1}{\rho}\, \gamma^*_{CD}{}^A{}_B + \check{\gamma}_{CD}{}^A{}_B \tag{5.19}$$

with

$$\gamma^*{}_{ABCD} = \frac{1}{2}(\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}), \quad \check{\gamma}_{01CD} = 0, \quad \check{\gamma}_{ABCD} = O(\rho).$$

The smoothness of the 1-forms and the vector fields $\check{H}$, X_+, X_- implies that the vector fields $\rho\, c_{CD}$ and the functions

$$c^1{}_{CD}, \quad \rho\, c^+{}_{CD}, \quad \rho\, c^-{}_{CD}, \quad \rho\, \gamma_{CDAB},$$

extend smoothly to all of $\mathcal{C}_e$.

A smooth function F on an open subset of $\mathcal{C}_e$ is said to have spin weight s if

$$X(F) = 2s\, F \tag{5.20}$$

on this set with $2s$ an integer. Any spinor-valued function induced by a spinor field on $\mathcal{B}_e$ has a well-defined spin weight, it holds, e.g.,

$$X\, \phi_{ABCD} = 2\,(2 - A - B - C - D)\, \phi_{ABCD}. \tag{5.21}$$

It follows from the construction of $\mathcal{C}_e$ that this is also true for the functions considered above, it turns out that

$$X\, c^1{}_{AB} = 2\,(1 - A - B)\, c^1{}_{AB}, \quad X\, c^\pm{}_{AB} = 2\,(1 - (\pm 1) - A - B)\, c^\pm{}_{AB},$$
$$X\, \gamma_{ABCD} = 2\,(2 - A - B - C - D)\, \gamma_{ABCD} \;\text{ for }\; A, B, C, D = 0, 1.$$

By our construction, equation (5.3), and the formula for $e_a(\delta)$ given above the vectors $T_\delta(\pi')\,(\sqrt{2}\,\check{H}(\delta)) = e_3(\delta)$ are tangent to and the frame $e_a(\delta(\rho, t))$ is parallely propagated along the geodesics $]-e, e[\ni \rho \to \pi'(\delta(\rho, t))$ through i. Thus we have constructed the type of frame field asked for in the beginning. Working on $\mathcal{C}_e$ has the advantage that ρ and $\sqrt{2}\,\check{H}$ define smooth fields and the smoothness of the various fields considered above can easily be discussed.

The transition from $\mathcal{B}_e$ to $\bar{\mathcal{C}}'_e$ respectively to $\bar{\mathcal{C}}^+_e$ amounts to a new choice of differential structure at space-like infinity. This change is reflected in the drop of rank of the map π' at the set $\mathcal{I}^0$. It follows from (5.11), (5.12), that at points over i the vectors X, $X_{\pm}$ project onto the zero vector while at points in $\pi'^{-1}(p)$ the real and imaginary parts of $\check{H}$, X_+, X_- have non-vanishing projections which span the tangent space $T_p\tilde{\mathcal{B}}_e$ if $p \in \tilde{\mathcal{B}}_e$. The relations (5.11), (5.12), (5.13), (5.14) show that the behavior of the map π' near $\mathcal{I}^0$ is encoded in the behavior of the solder and the connection form.

With the structures given above we can perform tensor calculations defined on $\tilde{\mathcal{B}}_e$ now also on $\mathcal{C}_e \setminus \mathcal{I}^0$ and they follow the 'usual' rules of the spin frame formalism. If F denotes the lift of a smooth function f on $\mathcal{B}_e$, the covariant differential Df is represented on $\mathcal{C}_e \setminus \mathcal{I}^0$ by the invariant function $D_{AB}f \equiv c_{AB}(F)$. In the following we shall use the same symbol for a function and its lift. If μ_{AB} is the invariant function induced by a spatial spinor field μ on $\tilde{\mathcal{B}}_e$ its covariant differential is given on $\mathcal{C}_e \setminus \mathcal{I}^0$ by the expression

$$D_{AB}\mu_{AD} = c_{AB}(\mu_{AD}) - \gamma_{AB}{}^{E}{}_{C}\,\mu_{ED} - \gamma_{AB}{}^{E}{}_{D}\,\mu_{CE}.$$

Analogous formulas hold for covariant differentials of spinor fields of higher valence.

In terms of ρ and $t = (t^A{}_B)$ on $\bar{\mathcal{C}}_e$ and the normal coordinates x^a satisfying (4.17) on $\mathcal{B}_e$, the projection π' has the local expression

$$\pi' : (\rho, t) \to x^a(\rho, t) = \rho\sqrt{2}\,\sigma^a{}_{CD}\,t^C{}_0\,t^D{}_1. \tag{5.22}$$

This can be used to pull back the functions Ω, U, and W, which are related by (4.29), to functions of spin weight zero on $\bar{\mathcal{C}}_e$. The metric in (4.13) is built into our formalism and the second fundamental form lifts to a symmetric spinor-valued function χ_{ABCD} which vanishes everywhere. Using the fields

$$\check{c}^{\pm}{}_{AB}, \qquad \check{\gamma}_{CDAB}, \qquad s_{ABCD}, \qquad r, \tag{5.23}$$

given by (5.16), (5.19), and (5.7), one can determine

$$D_{AB}\,D_{CD}\,\Omega, \tag{5.24}$$

on $\mathcal{C}_e^+$ and thus also the derived data (4.14), (4.15), (4.16).

In particular, a detailed expression for the rescaled conformal Weyl spinor ϕ_{ABCD} is obtained on $\mathcal{C}_e^+$ by using (4.16) and (4.29). It takes the form

$$\phi_{ABCD} = \phi'_{ABCD} + \phi^W_{ABCD}, \tag{5.25}$$

where

$$\begin{aligned}\phi'_{ABCD} &= \sigma^{-2}\left\{D_{(AB}\,D_{CD)}\,\sigma + \sigma\,s_{ABCD}\right\} \\ &= \frac{1}{\rho^4}\left\{U^2\,D_{(AB}\,D_{CD)}\,(\rho^2) - 8\,\rho\,U\,D_{(AB}\rho\,D_{CD)}U\right\} \\ &\quad - \frac{1}{\rho^2}\left\{2\,U\,D_{(AB}\,D_{CD)}\,U - 6\,D_{(AB}U\,D_{CD)}U - U^2\,s_{ABCD}\right\}\end{aligned} \tag{5.26}$$

is derived from $\sigma = \rho^2\,U^{-2}$ and thus from the local geometry near i, while

$$\phi^W_{ABCD} = \frac{1}{\rho^3}\left\{-6\,U\,W\,D_{(AB}\,\rho\,D_{CD)}\,\rho + U\,W\,D_{(AB}\,D_{CD)}\,(\rho^2)\right\} \tag{5.27}$$

$$+\frac{4}{\rho^2}\,(W\,D_{(AB}\rho\,D_{CD)}U - 3\,U\,D_{(AB}\rho\,D_{CD)}W)$$

$$-\frac{2}{\rho}\,(U\,D_{(AB}\,D_{CD)}\,W + W\,D_{(AB}\,D_{CD)}\,U - 6\,D_{(AB}U\,D_{CD)}\,W - U\,W\,s_{ABCD})$$

$$-2\,W\,D_{(AB}\,D_{CD)}\,W + 6\,D_{(AB}W\,D_{CD)}\,W + W^2\,s_{ABCD},$$

is the part of the rescaled conformal Weyl spinor which depends on the non-local information in W and which vanishes in the massless case. Observing that

$$D_{AB}\,\rho = x_{AB}, \quad D_{AB}\,D_{CD}\,(\rho^2) = -4\,\rho\,\check{\gamma}_{(AB}{}^{E}{}_{C}\,x_{D)E} = O(\rho^2), \quad D_{AB}\,U = O(\rho),$$

one finds that

$$\phi'_{ABCD} = O(\frac{1}{\rho^2}), \quad \phi^W_{ABCD} = -\frac{6\,m}{\rho^3}\,\epsilon^2_{ABCD} + O(\frac{1}{\rho^2}), \tag{5.28}$$

where we set $\epsilon^j_{ABCD} \equiv \epsilon_{(A}{}^{(E}\,\epsilon_B{}^F\,\epsilon_C{}^G\,\epsilon_{D)}{}^{H)_j}$ for $j = 0, \ldots, 4$.

5.1.2. Normal expansions at $\mathcal{I}^0$ and the functions $T_m{}^j{}_k$. To analyze in detail the behavior of the various fields near space-like infinity it is convenient to study a particular type of expansion. It will be discussed here for an unprimed spinor field, similar expansions hold for other fields. In terms of the normal coordinates x^a on $\mathcal{B}_e$ define the radial vector field $V = x^a\,\partial_{x^a}$. Let $\delta^* = \delta^*(x^a)$ be the smooth spin frame field on $\mathcal{B}_e$ which satisfies $D_V\,\delta^* = 0$ on $\mathcal{B}_e$ and coincides with the spin frame at i chosen as the starting point for our construction of $\mathcal{C}_e$. Denote by e^*_{AB} the orthonormal frame associated with δ^* and write $V = V^{AB}\,e^*_{AB}$.

Suppose ξ is a smooth spinor field on $\mathcal{B}_e$ which is given in terms of the spin frame field δ^* by $\xi^*_{A_1\ldots A_l} = \xi^*_{A_1\ldots A_l}(x^a)$. Then its Taylor expansion at i is of the form

$$\xi^*_{A_1\ldots A_l}(x^a) = \sum_{p=0}^{p=\infty}\frac{1}{p!}\,V^{B_pC_p}(x^a)\ldots V^{B_1C_1}(x^a)\,D_{B_pC_p}\ldots D_{B_1C_1}\xi^*_{A_1\ldots A_l}(i). \tag{5.29}$$

To determine the lift $\xi_{A_1\ldots A_l}$ of this field to $\mathcal{C}^+_e$ one has to observe its transformation behaviour $\xi^*_{A_1\ldots A_l} \to \xi^*_{A_1\ldots A_l}\,t^{B_1}{}_{A_1}\ldots t^{B_l}{}_{A_l}$ under changes of the frame and the fact that the pull-back of the functions V^{AB} are given in view of (5.22) by

$$V^{AB}(x^a(\rho,t)) = \sqrt{2}\,\rho\,t^{(A}{}_0\,t^{B)}{}_1. \tag{5.30}$$

If the expansion coefficients $D_{B_pC_p}\ldots D_{B_1C_1}\xi^*_{A_1\ldots A_l}(i)$ are then decomposed into products of ϵ_{ab}'s and symmetric spinors at i, the essential components $\xi_j = \xi_{(a_1\ldots a_l)_j}$, $0 \le j \le l$, $0 \le j \le l$, which are of spin weight $s = \frac{l}{2} - j$, are obtained as expansion of the form

$$\xi_j = \sum_{p=0}^{\infty}\xi_{j,p}\,\rho^p \tag{5.31}$$

where

$$\xi_{j,p} = \sum_{m=\max\{|l-2j|,l-2p\}}^{2p+l} \sum_{k=0}^{m} \xi_{j,p;m,k} \, T_m{}^k{}_{\frac{m-l}{2}+j} \tag{5.32}$$

with complex coefficients $\xi_{j,p;m,k}$ and functions $T_m{}^j{}_k$ of t as discussed below.

We refer to this type of expansion as to the *normal expansion of* ξ *at* I^0. In the case considered above the lift of ξ to $\mathcal{C}_e^+$ has smooth limits at $\mathcal{I}^0$. Corresponding expansions in terms of ρ^k, $k \in \mathbb{Z}$, can also be obtained for fields such as ϕ_{ABCD} on $\mathcal{C}_e^+$ which are given by algebraic expressions of regular fields but which become singular at $\mathcal{I}^0$.

The functions $T_m{}^j{}_k$, arise (apart from some normalizing factors) naturally by the procedure indicated above. They are matrix elements of unitary representations

$$SU(2) \ni t \to T_m(t) = (T_m{}^j{}_k(t)) \in SU(m+1),$$

which are given by

$$T_0{}^0{}_0(t) = 1, \quad T_m{}^j{}_k(t) = \binom{m}{j}^{\frac{1}{2}} \binom{m}{k}^{\frac{1}{2}} t^{(b_1}{}_{(a_1} \ldots t^{b_m)_j}{}_{a_m)_k},$$

$$j, k = 0, \ldots, m, \quad m = 1, 2, 3, \ldots.$$

The brackets with lower index now indicate symmetrization and taking 'essential components'. The expansions obtained above make sense under quite general assumptions; the functions $\sqrt{m+1}\, T_m{}^j{}_k(t)$ form a complete orthonormal set in the Hilbert space $L^2(\mu, SU(2))$ where μ denotes the normalized Haar measure on $SU(2)$.

Using the identification of $\mathcal{I}^0$ with $SU(2)$ built into our construction, we consider the $T_m{}^j{}_k$ as functions on $\mathcal{I}^0$ and extend them as ρ-independent functions to $\bar{\mathcal{C}}_e$. The vector fields $X_\pm$, X then act as left invariant vector fields and it holds

$$X \, T_m{}^k{}_j = (m-2j) \, T_m{}^k{}_j, \tag{5.33}$$

$$X_+ \, T_m{}^k{}_j = \beta_{m,j} \, T_m{}^k{}_{j-1}, \quad X_- \, T_m{}^k{}_j = -\beta_{m,j+1} \, T_m{}^k{}_{j+1} \tag{5.34}$$

for $0 \leq k, j \leq m$, $m = 0, 1, 2, \ldots$, with $\beta_{m,j} = \sqrt{j\,(m-j+1)}$. It follows that functions f with spin weight s have expansions of the form

$$f = \sum_{m \geq |2s|} \sum_{k=0}^{m} f_{m,k} \, T_m{}^k{}_{\frac{m}{2}-s}, \tag{5.35}$$

where the m's are even if s is an integer and odd if s is a half-integer. All functions considered in the following have integer spin weight.

5.2. The regularizing gauge for the evolution equations

To obtain definite expressions for the expansions of the data at i and because the terms of lower order are then simplified, it has been assumed in [37] that the metric h is given in a cn-gauge near i. This will be assumed also here, though the discussion of the static case given below will show that this is not necessary for our construction. The coordinates ρ, t and the frame field constructed above depend on the choice of scaling of the metric h on $\mathcal{S}$. Most important is the fact that $\Omega = O(\rho^2)$ near $\mathcal{I}^0$, it affects the definition of ρ in an essential way.

In analyzing the evolution of our data in time it turns out convenient to use a different conformal factor Θ which is related to the conformal factor Ω by

$$\Theta = \kappa^{-1}\,\Omega \quad \text{on} \quad \bar{\mathcal{C}}_e, \tag{5.36}$$

with a function

$$\kappa = \rho\,\kappa' \quad \text{with} \quad \kappa' \in C^{\infty}(\bar{\mathcal{C}}_e), \quad \kappa' > 0, \quad X\,\kappa' = 0, \quad \kappa'|_{\mathcal{I}^0} = 1. \tag{5.37}$$

The value of κ' on $\mathcal{I}^0$ is chosen for convenience here, nothing is gained in the following by requiring a different (positive) boundary value for it.

The change of the conformal factor implies a map $\Xi : \delta \rightarrow \kappa^{\frac{1}{2}}\,\delta$ which maps the set $\mathcal{C}_e^+$ bijectively onto a smooth submanifold $\mathcal{C}^*$ of the bundle of conformal spin frames over $\tilde{\mathcal{B}}_e$. We use the diffeomorphism Ξ to carry the coordinates ρ and t and the vector fields $\partial_\rho, X, X_+, X_-$ to $\mathcal{C}^*$. The projection of $\mathcal{C}^*$ onto $\tilde{\mathcal{B}}_e$ will be denoted again by π'.

Assuming a conformal Gauss system for the evolution in time as described in Section 2.1, the evolution of the spin frames constituting $\mathcal{C}^*$ defines in the the bundle of conformal frames over the space-time manifold $\tilde{\mathcal{M}}$ a smoothly embedded 5-dimensional manifold $\tilde{\mathcal{N}}$ which is again a $U(1)$ bundle over the space-time and whose projection onto $\tilde{\mathcal{M}}$ we denote again by π'. The manifold C^* represents a smooth hypersurface of $\tilde{\mathcal{N}}$.

By pushing forward the coordinates ρ, t and the vector fields ∂_ρ, X, $X_\pm$ with the flow of the conformal geodesics ruling $\tilde{\mathcal{N}}$, these structures can be extended to $\tilde{\mathcal{N}}$ such that $i\,X$ generates the kernel of π'. The parameter $x^0 \equiv \tau$ of the conformal geodesics defines a further independent coordinate with $x^0 = \tau = 0$ on $\mathcal{C}^*$, so that the tangent vector field of this congruence can be denoted by ∂_τ.

The reduced field equations (2.38), (2.39), (2.40), (2.42) (the latter specialization of (2.43) is chosen here for only definiteness) are now interpreted as equations on $\tilde{\mathcal{N}}$ by assuming that the $e_{AA'}$ are vector fields on $\tilde{\mathcal{N}}$ which are defined at a spin frame $\delta \in \tilde{\mathcal{N}}$ by the requirement that they project onto the frame defined by δ on $\tilde{\mathcal{M}}$, i.e., $T_\delta\,\pi'(e_{AA'}) = \delta_A\,\bar{\delta}_{A'}$, and whose X-component is fixed by requiring an expansion of the form

$$e_{AA'} = \frac{1}{\sqrt{2}}\,\tau_{AA'}\,\partial_\tau - \tau^B{}_{A'}\,e_{AB}, \tag{5.38}$$

with 'spatial vectors'

$$e_{AB} = e^0{}_{AB}\,\partial_\tau + e^1{}_{AB}\,\partial_\rho + e^+{}_{AB}\,X_+ + e^-{}_{AB}\,X_-. \tag{5.39}$$

The unknowns in the reduced field equations are then interpreted as spinor-valued functions on $\tilde{\mathcal{N}}$. It can be shown that spin weights are preserved under the evolution by the reduced system.

We have to express the initial data for the conformal field equations in terms of the new scaling. With κ, the fields (5.23), (5.24), and the associated covariant derivatives (carried over to $\mathcal{C}^*$, observing that the local expression of Ξ in the given coordinates is the identity) one gets for the curvature fields

$$\phi_{ABCD} = \frac{\kappa^3}{\Omega^2}\left(D_{(AB}\,D_{CD)}\Omega + \Omega\, s_{ABCD}\right), \tag{5.40}$$

$$\Theta_{AA'CC'} = -\kappa^2\left(\frac{1}{\Omega}\,D_{(AB}\,D_{CD)}\Omega + \frac{1}{12}\,r\,h_{ABCD}\right)\tau^B{}_{A'}\,\tau^B{}_{C'}. \tag{5.41}$$

For the frame (5.38), one gets by (5.16)

$$e^0{}_{AB} = 0, \quad e^1{}_{AB} = \rho\,\kappa'\,x_{AB}, \tag{5.42}$$

$$e^+{}_{AB} = \kappa'\,z_{AB} + \kappa\,\check{c}^+{}_{AB}, \quad e^-{}_{AB} = \kappa'\,y_{AB} + \kappa\,\check{c}^-{}_{AB}. \tag{5.43}$$

For the conformal factor Θ we get

$$\Theta = \Theta_* \equiv \kappa^{-1}\,\Omega = \frac{\rho}{\kappa'\,(U+\rho\,W)^2} \quad \text{on} \quad \mathcal{C}^*. \tag{5.44}$$

We assume that initial data for the 1-form f, which will be related in the end to $\tilde{f}$ by the relation $f = \tilde{f} - \Theta^{-1}\,d\,\Theta$, satisfy

$$\langle f, \partial_\tau\rangle = 0, \qquad \text{pull-back of } f \text{ to } \mathcal{C}^* = \kappa^{-1}\,d\,\kappa. \tag{5.45}$$

It follows then that from (2.24) that Θ takes the form

$$\Theta = \Theta_*\left(1 - \tau^2\,\frac{\kappa_*^2}{\omega_*^2}\right) \quad \text{on } \tilde{\mathcal{N}}, \tag{5.46}$$

with a function ω which is given by

$$\omega = \frac{2\Omega}{\sqrt{|D_a\Omega D^a\Omega|}} = \rho(U+\rho W)\{U^2 + 2\rho U x^{AB}D_{AB}U - \rho^2 D^{AB}U\,D_{AB}U \\ +2\rho^2 U x^{AB}D_{AB}W - 2\rho^3 D^{AB}U\,D_{AB}W - \rho^4 D^{AB}W\,D_{AB}W\}^{-\frac{1}{2}} \text{ on } \mathcal{C}^*. \tag{5.47}$$

Here the second member is given in the notation of Section 4.1 while the term on the right-hand side is given in the notation of Section 5.1.1. In (5.46) and in the following formulas the subscripts $*$ are saying that the corresponding functions are constant along the conformal geodesics.

For $d_{AA'}$ we get by (2.25) the explicit expression

$$d_{AA'} = \frac{1}{\sqrt{2}}\,\tau_{AA'}\,\dot{\Theta} - \tau^B{}_{A'}\,d_{AB} \quad \text{on } \tilde{\mathcal{N}}, \tag{5.48}$$

where the dot denotes the derivative with respect to τ and

$$d_{AB} = 2\rho \left(\frac{U x_{AB} - \rho D_{AB} U - \rho^2 D_{AB} W}{(U + \rho W)^3} \right)_*, \tag{5.49}$$

where the notation of Section 5.1.1 is used on the right-hand side.

If one uses (5.42) and (5.43) to write for a given smooth function μ on $\mathcal{C}^*$

$$\mu_{AB} \equiv \kappa^{-1} (e^1{}_{AB}\, \partial_\rho + e^+{}_{AB}\, X_+ + e^-{}_{AB}\, X_-)\, \mu,$$

one gets with the 1-form (5.45) and the spatial connection coefficients (5.19) the space-time connection coefficients in the form

$$\Gamma_{AA'CD} = \left(\frac{1}{2}\, \rho\, (\epsilon_{AC}\, \kappa'_{BD} + \epsilon_{BD}\, \kappa'_{AC}) - \rho\, \kappa'\, \check{\gamma}_{ABCD} + \frac{1}{2}\, \epsilon_{AB}\, \kappa_{CD} \right) \tau^B{}_{A'}. \tag{5.50}$$

Note that the $\hat{\Gamma}_{AA'BC}$ in the reduced equations can be expressed by (2.37) in terms of the $\Gamma_{AA'BC}$.

Most important for us is the observation that *the functions given by* (5.40), (5.41), (5.42), (5.43), (5.46), (5.48), (5.50) *have smooth limits as $\rho \to 0$ and can in fact be smoothly extended into the coordinate range $\rho \leq 0$*. For the unknowns in the new scaling we thus obtain normal expansion in terms of non-negative powers of ρ. In particular, one has

$$\phi_{ABCD} = \kappa^3 (\phi'_{ABCD} + \phi^W_{ABCD}), \tag{5.51}$$

with (5.26), (5.27) on the right-hand side. Pushing the expansion (5.28) a bit further and using (5.36) one gets in the cn-gauge (in which $h_{ab} = -\delta_{ab} + O(\rho^3)$)

$$\begin{aligned} \phi_{ABCD} = {} & -\kappa'^3\, 6\, m\, \epsilon^2_{ABCD} \\ & -\rho\, \kappa'^3\, 12\, \left(X_+ W_1\, \epsilon^1{}_{ABCD} + 3\, W_1\, \epsilon^2{}_{ABCD} - X_- W_1\, \epsilon^3{}_{ABCD} \right) \\ & -\frac{\rho^2 \kappa'^3}{2} \sum_{k,j=0}^{4} \binom{4}{j} \left(4 \sqrt{6 \binom{4}{j}}\, W_{2;4,k} - \frac{2-j}{3} \sqrt{2 \binom{4}{k}}\, b^*_k(i) \right) T_4{}^k{}_j\, \epsilon^j_{ABCD}, \\ & + O(\rho^3). \end{aligned} \tag{5.52}$$

It is assumed here that W is an arbitrary solution to $(\Delta_h - \frac{1}{8}\, r)\, W = 0$ on $\mathcal{B}_e$. Its normal expansion takes in the cn-gauge the form

$$W = \sum_{p=0}^{2} \rho^p\, W_p + O(\rho^3) = \sum_{p=0}^{2} \rho^p \, (\sum_{k=0}^{2p} W_{p;2p,k}\, T_{2p}{}^k{}_p) + O(\rho^3)$$

with

$$W_{0;0,0} = W(i) = \frac{m}{2}, \qquad W_{1;2,k} = \binom{2}{k}^{\frac{1}{2}} D_{(ab)_k} W^*(i),$$

$$W_{2;4,k} = \binom{4}{2}^{-\frac{1}{2}} \binom{4}{k}^{\frac{1}{2}} D_{(ab} D_{cd)_k} W^*(i).$$

In the case where κ' is constant the right-hand side of (5.52) provides the terms of a normal expansion up to the quadrupole term. If κ depends on ρ and t the terms given above need to be expanded further to obtain the normal expansion.

The transition (5.36) to the conformal factor Θ corresponds to a transition $h \to h' = \kappa^{-2}\,h$ of the metric on $\mathcal{B}_e$ (assuming that κ' arise as a lift of a smooth positive function on $\mathcal{B}_e$ with $\kappa'(i) = 1$) in the sense that then $\Omega^{-2}\,h = \tilde{h} = \Theta^{-2}\,h'$. The coordinate ρ is then not adapted to the geometry defined by the metric h'. To illustrate the situation assume that h is flat. Then

$$h' = -\kappa'^{-2}\,\rho^{-2}\,(d\,\rho^2 + \rho^2\,d\,\sigma^2) = -\kappa'^{-2}\,(d\,r^2 + d\,\sigma^2), \qquad (5.53)$$

with $r = -\log\rho$ near i. With respect to the new coordinate r, which is adapted to the geometry of h', the point i is shifted to infinity but the surface measure of any sphere around i remains finite and positively bounded from below. This behavior is reflected by the fact that the frame coefficient $e^1{}_{AB}$ in (5.42) vanishes while the frame coefficients $e^{\pm}{}_{AB}$ in (5.43) have finite and non-vanishing limits at $\mathcal{I}^0$. We shall keep the coordinate ρ because it ensures the finite coordinate representation of the boundary $\mathcal{I}^0$ as well as the smoothness of the data near $\mathcal{I}^0$.

With the gauge defined above the functions Θ and $d_{AA'}$ in equations (2.39), (2.40) are given by (5.46) and (5.48) and the finite regular initial value problem near space-like infinity for the reduced field equations (2.38), (2.39), (2.40), (2.42) is completely determined. We write this system schematically as system of equations for the unknown $u = (w, \phi)$ with $\phi = (\phi_{ABCD})$ and $w = (e_{AA'}, \hat{\Gamma}_{AA'BC}, \Theta_{AA'BB'})$ or, alternatively, $w = (e_{AA'}, \Gamma_{AA'BC}, \Theta_{AA'BB'})$. It takes the form

$$\partial_\tau\,w = F(x, w, \phi), \quad A^\mu\,\partial_{x^\mu}\,\phi = H(w)\,\phi, \qquad (5.54)$$

where the x-dependence in the first equation comes in here via the functions Θ and $d_{AA'}$.

Important for the following is that *with any choice of κ satisfying (5.37) the functions Θ and $d_{AA'}$ take smooth limits as $\rho \to 0$ and can be extended smoothly into a range where $\rho \le 0$.* With the smooth extensibility of the initial data observed before, we find that *the initial value problem for the reduced field equations with the data prescribed above can be extended smoothly into a range where $\rho \le 0$ so that the reduced equations form still a symmetric hyperbolic system.* It may be noted finally that the congruence of conformal geodesics (considered as point sets) underlying our gauge does not depend on the choice of κ, whereas the parameter τ depends on it in an essential way.

5.3. Specific properties of the regular finite initial value problem at space-like infinity

The nature of the initial value problem formulated above is conveniently discussed by considering certain subsets of $\mathbb{R} \times \mathbb{R} \times SU(2)$ which are defined by the range admitted for the coordinates (τ, ρ, t). We define 5-dimensional subsets

$$\tilde{\mathcal{N}} \equiv \{|\tau| < \frac{\omega}{\kappa},\ 0 < \rho < e,\ t \in SU(2)\},$$

$$\bar{\mathcal{N}} \equiv \{|\tau| \leq \frac{\omega}{\kappa},\ 0 \leq \rho < e,\ t \in SU(2)\},$$

where $\frac{\omega}{\kappa}$ is a function of ρ and t. It holds then

$$\bar{\mathcal{N}} = \tilde{\mathcal{N}} \cup \mathcal{J}^- \cup \mathcal{J}^+ \cup \mathcal{I} \cup \mathcal{I}^- \cup \mathcal{I}^+,$$

with 4-dimensional submanifolds

$$\mathcal{J}^\pm \equiv \{\tau = \pm\frac{\omega}{\kappa},\ 0 < \rho < e,\ t \in SU(2)\}, \quad \mathcal{I} \equiv \{|\tau| < 1,\ \rho = 0,\ t \in SU(2)\},$$

and 3-dimensional submanifolds

$$\mathcal{I}^\pm \equiv \{|\tau| \pm 1,\ \rho = 0,\ t \in SU(2)\}, \quad \mathcal{I}^0 \equiv \{\tau = 0,\ \rho = 0,\ t \in SU(2)\},$$

where it has been observed that $\frac{\omega}{\kappa} \to 1$ as $\rho \to 0$. We note that

$$\Theta > 0 \text{ on } \tilde{\mathcal{N}}, \quad \Theta = 0,\ d\Theta \neq 0 \text{ on } \mathcal{J}^- \cup \mathcal{J}^+ \cup \mathcal{I}, \quad \Theta = 0,\ d\Theta = 0 \text{ on } \mathcal{I}^\pm.$$

The set $C^* = \{\tau = 0,\ 0 < \rho < e,\ t \in SU(2)\}$ defines a hypersurface of $\tilde{\mathcal{N}}$. Its closure in $\bar{\mathcal{N}}$ is given by

$$\bar{C} \equiv \{\tau = 0,\ 0 \leq \rho < e,\ t \in SU(2)\} = C^* \cup \mathcal{I}^0.$$

Factoring out the group $U(1)$ implies projections (denoted again by π') onto subsets $\mathbb{R} \times \mathbb{R} \times S^2$ which are of one dimension lower than the sets above. In particular, $\tilde{\mathcal{N}}$ projects onto a set $\tilde{\mathcal{M}}$ which represents the 'physical space-time'. For convenience we will usually work with the manifolds above and use for them the same words as for the projections, so that $\tilde{\mathcal{N}}$ will be referred to as the 'physical space-time' etc.

For suitable $\epsilon > 0$ consider a smooth extension of the data given on C^* to the set $\bar{C}_{ext} = \{\tau = 0,\ -\epsilon < \rho < e,\ t \in SU(2)\}$ and an extension of the functions Θ, $d_{AA'}$ to the domain $\bar{\mathcal{N}}_{ext} = \{|\tau| < \frac{\omega}{\kappa},\ -\epsilon < \rho < e,\ t \in SU(2)\}$, so that the reduced conformal field equations (2.38), (2.39), (2.40), (2.42) still represent a symmetric hyperbolic system of the form (5.54). Then there exists a neighborhood $\mathcal{V}$ of $\bar{C}_{ext}$ in $\bar{\mathcal{N}}_{ext}$ on which there exists a unique smooth solution $e_{AA'}$, $\hat{\Gamma}_{AA'BC}$ (resp. $\Gamma_{AA'BC}$), $\Theta_{AA'BB'}$, ϕ_{ABCD} to our extended initial value problem which satisfies the gauge conditions (2.36).

It turns out, that *the restriction of this solution to the set* $\mathcal{V} \cap \bar{\mathcal{N}}$ *is uniquely determined by the data on* C^*. The data on C^* have a unique smooth extension to $\bar{C}$ and it follows from (5.38), (5.39), (5.42), and (5.43) that $e^1{}_{CC'} \to 0$ as $\rho \to 0$. Equations (2.38) imply in particular

$$\sqrt{2}\,\partial_\tau\, e^1{}_{CC'} = -\Gamma_{CC'}{}^{AA'}{}_{BB'}\, \tau^{BB'}\, e^1{}_{AA'}. \tag{5.55}$$

It follows that $e^1{}_{CC'} = 0$ on $\mathcal{V} \cap \mathcal{I}$ and as a consequence that the matrices A^μ in (5.54) are such that

$$A^1 = 0 \text{ on } \mathcal{I}, \tag{5.56}$$

if the solution extends far enough. One can apply to the system (5.54) on subsets of $\mathcal{V} \cap \tilde{\mathcal{N}}$ the standard method of deriving energy estimates. Without further information on the system the partial integration would yield contributions from boundary integrals over parts of $\mathcal{V} \cap \mathcal{I}$. Because of (5.56) these boundary integrals vanish and one obtains energy estimates which allow one to show the asserted uniqueness property. The extension above has been considered to simplify the argument. Alternatively, the space-time $\tilde{\mathcal{N}}$ can be thought of as a solution of a very specific 'maximally dissipative' initial boundary value problem where initial data are prescribed on $\bar{\mathcal{C}}$ and no data are prescribed on $\mathcal{I}$ because of (5.56) (cf. [42] and the existence theory in [48], [67]).

In the present gauge the set $\mathcal{I}$, which is generated from $\mathcal{I}^0$ by the extension and evolution process, can be considered as being obtained by performing limits of conformal geodesics. It represents a boundary of the space-time $\tilde{\mathcal{N}}$ which may be understood as a blow-up of the point i^0. We refer to it as the *cylinder at space-like infinity.*

Suppose that there exists on $\tilde{\mathcal{N}}$ a smooth solution $e_{AA'}$, $\hat{\Gamma}_{AA'BC}$ (resp. $\Gamma_{AA'BC}$), $\Theta_{AA'BB'}$, ϕ_{ABCD} of the reduced conformal field equations (2.38), (2.39), (2.40), (2.42) which satisfies the gauge conditions (2.36) on $\tilde{\mathcal{N}}$ and coincides on the initial hypersurface $C^* = \{\tau = 0\} \subset \tilde{\mathcal{N}}$ with the data given above. The projections $T\pi'(e_{AA'})$ then define a frame field on $\tilde{\mathcal{M}}$ for which exists a unique smooth metric g on $\tilde{\mathcal{M}}$ such that $g(T\pi'(e_{AA'}), T\pi'(e_{AA'})) = \epsilon_{AB}\,\epsilon_{A'B'}$. Denote by D' the domain of dependence in $\tilde{\mathcal{M}}$ with respect to g of the set $\pi'(C^*)$ and set $D = \pi'^{-1}(D')$. By the discussion above we can assume that the closure of D in $\tilde{\mathcal{N}}$ contains the set $\mathcal{I}$ and the solution extends smoothly to $\mathcal{I}$. It follows from the structure of the characteristics of the reduced equations, that the solution is determined on D uniquely by the data on C^* and it follows from the discussion in [36] and the fact that the data satisfy the constraints that the complete set (2.29), (2.30), (2.31), (2.32) of conformal field equations is satisfied on D. Since Θ has spin weight zero it descends to a function on $\tilde{\mathcal{M}}$ and $\tilde{g} = \Theta^{-2}\,g$ satisfies the vacuum field equations.

The restriction to D arises here because we only considered the data on C^*. Observing that the latter were obtained by restricting the data given on the initial hypersurface $\mathcal{S}$ to $\mathcal{B}_e$ it is reasonable to assume that the conformal field equations hold everywhere on $\tilde{\mathcal{N}} \cup \mathcal{I}$ and $\tilde{g}$ defines a solution to the vacuum field equations on $\tilde{\mathcal{M}}$.

Assume u is a solution of a (possibly non-linear) hyperbolic system of partial differential equations of first order on some manifold. A hypersurface of this manifold is then called a *characteristic* of that system (with respect to u), if the system implies for some components of u non-trivial interior differential equations on the hypersurface. These interior equations are called *transport equations* (cf. [22]).

Because of (5.56) the set $\mathcal{I}$ is then a characteristic of the extended field equations. It is in fact of a very special type (i.e., a *total characteristic*), because the system (5.54) reduces on $\mathcal{I}$ to an interior symmetric hyperbolic system of transport equations for the *complete* system of unknowns. Together with the data on $\mathcal{I}^0$ it allows us to determine $u = (v, \phi)$ on $\mathcal{I}$.

Suppose that the solution extends in a C^1 fashion to the sets $\mathcal{J}^{\pm}$. Since $\Theta = 0$, $d\Theta \neq 0$ on $\mathcal{J}^{\pm}$ the sets $\pi'(\mathcal{J}^{\pm})$ form (part of) the conformal boundary at null infinity for the vacuum solution $\tilde{g}$. Since ϕ_{ABCD} is C^1 one finds Sachs peeling. Of course, *it will be one of our main tasks to control under which assumptions the solutions will extend with a certain smoothness to the sets* $\mathcal{J}^{\pm}$.

As remarked before, we can expect the decision about the smoothness of the solution at null infinity to be made in the area where the latter 'touches' space-like infinity. This location has a precise meaning in the present setting. It is given by the *critical sets* $\mathcal{I}^{\pm}$, which can be considered either as boundaries of $\mathcal{J}^{\pm}$ or as the boundary components of $\mathcal{I}$. The nature of these sets is elucidated by studying *conformal Minkowski space* in the present setting.

We start with the line element given by (5.53) and choose $\kappa' = 1$. Since $\omega = \rho$ by (7.1) it follows that $\mathcal{J}^{\pm} = \{|\tau| = \pm 1,\ 0 < \rho < e,\ t \in SU(2)\}$ and $\mathcal{M} = \{|\tau| \leq 1,\ 0 \leq \rho < e\} \times S^2$. It will be useful to express the frames considered in the following in terms of the specific frame

$$v_0 = \partial_{\tau}, \quad v_1 = \rho\,\partial_{\rho}, \quad v_{\pm} = X_{\pm}. \tag{5.57}$$

The complete solution to the conformal field equations then is given by

$$e^{\star}_{AA'} = \frac{1}{\sqrt{2}}\Big\{\Big((1-\tau)\,\epsilon_A{}^0\,\epsilon_{A'}{}^{0'} + (1+\tau)\,\epsilon_A{}^1\,\epsilon_{A'}{}^{1'}\Big)\,v_0 \tag{5.58}$$

$$+(\epsilon_A{}^0\,\epsilon_{A'}{}^{0'} - \epsilon_A{}^1\,\epsilon_{A'}{}^{1'})\,v_1 - \epsilon_A{}^0\,\epsilon_{A'}{}^{1'}\,v_+ - \epsilon_A{}^1\,\epsilon_{A'}{}^{0'}\,v_-\Big\}$$

$$\Gamma^{\star}_{AA'BC} = -\frac{1}{2}\,\tau_{AA'}\,x_{BC}, \tag{5.59}$$

$$\Theta^{\star}_{AA'BB'} = 0, \tag{5.60}$$

$$\phi^{\star}_{ABCD} = 0. \tag{5.61}$$

The conformal factor and the metric g implied by $e^{\star}_{AA'}$ are given by

$$\Theta^{\star} = \rho\,(1-\tau^2), \qquad g^{\star} = d\tau^2 + 2\,\frac{\tau}{\rho}\,d\tau\,d\rho - \frac{1-\tau^2}{\rho^2}\,d\rho^2 - d\,\sigma^2. \tag{5.62}$$

With the coordinate transformation

$$r = \frac{1}{\rho\,(1-\tau^2)}, \qquad t = \frac{\tau}{\rho\,(1-\tau^2)}, \tag{5.63}$$

one gets in fact the standard Minkowski metric $\tilde{g} = \Theta^{-2}\,g^{\star} = dt^2 - dr^2 - r^2\,d\sigma^2$ in spherical coordinates. The flat metric corresponding to (3.4) is given by $\Omega^{\star 2}\tilde{g} = \rho^2\,g^{\star} = d(\tau\,\rho)^2 - d\rho^2 - \rho^2\,d\sigma^2$ with $\Omega^{\star} = \rho\,\Theta^{\star} = \rho^2 - (\tau\,\rho)^2$. For this metric the curves with constant coordinates ρ, θ, and ϕ are obviously conformal geodesics and because of their conformal invariance it follows that the corresponding curves for $g^{\star}$ are conformal geodesics with parameter τ. Equations (5.63) can be read as their parametrized version in Minkowski space.

The metric $g^\star$ given by (5.62) extends smoothly across null infinity but it has no reasonable limit at $\mathcal{I}$. Its contravariant version

$$g^{\star\sharp} = (1 - \tau^2)\,\partial_\tau^2 + 2\,\tau\,\partial_\tau\,(\rho\,\partial_\rho) - (\rho\,\partial_\rho)^2 - (d\,\sigma^2)^\sharp,$$

does extend smoothly to $\mathcal{I}$. While it drops rank in the limit, it does imply a smooth contravariant metric on $\mathcal{I}$ whose covariant version $l^\star = (1-\tau^2)^{-1}\,d\tau^2 - d\,\sigma^2$ defines a smooth conformally flat Lorentz metric on $\mathcal{I}$. The coordinate transformation $\tau = \sin\xi$ shows that this metric is not complete. The Killing fields of Minkowski space, which are conformal Killing fields for $g^\star$, extend smoothly to $\mathcal{I}$ such that they become tangent to $\mathcal{I}$, vanish there in the case of the translational Killing fields, and act as non-trivial conformal Killing fields for the metric $l^\star$ in the case of infinitesimal Lorentz transformations.

The fields (5.58), (5.59), (5.60) extend smoothly to all of $\bar{\mathcal{M}}$. The property (5.56) results from the fact that the fields $e^\star_{00'}$, $e^\star_{11'}$ become linear dependent on $\mathcal{I}$. Since they do not vanish there, this degeneracy does not cause any difficulties in the field equations. On $\mathcal{I}^+$ and $\mathcal{I}^-$ however, the field $e^\star_{00'}$ and $e^\star_{11'}$ respectively vanishes. This strong degeneracy has important consequences for the (extended) conformal field equations. To see this, we solve the transport equations on $\mathcal{I}$ to determine the matrices A^μ on $\mathcal{I}$ in the general case. Extending the data (5.40), (5.41), (5.42), (5.43), (5.50), one finds that they agree on $\mathcal{I}^0$, irrespective of the choice of κ' satisfying conditions of (5.37), with the implied Minkowski data. Since the extensions of the functions Θ and $d_{AA'}$ vanish on $\mathcal{I}$, the transport equations for the frame, connection, and Ricci tensor coefficients are independent of the choice of initial data. It follows that the restrictions of these coefficients to $\mathcal{I}$ agree with those of the Minkowski data given above. It follows in particular that $e^1{}_{AA'} = 0$ on $\mathcal{I}$. Applying formally the operator ∂_ρ to equation (5.55) (which is part of the reduced field equations), restricting to $\mathcal{I}$, and observing the data $\partial_\rho e^1{}_{AA'}|_{\mathcal{I}^0}$, one finds that $\partial_\rho e^1{}_{AA'} = \epsilon_A{}^0\,\epsilon_{A'}{}^{0'} - \epsilon_A{}^1\,\epsilon_{A'}{}^{1'}$ on $\mathcal{I}$. Writing

$$e_{AA'} = e^i{}_{AA'}\,v_i \quad \text{with} \quad i = 0, 1, +, -,$$

and assuming the summation rule, we find that *irrespective of the free datum h given on $\mathcal{S}$ and the choice of κ' the fields $e_{AA'}$, $\Gamma_{AA'BC}$, $\Theta_{AA'BB'}$ coincide at lowest order with the Minkowski fields above in the sense that $\Theta_{AA'BB'} = O(\rho)$ and*

$$e^i{}_{AA'} = e^{\star i}{}_{AA'} + \check{e}^i{}_{AA'}, \quad \Gamma_{AA'BC} = \Gamma^\star_{AA'BC} + \check{\Gamma}_{AA'BC}, \tag{5.64}$$

with

$$\check{e}^i{}_{AA'} = O(\rho), \quad \check{\Gamma}_{AA'BC} = O(\rho) \quad \text{as} \quad \rho \to 0. \tag{5.65}$$

Assuming $\kappa = \omega$ in the general case, which by (5.47) is consistent with (5.37) if e is chosen small enough, the similarity with the Minkowski case becomes even closer. Then $\Theta = f\,\Theta^\star$ with proportionality factor $f \equiv \frac{\Omega}{\rho\,\omega}$ which extends smoothly to $\bar{\mathcal{N}}$ such that $f \to 1$ on $\mathcal{I}$. The set $\mathcal{J}^\pm$ is given as in the Minkowski case above. The discussion below shows, however, that this particular choice of κ may not always be the most useful one.

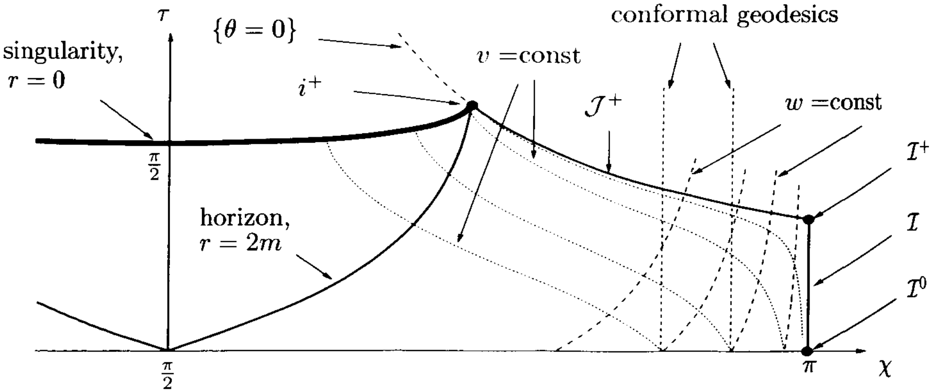

FIGURE 1. A Schwarzschild-Kruskal space-time in a conformal Gauss gauge. This is not a schematic picture but quantitatively correct. Each point in the figure corresponds to a 2-sphere. In Schwarzschild coordinates t and r the lower horizontal line, the initial hypersurface $\mathcal{S} \sim S^3$, corresponds for $\pi/2 < \chi < \pi$ to the hypersurface $\{t = 0\}$ of a Schwarzschild space-time. On $\mathcal{S}$ the coordinate χ satisfies $r = \tan(\chi/2)$, takes the value $\pi/2$ at the throat and the value π at one of the asymptotically flat ends. The parameter τ on the conformal geodesics vanishes on $\mathcal{S}$. With $\Omega = \sin^2\chi/2\,(1 + \sin\chi)$ the physical metric induced on $\tilde{\mathcal{S}}$ is $\tilde{h} = \Omega^{-2}\,d\omega^2$ with $d\omega^2$ the standard line element on S^3. The initial conditions of Section 5.2 are satisfied with $\kappa = \sin\chi$ so that $\Theta = \kappa^{-1}\,\Omega\,\{1 - \tau^2[\cos\chi(2 + \sin\chi)/2\,(1 + \sin\chi)]^2\}$. In the given gauge the maps $\chi \to \pi - \chi$ and $\tau \to -\tau$ are space-time symmetries. The rescaled space-time and the conformal Gauss gauge extend smoothly through null infinity, where $\Theta = 0$. The expression for Θ stops being meaningful when the conformal geodesics hit the singularity. The behavior of the hypersurfaces of constant retarded and advanced time w and v shows that along curves which approach the cylinder I the null cones collapse. Along curves on $\mathcal{J}^+$ which approach the critical set I^+ this behavior does not occur. This indicates a degeneracy at I^+ of the set of characteristics.

In the general case the first deviation from the Minkowski case is found in the value of the rescaled conformal Weyl spinor on I. On I^0 it is given by (5.52). Restricting the Bianchi equation to I and using the coefficients determined above one finds that

$$\phi_{ABCD} = -6\,m\,\epsilon^2_{ABCD} \quad \text{on } I. \tag{5.66}$$

The discussion above shows that the matrices A^μ are determined on I by the Minkowski data and the structure of the characteristics of the evolution equations for the rescaled conformal Weyl tensor agrees on $\mathcal{I}$ with that of the equations which are obtained by linearizing the Bianchi equation on Minkowski space. These (overdetermined) spin-2 equations take in the gauge above the form

$$(1+\tau)\,\partial_\tau\psi_k - \rho\,\partial_\rho\psi_k + X_+\,\psi_{k+1} + (2-k)\,\psi_k = 0, \tag{5.67}$$

$$(1-\tau)\,\partial_\tau\psi_{k+1} + \rho\,\partial_\rho\psi_{k+1} + X_-\,\psi_k + (1-k)\,\psi_{k+1} = 0, \tag{5.68}$$

where $k = 0,1,2,3$ and the ψ_j denote the essential components of the linearized conformal Weyl spinor.

The most conspicuous feature of these equations is the factor $(1+\tau)$ in (5.67), which vanishes on $\mathcal{J}^- \cup \mathcal{I}^-$, and the factor $(1-\tau)$ in (5.68), which vanishes on $\mathcal{J}^+ \cup \mathcal{I}^+$. On $\mathcal{J}^\pm$ these factors arise because the coordinate τ is constant on $\mathcal{J}^\pm$ and these sets are characteristics for the equations. By choosing κ' differently, this degeneracy can be removed on $\mathcal{J}^\pm$ (cf. [39]). At $\mathcal{I}^\pm$, however, this degeneracy cannot be removed in the present setting. Any symmetric hyperbolic system extracted from these equations, like, e.g.,

$$(1+\tau)\,\partial_\tau\,\psi_0 - \rho\,\partial_\rho\,\psi_0 + X_+\psi_1 = -2\,\psi_0,$$

$$(4+2\,\tau)\,\partial_\tau\,\psi_1 - 2\,\rho\,\partial_\rho\,\psi_1 + X_-\psi_0 + 3\,X_+\psi_2 = -4\,\psi_1,$$

$$6\,\partial_\tau\,\psi_2 + 3\,X_-\psi_1 + 3\,X_+\psi_3 = 0,$$

$$(4-2\,\tau)\,\partial_\tau\,\psi_3 + 2\,\rho\,\partial_\rho\,\psi_3 + 3\,X_-\psi_2 + X_+\psi_4 = 4\,\psi_3,$$

$$(1-\tau)\,\partial_\tau\,\psi_4 + \rho\,\partial_\rho\,\psi_0 + X_+\psi_1 = 2\,\psi_4.$$

must contain such factors at least in the equations for ψ_0 and ψ_4. Writing this in the form $A^\mu\,\partial_\mu\,\psi = H\,\psi$, and writing $\xi_\tau = \langle\partial_\tau,\xi\rangle$, $\xi_\rho = \langle\partial_\rho,\xi\rangle$, $\xi_\pm = \langle\xi, X_\pm\rangle$ we find

$$\det(A^\mu\,\xi_\mu) = 24\,\xi_\tau\,(g^{\mu\nu}\,\xi_\mu\,\xi_\nu)\,(3\,\xi_\tau^2 + g^{\mu\nu}\,\xi_\mu\,\xi_\nu)$$

with

$$g^{\mu\nu}\,\xi_\mu\,\xi_\nu = (1-\tau^2)\,\xi_\tau^2 + 2\,\tau\,\rho\,\xi_\tau\,\xi_\rho - \rho^2\,\xi_\rho^2 - \frac{1}{2}\,(\xi_+\,\xi_- + \xi_-\,\xi_+).$$

It follows that characteristics pertaining to the quadratic terms which start on $\mathcal{I}$, stay on $\mathcal{I}$ and that those starting in the physical space-time never end on $\mathcal{I} \cup \mathcal{I}^- \cup \mathcal{I}^+$ but always run out to $\mathcal{J}^\pm$. Most importantly however, and this also holds true for the general system (5.54), the quadratic form $g^{\mu\nu}\,\xi_\mu\,\xi_\nu$ degenerates at $\mathcal{I}^\pm$ and there is a loss of real characteristics (cf. Figure 1)[1]. This follows also directly from

$$\det(A^\tau) = 0 \ \text{ on } \mathcal{I}^\pm. \tag{5.69}$$

It appears that this *loss of hyperbolicity at the critical sets $I^\pm$*, is the key to the smoothness problem for the conformal structure at null infinity.

[1] I am grateful to Anil Zenginoglu for doing the calculations and preparing the figure for me.

5.4. The s-jet at space-like infinity

The relations (5.56) and (5.69) are the dominant features of the regular finite initial value problem at space-like infinity. The consequences of (5.69) are not deduced by the standard textbook analysis, we have to rely on the specific properties of our problem. It turns out that a considerable amount of information on the behavior of the solution near the critical sets can be obtained by exploiting (5.56). We know already that the solution is smooth in some neighborhood of $\bar{\mathcal{C}}$ in $\bar{\mathcal{N}}$ and that u can be calculated on $\mathcal{I}$ by solving intrinsic equations on $\mathcal{I}$. It will be shown in the following that a full formal expansion of u in terms of ρ can be calculated on $\mathcal{I}$ by solving certain transport equations.

The following notation will be convenient in the following. For $p = 0, 1, 2, \ldots$ and any sufficiently smooth (possibly vector-valued) function f defined on $\mathcal{N} \cup \mathcal{I}$ we write f^p for the restriction to $\mathcal{I}$ of the p-th radial derivative $\partial^p_\rho f$. The set of functions $f^0, f^1, \ldots, f^p$ on $\mathcal{I}$ will be denoted by $J^p_{\mathcal{I}}(f)$ and referred to as the *jet of order p of f on* $\mathcal{I}$ (and similarly with $\mathcal{I}$ replaced by $\mathcal{I}^0$.) If $u = (w, \phi)$ is a *solution* of equations (5.54) we refer to $J^p_{\mathcal{I}}(u)$ (respectively $J^p_{\mathcal{I}}(w)$, $J^p_{\mathcal{I}}(\phi)$) as to *the s-jet of u (resp. w, φ) of order p* and to the *data* $J^p_{\mathcal{I}^0}(u)$ (respectively $J^p_{\mathcal{I}^0}(w)$, $J^p_{\mathcal{I}^0}(\phi)$) on $\mathcal{I}^0$ as to *the d-jet of u (resp. w, φ) of order p*. A s-jet $J^p_{\mathcal{I}}(u)$ (respectively $J^p_{\mathcal{I}}(w)$, $J^p_{\mathcal{I}}(\phi)$) of order p will be called *regular on*

$$\bar{\mathcal{I}} \equiv \mathcal{I} \cup \mathcal{I}^- \cup \mathcal{I}^+,$$

(or simply *regular*) if the corresponding functions on $\mathcal{I}$ extend smoothly to the critical sets $\mathcal{I}^\pm$.

An initial data set on $\mathcal{S}$ will be called *asymptotically static of order p*, where $p \in \mathbb{N} \cup \{\infty\}$, if its d-jet $J^p_{\mathcal{I}^0}(u)$ coincides with the d-jet of order p of some static asymptotically flat data set defined on some neighborhood of i in $\mathcal{S}$. It will be seen later that *asymptotic staticity* (of order p) is an important feature of initial data sets.

Applying the operator ∂^p_ρ formally to the first of equations (5.54) and restricting to $\mathcal{I}$, one obtains for w^p an equation of the form

$$\partial_\tau w^p = G(\tau, t, w^0, \ldots w^{p-1}, w^p, \phi^0, \ldots, \phi^{p-1}), \;\; p = 1, 2, \ldots, \qquad (5.70)$$

where the right-hand side is an affine function of w^p. The functions ϕ^p do not appear here, because the rescaled conformal Weyl spinor occurs in the equations for the frame, connection, and Ricci coefficients with the factors Θ and d_{AB}, which vanish on $\mathcal{I}$. It follows that the s-jet $J^p_{\mathcal{I}}(w)$ can be determined by the integration of an (easily solvable) linear system of ODE's, if the s-jet $J^{p-1}_{\mathcal{I}}(u)$ and the d-jet $J^p_{\mathcal{I}^0}(w)$ are known.

With the notation (5.64) the Bianchi equation can be written

$$\nabla^{*F}{}_{A'}\,\phi_{BCDF} = -\phi_{A'BCD}, \qquad (5.71)$$

where

$$\phi_{A'BCD} = \phi_{A'(BCD)} \equiv \check{e}^{iF}{}_{A'}\,v_i(\phi_{BCDF}) - 4\,\check{\Gamma}^F{}_{A'}{}^E{}_{(B}\,\phi_{CDF)E}. \qquad (5.72)$$

Then

$$(\sqrt{2}\,\nabla^{\star F}{}_{A'}\,\phi_{BCDF})^p = -\sqrt{2}\,\phi^p_{A'BCD}, \tag{5.73}$$

provides equations with left-hand sides given by

$$(1+\tau)\,\partial_\tau\,\phi^p_j + X_+\,\phi^p_{j+1} + (2-j-p)\,\phi^p_j = \ldots, \tag{5.74}$$

$$(1-\tau)\,\partial_\tau\,\phi^p_{j+1} + X_-\,\phi^p_j + (1-j+p)\,\phi^p_{j+1} = \ldots, \tag{5.75}$$

where $j = 0, \ldots, 3$, and right-hand sides given by (cf. (5.65))

$$\phi^p_{A'BCD} = \sum_{i=0,+,-}\sum_{j=1}^{p}\binom{p}{j}(\check{c}^{iF}{}_{A'})^j\,v_i\,(\phi^{p-j}_{BCDF}) \tag{5.76}$$

$$+\sum_{j=1}^{p}(p-j)\binom{p}{j}(\check{c}^{1F}{}_{A'})^j\,\phi^{p-j}_{BCDF} - 4\sum_{j=1}^{p}\binom{p}{j}(\check{\Gamma}^F{}_{A'}{}^E{}_{(B})^j\,\phi^{p-j}_{CDF)E}.$$

We note that these expressions depend on $J^p_{\mathcal{I}}(w)$ but only on $J^{p-1}_{\mathcal{I}}(\phi)$. Thus, given these s-jets, the s-jet $J^p_{\mathcal{I}}(\phi)$ can be obtained by solving a linear system of ODE's, if $J^p_{\mathcal{I}^0}(\phi)$ is given. *Because the system is singular at the critical sets it is not clear a priori that $J^p_{\mathcal{I}}(\phi)$ is regular, even if $J^p_{\mathcal{I}}(w)$ and $J^{p-1}_{\mathcal{I}}(\phi)$ are regular.*

To obtain more detailed information on the solutions, it is useful to consider a system system of second order. From (5.71) follows

$$\nabla^\star_{EE'}\,\nabla^{\star EE'}\,\phi_{ABCD} = 2\,\nabla^{M\ E'}_{A}\,\nabla^{\star E}{}_{E'}\,\phi_{BCDE} = 2\,\nabla^{M\ E'}_{A}\,\phi_{E'BCD},$$

which is equivalent to

$$\nabla^\star_{EE'}\,\nabla^{\star EE'}\,\phi_{ABCD} = f_{ABCD} \equiv -2\,\nabla^\star_{E'(A}\,\phi^{E'}{}_{BCD)}, \tag{5.77}$$

$$0 = g_{BC} \equiv \nabla^{\star A'A}\,\phi_{A'ABC}. \tag{5.78}$$

While the right-hand side of

$$(\nabla^\star_{EE'}\,\nabla^{\star EE'}\,\phi_{ABCD})^p = f^p_{ABCD}, \tag{5.79}$$

depends again, similar to (5.76), on $J^p_{\mathcal{I}}(w)$ and $J^{p-1}_{\mathcal{I}}(\phi)$, the left-hand side takes the decoupled form

$$(1-\tau^2)\,\partial^2_\tau\,\phi^p_j + 2\,\{(p-1)\,\tau - j + 2\}\,\partial_\tau\,\phi^p_j + C\,\phi^p_j - p\,(p-1)\,\phi^p_j = \ldots \tag{5.80}$$

where the spin weight relations $X\,\phi_j = 2\,(2-j)\,\phi_j$ and the Casimir operator $C = -\frac{1}{2}\,(X_+\,X_- + X_-\,X_+) + \frac{1}{4}\,X^2$ on $SU(2)$ have been used to arrive at this expression.

The fields ϕ^p_j have expansions

$$\phi^p_j = \sum_{q=|2-j|}^{p}\phi^p_{j,q} \quad \text{where} \quad \phi^p_{j,q} = \sum_{k=0}^{2q}\phi^p_{j,q,k}\,T_{2q}{}^k{}_{q-2+j},$$

with coefficients $\phi^p_{j,q,k} = \phi^p_{j,q,k}(\tau)$. Since the Casimir operator satisfies

$$C\,(T_{2q}{}^k{}_{q-2+j}) = q\,(q+1)\,T_{2q}{}^k{}_{q-2+j},$$

equation (5.80) implies for $\phi^p_{j,q}$ ODE's of the form

$$D_{(n,\alpha,\beta)}\,\phi^p_{j,q} \equiv (1-\tau^2)\,\partial^2_\tau\,\phi^p_{j,q} + \{\beta - \alpha - (\alpha+\beta+2)\,\tau\}\,\partial_\tau\,\phi^p_{j,q} + n\,(n+\alpha+\beta+1)\,\phi^p_{j,q} = \dots \tag{5.81}$$

with

$$\alpha = j - p - 2,\quad \beta = -j - p + 2,\quad n = n_1 \equiv p + q \text{ or } n = n_2 \equiv p - q - 1.$$

The equations above allow us to calculate recursively a formal expansion of the solution $u = (w, \phi)$ to (5.54) in a series of the form

$$u = \sum_{n=0}^{\infty} \frac{1}{n!}\, u^p\, \rho^p, \tag{5.82}$$

on $\mathcal{I}$ (note the different meanings of the superscripts p) with coefficients $u^p = u^p(\tau, t) \in C^\infty(\mathcal{I})$. In some neighborhood of $\mathcal{I}^0$ in $\bar{\mathcal{N}}$ this series represents in fact the Taylor series of smooth functions and it converges near $\mathcal{I}^0$ if the datum h is real analytic. We shall try to deduce from it information on the behavior of u near the critical sets.

5.5. Behavior of the s-jets near the critical sets

Because $J^0_{\mathcal{I}}(u)$ is regular, the integration gives a regular s-jet $J^1_{\mathcal{I}}(w)$. The calculation of $J^1_{\mathcal{I}}(\phi)$ gives (in the cn-gauge and with $\kappa' = 1$) the regular solution

$$\phi^1_{ABCD} = -\{W_1\, 36\,(1-\tau^2) + m^2\,(18\,\tau^2 - 3\,\tau^4)\}\,\epsilon^2{}_{ABCD} - 12\,(1-\tau)^2\, X_+\, W_1\, \epsilon^1{}_{ABCD} + 12\,(1+\tau)^2\, X_-\, W_1\, \epsilon^3{}_{ABCD}. \tag{5.83}$$

Thus $J^2_{\mathcal{I}}(w)$ will again be regular. It turns out that $J^2_{\mathcal{I}}(\phi)$ will not necessarily be regular. The integration (cn-gauge, $\kappa' = 1$) gives

$$\phi^2_{ABCD} = \phi^{ih\,2}_{ABCD} + \breve{\phi}^{W\,2}_{ABCD} + \breve{\phi}'^{\,2}_{ABCD},$$

with

$$\phi^{2\,ih}_{(ABCD)_0} = 0,\quad \phi^{2\,ih}_{(ABCD)_2} = c_2(\tau)\, m\, W_1 + c_3(\tau)\, m^3,\quad \phi^{2\,ih}_{(ABCD)_4} = 0,$$
$$\phi^{2\,ih}_{(ABCD)_1} = c_1(\tau)\, m\, X_+ W_1,\quad \phi^{2\,ih}_{(ABCD)_3} = -c_1(-\tau)\, m\, X_- W_1,$$

where the $c_i(\tau)$ are polynomials in τ of order ≤ 8,

$$\breve{\phi}^{W\,2}_{(ABCD)_j} = -4\sqrt{6\binom{4}{j}}\,(1+\tau)^j\,(1-\tau)^{4-j} \sum_{k=0}^{4} W_{2;4,k}\, T_4{}^k{}_j, \tag{5.84}$$

and

$$\breve{\phi}'^{\,2}_{(ABCD)_j} = a_j(\tau)\,\frac{1}{3} \sum_{k=0}^{4} \sqrt{2\binom{4}{k}}\; b^*_{(EFGH)_k}\, T_4{}^k{}_j \tag{5.85}$$

with

$$a_0(\tau) = 2\,(1-\tau)^4\, K(-\tau) = -a_4(-\tau),$$
$$a_1(\tau) = 4\,(1-\tau)^3\,(1+\tau)\, K(-\tau) - \frac{3}{1-\tau} = -a_3(-\tau),$$

$$a_2(\tau) = \sqrt{6}\,\{\frac{2-\tau}{(1+\tau)^2} - 2\,(1-\tau)^2\,(1+\tau)^2\,K(\tau)\} = -a_2(-\tau),$$

$$K(\tau) = 1 - 3\int_0^\tau \frac{ds}{(1-s)\,(1+s)^5}.$$

While the first two terms extend smoothly to $\mathcal{I}^\pm$, the third term has logarithmic singularities at the critical sets unless the *regularity condition* $b_{ABCD}(i) = 0$ is satisfied (the quadrupole term W_2, which looks so innocent here, reappears in obstructions to smoothness at higher order [71], [72]).

It is thus clearly important to control the behavior of the s-jets at $\mathcal{I}^\pm$ at all orders. Equations $D_{(n,\alpha,\beta)}u = 0$ are well known from the theory of Jacobi polynomials and they have been used in [37] to derive a certain representation of the solutions in terms of polynomials built from the generalized Jacobi polynomials $P_n^{(\alpha,\beta)}(\tau)$ ([68]). By the overdeterminedness of the system (5.74), (5.75) the problem can be reduced to the integration of the functions $\phi^p_{0,q}$, $\phi^p_{4,q}$. The functions $\phi^p_{1,q}$, $\phi^p_{2,q}$, $\phi^p_{3,q}$ can be calculated from them algebraically.

One finds for $p \geq 3$ and $q = p$ the representation

$$\phi^p_{0,p} = (1-\tau)^{p+2}\,(1+\tau)^{p-2}\,\Big(\phi^p_{0,p\,*} + \tag{5.86}$$

$$\frac{(p+1)\,(p+2)}{4\,p}\,(\phi^p_{0,p\,*} - \phi^p_{4,p\,*})\int_0^\tau \frac{d\tau'}{(1+\tau')^{p-1}\,(1-\tau')^{p+3}}\Big),$$

$$\phi^p_{4,p} = (1+\tau)^{p+2}\,(1-\tau)^{p-2}\,\Big(\phi^p_{4,p\,*} - \tag{5.87}$$

$$\frac{(p+1)\,(p+2)}{4\,p}\,(\phi^p_{0,p\,*} - \phi^p_{4,p\,*})\int_0^{-\tau} \frac{d\tau'}{(1+\tau')^{p-1}\,(1-\tau')^{p+3}}\Big),$$

where the subscript $*$ indicates initial data on $\mathcal{I}^0$.

Denoting by $y_{p,q}$ the column vector formed from $\phi^p_{0,q}$, $\phi^p_{4,q}$, one obtains for $p \geq 3$ and $0 \leq q \leq p-1$

$$y_{p,q}(\tau) = X_{p,q}(\tau)\,\left(X^{-1}_{p,q\,*}\,y_{p,q\,*} + \int_0^\tau X_{p,q}(\tau')^{-1}\,B_{p,q}(\tau')\,d\tau'\right). \tag{5.88}$$

The functions $B_{p,q}$ are derived from the right-hand sides of (5.73) and (5.79) and can thus be calculated from $J^p_{\mathcal{I}}(w)$ and $J^{p-1}_{\mathcal{I}}(\phi)$. The matrix-valued functions $X_{p,q}$ are given by

$$X_{p,0} = \begin{pmatrix} (1+\tau)^{p-2}\,(p+\tau) & 0 \\ 0 & (1-\tau)^{p-2}\,(p-\tau) \end{pmatrix},$$

$$X_{p,1} = \begin{pmatrix} (1+\tau)^{p-2} & 0 \\ 0 & (1-\tau)^{p-2} \end{pmatrix},$$

$$X_{p,q} = \begin{pmatrix} Q_{1;p,q}(\tau) & (-1)^q\,Q_{3;p,q}(\tau) \\ (-1)^q\,Q_{3;p,q}(-\tau) & Q_{1;p,q}(-\tau) \end{pmatrix}, \quad 2 \leq q \leq p-1,$$

with polynomials

$$Q_{1;p,q}(\tau) = (\frac{1-\tau}{2})^{p+2} P_{q-2}^{(p+2,-p+2)}(\tau),$$

$$Q_{3;p,q}(\tau) = (\frac{1+\tau}{2})^{p-2} P_{q+2}^{(-p-2,p-2)}(\tau).$$

of degree $n_1 = p + q$.

The solutions to the transport equations can be calculated, order by order, explicitly. The only difficulty is the calculation of the functions

$$B_{p,q} = B_{p,q}\left[J_{\mathcal{I}}^{p}(w),\ J_{\mathcal{I}}^{p-1}(\phi)\right]$$

which become more and more complicated at each step.

The most conspicuous feature of these expressions is the occurrence of logarithmic singularities at $\mathcal{I}^{\pm}$. The latter can arise, as a consequence of the evolution process and the structure of the data, even under the strongest smoothness assumptions on the conformal datum h. We will have to discuss to what extent the occurrence of such singularities can be related to the structure of the initial data and whether it can be avoided by a judicious choice of the latter.

5.6. Regularity conditions

Expanding the integrals in (5.86), (5.87) one finds

$$\phi_{0,p}^{p} \approx (1-\tau)^{p+2}\,(1+\tau)^{p-2}\,\log(1-\tau) + \ analytic\ in\ \tau \quad \text{as} \quad \tau \to 1$$

and a similar behavior for $\phi_{4,p}^{p}$ as $\tau \to -1$, unless the initial data on $\mathcal{I}^0$ satisfy the condition

$$\phi_{0,p\,*}^{p} = \phi_{4,p\,*}^{p}.$$

(Note that the singularities get less severe with increasing p.) This raises the question whether data can be given which satisfy these conditions. By a lengthy recursion argument it can be shown ([37]) that *for given integer* $p_* \geq 0$ *the fields* $\phi_{j,p}^{p}$ *resulting from* (5.86), (5.87) *extend smoothly to* $\mathcal{I}^{\pm}$ *for* $2 \leq p \leq p_* + 2$ *if and only if the free datum* h *satisfies the regularity condition*

$$D_{(A_qB_q}\dots D_{A_1B_1}\,b_{ABCD)}(i) = 0, \quad q = 0,1,2,\dots,p_*. \tag{5.89}$$

By (4.46) these conditions are satisfied for static data with $p_* = \infty$. This allows one to construct a large class of data satisfying (5.89) by gluing with a partition of unity an asymptotically flat static end to a given time reflection symmetric data set and solving the Lichnerowicz equation.

Condition (5.89) has been observed as a regularity condition before. In [34] has been derived under the strong assumption that the solution be massless (cf. (4.30)) a *necessary and sufficient* condition on h that space-like infinity can be represented by a regular point i^0 in a smooth conformal space-time extension (so that $\mathcal{J}^{\pm}$ will be smooth near space-like infinity). This condition, referred to as *radiativity condition*, implies (5.89). It has been shown in [37] that these two conditions are in fact equivalent.

The first term in (5.88) is polynomial and thus regular. The second term is not so easy to handle. If (5.89) is not assumed the corresponding log-terms will enter the integral in a non-linear way and the solution will have at $\mathcal{I}^{\pm}$ polyhomogeneous expansions in terms of expressions $(1 \mp \tau)^k \log^j(1 \mp \tau)$ with $k, j \in \mathbb{N}_0$. We shall assume therefore that (5.89) holds with $p_* = \infty$.

From the expressions above it follows that the Wronskian $\det(X_{p,q})$ has a factor $(1-\tau^2)^{p-2}$. The regularity of the integrals in (5.88) thus depends on the precise structure of the functions $B_{p,q}(\tau)$, which get quite complicate with increasing p. It has been shown in [37] and [41] that $J^p_{\mathcal{I}}(u)$ is regular for $p \leq 3$ if (5.89) is satisfied with $p_* \leq 1$.

Because the functions $B_{p,q}$ are getting increasingly complicated with p, J.A. Valiente-Kroon studied the case where h is conformallly flat on $\mathcal{B}_e$ with the help of an algebraic computer program ([71]). In that case condition (5.89) is trivially satisfied but there still exists a large class of non-trivial data for which h is not conformally flat outside $\mathcal{B}_e$. In the conformal factor (5.44) one has $U = 1$ on $\mathcal{B}_e$ but W will be a non-trivial solution to the conformally covariant Laplace equation with $m = 2\,W(i) \neq 0$. It turns out that $J^4_{\mathcal{I}}(u)$ is again regular. For $J^5_{\mathcal{I}}(u)$ however, logarithmic terms are observed. They come with certain coefficients which depend on the data. Choosing the data such that these coefficients vanish, still new logarithmic terms are observed for $J^6_{\mathcal{I}}(u)$. Restricting to the axially symmetric case to keep the expressions manageable, new logarithmic terms crop up for $p = 7$ and $p = 8$.

The form of the conditions obtained at these orders suggests a general formula which needs to be satisfied to excluded logarithmic terms at any given order p ([71]). If this formula is correct, all derivatives of W must vanish at i if the logarithmic terms are required to vanish at all orders. As a consequence the solution must become asymptotically Schwarzschild at i (cf. Lemma 4.1). Since W is governed on $\mathcal{B}_e$ by an elliptic equation with analytic coefficients it would follow that the solution is precisely Schwarzschild near i.

How seriously do we need to take the singularities at $\mathcal{I}^{\pm}$? To answer this question one needs to control the evolution of the field in a full neighborhood of $\bar{\mathcal{I}}$ in $\tilde{\mathcal{M}}$. This has not been achieved yet. However, the analysis of the linearized setting, which is given by the spin-2 equations (5.67), (5.68) on Minkowski space in the gauge (5.58), gives some insight ([39]).

While the functions $B_{p,q}$ vanish in that case, the singularities arising from (5.86), (5.87) do in general survive the linearization process. The analysis then shows that for prescribed integer j the function

$$\psi_k - \sum_{p'=0}^{p-1} \frac{1}{p'!}\,\psi_k^{p'}\,\rho^{p'} \quad \text{on} \quad \tilde{\mathcal{M}}$$

extends to a function of class C^j on $\bar{\mathcal{M}}$, if one chooses $p \geq j + 6$ in the expansion above. Here $\psi_k^{p'}$, $p' = 0, 1, \ldots, p-1$, are understood as ρ-independent functions on $\tilde{\mathcal{M}} \cup \mathcal{I}$, which agree on $\mathcal{I}$ with the s-jet $J^{p-1}_{\mathcal{I}}(\psi)$ (defined by equations (5.67), (5.68)). Note that the sum above provides the first terms of an asymptotic expansion of the solution at $\mathcal{J}^{\pm}$.

It follows that the solution will extend smoothly to all of $\bar{\mathcal{M}}$ if the linearized version of (5.89) is satisfied with $p_* = \infty$. If the condition is satisfied only with some finite $p_* \geq 2$ but violated at $p = p_* + 1$, the solution will develop a logarithmic singularity at $\mathcal{I}^{\pm}$ which will be transported along the null generators of $\mathcal{J}^{\pm}$ so that the solution will be only in $C^{p_*-2}(\bar{\mathcal{M}})$. While it remains to be seen whether the solutions to the non-linear equations admit similar asymptotic expansions at $\mathcal{J}^{\pm}$, the discussion shows clearly that the regularity of the s-jets $J^p_{\mathcal{I}}(u)$ is a prerequisite for the smooth extensibility of the solutions to $\mathcal{J}^{\pm}$.

If the solutions to the non-linear equations show a singular behavior on $\mathcal{J}^{\pm}$ as indicated above, does it refer to something 'real' or to a failure of the gauge? If the underlying conformal structure where smooth at null infinity, the conformal geodesics should pass through $\mathcal{J}^{\pm}$ where $\Theta \to 0$ and the 1-form, the $\hat{\nabla}$-parallely transported frame, and therefore also the rescaled conformal Weyl spinor in that frame should be represented by smooth functions of τ along the conformal geodesics because these as well as their natural parameter τ depend only on the conformal structure. Singularities as indicated above therefore refer to intrinsic features of the underlying conformal structure.

The results of ([71]) show first of all that the regularity condition (5.89) with $p_* = \infty$ are *not sufficient* for the regularity of $J^p_{\mathcal{I}}(u)$, $p = 0, 1, 2, \ldots$. It appears that the Lichnerowicz equation, which breaks the conformal invariance by fixing the scaling of the physical metric $\tilde{h} = \Omega^{-2}\, h$, does play a role in the smoothness of the conformal structure at null infinity. This is remarkable because it shows that besides the local condition (5.89) there are other conditions to be observed which are 'not so local'. However, the Lichnerowicz equation is introduced only as a device to reduce the problem of solving the *underdetermined elliptic system* of constraints to an *elliptic* problem. The results of [16], [20], [21] exploit the underdeterminedness of the constraints in quite a different way. They teach us to be careful with the words 'local' and 'global' in the present context.

The main purpose of calculating $J^p_{\mathcal{I}}(u)$ for the first few p is to get an insight into (5.88) which would allow us to control the behavior of $J^p_{\mathcal{I}}(u)$ near $\mathcal{I}^{\pm}$ in dependence of the data given on $\mathcal{S}$. One may speculate that the results above are telling us that asymptotic staticity, or more generally asymptotic stationarity, at space-like infinity is of more importance in the present context than expected so far. Recent generalizations of the calculations in ([71]) to non-conformally flat data seem to support this view ([72]).

This raises the question whether the setting proposed in [37] is for static solutions as smooth as one would expect. This is far from obvious because of the loss of hyperbolicity at the critical sets. Giving an answer to this question for general static solutions will be the purpose of the following chapters.

6. Conformal extensions of static vacuum space-times

For static asymptotically flat vacuum solutions with positive ADM mass we shall construct in the following a conformal extension which will include null infinity and will also allow us to discuss the cylinder at space-like infinity. The extension will be defined in terms of explicitly given coordinates and conformal rescaling. In Section 7 will it be shown that it coincides with the extension (not the coordinates etc.) as defined in Section 5.3.

Because one expects usually 'not much to happen at space-like infinity' for static asymptotically flat solutions, one may wonder why the detailed discussion of the fields near space-like infinity should be so complicated. An obvious reason is that a gauge which is chosen to discuss space-like and null infinity must introduce a 'time dependence', it cannot be adapted to a Killing field whose flow lines run out to time-like infinity. However, the main reason is that the static field equations play an important role in discussing the regularity of the field near the critical sets; we will have to make extensive use of them.

The static vacuum solution is assumed in the form

$$\tilde{g} = v^2\,dt^2 + \Omega^{-2}\,h,$$

with $v = v(x^c)$, $h = h_{ab}(x^c)\,dx^a\,dx^b$ and a conformal factor $\Omega = \Omega(x^c)$, where we assume h-normal coordinates x^a which satisfy (4.17) and the conformal gauge which achieves (4.31) on the set $\mathbb{R} \times \mathcal{U}$, where $\mathcal{U} = \{|x| < \bar{\rho}_*\}$ with a sufficiently small $\bar{\rho}_* > 0$. We set

$$\Upsilon = |x|^2, \quad e^a = \frac{x^a}{|x|} = -\frac{1}{2}\Upsilon^{-1/2}\,D^a\,\Upsilon \quad \text{for} \quad |x| > 0, \quad \bar{\rho} = \sqrt{\sum_{a=0}^{3}(x^a)^2}.$$

Coordinates ψ^A, $A = 2, 3$, on the sphere $S^2 = \{|x| = 1\}$ can be used to parametrize e^a and we write then $e^a = e^a(\psi^A)$ and $d\,e^a = e^a{}_{,\psi^A}\,d\,\psi^A$. For convenience the coordinates ψ^A will be assumed in the following to be real analytic. If $x^a = \bar{\rho}\,e^a(\psi^A)$, the metric h takes the form

$$h = -d\,\bar{\rho}^2 + \bar{\rho}^2\,k,$$

with ($\bar{\rho}$-dependent) 2-metrics

$$k = k_{AC}\,d\psi^A\,d\,\psi^C \equiv h_{ac}(\bar{\rho}\,e^c)\,d\,e^a\,d\,e^c,$$

on the spheres $\bar{\rho} = \text{const.} > 0$. For $\bar{\rho} \to 0$ the metric k approaches the standard line element $d\,\sigma^2 = -k(0, \psi^A)$ on the 2-dimensional unit sphere in the coordinates ψ^A.

We write now $x^0 = t$ and $x^{0'} = \bar{\tau}$, $x^{1'} = \bar{\rho}$, $x^{A'} = \psi^A$ and consider the map $\Phi : x^{\mu'} \to x^{\mu}(x^{\mu'})$ defined by

$$t(x^{\mu'}) = \int_{\bar{\rho}\,(1-\bar{\tau})}^{\bar{\rho}} \frac{d\,s}{(v\,\Omega)(s\,e^a(\psi^A))}, \qquad x^a(x^{\mu'}) = \bar{\rho}\,(1 - \bar{\tau})\,e^a(\psi^A). \qquad (6.1)$$

It follows that the four differentials

$$d\,x^a = ((1-\bar\tau)\,d\,\bar\rho - \bar\rho\,d\,\bar\tau)\,e^a + \bar\rho\,(1-\bar\tau)\,d\,e^a, \tag{6.2}$$

$$dt = \left(\frac{1}{(v\,\Omega)(\bar\rho\,e^a)} - \frac{1-\bar\tau}{(v\,\Omega)(\bar\rho\,(1-\bar\tau)\,e^a)}\right) d\,\bar\rho + \frac{\bar\rho}{(v\,\Omega)(\bar\rho\,(1-\bar\tau)\,e^a)}\,d\,\bar\tau + l, \tag{6.3}$$

with

$$l = l_A\,d\,\psi^A, \qquad l_A = \int_{\bar\rho\,(1-\bar\tau)}^{\bar\rho} \left(\frac{1}{(v\,\Omega)(s\,e^a)}\right)_{,\psi^A} d\,s, \tag{6.4}$$

are independent for $0 \le \bar\tau < 1$ and $0 < \bar\rho < \bar\rho_*$ and we can consider the $x^{\mu'}$ as smooth coordinates on an open neighborhood of space-like infinity in $\{t \ge 0\}$. For $s > 0$ we set

$$h(s\,e^a) \equiv \frac{(v\,\Omega)(s\,e^a)}{s^2} = \frac{U(s\,e^a) - s\,\frac{m}{2}}{(U(s\,e^a) + s\,\frac{m}{2})^3}. \tag{6.5}$$

To indicate the different arguments replacing s in this and other functions of $s\,e^a$ or of s and ψ^A, we write out the argument replacing s explicitly but suppress the dependence on e^a or ψ^A. Thus $h(s)$ will be written for $h(s\,e^a)$ and $k(\bar\rho)$ for $k(\bar\rho, \psi^A)$, etc.

With this notation and the conformal factor

$$\Lambda = \Omega\,\Upsilon^{-1/2},$$

a conformal representation of $\tilde g$ is defined by

$$\begin{aligned} \bar g \equiv \Phi^*\left(\Lambda^2\,\tilde g\right) = 2\left(\frac{h(\bar\rho\,(1-\bar\tau))}{h(\bar\rho)}\,\frac{d\,\bar\rho}{\bar\rho} + \bar\rho\,h(\bar\rho\,(1-\bar\tau))\,l\right) d\,\bar\tau \\ -2\,(1-\bar\tau)\left(\frac{h(\bar\rho\,(1-\bar\tau))}{h(\bar\rho)}\,\frac{d\,\bar\rho^2}{\bar\rho^2} + \bar\rho\,h(\bar\rho\,(1-\bar\tau))\,l\,\frac{d\,\bar\rho}{\bar\rho}\right) \\ +(1-\bar\tau)^2\left(\frac{h(\bar\rho\,(1-\bar\tau))}{h(\bar\rho)}\,\frac{d\,\bar\rho}{\bar\rho} + \bar\rho\,h(\bar\rho\,(1-\bar\tau))\,l\right)^2 + k(\bar\rho\,(1-\bar\tau)). \end{aligned} \tag{6.6}$$

The new coordinates do not reflect the symmetries of the underlying space-time, but they are sufficient to discuss the part of the space-time in the future of the initial hypersurface $\{t = 0\}$. We replace $\mathcal{S}$ by the manifold with boundary $\bar{\mathcal{S}}$ introduced in Section 5.1.1. The points of $\partial\bar{\mathcal{S}}$ are thought of as ideal end points attached to the curves $\bar\rho \to x^a(\bar\rho) = \bar\rho\,e^a(\psi^A)$ in $\tilde{\mathcal{S}}$ as $\bar\rho \to 0$ for fixed value of ψ^A. The coordinates $\bar\rho$ and ψ^A extend (by definition) to analytic coordinates on $\bar{\mathcal{S}}$ with $\bar\rho = 0$ on $\partial\bar{\mathcal{S}}$. We set

$$\tilde{\mathcal{M}}' = \{0 \le \bar\tau < 1,\ 0 < \bar\rho\}, \quad \bar{\mathcal{M}}' = \tilde{\mathcal{M}}' \cup \mathcal{J}^{+'} \cup \mathcal{I}' \cup \mathcal{I}^{+'},$$

where it is understood that the unspecified coordinate systems ψ^A 'cover' the sphere S^2, and

$$\mathcal{J}^{+'} = \{\hat\tau = 1, \bar\rho > 0\}, \quad \mathcal{I}^{0'} = \partial\bar{\mathcal{S}} = \{\bar\tau = 0, \bar\rho = 0\},$$

$$\mathcal{I}' = \{0 \le \bar\tau < 1, \bar\rho = 0\}, \quad \mathcal{I}^{+'} = \{\bar\tau = 1, \bar\rho = 0\}, \quad \bar{\mathcal{I}}' = \mathcal{I}' \cup \mathcal{I}^{+'}.$$

While the notation alludes to related sets introduced in Section 5.3, the prime should warn the reader that the sets defined above differ in various aspects from those considered in 5.3. The range of $\bar{\rho}$ should be also bounded from above in these definitions. We leave this bound unspecified because its specific value is unimportant here, we will be concerned only with the behavior of the metric in a neighborhood of $\bar{\mathcal{I}}'$ in $\bar{\mathcal{M}}'$.

Important for the following are the observations:
(i) the function $h(s\, e^a(\psi^A))$ as given by the right-hand side of (6.5) and considered as function of s and ψ^A extends as a real analytic function into a domain where $s < 0$. This follows immediately from the values taken by U and its analyticity.
(ii) similarly, the 1-form l given by (6.4) extends as a real analytic function into a domain where $\bar{\rho} \leq 0$ and $\bar{\tau} \geq 1$. This follows from

$$\left(\frac{1}{v\,\Omega}\right)_{,\psi^A}(s) = \frac{1}{s^2}\,\frac{2\,(U(s) - s\,m)(U(s) + s\,\frac{m}{2})^2}{(U(s) - s\,\frac{m}{2})^2}\,U_{,\psi^A},$$

and (4.27) with $s^2 = \Upsilon$.

For the following it is convenient to slightly modify the frame (5.57) and set

$$v_0 = \partial_{\bar{\tau}},\; v_1 = \bar{\rho}\,\partial_{\bar{\rho}},\; v_A = \partial_{\psi^A}, \tag{6.7}$$

$$\alpha^0 = d\,\bar{\tau},\; \alpha^1 = \frac{1}{\bar{\rho}}\,d\,\bar{\rho},\; \alpha^B = d\,\psi^B, \quad A, B = 2, 3.$$

One then gets $\bar{g} = \bar{g}_{ik}\,\alpha^i\,\alpha^k$ with metric coefficients

$$\bar{g}_{00} = 0, \quad \bar{g}_{01} = \frac{h(\bar{\rho}\,(1-\bar{\tau}))}{h(\bar{\rho})}, \quad \bar{g}_{0A} = \bar{\rho}\,h(\bar{\rho}\,(1-\bar{\tau}))\,l_A,$$

$$\bar{g}_{11} = -(1-\bar{\tau})\,\frac{h(\bar{\rho}\,(1-\bar{\tau}))}{h(\bar{\rho})}\left(2 - (1-\bar{\tau})\,\frac{h(\bar{\rho}\,(1-\bar{\tau}))}{h(\bar{\rho})}\right),$$

$$\bar{g}_{1A} = -(1-\bar{\tau})\,\bar{\rho}\;h(\bar{\rho}\,(1-\bar{\tau}))\left(1 - (1-\bar{\tau})\,\frac{h(\bar{\rho}\,(1-\bar{\tau}))}{h(\bar{\rho})}\right)l_A,$$

$$\bar{g}_{AB} = \{\bar{\rho}\,(1-\bar{\tau})\;h(\bar{\rho}\,(1-\bar{\tau}))\}^2\,l_A\,l_B + k_{AB}(\bar{\rho}\,(1-\bar{\tau})).$$

In terms of the new coordinates the metric given by (6.6) extends analytically through the set $\mathcal{J}^{+'}$. The latter is a null hypersurface for the extended metric and represents future null infinity for the space-time defined by $\tilde{g}$. By contrast, the right-hand side of (6.6) does not extend smoothly to $\bar{\mathcal{I}}'$. However, the frame coefficients $\bar{g}_{ik}$ and their contravariant versions $\bar{g}^{ik}$ do extend analytically to all of $\bar{\mathcal{M}}'$. It will be shown later how $\mathcal{I}'$ relates to (part of) the cylinder at space-like infinity denoted in 5.3 by $\mathcal{I}$.

One has $\bar{g}_{ik} = g^{\star}_{ik} + O(\bar{\rho}^2)$ with

$$g^{\star}_{ik} = \begin{bmatrix} 0 & 1 + 2\,m\,\bar{\rho}\,\bar{\tau} & 0 & 0 \\ 1 + 2\,m\,\bar{\rho}\,\bar{\tau} & -(1-\bar{\tau})\,(1 + \bar{\tau} + 4\,m\,\bar{\rho}\,\bar{\tau}^2) & 0 & 0 \\ 0 & 0 & k_{22}(0) & k_{23}(0) \\ 0 & 0 & k_{32}(0) & k_{33}(0) \end{bmatrix}, \tag{6.8}$$

so that $\det(g^*_{ik}) < 0$ for $\bar\rho \geq 0$, $0 \leq \bar\tau \leq 1$ and $g^{ik} = g^{*ik} + O(\bar\rho^2)$ with

$$g^{*ik} = \begin{bmatrix} \frac{(1-\bar\tau)\,(1+\bar\tau+4\,m\,\bar\rho\,\bar\tau^2)}{(1+2\,m\,\bar\rho\,\bar\tau)^2} & \frac{1}{1+2\,m\,\bar\rho\,\bar\tau} & 0 & 0 \\ \frac{1}{1+2\,m\,\bar\rho\,\bar\tau} & 0 & 0 & 0 \\ 0 & 0 & k^{22}(0) & k^{23}(0) \\ 0 & 0 & k^{32}(0) & k^{33}(0) \end{bmatrix}. \tag{6.9}$$

Since the conformal factor Λ does not depend on t, the static Killing vector field represents a Killing field also for the metric $\bar g$. In the new coordinates it takes the form

$$K = \frac{(v\,\Omega)(\bar\rho)}{\bar\rho}\,\{(1-\bar\tau)\,\partial_{\bar\tau} + \bar\rho\,\partial_{\bar\rho}\} = \bar\rho\,h(\bar\rho)\,\{(1-\bar\tau)\,v_0 + v_1\}, \tag{6.10}$$

and extends smoothly to all of $\bar{\mathcal{M}}'$.

Denote by $\bar\nabla$ the Levi-Civita connection of $\bar g$. Since the commutators of the frame fields v_k vanish, the connection coefficients defined by $\bar\nabla_i v_j \equiv \bar\nabla_{v_i} v_j = \gamma_i{}^k{}_j\, v_k$ are given by the formula

$$\gamma_i{}^k{}_j = \frac{1}{2}\,\bar g^{kl}\,(v_j(\bar g_{il}) + v_i(\bar g_{lj}) - v_l(\bar g_{ij}))\,.$$

Again, *the connection coefficients $\gamma_i{}^k{}_j$ in the frame v_k extend analytically through $\{\bar\rho = 0\}$ and $\{\bar\tau = 1\}$*. One finds

$$\gamma_i{}^k{}_j = \frac{1}{2}\,g^{*kl}\,(v_j(g^*_{il}) + v_i(g^*_{lj}) - v_l(g^*_{ij})) + O(\bar\rho^2),$$

which implies

$$\gamma_i{}^k{}_j = \bar\tau\,\delta^k{}_0\,\{2\,\delta^0{}_{(i}\,\delta^1{}_{j)} - (1-\bar\tau^2)\,\delta^1{}_i\,\delta^1{}_j\} - \bar\tau\,\delta^k{}_1\,\delta^1{}_i\,\delta^1{}_j \quad \text{on} \quad \{\bar\rho = 0\}. \tag{6.11}$$

As a consequence of the behavior of $\bar g_{ij}$ and $\gamma_i{}^k{}_j$ the components of all tensor fields in the frame v_k which are derived by standard formulas from the metric and the connection coefficients, such as those of the Ricci tensor and the conformal Weyl tensor of $\bar g$, extend analytically through $\mathcal{J}^{+'}$ and $\bar{\mathcal{I}}'$, i.e., *the metric $\bar g$ and its connection $\bar\nabla$ imply in the frame v_i a smooth frame formalism on $\bar{\mathcal{M}}'$.*

It follows that the coordinate expressions of these tensor fields, such as $R_{\mu'\nu'}[\bar g] = R_{ik}\,\alpha^i{}_{\mu'}\,\alpha^k{}_{\nu'}$, and, by the argument given in [60] (cf. also [38]), the rescaled conformal Weyl tensor $W^{\mu'}{}_{\nu'\lambda'\rho'}[\bar g] = \Lambda^{-1}\,C^{\mu'}{}_{\nu'\lambda'\rho'}[\bar g]$ extend smoothly to $\mathcal{J}^{+'}$. Unfortunately, this does not give us the needed details about the components R_{jk} and it does not tell us anything about the behavior of the frame components $W^i{}_{jkl}[\bar g]$ of the rescaled conformal Weyl tensor on $\mathcal{I}'$ and the critical set $\mathcal{I}^{+'}$. This requires detailed calculations. Only the analyticity of h near i is required to control the smoothness of the fields near $\mathcal{J}^{+'}$. This follows from the *ellipticity* of the conformal static field equations near i. To deduce the desired behavior near $\bar{\mathcal{I}}'$, however, one will have to invoke at least, as discussed in Section 5.6, the regularity condition (5.89) with $p_* = \infty$. The detailed form of the conformal static field equations will thus become much more important.

With equations (4.36) and the relations

$$D_a\,\Omega = \Omega^{\frac{3}{2}}\,\sigma^{-\frac{3}{2}}\,D_a\,\sigma = (1+\sqrt{\mu\,\sigma})^{-3}\,D_a\,\sigma, \tag{6.14}$$

$$D_a\,D_b\,\Omega = \frac{1}{(1+\sqrt{\mu\,\sigma})^3}\,D_a\,D_b\,\sigma - \frac{3}{2}\sqrt{\frac{\mu}{\sigma}}\,\frac{1}{(1+\sqrt{\mu\,\sigma})^4}\,D_a\,\sigma\,D_b\,\sigma, \tag{6.15}$$

which are implied by (4.35), one gets

$$\Upsilon\,D^a\,N\,D_a\,\Lambda$$

$$= v\Omega\left\{\frac{2s(1-2\sqrt{\mu\sigma})}{(1-\sqrt{\mu\sigma})(1+\sqrt{\mu\sigma})^4} - \frac{(2-3\sqrt{\mu\sigma})\Upsilon^{-1}D^c\Upsilon\,D_c\sigma}{2(1-\sqrt{\mu\sigma})(1+\sqrt{\mu\sigma})^3} - \frac{1}{U^2(1+\sqrt{\mu\sigma})^2}\right\},$$

$$\Upsilon\,D^a\,\Lambda\,D_a\,\Lambda = \Omega\left\{\frac{2\,s}{(1+\sqrt{\mu\,\sigma})^4} - \frac{\Upsilon^{-1}\,D^c\,\Upsilon\,D_c\,\sigma}{(1+\sqrt{\mu\,\sigma})^3} - \frac{1}{U^2\,(1+\sqrt{\mu\,\sigma})^2}\right\},$$

$$\Upsilon^{1/2}\,D^*_a\,D^*_b\,\Lambda = \frac{s\,h_{ab} - \sigma(1-\mu\,\sigma)\,R_{ab}}{(1+\sqrt{\mu\,\sigma})^3} - \frac{3}{2}\,\frac{\sqrt{\mu\,\sigma}\;\sigma^{-1}\,D_a\sigma\,D_b\sigma}{(1+\sqrt{\mu\,\sigma})^4}$$
$$- \frac{D_a\,D_b\,\Upsilon}{2\,U^2\,(1+\sqrt{\mu\,\sigma})^2} + \frac{e_a\,e_b}{U^2\,(1+\sqrt{\mu\,\sigma})^2}$$
$$- h_{ab}\left(\frac{1}{U^2\,(1+\sqrt{\mu\,\sigma})^2} + \frac{\Upsilon^{-1}\,D^c\,\Upsilon\,D_c\,\sigma}{2\,(1+\sqrt{\mu\,\sigma})^3}\right),$$

which allow us to obtain the following expressions for the L_{jk}.

In the case of L_{00} there occurs a cancellation of the second terms in (6.12), (6.13) respectively, so that (with the understanding that $e^a \circ \Phi = e^a(\psi^A)$)

$$L_{00} = \bar{\rho}^2\,\left(\frac{L[\bar{g}]_{tt}}{(v\,\Omega)^2} + L[\bar{g}]_{ab}\,e^a\,e^b\right)\circ\Phi \tag{6.16}$$

$$= -\bar{\rho}^2\,\left(\frac{1}{v\,\Omega^2}\left\{\Upsilon\,D_a\,N\,D^a\,\Lambda + v\,\Omega\,\Upsilon^{1/2}\,D^*_a\,D^*_b\,\Lambda\,e^a\,e^b\right\}\right)\circ\Phi,$$

with

$$\Upsilon\,D_a\,N\,D^a\,\Lambda + v\,\Omega\,\Upsilon^{1/2}\,D^*_a\,D^*_b\,\Lambda\,e^a\,e^b$$

$$= v\,\Omega\,\left\{\frac{(1-4\sqrt{\mu\,\sigma}+\mu\,\sigma)\,(s+2)}{(1-\sqrt{\mu\,\sigma})\,(1+\sqrt{\mu\,\sigma})^4} - \frac{\sigma(1-\mu\,\sigma)\,R_{ab}\,e^a\,e^b}{(1+\sqrt{\mu\,\sigma})^3}\right.$$
$$- \frac{6\sqrt{\mu\,\sigma}}{(1+\sqrt{\mu\,\sigma})^4}\left[(\frac{1}{U} + \frac{D^a\,\Upsilon\,D_a\,U}{2\,U^2})^2 - 1\right]$$
$$\left. + \frac{(1-2\sqrt{\mu\,\sigma})}{(1-\sqrt{\mu\,\sigma})\,(1+\sqrt{\mu\,\sigma})^3}\left[\frac{2-2\,U^2}{U^2} + \frac{D^a\,\Upsilon\,D_a\,U}{U^3}\right]\right\}.$$

Since the term in curly brackets is of the order $O(\Upsilon)$, the function L_{00} extends smoothly to $\{\bar{\rho} = 0\}$ with $L_{00} \to 0$ as $\bar{\rho} \to 0$.

It holds

$$L_{01} = \frac{\bar{\rho}^2}{(v\,\Omega)(\bar{\rho})}\,\left(\frac{L[\bar{g}]_{tt}}{v\,\Omega}\right)\circ\Phi - (1-\bar{\tau})\,L_{00},$$

with

$$\frac{L[\bar g]_{tt}}{v\,\Omega} = \Omega^{-1}\,\Upsilon\,(\frac{v}{2}\,D_a\,\Lambda\,D^a\Lambda - D_a\,N\,D^a\,\Lambda)$$
$$= v\left\{\frac{1}{2\,U^2\,(1+\sqrt{\mu\,\sigma})^2} - \frac{(1-3\sqrt{\mu\,\sigma})\,s}{(1-\sqrt{\mu\,\sigma})\,(1+\sqrt{\mu\,\sigma})^4}\right.$$
$$\left. - \frac{1-2\sqrt{\mu\,\sigma}}{(1-\sqrt{\mu\,\sigma})\,(1+\sqrt{\mu\,\sigma})^3}\left[\frac{2}{U^2} + \frac{D^a\,\Upsilon\,D_a\,U}{U^3}\right]\right\},$$

so that L_{01} extends smoothly to $\{\bar\rho = 0\}$ with $L_{01} \to \frac{1}{2}$ as $\bar\rho \to 0$.

$$L_{0A} = \bar\rho\left(\frac{L[\bar g]_{tt}}{v\,\Omega}\right)\circ\Phi\;l_A - \bar\rho^2\,(1-\bar\tau)\,(L[\bar g]_{ab}\circ\Phi)\,e^a\,e^b_{\;,\psi^A},$$

with

$$L[\bar g]_{ab}\,e^a\,e^b_{\;,\psi^A}$$
$$= (1-\sqrt{\mu\,\sigma})\,R_{ab}\,e^a\,e^b_{\;,\psi^A} + \frac{\mu\,U_{,\psi^A}}{\sqrt{\mu\,\Upsilon\,\sigma}}\frac{1}{(1+\sqrt{\mu\,\sigma})^2}\left[\frac{1}{U^4} + 2\,\frac{\Upsilon\,D^a\Upsilon\,D_a\,U}{U^6}\right]$$

so that L_{0A} extends smoothly to $\{\bar\rho = 0\}$ with $L_{0A} \to 0$ as $\bar\rho \to 0$.

$$L_{11} = \left(\frac{\bar\rho^2\,(v\,\Omega)(\bar\rho(1-\bar\tau))}{(v\,\Omega)^2(\bar\rho)} - 2\,\frac{\bar\rho^2\,(1-\bar\tau)}{(v\,\Omega)(\bar\rho)}\right)\left(\frac{L[\bar g]_{tt}}{v\,\Omega}\right)\circ\Phi + (1-\bar\tau)^2\,L_{00},$$

extends smoothly to $\{\bar\rho = 0\}$ with $L_{11} \to -\frac{1-\bar\tau^2}{2}$ as $\bar\rho \to 0$.

$$L_{1A}$$
$$= \left(\frac{\bar\rho}{(v\,\Omega)(\bar\rho)} - \frac{\bar\rho\,(1-\bar\tau)}{(v\,\Omega)(\bar\rho(1-\bar\tau))}\right)L[\bar g]_{tt}\circ\Phi\;l_A - \bar\rho^2\,(1-\bar\tau)^2\,(L[\bar g]_{ab}\circ\Phi)\,e^a\,e^b_{\;,\psi^A},$$

extends smoothly to $\{\bar\rho = 0\}$ with $L_{1A} \to 0$ as $\bar\rho \to 0$.

$$L_{AB} = L[\bar g]_{tt}\circ\Phi\;l_A\,l_B - (\Gamma\,L[\bar g]_{ab})\circ\Phi\;e^a_{\;,\psi^A}\,e^b_{\;,\psi^B}.$$

extends smoothly to $\{\bar\rho = 0\}$ with $L_{AB} \to -\frac{1}{2}\,k_{AB}(0)$ as $\bar\rho \to 0$.

6.2. The rescaled conformal Weyl tensor of $\bar g$ near $\bar{\mathcal{I}}'$

In this section we shall make a few general observations concerning the rescaled conformal Weyl tensor and then specialize to the conformal static case. After a remark about the radiation field on $\mathcal{J}^{+'}$ we will analyze the smoothness of the rescaled conformal Weyl tensor near the set $\bar{\mathcal{I}}'$.

Let $\tilde g$ be a Lorentz metric and $\tilde S$ a space-like hypersurface with unit normal $\tilde n$ and induced metric $\tilde h_{\mu\nu} = \tilde g_{\mu\nu} - \tilde n_\mu\,\tilde n_\nu$. We set $\tilde p_{\mu\nu} = \tilde h_{\mu\nu} - \tilde n_\mu\,\tilde n_\nu$, $\tilde\epsilon_{\nu\lambda\rho} = \tilde n^\mu\,\tilde\epsilon_{\mu\nu\lambda\rho}$, and denote by $\tilde c_{\nu\rho} = C_{\mu\nu\lambda\rho}[\tilde g]\,\tilde n^\mu\tilde n^\lambda$ and $\tilde c^*_{\nu\rho} = C^*_{\mu\nu\lambda\rho}[\tilde g]\tilde n^\mu\,\tilde n^\lambda$ (the star on the right-hand side indicating the dual) the $\tilde n$-electric and the $\tilde n$-magnetic part of the conformal Weyl tensor respectively. The latter are symmetric, trace-free, and spatial, i.e., $\tilde n^\nu\,\tilde c_{\nu\rho} = 0$, $\tilde n^\nu\,\tilde c^*_{\nu\rho} = 0$. The conformal Weyl tensor of $\tilde g$ is then given in terms of its electric and the magnetic part by (cf. [42])

$$C_{\mu\nu\lambda\rho}[\tilde g] = 2\left(\tilde p_{\nu[\lambda}\,\tilde c_{\rho]\mu} - \tilde p_{\mu[\lambda}\,\tilde c_{\rho]\nu} - \tilde n_{[\lambda}\,\tilde c^*_{\rho]\delta}\,\epsilon^\delta{}_{\mu\nu} - \tilde n_{[\mu}\,\tilde c^*_{\nu]\delta}\,\epsilon^\delta{}_{\lambda\rho}\right). \qquad (6.17)$$

6.1. The Ricci tensor of $\bar{g}$ near $\bar{\mathcal{I}}'$

The tensor

$$L[\bar{g}]_{\rho'\nu'} = \frac{1}{2}\left(R[\bar{g}]_{\rho'\nu'} - \frac{1}{6}\,R[\bar{g}]\,\bar{g}_{\rho'\nu'}\right),$$

is needed to integrate the conformal geodesic equations which define the setting introduced in Section 5. The purpose of this section is to demonstrate

Lemma 6.1. *The frame components $L_{jk} = L[\bar{g}]_{\rho'\nu'}\,v^{\rho'}{}_j\,v^{\nu'}{}_k$ extend as real analytic functions to $\bar{\mathcal{I}}'$ with*

$$L_{0k} \to \frac{1}{2}\,\delta^1{}_k, \quad L_{11} \to -\frac{1-\bar{\tau}^2}{2}, \quad L_{1A} \to 0, \quad L_{AB} \to -\frac{1}{2}\,k_{AB}(0) \quad \textit{as} \quad \bar{\rho} \to 0.$$

Proof. Under the rescaling $\tilde{g} \to \bar{g} = \Lambda^2\,\tilde{g}$ the tensor

$$L[\tilde{g}]_{\rho\nu} = \frac{1}{2}\left(R[\tilde{g}]_{\rho\nu} - \frac{1}{6}\,R[\tilde{g}]\,\tilde{g}_{\rho\nu}\right),$$

transforms into

$$L[\bar{g}]_{\rho\nu} = L[\tilde{g}]_{\rho\nu} - \frac{1}{\Lambda}\,\bar{\nabla}_\rho\,\bar{\nabla}_\nu\,\Lambda + \frac{1}{2\,\Lambda^2}\,\bar{\nabla}_\mu\,\Lambda\,\bar{\nabla}^\mu\,\Lambda\;\bar{g}_{\rho\nu}.$$

Suppose $\tilde{g} = v^2\,dt^2 + \tilde{h}$ is a static vacuum solution and $\bar{g} = \Lambda^2\,\tilde{g} = \Lambda^2\,(v^2\,dt^2 + \tilde{h}) = N^2\,dt^2 + h^*$ with

$$N = \Lambda\,v, \quad h^* \equiv \Lambda^2\,\tilde{h} = \Lambda^2\,\Omega^{-2}\,h = h^*_{ab}(x^c)\,dx^a\,dx^b,$$

$$\mu = \mu(x^a), \qquad v = v(x^a), \qquad \Omega = \Omega(x^a), \qquad \Lambda = \Lambda(x^a).$$

In the following the gauge (4.31) and coordinates satisfying (4.17) will be assumed. The connection coefficients of the metric $\bar{g}$ in the coordinates t, x^a are given by

$$\Gamma_a{}^b{}_c[\bar{g}] = \Gamma_a{}^b{}_c[h^*] \quad \text{(the Levi-Civita connection of } h^*\text{)},$$

$$\Gamma_t{}^a{}_t[\bar{g}] = 0, \quad \Gamma_t{}^a{}_t[\bar{g}] = -N\,h^{*ab}D_b\,N, \quad \Gamma_b{}^a{}_t[\bar{g}] = \Gamma_t{}^a{}_b[\bar{g}] = 0,$$

$$\Gamma_b{}^t{}_c[\bar{g}] = 0, \quad \Gamma_b{}^t{}_t[\bar{g}] = \Gamma_t{}^t{}_b[\bar{g}] = \frac{1}{N}\,D_b\,N.$$

and $L[\bar{g}]_{\rho\nu}$ is given by

$$L[\bar{g}]_{tt} = -\frac{v\,\Omega^2}{\Lambda^2}\,D_a\,N\,D^a\,\Lambda + \frac{v^2\,\Omega^2}{2\,\Lambda^2}\,D_a\,\Lambda\,D^a\Lambda, \tag{6.12}$$

$$L[\bar{g}]_{ta} = L[\bar{g}]_{at} = 0,$$

$$L[\bar{g}]_{ab} = -\frac{1}{\Lambda}\,D^*_a\,D^*_b\,\Lambda + \frac{1}{2\,\Lambda^2}\,D_c\,\Lambda\,D^c\,\Lambda\,h_{ab}, \tag{6.13}$$

where D and D^* denote the h- and h^*-Levi-Civita connections respectively. With $\Lambda = \Omega\,\Upsilon^{-1/2}$ and the map Φ defined by (6.1) one can determine from these formulas the frame coefficients

$$L_{ik} = \langle \Phi^*(L[\Lambda^2\,\tilde{g}]);\, v_i,\, v_k\rangle =$$

$$\langle (L[\bar{g}]_{tt} \circ \Phi)\,dt\,dt + (L[\bar{g}]_{ab} \circ \Phi)\,dx^a\,dx^b;\, v_i,\, v_k\rangle.$$

Suppose that $\tilde{g}$ is a solution to the vacuum field equations. Then the first and second fundamental form $\tilde{h}_{ab}$ and $\tilde{\chi}_{ab}$ induced by $\tilde{g}$ on $\tilde{\mathcal{S}}$ satisfy the Gauss and the Codazzi equation (expressing the pull-back of spatial tensors to $\tilde{\mathcal{S}}$ in terms of spatial coordinates x^a)

$$r_{ab}[\tilde{h}] = -\tilde{c}_{ab} + \tilde{\chi}_c{}^c\,\tilde{\chi}_{ab} - \tilde{\chi}_{ca}\,\tilde{\chi}_b{}^c, \tag{6.18}$$

$$\tilde{D}_b\,\tilde{\chi}_{d(a}\,\tilde{\epsilon}_{c)}{}^{bd} = -\tilde{c}^*_{ac}. \tag{6.19}$$

This allows us to express the conformal Weyl tensor in terms of $\tilde{h}_{ab}$ and $\tilde{\chi}_{ab}$. If $\tilde{\mathcal{S}}$ is a hypersurface of time reflection symmetry, so that $\tilde{\chi}_{ab} = 0$, these equations imply

$$r_{ab}[\tilde{h}] = -\tilde{c}_{ab}, \qquad \tilde{c}^*_{ac} = 0, \tag{6.20}$$

and the Weyl tensor assumes the form

$$C_{\mu\nu\lambda\rho}[\tilde{g}] = 2\left(\tilde{p}_{\nu[\lambda}\,\tilde{c}_{\rho]\mu} - \tilde{p}_{\mu[\lambda}\,\tilde{c}_{\rho]\nu}\right) \equiv -(\tilde{p} \oslash \tilde{c})_{\mu\nu\lambda\rho}, \tag{6.21}$$

where $\oslash$ denotes the bi-linear Kulkarni-Nomizu product of two symmetric 2-tensors (cf. [8]).

If Λ is an arbitrary conformal factor, the rescaled conformal Weyl tensor of $\bar{g} = \Lambda^2\,\tilde{g}$ is given by $W^\mu{}_{\nu\lambda\rho}[\bar{g}] = \Lambda^{-1}\,C^\mu{}_{\nu\lambda\rho}[\bar{g}]$. In view of the behavior of the conformal Weyl tensor under conformal rescalings, one gets (observe the index positions)

$$W_{\mu\nu\lambda\rho}[\bar{g}] = \Lambda\,C_{\mu\nu\lambda\rho}[\tilde{g}]. \tag{6.22}$$

Its electric part with respect to the $\bar{g}$-unit vector $\Lambda^{-1}\,\tilde{n}$ is then given by

$$w_{\mu\nu}[\bar{g}] = \Lambda^{-1}\tilde{c}_{\mu\nu}[\tilde{g}] \tag{6.23}$$

With $h = \Omega^2\,\tilde{h}$, the gauge (4.31), the general transformation law

$$r_{ab}[\tilde{h}] = r_{ab}[h] + \Omega^{-1}\,D_aD_b\Omega + h_{ab}\,(\Omega^{-1}\,D_cD^c\Omega - 2\,\Omega^{-2}\,D_c\Omega\,D^c\Omega),$$

and the equation $2\,\Omega\,\Delta_h\,\Omega = 3\,D_a\Omega\,D^a\Omega$, one gets from (6.20) and (6.23) in the general time reflection symmetric case

$$w_{ab}[\bar{g}] = -(\Lambda\,\Omega)^{-1}(D_a\,D_b\,\Omega - \frac{1}{3}\,h_{ab}\,D_c\,D^c\,\Omega + \Omega\,r_{ab}[h]) \quad \text{on} \quad \mathcal{S} = \tilde{\mathcal{S}} \cup \{i\}. \tag{6.24}$$

A conformal scaling which represents space-like infinity (with respect to the initial hypersurface $\tilde{\mathcal{S}}$ and with respect to the solution space-time) by a point is achieved by choosing $\Lambda = \Omega$ on $\mathcal{S}$. With this particular choice one has

$$\begin{aligned} w_{ab}[\bar{g}] &= -\Omega^{-2}(D_a\,D_b\,\Omega - \frac{1}{3}\,h_{ab}\,D_c\,D^c\,\Omega + \Omega\,r_{ab}[h]) \\ &= O(\Upsilon^{-3/2}) \quad \text{as} \quad \Upsilon \to 0 \quad \text{unless} \quad m = 0. \end{aligned} \tag{6.25}$$

We note that in the massless case the precise behavior depends on the freely prescribed metric h on $\mathcal{S}$ near i. In the massless case one has $\Omega = \sigma$ and the comparison of the expression for $w_{ab}[\bar{g}]$ with (4.36) shows that in the case where

h represents conformally static vacuum data one has

$$w_{ab}[\bar{g}] = -\mu \, r_{ab}[h], \tag{6.26}$$

i.e., the rescaled conformal Weyl tensor is smooth.

We return to the case where $\Lambda = \Omega \, \Upsilon^{-1/2}$. With (6.14), (6.15) we get then

$$w_{ab}[\bar{g}] = -\frac{\sqrt{\Gamma}}{\sigma^2} \left\{ (1 + \sqrt{\mu \, \sigma}) \left(D_a \, D_b \, \sigma - \frac{1}{3} \, \Delta_h \, \sigma \, h_{ab} \right) \right. \tag{6.27}$$

$$\left. -\frac{1}{2} \sqrt{\frac{\mu}{\sigma}} \left(3 \, D_a \, \sigma \, D_b \, \sigma - D_c \, \sigma \, D^c \, \sigma \, h_{ab} \right) + \sigma \, (1 + \sqrt{\mu \, \sigma})^2 \, r_{ab} \right\}$$

$$= \frac{m}{4} \frac{U}{\sigma^2} \left(3 \, D_a \, \sigma \, D_b \, \sigma - D_c \, \sigma \, D^c \, \sigma \, h_{ab} \right) - \frac{m}{2} \, U \, (1 + \sqrt{\mu \, \sigma})^2 \, r_{ab}[h]$$

$$-\frac{\sqrt{\Upsilon}}{\sigma^2} \, (1 + \sqrt{\mu \, \sigma}) \, \Sigma_{ab},$$

where we use Σ_{ab} as defined by the right-hand side of (4.36) without assuming h to be conformally static. If h is conformally static the electric part of the rescaled conformal Weyl tensor on $\mathcal{S}$ is given by the right-hand side of (6.27) with $\Sigma_{ab} = 0$. In the present conformal gauge, defined by (4.31), one has

$$\Sigma_{ab} = O(\Upsilon^{3/2}) \quad \text{near} \quad i,$$

for any time reflection symmetric initial data h.

If the solution is static and written again in the form $\tilde{g} = v^2 \, dt^2 + \tilde{h}$, then equations (6.17), (6.20) hold with $\tilde{n} = \frac{1}{v} \, \partial_t$ and t-independent fields for each slice $\tilde{\mathcal{S}} = \{t = t_*\}$ with $t_* = const$. The relations above then imply for all (t, x^a)

$$W_{\mu\nu\lambda\rho}[\bar{g}] = -\Upsilon^{-1} (p \oslash w)_{\mu\nu\lambda\rho} \tag{6.28}$$

with

$$p_{\mu\nu} = h_{\mu\nu} - n_\mu \, n_\nu, \quad n_\mu = \Omega \, \tilde{n}_\mu.$$

With $\otimes$ denoting the tensor product, we write for arbitrary 1-forms a, c

$$a \otimes_s c = a \otimes c + c \otimes a, \quad a^2 = a \otimes a,$$

and note that the Kulkarni-Nomizu product is symmetric, i.e.,

$$m \oslash n = n \oslash m, \tag{6.29}$$

for symmetric 2-tensors m, n, and satisfies for arbitrary 1-forms a, c, e

$$(a \otimes a) \oslash (a \otimes_s c) = 0, \quad (a \otimes_s e) \oslash (a \otimes_s c) = -(a \otimes a) \oslash (c \otimes_s e). \tag{6.30}$$

We show how it follows in the present setting that the *radiation field* vanishes on $\mathcal{J}^{+'}$. Since the extended Killing vector field K is tangent to the null generators of $\mathcal{J}^{+'}$ without vanishing there, the complete information on the radiation field is contained in the field

$$K^\nu\, K^\rho\, W_{\mu\nu\lambda\rho}[\bar g]\, d\,x^\mu\, d\,x^\lambda = -\Upsilon^{-1}\, K^\nu\, K^\rho\, (p \oslash w)_{\mu\nu\lambda\rho}\, d\,x^\mu\, d\,x^\lambda$$

$$= -\Upsilon^{-1}\, K^\nu\, K^\rho\, p_{\nu\rho}\, w_{ab}\, d\,x^a\, d\,x^b = -\Upsilon^{-1}\, (v\,\Omega)^2\, w_{ab}\, d\,x^a\, d\,x^b$$

$$= \frac{m}{4}\,\frac{(1-\sqrt{\mu\,\sigma})^2}{(1+\sqrt{\mu\,\sigma})^6}\left\{\frac{1}{U}\,(2\,s\,h_{ab} - 3\,\sigma^{-1}\,D_a\,\sigma\,D_b\,\sigma)\,d\,x^a\,d\,x^b \right.$$

$$\left. + \frac{2\,\sigma}{U}(1+\sqrt{\mu\,\sigma})^2\, r_{ab}[h]\, d\,x^a\, d\,x^b\right\}$$

with s as given in (4.32). Because of the relation

$$\sigma^{-1}\, D_a\,\sigma\, D_b\,\sigma = 4\,U^{-4}\,(U^2\,e_a\,e_b - U\,\Upsilon^{1/2}\,(e_a\,D_b\,U + e_b\,D_a\,U) + \Upsilon\,D_a\,U\,D_b\,U), \tag{6.31}$$

and the factor σ in the second term it follows that

$$K^\nu\, K^\rho\, W_{\mu\nu\lambda\rho}[\bar g]\, d\,x^\mu\, d\,x^\lambda \to -2\,m\,\bar\rho^2\,d\,\bar\tau^2 \quad \text{as } \bar\tau \to 1,\ \bar\rho > 0. \tag{6.32}$$

Thus, the pull-back of $K^\nu\, K^\rho\, W_{\mu\nu\lambda\rho}[\bar g]\, d\,x^\mu\, d\,x^\lambda$ to $\mathcal{J}^+$, which provides the radiation field up to a scaling, vanishes everywhere on $\mathcal{J}^+$.

Lemma 6.2. *The components $W_{ijkl}[\bar g] = \Lambda^{-1}\, C_{\mu'\nu'\lambda'\rho'}[\bar g]\, v^{\mu'}{}_i\, v^{\nu'}{}_j\, v^{\lambda'}{}_k\, v^{\rho'}{}_l$ of the rescaled conformal Weyl tensor of $\bar g$ in the frame v_k extend as analytic functions to $\bar{\mathcal{I}}'$.*

Proof. In the coordinates $x^{\mu'}$ given by (6.1) the rescaled conformal Weyl tensor is obtained as the product of

$$-(\Upsilon \circ \Phi)^{-1} = -(\bar\rho\,(1-\bar\tau))^{-2},$$

with the Nomizu-Kulkarni product of

$$p' = (h_{ab} \circ \Phi)\, d\,x^a\, d\,x^b - ((v\,\Omega)\circ\Phi)^2\, d\,t^2 = p'_1 + p'_2 + p'_3 + p'_4,$$

and $w' = w'_1 + w'_2 + w'_3 + w'_4 + w'_5$, where

$$p'_1 = -2\,((1-\hat\tau)\,d\,\bar\rho - \bar\rho\,d\,\bar\tau)^2,$$

$$p'_2 = ((1-\bar\tau)\,d\,\bar\rho - \bar\rho\,d\,\bar\tau) \otimes_s \left(\frac{(v\,\Omega)(\bar\rho\,(1-\bar\tau))}{(v\,\Omega)(\bar\rho)}\,d\,\bar\rho + (v\,\Omega)(\bar\rho\,(1-\bar\tau))\,l\right)$$

$$p'_3 = -\left(\frac{(v\,\Omega)(\bar\rho\,(1-\bar\tau))}{(v\,\Omega)(\bar\rho)}\,d\,\bar\rho + (v\,\Omega)(\bar\rho\,(1-\bar\tau))\,l\right)^2,$$

$$p'_4 = \bar\rho^2\,(1-\bar\tau)^2\,k,$$

$$w'_5 = -\left(\left\{\frac{m}{2}\,U\,(1+\sqrt{\mu\,\sigma})^2\, r_{ab}[h]\right\}\circ\Phi\right)d\,x^a\,d\,x^b,$$

and

$$w'_1 + w'_2 + w'_3 + w'_4 = -\left(\left\{\frac{m}{4}\,\frac{U}{\sigma}\,(2\,s\,h_{ab} - 3\,\sigma^{-1}\,D_a\,\sigma\,D_b\,\sigma)\right\}\circ\Phi\right)d\,x^a\,d\,x^b,$$

with

$$w'_1 = -\frac{m}{4}\left(\left\{\frac{U}{\sigma}(s + 6\,U^{-2})\right\} \circ \Phi\right) p'_1,$$

$$w'_2 = -\left(\left\{\frac{m}{2}\,U^3\,s\right\} \circ \Phi\right) k,$$

$$w'_3 = -\left(\left\{\frac{3\,m}{U\,\sigma}\,\Upsilon^{1/2}\right\} \circ \Phi\right)((1-\bar{\tau})\,d\,\bar{\rho} - \bar{\rho}\,d\,\bar{\tau}) \otimes_s j,$$

$$w'_4 = -\left(\left\{\frac{m}{U}\right\} \circ \Phi\right) j^2.$$

We used above the relation (6.31) and set

$$j = (D_a\,U \circ \Phi)\,d\,x^a.$$

The desired result on the behavior of the rescaled conformal Weyl tensor near $\bar{\mathcal{I}}'$ is obtained now by showing that for arbitrary frame vector fields v_n one has

$$\langle p' \oslash w';\, v_i,\, v_j,\, v_k,\, v_l, \rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2).$$

From (6.29) it follows that

$$p'_1 \oslash w'_1 = p'_1 \oslash w'_3 = p'_2 \oslash w'_1 = 0.$$

Observing (4.27) one finds by inspection

$$\langle p'_1;\, v_i,\, v_j\rangle = O(\bar{\rho}^2), \quad \langle p'_M;\, v_i,\, v_j\rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2), \quad \text{for } M = 2,3,4,$$

$$\langle w'_4;\, v_i,\, v_j\rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2), \quad \langle w'_N;\, v_i,\, v_j\rangle = O(1), \quad \text{for } N = 2,3,5,$$

and thus

$$\langle p'_M \oslash w'_N;\, v_i,\, v_j,\, v_k,\, v_l\rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2) \quad \text{for } M = 2,3,4, \quad N = 2,3,4,5,$$

$$\langle p'_1 \oslash w'_4;\, v_i,\, v_j,\, v_k,\, v_l\rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2).$$

The remaining term is given by

$$p'_1 \oslash (w'_2 + w'_5) + (p'_3 + p'_4) \oslash w'_1 = p'_1 \oslash m$$

with

$$m = w'_2 + w'_5 - \frac{m}{4}\left(\left\{\frac{U}{\sigma}(s + 6\,U^{-2})\right\} \circ \Phi\right)(p'_3 + p'_4)$$

$$= -\frac{m}{4}\left(\left\{U\,(3\,s\,U^2 + 6)\right\} \circ \Phi\right) k - \left(\left\{\frac{m}{2}\,U\,(1 + \sqrt{\mu\,\sigma})^2\,r_{ab}[h]\right\} \circ \Phi\right) d\,x^a\,d\,x^b$$

$$+\frac{m}{4}\left(\left\{s\,U^3 + 6\,U\right\} \circ \Phi\right)\left(\frac{(v\,\Omega)(\bar{\rho}\,(1-\bar{\tau}))}{\bar{\rho}\,(1-\bar{\tau})\,(v\,\Omega)(\bar{\rho})}\,d\,\bar{\rho} + \frac{(v\,\Omega)(\bar{\rho}\,(1-\bar{\tau}))}{\bar{\rho}\,(1-\bar{\tau})}\,l\right)^2$$

For the three summands to be considered here we get the following. From $3\,s\,U^2 + 6 = O(\Upsilon)$ it follows that

$$\langle p'_1 \oslash \left(\left\{U\,(3\,s\,U^2 + 6)\right\} \circ \Phi\right) k;\, v_i,\, v_j,\, v_k,\, v_l\rangle = O(\bar{\rho}^2\,(1-\bar{\tau})^2).$$

Because of

$$d\,x^{(a}\,d\,x^{b)} = \bar{\rho}^2\,(1-\bar{\tau})^2\,d\,e^{(a} \otimes d\,e^{b)}$$

$$-\frac{1}{2}\,p'_1\,e^a\,e^b + \bar{\rho}\,(1-\bar{\tau})\,e^{(a}\,d\,e^{b)} \otimes_s ((1-\hat{\tau})\,d\,\rho - \rho\,d\,\hat{\tau}),$$

it follows by (6.30) that

$$\langle p_1' \oslash \left(\left\{ \frac{m}{2} U (1 + \sqrt{\mu \sigma})^2 r_{ab}[h] \right\} \circ \Phi \right) d x^a \, d x^b ; v_i, v_j, v_k, v_l \rangle$$
$$= O(\bar{\rho}^2 (1 - \bar{\tau})^2).$$

It holds that $s U^3 + 6 U = O(1)$ and by inspection it follows that

$$\langle p_1' \oslash \left(\frac{(v \Omega)(\bar{\rho}(1 - \bar{\tau}))}{\bar{\rho}(1 - \bar{\tau})(v \Omega)(\bar{\rho})} d \bar{\rho} + \frac{(v \Omega)(\bar{\rho}(1 - \bar{\tau}))}{\bar{\rho}(1 - \bar{\tau})} l \right)^2 ; v_i, v_j, v_k, v_l \rangle$$
$$= O(\bar{\rho}^2 (1 - \bar{\tau})^2).$$

7. Static vacuum solutions near the cylinder at space-like infinity

The conformal extension considered in the previous section relies on specific features of static fields. We use it to show that the construction of the cylinder at space-like infinity in Section 5, which is based on general concepts and applies to general solutions, is for static vacuum solutions as smooth as can be expected.

Theorem 7.1. *For static vacuum solutions which are asymptotically flat the construction of Section* 5 *is analytic in the sense that in the frame* (5.57) *all conformal fields, including the rescaled conformal Weyl tensor, extend to analytic fields on some neighborhood $\mathcal{O}$ of $\bar{\mathcal{I}}$ in $\bar{\mathcal{N}}$. This statement does not depend on a particular choice of (analytic) scaling of the (analytic) free datum h on $\mathcal{S}$.*

This result will be obtained as a consequence of Lemmas 7.2, 7.3, and 7.4 below.

The construction of Section 5 will be discussed here for static solutions in terms of the initial data h and Ω in the gauge given by (4.31), and the field $\tilde{g}$ given on $\tilde{\mathcal{M}}'$ in the coordinates defined by (6.1). The effect of a rescaling of h will be discussed separately because it is of interest in itself.

The conformal factor Θ is assumed in the form (5.44), (5.46) with

$$\kappa = \omega = 2 \Omega \, |D_a \Omega D^a \Omega|^{-\frac{1}{2}} = 2 \Omega \, (1 + \sqrt{\mu \sigma})^{-3} \sqrt{2 \, |s| \, \sigma}. \tag{7.1}$$

It follows that $\omega = \Upsilon^{1/2} + O(\Upsilon)$ and $\Theta = 1/2 \, |D_a \Omega D^a \Omega|^{\frac{1}{2}} = \Upsilon^{1/2} + O(\Upsilon)$ so that

$$\text{on } \tilde{\mathcal{S}}: \quad \lim_{\bar{\rho} \to 0} \Upsilon^{1/2} \omega^{-1} = \lim_{\bar{\rho} \to 0} \omega \, \Lambda^{-1} = \lim_{\bar{\rho} \to 0} \Theta \, \Lambda^{-1} = 1. \tag{7.2}$$

The metric $\bar{h}$ induced by $g = \Theta^2 \tilde{g}$ on $\tilde{\mathcal{S}}$ is given by $\omega^{-2} h$.

The main ingredient of the gauge for the evolution equations used in Section 5.2 are the conformal geodesics generating the conformal Gauss system described in Section 2.1. We shall try to control their evolution on $\tilde{\mathcal{M}}'$ near $\bar{\mathcal{I}}'$. Following the prescription in Section 2.1, we assume that the tangent vectors $\dot{x} = dx/d\tau$ of the conformal geodesics with parameter τ satisfy

$$\dot{x} \perp \tilde{\mathcal{S}}, \quad \Theta^2 \tilde{g}(\dot{x}, \dot{x}) = 1 \quad \text{on} \quad \tilde{\mathcal{S}}. \tag{7.3}$$

With the frame (6.7) and the coordinates (6.1) this translates into the initial condition

$$\dot{x} = \frac{\omega(\bar{\rho})}{\bar{\rho}}\,\partial_{\bar{\tau}} + \omega(\bar{\rho})\,\partial_{\bar{\rho}} = \frac{\omega(\bar{\rho})}{\bar{\rho}}\,(v_0 + v_1) \equiv X^i\,v_i \quad \text{at} \quad \bar{\tau} = 0, \tag{7.4}$$

with

$$X^i = \delta^i{}_0 + \delta^i{}_1 + O(\bar{\rho}) \ \text{ as } \bar{\rho} \to 0.$$

For the following we need to observe besides $\bar{g} = \Lambda^2\,\tilde{g}$ the relations

$$g = \Pi^2\,\bar{g} = \Theta^2\,\tilde{g}, \quad \text{with} \quad \Pi \equiv \Lambda^{-1}\,\Theta.$$

For the connections $\tilde{\nabla}$, ∇, and $\bar{\nabla}$ of $\tilde{g}$, g, and $\bar{g}$ respectively we have relations

$$\hat{\nabla} = \tilde{\nabla} + S(\tilde{f}), \qquad \hat{\nabla} = \nabla + S(f), \qquad \hat{\nabla} = \bar{\nabla} + S(\bar{f}),$$

$$\nabla = \tilde{\nabla} + S(\Theta^{-1}\,d\,\Theta),, \qquad \bar{\nabla} = \tilde{\nabla} + S(\Lambda^{-1}\,d\,\Lambda).$$

The comparison gives $f = \tilde{f} - \Theta^{-1}\,d\Theta$ and $\bar{f} = \tilde{f} - \Lambda^{-1}\,d\,\Lambda$ which imply the relation

$$\bar{f} = f + \Theta^{-1}\,d\,\Theta - \Lambda^{-1}\,d\,\Lambda = f + \Pi^{-1}\,d\Pi, \tag{7.5}$$

between the 1-form $\bar{f}$ which is obtained if the conformal geodesic equations are written in terms of the metric $\bar{g}$, the 1-form f which is supplied by the conformal geodesic equations written in terms of the metric g, and the conformal factor which relates $\bar{g}$ to g.

By the choices of Section 5.2 we have $\langle f, \dot{x}\rangle = 0$ everywhere on the space-time and $\langle d\,\Theta, \dot{x}\rangle = 0$ on $\tilde{S}$. Since Λ has been chosen to be independent of t and ∂_t is orthogonal $\tilde{S}$, it follows that $\langle d\,\Lambda, \dot{x}\rangle = 0$ and thus $\langle \bar{f}, \dot{x}\rangle = 0$ on $\tilde{S}$. Observing the pull-back of f to $\tilde{S}$ given by (5.45) and the relation

$$\Pi = \Upsilon^{1/2}\,\omega^{-1} \quad \text{on} \quad \tilde{S}, \tag{7.6}$$

we find that the pull-back of $\bar{f}$ to $\tilde{S}$ is given by $1/2\ \Upsilon^{-1}\,d\,\Upsilon$. From this one gets in the frame (6.7) and the coordinates (6.1) with $\bar{\tau} = 0$

$$\bar{f} = (1/2\ \Upsilon^{-1}\,D_a\,\Upsilon) \circ \Phi\ dx^a = \bar{f}_i\,\alpha^i \quad \text{with} \quad \bar{f}_i = -\delta^0{}_i + \delta^1{}_i \quad \text{on} \quad \tilde{S}. \tag{7.7}$$

The relation $\langle f, \dot{x}\rangle = 0$ and equation (7.5) imply the ODE

$$\dot{\Pi} = \Pi\langle \bar{f}, \dot{x}\rangle, \tag{7.8}$$

along the conformal geodesics, which, together with (7.6), will allow one to determine Π once $\langle \bar{f}, \dot{x}\rangle$ is known.

7.1. The extended conformal geodesic equation on I

With respect to the metric (6.6) a solution to the conformal geodesic equations is given by a space-time curve $x^\mu(\tau) = (\bar{\tau}(\tau), \bar{\rho}(\tau), \psi^A(\tau))$ and along that curve a vector field $X(\tau)$ and a 1-form $\bar{f}(\tau)$ such that

$$\dot{x} = X,$$
$$\bar{\nabla}_X X = -2\langle \bar{f}, X\rangle X + \bar{g}(X,X)\,\bar{f}^\sharp,$$
$$\bar{\nabla}_X \bar{f} = \langle \bar{f}, X\rangle \bar{f} - \frac{1}{2}\,\bar{g}(\bar{f},\bar{f})\,X^\flat + L(X,\,.\,).$$

With the expansions $X = X^i\,v_i$, $\bar{f} = \bar{f}_i\,\alpha^i$, $\bar{g} = \bar{g}_{jk}\,\alpha^j\,\alpha^k$, $\bar{g}^\sharp = \bar{g}^{jk}\,v_j\,v_k$, $L = L_{jk}\,\alpha^j\,\alpha^k$, $e_k = e^i{}_k\,v_i$, the equations above take in the domain where $\bar{\rho} > 0$ the form

$$\frac{d}{d\tau}\,\bar{\tau} = X^0, \quad \frac{d}{d\tau}\,\bar{\rho} = \bar{\rho}\,X^1, \quad \frac{d}{d\tau}\,\psi^A = X^A,$$

which is the equation $\dot{x}^\mu = X^i\,v^\mu{}_i$, relating the coordinate to the frame expressions,

$$\frac{d}{d\tau}\,X^i + \gamma_j{}^i{}_k\,X^j\,X^k = -2\,\bar{f}_k\,X^k\,X^i + \bar{g}_{jk}\,X^j\,X^k\,\bar{g}^{il}\,\bar{f}_l,$$
$$\frac{d}{d\tau}\,\bar{f}_k - \gamma_j{}^i{}_k\,\xi^j\,\bar{f}_i = \bar{f}_l\,X^l\,\bar{f}_k - \frac{1}{2}\,\bar{g}^{lj}\,\bar{f}_l\,\bar{f}_j\,\bar{g}_{lk}\,X^l + L_{jk}\,X^j.$$

Note that the functions $\bar{g}_{jk}$, $\bar{g}^{il}$, $\gamma_j{}^i{}_l$, L_{jk} entering these equations extend by analyticity through $\bar{\mathcal{I}}'$ into a domain where $\bar{\rho} < 0$. Assuming such an extension, we get the *extended conformal geodesic equations.* Since also the data are analytic on $\bar{\mathcal{S}}$, it makes sense to consider these equations in a neighborhood of $\bar{\mathcal{I}}'$.

Lemma 7.2. *With the values of L_{jk} on $\bar{\mathcal{I}}'$ found in Lemma* 6.1, *the initial data $x = (0, 0, \psi^{A'})$ and (cf.* (7.4), (7.7)*) $X^i = \delta^i{}_0 + \delta^i{}_1$, $\bar{f}_i = -\delta^0{}_i + \delta^1{}_i$ on $\mathcal{I}^{0'}$ determine a solution $x(\tau)$, $X(\tau)$, $\bar{f}(\tau)$ of the extended conformal geodesic equations with $\tau = 0$ on $\mathcal{I}^{0'}$ and*

$$x(\tau) = (\bar{\tau}(\tau), \bar{\rho}(\tau), \psi^A(\tau)) = (\tau, 0, \psi^{A'}).$$

By analyticity it extends as a solution into a domain $0 \le \tau \le 1 + 2\,\epsilon$ for some $\epsilon > 0$. The extension to $\bar{\mathcal{I}}'$ of the conformal factor Π *which is determined by* (7.6) *and* (7.8) *takes the value* $\Pi = 1$ *on $\bar{\mathcal{I}}'$.*

Proof. With the ansatz $x(\tau) = (\bar{\tau}(\tau), 0, \psi^{A'})$, $X(\tau) = X^0(\tau)\,v_0 + X^1(\tau)\,v_1$, $\bar{f} = \bar{f}_0(\tau)\,\alpha^0 + \bar{f}_1(\tau)\,\alpha^1$ those of the extended conformal geodesic equations which are not identically satisfied because of (6.8), (6.9), (6.11) are given by

$$\frac{d}{d\tau}\,\bar{\tau} = X^0,$$

$$\frac{d}{d\tau}\,X^0 + 2\,\bar{\tau}\,X^0\,X^1 - \bar{\tau}\,(1-\bar{\tau}^2)\,X^1\,X^1$$
$$= -2\,(\bar{f}_0\,X^0 + \bar{f}_1\,X^1)\,X^0 + (2\,X^0\,X^1 - (1-\bar{\tau}^2)X^1\,X^1)\,((1-\bar{\tau}^2)\,\bar{f}_0 + \bar{f}_1),$$

$$\frac{d}{d\tau}\,X^1 - \bar{\tau}\,X^1\,X^1 = -2\,(\bar{f}_0\,X^0 + \bar{f}_1\,X^1)\,X^1 + (2\,X^0\,X^1 - (1-\bar{\tau}^2)X^1\,X^1)\,\bar{f}_0,$$

$$\frac{d}{d\tau}\bar{f}_0 - \bar{\tau}\,\bar{f}_0\,X^1 = (\bar{f}_0\,X^0 + \bar{f}_1\,X^1)\,\bar{f}_0 - \frac{1}{2}\left((1-\bar{\tau}^2)\,\bar{f}_0\,\bar{f}_0 + 2\,\bar{f}_0\,\bar{f}_1\right)X^1 + \frac{1}{2}\,X^1,$$

$$\frac{d}{d\tau}\bar{f}_1 - \bar{\tau}\,\bar{f}_0\,(X^0 - (1-\bar{\tau}^2)\,X^1) + \bar{\tau}\,\bar{f}_1\,X^1 = (\bar{f}_0\,X^0 + \bar{f}_1\,X^1)\,\bar{f}_1$$

$$-\frac{1}{2}\left((1-\bar{\tau}^2)\,\bar{f}_0\,\bar{f}_0 + 2\,\bar{f}_0\,\bar{f}_1\right)(X^0 - (1-\bar{\tau}^2)\,X^1) + \frac{1}{2}\,X^0 - \frac{1-\bar{\tau}^2}{2}\,X^1.$$

A calculation shows that the solution of this system for the prescribed initial is given by

$$\bar{\tau} = \tau, \quad X^0 = 1, \quad X^1 = \frac{1}{1+\bar{\tau}}, \quad \bar{f}_0 = -\frac{1}{1+\bar{\tau}}, \quad \bar{f}_1 = 1. \tag{7.9}$$

This proves the first assertion. With the solution above equation (7.8) reads $\dot{\Pi} = 0$ and we have $\Pi = 1$ on I^0 by (7.2). This proves the second assertion.

Remark: The ODE above is sufficiently complicated so that giving the solution explicitly deserves an explanation. In (5.62) is given the conformal factor and the conformal representation of Minkowski space which result from the general procedure of Section 5. In (5.63) is given the coordinate transformation which, together with the conformal factor, relates the conformal metric to the standard representation of Minkowski space in coordinates t and r.

If the Minkowski values $m = 0$, $U = 1$, $h_{ab} = -\delta_{ab}$ are assumed in Section 6 the metric $\bar{g}$ reduces by (6.8) to the metric $g^*_{ik}\,\alpha^i\,\alpha^k$ with $m = 0$. One can consider this as the lowest order (in $\bar{\rho}$) approximation of the general version of $\bar{g}$. Tracing back how the functions $\bar{\tau}$, $\bar{\rho}$, Λ in Section 6 are related in the flat case to t and r, one finds

$$r = \frac{1}{\bar{\rho}\,(1-\bar{\tau})}, \quad t = \frac{\bar{\tau}}{\bar{\rho}\,(1-\bar{\tau})}, \quad \Lambda = \frac{1}{r}.$$

The conformal factors in the conformal representations thus agree but the coordinates are related by the transformation

$$\bar{\tau} = \tau, \quad \bar{\rho} = \rho\,(1+\tau). \tag{7.10}$$

This implies

$$2\,\frac{d\bar{\rho}}{\bar{\rho}}\,d\bar{\tau} - (1-\bar{\tau}^2)\left(\frac{d\bar{\rho}}{\bar{\rho}}\right)^2 - d\sigma^2 = d\tau^2 + 2\,\tau\,\frac{d\rho}{\rho}\,d\tau - (1-\tau^2)\left(\frac{d\rho}{\rho}\right)^2 - d\sigma^2. \tag{7.11}$$

The left-hand side is the conformal Minkowski metric (6.8) with $m = 0$ while the right-hand side is the metric g^* given by (5.62). The conformal geodesics underlying (5.62) have tangent vector $X = \partial_\tau$ and 1-form $f = \frac{d\rho}{\rho}$. With (7.10) these transform into

$$X = \partial_{\bar{\tau}} + \frac{1}{1+\bar{\tau}}\,\bar{\rho}\,\partial_{\bar{\rho}} = v_0 + \frac{1}{1+\bar{\tau}}\,v_1,$$

$$f = \frac{d\bar{\rho}}{\bar{\rho}} - \frac{1}{1+\bar{\tau}}\,d\bar{\tau} = -\frac{1}{1+\bar{\tau}}\,\alpha_0 + \alpha_1,$$

from which one can read off (7.9).

The hypersurfaces $\{\bar\rho = \bar\rho_{\#} = \text{const.} > 0\}$ are in general time-like for the metric (6.6). The form of $\bar g^\natural$ suggests that these hypersurfaces approximate null hypersurfaces in the limit as $\bar\rho_{\#} \to 0$, but the conclusion is delicate because of the degeneracy of $\bar g^\natural$ on $\bar{\mathcal{I}}'$. The discussion above shows that they do become null asymptotically in the sense that for the metric on the left-hand side of (7.11) the hypersurfaces $\{\bar\rho = \text{const.}\}$ are in fact null. To some extent this explains why the coordinates given by (6.1) had a chance to extend smoothly to $\mathcal{J}^+$ and to provide a description of the cylinder at space-like infinity.

7.2. The smoothness of the gauge of Section 5 for static asymptotically flat vacuum solution near $\mathcal{I}$

Let $\bar{\mathcal{S}}_{\text{ext}}$ denote an analytic extension of $\tilde{\mathcal{S}}$ into a range where $\bar\rho < 0$ so that $\bar\rho$, ψ^A extend to analytic coordinates. If the set $\bar{\mathcal{S}}_{\text{ext}} \setminus \bar{\mathcal{S}}$ is sufficiently small, the following statements make sense. The initial conditions (7.4), (7.7) extend analytically to $\bar{\mathcal{S}}_{\text{ext}}$ and determine near $\bar{\mathcal{S}}_{\text{ext}}$ an analytic congruence of solutions to the extended conformal geodesic equations. It therefore follows from Lemma 7.2 and well-known results on ODE's that, with the ϵ of Lemma 7.2, there exists a $\rho_{\#} > 0$ such that for initial data $\bar\tau(0) = 0$, $\bar\rho(0) = \rho'$ with $|\rho'| < \rho_{\#}$, $\psi^A(0) = \psi^{A'}$ and those implied at these points by (7.4), (7.7) the solution

$$\bar\tau = \bar\tau(\tau, \rho', \psi^{A'}), \quad \bar\rho = \bar\rho(\tau, \rho', \psi^{A'}), \quad \psi^A = \psi^A(\tau, \rho', \psi^{A'}),$$

$$X^i = X^i(\tau, \rho', \psi^{A'}), \quad \bar f_k = \bar f_k(\tau, \rho', \psi^{A'}),$$

of the extended conformal geodesic equations exists for the values $0 \le \tau \le 1 + \epsilon$ of their natural parameter and the function Π is positive in the given range of ρ' and τ.

Taking a derivative of the equation satisfied by $\bar\rho$ and observing (7.9) gives

$$\frac{d}{d\tau}\left(\frac{\partial\bar\rho}{\partial\rho'}|_{\rho'=0}\right) = \left(\frac{\partial\bar\rho}{\partial\rho'}|_{\rho'=0}\right)\frac{1}{1+\bar\tau},$$

which implies by (7.9)

$$\left(\frac{\partial\bar\rho}{\partial\rho'}|_{\rho'=0}\right) = 1 + \tau \ge 1.$$

It follows that the Jacobian of the analytic map

$$(\tau, \rho', \psi^{A'}) \to x^\mu(\tau, \rho', \psi^{A'}),$$

takes the value $1 + \bar\tau$ on $\bar{\mathcal{I}}'$ and for sufficiently small $\rho_{\#} > 0$ the Jacobian does not vanish in the range $0 \le \tau \le 1 + \epsilon$, $|\bar\rho| \le \rho_{\#}$. The relations $\Lambda = \Theta\,\Pi^{-1}$, $\Pi > 0$, and $\Theta = (\omega^{-1}\,\Omega)_*\,(1 - \tau^2)$ imply that the curves with $\rho' > 0$ cross $\mathcal{J}^{+'}$ for $\tau = 1$. It follows that τ, ρ', and $\psi^{A'}$ define an analytic coordinate system in a certain neighborhood $\mathcal{O}'$ of $\bar{\mathcal{I}}'$ in $\mathcal{M}'$, such that (suppressing again the upper bounds for ρ') $\mathcal{O}' \cap \mathcal{J}^{+'} = \{\tau = 1,\ \rho' > 0\}$, $\mathcal{I}' = \{0 \le \tau < 1,\ \rho' = 0\}$, $\mathcal{I}^{+'} = \{\tau = 1,\ \rho' = 0\}$, and $\mathcal{O}'$ is ruled by conformal geodesics.

The metric $g = \Pi^2\,\bar{g}$, the connection coefficients of the connection $\hat{\nabla}$ and the tensor fields (cf. (2.11))

$$\hat{L}_{\mu\nu} = L_{\mu\nu}[\bar{g}] - \nabla_\mu\,\bar{f}_\nu + \bar{f}_\mu\,\bar{f}_\nu - \frac{1}{2}\,\bar{g}_{\mu\nu}\,\bar{f}_\lambda\,\bar{f}^\lambda,$$
$$f = \bar{f} - \Pi^{-1}\,d\,\Pi, \quad W_{\mu\nu\rho\lambda}[g] = \Pi\,W_{\mu\nu\rho\lambda}[\bar{g}].$$

in the frame (6.7) extend in the new coordinates as analytic fields to $\mathcal{O}'$.

Given these structures and the conformal geodesics on $\mathcal{O}'$, the construction of the manifold $\bar{\mathcal{N}}$ as described in Section 5 poses no problems. With the given analytic initial data on $\bar{\mathcal{S}}$ it only involves solving linear ODE's corresponding to (2.21), such as

$$\frac{d}{d\tau}\,e^i{}_k + \gamma_j{}^i{}_l\,X^j\,e^l{}_k = -\bar{f}_l\,X^l\,e^i{}_k - \bar{f}_l\,e^l{}_k\,X^i + \bar{g}_{jl}\,X^j\,e^l{}_k\,\bar{g}^{im}\,\bar{f}_m, \qquad (7.12)$$

or its spinor analogue, along the conformal geodesics. This allows us to conclude

Lemma 7.3. *Starting with static asymptotically flat initial data in the gauge* (4.31), *the construction of Section* 5 *leads to a conformal representation of the static vacuum space-time which is real analytic in a neighborhood* $\mathcal{O}$ *of the set* $\bar{\mathcal{I}}$ *in* $\bar{\mathcal{N}}$.

7.3. Changing the conformal gauge on the initial slice

It will be shown now how the construction described in Section 5 depends for static vacuum solutions on rescalings

$$\omega^{-2}\,h \to h' = \vartheta^2\,\omega^{-2}\,\bar{h}, \quad \Omega \to \Omega' = \vartheta\,\Omega \quad \text{on} \quad \mathcal{S},$$

with analytic, positive conformal factors ϑ.

There are harmless consequences such as the change of the normal coordinates $x^a \to x'^a = x'^a(x^c)$ with $x'^a(0) = 0$ and a related change $e_a \to e'_a = \vartheta^{-1}\,s^c{}_a\,e_c$ of the frame vector fields tangent to $\tilde{\mathcal{S}}$. Here $s^c{}_a$ denotes an analytic function on $\tilde{\mathcal{S}}$ with values in $SO(3)$ such that $s^c{}_a \to \delta^c{}_a$ as $\bar{\rho} \to 0$. These changes will simply be propagated along the new conformal geodesics.

Critical is the transition from the congruence of conformal geodesics related to Ω (the *Ω-congruence*) to the new one related to Ω' (the *Ω'-congruence*). If the curves are considered as point sets, the two families of curves will be different if $\Omega'^{-1}\,d\,\Omega' - \Omega^{-1}\,d\,\Omega = \vartheta^{-1}\,d\,\vartheta \neq 0$ (cf. [40]).

The rescaling above implies on $\tilde{\mathcal{S}}$ the transitions

$$||d\,\Omega||_h \to ||d\,\Omega'||_{h'} = \xi\;||d\,\Omega||_h,$$
$$\omega = \frac{2\,\Omega}{||d\,\Omega||_h} \to \omega' = \frac{2\,\Omega'}{||d\,\Omega'||'_h} = \frac{\omega\,\delta}{\xi},$$
$$\Theta|_{\tilde{\mathcal{S}}} = \omega^{-1}\,\Omega \to \Theta'|_{\tilde{\mathcal{S}}} = \omega'^{-1}\,\Omega' = \xi\;\Theta|_{\tilde{\mathcal{S}}},$$

with the function

$$\xi = \left|1 - 3\,\vartheta^{-1}\,\frac{D_a\Omega\,D^a\vartheta}{\Delta_h\Omega} - \frac{3}{2}\,\vartheta^{-2}\Omega\,\frac{D_a\vartheta\,D^a\vartheta}{\Delta_h\Omega}\right|^{\frac{1}{2}},$$

which extends to $\bar{\mathcal{S}}$ as an analytic function of $\bar{\rho}$ and ψ^A.

It follows from the initial conditions for the Ω-congruence that

$$\xi^{-1}\,\dot{x} \perp \tilde{\mathcal{S}}, \qquad \Theta'^{2}\,\tilde{g}(\xi^{-1}\,\dot{x}, \xi^{-1}\,\dot{x}) = 1, \tag{7.13}$$

and for the transformed 1-form that

$$\langle f', \dot{x} \rangle = 0, \qquad f_{\tilde{\mathcal{S}}} \to f'_{\tilde{\mathcal{S}}} = \omega'^{-1}\, d\omega' = f_{\tilde{\mathcal{S}}} + \vartheta^{-1}\, d\vartheta - \xi^{-1}\, d\xi,$$

where the subscripts indicate the pull-back to $\tilde{\mathcal{S}}$. These two lines give the initial data for the Ω'-congruence if the conformal geodesic equations are expressed with respect to the rescaled metric g' and its connection ∇'.

To compare the Ω'-congruence with the Ω-congruence we observe the conformal invariance of conformal geodesics (cf. [38]) and express the equations for the Ω'-congruence in terms of g and its connection ∇. The space-time curves, including their parameter τ', then remain unchanged. The 1-form is transformed because of $g = (\Theta\,\Theta'^{-1})^2\, g'$ according to $f' \to f^* = f' - (\Theta\,\Theta'^{-1})^{-1}\, d\,(\Theta\,\Theta'^{-1})$, which implies $\langle f^*, \dot{x} \rangle = 0$, $f^*_{\tilde{\mathcal{S}}} = \bar{f}_{\tilde{\mathcal{S}}} + \vartheta^{-1}\, d\vartheta$ on $\tilde{\mathcal{S}}$. If this 1-form is expressed in terms of the g-orthonormal frame e_k with $e_0 \perp \tilde{\mathcal{S}}$, one finds

$$f^*_0 \equiv \langle f^*, e_0 \rangle = 0, \qquad f^*_a \equiv \langle f^*, e_a \rangle = f_a + \vartheta^{-1} \langle d\vartheta, e_a \rangle \;\; a = 1, 2, 3. \tag{7.14}$$

The fields $\xi^{-1}\,\dot{x}$, f^*_k are the initial data for the Ω'-congruence in terms of g, e_k, and ∇. Since $\xi \to 1$ and $\langle d\vartheta, e_a \rangle = O(\bar{\rho})$ as $\bar{\rho} \to 0$, it follows that

$$\frac{\Theta'}{\Theta} \to 1, \qquad \dot{x} - \xi^{-1}\,\dot{x} \to 0, \qquad f^*_k - f'_k \to 0 \;\; \text{as } \bar{\rho} \to 0.$$

As a consequence, the initial data for the Ω'- and the Ω-congruence have coinciding limits on $\mathcal{I}^{0'}$ and the corresponding curves are identical on $\bar{\mathcal{I}}'$.

Assuming now the conditions of Section 7.2 and using arguments similar to the ones used there, we conclude that in a certain neighborhood $\mathcal{O}'$ of $\bar{\mathcal{I}}'$ in $\tilde{\mathcal{M}}'$ the gauge related to the Ω'-congruence is as smooth and regular as the one related to the Ω-congruence. Thus we have

Lemma 7.4. *In the case of static asymptotically flat space-times the construction of the set $\bar{\mathcal{I}}'$ is independent of the choice of Ω and the set $\mathcal{I}'$ introduced in Section* 6 *coincides with the projection $\pi'(\mathcal{I})$ of the cylinder at space-like infinity as defined in Section* 5.

We note that the comparison of the Ω'- with the Ω-congruence leads in the case where the solution is not static and thus not necessarily analytic still to similar results if the solution acquires a certain smoothness near $\mathcal{J}^{\pm} \cup \mathcal{I}^{\pm}$. In the case of low smoothness, however, the detailed behavior of the different congruences needs to be analyzed in the context of an existence theorem.

8. Concluding remarks

Concerning the regularity conditions we have now the following situation. For *static* asymptotically flat solutions with $m \neq 0$ the conformal extensions to $\bar{\mathcal{I}}$ are smooth (in the sense discussed above) and their data satisfy the regularity condition (5.89) with $p_* = \infty$. In the *massless case* condition (5.89) with $p_* = \infty$ is necessary and sufficient for space-like infinity to be represented by a regular point in a smooth conformal extension. In the *general time reflection case with* $m \neq 0$ conditions (5.89) are necessary but not sufficient for the s-jets $J^p_{\mathcal{I}}(u)$, $p \in \mathbb{N}$, to be regular at the critical sets $\mathcal{I}^{\pm}$. Thus, the mass $m = 2\,W(i)$ and also the derivatives of $\partial^{\alpha}_{x^a} W(i)$, $\alpha \in \mathbb{N}^3$, play a crucial role for the behavior of the $J^p_{\mathcal{I}}(u)$ at $\mathcal{I}^{\pm}$. The mechanism which decides on the smoothness remains to be understood.

Only the d-jet $J^p_{\mathcal{I}^0}(u)$ and the s-jets $J^{p-1}_{\mathcal{I}}(u)$ are needed to obtain $J^p_{\mathcal{I}}(u)$ by integrating the transport equations on $\mathcal{I}$. Since the left-hand sides of the transport equations are universal in the sense that they do not depend on the data, it follows that $J^p_{\mathcal{I}}(u)$ is uniquely determined by $J^p_{\mathcal{I}^0}(u)$ for $p \in \mathbb{N}$. In the static case the s-jets $J^p_{\mathcal{I}}(u)$ are regular. In [16] has been exhibited a class of data which are asymptotically static of order p for given $p \in \mathbb{N} \cup \{\infty\}$ and which are essentially arbitrary on given compact sets. It follows that for prescribed differentiability order p there exists a large class of data for which the s-jet $J^p_{\mathcal{I}}(u)$ is regular on $\mathcal{I}$.

We expect there to be a threshold in p beyond which the regularity of $J^p_{\mathcal{I}}(u)$ ensures peeling resp. asymptotic smoothness of a given order of differentiability and below which the singularity of $J^p_{\mathcal{I}}(u)$ implies a failure of peeling. This order is likely to be low enough such that the behavior of $J^q_{\mathcal{I}}(u)$ with $q \leq p$ can be controlled by a direct, though tedious, calculation. However, if asymptotic staticity does play a role here, one should try to understand the underlying mechanism. It would be quite a remarkable feature of Einstein's equations if asymptotic staticity could be *deduced* from asymptotic regularity at null infinity.

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