Preprint typeset in JHEP style - HYPER VERSION

AEI-2003-110
hep-th/0312215

Symmetry breaking, permutation D-branes on group manifolds: boundary states and geometric description

Gor Sarkissian

The Abdus Salam Centre ICTP,
Strada Costiera 11,
34014, Trieste, ITALY
gor@ictp.trieste.it

Marija Zamaklar

MPI/AEI für Gravitationsphysik
Am Mühlenberg 1
14476 Golm, GERMANY
marija.zamaklar@aei.mpg.de

ABSTRACT: We use the permutation symmetry between the product of several group manifolds in combination with orbifolds and T-duality to construct new classes of symmetry breaking branes on products of group manifolds. The resulting branes mix the submanifolds and break part of the diagonal chiral algebra of the theory. We perform a Langrangian analysis as well as a boundary CFT construction of these branes and find agreement between the two methods.

KEYWORDS: group manifolds, boundary states.

Contents

1. Introduction and summary

There are basically two complementary ways of studying D-branes on curved manifolds: the microscopic CFT description and the approach which uses various effective actions (DBI or supergravity). The relation between these two approaches is non-trivial and it is of interest to have both descriptions available. However, many branes constructed using one or the other method are still lacking the alternative description. In some cases, like for classes of D-branes on group manifolds or on coset spaces, both descriptions of branes are available. The aim of this paper is to extend this list to a class of new D-branes, on products of group manifolds. The construction which we present does not in general require these manifolds to be identical, though most of our explicit formulas and examples will be given for these cases.

There are two main tools which are used in our construction: the permutation symmetry between the group manifolds as it was used in [1, 2, 3, 4, 5, 6, 7][1] and the construction of symmetry breaking branes using orbifolds in combination with T-duality, as proposed in [8, 9]. The resulting branes of our construction are *symmetry breaking*, i.e. they preserve less than the diagonal part of the affine algebra, and are *non-factorisable* (or *permutation*) branes, i.e. they non-trivially mix the factor groups.

Let us explain the basic idea of our construction on the simplest example of two identical groups (i.e. $H_0 = H_1 = H$). A generalisation to several groups will be given in some cases in the main text, where we will also make the schematic exposition of the introduction mathematically precise. Our starting point is the *maximally symmetric, permutation* (or *non-factorisable*), brane of [2] and [1]. Let us consider the brane whose worldvolume is given by the submanifold of the $H \times H$ space defined by

$$(i): \quad (g_0, g_1)\Big|_{\text{brane}} = \Big\{ (h_0 f_0 h_1^{-1}, h_1 f_1 h_0^{-1}) \,\Big|\, \forall h_i \in H\,,\ (i = 0, 1) \Big\}\,, \tag{1.1}$$

where g_0 and g_1 denote an element of H_0 and H_1 respectively, $\Big|_{\text{brane}}$ denotes the restriction to the brane surface, and f_0, f_1 are arbitrary, but fixed, elements in H. One refers to this brane as a *maximally symmetric brane* since it preserves the currents

$$\bar{J}_0^a = J_1^a\,, \quad J_0^a = \bar{J}_1^a \quad (a = 1 \ldots \dim H)\,, \tag{1.2}$$

i.e. the full diagonal affine subalgebra $H_{\text{diag}}^{(\bar{1}),(2)} \times H_{\text{diag}}^{(1),(\bar{2})}$ of the total affine algebra $H_{(1)} \times \bar{H}_{(\bar{1})} \times H_{(2)} \times \bar{H}_{(\bar{2})}$. It is easy to see, using the Sugawara construction, that the boundary conditions (1.1) are conformal. The reason why these branes are called *permutation branes*, is because the gluing conditions (1.2) are obtained from the gluing conditions for a direct product of branes by permuting the chiral current of one group with the chiral current of the other group. This non-factorisable character of the gluing conditions leads to an effective geometry of the brane, which turn out to be a submanifold diagonally embedded in the product of the groups.

[1] In the case of a product of non-identical groups, the permutation symmetry is applied to the common subgroups of the factor groups.

The *permutation* symmetry between the subgroups of the target space is also used in writing the boundary state for these branes [2]. Namely, a schematic form of the boundary state for this brane is, in the case when the levels of both groups are the same, given by

$$|A\mu\rangle_{\mathcal{P}} = \sum_{\nu,N,M} c_\mu{}^\nu |\nu, N\rangle_0 \otimes \overline{|\nu, N\rangle}_1 \otimes |\nu, M\rangle_1 \otimes \overline{|\nu, M\rangle}_0 \,. \tag{1.3}$$

Here $c_\mu{}^\nu$ are constants and the subscript on the ket vectors denotes Hilbert spaces in the first and second group manifolds. This brane will play a role similar to that of the A-type brane in the construction of symmetry breaking branes of [9].

Another essential ingredient which we use in this paper is the construction of the symmetry breaking branes on a group H using the relation between the Z_k orbifold of H and its T-dual theory. Namely, it was shown in [9] that an application of T-duality on a Z_k invariant superposition of A-type branes on H leads to a brane which breaks (a fraction of) the affine algebra and preserves the currents

$$J^Y = -\bar{J}^Y\,, \quad J^\alpha = \bar{J}^\alpha\,. \tag{1.4}$$

Here J^Y is the $U(1)_Y$ current which we T-dualise, J^α are all the currents in H which commute with J^Y and the remaining currents are not preserved. This is the reason why these branes are called *symmetry breaking branes*. We will also sometimes call these branes B-type branes. It was furthermore shown in [10] that, using the group theory language, the symmetry breaking branes are described by conjugacy classes multiplied from the left (or right) by the $U(1)_Y$ subgroup of H,

$$(ii): \quad (g)\Big|_{\rm brane} = \Big\{ Lhfh^{-1} \Big| \forall h \in H\,, L = e^{i\alpha Y} \in U(1)_Y \Big\}\,. \tag{1.5}$$

From (1.4) we see that the boundary condition in the $U(1)_Y$ direction is Neumann, unlike in the case of the A-type brane from which this B-brane was derived. Correspondingly, there is an additional $U(1)$ fibre introduced at each point of a conjugacy class (i.e. A-brane) in (1.5).

A schematic form of the boundary states for the B-type branes is [11]

$$|B\mu\rangle = \sum_{\nu\in P/kQ^\vee} |\mu\nu\rangle \otimes |\nu\rangle'\,, \tag{1.6}$$

where P and $Q^\vee$ are the weight and coroot lattices of H, k is the level of the group and $|\mu\rangle'$ is a state in the chiral algebra $\mathcal{A}(T)$ of the maximal torus of H which is of B-type with respect to the J^L current and of A-type with respect to the remaining ones.

Starting from the A-type brane (1.1), we use the logic given above to write the following two non-factorisable, B-type permutation branes on a product of group manifolds,

$$\mathrm{I}: \ (g_0, g_1)\Big|_{\rm brane} = \Big\{ (h_0 f_0 h_1^{-1}, h_1 f_1 h_0^{-1} L_1) \Big| \forall h_i \in H\,, (i=0,1)\ L_1 \in U(1)_{Y_1} \Big\}\,, \tag{1.7}$$

$$\mathrm{II}: \ (g_0, g_1)\Big|_{\rm brane} = \Big\{ (h_0 f_0 h_1^{-1} L_0, h_1 f_1 h_0^{-1} L_1) \Big| \forall h_i, \in H\,,\ L_i \in U(1)_{Y_i},\ (i=0,1) \Big\}\,. \tag{1.8}$$

These branes can also be considered as product of twisted conjugacy classes as suggested in [7]. By checking the invariance of the WZW action with these boundary conditions we show that the symmetries preserved by branes I and II are generated by the currents

$$\mathrm{I}: \quad \bar{J}_0^A = J_1^A\,, \qquad J_0^{Y_1} = -\bar{J}_1^{Y_1}\,, \quad J_0^\alpha = \bar{J}_1^\alpha\,, \tag{1.9}$$

$$\mathrm{II}: \quad \bar{J}_0^{Y_0} = -J_1^{Y_0}\,, \quad \bar{J}_0^\alpha = J_1^\alpha\,, \qquad J_0^{Y_1} = -\bar{J}_1^{Y_1}\,, \quad J_0^\alpha = \bar{J}_1^\alpha\,. \tag{1.10}$$

Here A denotes all generators of H while α and Y are as stated below formula (1.4). We see that the first condition in (1.9) is the same as for the symmetric, permutational brane (1.2) while the second and the third conditions are "permuted" version of the B-type conditions (1.4). For the second brane, all preserved currents are of the "permuted" B-type. Hence to derive the boundary states for these branes we start from the boundary state for the maximally symmetric permutation brane (1.3). In order to reduce the symmetry, we apply the procedure of [8, 9]: perform the Z_k orbifold and a T-duality in the second group (for type I brane) or in both groups (for type II branes). The unbroken currents remain related in the permuted way.

The resulting boundary states will have the following schematic form for the branes (1.7) and (1.8) (in the case of group $H = SU(2)$),

$$|I\rangle \;=\; \sum_{\tilde{\mu},N,n} c_\mu{}^{\tilde{\mu}} |\tilde{\mu}N\rangle_0 \otimes \overline{|\tilde{\mu}N\rangle}_{\bar{1}} \otimes |\tilde{\mu}n\rangle\rangle^{PF}_{1\bar{0}} \otimes |n\rangle\rangle'^{U(1)}_{1\bar{0}} \tag{1.11}$$

$$|II\rangle \;=\; \sum_{\tilde{\mu},n,m} c_\mu{}^{\tilde{\mu}} |\tilde{\mu}n\rangle\rangle^{PF}_{0\bar{1}} \otimes |n\rangle\rangle'^{U(1)}_{0\bar{1}} \otimes |\tilde{\mu}m\rangle\rangle^{PF}_{1\bar{0}} \otimes |m\rangle\rangle'^{U(1)}_{1\bar{0}}\,. \tag{1.12}$$

That is, in the first case, the resulting brane is a product of permuted A-Ishibashi states with permuted B-Ishibashi states, while in the second case it is a product of two permuted B-Ishibashi states.[2] Is is easy to see that the boundary conditions (1.7) and (1.8) are conformal. The total diagonal energy momentum tensor is the same as for the direct product of A and B branes (i.e. two B-branes). The only effect of the boundary conditions is to permute (with respect to the case of a direct product) the way in which the different chiral components of the energy momentum tensor are related to the anti-chiral components.

Something interesting happens when the $U(1)$ groups in (1.8) are embedded in such a way that $L_0 = L_1^{-1}$. In this case the conditions on the currents with indices α are as for the maximally symmetric permutation brane (1.2), the conditions on the currents L are "unpermuted" as

$$\mathrm{III}: \quad \bar{J}_0^Y = J_0^Y\,, \quad J_1^Y = \bar{J}_1^Y\,, \tag{1.13}$$

while the remainder of the currents (with generators which do not commute with Y) get broken. The boundary state for this brane can be deduced from the boundary state for the maximally symmetric permutation brane (1.3). One first decomposes the permuted Ishibashi states $0\bar{1}$ and $1\bar{0}$ as a product of parafermions and $U(1)$ factors, and then "unpermutes" the $U(1)$ Ishibashi states. The resulting boundary state is

$$|\mathrm{III}\,\mu\rangle = \sum_{\tilde{\mu},n} c_\mu{}^{\tilde{\mu}} |\tilde{\mu}n\rangle\rangle^{PF}_{0\bar{1}} \otimes |\tilde{\mu}n\rangle\rangle^{PF}_{1\bar{0}} \otimes |n\rangle\rangle^{U(1)}_{0\bar{0}} \otimes |n\rangle\rangle^{U(1)}_{1\bar{1}}\,, \tag{1.14}$$

[2] For explicit expressions for these boundary states see equations (6.29) and (6.38) in the main text.

i.e. it is a product of permuted parafermionic A-type Ishibashi states with unpermuted $U(1)$ A-type Ishibashi states.

All branes we have considered so far share one common feature: they are all built from bulk CFT states labeled by the same primaries in the first and second group, or equivalently, they are all characterised by one independent label ($f = f_0 f_1$) in the effective description, as we will show. At the level of the boundary state this was reflected through the presence of the permuted parafermionic Ishibashi states, for all boundary states which we listed. The opposite situation occurs if one starts with a *direct* product of conjugacy classes and multiplies them with a diagonally embedded U(1) group [12, 7, 13]

$$\text{IV}: \quad (g_0, g_1)\Big|_{\text{brane}} = \Big\{ (L h_0 f_0 h_0^{-1}, L h_1 f_1 h_1^{-1}) \Big| \forall h_i \in H\,, (i=0,1)\ L \in U(1)_L \Big\}\,. \quad (1.15)$$

In this case the relations between the currents are given by

$$\text{IV}: \quad J_0^Y = -\bar{J}_1^Y\,, \quad \bar{J}_0^Y = -J_1^Y\,, \quad J_0^\alpha = \bar{J}_0^\alpha\,, \quad J_1^\alpha = \bar{J}_1^\alpha\,. \quad (1.16)$$

We see that the effect of multiplying with the diagonally embedded $U(1)$ group is to mix the corresponding $U(1)$ currents in a diagonal way, while keeping the other currents unpermuted. Since in this case mixing between the submanifolds occurs only through the $U(1)$ subgroups, one cannot redefine the group elements so that the brane is characterised by one label. Hence, in contrast to the previous set of branes, this brane is characterised by two *independent* labels f_0 and f_1.

In order to check the correctness of the boundary states presented here, we check all Cardy conditions, and derive the effective geometries of the branes using the boundary states. These are then compared with the corresponding effective geometries derived from the group-theoretical definitions of branes. As required we find agreement between the two approaches. We also briefly discuss the cases of non-identical groups in section 3.1. In these cases, any brane that is not a direct product of branes on subgroups automatically preserves less than the maximal diagonal affine subalgebra.

This paper is organised as follows. In sections 2–5 we present various types of permutation symmetry breaking branes on product of group manifolds. We analyse their properties, derive the expressions for the worldvolume fluxes and for some examples we also present explicit geometries. In the second part of the paper (sections 6 and 7) we construct boundary states for some examples considered in section 4, and reconstruct the effective geometries of the branes from the boundary states. We end with comments in section 8. Finally we attach four appendices, two of which contain technical details for some of the calculations and two of which collect useful formulas.

2. General construction of maximally symmetric, permutation branes

In this section we review the construction of the maximally symmetric, permutation branes (1.1) of [1] and derive some of their properties. A generalisation of this construction to *non-identical* groups leads to a total breaking of some of the diagonal affine symmetries, and will be discussed in section 3.1.

2.1 Definition of the brane

Let us consider a group manifold M, which is a product of $K+1$ copies of a group G: $M = G \times \cdots \times G$ [1]. We define the *maximally symmetric, permutational* brane by the following formula

$$
\begin{aligned}
(g_0, g_1, \cdots, g_K)|_{\text{brane}} = \Big\{ & (h_0 f_0 h_1^{-1}, h_1 f_1 h_2^{-1}, h_2 f_2 h_3^{-1} \cdots h_{K-1} f_{K-1} h_K^{-1}, h_K f_K h_{K+1}^{-1}) \\
& \Big| \quad h_0 = h_{K+1}\,, \forall h_i \in G\,, \quad (i = 1, \cdots, K+1) \Big\}\,.
\end{aligned}
\tag{2.1}
$$

where g_i denotes an element of the i'th copy of G in target space, and $|_{\text{brane}}$ denotes the restriction to the brane surface. It is easy to see that by redefinition of the elements h_i one can always bring an arbitrary brane $(f_0, \ldots, f_K)$ into the form $(f_0 f_1 \cdots f_K, e, \ldots e)$, where e is the identity element.[3] Hence, another, more convenient form of writing the equation (2.1) is

$$
\begin{aligned}
& (g_0, g_1, \cdots, g_K)|_{\text{brane}} = \\
& \Big\{ \big(h_0 f h_0^{-1} g_K^{-1} \cdots g_1^{-1}, g_1, \cdots, g_K\big) \Big| \; f \equiv f_0 f_1 \cdots f_k\,, \forall h_0, g_i \in G\,, \, (i = 1, \cdots, K) \Big\}.
\end{aligned}
\tag{2.3}
$$

The dimension of this brane can easily be determined by looking at the image of (2.1) under the map $m : M = G \times \cdots \times G \to G$, defined by $m(g_0, g_1, \ldots, g_K) = g_0 g_1 \ldots g_K \equiv g$ [1]. The map m maps the brane (2.1) to the conjugacy class

$$
m(g_0, g_1, \ldots, g_k)|_{\text{brane}} = \mathcal{C} = \left\{ C = h_0 f h_0^{-1} | h_0 \in G \right\}. \tag{2.4}
$$

Next, note that the inverse of a point under m is diffeomorphic to G^K. To see this, observe that for any element of the form

$$
h \equiv (f_0 h_1^{-1}, h_1 f_1 h_2^{-1}, h_2 f_2 h_3^{-1}, \ldots, h_K^{-1} f_K)\,, \quad (\forall h_i \in G) \tag{2.5}
$$

the relation $m(h) = m(f_0 \ldots f_K)$ holds. Hence altogether we see that the dimension D of a generic brane $(f_0, \ldots f_K)$ is given by

$$
D = \dim \mathcal{C} + K \, \dim G\,. \tag{2.6}
$$

Equation (2.1) does not fully specify the consistent brane embedding. Namely, in order to claim that the geometric embedding (2.1) solves the DBI action, one needs to turn on

[3] Note that there is a freedom of rigid ("zero mode") motion of the brane on the target space. For example, in the case of two groups, the equation (2.1) can be generalised to

$$
(g_0, g_1)\big|_{\text{brane}} = \left\{ \Big(h_0 f_0 h_1^{-1}, (x h_1 x^{-1}) f_1 (y^{-1} h_0^{-1} y) \Big) | \forall h_0, h_1 \in G \right\}. \tag{2.2}
$$

Here x and y are two arbitrary but *fixed* elements in G, reflecting a freedom in which one can relate elements in the first and second group. Different choices for x,y lead to configurations that are related by the "translations" on a group G, i.e. different choices of coordinate origin. In order to simplify the notation we choose to work in a frame $x = y = e$.

a gauge invariant worldvolume flux. Due to the complexity of the DBI action, we will not determine this flux directly from the equations of motion. Instead, let us recall that in order to have a well defined Lagrangian action of the WZW theory on a world-sheet with boundary, the restriction of the WZW three-form to the D-brane worldvolume has to belong to the trivial cohomology class [14].[4] More precisely, there should exist a globally well-defined two-form $\omega^{(2)}$ on the brane worldvolume, satisfying the equation

$$\omega^{\rm WZ}(g)\big|_{\rm brane} = {\rm d}\omega^{(2)} \,. \tag{2.7}$$

Given the two-form $\omega^{(2)}$, the worldsheet action can be written as

$$S = S(g,k) - \frac{k}{4\pi}\int_D \omega^{(2)} \,, \tag{2.8}$$

$$\begin{aligned} S(g,k) &= \frac{k}{4\pi}\int_\Sigma d^2z {\rm Tr}(\partial_z g \partial_{\bar z} g^{-1}) + \frac{k}{4\pi}\int_B \frac{1}{3}{\rm Tr}(g^{-1}dg)^3 \\ &\equiv \frac{k}{4\pi}\left[\int_\Sigma d^2z L^{\rm kin} + \int_B \omega^{\rm WZ}\right] . \end{aligned} \tag{2.9}$$

Here D is an auxiliary disc (mapped under the map g to the D-brane submanifold) and joined to the string worldsheet Σ along the boundary, completing it to the closed manifold. The manifold B is a three-manifold satisfying the condition $\partial B = \Sigma + D$. The two-form $\omega^{(2)}$ is precisely the antisymmetric part of the matrix appearing in the DBI action [16], [17],

$$S_{\rm DBI} = \int \sqrt{\det(G+\omega^{(2)})} \,, \quad \omega^{(2)} = \mathcal{F} = B + F \,. \tag{2.10}$$

Therefore, in order to determine $\omega^{(2)}$ let us restrict ω^{WZ} to the D-brane surface (2.3). Using the following (Polyakov-Wiegmann) identities

$$L^{\rm kin}(gh) = L^{\rm kin}(g) + L^{\rm kin}(h) - \Big({\rm Tr}\big(g^{-1}\partial_z g \partial_{\bar z} h h^{-1}\big) + {\rm Tr}\big(g^{-1}\partial_{\bar z} g \partial_z h h^{-1}\big)\Big) \,, \tag{2.11}$$

$$\omega^{\rm WZ}(gh) = \omega^{\rm WZ}(g) + \omega^{\rm WZ}(h) - {\rm d}\Big({\rm Tr}\big(g^{-1}{\rm d}g{\rm d}h h^{-1}\big)\Big) \,, \tag{2.12}$$

and the relation (2.4) it is easy to see that

$$\sum_{i=0}^{K}\omega^{\rm WZ}(g_i)\Big|_{\rm brane} = \omega^{WZ}(C) + {\rm d}\left({\rm Tr}\sum_{i=0}^{K-1} g_i^{-1}dg_i d(g_{i+1}\cdots g_K)(g_{i+1}\cdots g_K)^{-1}\right) . \tag{2.13}$$

where we have, as before, $C = g_0 \dots g_K$. In deriving this expression one uses the identity $\omega^{WZ}(g_i^{-1}) = -\omega^{WZ}(g_i)$, which is crucial for the cancellation of the Wess-Zumino terms involving the elements g_i. The first term in equation (2.13) can be rewritten as a total derivative in a parametrisation independent form, using the results of [18]

$$\omega^{\rm WZW}(C) = {\rm d}\left({\rm Tr}\Big(C^{-1}{\rm d}C\frac{1}{1-{\rm Ad}_C}C^{-1}{\rm d}C\Big)\right) . \tag{2.14}$$

[4] Generically, there are also additional global topological restrictions following from the requirement of independence of the action (2.8) of the actual position of the embedding of the auxiliary disk in the group manifold. These lead, for example, to the quantisation condition for the position of the S^2 brane in $SU(2)$. We will not discuss these kind of conditions here. The details can be found, for example, in [15].

Here the operator $(1 - \mathrm{Ad}_g)$ (where Ad_g denotes the adjoint action on G) is invertible when restricted to the vectors tangent to (2.4). Putting all ingredients together, we see that Wess-Zumino three-form reduces to a total derivative on the boundary, as advertised. The expression (2.13), together with (2.14), gives us an expression for a covariant world-volume flux $\omega^{(2)}$ on the worldvolume of the D-brane (2.1). Since this result is expressed only in terms of group elements, it is manifestly *reparametrisation invariant*. Using the parametrisation (2.1), this gauge-invariant form can be written as

$$\omega^{(2)} = \sum_{i=0}^{K} \mathrm{Tr}\left(f_i^{-1} h_i^{-1} \mathrm{d}h_i f_i h_{i+1}^{-1} \mathrm{d}h_{i+1} \right) , \tag{2.15}$$

which will be used in the following sections. Note also that using (2.13) and (2.14) one can determine $\omega^{(2)}$ only up to an exact two form, while the boundary equations of motion fully fix its form. However, even without solving these equations, one can show that $\omega^{(2)}$ given in (2.13) and (2.14) is the only choice compatible with the geometrical symmetries of the brane, explored in the following section.

2.2 Symmetries of the brane

Next we want to determine the symmetries preserved by the brane (2.1). The boundary conditions (2.1) are invariant under any transformation of the form

$$\begin{aligned} g_i &\to g_i k_i^{-1} , \quad & g_{i+1} &\to k_i g_{i+1} , \\ g_0 &\to k g_0 , & g_K &\to g_K k^{-1} , \end{aligned} \quad (k_i, k \in G) , \quad (i = 0, 1, \ldots , (K-1)) , \tag{2.16}$$

which in our parametrisation correspond to the transformations

$$h_{i+1} \to k_i h_{i+1} , \quad h_{k+1} \to k h_{k+1} , \tag{2.17}$$

respectively. We will now show that the full action

$$S = \sum_{i=0}^{k} S(g_i) - \frac{k}{4\pi} \int_D \omega^{(2)} \tag{2.18}$$

with boundary condition (2.1) and $\omega^{(2)}$ given in (2.13) is invariant under the following transformations

$$\begin{aligned} g_i(z, \bar{z}, r) &\to g_i(z, \bar{z}, r) k_{iR}^{-1}(\bar{z}, r) , \\ g_{i+1}(z, \bar{z}, r) &\to k_{iL}(z, r) g_{i+1}(z, \bar{z}, r) , \quad (k_i \in G) , \quad (i = 0, 1, \ldots , K-1) , \\ k_{iL}(z)\big|_{\text{boundary}} &= k_{iR}(\bar{z})\big|_{\text{boundary}} = k_i(\tau) , \end{aligned} \tag{2.19}$$

as well as

$$\begin{aligned} g_0(z, \bar{z}, r) &\to k_L(z) g_0(z, \bar{z}, r) , \\ g_k(z, \bar{z}, r) &\to g_k(z, \bar{z}, r) k_R^{-1}(\bar{z}, r) , \\ k_L(z)\big|_{\text{boundary}} &= k_R(\bar{z})\big|_{\text{boundary}} = k(\tau) , \quad (k \in G) . \end{aligned} \tag{2.20}$$

Here z and $\bar{z}$ are complex coordinates on the boundary ∂B and the coordinate r is parameterising the radial direction in the three-ball B. For fixed i, in order to determine the variation of the action, we only need to consider the following terms

$$S(g_i, g_{i+1}) = S(g_i) + S(g_{i+1}), \tag{2.21}$$

$$\omega^{(2)}(h_{i+1}) = \mathrm{Tr}\left(f_i^{-1}h_i^{-1}\mathrm{d}h_i f_i h_{i+1}^{-1}\mathrm{d}h_{i+1} + f_{i+1}^{-1}h_{i+1}^{-1}\mathrm{d}h_{i+1}f_{i+1}h_{i+2}^{-1}\mathrm{d}h_{i+2}\right). \tag{2.22}$$

The variation of the kinetic and Wess-Zumino terms in the action can be read off from (2.11) and (2.12). Using the fact that, due to the (anti-)holomorphicity, $\omega^{WZ}(k_{iR/L}) = 0$, one deduces that the variation of the Wess-Zumino term reduces to a surface integral over the disc D and the string worldsheet Σ. The integral over the string world sheet is canceled by the corresponding Σ integral coming from the variation of the kinetic term. The remaining integral over the disc is

$$\Delta(S(g_i, g_{i+1})) = -\frac{k}{4\pi}\int_D \mathrm{Tr}\left(k_i^{-1}\mathrm{d}k_i\big(g_i^{-1}\mathrm{d}g_i + \mathrm{d}g_{i+1}g_{i+1}^{-1}\big)\right). \tag{2.23}$$

Substituting $g_i = h_i f_i h_{i+1}^{-1}$ and $g_{i+1} = h_{i+1}f_{i+1}h_{i+2}^{-1}$ we obtain

$$\Delta(S(g_i, g_{i+1})) = \frac{k}{4\pi}\int_D \mathrm{Tr}\left(k_i^{-1}\mathrm{d}k_i(h_{i+1}f_{i+1}h_{i+2}^{-1}\mathrm{d}h_{i+2}f_{i+1}^{-1}h_{i+1}^{-1} - h_{i+1}f_i^{-1}h_i^{-1}\mathrm{d}h_i f_i h_{i+1}^{-1})\right). \tag{2.24}$$

This term is canceled by the variation of the two-form term in (2.8). Computing the change of (2.22) we find

$$\omega^{(2)}(k_i h_{i+1}) - \omega^{(2)}(h_{i+1}) = \mathrm{Tr}\left(k_i^{-1}\mathrm{d}k_i(h_{i+1}f_{i+1}h_{i+2}^{-1}\mathrm{d}h_{i+2}f_{i+1}^{-1}h_{i+1}^{-1} - h_{i+1}f_i^{-1}h_i^{-1}\mathrm{d}h_i f_i h_{i+1}^{-1})\right) \tag{2.25}$$

which cancels (2.24). The proof of the invariance of the action (2.18) under the variation (2.20) is similar.

Having determined the symmetries of the brane (2.1) we can now turn to the question of which bulk currents are preserved by this brane. The invariance of the manifold M under separate left/right group multiplication in each subgroup gets lifted, on the world sheet of a *closed string*, to a *local* infinite-dimensional symmetry group $M(z)\times M(\bar{z})$. The presence of these symmetries implies the existence of the conserved currents $J_i(z) = -\partial g_i g_i^{-1}$ and $\bar{J}_i(\bar{z}) = g_i^{-1}\bar{\partial}g_i$ $(i = 0, 1, \ldots, K)$. As we have seen, the symmetries under separate left/right group multiplication are, in the presence of the worldsheet boundary, reduced to symmetries under *simultaneous* multiplication (2.19) and (2.20). This implies the following relations between the currents,

$$\bar{J}_i^a = J_{i+1}^a\,, \quad (i = 0, \ldots K-1)\,, \tag{2.26}$$

$$J_0^a = \bar{J}_K^a\,, \quad \forall T^a \in \mathrm{Lie(G)}\,. \tag{2.27}$$

We see that all left-moving currents for all groups are identified with the right moving currents of the "neighboring" groups, hence preserving the diagonal subalgebras $G_{i,\overline{i+1}}$ of the affine algebras $G_i \times \bar{G}_{i+1}$. Note however that the number of preserved currents is equal to the dimension of the Lie algebra of the target space, in contrast to the case of non-identical groups that will be discussed in the next section.

3. The symmetry-breaking, permutation branes

Starting from the branes constructed in the previous section, we would now like to discuss various possibilities for deformations of these branes, in such a way that the resulting branes are physically acceptable. All deformations which will be considered lead to breaking of some of the diagonal affine symmetries of (2.1).

3.1 Deformations by restriction to subgroups

One simple way of breaking the symmetries of (2.3) is to impose a restriction on this formula, such that all elements g_i take value in particular subgroups of G. More precisely, let us choose subgroups $H_K \subset H_{K-1} \cdots H_1 \subset H_0 \equiv G$, and consider (2.3) with the restriction that $g_i \in H_i$, $(i = 0, 1, ..., K)$. We choose all H_i to be proper subgroups of G in order to simplify the following analysis. No conceptually new elements appear if some $H_i = G$, although some details of the analysis may be different. Similar type of constructions have previously appeared in [12, 7, 13].

Most of the calculations which were shown in the previous section, like the determination of the expression for $\omega^{(2)}$ or the dimension of the brane, go though in this case with small and straightforward modifications. The only more relevant change involves the question of preserved symmetries. It is easy to see that the restrictions which were imposed change the symmetries of the brane (i.e. the full action) from (2.16) to

$$\begin{aligned} &g_i \to g_i k_i^{-1}\,, \quad g_{i+1} \to k_i g_{i+1}\,, \quad (k_i \in H_{i+1})\,, \, (i = 1, \ldots, K-1) \\ &g_0 \to k g_0\,, \qquad\quad g_k \to g_k k^{-1}\,, \quad (k \in H_K)\,. \end{aligned} \tag{3.1}$$

There could be additional symmetries present, depending on whether there is a subgroup F in G which commutes with all subgroups H_i. If this is the case, then there is an additional symmetry

$$g_0 \to k g_0 k^{-1}\,, \quad g_i \to g_i\,, \quad \forall k \in F\,. \tag{3.2}$$

In order to write down the relations between the currents which follow from (3.1) and (3.2) we go to a basis of the Lie algebra of G adapted to the chain of subgroups H_i. We denote by T^{a_i} the set of all generators of the subgroup H_i, and with T^F the subset belonging to F. Invariance of the modified (2.1) under the transformations (3.1) implies that the following combinations of the right and left affine algebras are preserved,

$$\begin{aligned} \bar{J}_i^{a_{i+1}} &= J_{i+1}^{a_{i+1}}\,, \quad (i = 0, \ldots, K-1)\,, \quad a_i = 1, \ldots, \dim H_i \\ \bar{J}_k^{a_k} &= J_0^{a_k}\,. \end{aligned} \tag{3.3}$$

The symmetry (3.2) furthermore implies the relation

$$J^F = \bar{J}^F\,. \tag{3.4}$$

We see from (3.3) that all left-moving currents for all groups (except the $H_0 \equiv G$) are preserved (via appropriate identification) while the right moving currents $J_i^{\alpha_i}$ for ($\alpha_i =$ dim $H_{i+1} \ldots$ dim H_i) are completely removed.

3.2 Deformation by multiplication with the subgroup

Another way to break some of the affine algebra symmetries was proposed in [9]: the idea was to construct branes by applying a T-duality transformation on a Z_k invariant superposition of A-type (i.e. symmetry-preserving) D-branes. In the Lagrangian formulation this procedure was shown to amount to multiplication of the conjugacy classes (corresponding to the initial A-branes) by $U(1)$ group [10]. We will now apply this logic to the maximally symmetric permutation brane (2.1) in order to generate a new type of symmetry-breaking branes. Since the brane (2.1) has a structure more complicated than that of a conjugacy class, there will be several inequivalent ways in which we can implement this idea.

To illustrate the basic idea let us first consider the simplest case of two identical groups: $M = G \times G$. We define the boundary conditions of the type I brane as

$$\mathrm{I}: \quad (g_0, g_1)\Big|_{\text{brane}} = \left\{ (h_0 f_0 h_1^{-1},\, h_1 f_1 h_0^{-1} L) \Big| \,,\, \forall L \equiv e^{i\alpha Y} \in U(1)_Y \right\} \tag{3.5}$$

where Y is an arbitrary (but fixed) generator in the Cartan subalgebra of G. As before, in order to fully specify the consistent D-brane we need to determine the worldvolume two-form $\omega^{(2)}$. We can reduce this calculation to the one which we did for (2.1) by introducing variables $K_0 = h_0 f_0 h_1^{-1}$ and $K_1 = h_1 f_1 h_0^{-1}$. We have already shown that

$$\omega^{\mathrm{WZ}}(K_0)\Big|_{\text{brane}} + \omega^{\mathrm{WZ}}(K_1)\Big|_{\text{brane}} = \mathrm{d}\omega^{(2)}(h_0, h_1)\Big|_{\text{brane}} \tag{3.6}$$

with $\omega^{(2)}(h_0, h_1)$ given in (2.13). Using (2.12) and the property that $\omega^{\mathrm{WZ}}(L) = 0$ for abelian groups,[5] we further get that

$$\omega^{\mathrm{WZ}}(K_1 L)\Big|_{\text{brane}} = \omega^{\mathrm{WZ}}(K_1)\Big|_{\text{brane}} - \mathrm{Tr}(K_1^{-1}\mathrm{d}K_1 \mathrm{d}L L^{-1})\,. \tag{3.7}$$

Combining (3.6) and (3.7) we finally obtain

$$\omega^{\mathrm{WZ}}(g)\Big|_{\text{brane}} = \mathrm{d}\omega^{(2)}(h_0, h_1, L) = \mathrm{d}\Big(\omega^{(2)}(h_0, h_1) - \mathrm{Tr}(K_1^{-1}\mathrm{d}K_1 \mathrm{d}L L^{-1})\Big)\,. \tag{3.8}$$

To determine the symmetries preserved by the brane we first look for the symmetries preserved by the boundary (3.5):

1. $g_0 \to g_0 k^{-1}$, $g_1 \to k g_1$ for all $k \in G$; under this transformation $h_1 \to k h_1$ and $K_1 \to k K_1$.

2. $g_0 \to k g_0$, $g_1 \to g_1 k^{-1}$ for all $k \in G$, $k \notin U(1)_Y$ and $[k, L] = 0$. Under this transformation $h_0 \to k h_0$. This means that for example, in the case of $G = SU(N+1)$ we get that $k \in SU(N)$ generated with isospin generators commuting with Y.

[5] Actually, this condition can be relaxed to the condition that L in (3.5) is in the maximal torus of G, while the extension of L to a nonabelian subgroup is less clear.

3. $g_0 \to kg_0$, $g_1 \to g_1$ for all $k \in U(1)_Y$. Under this transformation $h_0 \to kh_0$ and $L \to kL$.

4. $g_0 \to g_0$, $g_1 \to g_1 k$ for all $k \in U(1)_Y$. Under this transformation $L \to Lk$.

When extending these transformations to transformations of the action (as in equations (2.19) and (2.20)) one can show that the *full* action (2.8) is invariant separately under the transformations 1 and 2. On the other hand, only the following *combination* of the transformations 3 and 4 is a real symmetry of the full action:

3'. $g_0 \to kg_0$, $g_1 \to g_1 k$ where $k \in U(1)_Y$. Under this transformation $h_0 \to kh_0$, $L \to kLk$.

We give details of all of these calculation in appendix A.1. The set of symmetries listed above implies that the D-brane (3.5) preserves the following set of currents,

$$\begin{aligned} \bar{J_0}^a &= J_1^a\,, \qquad \forall T^a \in \mathrm{Lie(G)} && (3.9)\\ J_0^a &= \bar{J_1}^a\,, \qquad \forall T^a \in \mathrm{Lie(G)} \quad \text{s.t.} \quad [\mathrm{T^a}, \mathrm{Y}] = 0\,, && (3.10)\\ J_0^Y &= -\bar{J}_1^Y\,. && (3.11) \end{aligned}$$

We see that multiplication of the second group with the $U(1)_L$ subgroup leads to a removal of some of the currents present in the symmetric brane (2.27) and, as expected, also flips the sign of the current in the Y-direction.

3.3 Generalised symmetry-breaking branes

The number of preserved affine symmetries can be further reduced by implementing the procedure from the previous section on both groups in $M = G \times G$. More precisely, let us consider the brane

$$\mathrm{II}: \quad (g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1} L_0, \quad h_1 f_1 h_0^{-1} L_1), \tag{3.12}$$

where L_0, L_1 belong to two *different* $U(1)$ groups in M: $L_0, \in U(1)_{Y_0}$, $L_1, \in U(1)_{Y_1}$, $L_0 = e^{i\beta Y_0}$, $L_1 = e^{i\alpha Y_1}$. It is easy to show that this brane preserves the currents (3.10). On the other hand, the equations (3.9) and (3.11) get modified in an obvious manner,

$$\begin{aligned} \bar{J_0}^a &= J_1^a\,, \qquad \forall T^a \in \mathrm{Lie(G)} \quad \text{s.t.} \quad [\mathrm{T^a}, \mathrm{Y_1}] = 0\,, && (3.13)\\ J_0^{Y_1} &= -\bar{J}_1^{Y_1}\,, \quad \bar{J}_0^{Y_0} = -J_1^{Y_0}\,. && (3.14) \end{aligned}$$

A special case occurs when both groups in M are multiplied by the same $U(1)$ group (i.e. when we take $Y_0 = Y_1$ and $\alpha = -\beta$),

$$\mathrm{III}: \quad (g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1} L, \quad h_1 f_1 h_0^{-1} L^{-1})\,. \tag{3.15}$$

In this case the action is again invariant under the symmetries 1 and 2 from the previous section (with the restriction in 1 to those k which commute with the $U(1)$ group). These symmetries lead to the preserved currents (3.10) and (3.13). The symmetry 3', however, is replaced by

3.” $g_0 \to k g_0 k^{-1}$, $g_1 \to g_1$, $k \in U(1)_Y$. Under this transformation $h_0 \to k h_0$ and $L \to k^{-1} L$.

4.” $g_0 \to g_0$, $g_1 \to k g_1 k^{-1}$, $k \in U(1)_Y$. Under this transformation $h_1 \to k h_1$, and $L \to kL$.

The details of the proof of the invariance of the full action under these transformations are given in appendix A.2. These new symmetries imply the following additional relations between the currents,

$$J_0^Y = \bar{J}_0^Y \,, \tag{3.16}$$

$$\bar{J}_1^Y = J_1^Y \,. \tag{3.17}$$

We see that the left and right currents in the (same) Y direction in both groups are separately related in each group. This is different from the situation with a diagonal identification of the currents which occurred in the case of non-identical groups. In that case the left current from one group (in Y direction) was related to the right current in the other group (3.14).

4. Some examples of non-factorisable branes in $AdS_3 \times S^3 \times S^3 \times S^1$ space

In this section we will analyse the geometry of the branes constructed in the previous sections for several explicit examples in the $AdS_3 \times S^3 \times S^3 \times S^1$ space. These effective geometries of branes will be matched in the subsequent sections with the geometry arising from the boundary state construction of branes.

Maximally symmetric, permutation branes:

Let us first consider an explicit example of the maximally symmetric permutation (non-factorisable branes) (2.1) for the case in which the group $G = SU(2)$. In this case the general formula (2.1) reduces to

$$(g_0, g_1)\Big|_{\text{brane}} = (h_0 f_0 h_1^{-1},\, h_1 f_1 h_0^{-1}) \,. \tag{4.1}$$

The preserved currents are

$$J_0^a + \bar{J}_1^a = 0 \,, \quad J_1^a + \bar{J}_0^a = 0 \quad (a = 1, 2, 3) \,. \tag{4.2}$$

The general expression for two form $\omega^{(2)}$ given in (2.13) reduces to

$$\omega^{(2)} = \mathrm{Tr}\left(h_0^{-1} \mathrm{d} h_0 (f_0 h_1^{-1} \mathrm{d} h_1 f_0^{-1} - f_1^{-1} h_1^{-1} \mathrm{d} h_1 f_1) \right) . \tag{4.3}$$

In order to write down the geometry of the brane we will use the coordinates given in (D.5) and those given in (D.8) for $SU(2)$. The image of a generic brane under the multiplication of elements of the first and second group is the conjugacy class (2.4). Therefore all elements on the brane surface satisfy the condition

$$\begin{aligned}\mathrm{Tr}(g_0g_1) = \mathrm{Tr}(f_0f_1) &= \cos\theta_0\cos\theta_1\cos(\tilde{\phi}_0+\tilde{\phi}_1) - \sin\theta_0\sin\theta_1\cos(\phi_0-\phi_1)\\ &= \cos\psi_0\cos\psi_1 - \sin\psi_0\sin\psi_1\sin\xi_0\sin\xi_1\cos(\eta_0-\eta_1)\\ &\quad + \sin\psi_0\sin\psi_1\cos\xi_0\cos\xi_1 = \text{const.}\,, \quad |\text{const}| \leq 1\,.\end{aligned} \tag{4.4}$$

When $f_0f_1 \neq e$, the conjugacy class (2.4) is two dimensional and (4.4) is the only equation for the brane surface. In other words, the brane is five dimensional and it is obvious from equation (2.4) that it is topologically equal to $S^2 \times S^3$; at each point of the image of the map m given in (2.4) there is an $SU(2) \sim S^3$ fibre [1]. However, when $f_0f_1 = e$, the embedding equations (4.4) are replaced by the stronger set of conditions $g_0 = g_1^{-1}$, which in the coordinates (D.5) and (D.8) can be written as

$$\tilde{\phi}_1 = -\tilde{\phi}_0\,, \quad \theta_1 = \theta_0\,, \quad \phi_1 = \phi_0 \pm \pi\,. \tag{4.5}$$

Here the sign in the last term depends on whether $-\pi \leq \phi_0 \leq 0$ or $0 \leq \phi_0 \leq \pi$ respectively. The equations (4.5) tell us that the brane is a three sphere embedded in the diagonal and symmetric way between the two $SU(2)$ groups. The two-form (4.3) vanishes in this case, in agreement with the observation of [1] that any Lie subgroup of the Lie group is totally geodesic submanifold.

The symmetry-breaking brane of type I:

Next we want to determine the geometry of the type I brane (3.5) on an $SU(2) \times SU(2)$ manifold,

$$(g_0, g_1)\Big|_{\text{brane}} = (h_0f_0h_1^{-1}, h_1f_1h_0^{-1}L), \tag{4.6}$$

where we will take L to be of the form $L = e^{i\alpha\frac{\sigma_3}{2}}$. In this case, the preserved currents (3.9)–(3.11) reduce to

$$J_0^3 = -\bar{J}_1^3\,, \quad \bar{J}_0^a = J_1^a\,, \quad (a = 1,2,3)\,. \tag{4.7}$$

Under the map m of (2.4), the type I brane gets mapped to the conjugacy class multiplied by the $U(1)_{\sigma_3}$ group: $\hat{g} \equiv g_0g_1 = h_0f_0f_1h_0^{-1}e^{i\alpha\frac{\sigma_3}{2}} \equiv \mathcal{C}L$. In what follows we will always denote with hats those quantities which appear in a product of group elements from the first and the second group. The geometry of the image can be determined as follows [19]. Using the fact that $\mathrm{Tr}\,\mathcal{C} = \mathrm{Tr}f_0f_1 = \text{const} = 2\cos\psi_0$ we can write

$$\mathrm{Tr}\left(\hat{g}e^{-i\alpha\frac{\sigma_3}{2}}\right) = 2\cos\psi_0\,. \tag{4.8}$$

From here we see that the element $\hat{g}$ belongs to the image of the brane surface if and only if there is a $U(1)$ element ($e^{i\alpha\frac{\sigma_3}{2}}$) such that the equation (4.8) is satisfied. So let us determine for which $\hat{g}$ this equation admits solutions for α. Denoting with $\hat{\theta}$, $\hat{\tilde{\phi}}$ and $\hat{\phi}$ the coordinates of $\hat{g}$ in the parametrisation given in (D.5), the equation (4.8) takes the form

$$\cos\hat{\theta}\cos(\hat{\tilde{\phi}} - \frac{\alpha}{2}) = \cos\psi_0\,, \tag{4.9}$$

or equivalently,

$$0 \leq \cos^2(\hat{\hat{\phi}} - \frac{\alpha}{2}) = \frac{\cos^2\psi_0}{\cos^2\hat{\theta}} \leq 1\,. \tag{4.10}$$

Hence, equation (4.10) can be solved for α only when $\cos^2\hat{\theta} \geq \cos^2\psi_0$, or equivalently when

$$\cos\hat{\hat{\theta}} \geq \cos 2\psi_0\ , \quad \hat{\hat{\theta}} = 2\hat{\theta}\,. \tag{4.11}$$

We see that the image of the brane is a three-dimensional surface defined by the inequality (4.11). To determine the geometry of the full brane, let us denote the Euler angles for elements in g_0 and g_1 with "0" and "1" indices. Then the $\hat{\hat{\theta}}$ and $\hat{\hat{\phi}}$ angles of their product are given by [20]

$$\cos\hat{\hat{\theta}} = \cos\tilde{\theta}_0\cos\tilde{\theta}_1 - \sin\tilde{\theta}_0\sin\tilde{\theta}_1\cos(\chi_1+\varphi_0)\,, \tag{4.12}$$

$$e^{i\hat{\hat{\phi}}} = \frac{e^{i\frac{\chi_0+\varphi_1}{2}}}{\cos\frac{\hat{\hat{\theta}}}{2}}\left(\cos\frac{\tilde{\theta}_0}{2}\cos\frac{\tilde{\theta}_1}{2}e^{i\frac{\chi_1+\varphi_0}{2}} - \sin\frac{\tilde{\theta}_0}{2}\sin\frac{\tilde{\theta}_1}{2}e^{-i\frac{\chi_1+\varphi_0}{2}}\right)\,. \tag{4.13}$$

Substituting the expression for $\hat{\hat{\theta}}$ in the equation for the image of the brane, we see that a generic brane (3.5) is six dimensional and given by the inequality

$$\cos\hat{\hat{\theta}} = \cos\tilde{\theta}_0\cos\tilde{\theta}_1 - \sin\tilde{\theta}_0\sin\tilde{\theta}_1\cos(\chi_1+\varphi_0) \geq \cos 2\psi_0\,. \tag{4.14}$$

As before, the previous discussion was valid in the cases for which $f_0 f_1 \neq e$. If $\psi_0 = 0$, the conjugacy class $\mathcal{C}$ is a point and the total brane is four dimensional, given by the relations

$$\tilde{\theta}_0 = \tilde{\theta}_1, \quad \chi_1 + \varphi_0 = \pi\,. \tag{4.15}$$

The symmetry-breaking brane of type II:

As explained before, the symmetries preserved by the brane of type I can be broken further by multiplying its first term by a $U(1)$ subgroup,

$$(g_0, g_1)\Big|_{\text{brane}} = \left(h_0 f_0 h_1^{-1} e^{i\beta\frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{i\alpha\frac{\sigma_3}{2}}\right)\,. \tag{4.16}$$

Here we have taken both $U(1)$ groups to be along the same generator, but we take them to be parametrised by two *independent* parameters α and β. The symmetries of brane I, given in (4.7) are now reduced to

$$J_0^3 = -\bar{J}_1^3\,, \quad J_1^3 = -\bar{J}_0^3\,. \tag{4.17}$$

Note that the following equation holds

$$\mathrm{Tr}\left(g_0\, e^{-i\beta\frac{\sigma_3}{2}}\, g_1 e^{-i\alpha\frac{\sigma_3}{2}}\right) = \mathrm{Tr}\left(f_0 f_1\right)\,. \tag{4.18}$$

Using the same arguments which, in the previous case, led to the inequality (4.14), one now concludes that

$$\cos\tilde{\theta}_0\cos\tilde{\theta}_1 - \sin\tilde{\theta}_0\sin\tilde{\theta}_1\cos(\chi_1+\varphi_0-\beta) \geq \cos 2\psi_0\,, \tag{4.19}$$

where $\mathrm{Tr}(f_0 f_1) = 2\cos\psi_0$. As before, the elements g_0 and g_1 will belong to the brane surface if and only if this inequality admits a solution for the parameter β. This will happen if and only if the maximum of the left hand side of (4.19) is larger than $\cos 2\psi_0$. It is easy to see that this maximum is equal to $\cos(\tilde{\theta}_0 - \tilde{\theta}_1)$. Therefore, the generic brane (4.16) is six dimensional and given by an inequality

$$\cos(\tilde{\theta}_0 - \tilde{\theta}_1) \geq \cos 2\psi_0 \,. \tag{4.20}$$

When $\psi_0 = 0$ the brane is five dimensional and given by the equation $\tilde{\theta}_0 = \tilde{\theta}_1$.

The symmetry-breaking brane of type III:

This brane is derived from the previous one by imposing the restriction that the parameters α, and β to satisfy the relation $\alpha = -\beta$

$$(g_0, g_1)\Big|_{\text{brane}} = \left(h_0 f_0 h_1^{-1} e^{i\alpha\frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{-i\alpha\frac{\sigma_3}{2}}\right). \tag{4.21}$$

The elements g_0 and g_1 belong to the brane surface if the following equation admits a solution for the parameter α,

$$\mathrm{Tr}\left(g_0 e^{-i\alpha\frac{\sigma_3}{2}} g_1 e^{i\alpha\frac{\sigma_3}{2}}\right) = 2\cos\hat{\psi}_0 \,. \tag{4.22}$$

Using the formulae (4.12) and (4.13) we can rewrite this equation as

$$\cos\frac{\hat{\Theta}}{2}\cos(\gamma/2 - \xi/2 - \tilde{\phi}_0 - \tilde{\phi}_1) = \cos\hat{\psi}_0 \,, \tag{4.23}$$

where

$$\cos\hat{\Theta} = \cos\tilde{\theta}_0\cos\tilde{\theta}_1 - \sin\tilde{\theta}_0\sin\tilde{\theta}_1\cos\gamma \,, \tag{4.24}$$

and we have introduced new labels $\gamma = \chi_1 + \varphi_0 - \alpha$ and $\xi = \hat{\hat{\phi}} - \frac{\chi_0+\varphi_1}{2}$. The variables ξ and γ are related to each other by the equation

$$e^{i\frac{\xi}{2}} = \frac{1}{\cos\frac{\hat{\Theta}}{2}}\left(\cos\frac{\tilde{\theta}_0}{2}\cos\frac{\tilde{\theta}_1}{2}e^{i\frac{\gamma}{2}} - \sin\frac{\tilde{\theta}_0}{2}\sin\frac{\tilde{\theta}_1}{2}e^{-i\frac{\gamma}{2}}\right). \tag{4.25}$$

Hence the brane consists of those points for which equation (4.23) admits a solution for α. For $\psi_0 = 0$ there are again additional constraints, which imply that in this case the brane is four dimensional and given by the equations

$$\tilde{\theta}_0 = \tilde{\theta}_1, \quad \tilde{\phi}_0 + \tilde{\phi}_1 = \pi \,. \tag{4.26}$$

Restriction to the subgroups:

As was explained in the previous section, the restriction to subgroups of G for different factors in (2.1) leads to total breaking of some of the affine symmetries. In particular, let us consider an $SU(2)\times U(1)$ manifold and a brane

$$(g_0, g_1)\Big|_{\text{brane}} = \left(h_0 f_0 h_0^{-1} e^{-i\eta\frac{\sigma_3}{2}}, e^{i\eta\frac{\sigma_3}{2}}\right), \tag{4.27}$$

where $h_0, f_0 \in SU(2)$. Following the steps elaborated on around formula (4.12) one can deduce that the embedding of this brane in the case in which $f_0 \neq e$ is given by

$$\cos\theta\cos(\tilde{\phi} - \frac{\eta}{2}) = \cos(\psi_0)\,, \tag{4.28}$$

where $2\cos\psi_0 = \mathrm{Tr}\, f_0$, while θ and $\tilde{\phi}$ are coordinates on the S^3 in the parametrisation given in (D.5). The difference with respect to the previous cases is that η in this case is a coordinate in the $U(1)$ subgroup and not a parameter. Hence the brane surface is defined by equation (4.28) and it is three dimensional. When $f = e$ the brane is a one-dimensional $U(1)$ brane, diagonally wrapping a two torus which is the direct product of $U(1)_{\sigma_3} \times U(1)$.

Let us also consider a more complicated example of the restricted (2.1) branes on an $M = SU(2) \times SU(2) \times U(1)$ manifold. The brane is given by

$$(g_0, g_1, g_2)\Big|_{\text{brane}} = \left\{ \left(h_0 f h_1^{-1},\, h_1 h_0^{-1} e^{-i\eta\frac{\sigma_3}{2}},\, e^{i\eta\frac{\sigma_3}{2}}\right) \Big| h_0, h_1, f \in SU(2),\, g_2 \in U(1) \right\}, \tag{4.29}$$

with the two form $\omega^{(2)}$ given by (2.13) and (2.14). The conserved currents are

$$\begin{aligned} J_0^3 &= \bar{J}_2\,, \\ \bar{J}_0^a &= J_1^a \quad (a = 1,2,3)\,, \\ \bar{J}_1^3 &= J_2\,. \end{aligned} \tag{4.30}$$

To determine the geometry of this brane, we note that the first two factors in (4.29) define the brane of type I (4.6), which is connected with the $U(1)$ factor in a diagonal way, as in (4.27). When $f \neq e$, the brane is six dimensional with the geometry given in Euler coordinates by

$$\mathrm{Tr}(g_0 g_1 g_2) = \cos\hat{\theta}\cos(\hat{\tilde{\phi}} - \frac{\eta}{2}) = \cos\psi_0\,, \tag{4.31}$$

where $\hat{\theta}$ and $\hat{\tilde{\phi}}$ are defined in equations (4.12) and (4.13), respectively. In the special case in which $f = e$, the brane equation is $g_0 = (g_1 g_2)^{-1}$ which, when written in components, gives

$$\varphi_0 + \chi_1 = \pi\,, \quad \tilde{\theta}_0 = \tilde{\theta}_1\,, \quad \pi + \chi_0 + \varphi_1 = -\eta\,. \tag{4.32}$$

So we see that in this special case the brane is four dimensional.

5. Boundary states for maximally symmetric, permutation branes

In this and the following sections we will present boundary states for the D-branes considered in the previous sections. We will always work in the closed string channel, and in cases of several groups we will choose the levels of all groups to be the same. To set up the scene, we start the discussion by reviewing the construction of the boundary states for maximally symmetric, permutation branes presented in [2]. Using the constructed boundary states we then calculate the effective geometries of these branes, recovering the classical results from the previous sections. In the section 6, we then continue by constructing the boundary states and effective geometries for cases of symmetry-breaking non-factorisable branes.

5.1 Construction of the boundary state

To construct the boundary states for maximally symmetric, permutation branes on $SU(2)_k \times SU(2)_k$ manifold, one starts with the boundary state for a direct product of two maximally symmetric branes on $SU(2)$. These branes preserve the diagonal affine algebras in both groups *separately*

$$J_0^a + \bar{J}_0^a = 0\,, \quad J_1^a + \bar{J}_1^a = 0\,, \quad a = 1,2,3\,. \tag{5.1}$$

The boundary state is described by the tensor product of the corresponding Cardy states for the first and the second groups

$$|a_0, a_1\rangle = |a_0\rangle_C^{SU(2)_0} \otimes |a_1\rangle_C^{SU(2)_1}\,, \tag{5.2}$$

where the boundary states for each group are of the standard form

$$|a_i\rangle_C^{SU(2)_i} = \sum_{j_i} \frac{S_{a_i j_i}}{\sqrt{S_{0 j_i}}} |A, j_i\rangle\rangle_u^{SU(2)_i}\,, \quad (i = 0,1)\,, \tag{5.3}$$

and the Ishibashi states are given by

$$|A, j_i\rangle\rangle_u^{SU(2)_i} = \sum_N |j_i, N\rangle_i \otimes \overline{|j_i, N\rangle}_i\,, \quad (i = 0,1)\,. \tag{5.4}$$

As usual $S_i{}^j$ is the matrix of a modular transformation $\tau \to -\frac{1}{\tau}$ and $|j_i, N\rangle_i$ is an orthonormal basis of the irreducible representation j_i of $SU(2)_k$, $j_i = (0, \frac{1}{2}, 1, ...\frac{k}{2})$. We will use the following notation for the boundary states in the rest of the paper: the subscript u will indicate that the Ishibashi state is formed from the left and right states of the same theory, the subscript τ will denote that the state is formed from different theories, and a superscript will indicate which Hilbert spaces are used. All Cardy states will be denoted with $|\rangle_C$.

As discussed in section 2, one can construct the symmetry preserving, permutational brane (4.1) using a permutation symmetry ($\mathcal{P}(h_0, h_1) = (h_1, h_0)$) to twist the product of two conjugacy classes. In this simple case, the twisting changes the the relation between the currents (5.1) to

$$J_0^a + \bar{J}_1^a = 0\,, \quad \bar{J}_0^a + J_1^a = 0\,. \tag{5.5}$$

The preserved currents (5.5) imply that the allowed Ishibashi states for the permutational branes (denoted with the subscript $\mathcal{P}$) are of the form

$$|j_0, j_1\rangle\rangle_{\mathcal{P}} = |j_0\rangle\rangle_\tau^{SU(2)_0 \times SU(2)_{\bar{1}}} \otimes |j_1\rangle\rangle_\tau^{SU(2)_1 \times SU(2)_{\bar{0}}} \tag{5.6}$$

where

$$|j_0\rangle\rangle_\tau^{SU(2)_0 \times SU(2)_{\bar{1}}} = \sum_N |j_0, N\rangle_0 \otimes \overline{|j_0, N\rangle}_1\,, \tag{5.7}$$

$$|j_1\rangle\rangle_\tau^{SU(2)_1 \times SU(2)_{\bar{0}}} = \sum_M |j_1, M\rangle_1 \otimes \overline{|j_1, M\rangle}_0\,. \tag{5.8}$$

Note that not all of Ishibashi states (5.6) can be used to construct the boundary state. This is because the boundary state is part of a closed string Hilbert space, and hence, since the bulk partition function is diagonal, only Ishibashi states with $j_0 = j_1$ will be allowed. Notice however that the indices M and N in (5.7) and (5.8) are independent. The complete boundary state is a linear combination of allowed Ishibashi states. The requirement that Cardy's consistency conditions hold restrict the allowed linear combination to

$$|a\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} |j,j\rangle\rangle_{\mathcal{P}} = \sum_j \frac{S_{aj}}{S_{0j}} \sum_{N,M} |j,N\rangle_0 \otimes \overline{|j,N\rangle}_1 \otimes |j,M\rangle_1 \otimes \overline{|j,M\rangle}_0 \,. \tag{5.9}$$

The proof of the Cardy conditions for this brane is given in appendix B. Note also that although the permutational brane lives on a product of two manifolds, the boundary state is characterised by a *single* primary a, in contrast to the untwisted brane (5.2). This fact is in agreement with the statement that the brane (2.1) can always be put into the form (2.3), characterised by a *single* group element $g_0 g_1$.

5.2 The effective geometry of the brane

Given a boundary state, the shape of the brane can be deduced by considering the overlap of the boundary state with the localised bulk state $|\vec{\theta}\rangle$, with $\vec{\theta}$ denoting the three $SU(2)$ angles in some coordinate system [21, 9, 22]. As we will see, the boundary state wave function over the configuration space of all localised bulk states peaks precisely at those states which are localised at positions derived by the effective methods in the previous sections. In the large k limit, the eigen-position bulk state is given by

$$|\vec{\theta}\rangle = \sum_{j,m,m'} \sqrt{2j+1}\mathcal{D}^j_{mm'}(\vec{\theta})|j,m,m'\rangle \,, \tag{5.10}$$

where $\mathcal{D}^j_{mm'}$ are the Wigner $\mathcal{D}$-functions:

$$\mathcal{D}^j_{mm'} = \langle jm|g(\vec{\theta})|jm'\rangle, \quad \langle jm|jm'\rangle = \delta_{m,m'} \tag{5.11}$$

where $|jm\rangle$ are a basis for the spin j representation of $SU(2)$. To calculate the overlap with the boundary state, we will need the knowledge of S-matrix of $SU(2)$ at level k,

$$S_{aj} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2a+1)(2j+1)\pi}{k+2}\right) . \tag{5.12}$$

In the large-k limit the ratio of S-matrix elements appearing in the boundary state simplifies to

$$\frac{S_{aj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)} \sin[(2j+1)\hat{\psi}] \,, \tag{5.13}$$

where, to shorten the notation, we have introduced $\hat{\psi} = \frac{(2a+1)\pi}{k+2}$. Using these results, the overlap between the boundary state and the localised bulk state becomes

$$\langle\vec{\theta}_0, \vec{\theta}_1|a\rangle_{\mathcal{P}} \sim \sum_{j,m,n} \frac{(k+2)}{\pi} \sin[(2j+1)\hat{\psi}]\mathcal{D}^j_{nm}(g_0(\vec{\theta}_0))\mathcal{D}^j_{mn}(g_1(\vec{\theta}_1)) \,. \tag{5.14}$$

To simplify this expression we need the identity

$$\sum_m \mathcal{D}^j_{nm}(g_0(\vec{\theta}_0))\mathcal{D}^j_{mn'}(g_1(\vec{\theta}_1)) = \mathcal{D}^j_{nn'}(g_0(\vec{\theta}_0)g_1(\vec{\theta}_1))\,, \tag{5.15}$$

which follows from the fact that the matrices $\mathcal{D}^j_{nm}$ form a representation of the group. Finally, one needs the property of the Wigner D-functions that $\sum_n \mathcal{D}^j_{nn}(g) = \frac{\sin(2j+1)\psi}{\sin\psi}$, where ψ is the angle of the standard metric (D.8) and defined by the relation $\mathrm{Tr}g = 2\cos\psi$ (or in our case $\mathrm{Tr}(g_0 g_1) = 2\cos\psi$). The overlap (5.14) becomes

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle_{\mathcal{P}} \sim \frac{k+2}{\pi\sin\psi}\sum_j \sin[(2j+1)\hat{\psi}]\sin[(2j+1)\psi] \tag{5.16}$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle_{\mathcal{P}} \sim \frac{k+2}{4\sin\psi}\delta(\psi-\hat{\psi})\,. \tag{5.17}$$

Hence we see that the brane wave function is localised on $\psi =$ const. bulk states, which is indeed the same relation as the one obtained in the effective approach (4.4).

6. Boundary states for symmetry breaking branes

Starting from a direct product of maximally symmetric branes and using a permutation symmetry of the theory, we have in the previous section generated maximally symmetric permutation branes. In this section we will use these branes and apply the technique of [9] to generate symmetry breaking, permutation branes of type I and type II. In section 6.1 we will first review the essential steps of the MMS construction, which we will then apply in section 6.2 and 6.3. Finally, in section 6.4, we will use the permutation symmetry between the $U(1)$ subgroups of the $SU(2)\times SU(2)$ group in order to derive the boundary state for brane III, starting from the direct product of two $SU(2)$, A-type branes.

6.1 Background material

Let us start by reviewing the T-duality between a Lens space and the $SU(2)$ theory. Geometrically, a Lens space is obtained by quotienting the group manifold by the right action of the subgroup Z_k of the $U(1)$, and in the Euler coordinates it corresponds to the identification $\varphi\sim\varphi+\frac{4\pi}{k}$. In terms of the $SU(2)$ WZW model this is the orbifold $SU(2)/Z^R_k$, where Z^R_k is embedded in the right $U(1)$. The partition function for this theory is

$$Z = \sum_j \chi^{SU(2)}_j(q)\chi^{PF}_{jn}(\bar{q})\psi^{U(1)}_{-n}(\bar{q}) \tag{6.1}$$

and coincides with the one for the $SU(2)$ group, up to T-duality. This relation enables one to construct new D-branes in the $SU(2)$ theory starting from the known ones. As a first step one constructs the brane in the Lens theory. As is usual for orbifolds, this is achieved by summing over images of D-branes under the right Z_k multiplications. Performing then the

T-duality on the Lens theory brings us back to the $SU(2)$ theory and maps the orbifolded brane to a new $SU(2)$ brane.

As warm-up exercise let us recall how this procedure works in the case of a *single* $SU(2)$ group [9]. Our starting point is a maximally symmetric A-brane, preserving the symmetries (5.1). If we shift the brane by the right multiplication with some element $\omega^l = e^{\frac{2\pi li}{k}\sigma_3}$ of the Z_k^R group, then the symmetries preserved by this brane are

$$J^a + \omega^l \bar{J}^a \omega^{-l} = 0\,, \quad (a = 1, 2, 3)\,, \tag{6.2}$$

while the brane is described by the Cardy state with rotated Ishibashi state

$$|A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{6.3}$$

Summing over the images one obtains a Z_k^R invariant state, present in the Lens theory

$$\sum_{l=0}^{k} |A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_{l=0}^{k} \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{6.4}$$

To compute the sum of the Ishibashi states on the right-hand side, one next uses the orbifold decomposition of $SU(2)_k$

$$SU(2)_k = (\mathcal{A}^{PF(k)} \otimes U(1)_k)/Z_k\,. \tag{6.5}$$

This decomposition implies that Ishibashi states for the maximally symmetric A-brane (5.4) can be written as

$$|A, j\rangle\rangle^{SU(2)} = \sum_{n=1}^{2k} \frac{1 + (-1)^{2j+n}}{2} |A, j, n\rangle\rangle_u^{PF} \otimes |A, n\rangle\rangle_u^{U(1)}\,, \tag{6.6}$$

where

$$|A, j, n\rangle\rangle_u^{PF} = \sum_N |j, n, N\rangle \otimes \overline{|j, n, N\rangle}\,, \tag{6.7}$$

and

$$|Ar\rangle\rangle_u^{U(1)} = \exp\left[\sum_{n=1}^{\infty} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right] \sum_{l \in Z} |\frac{r + 2kl}{\sqrt{2k}}\rangle \otimes \overline{|\frac{r + 2kl}{\sqrt{2k}}\rangle}\,, \tag{6.8}$$

are the A-type Ishibashi states for the parafermion and $U(1)_k$ theories. If the Z_k^R subgroup lies in the $U(1)$ group appearing in the decomposition (6.5), then under the action of element $\omega^l \in Z_k^R$ the expression (6.6) transform as

$$|A, j\rangle\rangle^{SU(2)} \to \sum_{n=1}^{2k} \frac{1 + (-1)^{2j+n}}{2} \omega^{ln} |A, j, n\rangle\rangle_u^{PF} \otimes |A, n\rangle\rangle_u^{U(1)}\,. \tag{6.9}$$

Hence summing over images projects onto the Z_k^R-invariant Ishibashi states for which n is restricted to the two values 0 and k. Performing T-duality, flips the sign of the right moving $U(1)$ sector and one gets a B-type Ishibashi state of the original $SU(2)$ theory,

$$|B,j\rangle\rangle^{SU(2)} =$$
$$\left[\frac{1+(-1)^{2j}}{2}|A,j,0\rangle\rangle_u^{PF}\otimes|B,0\rangle\rangle_u^{U(1)}+\frac{1+(-1)^{2j+k}}{2}|A,j,k\rangle\rangle_u^{PF}\otimes|B,k\rangle\rangle_u^{U(1)}\right], \quad (6.10)$$

where

$$|Br\rangle\rangle_u^{U(1)} = \exp\left[-\sum_{n=1}^{\infty}\frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right]\sum_{l\in Z}|\frac{r+2kl}{\sqrt{2k}}\rangle\otimes\overline{|-\frac{r+2kl}{\sqrt{2k}}\rangle}\,, \quad (6.11)$$

is a B-type Ishibashi state of $U(1)_k$ theory satisfying the Neumann boundary conditions. Knowing the T-dual expression of the (6.4) allows one to write down the boundary state for the B-type brane

$$|B,a\rangle_C^{SU(2)} = \sum_{j\in Z}\frac{\sqrt{k}S_{aj}}{\sqrt{S_{0j}}}|Aj,0\rangle\rangle_u^{PF}\otimes(|B0\rangle\rangle_u^{U(1)}+\eta|Bk\rangle\rangle_u^{U(1)})\,. \quad (6.12)$$

where $\eta=(-1)^{2a}$. In deriving this expression one uses the field identification rule $(j,n)\sim(k/2-j,k+n)$ and the following property of the matrix of modular transformation (5.12)

$$S_{a,k/2-j} = (-1)^{2a}S_{aj}\,. \quad (6.13)$$

To derive the symmetries preserved by the B-brane, one observes from (6.2) that a Z_k^R invariant superposition of the A-branes preserves only the current $J^3+\bar{J}^3$ and breaks all other currents; namely, any two Z_k^R images only have this preserved current in common. Performing further T-duality in the $\bar{J}^3$ direction flips the relative sign between the two terms in this current and hence implies that the only current preserved by the B-brane is

$$J^3-\bar{J}^3 = 0\,. \quad (6.14)$$

We are now ready to apply this procedure to the brane (5.9) in order to generate symmetry breaking permutation branes. Applying the MMS procedure to one of the two $SU(2)$ groups will lead to brane I, while applying it to both groups will lead to the brane of type II. In this derivation we will need the following permuted Ishibashi states in the $U(1)_k\times U(1)_k$ and $PF_k\times PF_k$ theories.

The *permuted* Ishibashi states satisfy the conditions

$$J_0^3\pm\bar{J}_1^3=0\,,\quad J_1^3\pm\bar{J}_0^3=0\,. \quad (6.15)$$

and can be written as

$$|r\rangle\rangle_{\tau\pm}^{U(1)_0\times U(1)_{\bar{1}}} = \exp\left[\pm\sum_{n=1}^{\infty}\frac{\alpha^0_{-n}\tilde{\alpha}^1_{-n}}{n}\right]\sum_{l\in Z}|\frac{r+2kl}{\sqrt{2k}}\rangle_0\otimes\overline{|\pm\frac{r+2kl}{\sqrt{2k}}\rangle}_1 \quad (6.16)$$

$$|r'\rangle\rangle_{\tau\pm}^{U(1)_1\times U(1)_{\bar{0}}} = \exp\left[\pm\sum_{n=1}^{\infty}\frac{\alpha^1_{-n}\tilde{\alpha}^0_{-n}}{n}\right]\sum_{l'\in Z}|\pm\frac{r'+2kl'}{\sqrt{2k}}\rangle_1\otimes\overline{|\frac{r'+2kl'}{\sqrt{2k}}\rangle}_0\,. \quad (6.17)$$

Here the signs in (6.15) are both taken to be plus or both minus. Note that in all these expressions there is a sum over the l_0, l, l' labels, in order to preserve the full extended $U(1)_k$ symmetry algebra. For example, all states formed from (6.16) will preserve not only the first symmetry in (6.15) but also the conditions $\Gamma^+_{(0)} \pm \bar{\Gamma}^+_{(\bar{1})} = 0$ and $\Gamma^-_{(0)} \pm \bar{\Gamma}^-_{(\bar{1})} = 0$.

As for the abelian case, for the product of two parafermion theories $(\mathcal{A}_0^{PF(k)} \times \mathcal{A}_1^{PF(k)})$ we can define two kinds of permuted states

$$|j_1, n_1\rangle\rangle_\tau^{PF_0 \times PF_{\bar{1}}} = \sum_N |j_1, n_1, N\rangle_0 \otimes \overline{|j_1, n_1, N\rangle}_1 \,, \tag{6.18}$$

$$|j_2, n_2\rangle\rangle_\tau^{PF_1 \times PF_{\bar{0}}} = \sum_M |j_2, n_2, M\rangle_1 \otimes \overline{|j_2, n_2, M\rangle}_0 \,. \tag{6.19}$$

Here $j_i \in Z/2$, $n_i \in Z$ satisfy the constraint $2j + n = 0 \mod 2$ and an equivalence relation $(j, n) \sim (k/2 - j, k + n)$.

6.2 Boundary states for symmetry breaking type I branes

We now want to construct the boundary state for the brane of type I given in (1.7),

$$(g_0, g_1)\Big|_{\text{brane}} = \left(h_0 f_0 h_1^{-1}, h_1 f_1 h_0^{-1} e^{i\alpha \frac{\sigma_3}{2}}\right). \tag{6.20}$$

Recall that, as we have derived before using the Langrangian approach, this brane preserves the currents

$$J_0^3 \,-\, \bar{J}_1^3 = 0\,, \tag{6.21}$$

$$\bar{J}_0^a \,+\, J_1^a = 0\,, \quad (a = 1, 2, 3)\,. \tag{6.22}$$

To construct the boundary state, our starting point is the maximally symmetric permutation brane (5.9) which preserves the symmetries (5.5). In order to reduce these symmetries down to (6.21), we will now show that one should apply the procedure described in the previous section to the second $SU(2)$ group in which the permutation brane lives. Namely, let us shift the brane (5.9) by multiplying it from the right with an element $\omega^l_{(2)} = e^{\frac{2\pi l i}{k}\sigma_3}$ of the Z_k^R subgroup of the second $SU(2)$ group. The shifted brane preserves the symmetries

$$J_0^a \,+ \omega^l_{(2)} \bar{J}_1^a \omega^{-l}_{(2)} = 0\,, \tag{6.23}$$

$$\bar{J}_0^a \,+\, J_1^a = 0\,, \qquad (a = 1, 2, 3)\,, \tag{6.24}$$

and is given by the Cardy state

$$|a\rangle_C^{\omega^l_{(2)}} = \sum_j \frac{S_{aj}}{S_{0j}} |j\rangle\rangle_\tau^{SU(2)_1 \times SU(2)_{\bar{0}}} \otimes |j\rangle\rangle_\tau^{\omega^l_{(2)}}\,, \tag{6.25}$$

where

$$|j\rangle\rangle_\tau^{\omega^l_{(2)}} = \sum_N |j, N\rangle_0 \otimes (\omega^l_{(2)} \overline{|j, N\rangle}_{\bar{1}})\,. \tag{6.26}$$

As in the previous section, summing over the images and performing the T-duality in the right sector, will reduce the first set of currents (6.23) down to (6.21), as desired.

As far as the boundary state is concerned, summing over images will not touch the first Ishibashi state in (6.25) but will project the second Ishibashi state down to the Z_k^R invariant components. Using the permuted version of the decomposition (6.6)

$$|j\rangle\rangle^{SU(2)_0\times SU(2)_{\bar{1}}} = \sum_{n=1}^{2k} \frac{1+(-1)^{2j+n}}{2} |j,n\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}} \otimes |n\rangle\rangle_{\tau+}^{U(1)_0\times U(1)_{\bar{1}}} \tag{6.27}$$

and applying T-duality to it, one obtains the permuted B-type Ishibashi state of the initial $SU(2)$ theory,

$$\begin{aligned} |B,j\rangle\rangle_\tau^{0\bar{1}} = & \frac{1+(-1)^{2j}}{2} |j,0\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}} \otimes |0\rangle\rangle_{\tau-}^{U(1)_0\times U(1)_{\bar{1}}} \\ & + \frac{1+(-1)^{2j+k}}{2} |j,k\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}} |k\rangle\rangle_{\tau-}^{U(1)_0\times U(1)_{\bar{1}}} . \end{aligned} \tag{6.28}$$

Here the permutation $U(1)$ and the permutation parafermion Ishibashi states are given in formulas (6.16) and (6.18). Using this expression the Cardy state for a new brane can be written as:

$$|a\rangle_C^{(1)} = \sqrt{k} \sum_j \frac{S_{aj}}{S_{0j}} |j\rangle\rangle_\tau^{SU(2)_1\times SU(2)_{\bar{0}}} \otimes |B,j\rangle\rangle_\tau^{0\bar{1}} . \tag{6.29}$$

Note also that since the boundary state (6.29) is "derived" from the maximally symmetric boundary state (5.9), it is characterised with a *single* primary j as was the case for the brane (5.9). This is again related to the fact that in the effective description (6.20), there is only one independent parameter ($f \equiv f_0 f_1$).[6]

To check the consistency of the proposed boundary state, one should check, as usual, the Cardy condition. Since we are in a theory which admits several different types of branes, one should in principle check these conditions for the type I brane with any of the other branes in the spectrum. We have done the calculation involving two branes of type I, with one of type I and a permutational brane, and with a brane which is direct product of two $SU(2)$ A-branes. The tree-level amplitude between two Cardy states for two type I branes reduces, after the S-modular transformation reduces, to

$$Z_{a_1a_2} = \sum_{r,j',j''} \sum_{n_1,n_2} N^r_{a_1a_2} N^{j'}_{rj''} \chi_{j'}(q) \chi_{j'',n_1}(q) \psi_{n_2}(q) \frac{1+(-1)^{n_1+n_2}}{2} , \tag{6.30}$$

hence satisfying the Cardy requirement. The annulus amplitude between the type I and the maximally symmetric permutation brane (5.9) reduces, after the S-modular transformation, to

$$Z_{a_1a_2} = \sum_{r,j',j''} \sum_{n_1} N^r_{a_1a_2} N^{j'}_{rj''} \chi_{j'}(q) \chi_{j'',n_1}(q) \frac{q^{1/48}}{\prod_m (1-q^{m-1/2})} , \tag{6.31}$$

while the one between states of type I and the brane (5.2) reduces to

$$Z_{a,(a_0a_1)} = \sum_{r,j'} \sum_n N^r_{a_0a_1} N^{j'}_{ra} \chi_{j',n}(q^{1/2}) \frac{(q^{1/2})^{1/48}}{\prod_m (1-(q^{1/2})^{m-1/2})} . \tag{6.32}$$

[6] This can be easily be seen by changing coordinates as $h_0 \to h_0 f_1^{-1}$.

Here the factor $\frac{q^{1/48}}{\prod_m (1-q^{m-1/2})}$ is the partition function of a scalar with mixed Neumann-Dirichlet type boundary conditions. The details of calculations of (6.30), (6.31) and (6.32) can be found in the appendix B.

We will now show that the boundary state (6.29) reproduces the effective brane geometry (4.14). In the large k limit the second term in (6.28) can be ignored. As in section 5 one should compute the overlap $\langle \vec{\theta}_0, \vec{\theta}_1 | a\rangle_C^{(1)}$. We will again use the formula (5.10), but taking into account that the matrix $\mathcal{D}^{(1)}$ derived for the first group has left index 0 and the right index m, whereas the $\mathcal{D}^{(2)}$ matrix derived for the second group has left index m and the right index 0. Therefore, the overlap is again given by formula (5.14), but with n set to zero. Using furthermore (5.15) we arrive at the equation

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a\rangle_C^{(1)} \sim \sum_j \frac{k^{3/2}}{\pi} \sin[(2j+1)\hat{\psi}]\, \mathcal{D}^j_{00}(g_0(\vec{\theta}_0) g_1(\vec{\theta}_1)) \tag{6.33}$$

Next we will need the relation between the Wigner D-functions and the Legendre polynomials $P_j(\cos\tilde{\theta})$ given by $\mathcal{D}^j_{00} = P_j(\cos\tilde{\theta})$, as well as the formula for the generating function for Legendre polynomials

$$\sum_n t^n P_n(x) = \frac{1}{\sqrt{1-2tx+t^2}}\,. \tag{6.34}$$

Using these expressions equation (6.33) can be simplified to

$$\langle \vec{\theta}_0, \vec{\theta}_1 | a\rangle_C^{(1)} \sim \frac{\Theta(\cos\hat{\tilde{\theta}} - \cos 2\hat{\psi})}{\sqrt{\cos\hat{\tilde{\theta}} - \cos 2\hat{\psi}}}\,, \tag{6.35}$$

where Θ is the step function. This indeed coincides with the expression for the effective geometry (4.14).

6.3 Boundary states for symmetry breaking type II branes

Let us now turn to the type II brane (1.8)

$$(g_0, g_1)\Big|_{\text{brane}} = \left(h_0 f_0 h_1^{-1} e^{i\beta\frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{i\alpha\frac{\sigma_3}{2}}\right), \tag{6.36}$$

which preserves the currents

$$J_0^3 - \bar{J}_1^3 = 0\,, \quad \bar{J}_0^3 - J_1^3 = 0\,. \tag{6.37}$$

This brane has a structure which is very similar to the type I brane. It can be derived from this brane by applying the described procedure (with right cosetting) to the *first* $SU(2)$ group in which brane I lives. As for brane I, this procedure will reduce the currents (6.21) and (6.22) down to (6.37). At the level of the boundary state, the Ishibashi state in the $0\bar{1}$ sector will remain unchanged, while the $\bar{0}1$ Ishibashi state (5.7) will be projected down to a $Z^R_{k,1}$ invariant state (where subscript 1, indicates that this action is taken in the first $SU(2)$ group). Finally, applying the T-duality we obtain the boundary state

$$|a\rangle_C^{(2)} = k\sum_j \frac{S_{aj}}{S_{0j}} |B,j\rangle\rangle_\tau^{0\bar{1}} \otimes |B,j\rangle\rangle_\tau^{1\bar{0}} \tag{6.38}$$

where $|B,j\rangle\rangle_\tau^{1\bar{0}}$ is defined as in (6.28) with 0 and 1 exchanged, and the coefficients in the linear combination are fixed by the Cardy condition. For even k the tree-level amplitude between the states (6.38) reduces to

$$Z_{a_1a_2} = \sum_{r,j',j''}\sum_{n_1,n_2}\sum_{n_3,n_4} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j',n_1}(q)\chi_{j'',n_3}(q)\psi_{n_2}(q)\psi_{n_4}(q) \times \frac{(1+(-1)^{n_1+n_2})(1+(-1)^{n_3+n_4})}{4}\,. \quad (6.39)$$

For an odd k (6.38) can be simplified and written as

$$|a\rangle_C^{(2)} = k\sum_j \frac{S_{aj}}{S_{0j}}\left[\frac{1+(-1)^{2j}}{2}|j,0\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}}\otimes|j,0\rangle\rangle_\tau^{PF_1\times PF_{\bar{0}}}\otimes|0\rangle\rangle_{\tau-}^{U(1)_0\times U(1)_{\bar{1}}}\otimes|0\rangle\rangle_{\tau-}^{U(1)_1\times U(1)_{\bar{0}}}\right.$$
$$\left.+\frac{1+(-1)^{2j+k}}{2}|j,k\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}}\otimes|j,k\rangle\rangle_\tau^{PF_1\times PF_{\bar{0}}}\otimes|k\rangle\rangle_{\tau-}^{U(1)_0\times U(1)_{\bar{1}}}\otimes|k\rangle\rangle_{\tau-}^{U(1)_1\times U(1)_{\bar{0}}}\right]. \quad (6.40)$$

The tree level amplitude between the states (6.40) is

$$Z_{a_1a_2} = \sum_{r,j',j''}\sum_{n_1,n_2}\sum_{n_3,n_4} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j',n_1}(q)\chi_{j'',n_3}(q)\psi_{n_2}(q)\psi_{n_4}(q)\frac{1+\eta(-1)^{n_2+n_4}}{4} \quad (6.41)$$

where $\eta = (-1)^{2a_1+2a_2}$. The Cardy condition is satisfied in this case because, after taking into account field identification in the parafermionic sector, one can show that each state in the sum appears twice, and therefore all states appear with integer coefficient. The calculations leading to (6.39) and (6.41) follow closely to that of for (6.30) outlined in the appendix B.

To derive the effective geometry of this brane, one follows the same arguments we as presented for the type I brane. The overlap with the localised bulk probe is given by

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle_C^{(2)} \sim \sum_j \frac{k^2}{\pi}\sin[(2j+1)\hat{\psi}]P_j(\cos\tilde{\theta}_0)P_j(\cos\tilde{\theta}_1)\,. \quad (6.42)$$

Using now the formula [20]

$$P_j(\cos\tilde{\theta}_0)P_j(\cos\tilde{\theta}_1) = \frac{1}{\pi}\int_{|\tilde{\theta}_0-\tilde{\theta}_1|}^{\tilde{\theta}_0+\tilde{\theta}_1} P_j(\cos\theta)\frac{\sin\theta d\theta}{\sqrt{[\cos\theta-\cos(\tilde{\theta}_0+\tilde{\theta}_1)][\cos(\tilde{\theta}_0-\tilde{\theta}_1)-\cos\theta]}} \quad (6.43)$$

and equation (6.34) we obtain

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle_C^{(2)} \sim \frac{1}{\pi}\int_{|\tilde{\theta}_0-\tilde{\theta}_1|}^{\tilde{\theta}_0+\tilde{\theta}_1}\frac{\Theta(\cos\theta-\cos 2\hat{\psi})}{\sqrt{\cos\theta-\cos 2\hat{\psi}}}\frac{\sin\theta d\theta}{\sqrt{[\cos\theta-\cos(\tilde{\theta}_0+\tilde{\theta}_1)][\cos(\tilde{\theta}_0-\tilde{\theta}_1)-\cos\theta]}}\,. \quad (6.44)$$

The integral (6.44) is different from zero if $\cos(\tilde{\theta}_0-\tilde{\theta}_1) \geq \cos 2(\hat{\psi}_0+\hat{\psi}_1)$ which is precisely the condition (4.20).

6.4 Boundary states for symmetry breaking type III branes

Finally, let us consider the type III brane given in (4.21),

$$(g_0, g_1)\Big|_{\rm brane} = \left(h_0 f_0 h_1^{-1} e^{i\alpha\frac{\sigma_3}{2}}, h_1 f_1 h_0^{-1} e^{-i\alpha\frac{\sigma_3}{2}}\right), \tag{6.45}$$

with preserved currents

$$J_0^3 + \bar{J}_0^3 = 0\,, \quad J_1^3 + \bar{J}_1^3 = 0\,. \tag{6.46}$$

This brane has a structure which is different from the two previous branes. To derive the boundary state for this brane we start from the maximally symmetric permutation brane (5.9) and decompose the factors (5.7) and (5.8) according to the equation (6.27) and its 0 and 1 exchanged version. We then apply the permutational symmetry between the two $U(1)$ factors in the $0\bar{1}$ and $1\bar{0}$ subsectors to unpermute them. Since the final state has to be Z_k invariant (in accordance with the decomposition (6.5)),[7] one is forced to restrict to the subsector with $n = m$. The boundary state for the brane (6.45) can hence be written as

$$|a\rangle_C^{(3)} = \sqrt{2k}\sum_j\sum_n \frac{S_{aj}}{S_{0j}}\frac{1+(-1)^{2j+n}}{2}$$

$$|j,n\rangle\rangle_\tau^{PF_0\times PF_{\bar{1}}} \otimes |j,n\rangle\rangle_\tau^{PF_1\times PF_{\bar{0}}} \otimes |A,n\rangle\rangle_u^{U(1)_0} \otimes |A,n\rangle\rangle_u^{U(1)_1}\,. \tag{6.48}$$

As before, to check the consistency of the boundary state, one needs to ensure that the Cardy condition holds. The tree-level amplitude between two type III boundary states reduces, after the S-modular transformation, to

$$Z_{a_1a_2} = \sum_{r,j',j''}\sum_{n_1}\sum_{n_2,n_3} N_{a_1a_2}^r N_{rj''}^{j'} \chi_{j',n_1+n_2+n_3}(q)\chi_{j'',n_1}(q)\psi_{n_2}(q)\psi_{n_3}(q)\,, \tag{6.49}$$

and hence is consistent with the Cardy condition. Similarly, the annulus amplitude between brane III (6.48) and the maximally symmetric permutational brane (5.9) is

$$Z_{a_1a_2} = \sum_{r,j',j''}\sum_{n_1,n_2} N_{a_1a_2}^r N_{rj''}^{j'} \chi_{j',n_1+n_2}(q)\chi_{j'',n_1}(q)\psi_{n_2}(q^{1/2})\,. \tag{6.50}$$

and between type III and (5.2)

$$Z_{a,(a_0a_1)} = \sum_{j',r}\sum_{n_1,n_2} N_{a_0a_1}^r N_{ra}^{j'} \chi_{j',n_1+n_2}(q^{1/2})\psi_{n_1}(q)\psi_{n_2}(q)\,. \tag{6.51}$$

and type III and type I

$$Z_{a_1a_2} = \sum_{r,j',j''}\sum_{n,m} N_{a_1a_2}^r N_{rj''}^{j'} \chi_{j',n}(q)\chi_{j'',m}(q)\frac{(q^{1/2})^{1/48}}{\prod_m(1-(q^{1/2})^{m-1/2})}\,. \tag{6.52}$$

[7] Note that the action of the Z_k group in (6.5) is different from the one used in the construction of the Lens space Z_k^R. The action of $\omega \in Z_k$ on an arbitrary state $|j,n\rangle^{PF}|m\rangle^{U(1)}$ is given by

$$\omega|j,n\rangle^{PF}|m\rangle^{U(1)} = \omega^{n-m}|j,n\rangle^{PF}|m\rangle^{U(1)}\,. \tag{6.47}$$

The details of calculations (6.49)-(6.52) are provided in the appendix B.

Now we calculate the effective geometry corresponding to this boundary state. As before, to obtain the effective geometry, one should compute the overlap $\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle^{(3)}_C$. In the large-$k$ limit the overlap reduces to

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle^{(3)}_C \sim \sum_j\sum_n \sin(2j+1)\mathcal{D}^j_{nn}(g_0(\vec{\theta}_0))\mathcal{D}^j_{nn}(g_1(\vec{\theta}_1))\,. \tag{6.53}$$

One needs to use the fact that Wigner D-functions may be represented as a product of three functions, each of which depends only on one Euler coordinate,

$$\mathcal{D}^j_{nm}(g(\vec{\theta})) = e^{-i(n\chi+m\varphi)}d^j_{nm}(\cos\tilde{\theta})\,, \tag{6.54}$$

where d^j_{nm} are real functions satisfying the relation (note that there is no summation assumed for the repeated indices)

$$d^j_{nn}(\cos\tilde{\theta}_0)d^j_{nn}(\cos\tilde{\theta}_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(\gamma-\xi)}d^j_{nn}(\cos\hat{\Theta})d\gamma\,, \tag{6.55}$$

The functions $\hat{\Theta}$ and ξ are functions of $\tilde{\theta}_0, \tilde{\theta}_1$ and γ defined in equations (4.24) and (4.25). The overlap of the boundary state with the bulk probe can be written as

$$\begin{aligned}\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle^{(3)}_C &\sim \sum_j\sum_n\int_{-\pi}^{\pi}\sin(2j+1)e^{in(\gamma-\xi-2\tilde{\phi}_0-2\tilde{\phi}_1)}d^j_{nn}(\cos\hat{\Theta})d\gamma \\ &\sim \sum_j\sum_n\int_{-\pi}^{\pi}\sin(2j+1)D^j_{nn}(\hat{\Theta},\gamma/2-\xi/2-\tilde{\phi}_0-\tilde{\phi}_1,\gamma/2-\xi/2-\tilde{\phi}_0-\tilde{\phi}_1)d\gamma\end{aligned} \tag{6.56}$$

Now repeating the same steps as in (5.16) and (5.17) we get

$$\langle\vec{\theta}_0,\vec{\theta}_1|a\rangle^{(3)}_C \sim \int_{-\pi}^{\pi}\frac{\delta(\psi-\hat{\psi})}{\sin\hat{\psi}}d\gamma\,, \tag{6.57}$$

where

$$\cos\psi = \cos\frac{\hat{\Theta}}{2}\cos(\gamma/2-\xi/2-\tilde{\phi}_0-\tilde{\phi}_1) \tag{6.58}$$

From this equation it follows that the brane consist of all those points for which the expression in the argument of the δ function has a root for γ. This is the same condition as the one coming from equation (4.23), obtained in the Langrangian approach.

7. Discussion

In this paper we have presented several new types of symmetry breaking branes. While most of our analysis was focused on exploring different ways in which one can break part of the diagonal affine algebra by branes, the understanding of the properties of these branes was only briefly touched upon. In particular, issues concerning the embedding of these branes in supersymmetric models, their stability, a comparison of the spectra between the

CFT and the effective approaches and so forth should definitely still be explored further. A preliminary investigation of the supersymmetry of these branes using the probe kappa-symmetry approach indicates that most of the branes are non-supersymmetric. Only a special class of maximally symmetric, permutation branes seems to preserve a fraction of supersymmetry. While most of the required effective analysis as found in [23] or [24] is complicated when applied directly to a target that is a product of groups, simplifications might occur in the Penrose limit applied to branes along the diagonal geodesic which winds between the groups, as in [25]. We hope to address some of these issues in the future.

Acknowledgements

We would like to thank Pedro Bordalo, Annamaria Font, Kasper Peeters, Sakura Schäfer-Nameki and Volker Schomerus for useful discussions.

A. Some details of the calculations

A.1 Symmetries of the brane I

In this appendix we present a detailed calculation of the invariance of the string action (2.8) for the brane (3.5) under the symmetry transformations 1 and 2 of section 3.2. Under the first transformation the change of the bulk action is (2.23) with $g_0 = K_0$ and $g_1 = K_1 L$, while the change of the first term in (3.8) is

$$\omega^{(2)}(h_0, kh_1) - \omega^{(2)}(h_0, h_1) = -\,\mathrm{Tr}\left(k^{-1}\mathrm{d}k(K_0^{-1}\mathrm{d}K_0 + \mathrm{d}K_1 K_1^{-1})\right). \tag{A.1}$$

The change in the second term in (3.8) is given by

$$\Delta\Big(-\,\mathrm{Tr}\left(K_1^{-1}\mathrm{d}K_1\mathrm{d}LL^{-1}\right)\Big) = -\,\mathrm{Tr}\left(k^{-1}\mathrm{d}k(K_1\mathrm{d}LL^{-1}K_1^{-1})\right). \tag{A.2}$$

Hence putting all terms together, one gets that the change of $\omega^{(2)}(h_0, h_1, L)$ exactly cancels (2.23).

Under the second transformation in (3.2) the change in the bulk term in the action is

$$\Delta S = -\frac{k}{4\pi}\int_D \mathrm{Tr}\left(k^{-1}\mathrm{d}k\big(\mathrm{d}K_0K_0^{-1} + K_1^{-1}\mathrm{d}K_1 + L^{-1}dL\big)\right) \tag{A.3}$$

where we used that $[k, L] = 0$. The change in the first term in (3.8) is given by

$$\omega^{(2)}(kh_0, h_1) - \omega^{(2)}(h_0, h_1) = -\,\mathrm{Tr}\left(k^{-1}\mathrm{d}k\big(\mathrm{d}K_0K_0^{-1} + K_1^{-1}\mathrm{d}K_1\big)\right) \tag{A.4}$$

and in the change in the second term by

$$\Delta\Big(-\,\mathrm{Tr}\left(K_1^{-1}\mathrm{d}K_1\mathrm{d}LL^{-1}\right)\Big) = \mathrm{Tr}\left(k^{-1}\mathrm{d}k\mathrm{d}LL^{-1}\right), \tag{A.5}$$

again due to $[k, L] = 0$. Hence the total change of the action is,

$$\Delta\left(S - \frac{k}{4\pi}\int_D \omega^{(2)}(h_0, h_1, L)\right) = \frac{k}{2\pi}\int_D \mathrm{Tr}\left(k^{-1}\mathrm{d}k\mathrm{d}LL^{-1}\right), \tag{A.6}$$

which vanishes for all $[k, L] = 0$.

The change in the bulk part of the action under the transformations 3' is

$$\Delta S = -\frac{k}{4\pi}\int_D \mathrm{Tr}\left(k^{-1}\mathrm{d}k\big(\mathrm{d}K_0K_0^{-1} - K_1^{-1}\mathrm{d}K_1 - L^{-1}dL\big)\right), \tag{A.7}$$

while the change in the second term in (3.8) is

$$\Delta\Big(-\,\mathrm{Tr}\left(K_1^{-1}\mathrm{d}K_1\mathrm{d}LL^{-1}\right)\Big) = \mathrm{Tr}\left(k^{-1}\mathrm{d}k(2K_1^{-1}\mathrm{d}K_1 + \mathrm{d}LL^{-1}\right) \tag{A.8}$$

which altogether with (A.4) leads to a vanishing total variation of the action.

A.2 Symmetries of the brane III

In this section we prove invariance of the brane (3.15) under the transformations 3" and 4" in section 3.3. The gauge-invariant two-form for this brane is given by

$$\omega^{\mathrm{WZ}}(g)\Big|_{\mathrm{brane}} = \mathrm{d}\Big(\omega^{(2)}(h_0,h_1) - \mathrm{Tr}\big(K_0^{-1}\mathrm{d}K_0\mathrm{d}LL^{-1} - K_1^{-1}\mathrm{d}K_1\mathrm{d}LL^{-1}\big)\Big)\,. \tag{A.9}$$

Under the transformation 3" the bulk action changes by the amount

$$\begin{aligned}\Delta S &= \frac{k}{4\pi}\int_D \mathrm{Tr}\Big(k^{-1}\mathrm{d}k\big(g_0k^{-1}\mathrm{d}kg_0^{-1} - g_0^{-1}\mathrm{d}g_0 - \mathrm{d}g_0g_0^{-1}\big)\Big)\\ &= \frac{k}{4\pi}\int_D \mathrm{Tr}\Big(k^{-1}\mathrm{d}k\big(K_0k^{-1}\mathrm{d}kK_0^{-1} - K_0^{-1}\mathrm{d}K_0 - \mathrm{d}K_0K_0^{-1} - L^{-1}\mathrm{d}L\\ &\qquad\qquad - K_0\mathrm{d}LL^{-1}K_0^{-1}\big)\Big)\,.\end{aligned} \tag{A.10}$$

The change of the first term in the two-form (A.9) is given by (A.4) while the change of the second term is

$$\begin{aligned}&\Delta\Big(-\mathrm{Tr}\big(K_0^{-1}\mathrm{d}K_0\mathrm{d}LL^{-1} - K_1^{-1}\mathrm{d}K_1\mathrm{d}LL^{-1}\big)\Big) =\\ &\quad\mathrm{Tr}\Big(k^{-1}\mathrm{d}k\big(K_1^{-1}\mathrm{d}K_1 - K_0^{-1}\mathrm{d}K_0 - \mathrm{d}LL^{-1} + K_0\mathrm{d}kk^{-1}K_0^{-1} - K_0\mathrm{d}LL^{-1}K_0^{-1}\big)\Big)\,.\end{aligned} \tag{A.11}$$

Collecting all terms together, we see that the action is invariant under the transformation 3". Invariance of the action under the transformations 4" is proved in the same way.

B. Proof of the Cardy conditions

One of the necessary consistency conditions which any boundary state has to satisfy is the Cardy condition: for any pair of boundary states $|\alpha\rangle$, $|\beta\rangle$ in the theory, the closed string tree-level amplitude between them should, after a modular transformation, be expressible as the partition function of a CFT on a strip with boundary conditions α and β at each end of the strip. The requirement that the boundary state satisfies the Cardy condition restricts the coefficients in the Cardy state as follows. We present here the details of the calculation for (5.9) following [2]. The computation for other branes in section 6.2 is similar, so only the results are presented in the main text.

Let us consider the following tree amplitudes between the two permutational branes given in (5.9) and between the permutational and factorisable branes of (5.3),

$$Z_{a_1,a_2} = {}_{\mathcal{P}}\langle a_1|\exp(-\pi i H_{(cl)}/T)|a_2\rangle_{\mathcal{P}} = {}_{\mathcal{P}}\langle a_1|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|a_2\rangle_{\mathcal{P}}\,, \tag{B.1}$$

$$Z_{a,(a_0a_1)} = {}_{\mathcal{P}}\langle a|\exp(-\pi i H_{(cl)}/T)|a_0,a_1\rangle = {}_{\mathcal{P}}\langle a|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|a_0,a_1\rangle\,. \tag{B.2}$$

Here $\tilde{q}\equiv e^{-2\pi i/T}$ and $L_0 = L_0^{(0)} + L_0^{(1)}$, $\bar{L}_0 = \bar{L}_0^{(0)} + \bar{L}_0^{(1)}$. We want to show that after the S-modular transformation, these expressions can be written as

$$Z_{\alpha\beta} = \sum_i n^i_{\alpha\beta}\chi_i(q)\,. \tag{B.3}$$

Here α and β are conditions defined by the states (5.2) and (5.9), $q \equiv \exp(-2\pi i T)$ is the open string modular parameter, $n^i_{\alpha\beta}$ are some positive *integer* numbers, and $\chi_i(q)$ are characters for *some* conformal symmetry algebra. Note that, depending on the choice of the boundary conditions, the same initial bulk algebra leads to different open string conformal algebras.

The first expression (B.1) becomes

$$
\begin{aligned}
{}_{\mathcal{P}}\langle\langle j_1, j_1|(\tilde{q}^{1/2})^{L_0+\overline{L}_0-c/12}|j_2, j_2\rangle\rangle_{\mathcal{P}} = & \\
= \sum_{M,N,M'N'} \tilde{q}^{h(j_1,M)+h(j_2,N)-c/12} & \left({}_0\langle j_1, M|j_2, M'\rangle_0\right)\left(\overline{{}_1\langle j_1, M|j_2, M'\rangle_1}\right) \\
& \times \left(\overline{{}_0\langle j_1, N|j_2, N'\rangle_0}\right)\left({}_1\langle j_1, N|j_2, N'\rangle_1\right) \\
= \sum_{M,N} \tilde{q}^{h(j_1,M)+h(j_2,N)-c/12}\delta_{j_1,j_2} = & \chi_{j_1}(\tilde{q})\chi_{j_2}(\tilde{q})\delta_{j_1,j_2}\,.
\end{aligned}
\tag{B.4}
$$

Applying the modular transformations to this expression, and using the symmetry properties of the S-matrix ($S_{ij} = S_{ji}$) as well as the Verlinde formula, one obtains[8]

$$
\begin{aligned}
Z_{a_1,a_2} &= \sum_{j,k,l} \frac{S_{a_1 j}}{S_{0j}}\frac{S_{a_2 j}}{S_{0j}} S_{jk}S_{jl}\chi_k(q)\chi_l(q) \\
&= \sum_{r,k,l} N^r_{a_1 a_2} N^k_{rl}\chi_k(q)\chi_l(q)\,.
\end{aligned}
\tag{B.7}
$$

To put this expression in the form (B.3) we need to realise that the product of two characters in (B.7) corresponds to a *single* character of the total $SU(2)\times SU(2)$ group, since the primaries of the total group are labeled by two (rather then one number). Hence we can write $\chi_k(q)\chi_l(q) \equiv \chi_{(k,l)}(q)$. Also the sum of the two fusion matrices in (B.7) is obviously defining a set of positive integer numbers, which we can denote with $N^{k,l}_{a_0 a_1}$. Using all these ingredients we see that (B.7) is of the required form.

It is instructive to compare the partition function we have just computed with the the partition function between the boundary states for the direct product of two $SU(2)$ branes

$$
Z_{(a_0,a_1),(a_0',a_1')} = \sum_{r,k,l} N^r_{a_0 a_0'} N^k_{a_1 a_1'}\chi_r(q)\chi_l(q)\,. \tag{B.8}
$$

We see that the difference with respect to the (B.7) is contained in the form of the fusion coefficients; while in the formula (B.7) the coefficients appear as product of fusion matrices ($(N_i)_{jk} = N^k_{ij}$), they appear in the expression (B.8) in an "uncoupled" form.

[8] Recall that the Verlinde formula is given by

$$
N^k_{ij} = \sum_p \frac{S_{ip}S_{jp}\bar{S}_{pk}}{S_{0p}}\,, \tag{B.5}
$$

where $\bar{S} = CS$ is a complex conjugated S-matrix, N^k_{ij} are fusion matrices and the 0 index in S denotes the vacuum representation. In the case of a real S-matrix, this can be rewritten as

$$
\frac{S_{aj}S_{lj}}{S_{0j}} = \sum_k N^k_{al}S_{kj}\,. \tag{B.6}
$$

Next we turn to the computation of (B.2).

$$Z_{a,(a_0a_1)} = \sum_{j,j_1,j_2} \frac{S_{aj}}{S_{0j}} \frac{S_{a_0j_0}S_{a_1j_1}}{\sqrt{S_{0j_0}S_{0j_1}}} {}_{\mathcal{P}}\langle j,j|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|j_0\rangle\rangle_u^{SU(2)_0}|j_1\rangle\rangle_u^{SU(2)_1}\,. \tag{B.9}$$

In this case the overlap between the Ishibashi states is

$$\begin{aligned}
&{}_{\mathcal{P}}\langle j,j|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|j_0\rangle\rangle_u^{SU(2)_0}|j_1\rangle\rangle_u^{SU(2)_1} = \\
&\quad = \sum_{M,N,M'N'} \tilde{q}^{h(j_1,M)+h(j_2,N)-c/12} \left({}_0\langle j,M|j_0,M'\rangle_0\right)\left({}_1\overline{\langle j,M|j_1,N'\rangle_1}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \left({}_0\overline{\langle j,N|j_0,M'\rangle_0}\right)\left({}_1\langle j,N|j_1,N'\rangle_1\right) \\
&\quad = \sum_M \tilde{q}^{2h(j,M)-c/12}\delta_{j,j_0}\delta_{j,j_1} = \chi_j(\tilde{q}^2)\delta_{j,j_0}\delta_{j,j_1}\,.
\end{aligned} \tag{B.10}$$

After the modular transformations and manipulations similar to those we have already used, the partition function becomes

$$Z_{a,(a_0a_1)} = \sum_{j,k,r} N_{a_0a_1}^r S_{rj}\frac{S_{aj}}{S_{0j}}S_{jk}\chi_k(q^{1/2}) = \sum_{k,r} N_{a_0a_1}^r N_{ra}^k \chi_k(q^{1/2})\,. \tag{B.11}$$

We see that the partition function is expressible as a sum over characters in a twisted sector of a theory orbifolded by the permutation symmetry [26, 27].

For boundary states (6.29) we have

$$\begin{aligned}
&{}^{(1)}_{C}\langle a_1|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|a_2\rangle^{(1)}_C = \\
&\quad k\sum_j \frac{S_{a_1j}S_{a_2j}}{S_{0j}S_{0j}}\left(\frac{1+(-1)^{2j}}{2}\chi_j(\tilde{q})\chi_{j0}(\tilde{q})\psi_0(\tilde{q}) + \frac{1+(-1)^{2j+k}}{2}\chi_j(\tilde{q})\chi_{jk}(\tilde{q})\psi_k(\tilde{q})\right)
\end{aligned} \tag{B.12}$$

which using the Verlinde formula can be written as

$$\begin{aligned}
&{}^{(1)}_{C}\langle a_1|(\tilde{q}^{1/2})^{L_0+\overline{L_0}-c/12}|a_2\rangle^{(1)}_C = \\
&\quad k\sum_{j,r} N^r_{a_1,a_2}\frac{S_{rj}}{S_{0j}}\left(\frac{1+(-1)^{2j}}{2}\chi_j(\tilde{q})\chi_{j0}(\tilde{q})\psi_0(\tilde{q}) + \frac{1+(-1)^{2j+k}}{2}\chi_j(\tilde{q})\chi_{jk}(\tilde{q})\psi_k(\tilde{q})\right)
\end{aligned} \tag{B.13}$$

Now we will evaluate all four terms in parenthesis after modular transformation

$$\begin{aligned}
&\frac{k}{2}\sum_{j,r} N^r_{a_1,a_2}\frac{S_{rj}}{S_{0j}}\chi_j(\tilde{q})\chi_{j0}(\tilde{q})\psi_0(\tilde{q}) \\
&\quad = \frac{k}{2}\sum_{j,r} N^r_{a_1,a_2}\sum_{j',j'',n_1,n_2}\frac{S_{rj}S_{jj'}S_{jj''}}{S_{0j}}\chi_{j'}(q)\frac{\chi_{j''n_1}(q)\psi_{n_2}(q)}{\sqrt{2k}\sqrt{2k}} \\
&\quad = \frac{1}{4}\sum_r\sum_{j',j'',n_1,n_2} N^r_{a_1,a_2}N^{j'}_{r,j''}\chi_{j'}(q)\chi_{j''n_1}(q)\psi_{n_2}(q)
\end{aligned} \tag{B.14}$$

Here we used matrix of modular transformation of $U(1)_k$ theory given by formula (C.4).

$$\begin{aligned}
&\frac{k}{2}\sum_{j,r} N^r_{a_1,a_2}\frac{S_{rj}}{S_{0j}}(-1)^{2j}\chi_j(\tilde{q})\chi_{j0}(\tilde{q})\psi_0(\tilde{q}) = \\
&= \frac{k}{2}\sum_{j,r} N^r_{a_1,a_2}\sum_{j',j'',n_1,n_2}\frac{S_{rj}S_{jj'}S_{jj''}}{S_{0j}}(-1)^{2j}\chi_{j'}(q)\frac{\chi_{j''n_1}(q)\psi_{n_2}(q)}{\sqrt{2k}\sqrt{2k}} \\
&= \frac{1}{4}\sum_{j,r} N^r_{a_1,a_2}\sum_{j',j'',n_1,n_2}\frac{S_{rj}S_{jj'}S_{j,\frac{k}{2}-j''}}{S_{0j}}\chi_{j'}(q)\chi_{j''n_1}(q)\psi_{n_2}(q) \\
&= \frac{1}{4}\sum_{r}\sum_{j',j'',n_1,n_2} N^r_{a_1,a_2}N^{j'}_{r,\frac{k}{2}-j''}\chi_{j'}(q)\chi_{j''n_1}(q)\psi_{n_2}(q) \\
&= \frac{1}{4}\sum_{r}\sum_{j',j'',n_1,n_2} N^r_{a_1,a_2}N^{j'}_{r,\frac{k}{2}-j''}\chi_{j'}(q)\chi_{\frac{k}{2}-j'',k+n_1}(q)\psi_{n_2}(q) \\
&= \frac{1}{4}\sum_{r}\sum_{j',j'',n_1,n_2} N^r_{a_1,a_2}N^{j'}_{r,j''}\chi_{j'}(q)\chi_{j'',n_1}(q)\psi_{n_2}(q)
\end{aligned} \tag{B.15}$$

Here passing from the first to the second line we used the symmetry property (6.13) of the $SU(2)_k$ modular transformation matrix, and passing from the third to the forth line the field identification property of the parafermion primaries $(j,n) \sim (k/2-j, k+n)$. We see that contribution of the first and the second terms are equal, and therefore in the partition function calculations we can effectively replace the first projector by one. Using the same arguments (symmetries of S-matrix and the field identification) we can show that also the second projector can be effectively replaced by one. Taking this into account for the second part in the parenthesis we obtain:

$$\begin{aligned}
&k\sum_{j,r} N^r_{a_1,a_2}\frac{S_{rj}}{S_{0j}}\chi_j(\tilde{q})\chi_{jk}(\tilde{q})\psi_k(\tilde{q}) = \\
&\qquad = k\sum_{j,r} N^r_{a_1,a_2}\sum_{j',j'',n_1,n_2}\frac{S_{rj}S_{jj'}S_{jj''}}{S_{0j}}\chi_{j'}(q)\frac{\chi_{j''n_1}(q)\psi_{n_2}(q)}{\sqrt{2k}\sqrt{2k}}(-1)^{(n_1+n_2)} \\
&\qquad = \frac{1}{2}\sum_{r}\sum_{j',j'',n_1,n_2} N^r_{a_1,a_2}N^{j'}_{r,j''}\chi_{j'}(q)\chi_{j''n_1}(q)\psi_{n_2}(q)(-1)^{(n_1+n_2)}
\end{aligned} \tag{B.16}$$

Collecting (B.14), (B.15) and (B.16) we obtain (6.30).

For boundary states (5.9) and (6.29) using (6.27) we have

$$\begin{aligned}
&{}^{(1)}_{C}\langle a_1|(\tilde{q})^{L_0-c/24}|a_2\rangle_{\mathcal{P}} = \sqrt{k}\sum_{j,n}\frac{S_{a_1j}S_{a_2j}}{S_{0j}S_{0j}}\chi_j(\tilde{q})\frac{1+(-1)^{2j+n}}{2}\times \\
&{}^{0\bar{1}}\langle\langle Bj|(\tilde{q})^{L_0-c/24}|jn\rangle\rangle^{PF_0\times PF_{\bar{1}}}_{\tau}|n\rangle\rangle^{U(1)_0\times U(1)_{\bar{1}}}_{\tau+}
\end{aligned} \tag{B.17}$$

Let us recall that

$$ {}^{U(1)_0\times U(1)_{\bar{1}}}{}_{\tau-}\langle\langle n|(\tilde{q})^{L_0-c/24}|r\rangle\rangle^{U(1)_0\times U(1)_{\bar{1}}}_{\tau+} = \delta_{n,0}\delta_{r,0}\chi_{ND}(\tilde{q}) \tag{B.18}$$

where

$$\chi_{ND}(\tilde{q}) = (\tilde{q})^{-1/24}\mathrm{Tr}(P(\tilde{q})^{L_0^{U(1)}}) = \frac{1}{(\tilde{q})^{1/24}\prod(1+(\tilde{q})^n)} \tag{B.19}$$

where P takes X to $-X$. This can be derived noting that zero-modes do not contribute, and the overlap is given by the parity-weighted partition function receiving contribution only from higher oscillator states. Using (B.18) we obtain

$$\,^{(1)}_{C}\langle a_1|(\tilde{q})^{L_0-c/24}|a_2\rangle_{\mathcal{P}} = \sqrt{k}\sum_j \frac{S_{a_1 j}S_{a_2 j}}{S_{0j}S_{0j}}\frac{1+(-1)^{2j}}{2}\chi_j(\tilde{q})\chi_{j0}(\tilde{q})\chi_{ND}(\tilde{q}) \tag{B.20}$$

Performing modular transformation and using that

$$\chi_{ND}(\tilde{q}) = \sqrt{2}\frac{q^{1/48}}{\prod(1-q^{n-1/2})} \tag{B.21}$$

we obtain (6.31)

$$\,^{(1)}_{C}\langle a_1|(\tilde{q})^{L_0-c/24}|a_2\rangle_{\mathcal{P}} = \sqrt{k}\sum_r\sum_{j',j'',n} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j'}(q)\frac{\chi_{j''n}(q)}{\sqrt{2k}}\sqrt{2}\frac{q^{1/48}}{\prod(1-q^{n-1/2})} =$$
$$\sum_r\sum_{j',j'',n} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j'}(q)\chi_{j''n}(q)\frac{q^{1/48}}{\prod(1-q^{n-1/2})} \tag{B.22}$$

For boundary states (5.2) and (6.29), again using permuted decomposition (6.27) and similar for $\bar{0}1$ sector we can write

$$\,^{(1)}_{C}\langle a|(\tilde{q})^{L_0-c/24}|a_0,a_1\rangle = \sqrt{k}\sum_{j,j_0,j_1}\sum_{n_1,n_2,n_3=1}^{2k}\sum_{i=0,k}\frac{S_{aj}}{S_{0j}}\frac{S_{a_0j_0}S_{a_1j_1}}{\sqrt{S_{0j_0}S_{0j_1}}}\times$$
$$A^{j,j_0,j_1}_{n_1,n_2,n_3,i}B_{n_1,n_2,n_3,i}\frac{1+(-1)^{2j_0+n_1}}{2}\frac{1+(-1)^{2j_1+n_2}}{2}\frac{1+(-1)^{2j+n_3}}{2}\frac{1+(-1)^{2j+i}}{2} \tag{B.23}$$

where $A^{j,j_0,j_1}_{n_1,n_2,n_3,i}$ and $B_{n_1,n_2,n_3,i}$ are contributions from the parafermion and $U(1)_k$ parts respectively and given by the expressions

$$A^{j,j_0,j_1}_{n_1,n_2,n_3,i} = {}^{0\bar{1}}\langle\langle j,i|^{\bar{0}1}\langle\langle j,n_3|(\tilde{q})^{L_0^{PF}-c/24}|j_0,n_1\rangle\rangle_u^{PF_0}|j_1,n_2\rangle\rangle_u^{PF_1} \tag{B.24}$$

$$B_{n_1,n_2,n_3,i} = {}^{0\bar{1}}{}_{\tau-}\langle\langle i|^{\bar{0}1}{}_{\tau+}\langle\langle n_3|(\tilde{q})^{L_0^{U(1)}-c/24}|An_1\rangle\rangle_u^{U(1)_0}|An_2\rangle\rangle_u^{U(1)_1} \tag{B.25}$$

Following the same steps as in formula (B.10) for (B.24) we can write

$$A^{j,j_0,j_1}_{n_1,n_2,n_3,i} = \chi_{j,n}(\tilde{q}^2)\delta_{j,j_0}\delta_{j,j_1}\delta_{i,n_1}\delta_{i,n_2}\delta_{i,n_3} \tag{B.26}$$

Calculating (B.25) we again can follow the similar steps as in the formula (B.10), but presence of "$\tau-$" state in (B.25) will bring the following changes. For $U(1)_k$ boundary states as we can see from (6.8), (6.11), (6.16) and (6.17) the index labeling orthonormal basis of the $U(1)_k$ representations runs over zero-mode momentum part, and over α's created part. Denoting the zero-mode momenta in four states in (B.25) by $p^{0\bar{1}}$, $p^{\bar{0}1}$,$p^{0\bar{0}}$,$p^{1\bar{1}}$ respectively by the similar steps as in (B.10) we obtain:

$$\begin{aligned} p^{0\bar{1}} &= p^{0\bar{0}}\,, \quad -p^{0\bar{1}} = p^{1\bar{1}} \\ p^{\bar{0}1} &= p^{0\bar{0}} \quad\;\; p^{\bar{0}1} = p^{1\bar{1}}\,, \end{aligned} \tag{B.27}$$

where the minus in the second line comes from the "$\tau-$" type first Ishibashi state in (B.25). It is easy to see that these four conditions imply that all zero-modes momenta are zero. Therefore as before zero-modes do not contribute to (B.25). For the α's created part we again get as in (B.10) doubled energy in exponent but weighted with the sign coming from the '$\tau-$" type Ishibashi state:

$$B_{n_1,n_2,n_3,i} = (\tilde{q})^{-1/12}\mathrm{Tr}(P(\tilde{q})^{2L_0^{U(1)}})\delta_{n_1,0}\delta_{n_2,0}\delta_{n_3,0}\delta_{i,0} = \chi_{ND}(\tilde{q}^2)\delta_{n_1,0}\delta_{n_2,0}\delta_{n_3,0}\delta_{i,0} \tag{B.28}$$

Putting (B.26) and (B.28) in (B.23) we get:

$$\begin{aligned} &{}^{(1)}_{C}\langle a|(\tilde{q})^{L_0-c/24}|a_0,a_1\rangle = \sqrt{k}\sum_{j,r} N^r_{a_0a_1}S_{rj}\frac{S_{aj}}{S_{0j}}\frac{1+(-1)^{2j}}{2}\chi_{j,0}(\tilde{q}^2)\chi_{ND}(\tilde{q}^2) = \\ &\sqrt{k}\sum_{r,j,j',n} N^r_{a_0a_1}\frac{S_{rj}S_{aj}S_{jj'}}{S_{0j}}\frac{1+(-1)^{2j}}{2}\frac{\chi_{j',n}}{\sqrt{2k}}(q^{1/2})\sqrt{2}\frac{(q^{1/2})^{1/48}}{\prod(1-(q^{1/2})^{n-1/2})} = \\ &\sum_{r,j',n} N^r_{a_0a_1}N^{j'}_{ra}\chi_{j',n}(q^{1/2})\frac{(q^{1/2})^{1/48}}{\prod(1-(q^{1/2})^{n-1/2})} \end{aligned} \tag{B.29}$$

which is (6.32).

For boundary states (6.48) we have

$$\begin{aligned} &{}^{(3)}_{C}\langle a_1|(\tilde{q})^{L_0-c/24}|a_2\rangle^{(3)}_{C} = 2k\sum_{j,n}\frac{S_{a_1j}S_{a_2j}}{S_{0j}S_{0j}}\frac{1+(-1)^{2j+n}}{2}\times \\ &\chi_{j,n}(\tilde{q})\chi_{j,n}(\tilde{q})\psi_n(\tilde{q})\psi_n(\tilde{q}) \end{aligned} \tag{B.30}$$

Performing modular transformation we obtain (6.49)

$$\begin{aligned} Z_{a_1a_2} &= 2k\sum_{r,n,j,j',j''}\sum_{n_1,n_2,n_3,n_4} N^r_{a_1a_2}\frac{S_{rj}S_{jj'}S_{jj''}}{S_{0j}}\frac{\chi_{j',n_1}(q)\chi_{j'',n_2}(q)\psi_{n_3}(q)\psi_{n_4}(q)}{\sqrt{2k}\sqrt{2k}\sqrt{2k}\sqrt{2k}}e^{i\frac{\pi}{k}n(n_1+n_2+n_3+n_4)} = \\ &\sum_{r,j',j''}\sum_{n_2,n_3,n_4} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j',n_2+n_3+n_4}(q)\chi_{j'',n_2}(q)\psi_{n_3}(q)\psi_{n_4}(q) \end{aligned} \tag{B.31}$$

where we used that

$$\sum_n e^{i\frac{\pi}{k}nn'} = 2k\delta_{n',0} \tag{B.32}$$

Now consider partition function between states (5.9) and (6.48). We note that parafermions in both sides appear in the permuted way, therefore for them following to (B.4) we obtain $\chi_{j,n}(\tilde{q})\chi_{j,n}(\tilde{q})$. But we see that $U(1)_k$ states appear in permuted way in one side, and unpermuted way in other side, therefore for them following to (B.10) we obtain $\psi_n(\tilde{q}^2)$. Taking all this into account we get:

$$\,_C^{(3)}\langle a_1|(\tilde{q})^{L_0-c/24}|a_2\rangle_{\mathcal{P}} = \sqrt{2k}\sum_{j,n}\frac{S_{a_1j}S_{a_2j}}{S_{0j}S_{0j}}\frac{1+(-1)^{2j+n}}{2}\chi_{j,n}(\tilde{q})\chi_{j,n}(\tilde{q})\psi_n(\tilde{q}^2) \quad \text{(B.33)}$$

Performing modular transformation we obtain (6.50)

$$Z_{a_1a_2} = \sqrt{2k}\sum_{r,n,j,j',j''}\sum_{n_1,n_2,n_3} N^r_{a_1a_2}\frac{S_{rj}S_{jj'}S_{jj''}}{S_{0j}}\frac{\chi_{j',n_1}(q)\chi_{j'',n_2}(q)\psi_{n_3}(q^{1/2})}{\sqrt{2k}\sqrt{2k}\sqrt{2k}}e^{i\frac{\pi}{k}n(n_1+n_2+n_3)} =$$

$$\sum_{r,j',j''}\sum_{n_2,n_3} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j',n_2+n_3}(q)\chi_{j'',n_2}(q)\psi_{n_3}(q^{1/2}) \quad \text{(B.34)}$$

For partition function between the states (5.2) and (6.48), the situation is opposite. Now we have parafermions in the permuted way in one side, and unpermuted way in other side, therefore leading to $\chi_{j,n}(\tilde{q}^2)$. But we have $U(1)_k$ states in unpermuted way on the both sides, therefore leading to $\psi_n(\tilde{q})\psi_n(\tilde{q})$. Collecting all we get

$$\,_C^{(3)}\langle a|(\tilde{q})^{L_0-c/24}|a_0,a_1\rangle = \sqrt{2k}\sum_j\sum_n\frac{S_{aj}}{S_{0j}}\frac{S_{a_0j}S_{a_1j}}{S_{0j}}\times$$

$$\chi_{j,n}(\tilde{q}^2)\psi_n(\tilde{q})\psi_n(\tilde{q})\frac{1+(-1)^{2j+n}}{2} \quad \text{(B.35)}$$

Performing modular transformation we obtain (6.51)

$$Z_{a,a_0a_1} = \sqrt{2k}\sum_{j,r,n}\sum_{n_1,n_2,n_3} N^r_{a_0a_1}\frac{S_{rj}S_{aj}S_{jj'}}{S_{0j}}\frac{1+(-1)^{2j+n}}{2}\frac{\chi_{j',n_1}(q^{1/2})\psi_{n_2}(q)\psi_{n_3}(q)}{\sqrt{2k}\sqrt{2k}\sqrt{2k}}\times$$

$$e^{i\frac{\pi}{k}n(n_1+n_2+n_3)} = \sum_{r,j'}\sum_{n_2,n_3} N^r_{a_0a_1}N^{j'}_{ra}\chi_{j',n_2+n_3}(q^{1/2})\psi_{n_2}(q)\psi_{n_3}(q) \quad \text{(B.36)}$$

In the case of partition function between (6.29) and (6.48) we have parafermion in the permuted way on both sides, but $U(1)_k$ states in permuted way on one side, and unpermuted way in other side, and additionly we should take into account that in the $0\bar{1}$ sector of (6.29) we have "$\tau-$" state, whereas in (6.48) we have A type states. Therefore repeating the same steps as in derivation of (B.28) we get

$$Z_{a_1a_2} = \sqrt{2k}\sqrt{k}\sum_j\frac{S_{a_1j}S_{a_2j}}{S_{0j}S_{0j}}\frac{1+(-1)^{2j}}{2}\chi_{j,0}(\tilde{q})\chi_{j,0}(\tilde{q})\chi_{ND}(\tilde{q}^2) \quad \text{(B.37)}$$

Performing modular transformation we get (6.52)

$$Z_{a_1a_2} = \sqrt{2k}\sqrt{k}\sum_{j,r,j',j''}\sum_{n_1,n_2} N^r_{a_1a_2}\frac{S_{rj}S_{j'j}S_{jj''}}{S_{0j}}\frac{\chi_{j',n_1}(q)\chi_{j'',n_2}(q)}{\sqrt{2k}\sqrt{2k}}\sqrt{2}\frac{(q^{1/2})^{1/48}}{\prod(1-(q^{1/2})^{n-1/2})} =$$

$$\sum_{j',j'',r}\sum_{n_1,n_2} N^r_{a_1a_2}N^{j'}_{rj''}\chi_{j',n_1}(q)\chi_{j'',n_2}(q)\frac{(q^{1/2})^{1/48}}{\prod(1-(q^{1/2})^{n-1/2})} \quad \text{(B.38)}$$

C. Some facts about $U(1)_k$, $SU(2)_k$ and parafermion theories

In this section we briefly review some necessary facts about $U(1)_k$, $SU(2)_k$ and $\mathcal{A}^{PF(k)} = \frac{SU(2)_k}{U(1)_k}$ theories.

$U(1)_k$ theory:

The $U(1)_k$ chiral algebra ($k \in Z$) contains, besides the Gaussian $U(1)$ current $J = i\sqrt{2k}\partial X$, two additional generators

$$\Gamma^{\pm} = e^{\pm i\sqrt{2k}X} \tag{C.1}$$

of integer dimension k and charge $\pm 2k$. The primary fields of the extended theory are those vertex operators $e^{i\gamma X}$ whose OPEs with the generators (C.1) are local. This fixes γ to be

$$\gamma = \frac{n}{\sqrt{2k}}\,, \quad n \in Z \tag{C.2}$$

Their conformal dimension is $\Delta_n = \frac{n^2}{4k}$. For primary fields, the range of n must be restricted to the fundamental domain $n = -k+1, -k+2, \ldots, k$ since a shift of n by $2k$ in $e^{inX/\sqrt{2k}}$ amounts to an insertion of the ladder operator Γ^+, which thereby produces a descendant field.

From the point of view of the extended algebra the characters are easily derived. A factor $q^{\Delta_n - 1/24}/\eta(q)$ takes care of the action of the free boson generators. To account for the effect of the distinct multiple applications of the generators (C.1), which yield shifts of the momentum n by integer multiples of $2k$, we must replace n by $n + l2k$ and sum over l. The net result is

$$\psi_n(q) = \frac{1}{\eta(q)} \sum_{l \in Z} q^{k(l+n/2k)^2}\,. \tag{C.3}$$

The action of the modular transformation S on the characters (C.3) is

$$\psi_n(q') = \frac{1}{\sqrt{2k}} \sum_{n'} e^{\frac{-i\pi n n'}{k}} \psi_{n'}(q) \quad q = e^{2\pi i \tau} \quad \tau' = -\frac{1}{\tau}\,. \tag{C.4}$$

The parafermion $\mathcal{A}^{PF(k)} = \frac{SU(2)_k}{U(1)_k}$

The chiral algebra of this theory has a set of irreducible representations described by pairs (j,n) where $j \in \frac{1}{2}Z$, $0 \leq j \leq k/2$, and n is an integer defined modulo $2k$. The pairs are subject to a constraint $2j + n = 0 \mod 2$, and an equivalence relation $(j,n) \sim (k/2-j, k+n)$. The character of the representation (j,n), denoted by $\chi_{j,n}(q)$, is determined implicitly by the decomposition

$$\chi_j^{SU(2)}(q) = \sum_{n=-k}^{k+1} \chi^k_{j,n}(q)\psi_n(q)\,. \tag{C.5}$$

The action of modular group on the character is

$$\chi^k_{j,n}(q') = \sum_{(j',n')} S^{PF}_{(j,n),(j'n')} \chi^k_{j',n'}(q) \tag{C.6}$$

and the PF S-matrix is

$$S^{PF}_{(j,n),(j'n')} = \frac{1}{\sqrt{2k}} e^{\frac{i\pi n n'}{k}} S_{jj'}\,, \tag{C.7}$$

where $S_{jj'}$ defined in (5.12).

When combining left and right-movers, the simplest modular invariant partition function of the parafermion theory is obtained by summing over all distinct representations

$$Z = \sum_{(j,n)\in PF_k} |\chi_{j,n}|^2\,. \tag{C.8}$$

The parafermion theory has a global Z_k symmetry under which the fields $\psi_{j,n}$ generating the representation (j,n) transform as

$$g: \quad \psi_{j,n} \to \omega^n \psi_{j,n}, \quad \omega = e^{\frac{2\pi i}{k}}\,. \tag{C.9}$$

Therefore we can orbifold the theory by this group. Taking the symmetric orbifold by Z_k of (C.8) leads to the partition function

$$Z = \sum_{(j,n)\in PF_k} \chi_{j,n}\bar{\chi}_{j,-n}\,. \tag{C.10}$$

We see that effect of the orbifold is to change the relative sign between the left and right movers of the $U(1)$ group with which we orbifold. Therefore the Z_k orbifold of the parafermion theory at level k is T-dual to the original theory. This fact will be the basis of many constructions in the main text.

D. Various coordinate systems for the sphere and relations between them

A three-sphere S^3 is a group manifold of the $SU(2)$ group. A generic element in this group can be written as

$$g = X_0\sigma_0 + i(X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3) = \begin{pmatrix} X_0 + iX_3 & X_2 + iX_1 \\ -(X_2 - iX_1) & X_0 - iX_3 \end{pmatrix} \tag{D.1}$$

subject to condition that the determinant is equal to one

$$X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1\,. \tag{D.2}$$

The metric on S^3 can be written in the following three ways, which will be used in the main text. Firstly, using the Euler parametrisation of the group element we have

$$g = e^{i\chi\frac{\sigma_3}{2}} e^{i\tilde{\theta}\frac{\sigma_1}{2}} e^{i\varphi\frac{\sigma_3}{2}} \tag{D.3}$$

$$\mathrm{d}s^2 = \frac{1}{4}\Big((\mathrm{d}\chi + \cos\tilde{\theta}\mathrm{d}\varphi)^2 + \mathrm{d}\tilde{\theta}^2 + \sin^2\tilde{\theta}\mathrm{d}\varphi^2\Big)\,. \tag{D.4}$$

The ranges of coordinates are $0 \le \tilde{\theta} \le \pi$, $0 \le \varphi \le 2\pi$ and $0 \le \chi \le 4\pi$.

Secondly, we can use coordinates that are analogue to the global coordinate for AdS_3

$$X_0 + iX_3 = \cos\theta e^{i\tilde{\phi}}\,, \quad X_2 + iX_1 = \sin\theta e^{i\phi} \tag{D.5}$$

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\tilde{\phi}^2 + \sin^2\theta\mathrm{d}\phi^2\,. \tag{D.6}$$

The relation between the metrics (D.3) and (D.5) is given by

$$\chi = \tilde{\phi} + \phi\,, \quad \varphi = \tilde{\phi} - \phi\,, \quad \theta = \frac{\tilde{\theta}}{2}\,. \tag{D.7}$$

The ranges of coordinates are $-\pi \leq \tilde{\phi}, \phi \leq \pi$ and $0 \leq \theta \leq \frac{\pi}{2}$.

Thirdly, the standard metric on S^3 is given by ($\vec{n}$ is a unit vector on S^2)

$$g = e^{2i\psi \frac{\vec{n}\cdot\vec{\sigma}}{2}}\,, \quad \mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\xi^2 + \sin^2\xi \mathrm{d}\eta^2) \tag{D.8}$$

$$X_0 + iX_3 = \cos\psi + i\sin\psi\cos\xi\,, \quad X_2 + iX_1 = \sin\psi\sin\xi e^{i\eta}\,. \tag{D.9}$$

The ranges of the coordinates are $0 \leq \psi\,,\ \xi \leq \pi$ and $0 \leq \eta \leq 2\pi$.

References

[1] J. M. Figueroa-O'Farrill and S. Stanciu, *D-branes in AdS(3) x S(3) x S(3) x S(1)*, *JHEP* **04** (2000) 005, [hep-th/0001199].

[2] A. Recknagel, *Permutation branes*, *JHEP* **04** (2003) 041, [hep-th/0208119].

[3] M. R. Gaberdiel and S. Schafer-Nameki, *D-branes in an asymmetric orbifold*, *Nucl. Phys.* **B654** (2003) 177–196, [hep-th/0210137].

[4] J. Fuchs, I. Runkel, and C. Schweigert, *Boundaries, defects and frobenius algebras*, *Fortsch. Phys.* **51** (2003) 850–855, [hep-th/0302200].

[5] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. I: General theory*, *Nucl. Phys.* **B558** (1999) 419–483, [hep-th/9902132].

[6] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. II: More structures, examples*, *Nucl. Phys.* **B568** (2000) 543–593, [hep-th/9908025].

[7] T. Quella, *On the hierarchy of symmetry breaking D-branes in group manifolds*, *JHEP* **12** (2002) 009, [hep-th/0209157].

[8] J. Fuchs *et. al.*, *Boundary fixed points, enhanced gauge symmetry and singular bundles on K3*, *Nucl. Phys.* **B598** (2001) 57–72, [hep-th/0007145].

[9] J. M. Maldacena, G. W. Moore, and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *JHEP* **07** (2001) 046, [hep-th/0105038].

[10] G. Sarkissian, *Non-maximally symmetric D-branes on group manifold in the Lagrangian approach*, *JHEP* **07** (2002) 033, [hep-th/0205097].

[11] J. M. Maldacena, G. W. Moore, and N. Seiberg, *D-brane instantons and K-theory charges*, *JHEP* **11** (2001) 062, [hep-th/0108100].

[12] G. Sarkissian, *On D-branes in the Nappi-Witten and GMM models*, *JHEP* **01** (2003) 059, [hep-th/0211163].

[13] T. Quella and V. Schomerus, *Asymmetric cosets*, *JHEP* **02** (2003) 030, [hep-th/0212119].

[14] C. Klimcik and P. Severa, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B488** (1997) 653–676, [hep-th/9609112].

[15] K. Gawedzki, *Conformal field theory: A case study*, hep-th/9904145.

[16] P. Bordalo, S. Ribault, and C. Schweigert, *Flux stabilization in compact groups*, *JHEP* **10** (2001) 036, [hep-th/0108201].

[17] S. Stanciu, *A note on D-branes in group manifolds: Flux quantization and D0-charge*, *JHEP* **10** (2000) 015, [hep-th/0006145].

[18] A. Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D60** (1999) 061901, [hep-th/9812193].

[19] G. Sarkissian, *On DBI action of the non-maximally symmetric D-branes on SU(2)*, *JHEP* **01** (2003) 058, [hep-th/0211038].

[20] N. Vilenkin, *Special functions and the theory of group representations*, . Providence, R.I., American Mathematical Society (AMS), (1968) 613 p.

[21] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162–190, [hep-th/9909030].

[22] V. Schomerus, *Lectures on branes in curved backgrounds*, *Class. Quant. Grav.* **19** (2002) 5781–5847, [hep-th/0209241].

[23] P. Bain, P. Meessen, and M. Zamaklar, *Supergravity solutions for D-branes in Hpp-wave backgrounds*, *Class. Quant. Grav.* **20** (2003) 913–934, [hep-th/0205106].

[24] P. Bain, K. Peeters, and M. Zamaklar, *D-branes in a plane wave from covariant open strings*, *Phys. Rev.* **D67** (2003) 066001, [hep-th/0208038].

[25] G. Sarkissian and M. Zamaklar, *Diagonal D-branes in product spaces and their Penrose limits*, hep-th/0308174.

[26] A. Klemm and M. G. Schmidt, *Orbifolds by cyclic permutations of tensor product conformal field theories*, *Phys. Lett.* **B245** (1990) 53–58.

[27] L. Borisov, M. B. Halpern, and C. Schweigert, *Systematic approach to cyclic orbifolds*, *Int. J. Mod. Phys.* **A13** (1998) 125–168, [hep-th/9701061].