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Generalized Schrödinger equation in Euclidean field theory

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Abstract

We investigate the idea of a "general boundary" formulation of quantum field theory in the context of the Euclidean free scalar field. We propose a precise definition for an evolution kernel that propagates the field through arbitrary spacetime regions. We show that this kernel satisfies an evolution equation which governs its dependence on deformations of the boundary surface and generalizes the ordinary (Euclidean) Schrödinger equation. We also derive the classical counterpart of this equation, which is a Hamilton-Jacobi equation for general boundary surfaces.

1 Introduction

In quantum field theory (QFT) on Minkowski space, we can use the Schrödinger picture and have states associated to flat spacelike (hyper-)surfaces. The transition amplitude between an initial state and a final state is obtained by acting with the unitary evolution operator on the former and taking the inner product with the latter. The possibility of a Schrödinger picture has also been considered in QFT on curved spacetime [1]. In this case, states live on arbitrary spacelike Cauchy surfaces forming a foliation of spacetime. Evolution along these surfaces is non-unitary in general, as it does not correspond to a symmetry of the metric. In background independent quantum gravity, on the other hand, there is no fixed spacetime geometry; in this case, states live on arbitrary Cauchy surfaces and the requirement that the surface is spacelike is encoded in the state itself, which represents a quantum state of a spacelike geometry (see e.g. [2, 3]).

In all these cases, transition amplitudes are calculated for boundary states (i.e. an initial and final state) defined on spacelike boundaries. Recently, Oeckl has suggested that it could be possible to relax this restriction to *spacelike* boundaries in QFT [4, 5]. Oeckl offers heuristic arguments which suggest that transition amplitudes can be associated to a wider class of boundaries, as we do in topological quantum field theory [6]. These more general boundaries may include hypersurfaces which are partially timelike, that enclose a finite region of spacetime, or disjoint unions of such sets. This would imply, for instance, that in theories like QED or QCD, we could associate quantum "states" to a hypersphere, a hypercube or more exotic surfaces, and assign probability amplitudes to them. Similar suggestions were made in [3], with different motivations.

This "general boundary" approach to QFT could be interesting for several reasons. Firstly, finite closed boundaries represent the way real experiments are set up more directly than constant-time surfaces. A realistic experiment is confined to a finite region of spacetime. In particle colliders, for instance, the interaction region is enclosed by a finite outer region where state preparation and measurement take place. As sketched in Fig. 1, the walls and openings of a particle detector trace out a hypercube in spacetime. A

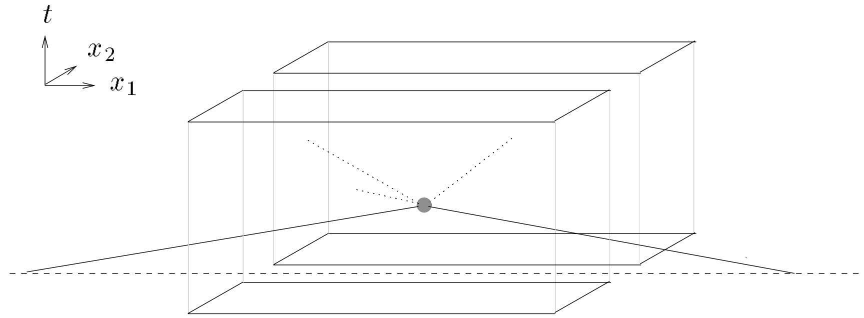

Figure 1: Spacetime diagram of particle scattering.

“state” on the hypercube’s surface would represent both incoming beams and jets of outgoing particles in a completely local fashion, without making any reference to inaccessible infinitely distant regions.

Secondly, in a quantum theory of gravity closed boundaries may provide a way to define scattering amplitudes, and help in solving the traditional interpretational difficulties of background independent QFT. This idea has been recently studied in [7], where it has also been used to propose an explicit way for computing the Minkowski vacuum state from a spinfoam model [8]-[11]. In a background independent theory the conventional spacelike states do not impose any constraint on the proper time lapsed between the initial and final states. As a result, the transition amplitude stems from a superposition of processes whose duration may range from microscopic to cosmic time scales. Fixing a timelike boundary can control the time lapsed during the experiment.

Furthermore, in spinfoam approaches, the introduction of general boundaries might open up the possibility of quantizing 3-geometries along time-like surfaces and clarify the physical meaning of Lorentzian spinfoams.

Finally, a general boundary formulation could give us a broader perspective on QFT: it would stress geometrical aspects of QFT by no longer singling out a special subclass of surfaces, and may shed some light on the holographic principle, which states that the complete information about a spacetime region can be encoded in its boundary.

As noted by Oeckl, a heuristic idea for adapting QFT to general boundaries is provided by Feynman’s sum-over-paths-picture. Given an arbitrary spacetime region V, bounded by a 3d hypersurface, the Feynman path integral over the spacetime region V, with fixed boundary value φ of the field, defines a functional $W[\varphi, V]$. This functional can be seen as a generalized evolution kernel, or a generalized field propagator. The path integral is therefore a natural starting point for developing a general boundary formalism.

The path to make these ideas precise is long. There are two types of problems. Firstly, the probabilistic interpretation of quantum theory and QFT must be adapted to this more general case. The physical meaning of states at fixed time and their relation to physical measurements are well established; the extension to arbitrary boundaries is probably doable, but far from obvious. It requires us to treat quantum state preparation and quantum measurement on the same ground, and to give a precise interpretation to the general probability amplitudes. Some steps in this direction can be found in [5] and [3].

Secondly, the mathematical apparatus of QFT, i.e. the path integral and operator formalism, needs to be extended to general spacetime regions. On a formal level, such a generalization appears natural for path integrals, but it is far from clear that it can be given a concrete and well-defined meaning.

In this paper we focus on the second of these issues: the definition of the field theoretical functional integral over an arbitrary region, and its relation to operator equations. We start to address the problem by considering the simplest system: Euclidean free scalar field theory. In this context, we propose an exact definition for the propagator kernel $W[\varphi, V]$, based on limits of lattice path integrals. Under a number of assumptions, we can show that the propagator satisfies a generalized Schrödinger equation, of the Tomonaga-Schwinger kind [12, 13]. The equation governs the way the propagator changes under infinitesimal deformations of V. It reduces to the ordinary Schrödinger equation in the case in which a boundary surface of V is a constant-time surface and the deformation is a global shift in time.

With this result, we provide a first step towards constructing an operator formalism for general boundaries. The derivation can be seen as a higher-dimensional generalization of Feynman's path integral derivation of the Schrödinger equation for a single particle [14]. The main assumption we need is the existence of a rotationally invariant continuum limit.

We also derive the classical counterpart of the evolution equation: a generalized version of the Euclidean Hamilton-Jacobi equation. At present, we have no prescription for Wick rotation, so we cannot give any Lorentzian form for the propagator or the evolution equation. Hints in this direction were given in [7].

If one continues along this line, the ultimate goal would be to construct a full general boundary formalism for background dependent QFTs, which incorporates Wick rotation, interactions and renormalization. While of interest in itself, such a project could be also viewed as a testing ground for the general boundary method, which would prepare us for applying it in the more difficult context of background free QFT: there, as indicated before, the use of generalized boundary conditions may not only be helpful, but also essential for understanding the theory.

Our technique for deriving the evolution equation could be of interest in view of the attempts to relate canonical and path integral formulations of quantum gravity, i.e. when deriving the Wheeler-DeWitt equation from a concrete realization of a sum over geometries. (For existing results on this problem, see e.g. [15].)

The paper is organized as follows. In section 2, we present some of the heuristic considerations about state functionals on general boundaries, and their associated evolution kernel. Section 3 deals with the classical case: we present two derivations of the generalized Hamilton-Jacobi equation. The lattice regularization of the quantum propagator is defined in section 4. There, we also state the assumptions which are then used in section 5 for deriving the generalized Schrödinger equation. Both Hamilton-Jacobi and Schödinger equation are given in their integral form. In the appendix we clarify the relation with the local notation in [3].

Notation. V is the spacetime domain over which the action and the path integrals are defined. Σ is the boundary of V. The letter ϕ denotes a real scalar field on V, while φ stands for its restriction to Σ, i.e. $\varphi = \phi|_\Sigma$. Depending on the context, ϕ can be a solution of the classical equations of motion or an arbitrary field configuration. The action associated to ϕ is written as $S[\phi, V]$. When boundary conditions (φ,Σ) determine a classical solution ϕ on V, we denote the corresponding value of the action by $S[\varphi, V]$. Thus, the functional $S[\varphi, V]$ can be viewed as a Hamilton function (see sec. 3.3 of [3]). Vector components carry greek indices $\mu, \nu, \ldots$ (e.g. $v = (v^\mu)$). The dimension of spacetime is d. The symbol $\int_V d^d x$ represents integrals over V, while integrals over Σ are indicated by $\int_\Sigma d\Sigma(x)$. The letter n denotes the outward pointing and unit normal vector of Σ. The normal derivative is written as ∂_n, while ∇_Σ is the gradient along Σ. Accordingly, the full gradient ∇ decomposes on Σ as

$$\nabla|_\Sigma = n\, \partial_n + \nabla_\Sigma\,. \tag{1}$$

In section 4, we introduce a lattice with lattice spacing a and regularize various continuum quantities. Their discrete analogues are designated by the index a: for example, φ, V and Σ become φ_a, V_a and Σ_a.

2 General Boundary Approach

What is the meaning of a state on a general surface which is not necessarilly spacelike? What does it mean to propagate fields along a general spacetime domain? Following [5], an intuitive answer to these questions is provided by the path integral approach to QFT. We illustrate this intuitive idea in this section, as a heuristic motivation for the more rigorous definitions and developments in the remainder of the article. For

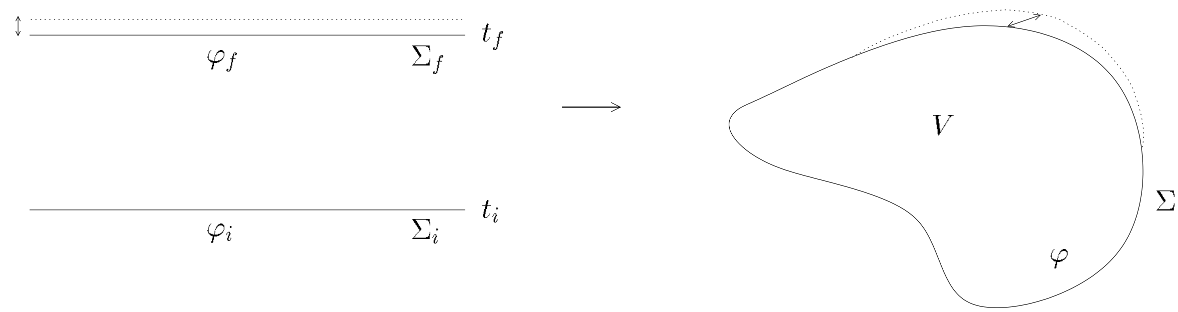

Figure 2: From V_{fi} to general V.

simplicity, we refer here to a scalar field theory, but similar considerations can be extended to various path integral formulations of QFT, including sum over metrics or spinfoam models in quantum gravity.

Consider Minkowskian scalar QFT in the Schrödinger picture. Let $|\Psi_i\rangle$ be an initial state at time t_i and $|\Psi_f\rangle$ a final state at time t_f. The transition amplitude between the two is

$$A = \langle \Psi_f | e^{-iH(t_f - t_i)/\hbar} | \Psi_i \rangle \, . \tag{2}$$

Using the functional representation, the amplitude (2) can be expressed as a convolution

$$A = \int D\varphi_f \int D\varphi_i \, \Psi_f^*[\varphi_f] \, W[\varphi_f, t_f; \varphi_i, t_i] \, \Psi_i[\varphi_i] \tag{3}$$

with the propagator kernel

$$W[\varphi_f, t_f; \varphi_i, t_i] := \langle \varphi_f | e^{-iH(t_f - t_i)/\hbar} | \varphi_i \rangle \, . \tag{4}$$

This field propagator is a functional of the field: it should not be confused with the Feynman propagator, which is a two-point function, and propagates particles. When rewritten as a path integral, this kernel takes the form

$$W[\varphi_f, t_f; \varphi_i, t_i] = \int\limits_{\substack{\phi(.,t_i) = \varphi_i \, , \\ \phi(.,t_f) = \varphi_f}} D\phi \; e^{iS[\phi, t_i, t_f]/\hbar} \, . \tag{5}$$

The action integral extends over the spacetime region $V_{fi} := \mathbb{R}^{d-1} \times [t_i, t_f]$ and the path integral sums over all field configurations ϕ on V_{fi} that coincide with the fields φ_f and φ_i on the boundary. The complete boundary consists of two parts: the hyperplane Σ_i at the initial time t_i, and the hyperplane Σ_f at the final time t_f. We call their union $\Sigma_{fi} := \Sigma_f \cup \Sigma_i$. If we view φ_f and φ_i as components of a single boundary field $\varphi_{fi} := (\varphi_f, \varphi_i)$ on Σ_{fi}, we can write the evolution kernel (5) more concisely as

$$W[\varphi_{fi}, V_{fi}] := \int_{\phi|_{\Sigma_{fi}} = \varphi_{fi}} D\phi \; e^{iS[\phi, V_{fi}]/\hbar} \, .$$

With this notation, it seems natural to introduce a propagator functional for more general spacetime regions V (see Fig. 2): we define it as

$$W[\varphi, V] := \int_{\phi|_\Sigma = \varphi} D\phi \; e^{iS[\phi, V]/\hbar} \, . \tag{6}$$

Here ϕ varies freely on the interior of V and is fixed to the value φ on the boundary Σ. Of course, this is only a formal expression, and it is not clear that it can be given mathematical meaning. Let us suppose for the moment that it *has* meaning and see what would follow from it.

Ordinary propagators satisfy convolution (or Markov) identities which result from the subdivision or joining of time intervals. If the functional W behaves the way our naive picture tells us, the splitting and

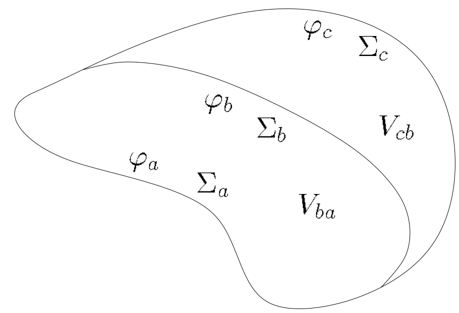

Figure 3: Splitting of V.

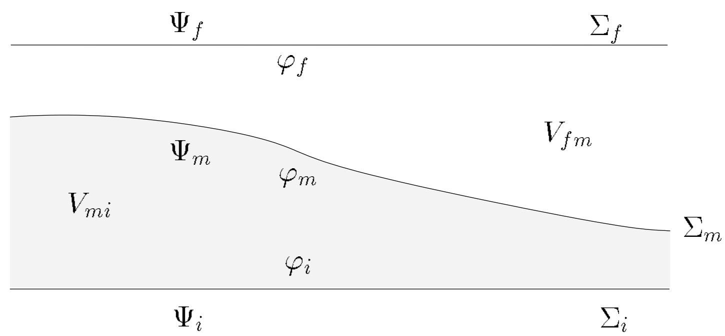

Figure 4: Splitting of V_{fi}.

joining of volumes should translate into analogous convolution relations. For instance, if V is divided as shown Fig. 3, the new regions V_{cb} and V_{ba} carry propagators

$$W[(\varphi_c, \varphi_b), V_{cb}] = \int_{\phi|_{\Sigma_{cb}}=(\varphi_c,\varphi_b)} D\phi \, e^{iS[\phi, V_{cb}]/\hbar}, \tag{7}$$

$$W[(\varphi_b, \varphi_a), V_{ba}] = \int_{\phi|_{\Sigma_{ba}}=(\varphi_b,\varphi_a)} D\phi \, e^{iS[\phi, V_c]/\hbar}. \tag{8}$$

When integrating the product of (7) and (8) over the field φ_b along the common boundary, one recovers the original propagator:

$$W[(\varphi_c, \varphi_a), V] = \int D\varphi_b \, W[(\varphi_c, \varphi_b), V_{cb}] \, W[(\varphi_b, \varphi_a), V_{ba}]. \tag{9}$$

Similarly, the infinite strip V_{fi} between t_f and t_i could be cut by a “middle” surface Σ_m as in Fig. 4, giving the new volumes V_{fm} and V_{mi}. Its kernel decomposes as

$$W[(\varphi_f, \varphi_i), V_{fi}] = \int D\varphi_m \, W[(\varphi_f, \varphi_m), V_{fm}] \, W[(\varphi_m, \varphi_i), V_{mi}].$$

Thus, the evolution of $|\Psi_i\rangle$ to the final time is divided into two steps: using the propagator on V_{mi}, we evolve up to the surface Σ_m and obtain the intermediate state

$$\Psi_m[\varphi_m] := \int d\varphi_i \, W[(\varphi_m, \varphi_i), V_{mi}] \, \Psi_i[\varphi_i].$$

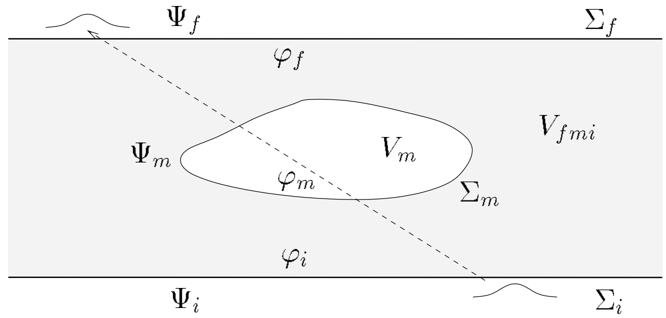

Figure 5: Evolution to a closed surface Σ_m.

The kernel $W[., V_{fm}]$ covers the remaining evolution and gives the original amplitude (2) when convoluted with Ψ_f^* and Ψ_m

$$A = \int D\varphi_f \int D\varphi_m \, \Psi_f^*[\varphi_f] \, W[(\varphi_f, \varphi_m), V_{fm}] \, \Psi_m[\varphi_m] \,. \tag{10}$$

Since the same amplitude can either be calculated from Ψ_f and Ψ_i or from Ψ_f and Ψ_m, the wave functional Ψ_m encodes all physical information about the intial state. On account of this property, we say that Ψ_m *is the state functional which results from evolving* Ψ_i *by the volume* V_{mi}. As for ordinary state functionals, one can think of Ψ_m as being an element $|\Psi_m\rangle$ in a Hilbert space, which we call $\mathcal{H}_{\Sigma_m}$. The latter consists of functionals of fields over Σ_m and has the inner product

$$\langle \Psi_2 | \Psi_1 \rangle := \int D\varphi_m \, \Psi_2^*[\varphi_m] \Psi_1[\varphi_m] \,, \qquad |\Psi_1\rangle \,, |\Psi_2\rangle \in \mathcal{H}_{\Sigma_m} \,.$$

It is important to note that the evolution map from $\mathcal{H}_{\Sigma_i}$ to $\mathcal{H}_{\Sigma_m}$ need not be unitary. The results of Torre and Varadarajan show, in fact, that in flat spacetime state evolution between curved Cauchy surfaces cannot be implemented unitarily [16]. Nevertheless, a probability interpretation *is* viable for states in H_{Σ_m}, as the meaning of amplitudes such as (10) can be traced back to that of the standard amplitude (3).

Consider now a more unconventional example. Cut out a bounded and simply connected set V_m from V_{fi} and denote the remaining volume by V_{fmi} (Fig. 5). This time we define the state functional Ψ_m by "evolving" both Ψ_f and Ψ_m to the middle boundary Σ_m, i.e.

$$\Psi_m[\varphi_m] := \int D\varphi_f \int D\varphi_i \, \Psi_f^*[\varphi_f] \, W[(\varphi_f, \varphi_m, \varphi_i), V_{fmi}] \, \Psi_i[\varphi_i] \,.$$

Clearly, the amplitude (3) is now equal to

$$A = \int D\varphi_m \, W[\varphi_m, V_m] \, \Psi_m[\varphi_m] \,. \tag{11}$$

Therefore, the functional Ψ_m contains the entire information needed to compute the transition amplitude between Ψ_i and Ψ_f.

To make this more concrete and more intuitive, suppose that the scalar field theory is free and that Ψ_i and Ψ_f are the initial and final one-particle states of a single, localized particle whose (smeared out) worldline passes through V_m. In both functionals, the presence of the particle appears as a local deviation from the vacuum, in the functional dependence. Likewise, it is natural to presume that the functional form of Ψ_m reflects where the worldline of the particle enters and exits the volume V_m.

How can we interpret the "state" Ψ_m and the associated amplitude (11)? To answer this, let us get back to equation (2). Notice that the amplitude A depends on the *couple* of states $(|\Psi_i\rangle, |\Psi_f\rangle)$. This couple represents a possible outcome of a measurement at time t_f as well as a state preparation at time t_i. A state preparation is itself a quantum measurement, therefore we can say that this couple represents a possible outcome of an ensemble of quantum measurements performed at times t_i and t_f. We may introduce a name to denote such a couple. We call it a *process*, since the two states $(|\Psi_i\rangle, |\Psi_f\rangle)$, taken together, represent the ensemble of data (initial and final) that we can gather about a physical process. A probability amplitude is associated to the entire process $(|\Psi_i\rangle, |\Psi_f\rangle)$. Now, it is clear that the functional Ψ_m represents a generalization of this idea of a process. It is tempting to presume that Ψ_m can be interpreted as representing a possible outcome of quantum measurements that can be made on Σ_m. In the example of the particle above, for instance, it will represent the detection of the incoming and outgoing particle.

The idea is that given an arbitrary closed surface, the possible results of the ensemble of measurements that we can make on it determines a space of generalized "states" which can be associated to the surface. Each such state represents a process whose probabilistic amplitude is provided by expression (11). The conventional formalism is recovered when the surface is formed by two parallell spacelike planes. For more details on the physical interpretation of general boundary states, see sec. 5.3 of [3].

2.1 Operator Formalism

If path integrals can be defined for general boundaries, how would a corresponding operator formalism look like? In particular, is there an operator that governs the dynamics, as the Hamiltonian does for rigid time translations? Recall that the Hamiltonian can be recovered from the path integral by considering an infinitesimal shift of the final time. For example, if we displace by a time interval Δt the final surface Σ_f in (5), keeping the same boundary field φ_f, the new propagator results from the convolution

$$W[\varphi_f, t_f + \Delta t; \varphi_i, t_i] = \int D\varphi\, W[\varphi_f, t_f + \Delta t; \varphi, t_f]\, W[\varphi, t_f; \varphi_i, t_i]\,. \tag{12}$$

For infinitesimal Δt, this gives the Schrödinger equation, which expresses the variation of W in terms of the Hamiltonian operator

$$\left(i\hbar \frac{\partial}{\partial t_f} - H[\varphi_f, -i\hbar \frac{\delta}{\delta\varphi_f}] \right) W[\varphi_f, t_f; \varphi_i, t_i] = 0\,, \tag{13}$$

where

$$H[\varphi_f, \pi_f] \;=\; \int_{\Sigma_f} d\Sigma\, \frac{1}{2} \left(\pi_f^2 + (\nabla_\Sigma \varphi_f)^2 + m^2 \varphi_f^2 \right)\,. \tag{14}$$

Similarly, if φ_f is displaced in a tangential direction $e_{||}$ along Σ_f, the variation of W is generated by the momentum operator

$$e_{||} \cdot P[\varphi_f, -i\hbar \frac{\delta}{\delta\varphi_f}]\,,$$

where

$$P[\varphi_f, \pi_f] = -\int_{\Sigma_f} d\Sigma(x)\, \nabla_\Sigma \varphi_f(x)\, \pi_f\,.$$

In the case of a general volume V, it is natural to expect that deformations of the boundary surface Σ lead to an analogous functional differential equation for the propagator. However, for a general shape of V there is no notion of preferred rigid displacement of the boundary. We must consider arbitrary deformations of the boundary surface, and we expect that the associated change in W is governed by a generalized Schrödinger equation (see Fig. 2). In the same way that H and P generate temporal and spatial shifts, the operators in such a Schrödinger equation could be seen as the generators for general boundary deformations of W. In a difformopshim invariant QFT, the analogous W-functional would be independent of Σ, and the generalized Schrödinger equation reduces to the Wheeler-DeWitt equation.

In the present paper, we consider only Euclidean field theory, so we seek to define the Euclidean form

$$W[\varphi, V] := \int_{\phi|_\Sigma = \varphi} D\phi\; e^{-S[\phi, V]/\hbar}\,. \tag{15}$$

of the propagator (6), and generalize the Euclidean version of the Schrödinger equation (13).

Before dealing with path integrals and deformations of their boundaries, however, we discuss the analagous problem in classical field theory. The classical counterpart of the Schrödinger equation is the Hamilton-Jacobi equation. The Hamilton function $S[\varphi_f, t_f, \varphi_i, t_i]$ is a function of the same arguments as the field propagator (4). It is defined as the value of the action of the classical field configuration which solves the equations of motion and has the given boundary values. It satisfies the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t_f} S[\varphi_f, t_f; \varphi_i, t_i] + H[\varphi_f, \frac{\delta S}{\delta \varphi_f}] = 0. \tag{16}$$

For more general regions V, the Hamilton function becomes a functional of V and the boundary field φ specified on Σ. In the next section we show that this functional satisfies a generalized Hamilton-Jacobi equation which governs its dependence on arbitrary variations of V.

3 Generalized Hamilton-Jacobi Equation

Let V be an open and simply connected subset of Euclidean d-dimensional space $\mathbb{R}^d$. We consider the Euclidean action

$$S[\phi, V] = \int_V d^dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right], \tag{17}$$

where U is some polynomial potential in ϕ. In the classical case, unlike in the quantum case, an interaction term can be added without complicating the derivation that follows. The equations of motion are

$$\Box\phi - m^2\phi - \frac{\partial U}{\partial \phi} = 0\,. \tag{18}$$

The Hamilton function $S[\varphi, V]$ is defined by $S[\varphi, V] = S[\phi, V]$, where ϕ solves (18) and $\phi|_\Sigma = \varphi$. It is defined for all values (φ, Σ) where this solution exists and is multivalued if this solution is not unique.

We now study the change in $S[\varphi, V]$ under a local variation of V. To make this precise, consider a vector field $N = (N^\mu)$ over $\mathbb{R}^d$. N induces a flow on $\mathbb{R}^d$ which we denote by $\sigma : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Define the transformed volume as $V^s := \sigma(s, V)$. Likewise, $\Sigma^s := \sigma(s, \Sigma)$.

To define the change in $S[\varphi, V]$ under a variation of V, we need to specify what value the boundary field should take on the new boundary Σ^s. We choose it to be the pull-forward by $\sigma_s \equiv \sigma(s, .)$, i.e. $\varphi^s := \varphi \circ \sigma_s^{-1} \equiv \sigma_{s*}\varphi$. Let us assume that the point (φ, Σ) is regular in the space of boundary conditions, in the sense that slightly deformed boundary conditions (φ^s, Σ^s) give a new unique solution ϕ^s on V^s, close to the previous one. In this case, the number $S[\varphi^s, V^s]$ is well-defined and we can write down the differential quotient

$$L_N S[\varphi, V] := \lim_{s \to 0} \frac{1}{s} \left(S[\varphi^s, V^s] - S[\varphi, V]\right), \tag{19}$$

with the vector field N as a parameter. As we show below, this limit exists and the map L_N is a functional differential operator. The local form of this differential operator is given in the appendix.

We decompose the restriction of N to Σ into its components normal and tangential to Σ,

$$N_{|\Sigma} = N_\perp n + N_{||}\,,$$

where the scalar field $N_\perp$ is defined as

$$N_\perp := n_\mu N^\mu\,.$$

Observe that under a small variation $\delta\varphi$ of the boundary field, we have

$$\begin{aligned} \delta S[\varphi, V] &= S[\delta\varphi, V] = S[\delta\phi, V] \\ &= \int_V d^dx \left[\nabla^\mu \delta\phi\, \nabla_\mu \phi + m^2 \phi\, \delta\phi + \frac{\partial U}{\partial \phi}\delta\phi\right] \end{aligned}$$

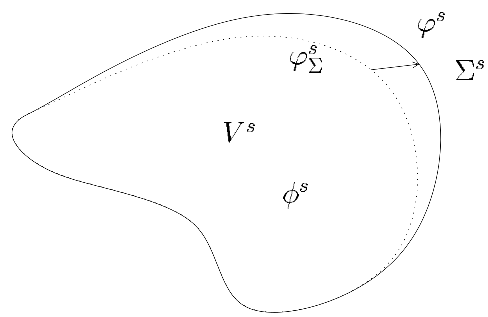

Figure 6: Definition of φ^s_Σ.

$$= \int_V d^dx \left[\nabla_\mu(\nabla_\mu\phi\,\delta\phi) + \delta\phi\underbrace{\left(-\Box\phi + m^2\phi + \frac{\partial U}{\partial\phi}\right)}_{=0}\right]$$
$$= \int_\Sigma d\Sigma\,\partial_n\phi\,\delta\phi = \int_\Sigma d\Sigma\,\partial_n\phi\,\delta\varphi\,.$$

Therefore, we have

$$\frac{\delta S}{\delta\varphi(x)}[\varphi, V] = \partial_n\phi(x)\,. \tag{20}$$

3.1 Direct Derivation

Suppose for the moment that V is only extended by the deformation (i.e. $V \subset V^s$ for every s). Then, the most direct derivation of the Hamilton-Jacobi equation can be obtained by considering the restriction φ^s_Σ of ϕ^s to Σ: that is, the value of the classical solution on Σ when the boundary condition φ^s is specified on Σ^s (see Fig. 6).

Note that $\varphi^0_\Sigma = \varphi^0 = \varphi$. By inserting $S[\varphi^s_\Sigma, V]$ into the difference, i.e.

$$S[\varphi^s, V^s] - S[\varphi, V] = S[\varphi^s, V^s] - S[\varphi^s_\Sigma, V] + S[\varphi^s_\Sigma, V] - S[\varphi, V]\,,$$

the differential quotient becomes a sum of two limits:

$$\lim_{s\to 0}\frac{S[\varphi^s, V^s] - S[\varphi, V]}{s} = \lim_{s\to 0}\frac{S[\varphi^s, V^s] - S[\varphi^s_\Sigma, V]}{s} + \lim_{s\to 0}\frac{S[\varphi^s_\Sigma, V] - S[\varphi, V]}{s}$$

As φ^s and φ^s_Σ are part of the same solution, the first limit is easily seen to be

$$\begin{aligned}\lim_{s\to 0}\frac{S[\varphi^s, V^s] - S[\varphi^s_\Sigma, V]}{s} &= \int_\Sigma d\Sigma\; N_\perp\left(\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + \frac{1}{2}m^2\phi^2 + U(\phi)\right) \\ &= \int_\Sigma d\Sigma\; N_\perp\left(\frac{1}{2}\left(\frac{\delta S}{\delta\varphi}\right)^2 + \frac{1}{2}(\nabla_\Sigma\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right)\,.\end{aligned}$$

In the last line, we used the decomposition (1) and equation (20). The second differential quotient gives

$$\begin{aligned}\lim_{s\to 0}\frac{S[\varphi^s_\Sigma, V] - S[\varphi, V]}{s} &= \int_\Sigma d\Sigma\,\frac{\delta S}{\delta\varphi}\,\frac{d}{ds}\varphi^s_\Sigma|_{s=0} \\ &= \int_\Sigma d\Sigma\,\frac{\delta S}{\delta\varphi}\left(-N_\perp\partial_n\phi - N_{||}\cdot\nabla_\Sigma\varphi\right)\,.\end{aligned}$$

Altogether one has

$$L_N S[\varphi, V] = \int_\Sigma d\Sigma\left\{N_\perp\left[-\frac{1}{2}\left(\frac{\delta S}{\delta\varphi}\right)^2 + \frac{1}{2}(\nabla_\Sigma\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right] - N_{||}\cdot\nabla_\Sigma\varphi\frac{\delta S}{\delta\varphi}\right\}\,. \tag{21}$$

We started from the assumption that $V \subset V^s$ for all s, but it is easy to see that the previous argument can be adapted to the general case where the volume V is partly extended and partly decreased.

If we introduce the quantities

$$\begin{aligned} H_N[\varphi,\pi,V] &:= \int_\Sigma d\Sigma\, N_\perp \Big(-\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla_\Sigma\varphi)^2 + \frac{1}{2}m^2\varphi^2 + U(\varphi)\Big)\,, \\ P_N[\varphi,\pi,V] &:= -\int_\Sigma d\Sigma\, N_{||}\cdot\nabla_\Sigma\varphi\,\pi\,, \end{aligned}$$

equation (21) takes the form

$$L_N S[\varphi,V] = H_N[\varphi,\frac{\delta S}{\delta\varphi},V] + P_N[\varphi,\frac{\delta S}{\delta\varphi},V]\,. \tag{22}$$

When V is a strip of spacetime between times t_i and t_f, and $N_\perp|_{t_f} = 1$, $N_\perp|_{t_i} = 0$, $N_{||} = 0$, equation (22) reduces to the usual Hamilton-Jacobi equation

$$\frac{\partial}{\partial t_f} S[\varphi_f,t_f;\varphi_i,t_i] = \int_{\Sigma_f} d\Sigma\, \left(-\frac{1}{2}\left(\frac{\delta S}{\delta\varphi_f}\right)^2 + \frac{1}{2}(\nabla_\Sigma\varphi_f)^2 + \frac{1}{2}m^2\varphi_f^2 + U(\varphi_f)\right)$$

of Euclidean field theory. Hence (22) can be seen as a geometric generalization of the Hamilton-Jacobi equation.

3.2 Alternative Derivation

Let us describe another way of evaluating the "deformation derivative" $L_N S[\varphi,V]$. The spacetime metric tensor g enters in the definition of the action and therefore in the definition of $S[\varphi,V]$. Let us write this dependence explicitely as $S[\varphi,g,V]$. A diffeomorphism that acts on ϕ, the boundary Σ *and* the metric g, leaves the action invariant. Therefore

$$S[\varphi^s,g^s,V^s] = S[\varphi,g,V]\,.$$

Equivalently,

$$S[\varphi^s,g,V^s] = S[\varphi,g^{-s},V].$$

Plugging this into the definition of the operator (19) gives

$$L_N S[\varphi,g,V] = \lim_{s\to 0}\frac{1}{s}\left(S[\varphi,g^{-s},V] - S[\varphi,g,V]\right)\,, \tag{23}$$

which is a variation of the action w.r.t. the metric *only*. Now we can use the definition of the energy-momentum tensor to obtain

$$\begin{aligned} L_N S[\varphi,g,V] &= \frac{1}{2}\int_V d^dx\, T^{\mu\nu}\frac{d}{ds}g^{-s}_{\mu\nu}|_{s=0} = \frac{1}{2}\int_V d^dx\, T^{\mu\nu}(-\nabla_\mu N_\nu - \nabla_\nu N_\mu) \\ &= -\int_V d^dx\, T^{\mu\nu}\nabla_\mu N_\nu = -\int_\Sigma d\Sigma\, n_\mu T^{\mu\nu}N_\nu + \int_V d^dx\, \underbrace{\nabla_\mu T^{\mu\nu}}_{=0} N_\nu \\ &= -\int_\Sigma d^dx\, n_\mu T^{\mu\nu}N_\nu \end{aligned} \tag{24}$$

In the last two steps we used Stoke's theorem and the equations of motion respectively. On the other hand, we know that

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \nabla^\mu\phi\nabla^\nu\phi$$

and

$$
\begin{aligned}
n_\mu N_\nu T^{\mu\nu} &= -N_\perp\left[\frac{1}{2}(\partial_n\phi)^2 + \frac{1}{2}(\nabla_\Sigma\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right] + \partial_n\phi(N_\perp\partial_n\phi + N_{||}\cdot\nabla_\Sigma\phi) \\
&= -N_\perp\left[-\frac{1}{2}(\partial_n\phi)^2 + \frac{1}{2}(\nabla_\Sigma\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right] + N_{||}\cdot\nabla_\Sigma\phi\,\partial_n\phi
\end{aligned} \tag{25}
$$

Inserting (25) in (24) and using (20), we arrive again at the generalized Hamilton-Jacobi equation

$$
L_N S[\varphi, V] = \int_\Sigma d\Sigma\left\{N_\perp\left[-\frac{1}{2}\left(\frac{\delta S}{\delta\varphi}\right)^2 + \frac{1}{2}(\nabla_\Sigma\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi)\right] - N_{||}\cdot\nabla_\Sigma\varphi\frac{\delta S}{\delta\varphi}\right\}.
$$

4 Definition of the Evolution Kernel

In this section, we define a Euclidean free field propagator for arbitrary spacetime domains V. Limits of lattice path integrals are used to give a precise meaning to the expression (15). We begin by considering the case $V = V_{fi}$ and derive the lattice path integral from the operator formalism. Then, we propose a way to extend this expression to more general volumes V.

4.1 From Operators to Path Integrals

The transition from operator formalism to path integral is a standard procedure. We repeat it here, since treatments of lattice field theory usually omit normalization factors. There, constant factors drop out when dividing by the partition function Z. In our case, their precise form will be crucial for the definition of the propagator.

In the Schrödinger picture, the space of states $\mathcal{H}$ is associated to the manifold $\mathbb{R}^{d-1}$: we regularize it by a finite lattice

$$
S_a := \{\vec{x} \in a\mathbb{Z}^{d-1} \mid -Ma \leq |x_i| \leq Ma\,,\ i = 1,\ldots,d-1\}
$$

with lattice constant $a > 0$ and edge length $2aM$, $M \in \mathbb{N}$. e_i is the unit vector in the ith direction. For a scalar function f on S_a, the forward derivative is

$$
\nabla_i f(\vec{x}) := \frac{\phi(\vec{x}+ae_i) - \phi(\vec{x})}{a}\,,
$$

and we set $\phi(\vec{x}+ae_i) := \phi(\vec{x}-aMe_i)$ when $x_i = aM$. Let $\{\hat{\phi}(\vec{x})\}$, $\{\hat{\pi}(\vec{x})\}$ be canonical operators with eigenstates $\{|\phi\rangle\}$, $\{|\pi\rangle\}$ such that

$$
\hat{\phi}(\vec{x})\,|\,\phi\rangle = \phi(\vec{x})|\phi\rangle, \qquad \hat{\pi}(\vec{x})|\pi\rangle = \pi(\vec{x})|\pi\rangle\,, \qquad \vec{x}\in S_a\,, \tag{26}
$$

and

$$
[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \frac{i\hbar}{a^{d-1}}\delta(\vec{x}-\vec{y})\,, \qquad \vec{x},\vec{y}\in S_a\,. \tag{27}
$$

The eigenstates are normalized as

$$
\langle\phi,\phi'\rangle = \prod_{\vec{x}\in S_a}\delta(\phi(\vec{x})-\phi'(\vec{x}))\,, \qquad \langle\pi,\pi'\rangle = \prod_{\vec{x}\in S_a}\delta(\pi(\vec{x})-\pi'(\vec{x}))\,, \tag{28}
$$

and give rise to completeness relations

$$
\left(\prod_{\vec{x}\in S_a}\int_{-\infty}^{\infty} d\phi(\vec{x})\right)|\phi\rangle\langle\phi| = \mathbb{1}\,, \qquad \left(\prod_{\vec{x}\in S_a}\int_{-\infty}^{\infty} d\pi(\vec{x})\right)|\pi\rangle\langle\pi| = \mathbb{1}\,. \tag{29}
$$

From (26), (27) and (28), it follows that

$$\hat{\pi}(\vec{x}) = -\frac{i\hbar}{a^{d-1}}\frac{\partial}{\partial\phi(\vec{x})} \tag{30}$$

and

$$\langle\phi,\pi\rangle = \left(\prod_{\vec{x}\in S_a}\sqrt{\frac{a^{d-1}}{2\pi\hbar}}\right)\exp\left(\frac{i}{\hbar}\sum_{\vec{x}\in S_a} a^{d-1}\phi(\vec{x})\pi(\vec{x})\right) \tag{31}$$

The Hamiltonian operator is

$$\begin{aligned} H[\hat{\phi},\hat{\pi}] &:= \sum_{\vec{x}\in S_a} a^{d-1}\left[\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}(\vec{\nabla}\hat{\phi})^2(\vec{x}) + \frac{1}{2}m^2\hat{\phi}^2(\vec{x})\right] \\ &\equiv T[\hat{\pi}] + V[\hat{\phi}]\,. \end{aligned}$$

We rewrite the Euclidean propagator

$$\langle\varphi_f|e^{-H(t_f-t_i)/\hbar}|\varphi_i\rangle\,, \qquad t_f - t_i = na\,,$$

by inserting repeatedly the completeness relations (29):

$$\begin{aligned} \langle\varphi_f|e^{-naH/\hbar}|\varphi_i\rangle = &\left(\prod_{k=1}^{n-1}\prod_{\vec{x}}\int d\phi_k(\vec{x})\right)\left(\prod_{k=0}^{n-1}\prod_{\vec{x}}\int d\pi_k(\vec{x})\right)\times \\ &\times\langle\varphi_f|\pi_{n-1}\rangle\langle\pi_{n-1}|e^{-aH/\hbar}|\phi_{n-1}\rangle\langle\phi_{n-1}|\pi_{n-2}\rangle\langle\pi_{n-2}|e^{-aH/\hbar}|\phi_{n-2}\rangle\cdots\langle\phi_1|\pi_0\rangle\langle\pi_0|e^{-aH/\hbar}|\varphi_i\rangle \end{aligned}$$

After making the replacement

$$e^{-aH/\hbar} = e^{-aT/\hbar}e^{-aV/\hbar} + O(a^2) \;\rightarrow\; e^{-aT/\hbar}e^{-aV/\hbar}$$

and using (31), we obtain

$$\begin{aligned} &\left(\prod_{k=1}^{n-1}\prod_{\vec{x}}\int d\phi_k(\vec{x})\right)\left(\prod_{k=0}^{n-1}\prod_{\vec{x}}\int d\pi_k(\vec{x})\right)\langle\varphi_f|\pi_{n-1}\rangle\langle\pi_{n-1}|\phi_{n-1}\rangle\langle\phi_{n-1}|\pi_{n-2}\rangle\langle\pi_{n-2}|\phi_{n-2}\rangle\cdots \\ &\qquad\cdots\langle\phi_1|\pi_0\rangle\langle\pi_0|\varphi_i\rangle\exp\left(\frac{1}{\hbar}\sum_{k=0}^{n-1} aH[\phi_k,\pi_k]\right)\Bigg|_{\phi_0=\varphi_i} \\ = &\left(\prod_{k=1}^{n-1}\prod_{\vec{x}}\int d\phi_k(\vec{x})\right)\left(\prod_{k=0}^{n-1}\prod_{\vec{x}}\int d\pi_k(\vec{x})\right)\left(\prod_{\vec{x}}\sqrt{\frac{a^{d-1}}{2\pi\hbar}}\right)^{2n}\times \\ &\times\exp\left\{\frac{1}{\hbar}\sum_{k=0}^{n-1} a\sum_{\vec{x}} a^{d-1}\left[i\frac{\phi_{k+1}(\vec{x})-\phi_k(\vec{x})}{a}\pi_k(\vec{x}) - \frac{1}{2}\pi_k^2(\vec{x}) - \frac{1}{2}(\vec{\nabla}\phi_k)^2(\vec{x}) - \frac{1}{2}m^2\phi_k^2(\vec{x})\right]\right\}\Bigg|_{\substack{\phi_0=\varphi_i\,,\\ \phi_n=\varphi_f}}\,. \end{aligned}$$

Integration over the momenta yields the path integral

$$\begin{aligned} &\left(\prod_{k=1}^{n-1}\prod_{\vec{x}}\int d\phi_k(\vec{x})\right)\left(\prod_{\vec{x}}\left(\frac{a^{d-1}}{2\pi\hbar}\right)^n\left(\frac{2\pi\hbar}{a^d}\right)^{n/2}\right)\times \\ &\quad\times\exp\left\{-\frac{1}{\hbar}\sum_{k=0}^{n-1} a\sum_{\vec{x}} a^{d-1}\left[\frac{1}{2}\left(\frac{\phi_{k+1}(\vec{x})-\phi_k(\vec{x})}{a}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi_k)^2(\vec{x}) + \frac{1}{2}m^2\phi_k^2(\vec{x})\right]\right\}\Bigg|_{\substack{\phi_0=\varphi_i\,,\\ \phi_n=\varphi_f}}\,. \end{aligned} \tag{32}$$

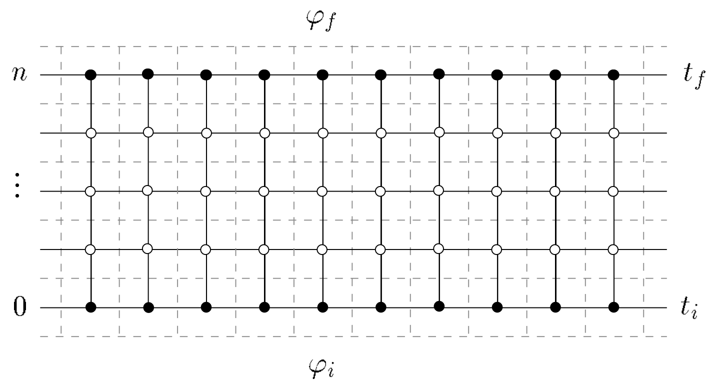

Figure 7: Lattice diagram for path integral on V_{fi}.

In the zeroth and nth layer, ϕ is fixed to the initial and final values φ_i and φ_f respectively, while it is integrated over from layer 1 to $n-1$, weighted by the exponentiated action.

We can make this formula more symmetric with respect to the boundaries t_i and t_f, if we add potential terms to the nth layer, writing

$$\left(\prod_{k=1}^{n-1}\prod_{\vec{x}}\int d\phi_k(\vec{x})\right)\left(\prod_{\vec{x}}\left(\frac{a^{d-2}}{2\pi\hbar}\right)^{n/2}\right)\times$$
$$\times\exp\left\{-\frac{1}{\hbar}\sum_{\vec{x}} a^d\left[\sum_{k=0}^{n-1}\frac{1}{2}\left(\frac{\phi_{k+1}(\vec{x})-\phi_k(\vec{x})}{a}\right)^2+\sum_{k=0}^{n}\left(\frac{1}{2}(\vec{\nabla}\phi_k)^2(\vec{x})+\frac{1}{2}m^2\phi_k^2(\vec{x})\right)\right]\right\}\Bigg|_{\substack{\phi_0=\varphi_i,\\ \phi_n=\varphi_f}}. \quad (33)$$

Clearly, such a change does not affect the continuum limit. We also rewrite the normalization factors: in (33), there are n factors of

$$C_a := \sqrt{\frac{a^{d-2}}{2\pi\hbar}} \quad (34)$$

for each $\vec{x}\in S_a$. We express this in a more geometric fashion by attributing a factor C_a to every spacetime point $x=(\vec{x},t_i+ka)$ for which $\phi_k(\vec{x})$ is integrated over, and by associating a factor $\sqrt{C_a}$ to each point in the initial and final layer:

$$W_a[\varphi_f,t_f;\varphi_i,t_i] := \left(\prod_{\vec{x}\in S_a}\sqrt{C_a}\right)^2\left(\prod_{k=1}^{n-1}\prod_{\vec{x}\in S_a}\int C_a\, d\phi_k(\vec{x})\right)\times$$
$$\times\exp\left\{-\frac{1}{\hbar}\sum_{\vec{x}\in S_a} a^d\left[\sum_{k=0}^{n-1}\frac{1}{2}\left(\frac{\phi_{k+1}(\vec{x})-\phi_k(\vec{x})}{a}\right)^2+\sum_{k=0}^{n}\left(\frac{1}{2}(\vec{\nabla}\phi_k)^2(\vec{x})+\frac{1}{2}m^2\phi_k^2(\vec{x})\right)\right]\right\}\Bigg|_{\substack{\phi_0=\varphi_i,\\ \phi_n=\varphi_f}}. \quad (35)$$

In Fig. 7, this is represented diagrammatically for the case $d=2$: open points stand for an integration over the associated field variable and carry a factor C_a. Boundary points are solid and contribute a factor $\sqrt{C_a}$. For each point there is a mass term in the action, and each link between points gives a term with the corresponding lattice derivative. The dual lattice is drawn shaded.

4.2 General Definition

By applying the same rules to more complicated arrangements of points, we can define a path integral regularization for general volumes V. Let $V\subset\mathbb{R}^d$ be open and its boundary Σ piecewise smooth. We use

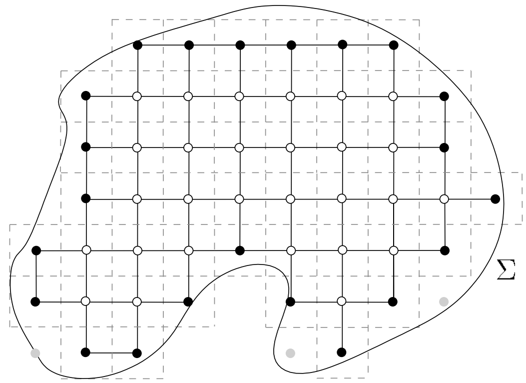

Figure 8: Lattice diagram for a general volume V.

hypercubic lattices

$$L_a := \{x \in a\mathbb{Z}^d \mid -aM \leq |x_\mu| \leq aM\,,\ \mu = 1,\ldots,d\}$$

with lattice constant $a > 0$ and edge length $2aM$, $M \in \mathbb{N}$. e_μ is the unit vector in the μth direction. A lattice point x and a direction μ define a link

$$l \equiv (x, \mu)\,.$$

The associated lattice gradient is

$$\nabla_l f \equiv \nabla_\mu f(x) := \frac{f(x+e_\mu) - f(x)}{a}\,.$$

Given a subset $P \subset L_a$, $l(P)$ denotes the set of links that connect points within P. Let

$$\tilde{V}_a := L_a \cap V$$

be the intersection of V with the lattice. The points of $\tilde{V}_a$ fall into three categories (Fig. 8): we call a point *interior* if it has $2d$ links to points of $\tilde{V}_a$. If a point is linked to less than $2d$ points of $\tilde{V}_a$, but connected to at least one interior point, it is a *boundary point*. The remaining points of $\tilde{V}_a$ have only links to boundary points and we will not use them when representing the path integral on the lattice (they are drawn shaded in Fig. 8). The set of relevant points is therefore

$$V_a := I_a \cup \Sigma_a\,,$$

where I_a and Σ_a denote the set of interior and boundary points respectively.

On the lattice, the path integral becomes a summation over scalar fields $\phi : V_a \to \mathbb{R}$ on V_a. The action consists of contributions from links in $l(V_a)$ and points in V_a:

$$S[\phi, V_a] := \sum_{l \in l(V_a)} a^d \, \frac{1}{2}(\nabla_l \phi)^2 + \sum_{x \in V_a} a^d \, \frac{1}{2} m^2 \phi(x)$$

Given a continuous boundary field φ on Σ, one has to translate it into boundary data for V_a. We do so by defining the discrete boundary field

$$\varphi_a : \Sigma_a \to \mathbb{R}\,, \quad \varphi_a(x) = \varphi(pmd_\Sigma(x))\,,$$

The function pmd_Σ (pmd stands for "point of minimal distance") returns a point on Σ which has minimal distance to x. Now, we have all the necessary notation to give the regularized form of the propagator

$W[\varphi, V]$: we specify it as

$$W_a[\varphi_a, V_a] := \left(\prod_{x\in\Sigma_a} \sqrt{C_a}\right) \left(\int \prod_{x\in I_a} C_a \, d\phi(x)\right) \exp\left(-\frac{1}{\hbar} S[\phi, V_a]\right)\Big|_{\phi|_{\Sigma_a}=\varphi_a}, \tag{36}$$

with factors C_a as in (34). The continuum propagator $W[\varphi, V]$ is then defined by the limit of vanishing lattice constant and infinite lattice size:

$$W[\varphi, V] := \lim_{a\to 0} \lim_{M\to\infty} W_a[\varphi_a, V_a]\,.$$

To simplify notation, we omit the $\lim\limits_{M\to\infty}$ –symbol in the remainder of the text. That is, the limit of infinite lattice size (for constant a) is implicit in all subsequent formulas.

We now make a number of unproven assumptions about the regularization (36):

(A1) The propagator (36) has a continuum limit: that is, there is a non-trivial space $F(\Sigma)$ of boundary fields on Σ such that for each $\varphi \in F(\Sigma)$ the limit

$$W[\varphi, V] := \lim_{a\to 0} W_a[\varphi_a, V_a]$$

is well-defined.

(A2) W reproduces the conventional propagator: for $V_{fi} = \mathbb{R}^{d-1} \times [t_i, t_f]$ and appropriate boundary conditions at spatial infinity,

$$W[(\varphi_f, \varphi_i), V_{fi}] = \langle \varphi_f | e^{-H(t_f - t_i)/\hbar} | \varphi_i \rangle\,.$$

(A3) $W[\varphi, V]$ is translation and rotation invariant: i.e. under an isometry $f : \mathbb{R}^d \to \mathbb{R}^d$,

$$W[\varphi \circ f^{-1}, f(V)] = W[\varphi, V]\,.$$

(A4) There is a functional derivative $\dfrac{\delta}{\delta\varphi}$ on $F(\Sigma)$ whose action on $W[\varphi, V]$ can be approximated as follows:

$$\sum_{x\in\Sigma_a} a^{d-1} \frac{\partial^n W_a}{\partial(a^{d-1}\varphi_a(x))^n}[\varphi_a, V_a] \quad \stackrel{a\to 0}{\longrightarrow} \quad \int_\Sigma d\Sigma(x) \frac{\delta^n W}{\delta\varphi(x)^n}[\varphi, V]\,.$$

To evaluate the path integral (36), it is useful to arrange the field variables from each point in vectors ϕ and write the action as

$$S[\phi, V_a] = \frac{1}{2}\phi \cdot B_a \, \phi + c_a \cdot \phi + d_a\,. \tag{37}$$

The boundary fields φ_a are contained in the vectors c_a and d_a respectively. The action is bounded from below, so for each φ_a, there is at least one solution ϕ_{cl} of the Euclidean equations of motion

$$\frac{\partial S}{\partial \phi}[\varphi_a, V_a] = B_a \, \phi + c_a = 0\,.$$

If B_a is non-degenerate, the solution is unique and one can define the Hamilton function

$$S[\varphi_a, V_a] := S[\phi_{cl}, V_a]$$

for the discrete Euclidean system. We assume, in fact, that

(A5) The matrix B_a is non-degenerate and the Hamilton function $S[\varphi_a, V_a]$ is analytic in φ_a.

The change of variables $\xi := \phi - \phi_{cl}$ renders the action (37) quadratic:

$$S[\xi, V_a] = \frac{1}{2}\xi \cdot B_a\, \xi + S[\varphi_a, V_a]$$

The integral (36) becomes Gaussian and gives

$$W_a[\varphi_a, V_a] = \left(\prod_{x\in\Sigma_a} \sqrt{C_a}\right)\left(\int \prod_{x\in I_a} \sqrt{2\pi}\, C_a\right) \frac{1}{\sqrt{\det B_a}} \exp\left(-\frac{1}{\hbar} S[\varphi_a, V_a]\right). \tag{38}$$

Therefore, by **(A5)**, the regularized kernel $W_a[\varphi_a, V_a]$ must be analytic in φ_a, which will be used in section 5.1 when deriving the Schrödinger equation.

Remark: In **(A1)** and **(A5)** we have formulated the continuum limit in terms of pointwise convergence, i.e. by separate convergence for each boundary field φ in $F(\Sigma)$. According to (38), the field dependence of the regularized kernel resides only in the Hamilton function $S[\varphi_a, V_a]$. The latter converges against the continuum function $S[\varphi, V]$, which is defined pointwise. Thus, it is plausible to assume that the continuum propagator $W[\varphi, V]$, too, is a pointwise function on $F(\Sigma)$. When further developing the formalism, pointwise convergence is likely to be replaced by convergence in a Hilbert space norm or other measures which only distinguish between equivalence classes of boundary fields. For the purpose of this article, however, it is sufficient and simplifies notation if we use convergence on single fields.

5 Generalized Schrödinger Equation

The propagator W depends on a spacetime region V and a field φ specified on the boundary Σ. Thus, as for the Hamilton function in section 3, one can define a deformation derivative: using the same notation as there, we set

$$L_N W[\varphi, V] := \lim_{s\to 0} \frac{W[\varphi^s, V^s] - W[\varphi, V]}{s}.$$

In this section, we derive that

$$\hbar L_N W[\varphi, V] = \left(-H_N[\varphi, \hbar\frac{\delta}{\delta\varphi}, V] + P_N[\varphi, \hbar\frac{\delta}{\delta\varphi}, V]\right) W[\varphi, V], \tag{39}$$

where

$$\begin{aligned} H_N[\varphi, \pi, V] &:= \int_\Sigma d\Sigma\, N_\perp\left(-\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla_\Sigma\varphi)^2 + \frac{1}{2}m^2\varphi^2\right), \\ P_N[\varphi, \pi, V] &:= -\int_\Sigma d\Sigma\, N_{||}\cdot \nabla_\Sigma\varphi\, \pi. \end{aligned}$$

When $V = \mathbb{R}^{d-1} \times [t_i, t_f]$ and $N_\perp|_{t_f} = 1$, $N_\perp|_{t_i} = 0$, $N_{||} = 0$, this yields the ordinary Euclidean Schrödinger equation

$$\left(\hbar\frac{\partial}{\partial t_f} + H[\varphi_f, \hbar\frac{\delta}{\delta\varphi_f}]\right) W[\varphi_f, t_f; \varphi_i, t_i] = 0.$$

The strategy of the derivation: using assumption **(A5)** and rotation invariance **(A3)**, we show that the regularized propagator satisfies a lattice version of equation (39) when V is deformed along flat parts of its boundary. The central step is analogous to the calculation Feynman used when deriving the Schrödinger equation from the path integral of a point particle [14] (see also chap. 4, [17]). Due to **(A1)** and **(A4)**, the

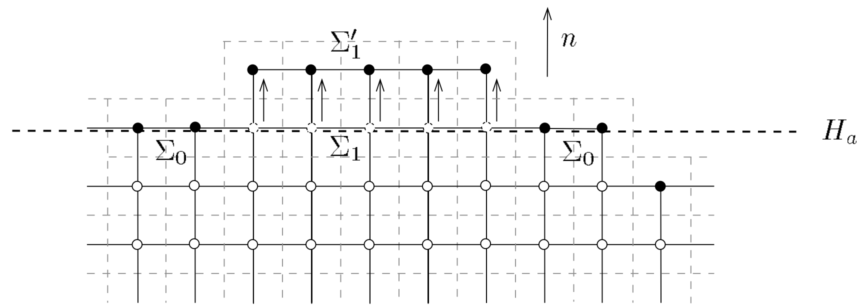

Figure 9: Addition of a single layer.

discrete equations have the continuum limit (39). To cover also the case, when deformations are applied to curved sections of Σ, we approximate Σ by a triangulation, apply (39) to each triangle and let the fineteness of the triangulation go to zero.

For simplicity, the argument is formulated for *bounded* volumes below. The generalization to infinitely extended V is straightforward.

5.1 Discrete Schrödinger Equation

Consider a lattice diagram in which part of Σ_a coincides with a hypersurface H_a of the lattice L_a. Let n denote the normal vector of H_a. The simplest way of modifying such a diagram is to add a $(d-1)$-dimensional layer of points along $H_a \cap \Sigma_a$ (see Fig. 9). The old boundary points adjacent to the layer become interior points. We describe this by a lapse function $N_a : \Sigma_a \to \{0,1\}$ which indicates for any given point of the boundary if a new point will be linked to it or not. Then, the function

$$\sigma_a : \Sigma_a \to L_a\,, \quad x \mapsto x + aN_a(x)\, n$$

is the discrete flow associated to the deformation of the boundary Σ_a. Define the new diagram and its boundary by

$$V'_a := V_a \cup \sigma_a(\Sigma_a)\,, \qquad \Sigma'_a := \sigma_a(\Sigma_a)\,.$$

The set

$$\Sigma_1 := N_a^{-1}(1)$$

is the part of Σ_a which is moved and becomes $\Sigma'_1 := \sigma_a(\Sigma_1)$, while

$$\Sigma_0 := N_a^{-1}(0)$$

remains unchanged. As in the continuous case, we choose the new boundary field to be the pull-forward of the old one, that is,

$$\varphi'_a := \varphi_a \circ \sigma_a^{-1}\,.$$

The resulting path integral is

$$\begin{aligned} W_a[\varphi'_a, V'_a] = & \left(\prod_{x\in\Sigma_0\cup\Sigma'_1} \sqrt{C_a}\right)\left(\prod_{x\in I_a\cup\Sigma_1} \int C_a\, d\phi(x)\right) \times \\ & \times \exp\left\{-\frac{a^d}{\hbar}\left[\sum_{x\in\Sigma_1}\frac{1}{2}\left(\frac{\varphi'_a(\sigma_a(x)) - \phi(x)}{a}\right)^2 + \sum_{l\in l(\Sigma'_1)}\frac{1}{2}(\nabla_l\varphi'_a)^2 + \sum_{x\in\Sigma'_1}\frac{1}{2}m^2{\varphi'_a}^2(x)\right]\right. \\ & \left.\left. -\frac{1}{\hbar}S[\phi, V_a]\right\}\right|_{\phi|_{\Sigma_0} = \varphi_a|_{\Sigma_0}} \end{aligned}$$

(Recall that $l(\Sigma_1')$ is the set of links between points of Σ_1'.) The same can also be written as a convolution with the original propagator:

$$\left(\prod_{x\in\Sigma_1}\int C_a\,d\phi(x)\right)\exp\left\{-\frac{a^d}{\hbar}\left[\sum_{x\in\Sigma_1}\frac{1}{2}\left(\frac{\varphi_a(x)-\phi(x)}{a}\right)^2 + \sum_{l\in l(\Sigma_1)}\frac{1}{2}(\nabla_l\varphi_a)^2+\sum_{x\in\Sigma_1}\frac{1}{2}m^2\varphi_a^2(x)\right]\right\} W_a[(\varphi_a|_{\Sigma_0},\phi),V_a]$$

Following Feynman's derivation of the Schrödinger equation, we introduce new variables

$$\xi(x) := \sqrt{\frac{a^{d-2}}{\hbar}}(\phi(x)-\varphi_a(x))\,, \quad x\in\Sigma_1$$

and get

$$\begin{aligned} W_a[\varphi_a',V_a'] &= \left(\prod_{x\in\Sigma_1}\int \overbrace{C_a\sqrt{\frac{\hbar}{a^{d-2}}}}^{=1/\sqrt{2\pi}}\,d\xi(x)\right)\exp\left\{-\frac{1}{2}\sum_{x\in\Sigma_1}\frac{\xi^2(x)}{a}-\frac{a^d}{\hbar}\left[\sum_{l\in l(\Sigma_1)}\frac{1}{2}(\nabla_l\varphi_a)^2+\sum_{x\in\Sigma_1}\frac{1}{2}m^2\varphi_a^2(x)\right]\right\} \\ &\quad\times W_a\left[\left(\varphi_a|_{\Sigma_0},\varphi_a|_{\Sigma_1}+\sqrt{\hbar/a^{d-2}}\,\xi\right)\,,\,V_a\right] \end{aligned}$$

Next we apply Laplace's method to obtain an asymptotic expansion of this expression (see e.g. chap. 11, [17]): the dominant contribution to the Gaussian integral comes from an a-dependent interval $[-\varepsilon_a,\varepsilon_a]^{|\Sigma_1|}$ around $\xi=0$. ($|\Sigma_1|$ denotes the number of points in Σ_1.) The integral outside is exponentially damped for $a\to 0$ and neglected. Within the interval, one can Taylor expand W_a in ξ and reverse the order of integration and Taylor expansion. To evaluate the integration for each term, the integration range is extended back to its full size: this introduces an error in each term of the sum and convergence is lost, but the expansion is still valid asymptotically for $a\to 0$.

For the integrations and estimates, we use the following formulas:

$$\int_{-\infty}^{\infty} dy\,y^n\,e^{-y^2/2} = \begin{cases} (n-1)(n-3)\cdots 3\cdot 1\cdot\sqrt{2\pi} & , \quad n\geq 0 \text{ and even}\,, \\ 0 & , \quad n \text{ odd}\,, \end{cases} \tag{40}$$

$$\int_{\pm\varepsilon_a}^{\pm\infty} dy\,y^n\,e^{-y^2/2} = O(\varepsilon_a^{n-1}\,e^{-\varepsilon_a^2/2}) \quad \text{as } a\to 0\,. \tag{41}$$

We set $\epsilon_a = 1/a$. Consider first the integral outside the chosen interval:

$$\begin{aligned} &\left(\prod_{x\in\Sigma_1}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}\setminus[-1/a,1/a]} d\xi(x)\right)\exp\left(-\frac{1}{2}\sum_{x\in\Sigma_1}\frac{\xi^2(x)}{a}\right)\exp\left\{-\frac{a^d}{\hbar}\left[\sum_{l\in l(\Sigma_1)}\frac{1}{2}(\nabla_l\varphi_a)^2+\sum_{x\in\Sigma_1}\frac{1}{2}m^2\varphi_a^2(x)\right]\right\} \\ &\quad\times W_a\left[\left(\varphi_a|_{\Sigma_0},\varphi_a|_{\Sigma_1}+\sqrt{\hbar/a^{d-2}}\,\xi\right)\,,\,V_a\right] \end{aligned} \tag{42}$$

The second exponent vanishes in the continuum limit. For W_a, we employ formula (38) and replace $\exp(-S[\ldots,V_a])$ by 1, as the action is positive. The determinant and C_a-factors are together of order

$O(1)$, since, by assumption, (38) approaches a finite continuum limit. Thus, the modulus of (42) is smaller than

$$\left(\prod_{x\in\Sigma_1}\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}\setminus[-1/a,1/a]} d\xi(x)\, e^{-\xi^2(x)/2}\right)\cdot O(1) \quad \overset{(41)}{\leq} \quad O\left(\frac{1}{a}e^{-\frac{|\Sigma_1|}{2a^2}}\right) \quad \text{as } a\to 0\,.$$

In the integral over $[-1/a, 1/a]^{|\Sigma_1|}$, we Taylor expand W_a in ξ:

$$\begin{aligned}&\left(\prod_{x\in\Sigma_1}\frac{1}{\sqrt{2\pi}}\int_{-1/a}^{1/a} d\xi(x)\right)\exp\left(-\frac{1}{2}\sum_{x\in\Sigma_1}\frac{\xi^2(x)}{a}\right)\exp\left\{-\frac{a^d}{\hbar}\left[\sum_{l\in l(\Sigma_1)}\frac{1}{2}(\nabla_l\varphi_a)^2+\sum_{x\in\Sigma_1}\frac{1}{2}m^2\varphi_a^2(x)\right]\right\}\\ &\times\left(W_a[\varphi_a,V_a]+\sum_{x\in\Sigma_1}\frac{\partial W_a}{\partial\varphi_a(x)}\sqrt{\frac{\hbar}{a^{d-2}}}\,\xi(x)+\frac{1}{2}\sum_{x,y\in\Sigma_1}\frac{\partial^2 W_a}{\partial\varphi_a(x)\partial\varphi_a(y)}\frac{\hbar}{a^{d-2}}\,\xi(x)\xi(y)+\ldots\right)\end{aligned}$$

By assumption **(A5)**, W_a is analytic in the field variable, so the Taylor expansion converges uniformly and we are allowed to integrate each term of the series separately. We also set the limits of integration back to plus and minus infinity. This does not affect the *asymptotic* property of the series, since for each term the resulting error is only exponentially small: for example , the linear term yields

$$\left|\left(\prod_{x\in\Sigma_1}\frac{1}{\sqrt{2\pi}}\int_{-1/a}^{1/a} d\xi(x)\right)e^{-\xi^2(x)/2}\sum_{x\in\Sigma_1}a^{d-1}\frac{\partial W_a}{\partial(a^{d-1}\varphi_a(x))}\sqrt{\frac{\hbar}{a^{d-2}}}\,\xi(x)\right| \quad\leq\quad O\left(\frac{|\Sigma_1|}{\sqrt{a^{d-2}}}\,e^{-\frac{1}{2a^2}}\right),$$

because of (41) and **(A4)**. Then, we can use equation (40) to do the Gaussian integration in each term of the asymptotic series. Each integration, that is, each point $x\in\Sigma_1$, leaves an overall factor $\sqrt{2\pi}$. Terms with an uneven number of ξ variables (of the same point) vanish. We obtain

$$\begin{aligned}W_a[\varphi'_a,V'_a] \;\sim\; &\exp\left\{-\frac{a^d}{\hbar}\left[\sum_{l\in l(\Sigma_1)}\frac{1}{2}(\nabla_l\varphi_a)^2+\sum_{x\in\Sigma_1}\frac{1}{2}m^2\varphi_a^2(x)\right]\right\}\\ &\times\left(W_a[\varphi_a,V_a]+\sum_{x\in\Sigma_1}\frac{\hbar}{2}\frac{\partial^2 W_a}{\partial\varphi_a(x)^2}\cdot\frac{1}{a^{d-2}}+\sum_{n=2}^{\infty}\sum_{x\in\Sigma_1}c(n)\frac{\partial^{2n}W_a}{\partial\varphi_a(x)^{2n}}\cdot\frac{1}{(a^{d-2})^n}\right),\end{aligned}$$

where the $c(n)$'s are numerical coefficients. If we write φ_a as the pull-back $\varphi'_a\circ\sigma_a\equiv\sigma_a^*\varphi'_a$ of φ'_a and use also the lapse function N_a, the final result becomes

$$\begin{aligned}W_a[\varphi'_a,V'_a] \;\sim\; &\exp\left\{-\frac{1}{\hbar}\left[\sum_{l\in l(\Sigma_a)}a^{d-1}\,aN_a\,\frac{1}{2}(\nabla_l\sigma_a^*\varphi'_a)^2+\sum_{x\in\Sigma_a}a^{d-1}\,aN_a\frac{1}{2}m^2(\sigma_a^*\varphi'_a)^2(x)\right]\right\}\\ &\times\left\{W_a[\sigma_a^*\varphi'_a,V_a]+\sum_{x\in\Sigma_a}a^{d-1}\,aN_a\left(\frac{\hbar}{2}\frac{\partial^2 W_a}{\partial(a^{d-1}\varphi_a(x))^2}[\sigma_a^*\varphi'_a,V_a]\right.\right.\\ &\left.\left.+\sum_{n=2}^{\infty}a^{d(n-1)}\,c(n)\frac{\partial^{2n}W_a}{\partial(a^{d-1}\varphi_a(x))^{2n}}[\sigma_a^*\varphi'_a,V_a]\right)\right\}. \qquad (43)\end{aligned}$$

Iteration

Suppose now that the deformed set V'_a does not arise from the addition of a single layer, but from a continuous deformation V^s of the orginial volume V. That is, we want to compare $W_a[\varphi_a^s,V_a^s]$ against $W_a[\varphi_a,V_a]$. To make the calculation tractable, we require that the vector field N vanishes outside a neighbourhood U of a boundary point $x\in\Sigma$, and that within U the boundary Σ is flat. Denote this part of Σ by $\Sigma_U:=\Sigma\cap U$.

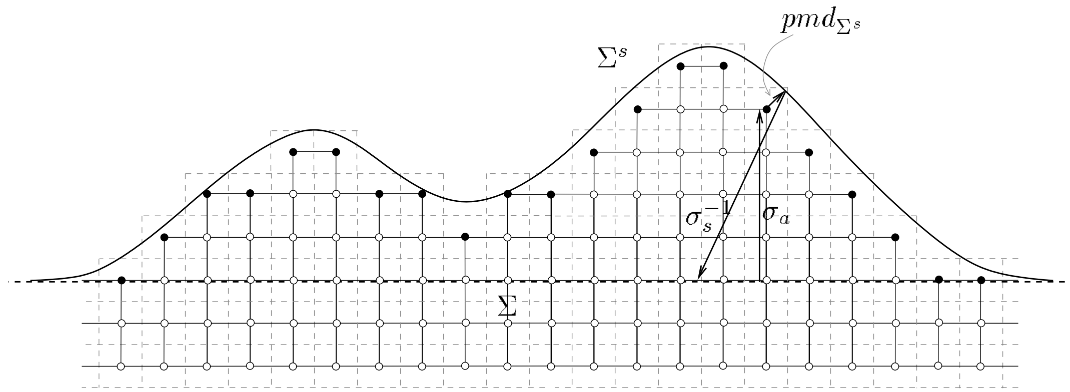

Figure 10: Diagram for V_a^s.

By rotation and translation invariance ((**A3**)), we can orient V such that Σ_U coincides with a hyperplane of the lattice L_a. Let us begin by considering the case where the lapse $N_\perp$ is positive, that is, $V \subset V^s$. For small enough s, the typical diagram for $W_a[\varphi_a^s, V_a^s]$ looks like Fig. 10 (or its higher-dimensional equivalent), where along the normal direction n each point of Σ_a is in one-to-one correspondence with a point of Σ_a^s. (Note that in the limit $s \to 0$, the slope of Σ^s against Σ becomes arbitrarily small.) The new boundary Σ_a^s can be built from Σ_a by repeatedly adding single layers as described previously. Thus, we can iterate formula (43) to obtain a relation between $W_a[\varphi_a^s, V_a^s]$ and $W_a[\sigma_a^*\varphi_a^s, V_a]$ where now, σ_a is the concatenation of all single-step flows. When collecting the various terms of the iteration, the lapse functions for each step add up to the *total* lapse function N_a. We order the result in powers of aN_a and a:

$$\begin{aligned} W_a[\varphi_a^s, V_a^s] &= W_a[\sigma_a^*\varphi_a^s, V_a] + \sum_{x\in\Sigma_a} a^{d-1}\, aN_a \frac{\hbar}{2}\frac{\partial^2 W_a}{\partial(a^{d-1}\varphi_a(x))^2}[\sigma_a^*\varphi_a^s, V_a] \\ &\quad - \frac{1}{\hbar}\left[\sum_{l\in l(\Sigma_a)} a^{d-1}\, aN_a\,\frac{1}{2}(\nabla_l\sigma_a^*\varphi_a^s)^2 + \sum_{x\in\Sigma_a} a^{d-1}\, aN_a \frac{1}{2} m^2 (\sigma_a^*\varphi_a^s)^2(x)\right] \\ &\quad + O(a^d(aN_a)) + O((aN_a)^2) \qquad \text{as } a, s \to 0\,. \end{aligned} \tag{44}$$

Note that the displacement vector aN_a approaches $sN_\perp$ when both a and s become small, i.e.

$$aN_a = sN_\perp + O(s^2) + O(a)\,. \tag{45}$$

If N is normal to Σ_U, the discrete flow σ_a approximates the continuous one, σ_s, and

$$\begin{aligned} \sigma_a^*\varphi_a^s &= \varphi_a^s \circ \sigma_a = \varphi^s \circ pmd_{\Sigma_s} \circ \sigma_a \\ &= \varphi \circ \sigma_s^{-1} \circ pmd_{\Sigma_s} \circ \sigma_a \\ &= \varphi_a + O(a)\,. \end{aligned}$$

In general, N has also a tangential component, so

$$\sigma_a^*\varphi_a^s = \varphi_a - sN_{\|}^\mu \nabla_\mu \varphi_a + O(s^2) + O(a)\,, \tag{46}$$

as can be seen from the arrow diagram in Fig. 10. Plugging (45) and (46) into (44), one arrives at a regularized form of the Euclidean Schrödinger equation:

$$\frac{W_a[\varphi_a^s, V_a^s] - W_a[\varphi_a, V_a]}{s} = \hat{O}_a W_a[\varphi_a, V_a] + O(s) + O(a) + O(a/s)\,, \tag{47}$$

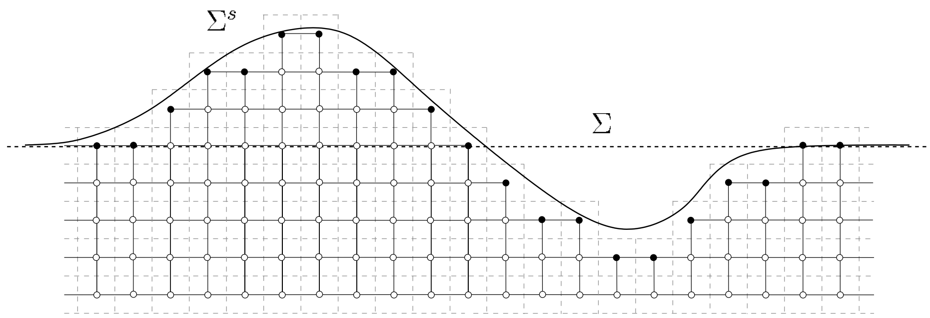

Figure 11: Lapse with positive and negative sign.

where $\hat{O}_a$ is the operator

$$\hat{O}_a \;:=\; -\frac{1}{\hbar}\sum_{x\in\Sigma_a} a^{d-1}\left[N_\perp(x)\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial(a^{d-1}\varphi_a(x))^2}+\frac{1}{2}m^2\varphi_a^2(x)\right)+N_{||}^\mu(x)\nabla_\mu\varphi_a(x)\,\hbar\frac{\partial}{\partial(a^{d-1}\varphi_a(x))}\right] - \frac{1}{\hbar}\sum_{l\in l(\Sigma_a)} a^{d-1}N_\perp(x)\,\frac{1}{2}(\nabla_l\varphi_a)^2 \tag{48}$$

An analogous argument applies to the case of negative lapse $N_\perp$. For mixed diagrams as in Fig. 11, both types of calculations can be combined to give (47) for lapses of arbitrary sign.

5.2 Continuous Schrödinger Equation

Choose N as before, i.e. with support on a neighbourhood U of $x\in\Sigma$ where $\Sigma\cap U$ is flat. We want to show that

$$L_N W[\varphi,V] = \lim_{s\to 0}\frac{W[\varphi^s,V^s]-W[\varphi,V]}{s} = \hat{O}W[\varphi,V]\,, \tag{49}$$

where

$$\hat{O} \;:=\; -\frac{1}{\hbar}\int_\Sigma d\Sigma(x)\left[N_\perp(x)\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\varphi(x)^2}+\frac{1}{2}(\nabla_\Sigma\varphi)^2(x)+\frac{1}{2}m^2\varphi^2(x)\right) + N_{||}(x)\cdot\nabla_\Sigma\varphi(x)\,\hbar\frac{\delta}{\delta\varphi(x)}\right].$$

Stated more explicitly, this means that for any $\epsilon>0$ there is an $s_0>0$ such that

$$\left|\frac{W[\varphi^s,V^s]-W[\varphi,V]}{s}-\hat{O}W[\varphi,V]\right| \;\le\; \epsilon \qquad \text{for all } s<s_0\,. \tag{50}$$

To obtain an upper estimate on the left-hand side, we insert regularized propagators and operators in a suitable way, and then apply the triangle inequality:

$$\begin{aligned}
\text{LHS of (50)} \;=\;& \left|\frac{1}{s}\Big(W[\varphi^s,V^s]-W_a[\varphi_a^s,V_a^s]+W_a[\varphi_a^s,V_a^s]-W_a[\varphi_a,V_a]+W_a[\varphi_a,V_a]-W[\varphi,V]\Big)\right.\\
&\left.-\hat{O}_aW_a[\varphi_a,V_a]+\hat{O}_aW_a[\varphi_a,V_a]-\hat{O}W[\varphi,V]\right|\\
\le\;& \frac{1}{s}\Big|W[\varphi^s,V^s]-W_a[\varphi_a^s,V_a^s]\Big|+\frac{1}{s}\Big|W[\varphi,V]-W_a[\varphi_a,V_a]\Big|\\
&+\Big|\hat{O}W[\varphi,V]-\hat{O}_aW_a[\varphi_a,V_a]\Big|\\
&+\left|\frac{W_a[\varphi_a^s,V_a^s]-W_a[\varphi_a,V_a]}{s}-\hat{O}_aW_a[\varphi_a,V_a]\right|.
\end{aligned}$$

By assumption **(A1)** (existence of the continuum limit), the first two terms become smaller than $\epsilon/4$ when the lattice constant a is smaller than some $a_s > 0$. The partial derivatives and potential terms in (48) approach their continuum analogues as $a \to 0$, so there is also an $a_0 > 0$ such that

$$\left|\hat{O}W[\varphi, V] - \hat{O}_a W_a[\varphi_a, V_a]\right| < \frac{\epsilon}{4} \qquad \text{for all } a < a_0 \,.$$

The regularized Schrödinger equation tells us that for s smaller than some s_0, there is an $a'_s > 0$ such that

$$\left|\frac{W_a[\varphi_a^s, V_a^s] - W_a[\varphi_a, V_a]}{s} - \hat{O}_a W_a[\varphi_a, V_a]\right| < \frac{\epsilon}{4} \qquad \text{for all } a < a'_s \,.$$

Thus, for any $s < s_0$, we can choose $a < \min\{a_s, a_0, a'_s\}$ and the left-hand side of (50) must be smaller than ϵ. □

5.3 Curved Boundaries

As it is based on the lattice equation (47), the previous derivation applies only when flat sections of the boundary Σ are deformed. We do not know how to extend the lattice calculation to the case where both initial and deformed surface are curved. Below we give an argument which circumvents this difficulty, but requires additional assumptions. The idea is to approximate the curved boundary by a triangulation, apply the variation to each of the flat triangles and add up the contributions.

Let T_δ be a triangulation of Σ with fineness δ: that is, when two 0-simplices are connected by a 1-simplex, their metric distance is at most δ. Let $\{\Sigma_\alpha\}$ denote the set of $(d-1)$-simplices $\Sigma_\alpha \subset \Sigma$ of the triangulation. The corner points of each such simplex Σ_α define a $(d-1)$-simplex in $\mathbb{R}^d$ which we call Σ_{Δ_α}. The hypersurface $\Sigma_\Delta := \cup_\alpha \Sigma_{\Delta_\alpha}$ approximates Σ and encloses the volume V_Δ. We can view V as a deformation of V_Δ and find a flow

$$\rho : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\,, \quad (t, x) \mapsto \rho(t, x) \equiv \rho_t(x)$$

such that $\rho_1(V_\Delta) = V$ and $\rho_1(\Sigma_{\Delta_\alpha}) = \Sigma_\alpha$. We equip Σ_Δ with the boundary field $\varphi_\Delta := \rho_1^* \varphi = \varphi \circ \rho_1$, the pull-back of φ under this flow. Motivated by equation (49) for flat surfaces, we assume that the difference between $W[\varphi, V]$ and $W[\varphi_\Delta, V_\Delta]$ is of the order of the volume difference between V and V_Δ:

$$\begin{aligned} W[\varphi, V] &= W[\rho_{1*}\varphi_\Delta, \rho_1(V_\Delta)] \\ &= W[\varphi_\Delta, V_\Delta] + O(|V - V_\Delta|)\,. \end{aligned} \qquad (51)$$

Next we introduce "characteristic" functions $\chi_\alpha : \mathbb{R}^d \to \mathbb{R}$ with the property that

$$\begin{aligned} \chi_\alpha(x) = 1 \quad &\text{for} \quad x \in \Sigma_{\Delta_\alpha}\,, \\ \chi_\alpha(x) = 0 \quad &\text{for} \quad x \in \Sigma_{\Delta_\beta}\,,\ \alpha \neq \beta\,, \\ \text{and} \quad \textstyle\sum_\alpha \chi_\alpha(x) = 1 \ &\text{for all} \ \ x \in \mathbb{R}^d\,. \end{aligned}$$

Using these functions, we can decompose the deformation field N according to

$$N = \sum_\alpha \chi_\alpha N \equiv \sum_\alpha N_\alpha\,.$$

Each component N_α is a discontinuous vector field and gives rise to a discontinuuous flow within $\mathbb{R}^d$. Suppose that by a suitable limiting procedure, one can define L_{N_α} such that equation (49) holds and

$$L_N = \sum_\alpha L_{N_\alpha}\,.$$

Then, equation (51) becomes

$$L_N W[\varphi, V] = \sum_\alpha L_{N_\alpha} W[\varphi_\Delta, V_\Delta] + O(|V - V_\Delta|)\,.$$

By construction, the vector fields N_α are only nonzero on the flat simplices Σ_{Δ_α}. Therefore, our result for flat surfaces (equation (49)) is applicable and yields

$$\begin{aligned} & L_N W[\varphi, V] \\ = \; & \sum_\alpha \left\{ -\frac{1}{\hbar}\int_\Sigma d\Sigma \left[N_{\alpha\perp}\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\varphi^2} + \frac{1}{2}(\nabla_\Sigma \varphi_\Delta)^2 + \frac{1}{2}m^2\varphi_\Delta^2 \right) + N_{||}\cdot\nabla_\Sigma\varphi_\Delta\,\hbar\frac{\delta}{\delta\varphi}\right] W[\varphi_\Delta, V_\Delta] \right\} \\ & + O(|V - V_\Delta|) \\ = \; & -\frac{1}{\hbar}\int_\Sigma d\Sigma \left[N_\perp \left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\varphi^2} + \frac{1}{2}(\nabla_\Sigma\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right) + N_{||}\cdot\nabla_\Sigma\varphi\,\hbar\frac{\delta}{\delta\varphi}\right] W[\varphi, V] + O(|V - V_\Delta|) \end{aligned}$$

In the $\delta \to 0$ limit, $|V - V_\Delta|$ goes to zero and one recovers the generalized Schrödinger equation for curved boundaries.

6 Summary

We have proposed an exact definition for a Euclidean free scalar propagator $W[\varphi, V]$ which "evolves" wavefunctionals of fields along general spacetime domains V. Our main result is a derivation of the evolution equation

$$\hbar L_N W[\varphi, V] = \left(-H_N[\varphi, \hbar\frac{\delta}{\delta\varphi}, V] + P_N[\varphi, \hbar\frac{\delta}{\delta\varphi}, V] \right) W[\varphi, V]\,. \tag{52}$$

This equation describes how $W[\varphi, V]$ varies under infinitesimal deformations of V generated by a vector field N. The variation is given by the action of two operators: one is related to the field Hamiltonian and arises from normal deformations of the boundary $\Sigma = \partial V$. The second operator results from tangential deformations and generalizes the field momentum.

We showed also that the Hamilton function of the classical system satisfies an analogous Hamilton-Jacobi equation

$$L_N S[\varphi, V] = H_N[\varphi, \frac{\delta S}{\delta\varphi}, V] + P_N[\varphi, \frac{\delta S}{\delta\varphi}, V]\,. \tag{53}$$

When the boundary Σ consists of two infinite hyperplanes at fixed times, (52) and (53) reduce to the standard Schrödinger and Hamilton-Jacobi equation in their Euclidean form.

The derivation of eq. (52) is based on assumptions which we consider plausible, but are not proven. Most importantly, we have not shown that the proposed regularization of the propagator has a well-defined continuum limit. A description for converting the Euclidean to a Lorentzian propagator is missing. As described in section 2, we expect that an evolution equation analogous to (52) holds also for Lorentzian propagators. We emphasize that such state evolution may, in general, be non-unitary and nevertheless admit a physical interpretation.

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A Local Form of Hamilton-Jacobi and Schrödinger Equation

In the text we have presented the generalized Hamilton-Jacobi and Schrödinger equation in an integral form. They can be also expressed locally, and below we explain how the two representations are related. The local notation is used in [3] and [7].

Both the Hamilton function S and the propagator W depend on the volume V. The latter is enclosed by the boundary Σ. Consider a parametrization of Σ, i.e. a map

$$x : P \to \Sigma\,, \quad \tau \mapsto x(\tau)$$

from a $(d-1)$-dimensional manifold P to Σ. Provided it is clear on which "side" of Σ the volume lies, one can view S and W as functions of Σ, or equivalently, as functionals of the parametrizing map $\tau \mapsto x(\tau)$. The other variable of S and W is the boundary field $\varphi : \Sigma \to \mathbb{R}$: we may replace it by its pull-back $\tilde{\varphi}$ to P, so that S and W are completely expressed in terms of quantities on the parameter manifold P:

$$\begin{aligned}&\tilde{\varphi}(\tau) = \varphi(x(\tau))\,, \quad \tau \in P\,,\\ &S = S[\tilde{\varphi}(\tau), x(\tau)]\,, \quad W = W[\tilde{\varphi}(\tau), x(\tau)]\,.\end{aligned}$$

In section 3, we defined the deformation derivative L_N which acts by infinitesimal diffeomorphisms and pull-forwards of V and φ respectively. A moment's reflection shows that in the new notation the same effect is achieved by applying a variation

$$\delta x(\tau) = sN(x(\tau))$$

to the function $x(\tau)$ while leaving $\tilde{\varphi}(\tau)$ fixed. Therefore,

$$L_N \equiv \int_P d^{d-1}\tau\, N^\mu(x(\tau)) \frac{\delta}{\delta x^\mu(\tau)}\,. \tag{54}$$

Our explicit result for the Hamilton-Jacobi equation (p. 10, eq. (22)) can be rewritten as

$$\begin{aligned} L_N S[\varphi, V] \;=\; \int_P d^{d-1}\tau\, N^\mu(x(\tau)) \Bigg\{ & n_\mu(x(\tau)) \left[-\frac{1}{2} \left(\frac{\delta S}{\delta \varphi(\tau)} \right)^2 + \frac{1}{2}(\nabla\phi(\tau))^2 + \frac{1}{2} m^2 \phi^2(\tau) + U(\phi(\tau)) \right] \\ & - \nabla_\mu \varphi(\tau) \frac{\delta S}{\delta \varphi(\tau)} \Bigg\}\,, \end{aligned} \tag{55}$$

where on the right-hand side S is a functional of the new variables. Comparison with (54) gives the equation

$$\frac{\delta S}{\delta x^\mu(\tau)} = n_\mu(x(\tau)) \left[-\frac{1}{2} \left(\frac{\delta S}{\delta \varphi(\tau)} \right)^2 + \frac{1}{2}(\nabla\phi(\tau))^2 + \frac{1}{2} m^2 \phi^2(\tau) + U(\phi(\tau)) \right] - \nabla_\mu \varphi(\tau) \frac{\delta S}{\delta \varphi(\tau)}\,. \tag{56}$$

It describes how S behaves under local variations of the boundary Σ. By the same reasoning, we arrive at a local Schrödinger equation for the kernel W:

$$\frac{\delta W}{\delta x^\mu(\tau)} = n_\mu(x(\tau)) \left[-\frac{\hbar^2}{2} \frac{\delta^2 W}{\delta \varphi(\tau)^2} + \frac{1}{2}(\nabla\phi(\tau))^2 + \frac{1}{2} m^2 \phi^2(\tau) \right] - \hbar \nabla_\mu \varphi(\tau) \frac{\delta W}{\delta \varphi(\tau)}\,. \tag{57}$$

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