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Lectures on the Plane–Wave String/Gauge Theory Duality

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Abstract: These lectures give an introduction to the novel duality relating type IIB string theory in a maximally supersymmetric plane-wave background to $\mathcal{N} = 4$, $d = 4$, $U(N)$ Super Yang-Mills theory in a particular large N and large R-charge limit due to Berenstein, Maldacena and Nastase. In the first part of these lectures the duality is derived from the AdS/CFT correspondence by taking a Penrose limit of the $AdS_5 \times S^5$ geometry and studying the corresponding double-scaling limit on the gauge theory side. The resulting free plane-wave superstring is then quantized in light-cone gauge. On the gauge theory side of the correspondence the composite Super Yang-Mills operators dual to string excitations are identified, and it is shown how the string spectrum can be mapped to the planar scaling dimensions of these operators. In the second part of these lectures we study the correspondence at the interacting respectively non-planar level. On the gauge theory side it is demonstrated that the large N large R-charge limit in question preserves contributions from Feynman graphs of all genera through the emergence of a new genus counting parameter – in agreement with the string genus expansion for non-zero g_s. Effective quantum mechanical tools to compute higher genus contributions to the scaling dimensions of composite operators are developed and explicitly applied in a genus one computation. We then turn to the interacting string theory side and give an elementary introduction into light-cone superstring field theory in a plane-wave background and point out how the genus one prediction from gauge theory can be reproduced. Finally, we summarize the present status of the plane-wave string/gauge theory duality.

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1 Introduction

Ever since the work of 't Hooft in 1974 [1] it has been widely suspected that there should exist a dual description of large N gauge theories in terms of string theories. In these lectures we will discuss a very concrete realization of such a duality relating type IIB superstrings in a maximally supersymmetric plane-wave background to four dimensional $\mathcal{N} = 4$ Super Yang-Mills Theory in a particular double scaling limit, which was initiated by the influential paper of Berenstein, Maldacena and Nastase [2] in 2002. This duality may be viewed as a "corollary" of the well studied Anti-de-Sitter/Conformal Field Theory (AdS/CFT) duality [3], which asserts a dual description of the IIB superstring moving in a $AdS_5 \times S^5$ spacetime-background in terms of the four dimensional maximally supersymmetric $U(N)$ gauge theory. It is probably the most concrete example of a string/gauge theory duality ever established, as it is the first to truly probe the "stringy" regime of the correspondence in terms of higher mode excitations in the free string theory, as well as higher genus worldsheet interactions in the interacting string theory – a regime which has so far been technically inaccessible in the AdS/CFT correspondence. In these lecture notes we shall assume a basic knowledge of string theory, gauge theories and the AdS/CFT duality for which a number of reviews already exists [4].

The key developments in establishing the plane-wave string/gauge theory correspondence began with the discovery of Blau, Figueroa-O'Farrill, Hull and Papadopoulos [5], that the type IIB supergravity solution of a gravitational plane-wave with a constant, null five-form field strength constitutes a maximally supersymmetric background for the IIB string. As such it is a distinguished background of IIB string theory, as there exist only two additional maximally supersymmetric backgrounds of IIB string theory: The well studied cases of flat Minkowski and $AdS_5 \times S^5$ spaces [6]. In addition it turns out that the plane-wave string reduces to a free, massive two dimensional model once one goes to the light-cone gauge, as noticed by Metsaev and Tseytlin [7, 8]. It is therefore as straightforwardly quantized as the superstring in a flat background and in this respect strongly distinct to the $AdS_5 \times S^5$ string, which is given by a non-linear two dimensional field theory, whose quantization has not been achieved to date. On the other hand, the plane-wave geometry is obtained through a limit of the $AdS_5 \times S^5$ geometry [9]. This is particularly interesting as by virtue of the AdS/CFT duality this limit must entail a dual description of the plane-wave string in terms of the supersymmetric gauge theory in a corresponding limit [2]. Surprisingly the pane-wave string/gauge theory duality turns out to be perturbatively accessible from both sides of the correspondence – in contradistinction to the strong/weak coupling duality in AdS/CFT. It is then possible to set up a concrete "dictionary" relating string states to operators in the Super Yang-Mills theory and to compare their spectra in a perturbative expansion on both sides of the correspondence.

The limit to be taken on the gauge theory side is a novel type of double scaling limit [2], in which not only the rank N of the gauge group is taken to infinity, but one is led to simultaneously only considers correlation functions of operators with a diverging R-charge $J \sim \sqrt{N}$. In particular this limit is of non 't Hooftian type, as the 't Hooft coupling constant $\lambda := g_{\text{YM}}^2 N$ diverges. Indeed the limit maintains contributions from graphs of all genera [10, 11], due to a combinatorial abundance of non-planar graphs growing with J. Not surprisingly the non-planar sector of the gauge theory is found to be dual to plane-wave string interactions, which opens up the possibility of studying string interactions in the framework of light-cone string field theory via methods of perturbative large N and J gauge theory. This we shall do in detail in these lectures. On the string theory side interactions have been studied with the methods of light-cone superstring field theory [12–14] and shown to agree with the gauge theory predictions under certain assumptions, which will be the subject of the last section.

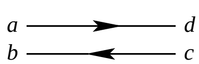
In summary this novel duality represents itself as an interesting and very concrete model to study the complementarity of string and gauge theories. By doing so one may hope to develop novel and wider accessible tools which could become useful in the study of phenomenologically more interesting systems in the future.

1.1 Strings and large *N* gauge theories

The expectation that there should exist a close relationship between gauge theory and strings is based, among other observations, on the analysis of the perturbation expansion of a $U(N)$ gauge theory in the large N limit. To understand this in some detail let us look at the following schematic action of $N \times N$ hermitian matrix fields $(\phi_i)_{ab}(x)$

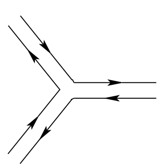
$$\mathcal{S} = \frac{1}{g_{\rm YM}^2} \int d^4x \Big[\operatorname{Tr}(\partial_\mu \phi_i \, \partial^\mu \phi_i) + c^{ijk} \operatorname{Tr}(\phi_i \, \phi_j \, \phi_k) + d^{ijkl} \operatorname{Tr}(\phi_i \, \phi_j \, \phi_k \, \phi_l) \Big] . \qquad (1)$$

This action mimics a $U(N)$ Yang-Mills model as well as possible couplings of scalar fields in the adjoint representation. The propagators of the matrix valued fields may be represented by "fat" graphs

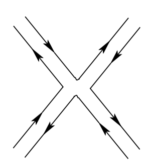


$$\sim g_{\rm YM}^2 : \qquad \langle\, (\phi_i)_{ab}(x)\, (\phi_j)_{cd}(0)\, \rangle = \frac{g_{\rm YM}^2}{8\pi^2\, x^2}\, \delta_{ij}\, \delta_{ad}\, \delta_{bc} \qquad (2)$$

One immediately reads off from the Lagrangian (1) that the vertices scale uniformly with $1/g_{\rm YM}^2$. Their fat graph structure may be depicted as follows:

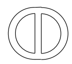


$$\sim \frac{1}{g_{\rm YM}^2}$$

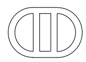


$$\sim \frac{1}{g_{\rm YM}^2}$$

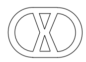
By making use of the double line notation for propagators any Feynman diagram in the perturbative expansion of (1) may be viewed as a simplicial decomposition of a surface with V vertices, E edges and F faces. Here the total number of propagators in the Feynman graph corresponds to E and F simply counts the number of index loops occurring in the graph. As an example let us count the factors of $g_{\rm YM}$ and N for the following vacuum graphs:



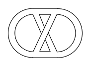
$$\hat{=}\ (g_{\rm YM}^2)^{3-2}\, N^3 = \ (g_{\rm YM}^2\, N)\, N^2$$



$$\hat{=}\ (g_{\rm YM}^2)^{6-4}\, N^4 = \ (g_{\rm YM}^2\, N)^2\, N^2$$



$$\hat{=}\ (g_{\rm YM}^2)^{8-5}\, N^5 = \ (g_{\rm YM}^2\, N)^3\, N^2$$



$$\hat{=}\ (g_{\rm YM}^2)^{6-4}\, N^2 = \ (g_{\rm YM}^2\, N)^2 \qquad (3)$$

We observe that the first three graphs are planar, i.e. they may be drawn on a plane without crossing of propagators, whereas the last one is non-planar. Also the combination $\lambda = g_{\rm YM}^2\, N$ emerges as a quantum loop counting parameter known as the 't Hooft coupling constant. Non-planar graphs are suppressed by powers of $1/N^2$ with respect to planar ones. In general it is easy to see that for a graph with V vertices, E propagators and F index loops one has

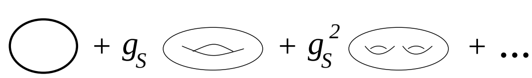
$$N^F\,(g_{\rm YM}^2)^{E-V} = N^{V-E+F}\,(g_{\rm YM}^2\,N)^{E-V}$$

Now the Euler number χ of a simplicial manifold is given by $\chi = V - E + F = 2 - 2\,g$, where g denotes the genus of the manifold, corresponding to its number of handles. Hence the perturbative expansion of (1) or a general $U(N)$ gauge theory is organized as a double expansion in $\lambda := g_{\rm YM}^2\, N$ and $1/N^2$, counting the number of quantum loops and handles respectively, i.e. the free energy F will decompose as

$$N^2\, F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} c_{g,n}\, \lambda^n \qquad (4)$$

This implies that there is a consistent way of performing a large N limit of a gauge theory - due to 't Hooft - by taking $N \to \infty$ while keeping λ fixed, i.e. a simultaneous scaling of $g_{\rm YM} \sim 1/\sqrt{N} \to 0$. Note that in the strict 't Hooft limit all non-planar graphs are suppressed and the gauge theory reduces to its planar limit.

The structure of the genus expansion of (4) strongly resembles the perturbative expansion of string theory as a sum over worldsheets of growing genus, with the role of $1/N^2$ played by the string coupling constant g_s



As non-planar graphs are suppressed in the strict 't Hooft limit, one expects that the large N limit of the gauge theory should correspond to a non-interacting ($g_s = 0$) string model. Of course finding the associated string theory to a given four dimensional gauge theory has been very hard.

1.2 The AdS/CFT Correspondence

The first concrete proposal of such a string/gauge theory duality pair emerged more than 20 years after its first suggestion with the AdS/CFT duality conjecture due to Maldacena in 1997 [3]. In its simplest form it states that the dual string model of the maximally supersymmetric gauge theory in four dimensions is the type IIB superstring propagating in the ten dimensional $AdS_5 \times S^5$ background geometry. The duality conjecture surprisingly relates a four dimensional gauge theory to a higher dimensional string model, which represents a manifestation of the so called holographic principle [15] indicating that the entire degrees of freedom of a quantum theory of gravity reside on the boundary of the space-time region in question. The boundary of $AdS_5 \times S^5$ is four dimensional and this is where the $\mathcal{N} = 4$ Super Yang-Mills theory lives. This picture is most transparent in the calculation of Wilson loops in the dual string model: The contour of the Super Yang-Mills Wilson loop represents the ends of an open string attached to the boundary of $AdS_5 \times S^5$ extending into the bulk of the anti-de-Sitter space. The expectation value of the Wilson loop operator is semi-classically nothing but the minimal surface of the AdS-string, which due to the curvature extends into the bulk.

The central relations in the AdS/CFT duality conjecture relate the gauge theory parameters g_{YM} and N to the string theory parameters α' (string tension), g_s (string coupling constant) and the radius R of the AdS_5 and S^5 spaces, via

$$\frac{R^4}{\alpha'^2} = g_{\text{YM}}^2\, N \qquad \text{and} \qquad 4\pi g_s = g_{\text{YM}}^2\,. \tag{5}$$

Unfortunately though, even the free ($g_s = 0$) $AdS_5 \times S^5$ string is a rather complicated two dimensional field theory, whose quantization remains a very challenging open problem. Hence the string theory side of the duality conjecture could so far only be addressed by studying its low energy effective description in terms of type IIB supergravity. This approximation to string theory is only meaningful as long as the curvature of the background is small compared to the string scale, i.e. the radius R in string units needs to be very large

$$1 \ll \frac{R^4}{\alpha'^2} = \lambda \tag{6}$$

This domain is perfectly incompatible to the perturbatively accessible regime of $\mathcal{N} = 4$ Super Yang-Mills, which requires $\lambda \ll 1$! One is hence dealing with a duality relating a weakly coupled to a strongly coupled theory and vice versa. So if the duality conjecture is indeed true we have a fascinating new tool at hand for studying the strongly coupled sector of a gauge or string theory. But in the same instance a proof of the duality conjecture is very hard if not impossible, as it requires solving the string or gauge theory non-perturbatively.

As we shall see in the novel plane-wave string/gauge theory duality to be discussed, this situation has improved, as both theories turn out to possess an overlapping perturbative regime. Therefore the correspondence may here be tested beyond the supergravity regime into the realm of true stringy effects. As such the first massive string excitations may be reproduced in the gauge theory. Moreover the interacting string sector turns out to correspond to non-planar gauge theory effects in a novel double-scaling limit. We hence have a very concrete and testable example of a string/gauge theory duality, which at the least represents an interesting toy model to hopefully develop new tools for the study of more complicated and phenomenologically relevant string/gauge theory systems.

We should mention that there is a price to pay here. Firstly the implementation of the holographic principle in the plane-wave/gauge theory duality is not well understood at this point[2]. The boundary of the plane-wave geometry is one dimensional, which hints at an effectively one dimensional dual gauge model - yet undiscovered. However, we shall encounter first traces of such a reduced quantum mechanical model in section 5. Secondly the new gauge theory limit to be discussed leads us to a vastly reduced sector of $\mathcal{N}=4$ Super Yang-Mills: Only a restricted class of operators survive the limit and turn out to directly correspond to the free string excitations. Moreover only two and three-point functions of these gauge theory operators turn out to exist in the scaling limit – higher point functions simply diverge [27]! Thus the number of possible observables is strongly reduced. This indeed should be taken as an indication of an effective lower dimensional description of the Berenstein-Maldacena-Nastase (BMN) sector of $\mathcal{N}=4$ Super Yang-Mills.

2 The plane-wave geometry as a Penrose limit of $AdS_5 \times S^5$

In this section we shall review the emergence of the plane-wave geometry as a limit of the $AdS_5 \times S^5$ background. The idea is to zoom into the geometry seen by a particle moving on a light-like geodesic along a great circle of the S^5 sphere [9]. Such a limit is possible for any space-time geometry and leads to a plane-wave metric as pointed out by Penrose [18]. In global coordinates the $AdS_5 \times S^5$ metric is given by

$$ds^2_{AdS_5\times S^5} = R^2\,[\,-dt^2\,\cosh^2\rho + d\rho^2 + \sinh^2\rho\, d\Omega_3{}^2 + d\psi^2\,\cos^2\theta + d\theta^2 + \sin^2\theta\, d\Omega_3'^2\,]\,. \quad (7)$$

One now considers the light-like trajectory parametrized by λ along

$$\rho = 0, \qquad \theta = 0, \qquad t = t(\lambda), \qquad \psi = \psi(\lambda) \qquad (8)$$

The relativistic particle moving along this geodesic is governed by the action

$$S = \tfrac{1}{2}\int d\lambda\,(e^{-1}\,g_{\mu\nu}(x)\,\dot{x}^\mu\,\dot{x}^\nu - e\,m^2) = \tfrac{R^2}{2}\int d\lambda\, e^{-1}\,(-\dot{t}^2 + \dot{\psi}^2) \qquad (9)$$

with e denoting the "einbein". In the second equality we have specialized to the zero mass case $m=0$ and the path parametrized in (8). Clearly then upon introducing light-cone coordinates $\tilde{x}^\pm = \frac{1}{2}(t\pm\psi)$ the light-like trajectory given by $\tilde{x}^- = \lambda$ and $\tilde{x}^+ =$ const. solves the equations of motion arising from (9). In order to study the geometry near this trajectory we introduce the new coordinates $x^\pm, r, y$ in the particular scaling limit $R\to\infty$

$$x^+ = \frac{\tilde{x}^+}{\mu}, \qquad x^- = \mu\,R^2\,\tilde{x}^-, \qquad \rho = \frac{r}{R}, \qquad \theta = \frac{y}{R} \qquad (10)$$

where μ is a new mass parameter introduced in order to maintain canonical length dimensions for $x^\pm, r$ and y. If one now performs this change of variables in (7) one sees that the terms of order R^2 cancel out and the leading contributions are R independent

$$\begin{aligned} ds^2_{AdS_5\times S^5} &= R^2\,[-\mu^2\,(dx^+)^2 + \mu^2\,(dx^+)^2] + [-2\,dx^+\,dx^- - \mu^2\,r^2\,(dx^+)^2 + dr^2 + r^2\,d\Omega_3{}^2 \\ &\quad -2\,dx^+\,dx^- - \mu^2\,y^2\,(dx^+)^2 + dy^2 + y^2\,d\Omega_3'^2] + \mathcal{O}(R^{-2}) \\ &= -4\,dx^+\,dx^- - \mu^2\,(\vec{y}^2 + \vec{r}^2)\,(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2 + \mathcal{O}(R^{-2}) \end{aligned} \qquad (11)$$

[2]For work along these lines see [16, 17].

where we have introduced the four-vectors $\vec{r}$ and $\vec{y}$ in the last step. Hence in the Penrose limit $R \to \infty$ the $AdS_5 \times S^5$ metric approaches the plane-wave metric

$$ds^2_{AdS_5 \times S^5} \to ds^2_{\text{pw}} = -4\, dx^+\, dx^- - \mu^2\, (x^i)^2\, (dx^+)^2 + (dx^i)^2 \qquad i = 1, \ldots, 8 \tag{12}$$

Similar considerations may be applied to the non-vanishing self-dual five-form to yield

$$F_{+1234} = F_{+5678} = 4\mu \tag{13}$$

in the Penrose limit. Therefore the transverse $SO(8)$ invariance of the metric (12) is broken to a $SO(4) \times SO(4)$ subgroup by the five-form field strength. In the light-cone string action this breaking will manifest itself in the fermionic mass term. We also observe that by taking the mass parameter μ to zero the plane-wave geometry contracts to flat Minkowski space-time. Hence, at least on the string theory side all results should limit to the well known flat background scenario upon taking μ to zero. In the dual gauge theory we shall see that $\mu \to 0$ corresponds to the strict strong coupling limit.

How does the Penrose limit, $R \to \infty$, translate into the dual gauge theory? For this it is instructive to study how the energy $E = i\partial_t$ and angular momentum $J = -i\partial_\psi$ conjugate to the global coordinates t and ψ relate to the newly introduced light-cone quantities $x^\pm$ and their conjugate momenta,

$$\begin{aligned} \mathcal{H}_{\text{lc}} := 2\,p^- &= i\partial_{x^+} = \mu\, i(\partial_t + \partial_\psi) = \mu\,(E - J) \\ 2\,p^+ &= i\partial_{x^-} = \frac{1}{\mu\, R^2} i(\partial_t - \partial_\psi) = \frac{E + J}{\mu\, R^2}\,, \end{aligned} \tag{14}$$

where we identified the light-cone Hamiltonian $\mathcal{H}_{\text{lc}}$ with $2\,p^-$. In the limit $R \to \infty$ we see that generic excitations (corresponding to string states in this background) will have vanishing p^+ momenta, unless the angular momentum J of such a state grows with R as $J \sim R^2$ in a correlated manner. In order to maintain a finite light-cone momentum for such a state, one deduces the further requirement from (14) that $E \approx J$ in the Penrose limit[3]. As was pointed out in the initial paper of Berenstein, Maldacena and Nastase [2] the standard AdS/CFT correspondence linking $\mathcal{N} = 4$ Super Yang-Mills to type IIB strings in $AdS_5 \times S^5$ must entail a new duality relating plane-wave strings to an adequate limit of the $\mathcal{N} = 4$ gauge theory, corresponding to the discussed Penrose limit. The nature of this limit may be identified by translating the gravity quantities E and J to Super Yang-Mills variables. Here the energy E in global coordinates is identified with the scaling dimension Δ of a composite Super Yang-Mills operator. The angular momentum J on the other hand corresponds to the charge of a $U(1)$ subgroup of the $SO(6)$ R symmetry group of $\mathcal{N} = 4$ Super Yang-Mills, which we shall discuss in more detail later on. Therefore the first relation of (14) may be rephrased from the gauge theory perspective as

$$\frac{\mathcal{H}_{\text{lc}}}{\mu} \mathrel{\hat{=}} \Delta - J \tag{15}$$

which is the central relation in the BMN correspondence. Due to the AdS/CFT relation $R^4 = \alpha'^2\, g^2_{\text{YM}}\, N$ the Penrose limit $R \to \infty$ with $J \sim R^2$ translates into the gauge theory limit

$$N \to \infty, \qquad J \sim \sqrt{N}, \qquad g_{\text{YM}} \text{ held fixed} \tag{16}$$

[3]Note that $p^\pm$ are non-negative due to the BPS condition $E \geq |J|$.

Holding $g_{\rm YM}$ fixed in this limit corresponds to a finite value of the string coupling constant $g_s = g_{\rm YM}^2/4\pi$ on the dual string side. Moreover the finite light-cone energy requirement $E \approx J$ discussed above tells us that in the gauge theory limit only Super Yang-Mills operators with

$$\Delta \approx J \tag{17}$$

will survive and correspond to finite light-cone energy states on the string side. We will return to this gauge theory limit in section four.

3 Light-cone quantization of the type IIB plane-wave string

Let us now discuss the quantization of the type IIB superstring in the plane-wave background (12) and (13). Due to the non-vanishing Ramond-Ramond background field strength it is necessary to work with the Green-Schwarz formulation of the superstring, defined through the worldsheet fields $X^\mu(\tau,\sigma)$ and $\theta^A_\alpha(\tau,\sigma)$ with $\mu = 0,\dots,9$, $A = 1,2$ and $\alpha = 1,\dots,16$ being 10d space-time vectors and two Majorana-Weyl spinors of same chirality respectively[4]. The resulting covariant action in the plane-wave background was worked out by Metsaev [7] and takes a very complicated form consisting of terms up to order $\mathcal{O}(\theta^{16})$ in the fermionic sector. The model becomes tractable, however, in the light-cone gauge [7, 8]. For this one uses the 2d diffeomorphism invariance to go to the conformal gauge for the worldsheet metric $g_{ab} = e^\phi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The freedom of performing residual conformal transformations is then used to set

$$X^+(\tau,\sigma) = p^+\,\tau \tag{18}$$

in complete analogy to the light-cone quantization in flat Minkowski background. In the fermion sector the local fermionic κ-symmetry is employed to gauge away one half of the fermionic degrees of freedom via the condition $\Gamma^+\,\theta^A = 0$, again as is done in the flat background [19]. This gauge choice dramatically simplifies the fermionic sector of the model and only terms up to quadratic order in fermions survive[5]. Explicitly one obtains a free quadratic model with action

$$\begin{aligned} S \;=\; \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi\alpha' p^+} d\sigma \Big(& \tfrac{1}{2}\,\partial_a X^I\,\partial_a X^I - \tfrac{1}{2}\,\mu^2 (X^I)^2 \\ & + i\,\theta^1(\partial_\tau+\partial_\sigma)\theta^1 + i\,\theta^2(\partial_\tau-\partial_\sigma)\theta^2 - 2\mu\,\theta^1\Gamma_{1234}\theta^2\Big)\,. \end{aligned} \tag{19}$$

The bosonic mass term stems from the worldsheet coupling of the background space-time metric in the ++ direction, i.e.

$$g_{++}\,\partial_a X^+ \partial_a X^+ = -\mu^2\,(X^I)^2\,, \tag{20}$$

due to the light-cone-gauge condition (18). Similarly the fermionic mass term arises from the fermion bilinear coupling to the Ramond-Ramond field strength F_{+1234}, which

[4]For an introduction to superstrings in the Green-Schwarz formulation see chapter five of [19].

[5]The emergence of a quadratic string action in the Penrose limit may also be traced back to the semi-classical quantization of the full $AdS_5 \times S^5$ superstring action around a point-like solution of the classical string equations of motion propagating on a geodesic along the S^5, see [20, 21] for details.

manifestly breaks the transverse $SO(8)$ to $SO(4) \times SO(4)$. Also we see that in the limit $\mu \to 0$ one recovers the standard flat space model [19].

The bosonic equations of motion following from (19) take the form

$$(\partial_\tau^2 - \partial_\sigma^2 + \mu^2)\, X^I = 0 \tag{21}$$

subject to the closed string boundary condition $X^I(\tau, \sigma + 2\pi\, \alpha'\, p^+) = X^I(\tau, \sigma)$. Its general solution in an oscillator mode decompositions reads

$$X^I = \cos(\mu\tau)\, \frac{x_0^I}{\mu} + \sin(\mu\tau)\, \frac{p_0^I}{\mu} + \sum_{n\neq 0} \frac{i}{\sqrt{2\,\omega_n}}\, (\, \alpha_n^I\, e^{-i(\omega_n \tau - k_n \sigma)} + \tilde{\alpha}_n^I\, e^{-i(\omega_n \tau + k_n \sigma)}\,)\,, \tag{22}$$

where $\omega_n = \text{sign}(n)\, \sqrt{k_n^2 + \mu^2}$ and $k_n = n/(\alpha'\, p^+)$. The corresponding canonical momentum, $P^I = \dot{X}^I$, reads

$$P^I = \cos(\mu\tau)\, p_0^I - \sin(\mu\tau)\, x_0^I + \sum_{n\neq 0} \sqrt{\frac{\omega_n}{2}}\, (\, \alpha_n^I\, e^{-i(\omega_n \tau - k_n \sigma)} + \tilde{\alpha}_n^I\, e^{-i(\omega_n \tau + k_n \sigma)}\,)\,. \tag{23}$$

Moreover the coordinate X^- is expressed in terms of the transverse degrees of freedom via the Virasoro constraint

$$P^+\, \partial_\sigma X^- + P^I\, \partial_\sigma X^I + i\theta^1\, \partial_\sigma \theta^1 + i\theta^2\, \partial_\sigma \theta^2 = 0\,, \tag{24}$$

which arises as a consequence of the conformal gauge choice. Next to determining $X^-(\tau, \sigma)$, the above equation also implies a constraint on the transverse degrees of freedom upon integrating it over σ:[6]

$$\int_0^{2\pi\alpha' p^+} d\sigma\, [\, P^I\, \partial_\sigma X^I + i\theta^1\, \partial_\sigma \theta^1 + i\theta^2\, \partial_\sigma \theta^2\,] = 0\,. \tag{25}$$

The plane-wave string is now readily quantized by replacing Poisson brackets by commutators in the standard fashion $\{.,.\}_{\text{P.B.}} \to i\,[.,.]$. From the canonical commutation relations one then deduces the commutation relations for the modes

$$[p_0^I, x_0^I] = -i\delta^{IJ}, \qquad [\alpha_m^I, \tilde{\alpha}_n^J] = 0, \qquad [\alpha_m^I, \alpha_n^J] = \delta_{n+m,0}\, \delta^{IJ}, \qquad [\tilde{\alpha}_m^I, \tilde{\alpha}_n^J] = \delta_{n+m,0}\, \delta^{IJ}\,. \tag{26}$$

Similar expressions arise for the fermionic modes, for details see [8]. The Hamiltonian then takes the form

$$\mathcal{H}_{\text{lc}} = \frac{1}{p^+} \int d\sigma\, \Big[\, (P^I)^2 + (\partial_\sigma X^I)^2 + \mu^2\, (X^I)^2 + \text{fermions} \Big]\,. \tag{27}$$

It is useful to introduce modes in the zero mode sector of x_0^I and p_0^I as well via

$$\alpha_0^I = \frac{1}{\sqrt{2\,\mu}}\, (\, p_0^I + i\,\mu\, x_0^I\,), \tag{28}$$

[6] Note that P^+ is a constant due to the light-cone gauge condition (18). Also we demand $X^-(\tau, \sigma + 2\pi\alpha' p^+) = X^-(\tau, \sigma)$.

in order to write down the Hamiltonian (27) in terms of oscillator modes

$$\mathcal{H}_{\rm lc} = \mu \, (\alpha_0^{\dagger\, I}\, \alpha_0^I + \theta_0^\dagger\, \theta_0\,) + \frac{1}{\alpha'\, p^+} \sum_{n=1}^{\infty} \sqrt{n^2 + (\alpha'\, p^+\, \mu)^2} \left[\, \alpha_{-n}^I\, \alpha_n^I + \tilde\alpha_{-n}^I\, \tilde\alpha_n^I + \theta_{-n}^1\, \theta_n^1 + \theta_{-n}^2\, \theta_n^2 \,\right] \tag{29}$$

where we have now also included the fermion modes in the conventions of [8]. An immediate observation is that upon taking $\mu \to 0$ the flat space light-cone Hamiltonian arises. This is indeed necessary, as for $\mu = 0$ the plane-wave geometry (12) turns into flat Minkowski space. The most pronounced difference of this Hamiltonian compared to the flat space situation is the fact that also the zero-mode sector is massive and governed by a harmonic oscillator spectrum. Therefore there are no asymptotically free states in the transverse direction anymore, rendering the concept of an S-matrix for the plane-wave string problematic: *All* transverse excitations are bound in a harmonic well.

One furthermore defines the Fock-vacuum $|0, p^+\rangle$ to be annihilated by the positive modes

$$\begin{aligned} \alpha_0\, |0,p^+\rangle = 0, \quad \alpha_n\, |0,p^+\rangle = 0, \quad \tilde\alpha_n\, |0,p^+\rangle = 0, \qquad n \geq 1 \\ \theta_0\, |0,p^+\rangle = 0, \quad \theta_n^1\, |0,p^+\rangle = 0, \quad \theta_n^2\, |0,p^+\rangle = 0, \qquad\qquad \end{aligned} \tag{30}$$

The physical states are subject to the Virasoro constraint arising from (25)

$$(N - \tilde N)\, |\text{phys}\rangle = 0 \qquad \text{with} \quad N = \sum_{n=1}^{\infty} (\, \alpha_{-n}^I\, \alpha_n^I + \theta_{-n}^1\, \theta_n^1\,) \quad \text{and} \quad \tilde N = \sum_{n=1}^{\infty} (\, \tilde\alpha_{-n}^I\, \tilde\alpha_n^I + \theta_{-n}^2\, \theta_n^2\,) \tag{31}$$

requiring a balanced excitation structure of the two sets of modes. The resulting spectrum then takes the simple form

$$E_{\rm lc} = \mu\, N_0 + \mu\, (\, N_n + \tilde N_n\,) \sqrt{1 + \frac{n^2}{(\alpha'\, p^+\, \mu)^2}}\, . \tag{32}$$

Let us now write down the lightest bosonic excitations. In the zero-mode or supergravity sector one has

$$\begin{aligned} |0,p^+\rangle &\qquad E_{\rm lc} = 0 \\ \alpha_0^{\dagger\, I}\, |0,p^+\rangle &\qquad E_{\rm lc} = \mu \\ \theta_0^\dagger\, |0,p^+\rangle &\qquad E_{\rm lc} = \mu \\ \alpha_0^{\dagger\, I_1} \ldots \alpha_0^{\dagger\, I_N}\, |0,p^+\rangle &\qquad E_{\rm lc} = N \cdot \mu \end{aligned} \tag{33}$$

and the first true stringy excitations - paying attention to the level matching condition

(31) - read

$$
\begin{aligned}
&\alpha^I_{-n}\,\tilde\alpha^J_{-n}|0,p^+\rangle \qquad && E_{\rm lc} = 2\,\mu\,\sqrt{1+\frac{n^2}{(\alpha'\,p^+\,\mu)^2}} \\
&\alpha^I_{-n_1}\,\alpha^J_{-n_2}\,\tilde\alpha^K_{-n_3}|0,p^+\rangle \qquad && E_{\rm lc} = \sum_{i=1}^{3}\mu\,\sqrt{1+\frac{n_i^2}{(\alpha'\,p^+\,\mu)^2}} \qquad \text{with} \quad n_1+n_2=n_3 \\
& && \vdots
\end{aligned}
\tag{34}
$$

Hence the spectrum of free, non-interacting, plane-wave string theory is under complete control. In the following sections 4 and 5 we shall see how it is reproduced from the dual gauge theory in terms of the scaling dimensions of the associated Super Yang-Mills operators.

String interactions may be described by methods of light-cone string field theory in the plane-wave background and will be the subject of section 6.

4 Plane-wave strings from $\mathcal{N}=4$ Super Yang-Mills

As was pointed out in the initial paper of Berenstein, Maldacena and Nastase [2] the discussed Penrose limit of $AdS_5\times S^5$ leading to the plane-wave geometry must entail – by virtue of the AdS/CFT correspondence – a dual gauge theory description of the plane-wave string model. In this section we shall determine the precise nature of this dual gauge theory limit.

The dual gauge theory of the $AdS_5\times S^5$ superstring is the maximally supersymmetric ($\mathcal{N}=4$) Yang-Mills theory in four dimensions [22]. Its field content is comprised of a gluon field, six scalars as well as 4 Majorana gluinos, which we choose to write as a 16 component 10d Majorana-Weyl spinor. All fields are in the adjoint representation of $U(N)$. Explicitly we have

$$
A_\mu(x), \qquad \phi_i(x), \quad i=1,\dots,6 \qquad \chi_\alpha(x), \quad \alpha=1,\dots 16 \tag{35}
$$

given by $N\times N$ hermitian matrices. The action of $\mathcal{N}=4$ Super Yang-Mills is uniquely determined by two parameters, the coupling constant $g_{\rm YM}$ and the rank of the gauge group N, to be

$$
S=\frac{2}{g^2_{\rm YM}}\int d^4x\ {\rm Tr}\Big\{\frac{1}{4}\,(F_{\mu\nu})^2+\frac{1}{2}\,(D_\mu\phi_i)^2-\frac{1}{4}\,[\phi_i,\phi_j]\,[\phi_i,\phi_j]+\frac{1}{2}\,\bar\chi\not{D}\,\chi-\frac{i}{2}\,\bar\chi\,\Gamma_i\,[\phi_i,\chi]\Big\} \tag{36}
$$

with the covariant derivative defined as $D_\mu=\partial_\mu-i[A_\mu,\ \ ]$. Furthermore, (Γ_μ,Γ_i) are the ten dimensional Dirac matrices.

This model displays a global $SO(6)$ symmetry group, called R-symmetry, acting as internal rotations on the six scalars and four spinors. Moreover due to the large amount of supersymmetry present, the conformal invariance of the classical field theory survives the quantization procedure: The coupling constant $g_{\rm YM}$ is not renormalized and its β-function is believed to vanish to all orders in perturbation theory [23]. This is why one often refers to $\mathcal{N}=4$ Super Yang-Mills as a "finite" quantum field theory. The conformal invariance group in four dimensions consisting of the Poincare group, dilatations and special conformal transformations is $SO(2,4)$. The full bosonic symmetry group of (36) is hence given by the product $SO(2,4)\times SO(6)_R$ – matching precisely with the isometry groups of the $AdS_5\times S^5$ geometry.

The vertices and propagators can be read off from the action (36). We will work in the Feynman gauge, where the free field limit of the gluon propagators in matrix notation is

$$\langle\, (A_\mu)_{ab}(x)\, (A_\nu)_{cd}(y)\, \rangle_0 = \frac{g_{\rm YM}^2\, \delta_{\mu\nu}}{8\,\pi^2\,(x-y)^2}\, \delta_{ad}\, \delta_{bc}\, , \tag{37}$$

similarly the propagators for the scalars read

$$\langle\, (\phi_i)_{ab}(x)\, (\phi_j)_{cd}(y)\, \rangle_0 = \frac{g_{\rm YM}^2\, \delta^{ij}}{8\,\pi^2\,(x-y)^2}\, \delta_{ad}\, \delta_{bc}\, . \tag{38}$$

The observables of interest to us are local, composite, gauge invariant operators, i.e. traces of products of fundamental fields at a given space-point, e.g. $\mathcal{O}_{i_1\ldots i_k}(x) = \mathrm{Tr}[\phi_{i_1}(x)\,\phi_{i_2}(x)\ldots\phi_{i_k}(x)]$. A central class of operators in a general conformal field theory are the conformal primary operators, which possess a definite scaling dimension. Their two point functions are determined by the conformal symmetry to be diagonal and to take the form

$$\langle \mathcal{O}_A(x)\, \mathcal{O}_B(y)\rangle = \frac{\delta_{AB}}{(x-y)^{2\,\Delta_{\mathcal{O}_A}}} \tag{39}$$

where $\Delta_{\mathcal{O}_A}$ is the scaling dimension of the composite operator $\mathcal{O}_A$. Classically these scaling dimensions are simply the sum of the individual dimensions of the constituent fields ($[\phi_i]=[A_\mu]=1$ and $[\chi]=3/2$). In quantum theory the scaling dimensions receive radiative correction, organized in a double expansion in $\lambda = g_{\rm YM}^2\, N$ (loops) and $1/N^2$ (genera)

$$\Delta = \Delta_0 + \sum_{l=1}^{\infty} \lambda^l \sum_{g=0}^{\infty} \frac{1}{N^{2\,g}}\, \Delta_{l,g}\, , \tag{40}$$

as discussed in section one. For example the simplest conformal primary operator of $\mathcal{N}=4$ Super Yang-Mills is the Konishi field $\mathcal{O}_K = \mathrm{Tr}[\phi_i\,\phi_i]$, whose planar scaling dimension is known up to two loop order $\Delta_{\mathcal{O}_K} = 2 + \frac{3\,\lambda}{4\pi^2} - \frac{3\,\lambda^2}{16\pi^4}$ [24, 25][7].

A remarkable feature of the $\mathcal{N}=4$ gauge theory is the existence of a class of operators, referred to as chiral primary or 1/2 BPS operators, whose scaling dimensions do *not* receive any radiative corrections. They can be conformal primaries or descendants thereof. In the scalar sector these protected operators are given by

$$\mathcal{O}^k_{\rm CPO} = C_{i_1 i_2\ldots i_k}\ \mathrm{Tr}[\phi_{i_1}\,\phi_{i_2}\ldots\phi_{i_k}] \tag{41}$$

with $C_{i_1 i_2\ldots i_k}$ being a symmetric traceless rank k tensor. The claim then is that the *exact* scaling dimension of $\mathcal{O}^k_{\rm CPO}$ is given by its classical value $\Delta_{\mathcal{O}^k_{\rm CPO}} = k$.

[7] The three loop result has been recently conjectured to be $\frac{21\,\lambda^3}{256\pi^6}$ [26]. Strictly speaking the 't Hooft expansion of anomalous dimensions (40) can also contain odd powers of $1/N$ due to operator mixing effects, see e.g. [26] for a discussion.

	Z	$\bar{Z}$	$\phi_{i=1,2,3,4}$	A_μ	ψ_A	$\tilde{\psi}_{\dot{A}}$
Δ_0	1	1	1	1	3/2	3/2
J	1	-1	0	0	1/2	$-1/2$
$\Delta_0 - J$	0	2	1	1	1	2

Table 1: Scaling dimensions and J charges of the fundamental fields.

Conformal symmetry moreover constrains the three-point functions of conformal primary operators. Their space-time dependence is completely determined by the scaling dimensions Δ_i of the participating operators

$$\langle\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\,\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}\,|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{42}$$

and the only new quantity emerging is the structure constant C_{123}. Three-point functions of protected operators are also protected from radiative corrections, four and higher-point functions do receive radiative corrections.

In order to address the BMN limit of the $\mathcal{N} = 4$ model we need to identify the $U(1)$ charge J in the gauge theory corresponding to the angular momentum J along the equator of the S^5 on the dual string side. It is given by the charge associated to the complex combination of two scalars, say ϕ_5 and ϕ_6

$$Z = \frac{1}{\sqrt{2}}\,(\phi_5 + i\,\phi_6)\,. \tag{43}$$

The classical scaling dimensions and J charges of the fundamental fields are summarized in table 1. Note that the fermions χ_α split into two components ψ_A and $\tilde{\psi}_{\dot{A}}$ of opposite J charge [2].

As discussed in section 2 the limit in question is then

$$N \to \infty \quad \text{and} \quad J \to \infty \quad \text{with} \quad \frac{J^2}{N} \quad \text{and} \quad g_{\text{YM}} \quad \text{fixed} \tag{44}$$

Let us stress that this limit is *distinct* to the standard 't Hooft large-N limit, where one takes $N \to \infty$ while keeping $\lambda = g_{\text{YM}}^2\, N$ fixed. In the limit of (44) λ diverges, which seems disastrous from a perturbative point of view, as the quantum loop corrections in the gauge theory appear as an expansion in λ. However, there is one central point we have not yet addressed, which was raised in the discussion at the end of section two, namely the necessity of restricting one's attention to the set of operators whose scaling dimensions are of the order of J, i.e. $\Delta \sim J$. Therefore the operators obeying this rule are made out of a long string of Z's, compare table 1.

For the class of protected operators (41) the strong coupling nature of the BMN limit (44) is not visible as their two and three-point functions do not receive any quantum corrections. Examples obeying $\Delta \sim J$ are

$$\text{Tr}(Z^J) \quad \text{and} \quad \text{Tr}(\phi_i\, Z^J)\,. \tag{45}$$

But one can do slightly better than that: The crucial insight of Berenstein, Maldacena and Nastase was to violate the "protectedness" of these operators in a small and controlled fashion, by inserting in the string of J Z's a small number of impurities in form of operators with $\Delta - J = 1$

$$\mathrm{Tr}(\,\phi_i\, Z \dots Z\, \phi_j\, Z \dots Z\, D_\mu Z\, Z \dots Z\, \psi_\alpha\, Z \dots Z\,)\,,$$

for a generic "BMN-operator"[8]. As we will show in the following two and three-point functions of these type of operators receive quantum correction through an *effective* loop counting parameter

$$\lambda' := \frac{g_{\mathrm{YM}}^2\, N}{J^2} \tag{46}$$

which remains finite and tunable in the BMN limit (44). Hence, even though the scaling dimensions of generic operators in the $\mathcal{N} = 4$ model diverge in the limit (44), there remains a perturbatively accessible sector comprised of BMN-operators with dimensions expressed in terms of the new effective coupling constant λ'.

Notably this effective weak coupling sector breaks down once one moves to four and higher point functions. As was demonstrated in [27] four-point functions of $\mathrm{Tr}(Z^J)$ diverge with J in the BMN-limit. This shows that the BMN limit (44) represents an extreme reduction of the quantum field theory whose precise nature remains to be understood.

4.1 The plane-wave string state/gauge theory operator dictionary

Can we identify the gauge theory operators which are dual to the plane-wave string states constructed in section 3? The guiding principle is the value of $\Delta - J$ to be identified with E_{lc}/μ of (32) by the central relation (15). For the string groundstate $|0, p^+\rangle$ with vanishing groundstate energy there is a unique single trace operator with vanishing $\Delta - J$, i.e.

$$E_{\mathrm{lc}} = 0 \qquad |0, p^+\rangle \;\hat{=}\; \tfrac{1}{\sqrt{J\, N^J}}\ \mathrm{Tr}\ Z^J \qquad \Delta - J = 0 \tag{47}$$

As $\mathrm{Tr}\, Z^J$ is a protected operator, its scaling dimension equals J to all orders in λ' in the full quantum theory. This is indeed necessary for the above identification to make sense. The normalization of $\mathrm{Tr}\, Z^J$ is chosen in order to have unit weight in the two point function (39) at leading order in N.

Let us now move on to the supergravity modes in the plane-wave string spectrum (33). Here we need $\Delta - J = 1$ for the first excitations. This is realized by the operators

$$E_{\mathrm{lc}} = \mu \qquad \alpha_0^{\dagger\, i}\, |0, p^+\rangle \quad \hat{=} \quad \tfrac{1}{\sqrt{N^J}}\ \mathrm{Tr}\,(\phi_i\, Z^J) \tag{48}$$

$$E_{\mathrm{lc}} = \mu \qquad \alpha_0^{\dagger\, \mu}\, |0, p^+\rangle \quad \hat{=} \quad \tfrac{1}{\sqrt{N^J}}\ \mathrm{Tr}\,(D_\mu Z\, Z^{J-1}) \tag{49}$$

$$E_{\mathrm{lc}} = \mu \qquad \theta^{\dagger}_{0\,A}\, |0, p^+\rangle \quad \hat{=} \quad \tfrac{1}{\sqrt{N^J}}\ \mathrm{Tr}\,(\psi_A\, Z^J) \tag{50}$$

corresponding to the 8+8 bosonic and fermionic excitations of the string. Note that while $\mathrm{Tr}\,(\phi_i\, Z^J)$ is a conformal primary operator, $\mathrm{Tr}\,(D_\mu Z\, Z^{J-1})$ is a descendant of the groundstate operator $\mathrm{Tr}\ Z^J$ obtained by acting with D_μ. Similarly $\mathrm{Tr}\,(\psi_A\, Z\, Z^J)$ is a superdescendant of the groundstate operator[9]. All three operators are again protected

[8] This is actually not entirely correct: $SO(4)$ singlet operators also require the compensating insertion of $\bar{Z}$'s with $\Delta - J = 2$ [27].

[9] For a detailed discussion of BMN operators and superconformal symmetry see [28].

and have a total $\Delta - J = 1$ exactly, matching the string spectrum $E_{\rm lc} = \mu$. Higher zero mode excitation are modeled by symmetrized insertions of ϕ_i, $D_\mu Z$ and ψ_A such as

$$E_{\rm lc} = 2\mu \qquad \alpha_0^{\dagger\, i}\, \alpha_0^{\dagger\, j}\, |0,p^+\rangle \;\hat{=}\; \tfrac{1}{\sqrt{J\, N^J}} \sum_{l=0}^{J} {\rm Tr}\, (\phi_i\, Z^l\, \phi_j\, Z^{J-l}) \qquad \Delta - J = 2 \tag{51}$$

The generalization to higher modes should be clear by now.

Now obviously the true challenge lies in reproducing the stringy mode spectrum through non-protected Super Yang-Mills operators. At the first stringy mode level we are searching for a dual gauge theory object $\mathcal{O}_n^{ij}$ with

$$E_{\rm lc} = 2\mu \sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}} \qquad \alpha^i_{-n}\, \tilde{\alpha}^j_{-n}\, |0,p^+\rangle \;\hat{=}\; \mathcal{O}_n^{ij} \tag{52}$$

The operator $\mathcal{O}_n^{ij}$ should carry two impurities ϕ_i and ϕ_j in order to reproduce the $SO(4)$ index structure of the dual string state. Moreover for $n \to 0$ it should reduce to the protected operator of (51). The simplest ansatz is

$$\mathcal{O}_n^{ij} = \tfrac{1}{\sqrt{J\, N^J}} \sum_{l=0}^{J} {\rm Tr}\, (\phi_i\, Z^l\, \phi_j\, Z^{J-l})\, f(n,l) \tag{53}$$

with a suitable function $f(n,l)$ obeying $f(0,l) = 1$. It turns out that the correct choice is

$$f(n,l) = e^{2\,\pi\, i\, n\, l/J} \tag{54}$$

and we will show in the next chapter why this is the case. Let us at this point just state that a computation of the *planar* scaling dimension with this choice of $f(n,l)$ up to one loop order yields the result [2]

$$\Delta_{\mathcal{O}_n^{ij}} = J + 2 + \frac{g_{\rm YM}^2\, N}{J^2}\, n^2 + \mathcal{O}(g_{\rm YM}^2) \tag{55}$$

Note the emergence of the promised effective coupling constant λ' in the BMN limit (44). In order to compare this result to the string light-cone energy we need to convert the string parameters α', p^+, μ and R to $g_{\rm YM}$ and N. From (14) we have

$$2p^+ = \frac{E + J}{\mu\, R^2} \sim \frac{2\, J}{\mu\, R^2} \quad \Rightarrow \quad (\mu p^+)^2 = \frac{J^2}{R^4} \tag{56}$$

which upon making use of the AdS/CFT relation $R^4 = g_{\rm YM}^2\, \alpha'^2\, N$ leads to

$$\frac{1}{(\alpha' p^+ \mu)^2} = \frac{g_{\rm YM}^2\, N}{J^2} =: \lambda' \tag{57}$$

Therefore the perturbative gauge theory expansion around $\lambda' = 0$ corresponds to a $\mu \to \infty$ expansion on the string side, exactly opposite to the flat Minkowski space regime $\mu = 0$.

In this domain the square root of the string state energy (52) may be expanded out to yield

$$\frac{E_{\rm lc}}{\mu} = 2\sqrt{1 + n^2\, \lambda'} = 2 + \lambda'\, n^2 + \mathcal{O}(\lambda'^2) \tag{58}$$

matching precisely with the Super Yang-Mills result (55)! Recovering the complete structure of the free plane-wave string spectrum therefore necessitates the summation of the complete planar perturbation series on the gauge theory side. The consistency of the Super Yang-Mills planar two loop result for the scaling dimension with the string theoretic square root was demonstrated in [29] and a mechanism for the all loop result was presented. Relying on certain assumptions a proof of the full square root structure of the gauge theory scaling dimensions was obtained thereafter in [30].

One therefore has strong evidence that the planar sector of BMN gauge theory scaling dimensions indeed reproduces the free plane-wave string spectrum, enabling one to write down a concrete dictionary linking string states to gauge theory operators.

4.2 The non-planar sector of BMN gauge theory

If the AdS/CFT duality is to hold in its strong version, i.e. implying the exact equivalence of the full interacting $AdS_5 \times S^5$ string theory to $\mathcal{N}=4$ Super Yang-Mills, one should be able to go beyond the free plane-wave string theory discussed above. How can one then recover the plane-wave string interactions on worldsheets of higher genera from the $\mathcal{N}=4$ gauge model in the BMN limit? The natural place to look for is the non-planar sector of the dual gauge theory, however, as was discussed in section one, non-planar graphs are expected to be suppressed in the large N limit. The surprising fact is that the BMN limit $N \sim J^2 \to \infty$ represents a *novel* double scaling limit of the gauge theory under which graphs of all genera survive [10,11]. Here the suppression of non-planar graphs with $1/N^2$ is balanced by the growing combinatorics of the diagrams involved with $J \to \infty$. Next to the effective coupling constant λ' a new *effective* genus counting parameter J^2/N arises which remains finite and tunable in the BMN limit (44). Hence the BMN gauge theory is controlled by two independent parameters

$$\lambda' := \frac{g_{\rm YM}^2 N}{J^2} \qquad \text{and} \qquad g_2 := \frac{J^2}{N} \tag{59}$$

allowing for a double expansion.

The emergence of g_2 is most transparent in the correlation function of two protected groundstate operators Tr Z^J, this being the simplest two-point function in BMN gauge theory. Here the exact coordinate and $g_{\rm YM}$ dependence is trivial, as there are no loop-corrections, and it remains to solve the combinatorial problem of taking into account all possible contractions of free field propagators. This may be efficiently summarized through a correlator in a Gaussian complex matrix model,

$$\langle \text{Tr } Z^J(x) \text{ Tr } \bar{Z}^J(0)\rangle = \left(\frac{g_{\rm YM}^2}{8\pi^2|x|^2}\right)^J \int dZ\, d\bar{Z} \text{ Tr } Z^J \text{ Tr } \bar{Z}^J\, e^{-\,\text{Tr}(Z\bar{Z})}\,. \tag{60}$$

Here the measure is an abbreviation of

$$dZ\, d\bar{Z} = \prod_{a,b=1}^{N} \frac{d\text{Re}Z_{ab}\, d\text{Im}Z_{ab}}{\pi} \qquad \text{and} \qquad \langle\, \mathcal{O}\, \rangle_{\rm MM} := \int dZ\, d\bar{Z}\, \mathcal{O}\, e^{-\,\text{Tr}(Z\bar{Z})} \tag{61}$$

ensuring $\langle\, 1\, \rangle_{\rm MM} = 1$. This matrix model captures the correct gauge theory combinatorics by virtue of its propagator

$$\langle\, Z_{ab}\, \bar{Z}_{cd}\, \rangle_{\rm MM} = \delta_{ad}\, \delta_{bc}\,, \tag{62}$$

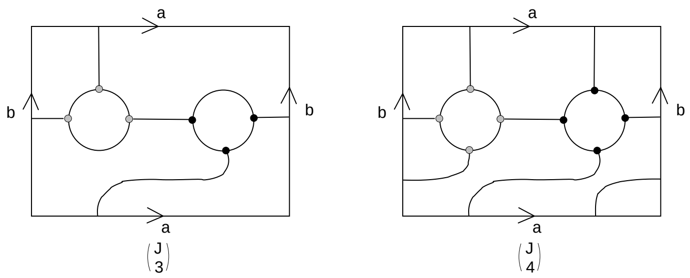


Figure 1: The irreducible genus one graphs with their combinatorial weight.

compare (38). As a matter of fact the correlator $\langle \operatorname{Tr} Z^J \operatorname{Tr} \bar{Z}^J \rangle_{\rm MM}$ can be computed exactly for finite N and J using matrix model techniques [10]. The result may be expanded as a series in $\frac{1}{N^2}$ and one extracts, for general J, the corrections to the (trivial) planar result JN^J:

$$\begin{aligned} \langle \mathrm{Tr} Z^J \ \mathrm{Tr} \bar{Z}^J \rangle_{\rm MM} &= J\ N^J \Bigg\{ 1 + \left[\binom{J}{4} + \binom{J}{3} \right] \frac{1}{N^2} \\ &\quad + \left[21 \binom{J}{8} + 49 \binom{J}{7} + 36 \binom{J}{6} + 8 \binom{J}{5} \right] \frac{1}{N^4} + \ldots \Bigg\} \end{aligned} \tag{63}$$

The structure of this result is easy to understand combinatorially. We have to find the possible ways of connecting two necklaces with J white (Z's) and J black ($\bar{Z}$'s) beads respectively, according to the following rules: (a) each connection has to link a black to a white bead, (b) in order to find the $\mathcal{O}(N^{J-2h})$ contribution the connections have to be drawn without crossing on a genus h surface such that no handle of the surface can be collapsed without pinching a connection. Let us call all connections that run (possibly after topological deformation) parallel to another connection "reducible". Eliminate all reducible connections. This will lead to a number of inequivalent, irreducible graphs on the genus h surface. There are two such irreducible graphs for the toroidal contribution carrying combinatorial weights indicated in figure 1 and appearing in the terms of order $1/N^2$ in (63). They arise from distributing the J beads of one necklace into three respectively four bins.

Upon taking the BMN limit (44) of the correlator (63) one is left with

$$\frac{1}{J\,N^J} \langle \mathrm{Tr} Z^J \ \mathrm{Tr} \bar{Z}^J \rangle_{\rm MM} \overset{J,N\to\infty}{\longrightarrow} 1 + \frac{1}{24}\frac{J^4}{N^2} + \frac{21}{8!}\frac{J^8}{N^4} + \ldots = \frac{2\,N}{J^2} \sinh(\frac{1}{2}\frac{J^2}{N}) \tag{64}$$

where we have inserted the all genus result obtained in [10] in the last step. The central observation here is that non-planar graphs are not fully suppressed in this novel type of large N limit and the new *effective* genus counting parameter $g_2 := J^2/N$ arises. The class of non-planar graphs surviving the BMN limit is distinctively smaller than the class of all non-planar graphs, as is seen from the above toroidal example. Only graphs with "handles" made of sufficiently many beads contribute in the double scaling limit (44). This structure gives rises to a discretized closed string interpretation of the gauge theory. Here a necklace of J beads corresponds to a closed string made out of J bits and the

$J \to \infty$ limit is a continuum limit from the discrete string point of view. String splitting is suppressed with $\frac{1}{N^2}$. The scaling law $J^2 \sim N$ ensures that only worldsheets with "fat" handles, i.e. made out of a sufficiently large number of string bits, survive the double scaling limit. Smooth, macroscopic surfaces are formed in this limiting process in great similarity to the double scaling limits of the "old matrix models" relevant to describing two-dimensional quantum gravity [31].

Let us also stress that the chosen scaling of J and N indeed represents a delicate balance between string splitting and string bit proliferations: Had one chosen a scaling law as $J^p \sim N$ in the large N limit, then for $p < 2$ non-planar graphs would actually dominate over planar ones, whereas for $p > 2$ only planar graphs would have survived. The BMN limit choice $p = 2$ is hence fine tuned to yield a full fleshed genus expansion and justifies the name "double scaling limit".

5 The computation of planar and non-planar corrections to Super Yang-Mills scaling dimensions

After these general remarks and observations we now move on to the explicit computation of scaling dimensions of BMN operators. In these lectures we shall confine our attention to the scaling dimension of the two impurity single trace operator

$$\mathcal{O}_p^J(x) = \mathrm{Tr}\,[\,\phi_1\, Z^p\, \phi_2\, Z^{J-p}\,](x) \tag{65}$$

up to one quantum loop and genus one. For this we need to study the two-point function

$$\langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\,\rangle = \langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\,\rangle_{\text{classical}} + \langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\,\rangle_{1-\text{loop}}\,. \tag{66}$$

Here it will prove useful to again employ the matrix model techniques encountered in our discussion of the protected operator $\mathrm{Tr}\, Z^J$. The classical piece of (66) is then

$$\langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\,\rangle_{\text{classical}} = \left(\frac{g_{\text{YM}}^2}{8\pi^2|x|^2}\right)^{J+2} \langle\, \mathcal{O}_p^J\, \bar{\mathcal{O}}_q^J\,\rangle_{\text{M M}} \tag{67}$$

with

$$\langle\, \mathcal{O}_p^J\, \bar{\mathcal{O}}_q^J\,\rangle_{\text{M M}} = \int dZ\, d\bar{Z}\ \mathrm{Tr}[Z^p\, \bar{Z}^q]\ \mathrm{Tr}[\, Z^{J-p}\, \bar{Z}^{J-q}]\, e^{-\mathrm{Tr}\,(Z\bar{Z})} \tag{68}$$

where we have contracted the two scalars ϕ_1 and ϕ_2 by making use of the $U(N)$ fission and fusion rules

$$\mathrm{Tr}[\underbrace{\phi^+ A\, \phi^-}\, B] = \mathrm{Tr}[A]\ \mathrm{Tr}[B] \qquad \mathrm{Tr}[\underbrace{\phi^+ A]\ \mathrm{Tr}[\phi^-}\, B] = \mathrm{Tr}[A\, B] \tag{69}$$

with $\langle \underbrace{\phi_{ab}^+\, \phi_{cd}^-} \rangle = \delta_{bc}\, \delta_{ad}$, an immediate consequence of (38). The remaining correlator (68) has been worked out explicitly in [10, 11] up to genus one. One finds[10]

$$\begin{aligned}
\langle\, \mathcal{O}_p^J\, \bar{\mathcal{O}}_q^J\,\rangle_{\text{M M}} &= \langle\, \mathrm{Tr}[Z^p\, \bar{Z}^q]\ \mathrm{Tr}[\, Z^{J-p}\, \bar{Z}^{J-q}]\,\rangle_{\text{M M}} = \\
&\delta_{p,q}\, N^{J+2} + N^J \Big[\delta_{p,q}\, \big[\tbinom{J-p+2}{4} + \tbinom{p+2}{4}\big] + \tfrac{1}{6}\, p\, (p+1)\, (3J + 1 - p - 3q) \\
&\qquad + (q-p)\, (p+1)\, (J - q + 1)\Big] + \mathcal{O}(N^{J-2})
\end{aligned} \tag{70}$$

[10] This formula is valid only for $(q > p\,,\ J - q > p)$ in the non-diagonal part. The other regions of p and q are determined by the two obvious symmetries $(p \leftrightarrow q)$ and $(p \to J-p, q \to J-q)$ of the correlator.

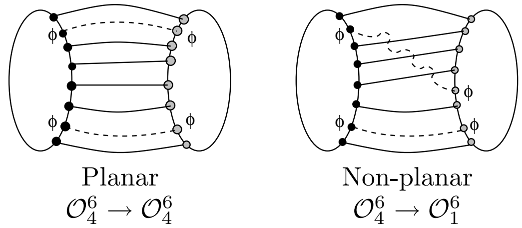
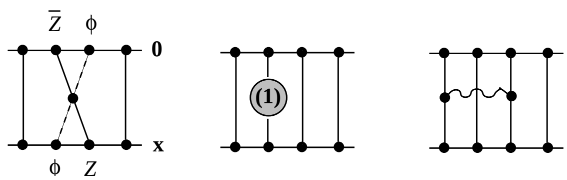


Figure 2: "Hopping" is induced by non-planar graphs.

For higher genus results see [32]. The important thing to note here is not the precise form of this correlator but rather the fact that the classical contribution to the two-point function (66) becomes non-diagonal in p and q at the toroidal level, i.e. the non-diagonal $\mathcal{O}(N^J)$ terms in the above. The source of this non-diagonality are the non-planar "hopping" graphs depicted in figure 2.

Moving on to the one loop radiative corrections to the correlator we need to include the scalar four point interaction, one loop self energy insertion and gluon exchange graph,



(71)

in the diagrams of figure 2. Now the "hopping" already occurs at the planar level, as seen in the above first diagram, which exchanges the position of a Z and ϕ_i field through the scalar self interaction term $\mathrm{Tr}[\phi_i, Z]\,[\phi_i, \bar{Z}]$ of the Lagrangian (36). Therefore

$$\langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\,\rangle_{1-\mathrm{loop}} \neq \delta_{p,q} \tag{72}$$

and one needs to rediagonalize the set of BMN operators $\mathcal{O}_p^J(x)$ in order to determine their scaling dimension from (39) already at the planar level.

How can one now incorporate the loop effects (71) in an effective matrix model vertex? As the matrix model is a zero dimensional field theory we need to keep track of the positions (0 or x) of the individual fields in the two-point function in form of two distinct matrix model fields. This was automatically achieved via complexification for Z (sitting at x) and $\bar{Z}$ (sitting at 0), for the remaining four real scalar fields ϕ_i, however, we store this information in two different matrix model fields denoted by $\Phi_i^\pm$

$$\phi_i(0) \to \Phi_i^- \qquad \phi_i(x) \to \Phi_i^+ \tag{73}$$

with zero dimensional "propagators"

$$\langle\, (\Phi_i^-)_{ab}\, (\Phi_j^+)_{cd}\,\rangle_{\mathrm{MM}} = \delta_{ad}\,\delta_{bc}\,\delta_{ij} \qquad \langle\, \Phi_i^+\, \Phi_j^+\,\rangle_{\mathrm{MM}} = 0 = \langle\, \Phi_i^-\, \Phi_j^-\,\rangle_{\mathrm{MM}}\,. \tag{74}$$

In doing so we are able to disentangle the space-dependence of the correlation function (72) from the more challenging combinatorics. The combinatorics of the scalar self interaction insertion is then given by the effective matrix model vertex

$$\tfrac{1}{2}\,\mathrm{Tr}[\phi_i, \phi_j]^2(x) \to \mathrm{Tr}[\Phi_i^+, \Phi_j^-]\,[\Phi_i^+, \Phi_j^-] + \mathrm{Tr}[\Phi_i^+, \Phi_j^-]\,[\Phi_i^-, \Phi_j^+] + \mathrm{Tr}[\Phi_i^+, \Phi_j^+]\,[\Phi_i^-, \Phi_j^-] \tag{75}$$

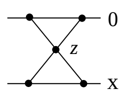
as two fields need to be contracted with the operator at 0 and the other two with the operator at x. Note that in the above we have temporarily reverted to the index range $i = 1, \ldots, 6$. By making use of the Jacobi identity this vertex may be reorganized into the more convenient form

$$\tfrac{1}{2}\,\mathrm{Tr}[\phi_i, \phi_j]^2(x) \to V_D + V_F + V_k \tag{76}$$

where

$$\begin{aligned} V_D &= \tfrac{1}{2}\,\mathrm{Tr}[\Phi_i^+, \Phi_i^-]\,[\Phi_j^+, \Phi_j^-] && \text{symmetric piece} \\ V_F &= -\,\mathrm{Tr}[\Phi_i^+, \Phi_j^+]\,[\Phi_i^-, \Phi_j^-] && \text{anti-symmetric piece} \\ V_K &= -\tfrac{1}{2}\,\mathrm{Tr}[\Phi_i^+, \Phi_j^-]\,[\Phi_i^+, \Phi_j^-] && \text{trace piece} \end{aligned} \tag{77}$$

couple to the symmetric, anti-symmetric and trace pieces respectively of an operator made entirely from the minus-valued matrix fields Φ_k^- upon contraction with the rules (69). In order to reproduce the field theoretic result this effective matrix model vertex needs to be augmented by a space dependent factor arising from the logarithmically divergent integral



$$= -\frac{g_{\mathrm{YM}}^4\, x^4}{64\pi^4} \int \frac{d^4 z}{(z-x)^4\, z^4} = \frac{g_{\mathrm{YM}}^4\, L}{32\pi^2} \tag{78}$$

with

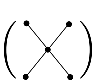
$$L := \log x^{-2} - (\tfrac{1}{\epsilon} + \gamma + \log\pi + 2) \tag{79}$$

in dimensional regularization. This logarithmic dependence on x is consistent with the one loop anomalous dimension Δ_1 term arising in the $\lambda' \ll 1$ expansion of the conformal field theory two-point function (39)

$$\frac{1}{x^{2(\Delta_0+\Delta_1)}} = \frac{1}{x^{2\,\Delta_0}}\,\big(1 + \Delta_1\,\log(\Lambda\, x)^{-2} + \mathcal{O}(\lambda'^2)\big) \tag{80}$$

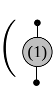
where we have introduced the scale factor Λ. The divergent piece of the integral (78) is canceled by an appropriate renormalization of the operators $\mathcal{O}_p^J(x)$. Hence, although the coupling constant g_{YM} is not renormalized in the $\mathcal{N}=4$ model, composite operators like $\mathcal{O}_p^J(x)$ are.

In summary the effective matrix model vertex reflecting the scalar self interaction takes the form



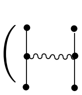
$$\Big(\;\Big)_{\text{Matrix Model}} = \frac{g_{\mathrm{YM}}^2\, L}{16\pi^2}\,(:V_D: + :V_F: + :V_K:) \tag{81}$$

to be inserted in a matrix model correlator. Here the colons ":" denote normal ordering, disallowing self-contractions of fields within one vertex. The vertices for scalar self-energy



$$\Big(\;\Big)_{\text{Matrix Model}} = \frac{g_{\mathrm{YM}}^2 (L+1)}{8\pi^2}\Big(N\ :\mathrm{Tr}(\Phi_i^-\Phi_i^+): - :\mathrm{Tr}\,\Phi_i^-\ \mathrm{Tr}\,\Phi_i^+:\Big) \tag{82}$$

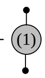
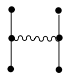
and gluon-exchange



$$\Big(\;\Big)_{\text{Matrix Model}} = \frac{g_{\mathrm{YM}}^2 (L+2)}{32\pi^2}\Big(:\mathrm{Tr}[\Phi_i^+, \Phi_i^-][\Phi_j^+, \Phi_j^-]:\Big), \tag{83}$$

are obtained in a similar fashion [27].

Remarkably the term V_D in the scalar interaction cancels against the gluon-exchanges and the scalar self-energies [11, 27]. The proof goes as follows. The sum of these terms can be written without normal-orderings in the following way

$\Big(: V_D : +$  $+$  $\Big)_{\text{Matrix Model}}$ (84)

$$= \frac{g_{\rm YM}^2 (L+1)}{8\pi^2} \big(\tfrac{1}{2} \operatorname{Tr}[\Phi_i^+, \Phi_i^-][\Phi_j^+, \Phi_j^-] - N \operatorname{Tr} \Phi_j^+ \Phi_j^- + \operatorname{Tr} \Phi_j^+ \operatorname{Tr} \Phi_j^- \big).$$

It is easy to see that the contraction of Φ_i^+ in the above vertex with an arbitrary trace of scalars $\Phi_{i_k}^-$ vanishes

$$\begin{aligned} &\Big(\operatorname{Tr}([\Phi_i^+, \Phi_i^-][\Phi_j^+, \Phi_j^-]) \ \operatorname{Tr}(\Phi_{i_1}^- \Phi_{i_2}^- \cdots \Phi_{i_n}^-) \Big)_{\Phi_i^+ \leftrightarrow \Phi_{i_k}^-} \\ =\ & \operatorname{Tr}\Big(\Big[\Phi_{i_1}^-, [\Phi_j^+, \Phi_j^-] \Big] \Phi_{i_2}^- \cdots \Phi_{i_n}^- \Big) + \operatorname{Tr}\Big(\Phi_{i_1}^- \Big[\Phi_{i_2}^-, [\Phi_j^+, \Phi_j^-] \Big] \Phi_{i_3}^- \cdots \Phi_{i_n}^- \Big) + \ldots \\ & + \operatorname{Tr}\Big(\Phi_{i_1}^- \cdots \Phi_{i_{n-1}}^- \Big[\Phi_{i_n}^-, [\Phi_j^+, \Phi_j^-] \Big] \Big) \\ =\ & - \operatorname{Tr}\Big([\Phi_j^+, \Phi_j^-] \Phi_{i_1}^- \Phi_{i_2}^- \cdots \Phi_{i_n}^- \Big) + \operatorname{Tr}\Big(\Phi_{i_1}^- \Phi_{i_2}^- \cdots \Phi_{i_n}^- [\Phi_j^+, \Phi_j^-] \Big) = 0. \end{aligned} \tag{85}$$

due to a telescoping sum and cyclicity of the trace. Furthermore, terms resulting from contracting the Φ_i^+ with one of the Φ_k^- inside the same vertex cancel against the remaining quadratic terms in (84). Thus the combination (84) does not give any contribution to two-point correlators of scalar fields.

In summary the total one-loop contribution to *any* scalar 2-point function in $\mathcal{N} = 4$ Super Yang-Mills may be obtained through the insertion of a simple effective matrix model vertex into a Gaussian matrix model correlator

$$\langle\, \mathcal{O}_1(x)\, \bar{\mathcal{O}}_2(0)\, \rangle_{1-\text{loop}} = \frac{g_{\rm YM}^2\, L}{16\, \pi^2} \Big\langle \mathcal{O}_1^+ \, (: V_F : + : V_K :)\, \bar{\mathcal{O}}_2^- \Big\rangle_{\rm M\,M} . \tag{86}$$

An immediate consequence of this result is the quoted non-renormalization of the BPS operators (41): Due to their symmetric/traceless structure they do not couple to V_F or V_K and therefore the one-loop contribution to *any* two-point function involving a chiral primary operator $\mathcal{O}_{\rm CPO}$ (41) vanishes. Recently this construction has been generalized to the case of an arbitrary (scalar, vector or spinor) two-point function in $\mathcal{N} = 4$ Super Yang-Mills in [33].

Let us now apply this insight to the computation of our two-impurity BMN operator $\mathcal{O}_p^J := \operatorname{Tr}(\phi_1 Z^p \phi_2 Z^{J-p})$ where we were interested in

$$\langle\, \mathcal{O}_p^J(x)\, \bar{\mathcal{O}}_q^J(0)\, \rangle = \left(\frac{g_{\rm YM}^2}{8\pi^2 |x|^2} \right)^{J+2} \Big(S_{pq} + T_{pq} \, \log |x\, \Lambda|^{-2} \Big) \tag{87}$$

with $S_{pq} = \langle\, \mathcal{O}_p^J \bar{\mathcal{O}}_q^J \,\rangle_{\rm M\,M}$ given in (70) and

$$T_{pq} = -\tfrac{g_{\rm YM}^2}{8\pi^2} \Big\langle \operatorname{Tr}(\phi_1^+ Z^p \phi_2^+ Z^{J-p}) \ : \operatorname{Tr}[\bar{Z}, \phi_i^-]\, [Z, \phi_i^+] : \ \operatorname{Tr}(\phi_1^- \bar{Z}^p \phi_2^- \bar{Z}^{J-p}) \Big\rangle_{\rm M\,M}$$

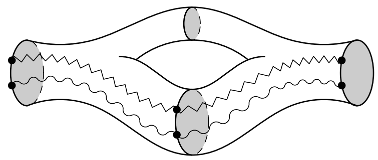


Figure 3: Cutting the torus of a correlator between two single trace operators yields double trace operators. The wiggly lines represent the impurity insertions.

having inserted the relevant terms of V_F in the $(Z, \bar{Z}, \phi_{i=1,2,3,4})$ basis. Note that the trace piece V_K of the effective vertex does not contribute here. Evaluating this correlator is a straightforward yet tedious problem and we shall see soon that this is actually not necessary for the determination of Δ.

Before proceeding, there is an additional complication we have to face, known as operator mixing [34]. In principle one could diagonalize the two-point functions of the $\mathcal{O}_p^J(x)$ of (87), this, however, is *not* correct. The reason is easy to understand pictorially: Considering the torus correction to a two-point function, we see from figure 3 that double-trace operators appear in intermediate channels. And indeed the overlap between such double-trace operators and the single-trace BMN operators is of $\mathcal{O}(g_2)$. It therefore affects the $\mathcal{O}({g_2}^2)$ anomalous dimension upon diagonalization [27, 35]. We conclude that we have to consider the enlarged set of multi-trace BMN-operators of the form

$$\mathcal{O}_p^{J_0,J_1,\ldots,J_k} = \mathrm{Tr}\,[\,\phi_1\, Z^p\, \phi_2\, Z^{J_0-p}\,]\ \mathrm{Tr}\ Z^{J_1} \ldots \mathrm{Tr}\ Z^{J_k} \tag{88}$$

with $J_0 + J_1 + \ldots + J_k = J$ in our computation of two-point functions. The reader might wonder why one does not need to include a multi-trace operator with a single impurity insertion in the first and the second trace of the form

$$\tilde{\mathcal{O}}^{J_0,J_1,\ldots,J_k} = \mathrm{Tr}\,[\,\phi_1\, Z^{J_0}\,]\ \mathrm{Tr}\,[\,\phi_2 Z^{J_1}\,]\ \mathrm{Tr}\, Z^{J_2} \ldots \mathrm{Tr}\ Z^{J_k}\ . \tag{89}$$

We shall see in the following that there is not mixing with these types of states[11].

5.1 BMN-gauge theory as a quantum mechanical system

We will now review the paper [36] in which an efficient and simple method for the computation of anomalous contributions to the scaling dimension Δ was introduced. For the general mixing situation at hand we need to study the two-point functions of multi-trace operators up to (say) one-quantum loop

$$\langle\, \mathcal{O}_\alpha(x)\, \bar{\mathcal{O}}_\beta(0)\, \rangle = \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2|x|^2}\right)^{J+2}\ \Big(\, S_{\alpha\beta} + T_{\alpha\beta}\, \log|x\,\Lambda|^{-2}\,\Big) \tag{90}$$

with α, β being multi-indices running over the set of multi-trace operators introduced in (88). $S_{\alpha\beta}$ denotes the tree-level and $T_{\alpha\beta}$ the one-loop matrix model correlators

$$S_{\alpha\beta} = \langle\, \mathcal{O}_\alpha\, \bar{\mathcal{O}}_\beta\, \rangle_{\mathrm{MM}} \qquad \text{and} \qquad T_{\alpha\beta} = \langle\, \mathcal{O}_\alpha\, H\, \bar{\mathcal{O}}_\beta\, \rangle_{\mathrm{MM}} \tag{91}$$

[11]The reason for this is that $\tilde{\mathcal{O}}^{J_0,J_1,\ldots,J_k}$ is a protected operator, which does not mix with the generically unprotected operators $\mathcal{O}_p^{J_0,J_1,\ldots,J_k}$ of (88).

with $H := -\frac{g_{\rm YM}^2}{8\pi^2} :{\rm Tr}[Z,\Phi_i^+][\bar{Z},\Phi_i^-]:$. We are seeking a basis transformation to a new set of operators $\mathcal{O}'_A$

$$\mathcal{O}_\alpha = V_{\alpha A}\,\mathcal{O}'_A \tag{92}$$

which possess definite scaling dimensions Δ_A read off from the *diagonal* two-point functions

$$\langle\,\mathcal{O}'_A(x)\,\bar{\mathcal{O}}'_B(0)\,\rangle = \frac{\delta_{AB}\,c_A}{|x|^{2(J+2+\Delta_A)}}\,, \tag{93}$$

where c_A is a normalization constant. Put differently the operators $\mathcal{O}'_A$ are eigenstates of the anomalous piece of the dilatation operator $\mathcal{D}$ with eigenvalues Δ_A, i.e. $\mathcal{D}\circ\mathcal{O}'_A = \Delta_A\,\mathcal{O}'_A$. Re-expressed in the original basis we have

$$\mathcal{D}\circ\mathcal{O}_\alpha = (V_{\alpha A}\,\Delta_A\,V^{-1}_{A\beta})\,\mathcal{O}_\beta \tag{94}$$

giving us the matrix elements $\mathcal{D}_{\alpha\beta}$ of the anomalous piece of the dilatation operator in the original, non-diagonal basis. Hence

$$\begin{aligned}
\langle\,\mathcal{O}_\alpha(x)\,\bar{\mathcal{O}}_\beta(0)\,\rangle &= V_{\alpha A}\,V^*_{\beta B}\,\langle\,\mathcal{O}'_A(x)\,\bar{\mathcal{O}}'_B(0)\,\rangle = V_{\alpha A}\,V^*_{\beta B}\,\frac{\delta_{AB}\,c_A}{|x|^{2(J+2+\Delta_A)}}\\
&= V_{\alpha A}\,V^*_{\beta B}\,\frac{\delta_{AB}\,c_A}{|x|^{2(J+2)}}\Big(1+\Delta_A\,\log|x\Lambda|^{-2}\Big)\\
&= \frac{1}{|x|^{2(J+2)}}\Big[\underbrace{(V\,C\,V^\dagger)_{\alpha\beta}}_{S_{\alpha\beta}}+\underbrace{(V\,C\,\Delta\,V^\dagger)_{\alpha\beta}}_{T_{\alpha\beta}}\,\log|x\Lambda|^{-2}\Big]
\end{aligned} \tag{95}$$

where we introduced the diagonal matrices $C_{AB} = \delta_{AB}\,c_A$ and $\Delta_{AB} = \delta_{AB}\,\Delta_A$. We conclude that the dilatation matrix $\mathcal{D}_{\alpha\beta}$ of (94) may be expressed as [37]

$$T_{\alpha\gamma}\,(S^{-1})_{\gamma\beta} = (V\,C\,\Delta\,V^\dagger\,V^{\dagger\,-1}\,C^{-1}\,V^{-1})_{\alpha\beta} = (V\,\Delta\,V^{-1})_{\alpha\beta} = \mathcal{D}_{\alpha\beta}\,, \tag{96}$$

the diagonal matrices Δ and C commute. Now consider the action of the effective vertex (or Hamiltonian) $H = -\frac{g_{\rm YM}^2}{8\pi^2} :{\rm Tr}[Z,\Phi_i^+][\bar{Z},\Phi_i^-]:$ on the left-hand-side operator (or state) $\mathcal{O}_\alpha(Z,\Phi_i^+)$ by Wick contraction

$$H\circ\mathcal{O}_\alpha = H_{\alpha\beta}\,\mathcal{O}_\beta \qquad \text{with} \qquad H_{\alpha\beta} : c\text{-numbers}\,. \tag{97}$$

This relation is to be understood as follows. Contract the two minus-indexed fields $\bar{Z}$ and Φ_i^- of H with the plus-indexed fields of the operator $\mathcal{O}_\alpha(Z,\Phi_i^+)$ according to the rules (69). The result will be a linear combination of operators of the type $\mathcal{O}_\alpha(Z,\Phi_i^+)$ again with coefficients $H_{\alpha\beta}$. In the correlation function one then simply has

$$T_{\alpha\beta} = \langle\,H\circ\mathcal{O}_\alpha\,\bar{\mathcal{O}}_\beta\,\rangle_{\rm MM} = H_{\alpha\gamma}\,\langle\,\mathcal{O}_\gamma\,\bar{\mathcal{O}}_\beta\,\rangle_{\rm MM} = H_{\alpha\gamma}\,S_{\gamma\beta} \tag{98}$$

and therefore the one-loop anomalous piece of the dilatation operator in the non-diagonal basis is simply given by the matrix elements $H_{\alpha\beta}$ upon using (96):

$$D_{\alpha\beta} = (J+2)\,\delta_{\alpha\beta} + H_{\alpha\beta} \tag{99}$$

Hence, the knowledge of tree-level mixing matrix $S_{\alpha\beta}$, whose involved structure as an infinite expansion in g_2 we already encountered in (70), is *completely unessential* for the determination of the anomalous dimensions Δ_A! These simply correspond to the eigenvalues of the matrix $H_{\alpha\beta}$. Let us note that while $T_{\alpha\beta}$ and $S_{\alpha\beta}$ are Hermitian by definition, $H_{\alpha\beta}$ is *not*, but rather

$$H^\dagger = S^{-1}\, H\, S \tag{100}$$

easily deduced from (98).

As we will discuss in the following section the dilatation operator with matrix elements $(J+2)\delta_{\alpha\beta} + H_{\alpha\beta}$ finds its natural dual in the string field theory Hamiltonian $\widehat{H}_{\rm string}$ of the full interacting plane-wave string theory. One is lead to propose the *operator* correspondence [38]

$$\frac{\widehat{H}_{\rm string}}{\mu} \mathrel{\hat{=}} 2\,\delta_{\alpha\beta} + H_{\alpha\beta} \tag{101}$$

where the left-hand side acts in the interacting plane-wave string theory and the right hand side acts in BMN gauge theory.

Let us now compute $H_{\alpha\beta}$. Acting on H with the single-trace operator $\mathcal{O}_p^J$ yields again single-trace operator terms of the same type, but also double-trace operators:

$$\begin{aligned} H\circ\mathcal{O}_p^J \;&=\; \tfrac{g_{\rm YM}^2}{4\pi^2}\Big[N(2\,\mathcal{O}_p^J - \mathcal{O}_{p-1}^J - \mathcal{O}_{p+1}^J) \\ &\quad + \sum_{l=1}^{p-1}\{\mathcal{O}_{p-l}^{J-l,l} - \mathcal{O}_{p-l-1}^{J-l,l}\} + \sum_{l=1}^{J-p-1}\{\mathcal{O}_{p}^{J-l,l} - \mathcal{O}_{p+1}^{J-l,l}\}\Big] \end{aligned} \tag{102}$$

This shows that the action of H on single-trace operators does not close. Acting with a double-trace operator leads to double, triple and single trace operators

$$\begin{aligned} H\circ\mathcal{O}_p^{J_0,J_1} \;&=\; \tfrac{g_{\rm YM}^2}{4\pi^2}\Big[N(2\,\mathcal{O}_p^{J_0,J_1} - \mathcal{O}_{p-1}^{J_0,J_1} - \mathcal{O}_{p+1}^{J_0,J_1}) \\ &\quad + \sum_{l=1}^{p-1}\{\mathcal{O}_{p-l}^{J_0-l,J_1,l} - \mathcal{O}_{p-l-1}^{J_0-l,J_1,l}\} + \sum_{l=1}^{J_0-p-1}\{\mathcal{O}_{p}^{J_0-l,J_1,l} - \mathcal{O}_{p+1}^{J_0-l,J_1,l}\} \\ &\quad + J_1\,(\mathcal{O}_{J_1+p}^J - \mathcal{O}_{J_1+p-1}^J + \mathcal{O}_p^J - \mathcal{O}_{p+1}^J)\Big] \end{aligned} \tag{103}$$

Here we have neglected boundary terms which are irrelevant in the BMN limit. Hence the Hamiltonian decomposes as $H = H_0 + H_+ + H_-$ where H_0 maps n-trace operators into n-trace operators and H_+ raises and H_- lowers the number of traces by one. Clearly the $H_\pm$ pieces correspond to string splitting and joining interactions in a dual string model.

One can directly rephrase (102) and (103) in the BMN limit, amounting to a continuum limit of the discretized string picture. For this upon taking $J\to\infty$ one introduces the continuum variables

$$x := \frac{p}{J}\quad\text{with } x\in[0,r_0];\quad r_i := \frac{J_i}{J}\quad\text{with } r_i\in[0,1];\quad\text{and}\quad \sum_{l=0}^{k} r_l = 1\,. \tag{104}$$

Then the discrete multi-trace operators are replaced by continuum states

$$\begin{aligned} \mathcal{O}_p^J \;&\to\; |x\rangle \\ \mathcal{O}_p^{J_0,J_1} \;&\to\; |x;r_1\rangle \\ \mathcal{O}_p^{J_0,J_1,\dots,J_k} \;&\to\; |x;r_1,\dots,r_k\rangle \end{aligned} \tag{105}$$

where x may be interpreted as the coordinate of a particle moving on a circle of circumference r_0, as one identifies $|x; r_1, \ldots, r_k\rangle \sim |x + r_0; r_1, \ldots, r_k\rangle$. The discrete action of $H \circ \mathcal{O}_p^J$ may now be rephrased as the action of a quantum mechanical Hamiltonian on $|x; r_1, \ldots, r_k\rangle$:

$$\hat{H}\,|x\rangle \;=\; \frac{\lambda'}{4\pi^2}\Big[-\partial_x^2|x\rangle + g_2\int_0^x dr\,\partial_x|x-r;r\rangle - g_2\int_0^{1-x} dr\,\partial_x|x;r\rangle\Big] \qquad (106)$$

from (102) with the genus counting parameter $g_2 = J^2/N$ appearing and

$$\begin{aligned}\hat{H}\,|x;r_1\rangle \;=\; \frac{\lambda'}{4\pi^2}\Big[&-\partial_x^2|x;r_1\rangle + g_2\,r_1(\partial_x|x+r_1\rangle - \partial_x|x\rangle)\\ &+g_2\int_0^x dr_2\,\partial_x|x-r_2;r_1,r_2\rangle - g_2\int_0^{r_0-x} dr_2\,\partial_x|x;r_1,r_2\rangle\Big]\end{aligned} \qquad (107)$$

from (103). The eigenvalues of $\hat{H}$ now correspond to the anomalous dimensions of the BMN operators. Let us stress the remarkable fact that the interaction terms of $\hat{H}$ terminate at order g_2. If we were to succeed in diagonalizing $\hat{H}$ exactly, we would have found an all genus prediction for interacting plane-wave string! Unfortunately one does not (yet) know how to do this and we have to revert to perturbation theory in g_2.

The planar ($g_2 = 0$) sector is easily diagonalized. We have $\hat{H}_0 = -\frac{\lambda'}{4\pi^2}\partial_x^2$ whose single-trace eigenstate reads (with n integer due to periodicity)

$$|n\rangle = \int_0^1 dx\,e^{2\pi i\,n\,x}\,|x\rangle \qquad \text{with} \qquad \hat{H}_0\,|n\rangle = \lambda'\,n^2\,|n\rangle\,. \qquad (108)$$

This result yields the promised one-loop planar anomalous dimension of the two impurity BMN operator $\Delta_1 = \lambda'\,n^2$ as advertised in (55). It also shows that the correct discrete BMN operator indeed is

$$\alpha^i_{-n}\,\tilde{\alpha}^j_{-n}\,|0,p^+\rangle \quad \hat{=} \quad \mathcal{O}_n = \sum_{l=0}^{J} e^{2\pi i\,n\,l/J}\,\mathcal{O}_p^J \qquad (109)$$

as claimed in (54). The generalization of (108) to multi-trace eigenstates is obvious

$$|n;r_1,\ldots,r_k\rangle = \tfrac{1}{\sqrt{r_0}}\int_0^{r_0} dx\,e^{2\pi i\,n\,x/r_0}\,|x;r_1,\ldots,r_k\rangle$$

with

$$\hat{H}|n;r_1,\ldots,r_k\rangle = \lambda'\,\frac{n^2}{r_0^2}\,|n;r_1,\ldots,r_k\rangle\,. \qquad (110)$$

Note that whereas the spectrum of single-trace states is discrete, the spectrum of multi-trace states is continuous due to $r_0 \in [0,1]$.

Higher genus corrections to the anomalous scaling dimension Δ may now be computed from simple quantum mechanical perturbation theory. The genus one correction to (108) reads

$$E^{(1)}_{|n\rangle} = \langle n|\hat{H}_{\text{int}}\,\frac{1}{E^{(0)}_{|n\rangle} - \hat{H}_0}\,\hat{H}_{\text{int}}|n\rangle \qquad \text{where} \qquad E^{(0)}_{|n\rangle} = \lambda'\,n^2 \qquad (111)$$

and where $\hat{H}_{\rm int}$ denote the g_2 dependent terms in (106) and (107). In order to perform this computation one needs the matrix element of $H_{\rm int}$ in the momentum basis,

$$\langle m,r|\hat{H}_{\rm int}|n\rangle = -\frac{1}{(1-r)^{3/2}}\,\frac{4m}{n-\frac{m}{1-r}}\,\sin^2(\pi n r) \tag{112a}$$

$$\langle n|\hat{H}_{\rm int}|m,r\rangle = \frac{r}{(1-r)^{1/2}}\,\frac{4n}{n-\frac{m}{1-r}}\,\sin^2(\pi n r)\,. \tag{112b}$$

Note the non-hermiticity of $\hat{H}_{\rm int}$ as discussed in (100). It is now a one line computation to compute the genus one energy shift to be

$$E^{(1)}_{|n\rangle} = \int_0^1 dr \sum_{m=-\infty}^{\infty} \langle n|\hat{H}_{\rm int}|m;r\rangle\,\frac{1}{4\pi^2(n^2-\frac{m^2}{(1-r)^2})}\,\langle m;r|\hat{H}_{\rm int}|n\rangle = \frac{1}{12}+\frac{35}{32\,\pi^2 n^2}\,. \tag{113}$$

Summarizing, the gauge theory prediction for the higher genus corrections to the plane-wave string spectrum read

$$\begin{aligned}\frac{E_{|n\rangle}}{\mu} &= 2+\lambda'\Big[n^2+{g_2}^2\frac{1}{4\pi^2}\,(\frac{1}{12}+\frac{35}{32\pi^2n^2})+\\ &{g_2}^4\tfrac{1}{4\pi^2}\,(-\tfrac{11}{46080}\,\tfrac{1}{\pi^2n^2}+\Big(\tfrac{521}{12288}-\tfrac{\zeta(3)}{128}\Big)\tfrac{1}{\pi^4n^4}+\Big(-\tfrac{5715}{16384}-\tfrac{45\,\zeta(3)}{512}+\tfrac{15\,\zeta(5)}{128}\Big)\tfrac{1}{\pi^6n^6})\Big]\end{aligned} \tag{114}$$

where we have also spelled out the more involved result of the genus two computation [36]. There is an important subtlety concerning the use of *non*-degenerate perturbation theory in the diagonalization of H. This is a priori not justified as single-trace states $|n\rangle$ of (108) are degenerate in energy with multi-trace states $|m;s_1,\ldots,s_k\rangle$ of (110) for $n\cdot s_0=\pm m$ [12]. It turns out that this degeneracy does not lead to problems in the genus one computation (113), as the overlaps of degenerate single with double-trace states of (112) vanish[13]. However, degeneracy of single with triple-trace operators leads to a breakdown of non-degenerate perturbation theory from genus two on. This may be interpreted in the dual string theory as an instability of excited single string states to decay into the continuum of degenerate triple-string states. See [39] for a discussion of this, a confirmation of the genus two result (114) and the computation of the corresponding decay width.

In the above we have only considered the insertion of two scalar impurities of distinct type. The case of two general scalar impurities [27] (at planar and toroidal level), mixed vector-scalar impurities [40] and two vector impurities [41] has been analyzed as well, leading to identical planar anomalous dimension $\Delta_0=\lambda'\cdot n^2$, in agreement with the free string spectrum. The necessity of this observed degeneracy of general two impurity states (to all orders in g_2) was proved in [28] by employing the underlying superconformal symmetry.

Recently the structure of the dilatation operator at the two loop level, i.e. at order λ'^2 was established [26]. The result at two-loops led the authors to conjecture that the

[12] Actually the $n=1$ state is non-degenerate as $s_0=1$ would be required, from which $s_i=0$ follows, turning the multi-trace state into a single-trace one. But $n=2$ is degenerate with $m=\pm1$, $s_0=1/2$, $n=3$ is degenerate with $m=\pm2$, $s_0=2/3$ and $m=\pm1$, $s_0=1/3$ and so forth.

[13] I.e. for degeneracy one has $n\cdot(1-r)=m$ leading to poles in (112). This however implies $n\cdot r$ being integer letting the numerator vanish to yield a vanishing overlap.

Hamiltonian of BMN gauge theory is given to all loop orders by

$$H_{\text{full}} = 2\sqrt{1 + \lambda' H} \tag{115}$$

with $H = H_0 + g_2 H_+ + g_2 H_-$ being the one-loop Hamiltonian discussed above. This result is intriguingly simple and consistent with the planar, all-loop result of Santambrogio and Zanon [30] and agrees with the free string spectrum. It is firm, however, only at order λ'^2. Let us also note that the diagonalization of H_{full} at higher orders in λ' is thus reduced to the one loop diagonalization of H, which we have performed perturbatively in (114). Knowing the spectrum of the one-loop Hamiltonian H would yield the exact interacting string spectrum to all orders in g_2 and λ'.

6 Interacting plane-wave superstrings

After having established the gauge theory predictions for higher genus corrections to the plane-wave string spectrum let us now study how these corrections can be obtained in a string theory computation. For this one needs to study string interactions arising from higher genus worldsheets. It turns out that the standard vertex operator methods for the computation of string scattering amplitudes are not easily generalized to the plane-wave background. Instead the methods of light-cone string field theory, developed in 1973-75 [42], have been successfully reformulated for the plane-wave superstring and used to compute the genus one corrections to the spectrum. This subject is highly technical in nature and we do not have the space in these lecture notes to develop it in detail – we have to refer the reader to the original literature as we proceed, also see [43] for a recent review. Instead our emphasis in this section will be to display the key points and structural issues of this technically involved subject.

To begin with we shall rewrite the Hamiltonian (29) of the free, light-cone superstring in the plane-wave background in the following unified notation [12, 14]

$$H_2 = p^- = \tfrac{1}{\alpha' p^+} \sum_{n\in\mathbb{Z}} \omega_n \, (a_n^{\dagger I}\, a_n^I + b_n^\dagger\, b_n) \qquad \text{where} \qquad \omega_n := \sqrt{n^2 + (\alpha' p^+ \mu)^2} \tag{116}$$

with $[a_n^I, a_m^{\dagger J}] = \delta^{IJ}\,\delta_{n,m}$ and $\{b_n^a, b_m^{\dagger b}\} = \delta^{ab}\,\delta_{n,m}$ where $a, b = 1, \ldots, 8$. The a_n^I oscillators are related to the α_n^I and $\tilde{\alpha}_n^I$ oscillators of section three via (suppressing the space index I)

$$a_n^\dagger = \begin{cases} \alpha_{-n} & n > 0 \\ \tilde{\alpha}_{-|n|} & n < 0 \\ \alpha_0^\dagger & n = 0 \end{cases} \qquad\qquad a_n = \begin{cases} \alpha_n & n > 0 \\ \tilde{\alpha}_{|n|} & n < 0 \\ \alpha_0 & n = 0 \end{cases} \tag{117}$$

In a similar fashion the fermionic modes $\theta_n^{(1,2)}$ of (29) combine into the complex fermionic oscillators $b_m^\dagger$ and b_m. The single string Hilbert space is then built on the vacuum-state $|0\rangle$ subject to

$$a_n^I |0\rangle = 0 \qquad b_n|0\rangle = 0 \qquad n \in \mathbb{Z} \tag{118}$$

and physical states have to satisfy the level-matching condition

$$\sum_{n\in\mathbb{Z}} n(a_n^{I\dagger}\, a_n^I + b_n^\dagger\, b_n)\, |\text{phys}\rangle = 0\,. \tag{119}$$

A central role in the construction of the interacting string field theory is the structure of the plane-wave superalgebra, which may be obtained by a contraction of the $AdS_5 \times S^5$ symmetry algebra as a consequence of the Penrose limit. The isometries of the plane-wave metric (12) are generated by the Hamiltonian $H = P^-$ and momentum operators P^I, P^+ as well as the angular momentum operators J^{+I}, J^{ij} and $J^{i'j'}$, where we denote $I = (i, i')$ with $i = 1, 2, 3, 4$ and $i' = 5, 6, 7, 8$. Let us stress that there are no isometry generators J^{-+} and J^{-I} present in the algebra, a manifestation of the broken Lorentz symmetry in the plane-wave background. Additionally the 32 supersymmetries are generated by the supercharges Q^+ and Q^-. Explicitly these read [7] (at $\tau = 0$)

$$Q^+ = \int d\sigma\, \sqrt{2}\, \lambda \qquad Q^- = \int d\sigma\, [2\pi\alpha' \not{p}\, \lambda - i\partial_\sigma \not{x}\, \lambda - i\mu \not{x}\, \Gamma_{1234}\, \lambda] \tag{120}$$

in the free ($g_s = 0$) theory and where λ is the conjugate momentum to $\theta := \theta^1 + i\theta^2$. Similarly the bosonic generators P^I and J^{I+} are given by (at $\tau = 0$)

$$P^I = \int d\sigma p^I \qquad J^{+I} = \tfrac{1}{2\pi\alpha'} \int d\sigma x^I\,. \tag{121}$$

The relevant (anti)-commutators of the plane-wave supersymmetry algebra are [5, 7]

$$[H, P^I] = i\, \mu^2\, J^{+I} \tag{122a}$$
$$[H, Q^+] = \mu\, \Gamma_{1234}\, Q^+ \tag{122b}$$
$$[P^I, Q^-] = \mu\, \Gamma_{1234}\, \Gamma^I\, Q^- \tag{122c}$$
$$\{Q^-_a, \bar{Q}^-_b\} = 2\delta_{ab}\, H - i\mu\, (\Gamma_{ij}\, \Gamma_{1234})_{ab}\, J^{ij} + i\mu\, (\Gamma_{i'j'}\, \Gamma_{1234})_{ab}\, J^{i'j'}\,. \tag{122d}$$

plus the remaining, unmodified super-Poincare algebra (anti)-commutators, with all commutators involving J^{+-} and J^{I-} omitted. The free ($g_s = 0$) string generators given in (116), (120) and (121) obey these (anti)-commutation relations.

Conventionally, string interactions are introduced in light-cone gauge quantization via vertex operators [19]. These are constructed by demanding covariant transformation properties under supersymmetry transformations. In flat space they then take a universal structure of the form

$$\hat{V}(k^+, k^I) = [\,\text{polarization dependent terms}\,] \cdot e^{i\,[k^+\, \hat{x}^-(\sigma) + k^I\, \hat{x}^I(\sigma)]} \tag{123}$$

corresponding to a string excitation with null momentum k^+ and transverse momentum k^I. To compute an N-particle scattering-amplitude one evaluates the light-cone string theory path-integral with N vertex operator insertions. In the light-cone gauge $\hat{x}^-(\sigma)$ is given in terms of a quadratic function of the transverse degrees of freedom $\hat{x}^I(\sigma)$, the analogue of (25), introducing quadratic terms into the exponential of (123). This leads to technical problems in the path integral evaluation of the correlators. A standard trick [19] to circumvent this problem for not too many external particles lies in going to a Lorentz frame where $k^+ = 0$. Then the amplitude may be calculated easily and the obtained result is simply covariantized by making the replacement $k^I \to k^\mu$ [19]. In the plane-wave background two complications for such a procedure arise: First of all the transverse momentum k^I is not a valid quantum number any longer which could label asymptotic

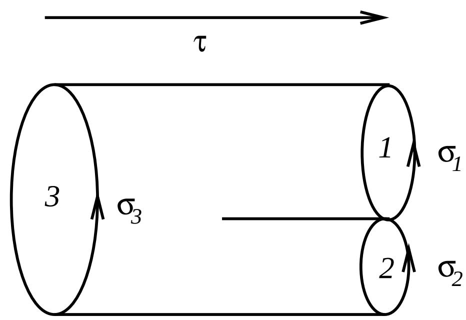


Figure 4: The worldsheet of the three-string interaction vertex $\widehat{H}_3$.

states: P^I does not commute with the Hamiltonian (122a). Moreover the absence of the J^{i-} generators forbids one to move to a Lorentz-frame with vanishing P^+ momentum. Hence this approach seems to fail.

As an alternative approach light-cone string field theory suggests itself [42][14]. Here one works in a multi-string Hilbert space built upon the multi-string vacuum:

$$||0\rangle = |0\rangle_{(1)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(3)} \otimes \dots \tag{124}$$

upon which one acts with the creation operators $a^\dagger_{n(r)}$ and $b^\dagger_{n(r)}$ of the individual strings, where r labels the corresponding string-number. The string field theory Hamiltonian acting in the multi-string Hilbert space is then defined as an infinite series in the string coupling constant g_s

$$\widehat{H} := \sum_r \widehat{H}_{2(r)} + g_s\,\widehat{H}_3 + g_s^2\,\widehat{H}_4 + \dots \tag{125}$$

The leading term $\widehat{H}_{2(r)}$ is the free string Hamiltonian (116) of the r'th string. Summing over r yields the free piece of the string Hamiltonian which conserves the number of strings. The first interaction term $\widehat{H}_3$ corresponds to the three-string interaction vertex describing a string splitting and joining process as depicted in figure 4. A general n-string state is mapped by $\widehat{H}_3$ to the sum of an $(n+1)$-string and an $(n-1)$-string state.

What determines the structure of the interacting pieces of the Hamiltonian $\widehat{H}_{n>2}$? In the simplest case of the bosonic string in a flat background the structure of $\widehat{H}_3$ is determined by demanding worldsheet continuity: In the interaction process the two worldsheets of string 1 and 2 are smoothly glued together to form string 3. This is formally achieved by the delta-functional expression

$$\Delta[(X_{(1)}(\sigma_1) + X_{(2)}(\sigma_2) - X_{(3)}(\sigma_3)]\,. \tag{126}$$

In practice such a delta-functional is represented by an infinite product of individual delta-functions for each Fourier-mode of the three $X_{(r)}(\sigma_r)$ involved. In light-cone string theory the total string length given by $\alpha'\,p^+$ is conserved in an interaction, which is pictorially obvious from figure 4 and an algebraic consequence of the commutator relation $[H_3, P^+] =$

[14]This subject is also developed in chapter 11 of [19].

0. This requirements yields the additional delta-function contribution $\delta(p^+_{(1)} + p^+_{(2)} - p^+_{(3)})$ to $\widehat{H}_3$. So

$$\widehat{H}_3^{\text{bosonic}} = \delta(p^+_{(1)} + p^+_{(2)} - p^+_{(3)})\, \Delta[(X_{(1)}(\sigma_1) + X_{(2)}(\sigma_2) - X_{(3)}(\sigma_3)] \,. \tag{127}$$

It is convenient to represent the operator $\widehat{H}_3$ as a state in a 3-string Hilbert space via

$$\langle\, \phi(3)\,|\, \widehat{H}_3 \,|\, \phi(1)\rangle \,|\, \phi(2)\,\rangle =: \langle\, \phi(3)\,|\, \langle\, \phi(2)\,|\, \langle\, \phi(1)\,|\, H_3\,\rangle \,. \tag{128}$$

It can be shown that the bosonic cubic interaction vertex is represented by the coherent three-string state [42]

$$|H_3^{\text{bosonic}}\rangle \sim \exp\{\tfrac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n\in\mathbb{Z}} a^\dagger_{m(s)}\, \bar{N}^{rs}_{mn}\, a^\dagger_{n(r)}\,\}\, |0\rangle_{123} \tag{129}$$

where $\bar{N}^{rs}_{mn}$ are known as the Neumann-matrices which are c-number valued entities following from (127). Moreover we have defined $|0\rangle_{123} = |0\rangle_{(1)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(3)}$.

We are interested in the generalization of this to the light-cone superstring in a plane-wave background. Light-cone superstring field theory in a flat background was developed in the eighties by Green, Schwarz and Brink [44]. This has been generalized in a number of papers to the plane-wave background [12–14,45–50], which we will review in the following. The construction principle in the supersymmetric case is the requirement that the superalgebra of (122) should be realized in the full interacting string field theory, i.e. in the presence of g_s corrections[15]. One hopes that this uniquely determines the form of the higher g_s corrections to all generators of the plane-wave superalgebra. In particular these generators may be divided into two sets reflecting the occurrence of g_s corrections:

- **Kinematical generators:** P^+, P^I, J^{+I}, J^{ij}, $J^{i'j'}$, Q^+
  These do not contain derivatives in $\partial_{x^+} = \partial_\tau$, they act at fixed light-cone time. These operators are not corrected by interactions and preserve the string-number.
- **Dynamical generators:** H, Q^-
  These operators are corrected by g_s terms and are the objects we are after. They create and destroy strings.

The requirement that the supersymmetry algebra (122) is preserved under interactions now leads to two sets of constraints: *Kinematical* constraints arise from the (anti)-commutation relations of dynamical with kinematical operators, $[D, K] = K$, and *dynamical* constraints arise from the (anti)-commutation relations of dynamical operators alone, $[D, D] = D + K$.

Let us first study the kinematical constraints. In the bosonic sector the commutation relation of the full Hamiltonian with the kinematical generators P^I reads $[H, P^I] = i\mu^2\, J^{+I}$. As only H is corrected by interactions one immediately concludes that the interaction pieces $H_{n>2}$ *commute* with P^I and hence conserve transverse momentum as in flat space

$$[H_3, P^I] = 0 \quad \Longrightarrow \quad \sum_{r=1}^{3} \widehat{P}_{(r)}(\sigma_r)\, |H_3\rangle = 0\,. \tag{130}$$

[15]The principle of worldsheet continuity in the bosonic case may also be viewed as a preservation of the Poincare algebra under g_s corrections.

Similarly $[H, J^{+i}] = -iP^I$ leads to

$$[H_3, J^{+I}] = 0 \quad \Longrightarrow \quad \Big(\widehat{X}_{(3)}(\sigma_3) - \widehat{X}_{(1)}(\sigma_1) - \widehat{X}_{(2)}(\sigma_2)\Big)\,|H_3\rangle = 0\,, \tag{131}$$

the worldsheet continuity condition. In the fermionic sector the commutator $[H, Q^+] = \mu\,\Gamma_{1234}\,Q^+$ yields the kinematical constraint

$$[H_3, Q^+] = 0 \quad \Longrightarrow \quad \sum_{r=1}^{3} \lambda_{(r)}(\sigma_r)\,|H_3\rangle = 0\,. \tag{132}$$

with $\lambda_{(r)}$ being the conjugate momentum to $\theta_{(r)}$. These kinematical constraints can be solved through the following state [12, 47]

$$|H_3\rangle \sim |E_a\rangle\,|E_b\rangle\,\delta(p^+_{(3)} - p^+_{(1)} - p^+_{(2)}) \;=: |V\rangle \tag{133}$$

where

$$|E_a\rangle = \exp\{\tfrac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n\in\mathbb{Z}} a^\dagger_{m(r)}\,\bar{N}^{rs}_{mn}(\mu)\,a^\dagger_{n(s)}\,\}\,|0\rangle_{123} \tag{134a}$$

$$|E_b\rangle = \exp\{\sum_{r,s=1}^{3}\sum_{m,n>0} b^\dagger_{-m(r)}\,Q^{rs}_{mn}(\mu)\,b^\dagger_{n(s)} - \sqrt{2}\,\Lambda\sum_{m>0} Q^r_m\,b^\dagger_{-m(r)}\,\}\,|E^0_b\rangle \tag{134b}$$

Here $\bar{N}^{rs}_{mn}(\mu)$ are the generalized Neumann matrices for the plane-wave background, which are explicitly known functions of μ [49]. Similarly Q^{rs}_{mn} and Q^r_m may be expressed in terms of $\bar{N}^{rs}_{mn}(\mu)$ [47]. Moreover Λ is a linear function in the fermion zero modes $\Lambda = \alpha'(p^+_{(1)}\,\lambda_{0(2)} - p^+_{(2)}\,\lambda_{0(1)})$ and $|E^0_b\rangle = \prod_{a=1}^{8}\Big[\sum_{r=1}^{3}\lambda^a_{0(r)}\Big]\,|0\rangle_{123}$ is the pure zero-mode part of the fermion vertex[16].

As we see the emerging structures become rather complicated, however, this is not the end of the story as we still need to implement the dynamical constraint to completely fix $|H_3\rangle$. For this the central anti-commutator is (122d) which one rewrites by defining the linear combinations of supercharges ($\eta = e^{i\pi/4}$)

$$\sqrt{2}\,\eta\,Q := Q^- + i\bar{Q}^- \qquad \sqrt{2}\,\bar{\eta}\,\tilde{Q} := Q^- - i\bar{Q}^- \tag{135}$$

to obtain from (122d)

$$\begin{aligned}\{Q_a, \tilde{Q}_b\} &= -i\mu(\Gamma_{ij}\,\Gamma_{1234})_{ab}\,J^{ij} + i\mu(\Gamma_{i'j'}\,\Gamma_{1234})_{ab}\,J^{i'j'}\\ \{Q_a, Q_b\} &= \{\tilde{Q}_a, \tilde{Q}_b\} = 2\delta_{ab}\,H\,.\end{aligned} \tag{136}$$

As the angular momentum generators are kinematical the following dynamical constraint

[16]Note that there is a subtlety in the fermionic zero mode structure of $|0\rangle_{123}$ related to its $\mathbb{Z}_2$ parity under the exchange of the two transverse $\mathbb{R}^4$ spaces. The choice of [45, 48, 50] differs from the one we employ in these lectures used in [12–14, 46, 47, 49]. Yet, in [50] evidence is presented that the two choices are physically equivalent.

equations follow from this

$$\sum_{r=1}^{3}\Big(\tilde{Q}_{2\,a(r)}\,|Q_{3\,b}\rangle+Q_{2\,b(r)}\,|\tilde{Q}_{3\,a}\rangle\Big)=0$$
$$\sum_{r=1}^{3}\Big(Q_{2\,a(r)}\,|Q_{3\,b}\rangle+Q_{2\,b(r)}\,|Q_{3\,a}\rangle\Big)=2\delta_{ab}\,|H_3\rangle$$
$$\sum_{r=1}^{3}\Big(\tilde{Q}_{2\,a(r)}\,|\tilde{Q}_{3\,b}\rangle+\tilde{Q}_{2\,b(r)}\,|\tilde{Q}_{3\,a}\rangle\Big)=2\delta_{ab}\,|H_3\rangle\;. \tag{137}$$

Here we have introduced the states $|Q_{3\,a}\rangle$ and $|\tilde{Q}_{3\,a}\rangle$ corresponding to the $\mathcal{O}(g_s)$ corrections of the dynamical supercharges (135) in analogy to (128). These constraints will be satisfied by augmenting the kinematical part of the vertex $|V\rangle$ of (133) with prefactors $h_3, q_3, \tilde{q}_3$ being polynomials in the oscillators $a^{\dagger\,I}_{m(r)}$ and $b^{\dagger}_{m(r)}$

$$|H_3\rangle=h_3\,|V\rangle\qquad|Q_3\rangle=q_3\,|V\rangle\qquad|\tilde{Q}_3\rangle=\tilde{q}_3\,|V\rangle\;. \tag{138}$$

These prefactors are constructed in such a fashion that they commute with the kinematical constraints in order to not spoil the achievements of $|V\rangle$ and at the same time enforce the dynamical constraints (137). It turns out that there are three functions K^I, $\tilde{K}^I$ and Y_a which have the property of commuting with the kinematical constraints. They are linear functions in the oscillators

$$K^I=\sum_{r=1}^{3}(\sum_{n\geq 0}F_{n(r)}\,a^{\dagger\,I}_{n(r)}+\sum_{n>0}F_{-n(r)}\,a^{\dagger\,I}_{-n(r)})$$
$$\tilde{K}^I=\sum_{r=1}^{3}(\sum_{n\geq 0}(F_{n(r)}\,a^{\dagger\,I}_{n(r)}-\sum_{n>0}F_{-n(r)}\,a^{\dagger\,I}_{-n(r)})$$
$$Y=\sum_{r=1}^{2}G_{0\,(r)}\,\lambda_{0\,(r)}+\sum_{r=1}^{3}\sum_{m>0}G_{m(r)}\,b^{\dagger}_{m(r)}\;. \tag{139}$$

The explicit form of the functions F_n and G_n entering in the above may be found in [43]. They obey relations of the form

$$-\sqrt{2}\eta\kappa\sum_{r=1}^{3}Q_{2\,(r)}\,|V\rangle=K^I\,\Gamma^I\,Y\,|V\rangle\qquad-\sqrt{2}\bar{\eta}\kappa\sum_{r=1}^{3}\tilde{Q}_{2\,(r)}\,|V\rangle=\tilde{K}^I\,\Gamma^I\,Y\,|V\rangle\,, \tag{140}$$

with $\kappa:=\alpha'^2p^+_{(1)}p^+_{(2)}p^+_{(3)}$. This in view of (137) makes them ideal building blocks for an ansatz for the prefactors h_3, q_3 and $\tilde{q}_3$. Utilizing these tools the final form of the dynamical supercharges and three-string vertex was derived in [13, 14]

$$|H_3\rangle=\Big(\tilde{K}^I\,K^J-\mu\,\kappa\,\delta^{IJ}\Big)\,v_{IJ}(Y)\,|V\rangle,$$
$$|Q_{3\,a}\rangle=\tilde{K}^I\,s^I_a(Y)\,|V\rangle,\qquad|\tilde{Q}_{3\,a}\rangle=K^I\,\tilde{s}^I_a(Y)\,|V\rangle\;, \tag{141}$$

where

$$
\begin{aligned}
v_{IJ} &= \delta_{IJ} - \tfrac{i}{2\kappa}(Y\Gamma_{IJ}Y) + \tfrac{1}{4!\,\kappa^2}(Y\Gamma_{IK}Y)(Y\Gamma_{JK}Y) \\
&\quad -\tfrac{1}{2\cdot 6!\,\kappa^3}(\Gamma_{IJ})_{ab}\,\epsilon^{ab}{}_{cdefgh}Y^c\ldots Y^h + \tfrac{1}{8!\,\kappa^4}\delta_{IJ}\,\epsilon_{abcdefgh}Y^a\ldots Y^h \\
s^I_a &= -i2\sqrt{2}\Big(\eta(Y\Gamma^I)_a - \tfrac{\bar\eta}{3!\,\kappa}(Y\Gamma_{IJ}Y)(Y\Gamma_J)_a - \tfrac{\eta}{6!\,\kappa^2}(\Gamma_{IJ})_{bc}(\Gamma_J)_{da}\epsilon^{bcd}{}_{efghi}Y^e\ldots Y^i \\
&\quad +\tfrac{\bar\eta}{7!\,\kappa^3}(\Gamma_I)_{ba}\,\epsilon^b{}_{cdefghi}Y^c\ldots Y^i\Big)
\end{aligned} \tag{142}
$$

and a similarly intricate expression for $\tilde{s}^I_a$. As a matter of fact the overall normalization of $|H_3\rangle$, $|Q_{3\,a}\rangle$ and $|\tilde{Q}_{3\,a}\rangle$ is not fixed by closing the plane-wave superalgebra. It is a priori an arbitrary function of μ, α' and the $p^+_{(r)}$'s. This ambiguity is due to the missing dynamical Poincaré generators J^{I-} in the algebra which fixes these normalizations in flat space.

6.1 Comparison to gauge theory at non-zero g_2

Having obtained these results one is in the position to explicitly evaluate matrix elements of the three string interaction vertex and compare the result with the Super Yang-Mills dilatation operator. Let us look at matrix elements involving the first stringy bosonic excitation $|n;p^+\rangle := a^{\dagger\,1}_n\,a^{\dagger\,2}_{-n}\,|0;p^+\rangle$ to find

$$
g_s\,\langle n;p^+|\,\widehat{H}_3\,|m;r\,p^+\rangle\,|0;(1-r)\,p^+\rangle = \mu\,g_s\,\lambda'\,(1-r)\,\frac{\sin^2(n\,\pi\,r)}{2\pi^2} + \mathcal{O}(1/\mu)\,, \tag{143}
$$

with $r\in[0,1]$ parameterizing the fraction of p^+ momentum and where we have displayed only the leading term in the $\mu\to\infty$ limit.

Note the complementarity of the string theory and gauge theory perturbation expansions: On the string theory side one has an expansion in g_s with a non-trivial (in principle known) dependence on μ for every term. Perturbative gauge theory on the other hand is organized in an expansion in $\lambda'\sim 1/\mu$ with a non-trivial (in principle known) dependence on g_2 for every term in the quantum-loop expansion.

If one now naively compares the string result (143) to the "corresponding" one-loop matrix element of the Super Yang-Mills dilatation operator of (112b)

$$
\lambda'\,g_s\,\langle n|\hat{H}_{\rm int}|m,r\rangle = \lambda'\,g_s\,\frac{r}{(1-r)^{1/2}}\,\frac{4n}{n-\frac{m}{1-r}}\,\sin^2(\pi n r) \tag{144}
$$

one sees that they do not agree! However, this should not be too surprising, as we are dealing with an operator correspondence here

$$
\frac{\widehat{H}_{\rm string}}{\mu} \mathrel{\hat{=}} \mathcal{D} - J\cdot\mathbf{1}\,, \tag{145}
$$

relating two operators which act in two distinct Hilbert-spaces. Hence if one wants to compare matrix elements of $\widehat{H}_{\rm string}$ and $\mathcal{D}$ one also has to provide an isomorphism between the string and the gauge theory Hilbert-spaces [38, 51].

An immediate consequence of the relation (145) irrespective of the isomorphism question is the agreement of the operator eigenvalues. On the gauge theory side we have seen how to compute them up to genus two in eq. (114). In string field theory the corresponding calculation has been performed in [52,53] up to genus one. To compute the mass shift of the single-string state $|n\rangle := a_n^{\dagger\,1}\, a_{-n}^{\dagger\,2}\,|0\rangle$ one uses non-degenerate perturbation theory, in analogy to the gauge theory computation. At leading order ($\mathcal{O}(g_s^2)$) the correction to the eigenvalue of $|n\rangle$ comes from a one-loop diagram involving $\widehat{H}_3$ and a contact term involving $\widehat{H}_4$, of which only the $\{Q_{3\,a}, Q_{3\,a}\}$ piece contributes due to the necessity of having intermediate two-string states only[17]

$$\delta E^{(1)}_{|n\rangle} = {g_2}^2 \sum_{|\alpha\rangle\,\in\,2-\text{string states}} \Big\{ \frac{\langle n|\widehat{H}_3|\alpha\rangle\,\langle\alpha|\widehat{H}_3|n\rangle}{E^{(0)}_n - E^{(0)}_\alpha} + \tfrac{1}{8}\langle n|Q_{3\,a}|\alpha\rangle\,\langle\alpha|Q_{3\,a}|n\rangle\,\Big\}\,. \tag{146}$$

From the final results sketched in (141) it is now possible to evaluate the above mass shift. At this point, however, it has only been possible to technically handle this computation by restricting the intermediate two-string channel $|\alpha\rangle$ to the "impurity-conserving" sector, i.e. the sector containing precisely two oscillators acting on the two-string vacuum. This truncation of the computation parallels the computation performed on the gauge theory side in (111). It is far from obvious at this point why such a truncation should be consistent, as there are contributions to the "impurity-non-conserving" channel, e.g. the matrix element

$$\langle 0; r\, p^+|\,\langle 0;(1-r)\,p^+|\widehat{H}_3\,|n\rangle \neq 0 \tag{147}$$

does not vanish and moreover is proportional to $\sqrt{\lambda'}$ [54]. It remains to be seen whether it is consistent to suppress the "impurity-non-conserving" channel, presently the contributions from this sector appear to diverge [52], which may be due to an order of limits problem.

Once one is willing to perform this restriction, however, the genus one correction to the mass shift

$$\delta E^{(1)}_{|n\rangle} = \frac{\mu\,{g_2}^2\,\lambda'}{4\pi^2}\,(\frac{1}{12} + \frac{35}{32\,n^2\,\pi^2}) \tag{148}$$

follows, in perfect agreement with the gauge theory result (113) - a further strong confirmation of the plane-wave string/Super Yang-Mills correspondence.

Let us now discuss the issue raised above concerning the existence of an isomorphism relating the string field theory and the gauge theory Hilbert-spaces. The natural bases we have been working with are the multi-string Fock-space of (124) and the multi-trace states of (88). Both come with a natural scalar product: The string field theory one is obvious and induced by the single-string scalar-product, with orthogonality on single and multi-string states

$$\langle s_\alpha|s_\beta\rangle = \delta_{\alpha\beta}\,. \tag{149}$$

On the gauge theory side it appears natural to use the free, tree-level ($\lambda'=0$) two-point functions as the scalar product of multi-trace operators,

$$\langle\,\mathcal{O}_\alpha(x)\,\bar{\mathcal{O}}_\beta(0)\,\rangle_{\text{tree-level}} = \left(\frac{g^2_{\text{YM}}}{8\pi^2|x|^2}\right)^{J+2} S_{\alpha\beta}\,, \qquad \Rightarrow \qquad \langle\mathcal{O}_\alpha|\mathcal{O}_\beta\rangle := S_{\alpha\beta}\,. \tag{150}$$

[17] The term $\{Q_{2\,a}, Q_{4\,a}\}$ does not contribute as Q_4 takes a single-string state to a triple-string state, while Q_2 preserves the string number.

But as we have discussed at length in section four, $S_{\alpha\beta}$ is not diagonal at higher orders in g_2 on single and multi-trace states, due to the non-planar hopping graphs of figure 2 on page 19

$$S_{\alpha\beta} = \delta_{\alpha\beta} + g_2\, S^{(1)}_{\alpha\beta} + {g_2}^2\, S^{(2)}_{\alpha\beta} + \dots \tag{151}$$

When $g_2 = 0$ the string field theory and gauge theory bases coincide: single string-states are to be identified with single-trace operators according to the BMN dictionary of section 4.1. Similarly multi-string states are identified with multi-trace operators. Taking into account corrections in g_2 the Super Yang-Mills trace-operators start mixing, whereas the string-states remain orthogonal. In order to remedy this situation one seeks a transformation matrix $U_{\alpha\beta}$ of the gauge theory states which diagonalizes $S_{\alpha\beta}$, [18]

$$|\tilde{\mathcal{O}}_\alpha\rangle = \Gamma_{\alpha\beta}\, |\mathcal{O}_\beta\rangle \qquad \Rightarrow \qquad \Gamma\, S\, \Gamma^\dagger = \mathbf{1}\,. \tag{152}$$

Clearly Γ is given in terms of an expansion in g_2

$$\Gamma = \mathbf{1} + g_2\, \Gamma^{(1)} + {g_2}^2\, \Gamma^{(2)} + \dots\,. \tag{153}$$

In the new $|\tilde{\mathcal{O}}_\alpha\rangle$ basis the Super Yang-Mills dilatation operator $\tilde{\mathcal{D}}$ is directly given by the one-loop piece $\tilde{T}_{\alpha\beta} = (U\, T\, U^\dagger)_{\alpha\beta}$ of the two point function (90) and its matrix elements should directly correspond to the matrix elements of the string field theory Hamiltonian $\widehat{H}_{\text{string}}$ [38, 51]

$$\tilde{\mathcal{D}}_{\alpha\beta} - J\cdot\delta_{\alpha\beta} = \underbrace{\tilde{T}_{\alpha\beta}}_{(\Gamma\, T\, \Gamma^\dagger)_{\alpha\beta}} - J\cdot\delta_{\alpha\beta} \quad \hat{=} \quad \langle s_\alpha|\widehat{H}_{\text{string}}|s_\beta\rangle\,. \tag{154}$$

However, there is a crucial problem with this proposal: The transformation matrix $\Gamma_{\alpha\beta}$ is not unique: As a matter of fact the Hermitian matrix S is diagonalized by

$$\Gamma := U\cdot S^{-1/2} \tag{155}$$

with U an arbitrary unitary matrix. But the form of U will affect the matrix elements of the dilatation operator in (154). In [38] it was proposed to fix this ambiguity by simply *demanding* agreement of three-string vertex and gauge theory dilatation operator matrix elements. This situation is not satisfactory, as one looses the predictive power of the duality conjecture. The non-trivial statement would then solely lie in the mere existence of a transformation matrix Γ, which diagonalizes S and lets (154) hold. This statement is equivalent to the match of eigenvalues of the operators $\widehat{H}_{\text{string}}$ and $\mathcal{D}$. In [51] it was noted that the agreement of the string and gauge theory matrix elements precisely occurs if Γ is a symmetric, real matrix, i.e. if $U = \mathbf{1}$. The authors of [51] proposed to introduce this as a fundamental property of Γ and to diagonalize with $S^{-1/2}$. Using this prescription to date all comparisons of matrix elements of the string field theory Hamiltonian with the corresponding matrix elements of the Super Yang-Mills dilatation operator have been shown to agree at the one-loop level[19]. It is clearly highly desirable to obtain an understanding why this particular change of basis is singled out.

[18] $\Gamma_{\alpha\beta}$ *only* diagonalizes the tree-level two-point function, not to be confused with $V_{\alpha A}$ of (92) diagonalizing the all loop two point-function.

[19] Note, however, the discrepancies of a factor of 2 mentioned at order $(\lambda')^2$ in [56].

The structure of the tree-level mixing matrix $S_{\alpha\beta}$ is very intriguing. There are good indications that the full g_2 expansion of (151) simply exponentiates, i.e.

$$S = \exp[\, g_2\, S^{(1)}] \,. \tag{156}$$

This has been shown to be the case in the ground state sector $\mathrm{Tr}\, Z^J$ to all orders in g_s and in the two-impurity sector to order ${g_2}^2$ in [57]. Moreover, the conjugation relation of the effective quantum mechanical Hamiltonian in (100) has been shown to hold for this particular choice of S in the space of two-impurity states in [56]. The ansatz (156) is motivated by a discretized string model due to H. Verlinde and collaborators [58, 57, 59] on which we shall briefly comment. We have seen in section 4.2 how a discretized string picture naturally emerges from the gauge theoretic considerations, with J corresponding to the number of string "bits". In [58, 60, 57, 59] a direct discretization of the space-like component of the plane-wave superstring world-sheet in light-cone gauge was studied, resulting in a quantum mechanical Hamiltonian of J constituent bits. String interactions weighted by g_2 can be included in this model and the diagonalization of the Hamiltonian to order ${g_2}^2$ has been performed in a hybrid string bit/string field theory computation in [59] finding agreement with the gauge theory prediction (113). The string bit formalism lies "in-between" the continuum string field theory and BMN gauge theory. The conjectured exponential structure of the tree-level mixing matrix S in (156) can be understood combinatorially in the string bit model where $S^{(1)}$ is identified as a string bit permutation operator [57]. In [56] a nice direct connection of the effective BMN quantum mechanics discussed in section 5.1 and the string bit model was demonstrated. In [61] it was shown that the string bit model is plagued by the familiar problem of fermion doubling, in [62] a way to evade this problem was pointed out.

Finally let us mention that there exists an alternative approach to study string interactions from perturbative gauge theory in the BMN limit based on the collective field method [63].

6.2 Summary of the performed tests of the duality

Let us give an account of the tests of the plane-wave string/gauge theory duality performed so far organized by the order of g_2 (genus counting parameter) and λ' (effective gauge theory quantum loop counting parameter) that was probed in the operator relation (145):

- $\boldsymbol{{g_2}^0\, \lambda'^n}$
  On the string theory side the result is known to all orders in n, as discussed in (32). For two scalar impurities the gauge theory result is known (based on certain assumptions) to all orders in n as well and agrees [30]. Checks for $n = 1$ exist for scalar-vector impurities [40] and for two vector impurities [41]. In [28] it is proved that *all* two impurity (scalar, fermion, vector) gauge theory excitation are degenerate in scaling dimensions with the two scalar impurity state to all orders in λ' and g_2.

- $\boldsymbol{{g_2}^1\, \lambda'^n}$
  Here again the string theory result is known in principle to all orders in n from the explicit form of $\widehat{H}_3$. Tests for $n = 1$ employing the map $\Gamma = S^{-1/2}$ of the gauge theory basis to the string theory basis have been performed in the two impurity sector for

scalar impurities in [51] and vector-scalar and pure vector impurities in [55]. Tests for an arbitrary number of scalar impurities were perfomed in [55] at $n = 1$.

- $\boldsymbol{g_2{}^2\,\lambda'}$
  At order ${g_2}^2$ the eigenvalues of the two operators in (145) are compared, which is a basis independent statement. The string theory result for all two scalar impurities was computed in [52, 53] by truncating to the impurity conserving channel as discussed in the previous section. The result matches the gauge theory result of [27,35]. Again due to [28] all two impurity excitations (bosonic and fermionic) are known to have identical scaling dimensions.

- $\boldsymbol{g_2{}^2\,\lambda'^2}$
  Here the two loop gauge theory analysis was carried out in [26] for two impurity states. The string theory result has been given in [56] and a *disagreement* by a factor of 2 was reported in a comparison of $\widehat{H}_3$ matrix elements employing the map $\Gamma = S^{-1/2}$.

- $\boldsymbol{g_2{}^4\,\lambda'}$
  Here only the gauge theory result for the eigenvalue of the dilatation operator is known for two impurity states [36] and non-degenerate perturbation theory breaks down [39].

Clearly the worrisome point is the mismatch at order ${g_2}^2\,\lambda'^2$ and should be clarified. Let us also note once more that in principle in the two-impurity sector the all orders result ${g_2}^m\,\lambda'^n$ in gauge theory follows from the diagonalization of the effective quantum mechanical Hamiltonian $H_{\rm full}$ of (115) – if one is willing to extrapolate the λ'^2 result to the square root expression.

6.3 An alternative ansatz for plane-wave string field theory

We should mention that there exists an alternative proposal for the construction of the three-string vertex due to Di Vecchia, Petersen, Petrini, Russo and Tanzini [64] differing from the approach discussed above. This approach departs from the one of Spradlin, Volovich, Pankiewicz, Stefanski and others in the construction of the prefactors needed to satisfy the dynamical constraint. The alternative ansatz of [64] to solve the dynamical constraints (137) is remarkably simple,

$$|H_3\rangle = \sum_{r=1}^{3} \widehat{H}_{2\,(r)}|V\rangle\,, \qquad |Q_{3\,a}\rangle = \sum_{r=1}^{3} Q_{2\,a(r)}|V\rangle\,, \qquad |\tilde{Q}_{3\,a}\rangle = \sum_{r=1}^{3} \tilde{Q}_{2\,a(r)}|V\rangle\,. \tag{157}$$

the prefactors are produced by acting with the free Hamiltonian and supercharges on the kinematical vertex $|V\rangle$[20]. With this ansatz the dynamical constraints are automatically fulfilled (137) as a consequence of the plane-wave superalgebra (122)[21]. One then needs to check that this alternative ansatz (157) does not ruin the kinematical constraints, which the authors of [64] demonstrate. In contradistinction to the vertex discussed in the

[20]Here $|V\rangle$ is the kinematical vertex built upon the vacuum state choice of [45, 48, 50] compare footnote 16.

[21]Note that the kinematical vertex $|V\rangle$ is annihilated by the angular-momenta J^{ij} and $J^{i'j'}$, as it is $SO(4)\times SO(4)$ invariant.

previous sections this alternative vertex does *not* lead to the flat space vertex in the $\mu \to 0$ limit. As a matter of fact the ansatz (157) could equally well be written down for the flat background string. It always has the property of yielding trivial on-shell amplitudes: The amplitude for energy conserving processes vanishes.

The new vertex does not satisfy the tests reported in the previous sections upon making use of the transformation matrix $\Gamma = S^{-1/2}$. The authors argue, however, that there exists a different matrix Γ in (155) for which the gauge theory results can be made to agree with their alternative proposal for $|H_3\rangle$. In any case this is of no direct concern to the authors of [64] as they propose to abandon the operator correspondence (145) at the non-planar level that we have been working with. Instead they propose to relate their three-string vertex to the gauge theory three-point functions in the *naive* basis (for which the Super Yang-Mills two-point functions will *not* be diagonal and the proposed dual operators do not have a well defined scaling dimension). They demonstrate the validity of their proposal at leading order in λ' through a number of checks. Their proposed correspondence only makes sense at tree-level (due to non-canonical coordinate dependencies of the Super Yang-Mills three-point function at higher loops from not resolving the operator mixing) and it is unclear how it could be generalized to higher point functions (due to the non-existence of four point functions in the BMN limit [27]). From our point of view this makes this proposal deficient to the one discussed in these lectures.

But irrespective of the dual gauge theory matter the alternative ansatz indicates the non-uniqueness of the construction of the three-string vertex from the plane-wave super-algebra. This situation is very worrying and we hope that this problem will be settled in future work.

7 Outlook

In these lectures we have developed the duality of type IIB string theory in a maximally supersymmetric plane-wave background and the BMN limit of $\mathcal{N} = 4$, $d = 4$ Super Yang-Mills at the free and interacting string level. The key relation in this correspondence is the identification of the string field Hamiltonian with the dilatation operator of the Super Yang-Mills model minus the $U(1)$ R-charge generator. The remarkable feature of this novel string/gauge theory duality is its apparent perturbative structure on both sides of the correspondence due to the emergence of the effective coupling constants λ' and g_2 in the BMN limit. This enabled us to perform string computations by means of perturbative Yang-Mills computations, which could even be further simplified to an effective quantum mechanical description. A wealth of perturbative tests has been performed, probing higher genus string theory and high quantum loop orders in Super Yang-Mills, as was summarized in these lectures. There are indications, however, of a breakdown of this perturbative correspondence, which occur in impurity non-conserving matrix elements of the string Hamiltonian of order $\sqrt{\lambda'}$. These have no counterpart in perturbative Super Yang-Mills and might be related to strong coupling effects in the gauge theory which remain invisible in the effective weak coupling computations we have performed[22]. More work along these lines is needed, which could also lead to a deeper understanding of why the truncation of

[22]Note that in AdS/CFT strong coupling predictions from string theory/supergravity typically scale as $\sqrt{\lambda}$.

the one-loop string field theory calculation to the "impurity-conserving" channel agrees with the perturbative gauge theoretic answer.

A further important question is what the effective quantum mechanics describing the BMN sector of the $\mathcal{N} = 4$ gauge theory really is. It was argued in [17] that it should be given by a dimensional reduction of $\mathcal{N} = 4$ Super Yang-Mills on a three sphere. This was studied in [65] where it was shown that a consistent reduction on S^3 exists (and is actually nothing but the plane-wave matrix model of [2] related to M-theory on a plane-wave). However, this quantum mechanical model fails to reproduce the eigenvalues of the BMN dilatation operator at two loop order. So the nature of "BMN quantum mechanics" remains an enigma. By working our way up perturbatively we have seen first traces of an effective quantum mechanics for BMN gauge theory in the two-impurity sector in our discussion in section 5.1. But a deeper insight into its inner workings is still necessary.

This problem is closely related to the unsettled question of how the holographic principle is realized in the plane-wave string/gauge theory duality. For work along these lines see [16].

Instanton effects in the BMN gauge theory have not been addressed so far. These should correspond to D-instantons of the dual plane-wave superstring and it would be very interesting to see whether their effects survive the BMN limit.

Let us mention in closing that there are a number of topics closely related to the material covered in these lectures, which we could not address: Corrections in $1/R^2$ to the plane-wave geometry from the Penrose limit (compare (11)) have been considered. In the free ($g_2 = 0$) string theory they give rise to perturbative corrections in $1/R$ of the spectrum [21] and these corrections have been reported to agree with planar $1/J$ corrections of the scaling dimensions on the gauge theory side [66]. A similar duality to the one studied here exists for open strings in plane-wave backgrounds [67], for which the interacting ($g_2 \neq 0$) theory has been explored on the dual $\mathcal{N} = 2$ $Sp(N)$ gauge theory side in [68] and in open string field theory in [69]. Situations with less supersymmetries have been studied as well, e.g. by orbifolding the plane-wave background resulting in $\mathcal{N} = 2$ $[U(N)]^M$ quiver gauge theory [70]. Moreover there is a lot of work on D-branes of the maximally supersymmetric plane-wave superstring [71] which we did not discuss and first steps in exploring their gauge theory duals have been undertaken in [72].

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