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The Complete One-Loop Dilatation Operator of $\mathcal{N}=4$ Super Yang-Mills Theory

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Abstract

We continue the analysis of hep-th/0303060 in the one-loop sector and present the complete $\mathfrak{psu}(2,2|4)$ dilatation operator of $\mathcal{N}=4$ Super Yang-Mills theory. This operator generates the matrix of one-loop anomalous dimensions for all local operators in the theory. Using an oscillator representation we show how to apply the dilatation generator to a generic state. By way of example, we determine the planar anomalous dimensions of all operators up to and including dimension 5.5, where we also find some evidence for integrability. Finally, we investigate a number of subsectors of $\mathcal{N}=4$ SYM in which the dilatation operator simplifies. Among these we find the previously considered $\mathfrak{so}(6)$ and $\mathfrak{su}(2)$ subsectors, a $\mathfrak{su}(2|4)$ subsector isomorphic to the BMN matrix model at one-loop, a $\mathfrak{u}(2|3)$ supersymmetric subsector of nearly eighth-BPS states and, last but not least, a non-compact $\mathfrak{sl}(2)$ subsector whose dilatation operator lifts uniquely to the full theory.

1 Introduction and conclusions

The AdS/CFT correspondence [1] and some of its recent advances involving plane waves and spinning string solutions [2–4] have attracted attention to the investigation of anomalous dimensions in $\mathcal{N}=4$ Super Yang-Mills theory [5]. The virtue of this particular theory is that it has a conformal phase which is not spoiled by quantum effects [6]. In fact, supersymmetry merges with conformal symmetry to the symmetry group $\mathrm{PSU}(2,2|4)$. Composite local operators of the $\mathcal{N}=4$ gauge theory therefore arrange into multiplets or modules of the algebra $\mathfrak{psl}(4|4)$ [1] [7]. The modules are characterised by a set of numbers which determine the transformation properties under $\mathfrak{psl}(4|4)$. All of these numbers are (half-)integer valued, except the scaling dimension, which may receive corrections due to quantum effects, the so-called anomalous dimensions.

In the free theory, the action of the algebra on the space of local operators closes in a rather trivial manner. When quantum corrections are switched on, however, the transformation properties of operators change in such a way that the closure of the algebra is preserved. This puts tight constraints on consistent deformations of the algebra generators. In fact, these constraints have been employed to determine the first few anomalous dimensions [8]. A similar technique has later on been applied to obtain a remarkable all-loop result for an anomalous dimension [9]. There is some hope that consistency requirements uniquely fix the deformations to a one-parameter family representing different values of the coupling constant g_{YM}.

One way to obtain a consistent and interesting deformation of generators clearly is to evaluate the deformations due to loop effects in $\mathcal{N}=4$ SYM. In a perturbative quantum field theory this is however a non-trivial issue as scaling dimensions are superficially fixed to their free theory values due to an absence of a compensating scale. Nevertheless, this scale must be introduced in any attempt to regulate the divergencies of quantum field theories. In the regularised theory, the infinities give rise to logarithms, which can be interpreted as small shifts of scaling dimensions. In that sense anomalous dimensions are intimately related to divergencies. Although the divergencies are the source for anomalous dimensions, they are also a major complication in higher loop computations.

The first few anomalous dimensions that have been calculated directly in $\mathcal{N}=4$ SYM and up to two-loops [10] turned out to be zero: The considered operators were BPS operators whose scaling dimensions are required to be protected in a $\mathfrak{psl}(4|4)$ symmetric theory, thus confirming the superconformal nature of $\mathcal{N}=4$ SYM. Besides the BPS operators there are further operators for which exceptional non-renormalisation theorems have been found [11]. Non-vanishing anomalous dimensions up to two-loops have subsequently been calculated for the most simple, non-protected operator, the Konishi operator [12, 11]. The obtained anomalous dimension was in agreement with the earlier, purely algebraic results. The complete tower of twist-two operators of which the Konishi operator is the lowest example was investigated in [13]. Their one-loop anoma-

[1] The conjugation property of the algebras will not be relevant here and we will always consider the complex versions. When we consider unitarity bounds, we will tacitly refer to the real form $\mathfrak{psu}(2,2|4)$.

lous dimensions were found to be $2n + (g_{\rm YM}^2 N/2\pi^2)h(2n)$, where $h(j)$ are the harmonic numbers

$$h(j) := \sum_{k=1}^{j} \frac{1}{k} = \Psi(j+1) - \Psi(1), \tag{1.1}$$

which can also be expressed in terms of the digamma function $\Psi(x) = \partial \log \Gamma(x)/\partial x$. Using methods of QCD deep inelastic scattering, the DGLAP and BFKL equations, as well as by means of computer algebra systems developed for higher loop computations in the standard model, this result has been generalised to two-loops [14]. Systematic means to obtain anomalous dimensions involving four-point functions and superspace techniques were worked out in, e.g. [15, 16]. Finally, multi-trace operators and operators which mix in a non-trivial way have been investigated in [17].

With the advent of the BMN correspondence [3] the attention has been shifted away from lower dimensional operators to operators with a large number of fields [18–20]. There, the complications are mostly of combinatorial nature. It was therefore desirable to develop efficient methods to determine anomalous dimensions without having to deal with artefacts of the regularisation procedure. In a purely algebraic way the dilatation operator [21, 22] generates the matrix of one-loop anomalous dimensions for any set of operators which are made out of the six scalars of $\mathcal{N} = 4$ SYM. What is more, the matrix of anomalous dimensions can be obtained exactly for all gauge groups and, in particular, for groups $\mathrm{SU}(N)$ with finite N. Even two or higher-loop calculations of anomalous dimensions, which are generelly plagued by multiple divergencies, are turned into a combinatorial excercise! Using the dilatation generator techniques many of the earlier case-by-case studies of anomalous dimensions were easily confirmed. They furthermore enabled a remarkable all-genus comparison between BMN gauge theory and plane-wave string theory [23].

Much progress has been made in recent months due to integrable structures discovered in planar gauge theory and free string theory. Minahan and Zarembo realised that the planar one-loop dilatation operator for scalar operators is isomorphic to the Hamiltonian of a $\mathfrak{so}(6)$ integrable spin chain [24]. Furthermore, there are indications that this remarkable result generalises to higher loops [22] potentially giving rise to a novel type of integrable model. Using the Bethe ansatz [24] is was possible to find states whose anomalous dimension can only be expressed in terms of elliptic functions [25]. Astonishingly, corresponding string states of the same energy could be found using semi-classical methods [26]. This matching might be related to an integrable structure found for free strings on $AdS_5 \times S^5$ [27]. It represents one of the strongest confirmations of the AdS/CFT correspondence so far.

Most investigations of anomalous dimensions have focussed on operators made out of scalars. For these the Feynman diagrams are comparatively easy to calculate. In a few exceptions [28–30] scalars with covariant derivatives have been considered. These calculations are notoriously difficult due to the complex index structures and terms that are required by conformal covariance of the correlators. Calculations involving fermions and field strengths have mostly been avoided, however, with the aid of computer algebra they are feasible [14]. All in all not much is known about such operators, except maybe by means of supersymmetry arguments [31].

In the current work we would like to address the issue of generic operators of $\mathcal{N}=4$ SYM and their one-loop anomalous dimensions. We derive the *complete* one-loop non-planar dilatation operator of $\mathcal{N}=4$ SYM (2.22). For simplicity we will refer to this as the *Hamiltonian*[2] H and define

$$D(g) = D_0 + \frac{g_{\scriptscriptstyle\mathrm{YM}}^2 N}{8\pi^2}\, H + \mathcal{O}(g^3). \tag{1.2}$$

First of all we will investigate the form of generic local operators and how the Hamiltonian acts on them. Considering the structure of one-loop Feynman diagrams it is easy to see that the Hamiltonian acts on no more than *two* fields within the operator at the same time. We could now go ahead and calculate these Feynman diagrams for all combinations of two fields. Most of these calculations would turn out to be redundant once superconformal symmetry is taken into account. We therefore first investigate the independent coefficients of a generic superconformally invariant function of two fields. These are in one-to-one correspondence to the irreducible multiplets in the tensor product of two field multiplets. The irreducible multiplets can be distinguished by their 'total spin' j. We claim that the 'Hamiltonian density' H_{12} representing the Hamiltonian H restricted to two fields $1,2$ is given by [3]

$$H_{12} = 2h(J_{12}), \qquad H = \sum_{k=1}^{L} H_{k,k+1}. \tag{1.3}$$

where J_{12} is an operator that measures the total spin j of the two fields and where $h(j)$ is defined in (1.1). The above mentioned anomalous dimensions of twist-two operators are a trivial consequence of this.

In principle this result allows the computation of all one-loop anomalous dimensions in $\mathcal{N}=4$ SYM: For every pair of fields within the operator, we decompose into components of definite total spin j. Each component we multiply by the harmonic number $2h(j)$ and add up all contributions. For non-planar corrections, in addition we have to compute the colour structure dictated by Feynman diagrammatics. In an explicit calculation this procedure has the drawback that the projection to total spin j is rather involved. Nevertheless, there is an alternative method to evaluate the action of the Hamiltonian density $2h(J_{12})$ which we will name the *harmonic action*. To describe the harmonic action we will represent fields of $\mathcal{N}=4$ in terms of excitations of a *supersymmetric harmonic oscillator*, see [33]. There is a natural way to do this such that each excitation corresponds to a spinor index of the field. The harmonic action describes how to shuffle the oscillators (or spinor indices) between the two involved fields in order to obtain $2h(J_{12})$. We will demonstrate the action in terms of a simple example and, as an application, we determine the spectrum of planar anomalous dimensions for single-trace operators of $\mathcal{N}=4$ SYM up to and including classical dimension 5.5, see Tab. 3. Here, we also see some signs of integrability of the complete planar dilatation generator. The issue of integrability will, however, not be discussed in detail in this work; this is discussed in [34].

[2] On the background $\mathbb{R}\times S^3$ it is in fact the one-loop part of the Hamiltonian.

[3] Similar Hamiltonians have appeared in the context of QCD energies (BFKL kernel) and one-loop anomalous dimensions (DGLAP kernel), see e.g. [32]. Note, however, that here the total spin operator J_{12} corresponds to the quadratic Casimir of $\mathfrak{psl}(4|4)$ instead of $\mathfrak{sl}(2)$.

Still, the harmonic action in its most general form requires some work to be applied. On the other hand, when we restrict to the $\mathfrak{so}(6)$ subsector of operators made out of scalar fields only, the action should simplify to the effective vertex of [19]. This vertex consists of only two terms and is straightforwardly applied. In the minimal $\mathfrak{su}(2)$ subsector it was even possible to derive the two-loop contribution to the dilatation generator [22]. It would therefore be desirable to investigate further subsectors within which the action of the dilatation generator closes. Here, one should distinguish between exactly closed subsectors (e.g. $\mathfrak{su}(2)$) and subsectors closed only at one-loop (e.g. $\mathfrak{so}(6)$). We will find a criterion for exactly closed subsectors and determine all such sectors. Among these we find the subsectors relevant to quarter-BPS and eighth-BPS operators. These are the above $\mathfrak{su}(2)$ sector and a new $\mathfrak{u}(2|3)$ sector. We also find a simple condition (5.13) for quarter-BPS and eighth-BPS operators which we use to determine the lowest-lying eighth-BPS operator. Among the one-loop subsectors we find the above $\mathfrak{so}(6)$ subsector, a non-compact $\mathfrak{so}(4,2)$ brother and a $\mathfrak{su}(2|4)$ subsector in which the Hamiltonian agrees fully with the one-loop Hamiltonian of the BMN matrix model [3,35].

Probably the most interesting exactly closed subsector is the non-compact brother of the $\mathfrak{su}(2)$ subsector, the $\mathfrak{sl}(2)$ subsector. The main difference between the two is that in the $\mathfrak{su}(2)$ sector there are only two fields, Z and ϕ, while in the $\mathfrak{sl}(2)$ sector there are infinitely many, $\mathcal{D}^n Z$. At first sight these sectors seem quite different, in terms of representations this is however not the case. In the complex form both algebras are the same, the fields Z, ϕ transform in the fundamental spin $\frac{1}{2}$ representations, whereas the fields $\mathcal{D}^n Z$ transform in the spin $-\frac{1}{2}$ representation. In this new subsector we are able to compute the dilatation generator (3.14) by field theoretic means. To demonstrate the usefulness of this simplified Hamiltonian we apply it to find a few anomalous dimensions, see Tab. 2. Finally, we will show that the Hamiltonian lifts uniquely to the full $\mathfrak{psl}(4|4)$ Hamiltonian, i.e. it fixes all independent coefficients of the most general $\mathfrak{psl}(4|4)$ invariant form.

This paper is organised as follows. We start by determining the most general form of the dilatation generator compatible with $\mathfrak{sl}(4|4)$ invariance in Sec. 2. In Sec. 3 we investigate the $\mathfrak{sl}(2)$ subsector, determine the Hamiltonian and show that it lifts uniquely to the full Hamiltonian. We then introduce the oscillator representation of fields and work out the action of the Hamiltonian in Sec. 4. Sec. 5 contains an investigation of various subsectors within $\mathcal{N} = 4$ SYM. We give an outlook in Sec. 6.

2 The form of the dilatation generator

We start by investigating the general form of the one-loop dilatation generator. We will see that representation theory of the symmetry group as well as Feynman diagrammatics puts tight constraints on the form. What remains is a sequence of undetermined coefficients, one for each value of 'total spin', which will turn out to be the harmonic series.

Letters and operators. Local operators $\mathcal{O}$ are constructed as gauge invariant combinations of 'letters' (irreducible covariant fields) W_A [4]

$$\mathcal{O} = \operatorname{Tr} W_* \cdots W_* \ \operatorname{Tr} W_* \cdots W_* \quad \ldots \tag{2.1}$$

The letters of $\mathcal{N}=4$ are the scalars, fermions and field strengths as well as their covariant derivatives. Here, traces and antisymmetries in derivative indices are excluded; such fields can be reexpressed as reducible products of letters via the equations of motion or Jacobi identities. The index A in W_A enumerates all such letters.

An alternative way of representing local operators is to use the state-operator map for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$, which is conformally equivalent to flat $\mathbb{R}^4$. When decomposing the fundamental fields into spherical harmonics on S^3 one gets as irreducible states precisely the same spectrum of letters W_A. In this picture the dilatation generator maps to the generator of shifts along $\mathbb{R}$, i.e. the Hamiltonian.

Tree-level algebra. Under the superconformal algebra $\mathfrak{psl}(4|4)$, c.f. App. A, consisting of the generators J_0, the letters transform among themselves

$$J_0 \, W_A = (J_0)^B{}_A W_B. \tag{2.2}$$

Classically, the local operators $\mathcal{O}$ transform in tensor product representations of (2.2). A generator J_0 of $\mathfrak{psl}(4|4)$ at tree-level can thus be written in terms of its action $(J_0)^B{}_A$ on a single letter W_A as

$$J_0 \, W_A \cdots W_B = (J_0)^C{}_A W_C \cdots W_B + \ldots + (J_0)^C{}_B W_A \cdots W_C. \tag{2.3}$$

Using a notation for variation with respect to fields [22]

$$\check{W}^A := \frac{\delta}{\delta W_A} = T^a \frac{\delta}{\delta W_A^a} \tag{2.4}$$

we can also write the tree-level generators as[5]

$$J_0 = (J_0)^B{}_A \operatorname{Tr} W_B \check{W}^A. \tag{2.5}$$

The variation will pick any of the letters within an operator and replace it by the transformed letter. In particular the tree-level dilatation generator is

$$D_0 = \sum\nolimits_A \dim(W_A) \ \operatorname{Tr} W_A \check{W}^A. \tag{2.6}$$

[4]For convenience we will restrict to the gauge group $\mathrm{U}(N)$. The letters W_A are $N\times N$ matrices, which can be parametrised by the generators T^a in the fundamental representation of $\mathrm{U}(N)$ as $W_A = T^a W_A^a$. Nevertheless, all results generalise straightforwardly to arbitrary gauge groups.

[5]Note the fusion and fission rules $\operatorname{Tr} X\check{W}^A \operatorname{Tr} Y W_B = \delta^A_B \operatorname{Tr} XY$, $\operatorname{Tr} X\check{W}^A Y W_B = \delta^A_B \operatorname{Tr} X \operatorname{Tr} Y$.

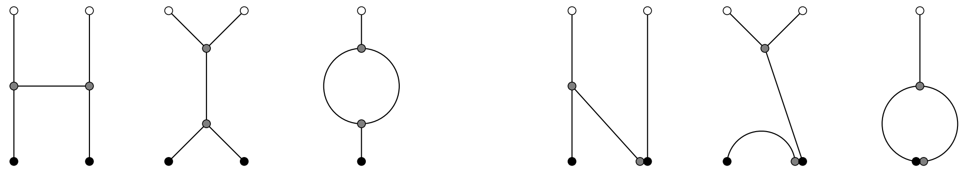

Figure 1: One-loop diagrams contributing to the anomalous dimension. The lines correspond to any of the fundamental fields of the theory.

Interacting algebra. When quantum corrections are turned on, the transformation properties of operators change. In perturbation theory we will therefore write the full generators J as a series in the coupling constant g,[6]

$$J(g) = \sum_{k=0}^{\infty} g^k J_k. \tag{2.7}$$

In this work we will concentrate on the first correction to the dilatation generator, D_2. It must be invariant under the tree-level algebra J_0. This follows from the interacting algebra identity

$$[D(g), J(g)] = [D_0, J_0 + gJ_1 + g^2 J_2] + g^2[D_2, J_0] + \mathcal{O}(g^3) = \dim(J)\, J(g) \tag{2.8}$$

for every operator J of the superconformal group. In perturbation theory the bare dimension D_0 of all generators J_k must be conserved

$$[D_0, J_k] = \dim(J)\, J_k. \tag{2.9}$$

Eq. (2.8) then implies for all J_0

$$[J_0, D_2] = 0. \tag{2.10}$$

In other words D_2 *is invariant under classical superconformal transformations*. It will turn out that D_2 is invariant under another, nontrivial generator B_0 of the algebra $\mathfrak{sl}(4|4) = \mathfrak{gl}(1) \ltimes \mathfrak{psl}(4|4)$, see e.g. [36]. We will refer to this additional $\mathfrak{gl}(1)$ hypercharge generator as 'chirality'. It is conserved at the one-loop level, but at higher loops it is broken due to the Konishi anomaly. In what follows we will therefore consider the classical $\mathfrak{sl}(4|4)$ algebra of generators J_0; the one-loop anomalous dilatation generator D_2 will be considered an independent object, the Hamiltonian H,

$$J(g) = J + \mathcal{O}(g), \qquad D(g) = D + \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} H + \mathcal{O}(g^3), \qquad [J, H] = 0. \tag{2.11}$$

Generic form. The Hamiltonian has the following generic form

$$\begin{aligned} H = {} & (C_{\mathrm{a}})^{AB}_{CD} :\mathrm{Tr}[W_A, \check{W}^C][W_B, \check{W}^D]: \\ & + (C_{\mathrm{b}})^{AB}_{CD} :\mathrm{Tr}[W_A, W_B][\check{W}^C, \check{W}^D]: \\ & + (C_{\mathrm{c}})^{A}_{B} :\mathrm{Tr}[W_A, T^a][T^a, \check{W}^B]:. \end{aligned} \tag{2.12}$$

[6]The odd powers of g are due to normalisation: Propagators are $\mathcal{O}(g^0)$, three-vertices are $\mathcal{O}(g)$.

field	Δ_0	$\mathfrak{sl}(2)\times\mathfrak{sl}(2)$	$\mathfrak{sl}(4)$	B
$\mathcal{D}^k\mathcal{F}$	$k+2$	$[k+2,k]$	$[0,0,0]$	$+1$
$\mathcal{D}^k\Psi$	$k+\frac{3}{2}$	$[k+1,k]$	$[0,0,1]$	$+\frac{1}{2}$
$\mathcal{D}^k\Phi$	$k+1$	$[k,k]$	$[0,1,0]$	± 0
$\mathcal{D}^k\bar\Psi$	$k+\frac{3}{2}$	$[k,k+1]$	$[1,0,0]$	$-\frac{1}{2}$
$\mathcal{D}^k\bar{\mathcal{F}}$	$k+2$	$[k,k+2]$	$[0,0,0]$	-1

Table 1: Components W_A of the $\mathcal{N}=4$ SYM field strength multiplet

These terms correspond to the three basic types of Feynman diagrams that arise at the one-loop level, see Fig. 1. The diagrams a,b,c correspond to bulk interactions, where the covariant derivatives $\mathcal{D}=\partial-ig\mathcal{A}$ within the fields W_A are reduced to a partial derivative. The boundary interactions a′,b′,c′ arise when one covariant derivative emits a gauge field, see also [28, 29]. Algebraically they have the same form as their bulk counterparts. Diagrams with two emitted gauge fields have no logarithmic dependence and do not contribute to the dilatation generator. We can transform the term of type c by means of gauge invariance, see [22] for a detailed description of the procedure. The generator of gauge transformations is

$$G^a = \mathrm{Tr}[W_A,\check W^A]T^a \tag{2.13}$$

and it annihilates gauge invariant operators, $G^a \mathrel{\widehat{=}} 0$. Therefore we can write

$$G^a\,\mathrm{Tr}[W_A,\check W^B]T^a = {:}\mathrm{Tr}[W_C,\check W^C][W_A,\check W^B]{:} + {:}\mathrm{Tr}[W_A,T^a][T^a,\check W^B]{:} \mathrel{\widehat{=}} 0, \tag{2.14}$$

which allows us to write the term of type c as a term of type a. Furthermore the term of type b can be transformed by means of a Jacobi-identity. We combine all coefficients into a single one of type a

$$C^{AB}_{CD} = -N\big((C_\mathrm{a})^{AB}_{CD} + (C_\mathrm{b})^{AB}_{CD} - (C_\mathrm{b})^{AB}_{DC} - \tfrac{1}{2}\delta^A_C(C_\mathrm{c})^B_D - \tfrac{1}{2}(C_\mathrm{c})^A_C\delta^B_D\big). \tag{2.15}$$

The total Hamiltonian is

$$H = -N^{-1}C^{AB}_{CD}\,{:}\mathrm{Tr}[W_A,\check W^C][W_B,\check W^D]{:}. \tag{2.16}$$

Symmetry. The combined coefficient C^{AB}_{CD} must be invariant under the tree-level superconformal algebra. Its independent components can be obtained by investigating the irreducible modules in the tensor product of two multiplets of fields. We list all letters and their transformation properties in Tab. 1. We have split up the fermions and field strengths into their chiral ($\Psi,\mathcal{F}$) and antichiral parts ($\bar\Psi,\bar{\mathcal{F}}$); the scalars ($\Phi$) are self-dual. In the table Δ_0 is the bare dimension and refers the $\mathfrak{gl}(1)$ of dilatations, $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ is the Lorentz algebra and $\mathfrak{sl}(4)$ is the flavour algebra. These are the manifestly realised parts of the full symmetry algebra $\mathfrak{psl}(4|4)$. We have also included the chirality B of the extended algebra $\mathfrak{sl}(4|4)$. In $\mathfrak{sl}(4|4)$ all letters W_A combine into one single, infinite-dimensional module which we will denote by V_F. The corresponding representation is also referred to as the ‘singleton’ representation. Its primary weight

$$w_\mathrm{F} = [\Delta_0; s_1, s_2; q_1, p, q_2; B, L] = [1;0,0;0,1,0;0,1] \tag{2.17}$$

corresponds to one of the six scalars at unit dimension. Here, $[q_1, p, q_2]$ are the Dynkin labels for a weight of $\mathfrak{sl}(4)$ and $[s_1, s_2]$ are *twice* the spins of $\mathfrak{sl}(2)^2$. The label L refers to the number of fields. This will be of importance later on, here it equals one by definition.

The tensor product of two V_{F} is given by [7]

$$V_{\mathrm{F}} \times V_{\mathrm{F}} = \sum_{j=0}^{\infty} V_j \tag{2.18}$$

where V_j are the modules with primary weights

$$\begin{aligned} w_0 &= [2; 0, 0; 0, 2, 0; 0, 2], \\ w_1 &= [2; 0, 0; 1, 0, 1; 0, 2], \\ w_j &= [j; j-2, j-2; 0, 0, 0; 0, 2]. \end{aligned} \tag{2.19}$$

Let $(P_j)^{AB}_{CD}$ project two field-strengths W_A, W_B to the module V_j. Then the most general invariant coefficients can be written as [8]

$$C^{AB}_{CD} = \sum_{j=0}^{\infty} C_j \, (P_j)^{AB}_{CD}. \tag{2.20}$$

In $\mathcal{N} = 4$ SYM we propose that the coefficients are given by the *harmonic numbers* $h(j)$ or equivalently by the digamma function Ψ

$$C_j = h(j) := \sum_{k=1}^{j} \frac{1}{k} = \Psi(j+1) - \Psi(1). \tag{2.21}$$

The one-loop dilatation generator of $\mathcal{N} = 4$ (2.11),(2.16) can thus be written as

$$D(g) = D - \frac{g_{\mathrm{YM}}^2}{8\pi^2} \sum_{j=0}^{\infty} h(j) \, (P_j)^{AB}_{CD} \, {:}\mathrm{Tr}[W_A, \check{W}^C][W_B, \check{W}^D]{:} + \mathcal{O}(g^3). \tag{2.22}$$

This is the principal result of this work. In Sec. 4 we will explain how to compute the action of D in practice.

Planar limit. At this point we can take the planar limit of (2.16) and act on a single-trace operator of L fields

$$H = \sum_{k=1}^{L} H_{k,k+1}, \qquad H_{k,k+1} = 2 C_j P_{k,k+1,j}, \tag{2.23}$$

where $P_{k,k+1,j}$ projects the fields at positions $k, k+1$ to the module V_j. We see that all coefficients C_j can be read off from this Hamiltonian. Therefore the Hamiltonian

[7] This statement can be proved by counting arguments using Polya theory (see e.g. [37]).

[8] In general, one might also consider Feynman diagrams in which two or three fields are transformed into one or vice versa. These terms, however, cannot contribute to the leading order Hamiltonian, which is $\mathfrak{psl}(4|4)$ invariant, because V_{F} is not included in $V_{\mathrm{F}} \times V_{\mathrm{F}}$ and $V_{\mathrm{F}} \times V_{\mathrm{F}} \times V_{\mathrm{F}}$.

density H_{12} generalises uniquely to the non-planar Hamiltonian H in (2.16),(2.20). In what follows we can safely restrict ourselves to the investigation of H_{12}. The proposed Hamiltonian density according to (2.22) is

$$H_{12} = \sum_{j=0}^{\infty} 2h(j)\, P_{12,j}. \tag{2.24}$$

Spin functions. To simplify some expressions, we define functions $f(J_{12})$ of the total spin J_{12} as

$$f(J_{12}) = \sum_{j=0}^{\infty} f(j)\, P_{12,j}. \tag{2.25}$$

In other words $f(J_{12})$ is a $\mathfrak{psl}(4|4)$ invariant operator which acts on V_j as

$$f(J_{12})\, V_j = f(j)\, V_j. \tag{2.26}$$

An arbitrary state with a dimension bounded by Δ_0 belongs to the direct sum of modules V_j with $j = 0, \ldots, \Delta_0$. The action of $f(J_{12})$ on such a state may be represented by a polynomial in the quadratic Casimir J_{12}^2, see (A.7). The eigenvalues of J_{12}^2, where $J_{12} = J_1 + J_2$ is the action of $\mathfrak{psl}(4|4)$ on the tensor product $V_\mathrm{F} \times V_\mathrm{F}$, are then given by

$$J_{12}^2 V_j = j(j+1) V_j. \tag{2.27}$$

This can be proved using the methods of Sec. 4.1. Note that all modules V_j can be distinguished by the value of the quadratic Casimir. We can therefore represent $f(J_{12})$ explicitly by an interpolating polynomial in J_{12}^2

$$f(J_{12}) = \sum_{j=0}^{\Delta_0} f(j) \prod_{\substack{k=0\\k\neq j}}^{\Delta_0} \frac{J_{12}^2 - k(k+1)}{j(j+1) - k(k+1)}. \tag{2.28}$$

We note that this action also preserves chirality B, because the quadratic Casimir J_{12}^2 does. Therefore any $\mathfrak{psl}(4|4)$ invariant function acting on $V_\mathrm{F} \times V_\mathrm{F}$ also conserves $\mathfrak{sl}(4|4)$. Clearly, this will not be the case for higher-loop corrections to the dilatation generator which act on more than two fields at the same time. At higher loops the Konishi anomaly mixes operators of different chirality. The same points also hold for the length L of a state. Nevertheless it makes perfect sense to speak of the leading order chirality and length to describe a state. Mixing with states of different chiralities or lengths is subleading, because the one-loop dilatation generator conserves these. Using the short notation the Hamiltonian density becomes simply

$$H_{12} = 2h(J_{12}). \tag{2.29}$$

Examples. A straightforward exercise is to determine the spectrum of operators of length $L = 2$. These so-called twist-two operators can be conveniently written as

$$\mathcal{O}_{j,AB} = (P_j)_{AB}^{CD} \operatorname{Tr} W_C W_D \sim P_{12,j} \operatorname{Tr} W_1 W_2. \tag{2.30}$$

This operator vanishes for odd j due to antisymmetry. Using (2.22),(2.23) or (2.29) we find

$$E = 4h(j), \qquad \delta D = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\, E = \frac{g_{\mathrm{YM}}^2 N}{2\pi^2}\, h(j) \tag{2.31}$$

in agreement with the results of [13]. In fact the result of [13] could be used to fix the even coefficients C_{2k} (2.21).

3 The non-compact $\mathfrak{sl}(2)$ closed subsector

In this section we will consider a closed subsector of states in $\mathcal{N} = 4$ SYM. We derive the relevant part of the Hamiltonian H' and show that it uniquely lifts to the full Hamiltonian H of $\mathcal{N} = 4$ SYM. Finally, we apply the Hamiltonian within this subsector and obtain a few anomalous dimensions.

Letters and operators. Let us consider single-trace operators in the planar limit which saturate the bound $\Delta_0 \geq L + n$, where n and L are the charges with respect to rotations in the spacetime 12-plane and flavour 56-plane, respectively. The operators consist only of the scalar field $\Phi_{5+i6} = \Phi_5 + i\Phi_6 = \Phi_{34}$ and derivatives $\mathcal{D}_{1+i2} = \ldots = \mathcal{D}_{11}$ acting on it.[9] A letter for the construction of operators is thus

$$(\mathbf{a}^\dagger)^n|0\rangle = \frac{1}{n!}(\mathcal{D}_{1+i2})^n \Phi_{5+i6}. \tag{3.1}$$

This subsector is closed, its operators do not mix with any of the other operators in $\mathcal{N} = 4$ SYM. This is similar to the subsector considered in [22] where both charges belong to $\mathfrak{sl}(4)$. We have collected further interesting subsectors in Sec. 5. The weight of a state with a total number of n excitations (derivatives) is given by

$$w = [L + n; n, n; 0, L, 0; 0, L]. \tag{3.2}$$

For $n \neq 0$ this weight is beyond the unitarity bounds and therefore it cannot be primary. The corresponding primary weight is

$$w = [L + n - 2; n - 2, n - 2; 0, L - 2, 0; L]. \tag{3.3}$$

Symmetry. The subsector is invariant under an $\mathfrak{sl}(2)$ subalgebra of the superconformal algebra (note that $L^1{}_1 = L^2{}_2 = \frac{1}{2}D - \frac{1}{2}L$ in this sector)

$$\begin{aligned}
J'_+ &= P_{11} = P_{1+i2},\\
J'_- &= K^{11} = K^{1+i2},\\
J'_3 &= \tfrac{1}{2}D + \tfrac{1}{2}\delta D + \tfrac{1}{2}L^1{}_1 + \tfrac{1}{2}L^2{}_2 = D + \tfrac{1}{2}\delta D - \tfrac{1}{2}L.
\end{aligned} \tag{3.4}$$

Here, the dilatation generator D is part of the algebra. At higher loops, one should keep in mind that only half of the anomalous piece appears. The $\mathfrak{sl}(2)$ subalgebra follows from the relations in App. A

$$[J'_+, J'_-] = -2J'_3, \qquad [J'_3, J'_\pm] = \pm J'_\pm. \tag{3.5}$$

[9] Depending on the situation, we take the freedom to use either spinor or vector index notation.

It has the quadratic Casimir operator

$$J'^2 = J_3'^2 - \tfrac{1}{2}\{J'_+, J'_-\}. \tag{3.6}$$

The algebra may be represented by means of oscillators $\mathbf{a}$, $\mathbf{a}^\dagger$

$$J'_- = \mathbf{a}, \quad J'_3 = \tfrac{1}{2} + \mathbf{a}^\dagger\mathbf{a}, \quad J'_+ = \mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}. \tag{3.7}$$

We define the canonical commutator as

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1 \tag{3.8}$$

and assume that the vacuum state $|0\rangle$ is annihilated by $\mathbf{a}$. Under this algebra the set of letters $(\mathbf{a}^\dagger)^n|0\rangle$ transforms in the infinite dimensional spin $j = -\frac{1}{2}$ representation V'_{F} with highest weight

$$w'_{\mathrm{F}} = [2j] = [-1]. \tag{3.9}$$

The Hamiltonian density H_{12} acts on two fields $(\mathbf{a}_1^\dagger)^k(\mathbf{a}_2^\dagger)^l|00\rangle$. Of particular interest is therefore the tensor product of two V'_{F}, it splits into modules of spin $-1-j$.

$$V'_{\mathrm{F}} \times V'_{\mathrm{F}} = \sum_{j=0}^{\infty} V'_j, \qquad \text{with} \quad w'_j = [-2-2j]. \tag{3.10}$$

The algebra acting on this tensor product is given by the generators $J'_{12} = J'_1 + J'_2$. The quadratic Casimir reads

$$J'^2_{12} = -(\mathbf{a}_1^\dagger - \mathbf{a}_2^\dagger)^2\mathbf{a}_1\mathbf{a}_2 + (\mathbf{a}_1^\dagger - \mathbf{a}_2^\dagger)(\mathbf{a}_1 - \mathbf{a}_2). \tag{3.11}$$

The highest weight state of V'_j, which is annihilated by $J'_{12,-}$, is

$$|j\rangle = (\mathbf{a}_1^\dagger - \mathbf{a}_2^\dagger)^j\,|00\rangle. \tag{3.12}$$

the eigenvalue of the quadratic Casimir in that representation is

$$J'^2_{12}\, V'_j = j(j+1)V'_j, \tag{3.13}$$

which intriguingly matches the $\mathfrak{psl}(4|4)$ counterpart (2.27).

The Hamiltonian. Using point splitting regularisation we find the action of the Hamiltonian density

$$H'_{12}\,(\mathbf{a}_1^\dagger)^k(\mathbf{a}_2^\dagger)^{n-k}|00\rangle = \sum_{k'=0}^{n}\left(\delta_{k=k'}\big(h(k)+h(n-k)\big) - \frac{\delta_{k\neq k'}}{|k-k'|}\right)(\mathbf{a}_1^\dagger)^{k'}(\mathbf{a}_2^\dagger)^{n-k'}|00\rangle, \tag{3.14}$$

see App. B for details. It is straightforward to verify that this H'_{12} is invariant under the generators J'_{12}. In analogy to (2.24) it is therefore clear that we can write H'_{12} as

$$H'_{12} = \sum_{j=0}^{\infty} 2C'_j\, P'_{12,j}. \tag{3.15}$$

where $P'_{12,j}$ projects a state $(\mathbf{a}_1^\dagger)^k(\mathbf{a}_2^\dagger)^{n-k}|00\rangle$ to the module V'_j. The coefficients C'_j are found by acting with H'_{12} on the highest weight states $|j\rangle$. Using the sum

$$\sum_{k=a+1}^{n} \frac{(-1)^{a+k+1}a!(n-a)!}{k!(n-k)!(k-a)} = h(n) - h(a) \tag{3.16}$$

we can show that

$$C'_j\,|j\rangle = H'_{12}\,|j\rangle = 2h(j)\,|j\rangle. \tag{3.17}$$

Note that the Hamiltonian density (3.15) has precisely the right eigenvalues (3.17) for total spin $-1-j$ in order to be integrable [38]. Thus we have found that the planar Hamiltonian in this subsector is the Hamiltonian of the $\mathrm{XXX}_{-1/2}$ Heisenberg spin chain. This is investigated in further detail in [34].

Lift to $\mathfrak{psl}(4|4)$. The interesting feature of this subsector is that there is a one-to-one correspondence between the modules V_j (2.19) and V'_j. The Hamiltonian in this subsector lifts to the full $\mathcal{N}=4$ Hamiltonian! Clearly, as $\mathfrak{sl}(2)$ is a subalgebra of $\mathfrak{psl}(4|4)$, the module V'_j is a submodule of some $V_{j'}$. Here, it turns out, that $j'=j$ and

$$V'_j \subset V_j. \tag{3.18}$$

In the $\mathfrak{psl}(4|4)$ algebra the state $|j\rangle$ has weight

$$w'_j = [j+2; j, j; 0, 2, 0; 0, 2]. \tag{3.19}$$

For $j=0$ this is the primary weight of the current multiplet $w_0 = [2;0,0;0,2,0;0,2]$. For $j>0$ the weight is beyond the unitarity bounds and cannot be primary. The corresponding primary weights are $w_j = [j; j-2, j-2; 0,0,0;0,2]$ or $w_1 = [2;0,0;1,0,1;0,2]$, respectively. These are exactly the primary weights of the modules V_j in (2.19). Using the fact that the two Hamiltonians must agree within the subsector we find

$$2C_j\,|j\rangle = H_{12}\,|j\rangle = H'_{12}\,|j\rangle = 2h(j)\,|j\rangle. \tag{3.20}$$

This shows that $C_j = h(j)$ in (2.20) and proves the claim (2.21).

Examples. The expression (3.14) can be used to calculate any one-loop anomalous dimension within this subsector. The generalisation to the non-planar sector is straightforward: We know that the structure of the non-planar Hamiltonian is given by (2.16). We can therefore act with the Hamiltonian density H_{kl} also on non-neighbouring sites k,l. The non-planar structure is then evaluated according to (2.16).

We go ahead and find some eigenstates of the Hamiltonian, see Tab. 2. Especially for the 'two-body' problems with two sites or two excitations (derivatives) one should be able to find exact eigenstates. For $L=2$ we know the answer already, it is $\mathrm{Tr}\,|j\rangle$, where the trace projects to cyclic states, effectively removing states with odd j. The spectrum is given by (2.31). Indeed, also for $n=2$ we find exact eigenstates

$$\mathcal{O}^L_n = 2\cos\frac{\pi n}{L+1}\,\mathrm{Tr}\,(\mathbf{a}_1^\dagger)^2\,|L\rangle + \sum_{p=2}^{L}\cos\frac{\pi n(2p-1)}{L+1}\,\mathrm{Tr}\,\mathbf{a}_1^\dagger\mathbf{a}_p^\dagger\,|L\rangle, \tag{3.21}$$

Δ_0	L	n	$\delta\Delta^P\ [g_{\mathrm{YM}}^2 N/\pi^2]$
4	2	2	$\frac{3}{4}^+$
5	3	2	$\frac{1}{2}^-$
6	4	2	$\frac{1}{8}(5\pm\sqrt{5})^+$
	3	3	$\frac{15}{16}^\pm$
	2	4	$\frac{25}{24}^+$
7	5	2	$\frac{1}{4}^-, \frac{3}{4}^-$
	4	3	$\frac{3}{4}^\pm$
	3	4	$\frac{3}{4}^-$

Δ_0	L	n	$\delta\Delta^P\ [g_{\mathrm{YM}}^2 N/\pi^2]$
8	6	2	$(64x^3-112x^2+56x-7)^+$
	5	3	$\frac{1}{32}(25\pm\sqrt{37})^\pm$
	4	4	$\frac{23}{24}^\pm, (768x^3-2336x^2+2212x-637)^+$
	3	5	$\frac{35}{32}^\pm$
	2	6	$\frac{49}{40}^+$
9	7	2	$\frac{1}{4}(2\pm\sqrt{2})^-, \frac{1}{2}^-$
	6	3	$(256x^3-608x^2+459x-108)^\pm$
	5	4	$\frac{1}{16}(13\pm\sqrt{41})^-, \frac{1}{96}(97\pm 7\sqrt{5})^\pm$
	4	5	$\frac{1}{96}(105\pm\sqrt{385})^\pm$
	3	6	$\frac{11}{12}^-, \frac{227}{160}^\pm$

Table 2: The first few states within the $\mathfrak{sl}(2)$ subsector. The weights of the corresponding primaries are $[L+n-2; n-2, n-2; 0, L-2, 0; 0, L]$. Cubic polynomials indicate three states with energies given by the roots of the cubic equation.

which are precisely the BMN operators with two symmetric-traceless vector indices [31, 29]. Their energy is

$$E = 8\sin^2\frac{\pi n}{L+1}, \qquad \delta D = \frac{g_{\mathrm{YM}}^2 N}{\pi^2}\sin^2\frac{\pi n}{L+1}. \tag{3.22}$$

In analogy to the special, unpaired three-impurity states found in [22], we might hope to find special states of three sites or with three excitations. It turns out, that there are no unpaired states of three excitations, but there are some for $L=3$. Empirically we find exactly one unpaired state for $2k$ excitations. This state has energy

$$E = 4h(k), \qquad \delta D = \frac{g_{\mathrm{YM}}^2 N}{2\pi^2}\,h(k). \tag{3.23}$$

and weight $[2k+3; 2k, 2k; 0, 3, 0; 0, 3]$. The corresponding superconformal primary has weight

$$[2k+1; 2k-2, 2k-2; 0, 1, 0; 0, 3]. \tag{3.24}$$

Interestingly, the spectrum comprises all harmonic numbers $h(k)$, whereas the $L=2$ spectrum (2.31) consists only of the even ones $h(2k)$.

4 The oscillator picture

In the last section we have made use of an oscillator representation for a subsector of operators in $\mathcal{N}=4$ SYM. In this section we will show how to represent a generic operator of $\mathcal{N}=4$ SYM in terms of excitations of (different) oscillators, see also [33]. We will then describe the action of the Hamiltonian on these states explicitly.

4.1 Oscillator representations

Let us explain the use of oscillators for fields and generators in terms of the algebra $\mathfrak{gl}(N)$: We write[10]

$$J^a{}_b = \mathbf{a}^\dagger_b \mathbf{a}^a, \qquad \text{with } a, b = 1, \ldots, N. \tag{4.1}$$

Using the commutators

$$[\mathbf{a}^a, \mathbf{a}^\dagger_b] = \delta^a_b, \qquad [\mathbf{a}^a, \mathbf{a}^b] = [\mathbf{a}^\dagger_a, \mathbf{a}^\dagger_b] = 0 \tag{4.2}$$

it is a straightforward exercise to show that J satisfies the $\mathfrak{gl}(N)$ algebra.

The $\mathfrak{gl}(N)$ invariant vacuum. Let us introduce a state $|0\rangle$ defined by

$$\mathbf{a}^a|0\rangle = 0 \qquad \text{for } a = 1, \ldots, N. \tag{4.3}$$

Then the states

$$\mathbf{a}^\dagger_{a_1} \ldots \mathbf{a}^\dagger_{a_k}|0\rangle \tag{4.4}$$

transform in the totally symmetrised product of k fundamental representations of $\mathfrak{sl}(N)$ and have $\mathfrak{gl}(1)$ central charge k.

A $\mathfrak{gl}(N)$ breaking vacuum. Another state $|n\rangle$ can be defined by

$$\mathbf{a}^a|n\rangle = 0, \quad \mathbf{a}^\dagger_{a'}|n\rangle = 0 \qquad \text{for } a = 1, \ldots, n, \quad a' = n+1, \ldots N. \tag{4.5}$$

In this case it is more convenient to write

$$\mathbf{b}^{\dot a} = \mathbf{a}^\dagger_{\dot a + n}, \quad \mathbf{b}^\dagger_{\dot a} = -\mathbf{a}^{\dot a + n}, \quad \text{for } \dot a = 1, \ldots, n' = N - n \quad \text{with } [\mathbf{b}^{\dot a}, \mathbf{b}^\dagger_{\dot b}] = \delta^{\dot a}_{\dot b} \tag{4.6}$$

and consider the subalgebra $\mathfrak{sl}(n) \times \mathfrak{sl}(n')$, $J^a{}_b = \mathbf{a}^\dagger_b \mathbf{a}^a$, $J^{\dot a}{}_{\dot b} = \mathbf{b}^\dagger_{\dot b} \mathbf{b}^{\dot a}$, under which $|n\rangle$ is invariant. Now as the off-diagonal part $J_{\dot a b} = \mathbf{a}^\dagger_b \mathbf{b}^\dagger_{\dot a}$ of the generator J consists of only creation operators, the state $|n\rangle$ transforms in an infinite-dimensional representation. In a generic state

$$\mathbf{a}^\dagger_{a_1} \ldots \mathbf{a}^\dagger_{a_k} \mathbf{b}^\dagger_{\dot b_1} \ldots \mathbf{b}^\dagger_{\dot b_{k'}}|n\rangle \tag{4.7}$$

the $\mathfrak{gl}(1)$ central charge $J^a{}_a - J^{\dot a}{}_{\dot a}$ is $k - k' - n'$ and it labels irreducible modules of $\mathfrak{sl}(N)$. The $\mathfrak{gl}(1)$ dilatation charge (dimension) $n' J^a{}_a + n J^{\dot a}{}_{\dot a}$ contained in $\mathfrak{sl}(N)$ equals $kn' + k'n + nn'$. Within each irreducible module there is a state with lowest dimension, it has either $k = 0$ or $k' = 0$ depending on the value of the central charge.

Fermionic oscillators. Instead of the bosonic commutators (4.2) we might choose fermionic oscillators with anticommutators

$$\{\mathbf{a}^a, \mathbf{a}^\dagger_b\} = \delta^a_b, \qquad \{\mathbf{a}^a, \mathbf{a}^b\} = \{\mathbf{a}^\dagger_a, \mathbf{a}^\dagger_b\} = 0. \tag{4.8}$$

In this case it is also straightforward to show that the algebra of $\mathfrak{gl}(N)$ is satisfied. The oscillator representation $J^a{}_b$ splits into $N+1$ totally antisymmetric products of the fundamental representation of $\mathfrak{gl}(N)$. This will also be the case for a $\mathfrak{gl}(N)$ breaking vacuum, though not manifestly.

[10] Strictly speaking, the oscillators $\mathbf{a}$ and $\mathbf{a}^\dagger$ are independent. Only in one of the real forms of $\mathfrak{gl}(N)$ they would be related by conjugation.

4.2 A representation of $\mathfrak{gl}(4|4)$

Keeping all this in mind, let us now have a look at Tab. 1. All representations of $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ are symmetric tensor products of the fundamental representation, while the representations of $\mathfrak{sl}(4)$ are antisymmetric. Using two bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ with $\alpha, \dot\alpha = 1, 2$ and one fermionic oscillator $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ with $a = 1, 2, 3, 4$ we can thus write [33]

$$\begin{aligned}
\mathcal{D}^k \mathcal{F} &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^0 |0\rangle, \\
\mathcal{D}^k \varPsi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^1 |0\rangle, \\
\mathcal{D}^k \varPhi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^k \quad (\mathbf{c}^\dagger)^2 |0\rangle, \\
\mathcal{D}^k \bar\varPsi &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+1} (\mathbf{c}^\dagger)^3 |0\rangle, \\
\mathcal{D}^k \bar{\mathcal{F}} &\mathrel{\widehat{=}} (\mathbf{a}^\dagger)^{k} \quad (\mathbf{b}^\dagger)^{k+2} (\mathbf{c}^\dagger)^4 |0\rangle.
\end{aligned} \tag{4.9}$$

Each of the oscillators $\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger_a$ carries one of the $\mathfrak{sl}(2)^2, \mathfrak{sl}(4)$ spinor indices of the fields, for example

$$\mathcal{D}_{\alpha\dot\beta}\, \mathcal{D}_{\gamma\dot\delta}\, \varPhi_{ab} = \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\gamma \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^\dagger_{\dot\delta} \mathbf{c}^\dagger_a \mathbf{c}^\dagger_b |0\rangle. \tag{4.10}$$

The statistics of the oscillators automatically symmetrises the indices in the desired way. The non-vanishing commutators of oscillators are taken to be

$$\begin{aligned}
[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] &= \delta^\alpha_\beta, \\
[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] &= \delta^{\dot\alpha}_{\dot\beta}, \\
\{\mathbf{c}^a, \mathbf{c}^\dagger_b\} &= \delta^a_b.
\end{aligned} \tag{4.11}$$

The canonical forms for the generators of the two $\mathfrak{sl}(2)$ and $\mathfrak{sl}(4)$ are

$$\begin{aligned}
L^\alpha{}_\beta &= \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \tfrac{1}{2} \delta^\alpha_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma, \\
\dot L^{\dot\alpha}{}_{\dot\beta} &= \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^{\dot\alpha} - \tfrac{1}{2} \delta^{\dot\alpha}_{\dot\beta} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}, \\
R^a{}_b &= \mathbf{c}^\dagger_b \mathbf{c}^a - \tfrac{1}{4} \delta^a_b \mathbf{c}^\dagger_c \mathbf{c}^c.
\end{aligned} \tag{4.12}$$

Under these the fields (4.9) transform canonically. We write the corresponding three $\mathfrak{gl}(1)$ charges as

$$\begin{aligned}
D &= 1 + \tfrac{1}{2} \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \tfrac{1}{2} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}, \\
C &= 1 - \tfrac{1}{2} \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \tfrac{1}{2} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} - \tfrac{1}{2} \mathbf{c}^\dagger_c \mathbf{c}^c, \\
B &= \qquad \tfrac{1}{2} \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma - \tfrac{1}{2} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma}.
\end{aligned} \tag{4.13}$$

Assuming that the oscillators $(\mathbf{a}, \mathbf{b}^\dagger, \mathbf{c})$ and $(\mathbf{a}^\dagger, \mathbf{b}, \mathbf{c}^\dagger)$ transform in fundamental and conjugate fundamental representations of $\mathfrak{gl}(4|4)$ we write down the remaining off-diagonal generators according to (4.1)

$$\begin{aligned}
Q^a{}_\alpha &= \mathbf{a}^\dagger_\alpha \mathbf{c}^a, & S^\alpha{}_a &= \mathbf{c}^\dagger_a \mathbf{a}^\alpha, \\
\dot Q_{\dot\alpha a} &= \mathbf{b}^\dagger_{\dot\alpha} \mathbf{c}^\dagger_a, & \dot S^{\dot\alpha a} &= \mathbf{b}^{\dot\alpha} \mathbf{c}^a, \\
P_{\alpha\dot\beta} &= \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta}, & K^{\alpha\dot\beta} &= \mathbf{a}^\alpha \mathbf{b}^{\dot\beta}.
\end{aligned} \tag{4.14}$$

Quite naturally the algebra $\mathfrak{gl}(4|4)$ is realised by the generators (4.12),(4.13)[11],(4.14). We have written this in a $\mathfrak{sl}(2)^2 \times \mathfrak{sl}(4)$ covariant way. In fact one combine the indices a and α into a superindex and obtain a manifest $\mathfrak{sl}(2) \times \mathfrak{sl}(2|4)$ notation. The generators with two lower or two upper indices, $P, \dot{Q}, K, \dot{S}$, together with the remaining charges complete the $\mathfrak{gl}(4|4)$ algebra.

First, we note that all fields (4.9) are uncharged with respect to the central charge C, it can therefore be dropped leading to $\mathfrak{sl}(4|4)$. Furthermore, in $\mathfrak{sl}(4|4)$ the generator B never appears in commutators and can be projected out, this algebra is $\mathfrak{psl}(4|4)$. The generators (4.12),(4.13),(4.14) form a specific irreducible representation of $\mathfrak{psl}(4|4)$. As they transform the fields (4.9) of $\mathcal{N} = 4$ SYM among themselves, this representation has primary weight (2.17)

$$w_{\mathrm{F}} = [1; 0, 0; 0, 1, 0; 0, 1]. \tag{4.15}$$

As an aside, in this representation it can be explicitly shown that the value of the quadratic Casimir operator (A.7) is zero

$$J^2 V_{\mathrm{F}} = 0. \tag{4.16}$$

In the same manner it can be shown that the value of the quadratic Casimir in the module V_j, (2.27), is $j(j+1)$.

Physical vacuum. As in the case of the conformal subalgebra $\mathfrak{sl}(4)$ we can split the flavour $\mathfrak{sl}(4)$ into $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$. In that case we define

$$\mathbf{d}_1^\dagger = \mathbf{c}^3, \quad \mathbf{d}_2^\dagger = \mathbf{c}^4, \qquad \mathbf{d}^1 = \mathbf{c}_3^\dagger, \quad \mathbf{d}^2 = \mathbf{c}_4^\dagger, \tag{4.17}$$

and use a vacuum $|Z\rangle$ annihilated by $\mathbf{a}_{1,2}, \mathbf{b}_{1,2}, \mathbf{c}_{1,2}, \mathbf{d}_{1,2}$. It is related to $|0\rangle$ by

$$|Z\rangle = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle. \tag{4.18}$$

The benefit of this vacuum is that it is not charged under the central charge C and thus physical. It corresponds to the primary weight $[1; 0, 0; 0, 1, 0; 0, 1]$ of the module V_{F}. Furthermore it is the vacuum used in the BMN correspondence [3] and one of the Bethe ansätze in [34]. The drawback is that it is not invariant under the full $\mathfrak{sl}(4)$ flavour algebra, but only under a subgroup $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$. The expressions for the $\mathfrak{gl}(4|4)$ generators thus complicate. However, if the indices (a, α) and $(\dot{a}, \dot{\alpha})$ are combined in two superindices, we have a manifest $\mathfrak{psl}(2|2) \times \mathfrak{psl}(2|2)$ covariance.

Weights and excitations. In this context it is useful to know how to represent an operator with a given weight

$$w = [\Delta_0; s_1, s_2; q_1, p, q_2; B, L] \tag{4.19}$$

in terms of excitations of the oscillators. We introduce a vacuum operator $|0, L\rangle$ which is the tensor product of L vacua $|0\rangle$. The oscillators $\mathbf{a}_{s,\alpha}^\dagger, \mathbf{b}_{s,\dot{\alpha}}^\dagger, \mathbf{c}_{s,a}^\dagger$. now act on site s,

[11] Note that a shift of B by a constant (-1) does not modify the algebra. Then the 1 in D, C, B can be absorbed into $1 + \frac{1}{2}\mathbf{b}_{\dot{\gamma}}^\dagger \mathbf{b}^{\dot{\gamma}} = \frac{1}{2}\mathbf{b}^{\dot{\gamma}} \mathbf{b}_{\dot{\gamma}}^\dagger$ to yield a canonical form.

where commutators of two oscillators vanish unless the sites agree. A generic state is written as

$$(\mathbf{a}^\dagger)^{n_\mathbf{a}}(\mathbf{b}^\dagger)^{n_\mathbf{b}}(\mathbf{c}^\dagger)^{n_\mathbf{c}}|0,L\rangle \tag{4.20}$$

By considering the weights of the oscillators as well as the central charge constraint, we find the number of excitations

$$n_\mathbf{a} = \begin{pmatrix} \frac{1}{2}\Delta_0 + \frac{1}{2}B - \frac{1}{2}L + \frac{1}{2}s_1 \\ \frac{1}{2}\Delta_0 + \frac{1}{2}B - \frac{1}{2}L - \frac{1}{2}s_1 \end{pmatrix}, \quad n_\mathbf{b} = \begin{pmatrix} \frac{1}{2}\Delta_0 - \frac{1}{2}B - \frac{1}{2}L + \frac{1}{2}s_2 \\ \frac{1}{2}\Delta_0 - \frac{1}{2}B - \frac{1}{2}L - \frac{1}{2}s_2 \end{pmatrix}. \tag{4.21}$$

and

$$n_\mathbf{c} = \begin{pmatrix} \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B + \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B + \frac{1}{2}p + \frac{1}{4}q_1 + \frac{3}{4}q_2 \end{pmatrix}. \tag{4.22}$$

If the physical vacuum $|Z\rangle$ is used instead of the $\mathfrak{sl}(4)$ invariant vacuum, the numbers of excitations of the $\mathfrak{sl}(4)$ oscillators are given by

$$n_\mathbf{c} = \begin{pmatrix} \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ \frac{1}{2}L - \frac{1}{2}B - \frac{1}{2}p + \frac{1}{4}q_1 - \frac{1}{4}q_2 \end{pmatrix}, \quad n_\mathbf{d} = \begin{pmatrix} \frac{1}{2}L + \frac{1}{2}B - \frac{1}{2}p - \frac{1}{4}q_1 + \frac{1}{4}q_2 \\ \frac{1}{2}L + \frac{1}{2}B - \frac{1}{2}p - \frac{1}{4}q_1 - \frac{3}{4}q_2 \end{pmatrix}. \tag{4.23}$$

4.3 The harmonic action

The $\mathfrak{psl}(4|4)$ invariant Hamiltonian density H_{12} is given by some function of J_{12}, see (2.25)

$$H_{12} = 2h(J_{12}). \tag{4.24}$$

We will now describe explicitly how H_{12} acts on a state of two-sites.

Invariant action. We will investigate the action of a generic function $f(J_{12})$ on two oscillator sites. As the dimension of any explicit state is a finite number, we can express $f(J_{12})$ as a polynomial in the quadratic Casimir, (2.28). Let us introduce a collective oscillator $\mathbf{A}^\dagger_A = (\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger_a)$. A general state in $V_\mathrm{F} \times V_\mathrm{F}$ can be written as

$$|s_1, \ldots, s_n; A\rangle = \mathbf{A}^\dagger_{s_1,A_1} \ldots \mathbf{A}^\dagger_{s_n,A_n}|00\rangle \tag{4.25}$$

subject to the central charge constraints $C_1|X\rangle = C_2|X\rangle = 0$. The label $s_k = 1,2$ determines the site on which the k-th oscillator acts. It is easily seen that the Casimir operator J_{12}^2 conserves the number of each type of oscillator; it can however move oscillators between both sites. Therefore the action of $f(J_{12})$ is

$$f(J_{12})|s_1, \ldots, s_n; A\rangle = \sum_{s'_1,\ldots s'_n} c_{s,s',A}\, \delta_{C_1,0}\delta_{C_2,0}\, |s'_1, \ldots, s'_n; A\rangle \tag{4.26}$$

with some coefficients $c_{s,s',A}$. The sums go over the sites $1,2$ and $\delta_{C_1,0}$, $\delta_{C_2,0}$ project to states where the central charge at each site is zero. In view of the fact that oscillators represent indices of fields, see (4.10), a generic invariant operators $f(J_{12})$ acts on two fields by moving indices between them.

Harmonic action. The action of the harmonic numbers within the Hamiltonian density $H_{12} = 2h(J_{12})$ turns out to be particularly simple. It does not depend on the types of oscillators A_k, but only on the number of oscillators which change the site

$$H_{12}|s_1, \ldots, s_n; A\rangle = \sum_{s'_1,\ldots s'_n} c_{n,n_{12},n_{21}}\, \delta_{C_1,0}\delta_{C_2,0}\, |s'_1, \ldots, s'_n; A\rangle. \tag{4.27}$$

Here n_{12}, n_{21} count the number of oscillators hopping from site 1 to 2 or vice versa. The coefficients $c_{n,n_{12},n_{21}}$ are given by

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}} \frac{\Gamma\bigl(\frac{1}{2}(n_{12}+n_{21})\bigr)\Gamma\bigl(1+\frac{1}{2}(n-n_{12}-n_{21})\bigr)}{\Gamma\bigl(1+\frac{1}{2}n\bigr)}. \tag{4.28}$$

In the special case of no oscillator hopping we find

$$c_{n,0,0} = h(\tfrac{1}{2}n), \tag{4.29}$$

which can be regarded as a regularisation of (4.28). We will refer to this action given by (4.27),(4.28),(4.29) as the 'harmonic action'.

Proof. To prove that H_{12} is given by (4.27),(4.28),(4.29) it suffices to show

$$[J_{12}, H_{12}] = 0, \qquad H_{12}V_j = 2h(j)V_j. \tag{4.30}$$

The invariance of H_{12} under the subalgebra $\mathfrak{sl}(2) \times \mathfrak{sl}(2|4)$ given by the generators $\dot{L}, L, R, D, Q, S$ is straightforward: These generators only change the types of oscillators, whereas the harmonic action does not depend on that. In contrast, the remaining generators $K, P, \dot{Q}, \dot{S}$ change the number of oscillators by two.

Let us act with $P_{12,\alpha\dot\beta}$ on a generic state

$$P_{12,\alpha\dot\beta}|s_1, \ldots, s_n; A\rangle = |1, 1, s_1, \ldots, s_n; A'\rangle + |2, 2, s_1, \ldots, s_n; A'\rangle, \tag{4.31}$$

and get a state with two new oscillators, $A' = (\alpha, \dot\beta, A)$. The action of the Hamiltonian density (4.27) yields eight terms. In two of these terms both new oscillators end up at site 1

$$c_{n+2,n_{12},n_{21}}|1, 1, s'_1, \ldots, s'_n; A'\rangle + c_{n+2,n_{12},n_{21}+2}|1, 1, s'_1, \ldots, s'_n; A'\rangle. \tag{4.32}$$

Here n_{12}, n_{21} refer only to the hopping of the old oscillators. Eq. (4.28) can be used to combine the two coefficients in one

$$c_{n+2,n_{12},n_{21}} + c_{n+2,n_{12},n_{21}+2} = c_{n,n_{12},n_{21}}. \tag{4.33}$$

We pull the additional two oscillators out of the state and get

$$\bigl(c_{n+2,n_{12},n_{21}} + c_{n+2,n_{12},n_{21}+2}\bigr)|1, 1, s'_1, \ldots, s'_n; A'\rangle = P_{1,\alpha\dot\beta}\, c_{n,n_{12},n_{21}}|s'_1, \ldots, s'_n; A\rangle \tag{4.34}$$

Summing up all contributions therefore yields $P_{1,\alpha\dot\beta}H_{12}|s_1, \ldots, s_n; A\rangle$. If both new oscillators end up at site 2 we get an equivalent result. It remains to be shown that the other four terms cancel. Two of these are

$$c_{n+2,n_{12},n_{21}+1}|1, 2, s'_1, \ldots, s'_n; A'\rangle + c_{n+2,n_{12}+1,n_{21}}|1, 2, s'_1, \ldots, s'_n; A'\rangle. \tag{4.35}$$

We see that the absolute values in (4.28) match for $c_{n+2,n_{12},n_{21}+1}$ and $c_{n+2,n_{12}+1,n_{21}}$, we sum up the signs

$$(-1)^{1+n_{12}n_{21}+n_{12}} + (-1)^{1+n_{12}n_{21}+n_{21}} = (-1)^{1+n_{12}n_{21}}\big((-1)^{n_{12}} + (-1)^{n_{21}}\big). \tag{4.36}$$

Now, oscillators always hop in pairs due to the central charge constraint. One of the new oscillators has changed the site, so the number of old oscillators changing site must be odd. The above two signs must be opposite and cancel in the sum. The same is true for the remaining two terms. This concludes the proof for $[P_{12,\alpha\dot{\beta}}, H_{12}] = 0$ and similarly, for invariance under $\dot{Q}$. To prove invariance under $K, \dot{S}$ we note that these generators remove two oscillators from one of the two sites. Assume it will remove the first two oscillators from a state (for each two oscillators that are removed, the argument will be the same). Now, the argument is essentially the same as the proof for $P_{12,\alpha\dot{\beta}}$ read in the opposite direction.

To prove that the eigenvalues of H_{12} are given by $2h(j)$ we act on a special state within V_j. We define

$$|j+2,k\rangle = \frac{(\mathbf{a}^\dagger_{1,1})^{k+2}(\mathbf{b}^\dagger_{1,1})^{k}(\mathbf{a}^\dagger_{2,1})^{j-k+2}(\mathbf{b}^\dagger_{2,1})^{j-k}}{(k+2)!k!(j-k+2)!(j-k)!}\,|00\rangle \tag{4.37}$$

which corresponds to $\mathcal{D}^k\mathcal{F}\,\mathcal{D}^{j-k}\mathcal{F}$. Then

$$|j+2\rangle = \sum_{k=0}^{j}(-1)^k\,|j+2,k\rangle \tag{4.38}$$

is a representative of V_{j+2}. It is therefore an eigenstate of H_{12} and we can choose to calculate only the coefficient of $|j+2,0\rangle$ in $H_{12}|j+2\rangle$. Using some combinatorics we find the coefficient

$$h(j+2) + \sum_{l=0}^{2}\sum_{k=0}^{j}(-1)^{1+k+l}\,\frac{\delta_{k+l\neq 0}}{k+l}\,\frac{2}{l!(2-l)!}\,\frac{j!}{k!(j-k)!} = 2h(j+2), \tag{4.39}$$

where l represents the number of oscillators $\mathbf{a}^\dagger_{1,2}$ hopping to site 1. For $j = 0, 1$ we define the states

$$\begin{aligned}
|j=0\rangle &= a^\dagger_{1,1}a^\dagger_{1,1}a^\dagger_{2,1}a^\dagger_{2,1}|00\rangle + a^\dagger_{2,1}a^\dagger_{2,1}a^\dagger_{1,1}a^\dagger_{1,1}|00\rangle - 2a^\dagger_{1,1}a^\dagger_{2,1}a^\dagger_{1,1}a^\dagger_{2,1}|00\rangle,\\
|j=1\rangle &= a^\dagger_{1,1}a^\dagger_{1,1}a^\dagger_{2,1}a^\dagger_{2,1}|00\rangle - a^\dagger_{2,1}a^\dagger_{2,1}a^\dagger_{1,1}a^\dagger_{1,1}|00\rangle.
\end{aligned} \tag{4.40}$$

It is a straightforward exercise to show that $H_{12}|j=0\rangle = 0$ and $H_{12}|j=1\rangle = 2|j=1\rangle$. This concludes the proof of (4.30).

Physical vacuum. We can also describe the harmonic action using the primary vacuum $|Z\rangle$. Then the collective oscillator is $\mathbf{A}^\dagger_A = (\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot{\alpha}}, \mathbf{c}^\dagger_a, \mathbf{d}^\dagger_{\dot{a}})$. A generic state in $V_\mathrm{F} \times V_\mathrm{F}$ is defined in analogy to (4.25). Interestingly, we find that the action of the Hamiltonian density is obtained using exactly the same expressions (4.27),(4.28),(4.29).

Invariance of H_{12} can be shown as above and the proof for the eigenvalues is very similar. We might use the states

$$|j,k\rangle = \frac{(\mathbf{a}^\dagger_{1,1})^k(\mathbf{b}^\dagger_{1,1})^k(\mathbf{a}^\dagger_{2,1})^{j-k}(\mathbf{b}^\dagger_{2,1})^{j-k}}{k!^2(j-k)!^2}\,|ZZ\rangle, \tag{4.41}$$

which correspond to the state $k!(j-k)!(\mathbf{a}^\dagger_1)^k(\mathbf{a}^\dagger_2)^{j-k}|00\rangle$ of Sec. 3. Then

$$|j\rangle = \sum_{k=0}^{j}(-1)^k\,|j,k\rangle \tag{4.42}$$

corresponds to $j!\,|j\rangle$ of (3.12) and belongs to the module V_j. As above we consider only the coefficient of $|j,0\rangle$ in $H_{12}|j\rangle$. It is given by

$$h(j) + \sum_{k=1}^{j}\frac{(-1)^{1+k}j!}{k\,k!(j-k)!} = 2h(j), \tag{4.43}$$

which proves that $H_{12} = 2h(J_{12})$.

4.4 Some examples

We will now determine the planar anomalous dimension of a couple of operators to demonstrate how to apply the above Hamiltonian.

Konishi. The Konishi operator has weight $[2;0,0;0,0,0;0,2]$. Using (4.21),(4.22) we find that we have to excite each of the four oscillators $\mathbf{c}$ once. There must be two oscillators on each site due to the central charge constraint and the three distinct configurations are

$$\begin{aligned}
|2112\rangle &= \mathbf{c}^\dagger_{2,1}\mathbf{c}^\dagger_{1,2}\mathbf{c}^\dagger_{1,3}\mathbf{c}^\dagger_{2,4}|00\rangle = \operatorname{Tr}\bar{X}X,\\
|1212\rangle &= \mathbf{c}^\dagger_{1,1}\mathbf{c}^\dagger_{2,2}\mathbf{c}^\dagger_{1,3}\mathbf{c}^\dagger_{2,4}|00\rangle = \operatorname{Tr}\bar{Y}Y,\\
|1122\rangle &= \mathbf{c}^\dagger_{1,1}\mathbf{c}^\dagger_{1,2}\mathbf{c}^\dagger_{2,3}\mathbf{c}^\dagger_{2,4}|00\rangle = \operatorname{Tr}\bar{Z}Z.
\end{aligned} \tag{4.44}$$

These can also be written in terms of three complex scalars X, Y, Z of $\mathcal{N}=4$ SYM and their conjugates. We can define them as

$$\begin{aligned}
X &= \mathbf{c}^\dagger_1\mathbf{c}^\dagger_4|0\rangle, \quad & Y &= \mathbf{c}^\dagger_2\mathbf{c}^\dagger_4|0\rangle, \quad & Z &= \mathbf{c}^\dagger_3\mathbf{c}^\dagger_4|0\rangle,\\
\bar{X} &= \mathbf{c}^\dagger_2\mathbf{c}^\dagger_3|0\rangle, \quad & \bar{Y} &= \mathbf{c}^\dagger_3\mathbf{c}^\dagger_1|0\rangle, \quad & \bar{Z} &= \mathbf{c}^\dagger_1\mathbf{c}^\dagger_2|0\rangle.
\end{aligned} \tag{4.45}$$

Let us now act with H_{12} on these states, we find

$$\begin{aligned}
H_{12}|1122\rangle &= c_{4,0,0}|1122\rangle + c_{4,1,1}|1221\rangle + c_{4,1,1}|1212\rangle\\
&\quad + c_{4,1,1}|2121\rangle + c_{4,1,1}|2112\rangle + c_{4,2,2}|2211\rangle\\
&= \tfrac{3}{2}|1122\rangle + \tfrac{1}{2}|1221\rangle + \tfrac{1}{2}|1212\rangle + \tfrac{1}{2}|2121\rangle + \tfrac{1}{2}|2112\rangle - \tfrac{1}{2}|2211\rangle\\
&= |2112\rangle + |1212\rangle + |1122\rangle
\end{aligned} \tag{4.46}$$

using (4.27),(4.28),(4.29) and cyclicity of the trace. Evaluating the Hamiltonian for the remaining two states $|1212\rangle$ and $|1221\rangle$ we find the energy matrix

$$H = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \tag{4.47}$$

the factor of 2 is due to $H = H_{12} + H_{21}$. One eigenstate is

$$\mathcal{K} = |2112\rangle + |1212\rangle + |1122\rangle = \operatorname{Tr} \bar{X}X + \operatorname{Tr} \bar{Y}Y + \operatorname{Tr} \bar{Z}Z. \tag{4.48}$$

with energy

$$E = 6, \qquad \delta\Delta = E \times \frac{g_{\scriptscriptstyle\mathrm{YM}}^2 N}{8\pi^2} = \frac{3g_{\scriptscriptstyle\mathrm{YM}}^2 N}{4\pi^2} \tag{4.49}$$

which clearly corresponds to the Konishi operator. The other two, $|2112\rangle - |1122\rangle$ and $|1212\rangle - |1122\rangle$, have vanishing energy and correspond to half-BPS operators.

Physical vacuum. Let us repeat this example using the physical vacuum $|Z\rangle$. Here, we can define

$$\begin{aligned} X &= \mathbf{c}_1^\dagger \mathbf{d}_1^\dagger |Z\rangle, \quad Y = \mathbf{c}_2^\dagger \mathbf{d}_1^\dagger |Z\rangle, \quad Z = |Z\rangle, \\ \bar{X} &= \mathbf{d}_2^\dagger \mathbf{c}_2^\dagger |Z\rangle, \quad \bar{Y} = \mathbf{c}_1^\dagger \mathbf{d}_2^\dagger |Z\rangle, \quad \bar{Z} = \mathbf{c}_1^\dagger \mathbf{c}_2^\dagger \mathbf{d}_2^\dagger \mathbf{d}_1^\dagger |Z\rangle. \end{aligned} \tag{4.50}$$

The three states are

$$\begin{aligned} |1212'\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{2,2}^\dagger \mathbf{d}_{1,1}^\dagger \mathbf{d}_{2,2}^\dagger |ZZ\rangle = \operatorname{Tr} \bar{X}X, \\ |1221'\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{2,2}^\dagger \mathbf{d}_{2,1}^\dagger \mathbf{d}_{1,2}^\dagger |ZZ\rangle = \operatorname{Tr} \bar{Y}Y, \\ |1111'\rangle &= \mathbf{c}_{1,1}^\dagger \mathbf{c}_{1,2}^\dagger \mathbf{d}_{1,1}^\dagger \mathbf{d}_{1,2}^\dagger |ZZ\rangle = -\operatorname{Tr} \bar{Z}Z. \end{aligned} \tag{4.51}$$

In the same way as above we get the energy matrix

$$H = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \tag{4.52}$$

The eigenstates and energies are the same as above, e.g.

$$\mathcal{K} = |1212'\rangle + |1221'\rangle - |1111'\rangle = \operatorname{Tr} \bar{X}X + \operatorname{Tr} \bar{Y}Y + \operatorname{Tr} \bar{Z}Z. \tag{4.53}$$

Pseudovacua. In the previous paragraph we have worked with the physical vacuum made from the scalar field $|Z\rangle$. This vacuum is the ground state, it is a protected half-BPS state with zero energy. There are, however, similar configurations where we assume the same field at each site. We find three possible fields to choose (including the above)

$$|Z\rangle = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle, \qquad |\mathit{\Psi}\rangle = \mathbf{a}_1^\dagger \mathbf{c}_4^\dagger |0\rangle, \qquad |\mathcal{F}\rangle = \mathbf{a}_1^\dagger \mathbf{a}_1^\dagger |0\rangle. \tag{4.54}$$

In each case the states of two sites $|ZZ\rangle, |\mathit{\Psi\Psi}\rangle, |\mathcal{FF}\rangle$ are eigenstates of the Hamiltonian density, the corresponding eigenvalues are $0, 2, 3$. Thus the energies of the vacua $|Z, L\rangle, |\mathit{\Psi}, L\rangle, |\mathcal{F}, L\rangle$ constructed from L such fields are

$$E_Z = 0, \qquad E_{\mathit{\Psi}} = 2L, \qquad E_{\mathcal{F}} = 3L, \tag{4.55}$$

where $|\Psi, L\rangle$ exists only for odd L. The $\mathfrak{sl}(4|4)$ weights of these vacua and the corresponding primaries are given by

$$
\begin{aligned}
&w_Z = [L; 0, 0; 0, L, 0; 0, L], && w_{Z,0} = [L; 0, 0; 0, L, 0; 0, L],\\
&w_\Psi = [\tfrac{3}{2}L; L, 0; 0, 0, L; \tfrac{1}{2}L, L], && w_{\Psi,0} = [\tfrac{3}{2}L - \tfrac{5}{2}; L-3, 0; 0, 0, L-3; \tfrac{1}{2}L - \tfrac{3}{2}, L-1],\\
&w_{\mathcal{F}} = [2L; 2L, 0; 0, 0, 0; L, L], && w_{\mathcal{F},0} = [2L-2; 2L-4, 0; 0, 0, 0; L-2, L].
\end{aligned}
\tag{4.56}
$$

Of these three states, only $|Z, L\rangle$ is stable, it does not receive corrections at higher loops. The other two states are expected to receive corrections to their form; to determine their higher-loop energy becomes a non-trivial issue.

Spectrum of operators with a low dimension. We have used the harmonic action to compute the planar one-loop spectrum of low-lying states in $\mathcal{N} = 4$ SYM. First of all we have determined the primary states using the algorithm proposed in [37]. In analogy to the sieve of Eratostene the algorithm subsequently removes descendants from the set of all states. What remains, are the primary states. For the primaries we have determined the number of oscillator excitations using (4.21),(4.22),(4.23). Next we have spread the oscillators on the sites in all possible distinct ways. The harmonic action (4.27),(4.28),(4.29) then yields an energy matrix that was subsequently diagonalised. For all the descendants that were removed in the sieve algorithm, we remove the corresponding energy eigenvalue. The remaining eigenvalue is the one-loop planar anomalous dimension of the primary operator.

The single-trace superconformal primaries of $\mathcal{N} = 4$ SYM for $\Delta_0 \leq 5.5$ and their planar one-loop anomalous dimensions are given in Tab. 3.[12] For a given primary weight we write the anomalous dimensions along with a parity P. Here, parity is defined such that for a $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$ gauge group the states with negative parity are projected out.[13] Parity $P = \pm$ indicates a pair of states with opposite parity but degenerate energy. Furthermore, we have indicated states with conjugate representations for which the order of $\mathfrak{sl}(2)^2$ and $\mathfrak{sl}(4)$ labels as well as the chirality B is inverted.

There are two important points to note looking at the energies in Tab. 3. Firstly, we note the appearance of paired states with $P = \pm$. As was argued in [22] this is an indication of integrability. Indeed, not only the planar $\mathfrak{so}(6)$ Hamiltonian of Minahan and Zarembo [24] is integrable, but also the *complete* planar $\mathfrak{sl}(4|4)$ Hamiltonian [34]! Secondly, we find some overlapping primaries in Tab. 2,4, clearly their energies do agree. What is more, we find that a couple of energies repeatedly occur. These are for example, $\frac{3}{4}, \frac{5}{4}, \frac{5}{8}, \frac{9}{8}$, but also $\frac{1}{8}(5 \pm \sqrt{5})$ and $\frac{1}{16}(13 \pm \sqrt{41})$. As these states are primaries transforming in different representations, they cannot be related by $\mathfrak{sl}(4|4)$. Of course, these degeneracies could merely be a coincidence of small numbers. Nevertheless the

[12] The table was computed as follows. A `C++` programme was used to determine all superconformal primary operators up to and including classical dimension 5.5 as well as their descendants. For the details of the implementation of the sieve algorithm see [39]. In a `Mathematica` programme all operators with a given number of excitations of the oscillators were collected; this involved up to hundreds of states for $\Delta_0 = 5.5$. In a second step, the harmonic action was applied to all these operators to determine the matrix of anomalous dimensions; this took a few minutes. The relevant eigenvalue corresponding to the primary operator was sorted out by hand.

[13] Therefore, our definition of parity differs from the one of [22] by a factor of $(-1)^L$.

Δ_0	$\mathfrak{sl}(2)^2$	$\mathfrak{sl}(4)$	B	L	$\delta\Delta^P\ [g^2_{\scriptscriptstyle\mathrm{YM}}N/\pi^2]$	
2	$[0,0]$	$[0,2,0]$	0	2	0^+	
	$[0,0]$	$[0,0,0]$	0	2	$\frac{3}{4}^+$	
3	$[0,0]$	$[0,3,0]$	0	3	0^-	
	$[0,0]$	$[0,1,0]$	0	3	$\frac{1}{2}^-$	
4	$[0,0]$	$[0,4,0]$	0	4	0^+	
	$[0,0]$	$[0,2,0]$	0	4	$\frac{1}{8}(5\pm\sqrt{5})^+$	
	$[0,0]$	$[1,0,1]$	0	4	$\frac{3}{4}^-$	
	$[0,0]$	$[0,0,0]$	0	4	$\frac{1}{16}(13\pm\sqrt{41})^+$	
	$[2,0]$	$[0,0,0]$	1	3	$\frac{9}{8}^-$	+ conj.
	$[1,1]$	$[0,1,0]$	0	3	$\frac{15}{16}^\pm$	
	$[2,2]$	$[0,0,0]$	0	2	$\frac{25}{24}^+$	
5	$[0,0]$	$[0,5,0]$	0	5	0^-	
	$[0,0]$	$[0,3,0]$	0	5	$\frac{1}{4}^-, \frac{3}{4}^-$	
	$[0,0]$	$[1,1,1]$	0	5	$\frac{5}{8}^\pm$	
	$[0,0]$	$[0,0,2]$	0	5	$\frac{1}{8}(7\pm\sqrt{13})^+$	+ conj.
	$[0,0]$	$[0,1,0]$	0	5	$\frac{5}{4}^-, \frac{5}{4}^-, \frac{1}{8}(5\pm\sqrt{5})^-$	
	$[2,0]$	$[0,0,2]$	1	4	$\frac{5}{4}^-$	+ conj.
	$[2,0]$	$[0,1,0]$	1	4	$\frac{1}{8}(8\pm\sqrt{2})^+$	+ conj.
	$[1,1]$	$[0,2,0]$	0	4	$\frac{3}{4}^\pm$	
	$[1,1]$	$[1,0,1]$	0	4	$\frac{5}{8}^\pm, \frac{5}{4}^\pm$	
	$[1,1]$	$[0,0,0]$	0	4	$\frac{9}{8}^\pm$	
	$[2,2]$	$[0,1,0]$	0	3	$\frac{3}{4}^-$	
5.5	$[1,0]$	$[0,2,1]$	$\frac{1}{2}$	5	$1^\pm$	+ conj.
	$[1,0]$	$[1,1,0]$	$\frac{1}{2}$	5	$\frac{1}{8}(8\pm\sqrt{2})^\pm$	+ conj.
	$[1,0]$	$[0,0,1]$	$\frac{1}{2}$	5	$\frac{1}{32}(35\pm\sqrt{5})^\pm$	+ conj.
	$[2,1]$	$[0,1,1]$	$\frac{1}{2}$	4	$\frac{9}{8}^\pm$	+ conj.
	$[2,1]$	$[1,0,0]$	$\frac{1}{2}$	4	$\frac{1}{32}(37\pm\sqrt{37})^\pm$	+ conj.
	$[3,2]$	$[0,0,1]$	$\frac{1}{2}$	3	$\frac{5}{4}^\pm$	+ conj.

Table 3: All one-loop planar anomalous dimensions of primary operators with $\Delta_0 \le 5.5$. Positive parity P indicates a state that survives the projection to gauge groups $\mathrm{SO}(N)$, $\mathrm{Sp}(N)$, parity $\pm$ indicates a degenerate pair. The label '+conj.' indicates conjugate states with $\mathfrak{sl}(2)^2, \mathfrak{sl}(4)$ labels reversed and opposite chirality B.

reappearance of e.g. $\frac{1}{16}(13 \pm \sqrt{41})$ is striking. This could hint at a further symmetry enhancement of the planar one-loop Hamiltonian. It might also turn out to be a consequence of integrability. Furthermore, one might speculate that it is some remnant of the broken higher spin symmetry of the free theory, see e.g. [40] and references in [37].

5 Subsectors

Especially in view of some recent advances of the AdS/CFT correspondence [3,4] it has become interesting to determine anomalous dimensions of specific operators in $\mathcal{N}=4$ SYM. In principle, the Hamiltonian (2.22),(2.24) provides the answer, but it is hard to apply. The harmonic action (4.27),(4.28),(4.29) is more explicit, but still requires a reasonable amount of combinatorics to be applied. Nevertheless, some operators corresponding to stringy states have special quantum numbers and often one can restrict to certain subsectors of $\mathcal{N}=4$, see e.g. [25]. For instance, in Sec. 3 we have investigated a subsector of $\mathcal{N}=4$ SYM. Within this subsector the number of letters as well as the symmetry algebra is reduced. This reduction of complexity leads to a simplification of the Hamiltonian (3.14) within the subsector. Thus, restricting to subsectors one can efficiently compute anomalous dimensions.

To construct subsectors, we note that the number of excitations in the oscillator picture, (4.21),(4.22), naturally puts constraints on the weights of operators. Certainly, there cannot be negative excitations. Furthermore, the oscillators $\mathbf{c}^\dagger$ are fermionic. Therefore there can only be one excitation on each site. In total we find twelve bounds

$$n_{\mathbf{a}} \geq 0, \quad n_{\mathbf{b}} \geq 0, \quad n_{\mathbf{c}} \geq 0, \quad L - n_{\mathbf{c}} \geq 0. \tag{5.1}$$

At the one-loop level all these excitation numbers are conserved. We can therefore construct 'one-loop subsectors' by considering operators for which several of these bounds are met. In certain cases these subsectors remain closed even at higher loops, we will refer to these as 'closed subsectors'.

5.1 Closed subsectors

Let us demonstrate this procedure for a rather trivial subsector. We will consider the subsector of operators with

$$n_{\mathbf{a}_{12}} = n_{\mathbf{b}^\dagger_{12}} = n_{\mathbf{c}_{12}} = 0, \quad n_{\mathbf{c}_{34}} = L. \tag{5.2}$$

In conventional language these operators consist only of the fields

$$Z = \mathbf{c}^\dagger_3 \mathbf{c}^\dagger_4 |0\rangle = |Z\rangle = \Phi_{34} = \Phi_{5+i6} = \Phi_5 + i\Phi_6. \tag{5.3}$$

These are the half-BPS operators $\operatorname{Tr} Z^L$ and its multi-trace cousins. The Hamiltonian within this subsector vanishes identically, as required by protectedness of BPS operators. Using (4.21),(4.22) the constraints (5.2) allows only the weight

$$w = [L; 0, 0; 0, L, 0; 0, L]. \tag{5.4}$$

So far we know only that this subsector is closed at one-loop. To prove closure at higher loops we consider a strictly positive combination of the bounds

$$n_{\mathbf{a}_1} + n_{\mathbf{a}_2} + n_{\mathbf{b}_1} + n_{\mathbf{b}_2} + n_{\mathbf{c}_1} + n_{\mathbf{c}_2} + (L - n_{\mathbf{c}_3}) + (L - n_{\mathbf{c}_4}) = 0. \tag{5.5}$$

Together with the bounds (5.1) this implies that each of the individual terms is zero as in (5.2). Using (4.21),(4.22) this combination implies

$$\Delta_0 = p. \tag{5.6}$$

The label p as well as the *bare* dimension Δ_0 is preserved in perturbation theory.[14] Therefore the condition (5.6) restricts to this subsector at all orders in perturbation theory which means that this subsector is exactly closed. We also have that $L = p$, which implies that the length is protected even at higher loops. Equivalently, the chirality is exactly zero.

Let us investigate all closed subsectors. For a closed subsector we need to find a positive linear combination of the bounds that is independent of B and L. Put differently, it must be independent of $L - B$ and $L + B$. The number of excitations $n_{\mathbf{b}}$ involves the combination $-B - L$. This can only be cancelled by $B + L$ in $L - n_{\mathbf{c}}$. Therefore, we can remove oscillators of type **b** only iff we also fully excite oscillators of type **c**. Equivalently, we can remove oscillators of type **a** only iff we also remove oscillators of type **c**. We find the following cases:

- If no oscillator is removed we get the full theory.
- If we set $n_{\mathbf{b}_2} = 0$ and $n_{\mathbf{c}_i} = L$, $i = k+1, \ldots, 4$, the remaining symmetry algebra will be $\mathfrak{u}(1,2|k)$. The non-compact form of the algebra is meant to indicate that there are infinitely many letters within this subsector. Furthermore, $B + L$ can be expressed in terms of p, q_1, q_2, it is therefore conserved.
- If we set $n_{\mathbf{b}_{1,2}} = 0$ then we should only set $n_{\mathbf{c}_4} = L$ in order to get a nontrivial subsector. The remaining symmetry algebra is $\mathfrak{u}(2|3)$ and will be discussed in Sec. 5.3. It has conserved $B + L$.
- If we set $n_{\mathbf{a}_2} = n_{\mathbf{b}_2} = n_{\mathbf{c}_i} = 0$, $i = 1, \ldots k$, and $n_{\mathbf{c}_j} = L$, $j = l+1, \ldots 4$ the remaining symmetry algebra will be $\mathfrak{u}(1,1|l-k)$. Here, B and L are conserved. For $k = l = 2$ we get the subsector considered in Sec. 3. For $k = l = 1, 3$ we get a similar subsector in which the spin of the letters equals -1 instead of $-\frac{1}{2}$.
- If we set $n_{\mathbf{a}_2} = n_{\mathbf{b}_{1,2}} = n_{\mathbf{c}_i} = 0$, $i = 1, \ldots k$, and $n_{\mathbf{c}_4} = L$ the remaining symmetry algebra will be $\mathfrak{u}(1|3-k)$. Here, B and L are conserved.
- If we set $n_{\mathbf{a}_{1,2}} = n_{\mathbf{b}_{1,2}} = n_{\mathbf{c}_1} = 0$, and $n_{\mathbf{c}_4} = L$ the remaining symmetry algebra will be $\mathfrak{su}(2)$. Here, B and L are conserved. This subsector will be discussed in Sec. 5.2.

[14] The scaling dimension is obviously not preserved. The bare dimension is, however, for mixing occurs only among operators of equal bare dimension.

5.2 The nearly quarter-BPS $\mathfrak{su}(2)$ subsector

In Sec. 5.1 we have found that we can express the half-BPS condition $\Delta_0 = p$ in terms of the excitation numbers of oscillators. We can do the same for the quarter BPS condition

$$\Delta_0 = p + q_1 + q_2. \tag{5.7}$$

This is equivalent to

$$n_{\mathbf{a}_1} + n_{\mathbf{a}_2} + n_{\mathbf{b}_1} + n_{\mathbf{b}_2} + 2n_{\mathbf{c}_1} + 2(L - n_{\mathbf{c}_4}) = 0, \tag{5.8}$$

which implies that this subsector is closed. The letters of this subsector are

$$Z = \mathbf{c}_3^\dagger \mathbf{c}_4^\dagger |0\rangle, \quad \phi = \mathbf{c}_2^\dagger \mathbf{c}_4^\dagger |0\rangle. \tag{5.9}$$

The weights are

$$w = [L + \delta\Delta; 0, 0; n, L - 2n, n; 0, L], \tag{5.10}$$

where n counts the number of ϕ's. The length can be expressed in terms of $\mathfrak{sl}(4)$ charges, $L = p + q_1 + q_2$, it is therefore protected within the subsector. The residual symmetry is $\mathfrak{su}(2) \times \mathfrak{u}(1)$. The $\mathfrak{su}(2)$ factor transforms Z and ϕ in the fundamental representation, whereas $\mathfrak{u}(1)$ measures the anomalous dimension $\delta\Delta$. A state with $\delta\Delta = 0$ is (at least) quarter-BPS, a generic state, however, will not be protected. Then the weight w is beyond the unitarity bounds and cannot be primary. The corresponding primary weight is (assuming a highest weight state of the residual $\mathfrak{su}(2)$)

$$w_0 = [L - 2 + \delta\Delta; 0, 0; n - 2, L - 2n, n - 2; 0, L - 2], \tag{5.11}$$

This $\mathfrak{su}(2)$ subsector was studied in [22], where also the two-loop contribution to the dilatation operator was found

$$\begin{aligned}
\delta D(g) = &- \frac{g_{\mathrm{YM}}^2}{8\pi^2} :\!\mathrm{Tr}[\phi, Z][\check{\phi}, \check{Z}]\!: \\
&- \frac{1}{2}\left(\frac{g_{\mathrm{YM}}^2}{8\pi^4}\right)^2 \Big(:\!\mathrm{Tr}\left[[\phi, Z], \check{Z}\right]\left[[\check{\phi}, \check{Z}], Z\right]\!: + :\!\mathrm{Tr}\left[[\phi, Z], \check{\phi}\right]\left[[\check{\phi}, \check{Z}], \phi\right]\!: \\
&\qquad + :\!\mathrm{Tr}\left[[\phi, Z], T^a\right]\left[[\check{\phi}, \check{Z}], T^a\right]\!: \Big) + \mathcal{O}(g^6).
\end{aligned} \tag{5.12}$$

Using this, the first few states and their planar anomalous dimensions were found, we list the one-loop part in Tab. 4.

Furthermore, it was confirmed that the quarter-BPS operators found in [17, 41] are annihilated by the two-loop anomalous dilatation operator $\delta D(g)$. Here, we make the observation that they are indeed annihilated by the much simpler operator $\mathcal{Q}$ (which takes values in the gauge algebra, $\mathcal{Q} = T^a \mathcal{Q}^a$)

$$\mathcal{Q} = [\check{Z}, \check{\phi}]. \tag{5.13}$$

As all terms of $\delta D(g)$ contain this operator, it is clear that $\delta D(g)$ annihilates those quarter-BPS operators. We conjecture that all quarter-BPS operators are in the kernel of $\mathcal{Q}$

$$\mathcal{Q}\, \mathcal{O}_{1/4-\mathrm{BPS}} = 0. \tag{5.14}$$

L	n	$\delta\Delta^P\,[g^2_{\mathrm{YM}}N/\pi^2]$
4	2	$\frac{3}{4}^{+}$
5	2	$\frac{1}{2}^{-}$
6	2	$\frac{1}{8}(5\pm\sqrt{5})^{+}$
	3	$\frac{3}{4}^{-}$
7	2	$\frac{1}{4}^{-}, \frac{3}{4}^{-}$
	3	$\frac{5}{8}^{\pm}$

L	n	$\delta\Delta^P\,[g^2_{\mathrm{YM}}N/\pi^2]$
8	2	$(64x^3-112x^2+56x-7)^{+}$
	3	$\frac{1}{2}^{\pm}, \frac{3}{4}^{-}$
	4	$(64x^3-160x^2+116x-25)^{+}$
9	2	$\frac{1}{4}(2\pm\sqrt{2})^{-}, \frac{1}{2}^{-}$
	3	$(512x^3-1088x^2+720x-147)^{\pm}$
	4	$\frac{5}{8}^{\pm}, \frac{1}{4}(3\pm\sqrt{3})^{-}$

Table 4: The first few states within the $\mathfrak{su}(2)$ subsector [22]. The weights of the corresponding primaries are $[L-2;0,0;n-2,L-2n,n-2;0,L-2]$. Cubic polynomials indicate three states with energies given by the roots of the cubic equation.

This should simplify the search for quarter BPS operators drastically. We furthermore conjecture that $\mathcal{Q}$ is an essential part in the first correction to the superboosts,

$$\delta S(g) \sim g\ \mathrm{Tr}\,\Psi\mathcal{Q}, \quad \delta\dot{S}(g) \sim g\ \mathrm{Tr}\,\bar{\Psi}\mathcal{Q}. \tag{5.15}$$

These generators would join the quarter-BPS multiplet at primary weight w with three semi-short multiplets into the long interacting multiplet at primary weight w_0. In the case of the a quarter-BPS operator, this does not happen and the multiplet remains short [7, 42].

5.3 The nearly eighth-BPS $\mathfrak{u}(2|3)$ subsector

In analogy to Sec. 5.2 we investigate the sector of nearly eighth-BPS operators. One of the two eighth-BPS conditions is (the other one would lead to a similar subsector)

$$\Delta_0 = p + \tfrac{1}{2}q_1 + \tfrac{3}{2}q_2. \tag{5.16}$$

This is equivalent to

$$n_{\mathbf{b}_1} + n_{\mathbf{b}_2} + 2(L - n_{\mathbf{c}_4}) = 0, \tag{5.17}$$

which again implies that this subsector is closed. The letters of this subsector are

$$\phi_a = \mathbf{c}^\dagger_a\mathbf{c}^\dagger_4|0\rangle, \qquad \psi_\alpha = \mathbf{a}^\dagger_\alpha\mathbf{c}^\dagger_4|0\rangle, \qquad (a=1,2,3,\ \alpha=1,2) \tag{5.18}$$

These transform in the fundamental representation of $\mathfrak{u}(2|3)$. The central charge of $\mathfrak{u}(2|3)$ is given by the anomalous dimension $\delta\Delta$. We also note that, although L and B are not protected in this subsector, the combination $L+B=\Delta_0$ is. The weights are

$$w = [L+B+\delta\Delta; s, 0; 2B+2L-2p-3q_2, p, q_2; B, L], \tag{5.19}$$

where the numbers of individual letters are given by

$$n_\phi = \begin{pmatrix} p+2q_2-2B-L, \\ L-p-q_2, \\ L-q_2, \end{pmatrix} \qquad n_\psi = \begin{pmatrix} B+\frac{1}{2}s, \\ B-\frac{1}{2}s. \end{pmatrix} \tag{5.20}$$

This subsector has only finitely many letters. The Hamiltonian would therefore have only a few terms and it could be applied easily. Also an investigation of the two-loop contribution to the dilatation operator along the lines of [22] seems feasible. Such an investigation would be very interesting for two reasons. On the one hand one could see effects of the Konishi anomaly. The $\mathcal{O}(g^3)$ ('3/2-loop') contribution to the dilatation operator transforms three scalars $\varepsilon^{abc}\phi_a\phi_b\phi_c$ into two fermions $\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta$ or vice versa. In other words it changes the length of the operator. On the other hand, this sector contains the subsector of Sec. 5.2. In [22] evidence for the integrability of the planar two-loop dilatation operator was found. Possibly the two-loop integrability extends to this subsector and maybe to the full theory.

The lowest-dimensional eighth-BPS operator is expected to be a triple-trace operator with weight

$$w = [6; 0, 0; 0, 0, 4; 0, 6]. \tag{5.21}$$

We find this operator, it is

$$\begin{aligned}\mathcal{O}_{1/8-\mathrm{BPS}} = \varepsilon^{abc}\varepsilon^{def}\big[&N(N^2-3)\,\mathrm{Tr}\,\phi_a\phi_d\,\mathrm{Tr}\,\phi_b\phi_e\,\mathrm{Tr}\,\phi_c\phi_f\\ &+6(N^2-1)\,\mathrm{Tr}\,\phi_a\phi_d\,\mathrm{Tr}\,\phi_b\phi_c\phi_e\phi_f\\ &-12N\,\mathrm{Tr}\,\phi_a\phi_b\phi_c\phi_d\phi_e\phi_f\\ &+8N\,\mathrm{Tr}\,\phi_a\phi_d\phi_b\phi_e\phi_c\phi_f\\ &+4\,\mathrm{Tr}\,\phi_a\phi_b\phi_c\,\mathrm{Tr}\,\phi_d\phi_e\phi_f\big].\end{aligned} \tag{5.22}$$

It is annihilated by the generalisation of (5.13)

$$\mathcal{Q}_a = \varepsilon_{abc}[\check\phi^b, \check\phi^c] \tag{5.23}$$

which implies that it is protected at one-loop. It is also annihilated by the operator

$$D_3 \sim \varepsilon_{abc}\varepsilon^{\alpha\beta}\,\mathrm{Tr}\,\psi_\alpha[\check\phi^a, [\check\phi^b, [\check\phi^c, \psi_\beta]]]. \tag{5.24}$$

An investigation of diagrams shows that the relevant 3/2-loop contribution to the dilatation generator should be proportional to this operator. This supports the claim that the operator (5.22) is eighth-BPS.

5.4 The $\mathfrak{so}(6)$ subsector

This is the subsector where the scalars are the only letters

$$n_{\mathbf{a}_{12}} = n_{\mathbf{b}_{12}} = 0. \tag{5.25}$$

The allowed weights are

$$w = [L; 0, 0; q_1, p, q_2; 0, L]. \tag{5.26}$$

The residual symmetry is $\mathfrak{sl}(4) = \mathfrak{so}(6)$. This subsector is a one-loop subsector. All bounds $n_\mathbf{a}, n_\mathbf{b}$ have a negative coefficient of L. Therefore all positive combinations of bounds involve L, which is broken at higher loops.

We will investigate the non-planar Hamiltonian in this sector. Two scalars $\varPhi_p, \varPhi_q$ can be symmetrised in three different ways, symmetric-traceless, antisymmetric and singlet. These correspond to the modules V_0, V_1, V_2, respectively. The projectors to these representations are

$$\begin{aligned}(P_0)^{pq}_{mn} &= \tfrac{1}{2}\delta^p_m\delta^q_n + \tfrac{1}{2}\delta^p_n\delta^q_m - \tfrac{1}{6}\delta_{mn}\delta^{pq},\\ (P_1)^{pq}_{mn} &= \tfrac{1}{2}\delta^p_m\delta^q_n - \tfrac{1}{2}\delta^p_n\delta^q_m,\\ (P_2)^{pq}_{mn} &= \tfrac{1}{6}\delta_{mn}\delta^{pq},\end{aligned} \tag{5.27}$$

The coefficient (2.20) of the Hamiltonian using the harmonic eigenvalues (2.21) is

$$C^{pq}_{mn} = 0\cdot(P_0)^{pq}_{mn} + 1\cdot\left(\tfrac{1}{2}\delta^p_m\delta^q_n - \tfrac{1}{2}\delta^p_n\delta^q_m\right) + \tfrac{3}{2}\cdot\left(\tfrac{1}{6}\delta_{mn}\delta^{pq}\right) \tag{5.28}$$

We substitute this in the Hamiltonian (2.16) and get

$$\begin{aligned}H &= N^{-1}\big(-\tfrac{1}{2}:\mathrm{Tr}[\varPhi_m,\check{\varPhi}^m][\varPhi_n,\check{\varPhi}^n]: + \tfrac{1}{2}:\mathrm{Tr}[\varPhi_m,\check{\varPhi}^n][\varPhi_n,\check{\varPhi}^m]: - \tfrac{1}{4}:\mathrm{Tr}[\varPhi_m,\check{\varPhi}^n][\varPhi_m,\check{\varPhi}^n]:\big)\\ &= N^{-1}\big(-\tfrac{1}{2}:\mathrm{Tr}[\varPhi_m,\varPhi_n][\check{\varPhi}^m,\check{\varPhi}^n]: - \tfrac{1}{4}:\mathrm{Tr}[\varPhi_m,\check{\varPhi}^n][\varPhi_m,\check{\varPhi}^n]:\big).\end{aligned} \tag{5.29}$$

After multiplication with $g^2_{\mathrm{YM}}N/8\pi^2$ this is exactly the effective vertex found in [19] and yields the dilatation generators in [24, 21, 22].

5.5 The $\mathfrak{so}(4,2)$ subsector

One might choose to set

$$n_{\mathbf{c}_{12}} = 0, \quad n_{\mathbf{c}_{34}} = L, \tag{5.30}$$

or, more conveniently, $n_{\mathbf{c}_{12}} = n_{\mathbf{d}_{12}} = 0$ using the primary vacuum. This is a generalisation of the subsector discussed in Sec. 3, which is, however, closed only at one-loop. In this subsector the letters are the scalars Z with any number of the four spacetime derivatives acting

$$(\mathbf{a}^\dagger_1)^{k_1}(\mathbf{a}^\dagger_2)^{k_2}(\mathbf{b}^\dagger_1)^{l_1}(\mathbf{b}^\dagger_2)^{k_1+k_2-l_1}|Z\rangle. \tag{5.31}$$

The weight of a state is given by

$$w = [\Delta_0; s_1, s_2; 0, L, 0; 0, L] \tag{5.32}$$

where the total numbers of excitations are

$$k_1 = \tfrac{1}{2}\Delta_0 - \tfrac{1}{2}L + \tfrac{1}{2}s_1, \quad k_2 = \tfrac{1}{2}\Delta_0 - \tfrac{1}{2}L - \tfrac{1}{2}s_1, \quad l_1 = \tfrac{1}{2}\Delta_0 - \tfrac{1}{2}L + \tfrac{1}{2}s_2. \tag{5.33}$$

This sector might be useful to investigate semiclassical strings spinning on one circle in S^5 and two circles in AdS^5.

5.6 The $\mathfrak{su}(2|4)$ one-loop BMN matrix model

The BMN matrix model [3] in the one-loop approximation [35] is obtained by setting

$$n_{\mathbf{b}_{12}} = 0. \tag{5.34}$$

Again, this yields only a one-loop subsector. The letters W''_A of the matrix model are

$$\mathbf{c}^\dagger_a\mathbf{c}^\dagger_b|0\rangle,\quad \mathbf{a}^\dagger_\alpha\mathbf{c}^\dagger_b|0\rangle,\quad \mathbf{a}^\dagger_\alpha\mathbf{a}^\dagger_\beta|0\rangle \tag{5.35}$$

corresponding to the $\mathfrak{so}(6)$ vectors, fermions and $\mathfrak{so}(3)$ vectors. The residual symmetry is $\mathfrak{su}(2|4)$. The multiplet of letters V''_{F} is given by the primary weight

$$V''_{\mathrm{F}} = [q_1, p, q_2]^{\Delta_0}_s = [0, 1, 0]^1_0. \tag{5.36}$$

The irreducible modules of two multiplets of letters are

$$V''_{\mathrm{F}} \times V''_{\mathrm{F}} = V''_0 + V''_1 + V''_2 \tag{5.37}$$

with

$$V''_0 = [0, 2, 0]^2_0,\quad V''_1 = [1, 0, 1]^2_0,\quad V''_2 = [0, 0, 0]^2_0. \tag{5.38}$$

In $\mathfrak{sl}(2)\times\mathfrak{sl}(4)$ these modules split into

$$\begin{aligned}
V''_0 &= [0,2,0]^2_0 + [0,1,1]^{2.5}_1 + [0,1,0]^3_2 + [0,0,2]^3_0 + [0,0,1]^{3.5}_1 + [0,0,0]^4_0,\\
V''_1 &= [1,0,1]^2_0 + [0,1,1]^{2.5}_1 + [1,0,0]^{2.5}_1 + [0,0,2]^3_2 + [0,1,0]^3_2 + [0,1,0]^3_0\\
&\quad + [0,0,1]^{3.5}_3 + [0,0,1]^{3.5}_1 + [0,0,0]^4_2,\\
V''_2 &= [0,0,0]^2_0 + [1,0,0]^{2.5}_1 + [0,1,0]^3_2 + [0,0,1]^{3.5}_3 + [0,0,0]^4_4.
\end{aligned} \tag{5.39}$$

In [35] it was found that the one-loop spectrum for the matrix model in the $\mathfrak{so}(6)$ vector sector matches the $\mathfrak{so}(6)$ sector of $\mathcal{N}=4$ SYM. The complete set of modules V''_j of the full $\mathfrak{su}(2|4)$ matrix model is already realised in the $\mathfrak{so}(6)$ subsector. Therefore the $\mathfrak{so}(6)$ subsector lifts uniquely to the full $\mathfrak{su}(2|4)$ matrix model. Thus the restriction of the one-loop $\mathcal{N}=4$ SYM dilatation operator to this subsector agrees with the complete matrix model Hamiltonian. In other words the matrix model Hamiltonian $H'' = MD''/2$ is given by

$$D''(1/M) = D''_0 - \frac{4}{M^3}\left((P''_1)^{AB}_{CD} + \tfrac{3}{2}(P''_2)^{AB}_{CD}\right) :\mathrm{Tr}[W''_A, \check{W}''^C][W''_B, \check{W}''^D]: + \mathcal{O}(M^{-4}), \tag{5.40}$$

where the precise form of the projectors $P''_{1,2}$ remains to be evaluated. Alternatively, the Hamiltonian density could be determined using (4.27),(4.28),(4.29).

6 Outlook

Here we have constructed the highly intricate, first radiative corrections to the (trivial) classical dilatation operator. One might also investigate radiative corrections to the other generators of $\mathfrak{psl}(4|4)$. The generators that receive corrections are the momenta and boosts, $P, K, Q, \dot{Q}, S, \dot{S}$. The closure of the algebra, see App. A, should put tight constraints on these as well as on the dilatation generator. In fact, these might determine the dilatation operator at one-loop or even higher! Let us demonstrate this using the related issue of multiplet splitting at the unitarity bounds [42]: We have constructed H as an invariant operator under the classical $\mathfrak{psl}(4|4)$. In the classical $\mathfrak{psl}(4|4)$, long multiplets at the unitarity bounds split up. In the interacting theory these multiplets must rejoin. This is, however, only possible if all submultiplets have degenerate anomalous dimensions. For instance, we find the following three energies for the Konishi submultiplets

$$E = 4C_2 = 6C_1 = 6C_1 - 2C_0, \tag{6.1}$$

where we have left the independent coefficients (2.20) unfixed. This consistency requirement determines C_0, C_1, C_2 up to an overall constant. A similar argument was used in [8] to determine some anomalous dimensions. One might hope that an investigation of the twist-two operators (2.30) might constrain *all* independent coefficients to $C_j = c\,h(j)$. If this works out, all one-loop anomalous dimensions can be obtained purely algebraically without evaluating a single Feynman diagram up to one overall constant! This constant can finally be fixed by a different consistency argument [8], which merely requires computing the quotient of two tree-level diagrams. In that spirit it would be very interesting to find out if these consistency arguments or, more generally, the closure of the interacting algebra, can be used to fix the higher-loop contributions to the dilatation generator as well. This is not inconceivable, given that the $\mathcal{N} = 4$ action is known to be unique.

It would be great to obtain higher-loop contributions to the dilatation generator. If the one-loop contribution turns out to be completely fixed by symmetry, only higher-loop anomalous dimensions could provide truly dynamical information about $\mathcal{N} = 4$ SYM and the dynamical AdS/CFT correspondence. Even better, one might find that *all* radiative corrections are somehow fixed by symmetry and thus kinematical. Also the question of higher-loop integrability raised in [22] could be addressed. As a starting point, one might restrict to the $\mathfrak{u}(2|3)$ subsector of Sec. 5.3 or the non-compact $\mathfrak{sl}(2)$ subsector of Sec. 3 to simplify the computations. Eventually a treatment of the complete dilatation generator at two-loops would be desirable. This might be feasible using computer algebra packages developed for higher-loop calculations within QCD.

The idea of investigating the dilatation operator can be generalised to a wider range of QFT's. For instance, a few theories with $\mathcal{N} = 2$ supersymmetry are conformal at the quantum level. For these the determination of the dilatation generator might shed some light on holographic dualities correspondence away from the well-studied case of $AdS_5 \times S^5$. Even in a QFT without conformal invariance the techniques developed in [21, 22] can be used to investigate logarithmic corrections to two-point functions and scattering amplitudes in a systematic way. In particular in QCD at large N_c and deep inelastic scattering, similar techniques are at use (see e.g. [32]). There, following pioneering work of Lipatov [43], methods of integrability have also had much impact.

An intriguing feature of Tab. 2,3,4 is that several states have degenerate energies, although they are not related by $\mathfrak{sl}(4|4)$ symmetry, see the discussion at the end of Sec. 4.4. The occurrence of parity pairs provides some evidence for integrability. Integrability of the full planar theory at one-loop and the corresponding Bethe ansatz equations are presented in [34], reproducing the anomalous dimensions in the above tables. Apart from these, we find more examples of degeneracies. Could this be the result of a further symmetry enhancement due to or beyond integrability?

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A The algebra $\mathfrak{gl}(4|4)$

The algebra $\mathfrak{gl}(4|4)$ consists of the generators $J = (Q, S, \dot{Q}, \dot{S}, P, K, L, \dot{L}, R, D, C, B)$. These are the (super)translations $Q, \dot{Q}, P$, the (super)boosts $S, \dot{S}, K$, the $\mathfrak{gl}(2) \times \mathfrak{gl}(2)$ rotations $L, \dot{L}$, the $\mathfrak{gl}(4)$ rotations R as well as the dilatation generator D, central charge C and chirality B.

Under the rotations $L, \dot{L}, R$, the indices of any generator J transform canonically according to

$$\begin{aligned} [L^\alpha{}_\beta, J_\gamma] &= \delta^\alpha_\gamma J_\beta - \tfrac{1}{2}\delta^\alpha_\beta J_\gamma, \quad & [L^\alpha{}_\beta, J^\gamma] &= -\delta^\gamma_\beta J^\alpha + \tfrac{1}{2}\delta^\alpha_\beta J^\gamma, \\ [\dot{L}^{\dot\alpha}{}_{\dot\beta}, J_{\dot\gamma}] &= \delta^{\dot\alpha}_{\dot\gamma} J_{\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta} J_{\dot\gamma}, \quad & [\dot{L}^{\dot\alpha}{}_{\dot\beta}, J^{\dot\gamma}] &= -\delta^{\dot\gamma}_{\dot\beta} J^{\dot\alpha} + \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta} J^{\dot\gamma}, \\ [R^a{}_b, J_c] &= \delta^a_c J_b - \tfrac{1}{2}\delta^a_b J_c, \quad & [R^a{}_b, J^c] &= -\delta^c_b J^a + \tfrac{1}{2}\delta^a_b J^c. \end{aligned} \tag{A.1}$$

The charges D, C, B of the generators are given by

$$[D, J] = \dim(J)\, J, \qquad [C, J] = 0, \qquad [B, J] = \mathrm{chi}(J)\, J \tag{A.2}$$

with non-vanishing dimensions

$$\dim(P) = -\dim(K) = 1, \quad \dim(Q) = \dim(\dot{Q}) = -\dim(S) = -\dim(\dot{S}) = \tfrac{1}{2} \tag{A.3}$$

and non-vanishing chiralities

$$\mathrm{chi}(Q) = -\mathrm{chi}(\dot{Q}) = -\mathrm{chi}(S) = \mathrm{chi}(\dot{S}) = \tfrac{1}{2}. \tag{A.4}$$

The translations and boosts commuting into themselves are given by

$$\begin{aligned} [S^\alpha{}_a, P_{\beta\dot\gamma}] &= \delta^\alpha_\beta \dot{Q}_{\dot\gamma a}, \qquad & [K^{\alpha\dot\beta}, \dot{Q}_{\dot\gamma c}] &= \delta^{\dot\beta}_{\dot\gamma} S^\alpha{}_c, \\ [\dot{S}^{\dot\alpha a}, P_{\beta\dot\gamma}] &= \delta^{\dot\alpha}_{\dot\gamma} Q^a{}_\beta, \qquad & [K^{\alpha\dot\beta}, Q^c{}_\gamma] &= \delta^\alpha_\gamma \dot{S}^{\dot\beta c}, \\ \{\dot{Q}_{\dot\alpha a}, Q^b{}_\beta\} &= \delta^b_a P_{\beta\dot\alpha}, \qquad & \{\dot{S}^{\dot\alpha a}, S^\beta{}_b\} &= \delta^a_b K^{\beta\dot\alpha}, \end{aligned} \tag{A.5}$$

while the translations and boosts commuting into rotations are given by

$$\begin{aligned} [K^{\alpha\dot\beta}, P_{\gamma\dot\delta}] &= \delta^{\dot\beta}_{\dot\delta} L^\alpha{}_\gamma + \delta^\alpha_\gamma \dot{L}^{\dot\beta}{}_{\dot\delta} + \delta^\alpha_\gamma \delta^{\dot\beta}_{\dot\delta} D, \\ \{S^\alpha{}_a, Q^b{}_\beta\} &= \delta^b_a L^\alpha{}_\beta + \delta^\alpha_\beta R^b{}_a + \tfrac{1}{2}\delta^b_a \delta^\alpha_\beta (D - C), \\ \{\dot{S}^{\dot\alpha a}, \dot{Q}_{\dot\beta b}\} &= \delta^a_b \dot{L}^{\dot\alpha}{}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta} R^a{}_b + \tfrac{1}{2}\delta^a_b \delta^{\dot\alpha}_{\dot\beta} (D + C). \end{aligned} \tag{A.6}$$

The chirality B never appears on the right hand side, it can be dropped. Furthermore, the central charge can be set to zero. The resulting algebra is $\mathfrak{psl}(4|4)$.

The quadratic Casimir of $\mathfrak{gl}(4|4)$ is

$$J^2 = \tfrac{1}{2}D^2 + \tfrac{1}{2}L^\gamma{}_\delta L^\delta{}_\gamma + \tfrac{1}{2}\dot{L}^{\dot\gamma}{}_{\dot\delta}\dot{L}^{\dot\delta}{}_{\dot\gamma} - \tfrac{1}{2}R^c{}_d R^d{}_c$$
$$- \tfrac{1}{2}[Q^c{}_\gamma, S^\gamma{}_c] - \tfrac{1}{2}[\dot{Q}_{\dot\gamma c}, \dot{S}^{\dot\gamma c}] - \tfrac{1}{2}\{P_{\gamma\dot\delta}, K^{\gamma\dot\delta}\} - BC. \qquad \text{(A.7)}$$

For $\mathfrak{psl}(4|4)$ the last term BC is absent.

B Calculation of diagrams

The two-point function of operators with vector indices is restricted by conformal symmetry to

$$\big\langle \mathcal{O}_{a,\mu_1\ldots\mu_m}(x)\, \mathcal{O}_{b,\nu_1\ldots\nu_n}(y)\big\rangle = \frac{\delta_{ab}(N_a + g^2 N'_a)}{(x-y)^{2\Delta_0 + 2g^2\delta\Delta_a}}\, J_{\mu_{i_1}\nu_{i_1}} J_{\mu_{i_2}\nu_{i_2}} \delta_{\mu_{i_3}\mu_{i_3}} \delta_{\nu_{i_4}\nu_{i_4}} \ldots \qquad \text{(B.1)}$$

The symbols

$$J_{\mu\nu} = \delta_{\mu\nu} - 2\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \qquad \text{(B.2)}$$

relate conformal tangent spaces at different space-time points. The vectors $x-y$ with open indices do not carry essential information, they can be discarded and reconstructed later by replacing $\delta_{\mu\nu}$ by $J_{\mu\nu}$ where appropriate. Also the corrections to the normalisation constants N' are irrelevant to the anomalous dimension at one-loop. Before diagonalisation we thus expect the correlator to have the following structure

$$\frac{N_{ac}(\delta^c_b + g^2 \delta\Delta^c{}_b \log|x-y|^{-2})}{(x-y)^{2\Delta_0}}\, \delta_{\mu_{i_1}\nu_{i_1}} \delta_{\mu_{i_2}\nu_{i_2}} \delta_{\mu_{i_3}\mu_{i_3}} \delta_{\nu_{i_4}\nu_{i_4}} \ldots \qquad \text{(B.3)}$$

We use the letters

$$Z_k = (\mathbf{a}^\dagger)^k|0\rangle = \frac{1}{k!}(\mathcal{D}_1 + i\mathcal{D}_2)^k(\Phi_5 + i\Phi_6). \qquad \text{(B.4)}$$

The expansion of the covariant derivative $\mathcal{D} = \partial - ig\mathcal{A}$ can move one 'bulk' vertex to the position of the field and become a 'boundary' vertex. At one-loop there can only be one boundary vertex, the contribution from two boundary vertices has no logarithmic behaviour. The relevant part of the letter Z_k is thus

$$Z_k = \frac{1}{k!}\,\partial^k\Phi - ig\sum_{j=1}^{k}\frac{1}{j!(k-j)!}\,[\partial^{j-1}\mathcal{A}, \partial^{k-j}\Phi]. \qquad \text{(B.5)}$$

We go ahead and calculate the relevant parts of the correlator

$$\big\langle Z_k(x_1)\, Z_{n-k}(x_2)\, \bar{Z}_{k'}(x_3)\, \bar{Z}_{n-k'}(x_4)\big\rangle. \qquad \text{(B.6)}$$

We restrict to the planar sector and use point splitting regulatisation. The matrix of anomalous dimensions $\delta\Delta$ is generated by the Hamiltonian, we which we assume the generic form

$$H_{12}\,(\mathbf{a}_1^\dagger)^k(\mathbf{a}_2^\dagger)^{n-k}|00\rangle = \sum_{k'=0}^{n} c_{n,k,k'}\,(\mathbf{a}_1^\dagger)^{k'}(\mathbf{a}_2^\dagger)^{n-k'}|00\rangle. \tag{B.7}$$

Each diagram contributes a set of constants $c_{n,k,k'}$, which we will list below. We list the contributions separately, they can be reused for a calculation within a different theory.

The diagrams Fig. 1a with an intermediate gluon (up to terms where one scalar line has been collapsed to a point by the equations of motion) yield the vertex

$$\frac{g_{\mathrm{YM}}^2 N}{32\pi^2}\,\frac{\partial_1^k\partial_2^{n-k}\partial_3^{k'}\partial_4^{n-k'}}{k!(n-k)!k'!(n-k')!}\,\frac{(r-s)\,\Phi(r,s)}{x_{13}^2x_{24}^2} \tag{B.8}$$

where r,s are the conformal cross ratios

$$r=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad s=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}. \tag{B.9}$$

We use the expansion [15] of Φ in the limit $x_2\to x_1$, $x_4\to x_3$ $(r\to 0, s\to 1)$

$$\Phi(r,s) = -\sum_{n,m=0}^{\infty}\frac{(n+m)!^2}{m!(1+2n+m)!}\,r^n(1-s)^m\log r+\ldots \tag{B.10}$$

and obtain the coefficient

$$c_{n,k,k'}=\frac{1}{2(n+1)}+\delta_{k=k'}\big(\tfrac12 h(k)+\tfrac12 h(n-k)\big)-\frac{\delta_{k\neq k'}}{2|k-k'|}. \tag{B.11}$$

Diagram Fig. 1a with a four-point interaction is given by

$$\frac{g_{\mathrm{YM}}^2 N}{32\pi^2}\,\frac{\partial_1^k\partial_2^{n-k}\partial_3^{k'}\partial_4^{n-k'}}{k!(n-k)!k'!(n-k')!}\,\frac{\Phi(r,s)}{x_{13}^2x_{24}^2} \tag{B.12}$$

and the corresponding coefficient is

$$c_{n,k,k'}=-\frac{1}{2(n+1)}. \tag{B.13}$$

Diagrams Fig. 1a with an intermediate gluon where one scalar line has been collapsed to a point by the equations of motion is given by

$$\frac{g_{\mathrm{YM}}^2 N}{32\pi^2}\,\frac{\partial_1^k\partial_2^{n-k}\partial_3^{k'}\partial_4^{n-k'}}{k!(n-k)!k'!(n-k')!}\,\frac{(s'-r')\Phi(r',s')}{x_{13}^2x_{24}^2}+3\text{ perm.} \tag{B.14}$$

with

$$r'=\frac{x_{12}^2}{x_{24}^2},\qquad s'=\frac{x_{14}^2}{x_{24}^2} \tag{B.15}$$

and the corresponding coefficient is

$$c_{n,k,k'} = \delta_{k=k'}\big(-\tfrac{1}{2}h(k+1) - \tfrac{1}{2}h(n-k+1)\big) - \frac{\delta_{k\neq k'}}{|k-k'|} + \frac{\delta_{k\neq k'}}{4(k+1)} + \frac{\delta_{k\neq k'}}{4(k'+1)} + \frac{\delta_{k\neq k'}}{4(n-k+1)} + \frac{\delta_{k\neq k'}}{4(n-k'+1)}. \tag{B.16}$$

Diagrams Fig. 1a′ yields

$$\frac{g_{\rm YM}^2 N}{32\pi^2} \sum_{j=1}^{k} \left(\frac{\partial_1^{k-j}\partial_3^{k'}}{(k-j)!k'}\,\frac{1}{x_{13}^2}\right)\left(\frac{\partial_1^{j-1}(\partial_1+2\partial_2)\partial_2^{n-k}\partial_4^{n-k'}}{j!(n-k)!(n-k')!}\,\frac{\Phi(r',s')}{x_{24}^2}\right) + 3\text{ perm.} \tag{B.17}$$

and the corresponding coefficient is

$$c_{n,k,k'} = \frac{\delta_{k\neq k'}}{2|k-k'|} - \frac{\delta_{k\neq k'}}{4(k+1)} - \frac{\delta_{k\neq k'}}{4(k'+1)} - \frac{\delta_{k\neq k'}}{4(n-k+1)} - \frac{\delta_{k\neq k'}}{4(n-k'+1)}. \tag{B.18}$$

The contributions from diagrams Fig. 1c with an intermediate gluon, with intermediate fermions and the contributions from Fig. 1c', respectively, are given by

$$\begin{aligned} c_{n,k,k'} &= -\delta_{k=k'}, \\ c_{n,k,k'} &= 2\delta_{k=k'}, \\ c_{n,k,k'} &= \delta_{k=k'}\big(\tfrac{1}{2}h(k) + \tfrac{1}{2}h(n-k) + \tfrac{1}{2}h(k+1) + \tfrac{1}{2}h(n-k+1) - 1\big). \end{aligned} \tag{B.19}$$

The sum of all coefficients is

$$c_{n,k,k'} = \delta_{k=k'}\big(h(k) + h(n-k)\big) - \frac{\delta_{k\neq k'}}{|k-k'|}. \tag{B.20}$$

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