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The Volume Operator in Spherically Symmetric Quantum Geometry

Martin Bojowald* and Rafal Swiderski†

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, D-14476 Golm, Germany

Abstract

The spherically symmetric volume operator is discussed and all its eigenstates and eigenvalues are computed. Even though the operator is more complicated than its homogeneous analog, the spectra are related in the sense that the larger spherically symmetric volume spectrum adds fine structure to the homogeneous spectrum. The formulas of this paper complete the derivation of an explicit calculus for spherically symmetric models which is needed for future physical investigations.

1 Introduction

In complicated theories such as general relativity or its possible quantization, loop quantum gravity [1], much information can usually be gleaned by studying simpler situations in symmetric contexts. Classically one thus obtains special exact solutions, which are relevant if perturbations around them are stable. In the quantum theory, symmetric models always present an approximation, whose relevance can be investigated by comparison with less symmetric situations. Examples for exact symmetric solutions were found early on, with applications in cosmology (isotropic space, [2, 3]) and black holes (spherical symmetry, [4]). Quantum gravity in the Wheeler–DeWitt form has, similarly, first been studied in isotropic models [5] followed later by inhomogeneous ones [6].

In loop quantum gravity there is a systematic procedure to derive symmetric models and to relate them to the full theory [7, 8]. Also here, applications were first done in the simplest, isotropic context followed by anisotropic models. Applications include a general non-singular behavior in those models [9, 10, 11, 12], new information about initial conditions [13], and several phenomenological scenarios [14, 15, 16].

Spherical symmetry presents an inhomogeneous model with infinitely many kinematical degrees of freedom, but also interesting physical applications for black holes. At the

*e-mail address: mabo@aei.mpg.de
†e-mail address: swiderski@aei.mpg.de

same time, results from homogeneous models can be checked by comparing with a more complicated, inhomogeneous context. In many respects, spherical symmetry yields the simplest inhomogeneous model. Classically, there are only finitely many physical degrees of freedom and the system is exactly soluble [17, 18]. The loop quantization also presents some features that can be expected to simplify calculations, such as the existence of a flux representation [19] which was helpful in homogeneous models. We will see here that also the volume spectrum can be obtained explicitly which is essential for calculations involving, e.g., the Hamiltonian constraint. However, the volume will already be more complicated than in homogeneous models, which will appear as special cases.

One of the aims of studying different symmetric models is to weaken the degree of symmetry step by step, in order to check the robustness of results. Here, our focus will be on the effect of level splitting (of volume eigenvalues; for the area operator see [20]) which is a consequence of weakening the symmetry and familiar from atomic or molecular spectroscopy. A difference in the case of quantum geometry is that the symmetry can only be reduced in discrete steps rather than by a continuous deformation. Mathematically this is implied by the fact that symmetric states are not ordinary states but distributions in a less symmetric model. Yet, the effect is visible, e.g. when comparing isotropy with anisotropy. It will be seen similarly, but in a much more finely structured way, when comparing homogeneity with inhomogeneity, even when only a single vertex is considered.

This effect illustrates properties of the volume and may also shed light on the full case. Further applications, which will be studied elsewhere, include, e.g., the Hamiltonian constraint, observables, horizons in quantum geometry. The calculations will be more cumbersome than in homogeneous models since the volume operator has eigenstates different from those of flux operators, and the transformation between the eigenbases is quite involved. We will present here some of the formulas which can be helpful for explicit investigations.

2 Spherical symmetry

The spherically symmetric model from the point of view of quantum geometry has been described in [19], where its states and basic operators have been derived from a symmetry reduction. Here we collect the information relevant for studying the volume operator.

2.1 Classical reduction

Spherically symmetric connections and densitized triads on a manifold $\Sigma \cong B \times S^2$ can be written as

$$A = A_x(x)\Lambda_3 \mathrm{d}r + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)\mathrm{d}\vartheta + (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)\sin\vartheta \mathrm{d}\varphi + \Lambda_3 \cos\vartheta \mathrm{d}\varphi \quad (1)$$

and

$$E = E^x(x)\Lambda_3 \sin\vartheta \frac{\partial}{\partial x} + (E^1(x)\Lambda_1 + E^2(x)\Lambda_2)\sin\vartheta \frac{\partial}{\partial \vartheta} + (E^1(x)\Lambda_2 - E^2(x)\Lambda_1)\frac{\partial}{\partial \varphi} \quad (2)$$

with real functions A_x, A_1, A_2, E^x, E^1 and E^2 on a one-dimensional, radial manifold B with coordinate x (in addition to polar coordinates ϑ and φ). The su(2)-matrices Λ_I are constant and identical to $\tau_I = -\frac{i}{2}\sigma_I$ (with Pauli matrices σ_I) or a rigid rotation thereof. The symplectic structure is given by

$$\Omega_B = \frac{1}{2\gamma G}\int_B \mathrm{d}x(\mathrm{d}A_x \wedge \mathrm{d}E^x + 2\mathrm{d}A_1 \wedge \mathrm{d}E^1 + 2\mathrm{d}A_2 \wedge \mathrm{d}E^2) \tag{3}$$

with the gravitational constant G and the Barbero–Immirzi parameter γ.

The reduced theory has residual U(1) gauge transformations corresponding to rotations around Λ_3 and generated by the Gauss constraint

$$G[\lambda] = \int_B \mathrm{d}x\lambda(E^{x\prime} + 2A_1E^2 - 2A_2E^1) \approx 0\,. \tag{4}$$

The components (A_1, A_2) and (E^1, E^2) transform in the defining SO(2)-representation, such that the components

$$A_\varphi(x) := \sqrt{A_1(x)^2 + A_2(x)^2}\,, \tag{5}$$
$$E_\varphi(x) := \sqrt{E^1(x)^2 + E^2(x)^2} \tag{6}$$

are gauge invariant. The corresponding internal directions

$$\Lambda^A_\varphi(x) := (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)/A_\varphi(x)\,, \tag{7}$$
$$\Lambda^E_\varphi(x) := (E^1(x)\Lambda_2 - E^2(x)\Lambda_1)/E^\varphi(x) \tag{8}$$

are rotated simultaneously such that

$$\cos\alpha(x) := \Lambda^A_\varphi(x)\cdot\Lambda^E_\varphi(x) \tag{9}$$

is also gauge invariant. Parameterizing

$$\Lambda^A_\varphi(x) =: \Lambda_1\cos\beta(x) + \Lambda_2\sin\beta(x)\,, \tag{10}$$
$$\Lambda^E_\varphi(x) =: \Lambda_1\cos(\alpha(x)+\beta(x)) + \Lambda_2\sin(\alpha(x)+\beta(x)) \tag{11}$$

introduces the angle $\beta(x)$ which is pure gauge.

In these variables which take into account the gauge structure we have the symplectic form

$$\begin{aligned}\Omega_B &= \frac{1}{2\gamma G}\int_B \mathrm{d}x\,(\mathrm{d}A_x\wedge\mathrm{d}E^x + 2\mathrm{d}A_\varphi\wedge\mathrm{d}(E^\varphi\cos\alpha) + 2\mathrm{d}\beta\wedge\mathrm{d}(A_\varphi E^\varphi\sin\alpha))\\ &= \frac{1}{2\gamma G}\int_B \mathrm{d}x\left(\mathrm{d}A_x\wedge\mathrm{d}E^x + \mathrm{d}A_\varphi\wedge\mathrm{d}P^\varphi + \mathrm{d}\beta\wedge\mathrm{d}P^\beta\right)\end{aligned} \tag{12}$$

with momenta

$$P^\varphi(x) := 2E^\varphi(x)\cos\alpha(x) \tag{13}$$

conjugate to A_φ and

$$P^\beta(x) := 2A_\varphi(x)E^\varphi(x)\sin\alpha(x) = A_\varphi(x)P^\varphi(x)\tan\alpha(x) \tag{14}$$

conjugate to β. The Gauss constraint then takes the form

$$G[\lambda] = \int_B \mathrm{d}x\lambda(E^{x\prime} - P^\beta) \approx 0 \tag{15}$$

which is easily solved by $P^\beta = E^{x\prime}$ while the function $A_x + \beta'$ is manifestly gauge invariant.

Thus, in this model flux variables (i.e., momenta of the connection), are not identical to triad components. In particular, some metric components in

$$\mathrm{d}s^2 = E^x(x)^{-1}E^\varphi(x)^2\mathrm{d}x^2 + E^x(x)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\,, \tag{16}$$

and in particular the volume $V(\mathcal{I}) = 4\pi\int_{\mathcal{I}}\mathrm{d}x\sqrt{|E^x|}E^\varphi$ of the shell $\mathcal{I}\times S^2 \subset \Sigma$ given by an interval $\mathcal{I} \subset B$, are rather complicated functions of the momenta as well as of connection components.

2.2 Loop quantization

States in the connection representation depend on the components A_x, A_φ and β. The loop representation is based on cylindrical states with respect to graphs g in the one-dimensional radial manifold B, consisting of a collection of non-overlapping edges e whose endpoints define the vertex set $V(g)$. We assume a given orientation of B and choose the same one for all edges. Each vertex $v \in V(g)$ then has one outgoing edge $e^+(v)$ and one incoming edge $e^-(v)$. An orthonormal basis of states is given by spin network states

$$T_{g,k,\mu} = \prod_{e\in g}\exp\left(\tfrac{1}{2}ik_e\int_e(A_x+\beta')\mathrm{d}x\right)\prod_{v\in V(g)}\exp(i\mu_v A_{\varphi(v)}) \tag{17}$$

in the gauge invariant form, with labels $k_e \in \mathbb{Z}$ for U(1)-representations on all edges $e \subset B$ and $\mu_v \in \mathbb{R}$ for all vertices $v \in B$. Thus, in addition to ordinary holonomies of A_x along edges there are vertices with point holonomies $\exp(i\mu_v A_{\varphi(v)})$ which can be thought of as representing orbital edges in the symmetry orbits S^2. Gauge invariance requires only that the differences $k_{e^+(v)} - k_{e^-(v)}$ for any two edges $e^+(v)$ and $e^-(v)$ meeting in a common vertex v must be even. Otherwise the edge and vertex labels are free.

Connection components are quantized via holonomies and act directly by multiplication. For instance, looking only at a single vertex v and suppressing all labels but μ_v, we obtain

$$\sin(\mu A_\varphi)T_{\mu_v} = -\tfrac{1}{2}i(T_{\mu_v+\mu} - T_{\mu_v-\mu}) \tag{18}$$

$$\cos(\mu A_\varphi)T_{\mu_v} = \tfrac{1}{2}(T_{\mu_v+\mu} + T_{\mu_v-\mu}) \tag{19}$$

as basic operators representing the transversal connection component A_φ which takes values in the Bohr compactification $\bar{\mathbb{R}}_{\rm Bohr}$ of the real line. Flux components act as

$$\hat{E}^x(x)T_{g,k,\mu} = \frac{\gamma\ell_{\rm P}^2}{8\pi}\frac{k_{e^+(x)}+k_{e^-(x)}}{2}T_{g,k,\mu} \tag{20}$$

$$\int_{\mathcal{I}}\hat{P}^\varphi T_{g,k,\mu} = \frac{\gamma\ell_{\rm P}^2}{4\pi}\sum_{v\in\mathcal{I}\cap V(g)}\mu_v T_{g,k,\mu} \tag{21}$$

$$\int_{\mathcal{I}}\hat{P}^\beta T_{g,k,\mu} = \frac{\gamma\ell_{\rm P}^2}{4\pi}\sum_{v\in\mathcal{I}\cap V(g)}k_v T_{g,k,\mu} \tag{22}$$

with $k_v := \frac{1}{2}(k_{e^+(v)}-k_{e^-(v)})\in\mathbb{Z}$, which shows that they all have discrete spectra (normalizable eigenstates).

The momenta P^φ and P^β transform as scalars of density weight one and thus have to be integrated over intervals $\mathcal{I}\subset B$ in order to yield well-defined flux operators. Alternatively, they can be regarded as operator valued distributions

$$\hat{P}^\varphi(x)T_{g,k,\mu} = \frac{\gamma\ell_{\rm P}^2}{4\pi}\sum_{v\in V(g)}\mu_v\delta(x,v)T_{g,k,\mu}\,.$$

3 Volume operator

Classically, the volume of a spherically symmetric region between two spherical shells located at points $x_1, x_2\in B$ is given by

$$V(\mathcal{I}) = 4\pi\int_{\mathcal{I}}{\rm d}x\sqrt{|E^x|}E^\varphi \tag{23}$$

where $\mathcal{I}=[x_1,x_2]$. Since $|\hat{E}^x|$ is a non-negative self-adjoint operator, we can directly use it in this expression. The triad component E^φ, on the other hand, is more complicated to quantize since it is related to the basic variables by (13) which also involves connection components through α. Using (13) and (14), we obtain

$$E^\varphi = \frac{P^\varphi}{2\cos\alpha} = \tfrac{1}{2}P^\varphi\sqrt{1+\tan^2\alpha} = \tfrac{1}{2}\sqrt{(P^\varphi)^2+A_\varphi^{-2}(P^\beta)^2}\,. \tag{24}$$

In order to quantize E^φ and diagonalize the resulting operator it suffices to look at the action on eigenstates of $\hat{E}^x$ and thus $\hat{P}^\beta$ since this does not restrict the A_φ-dependence of states. In the expression for E^φ we can then regard P^β as a constant $\gamma\ell_{\rm P}^2k/4\pi$ and use basic operators for P^φ and A_φ. Thus, $(E^\varphi)^2$ has the form of a classical mechanics Hamiltonian with potential k^2/A_φ^2. Taking this Hamiltonian at face value suggests that the volume spectrum is continuous since the potential does not have an absolute minimum at finite values. In fact, since E^φ depends on fluxes as well as connection components there is no a priori reason to expect a discrete spectrum. However, a continuous spectrum

would be difficult to reconcile with the full volume operator [21] as well as with results in homogeneous models [22, 10, 11].

This problem is resolved easily after noticing that there is no operator for A_φ directly in a loop representation, or on the Bohr Hilbert space in this case (see the discussion of [23] in the isotropic context). Then neither the quantization of A_φ^{-2} as a multiplication operator on a Schrödinger Hilbert space, as understood in the above argument, is available. In fact, the discreteness of geometric spectra depends crucially on characteristic features of the loop representation. Following a loop quantization of (24) and checking if the resulting spectrum will turn out to be discrete can thus be seen as a further test of the generality of loop results. Since the spherically symmetric volume is a rather complicated function of the basic variables, but not basic itself, one cannot expect a unique (up to smaller choices) quantization. But the sensitivity of general properties like the discreteness of its spectrum and its behavior can easily be studied with different versions and compared with the full theory as well as other models.

In a loop quantization only holonomies of the connection are represented as well-defined operators on the Hilbert space, not connection components directly. This is also true for connection components acting on a Bohr Hilbert space as in (18), (19). Since the states (17) are orthonormal, the μ-derivative at $\mu = 0$ of the operators $\sin(\mu A_\varphi)$ and $\cos(\mu A_\varphi)$ does not exist such that an operator for A_φ cannot be derived in this way. Instead, we have to use (18) and (19) with non-zero μ as our basic operators in every expression where A_φ appears and needs to be quantized. This in general is true for the Hamiltonian constraint which contains curvature components of the connection. A quantization can then be constructed by using holonomies of the connection [24], and in a homogeneous model or for orbital components of an invariant connection point holonomies [23]. Since in the spherically symmetric model the volume is more complicated in terms of basic variables, containing in particular connection components, we have to use the same technique here. We will exploit this fact in order to study consequences of replacing connection components by holonomies by means of comparing the spherically symmetric volume spectrum with that of homogeneous models where such a replacement is not necessary for the volume operator.

The connection components contain information about intrinsic curvature (via the spin connection) and extrinsic curvature. In a semiclassical regime, which we need to take into account to ensure that the resulting volume operator will have the correct classical limit, extrinsic curvature is small but the spin connection may not be small when we restrict ourselves to spherical coordinates which just leaves the freedom to transform x (under arbitrary coordinate transformations the spin connection can locally always be made as small as desired). For our purposes it is nevertheless sufficient to assume that the connection components, or at least A_φ, are small semiclassically since the spectrum will be invariant under shifting A_φ by a constant in the operator used later.

We thus replace A_φ by $\sin(\delta A_\varphi)/\delta$ for some $\delta \in \mathbb{R}\backslash\{0\}$. In the limit $\delta \to 0$ we would obtain the classical expression exactly, and for $\delta \neq 0$ we have small corrections in regions

where A_φ is small. This leads us to consider the Hamiltonian operator

$$\left(\frac{4\pi}{\gamma\ell_{\mathrm{P}}^2}\right)^2 \hat{H}_{k,\delta} = (4\pi\hat{P}^\varphi/\gamma\ell_{\mathrm{P}}^2)^2 + \delta^2 k^2 \sin(\delta A_\varphi)^{-2} \tag{25}$$

for integer k and non-zero real δ. It will later be used to quantize E^φ and the volume. At this point we can already see that replacing A_φ by the sine, which we are forced to do in the loop representation, leads to a discrete spectrum since the potential is now unbounded above at both sides. This fact does not depend on the explicit realization of the replacement and is thus robust to quantization ambiguities. In order to test the replacement we will later use more detailed properties of the spectrum and compare it with volume spectra in other models.

Before we continue we have to discuss the Hilbert space on which the operator (25) acts. Since it suffices to consider a single vertex and we already used the eigenvalue k of $\hat{P}^\beta$ it acts on a single copy of the Bohr Hilbert space with $A_\varphi \in \bar{\mathbb{R}}_{\mathrm{Bohr}}$. Thus, the Hamiltonian is not yet of the usual quantum mechanical form since so far it is not represented on the Schrödinger Hilbert space. However, it is clear from (18) that $\hat{H}_{k,\delta}$ leaves invariant any subspace spanned by states $\exp(i(n\delta+\epsilon)A_\varphi)$, $n \in \mathbb{Z}$ with fixed $\epsilon \in [0,\delta)$. Each such subspace is isomorphic to the Schrödinger Hilbert space on a circle where for $\epsilon \neq 0$ we have to use a non-trivial line bundle for the states. That the inner product on the subspace spanned by states $\exp(i(n\delta+\epsilon)A_\varphi)$, $n \in \mathbb{Z}$ is identical to that for states on a circle follows from

$$\begin{aligned}\int (e^{i(n_1\delta+\epsilon)A_\varphi})^* e^{i(n_2\delta+\epsilon)A_\varphi} \mathrm{d}\mu_{\mathrm{Bohr}}(A_\varphi) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i(n_1-n_2)\delta A_\varphi}\mathrm{d}A_\varphi \\ &= \lim_{m\to\infty}\frac{1}{2m}\int_{-m}^{m} e^{-2\pi i(n_1-n_2)y}\mathrm{d}y = \int_0^1 e^{-2\pi i(n_1-n_2)y}\mathrm{d}y \\ &= \delta_{n_1,n_2}\end{aligned}$$

using the Bohr measure and substituting $y = \delta A_\varphi/2\pi$ and $m = \delta T/2\pi$. Now we have an ordinary quantum mechanical system with Hamiltonian (25) whose spectrum can be determined with standard methods (from this perspective, ϵ plays the role of a θ-angle).

3.1 Spectrum of $\hat{H}$

Using $\hat{P}_\varphi \propto \mathrm{d}/\mathrm{d}A_\varphi$ and substituting δA_φ for A_φ, the operator family (25) labeled by δ satisfies the identity $\hat{H}_{k,\delta} = \delta^2 \hat{H}_{k,1}$ such that it is enough to determine the spectrum for $\delta = 1$. Moreover, since the sequence of states running through $\hat{P}_\varphi$-eigenvalue zero is most important, we will focus on the case $\epsilon = 0$, for which states are periodic functions on a circle. There are then standard techniques (see, e.g., [25], p. 732) to determine the spectrum of the operator

$$\hat{H}_\rho = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{\rho^2 - \frac{1}{4}}{\sin^2\xi}$$

where for later convenience we introduce $\rho := \sqrt{k^2 + \frac{1}{4}} > 0$ which for our case of integer k is always non-integer.

If we know an eigenstate ψ_n^ρ with eigenvalue λ_n labeled by some parameter n, i.e. $\hat{H}_\rho \psi_n^\rho = \lambda_n \psi_n^\rho$, we can find eigenstates of $\hat{H}_{\rho\pm1}$ by applying the operators

$$\hat{a}_\rho^\pm := \pm \frac{\mathrm{d}}{\mathrm{d}\xi} + (\rho - \tfrac{1}{2}) \cot \xi \tag{26}$$

to the state ψ_n^ρ. (For our values of ρ, $\rho \pm 1$ will not be of the form $\sqrt{k^2 + \frac{1}{4}}$ with integer k. Nevertheless, these values can be used in intermediate steps to construct the eigenstates.) This follows from the identities

$$\hat{a}_\rho^- \hat{a}_\rho^+ = \hat{H}_\rho - (\rho - \tfrac{1}{2})^2 \tag{27}$$

$$\hat{a}_{\rho+1}^+ \hat{a}_{\rho+1}^- = \hat{H}_\rho - (\rho + \tfrac{1}{2})^2 \tag{28}$$

which imply

$$\hat{H}_{\rho-1} \hat{a}_\rho^+ = (\hat{a}_\rho^+ \hat{a}_\rho^- + (\rho - \tfrac{1}{2})^2) \hat{a}_\rho^+ = \hat{a}_\rho^+ (\hat{a}_\rho^- \hat{a}_\rho^+ + (\rho - \tfrac{1}{2})^2) = \hat{a}_\rho^+ \hat{H}_\rho$$

and similarly

$$\hat{H}_{\rho+1} \hat{a}_{\rho+1}^- = \hat{a}_{\rho+1}^- \hat{H}_\rho \,.$$

Thus, if ψ_n^ρ is an eigenstate of $\hat{H}_\rho$ with eigenvalue λ_n, then $\hat{a}_\rho^+ \psi_n^\rho$ is an eigenstate of $\hat{H}_{\rho-1}$ and $\hat{a}_{\rho+1}^- \psi_n^\rho$ an eigenstate of $\hat{H}_{\rho+1}$ with the same eigenvalue λ_n.

In order to normalize the new eigenstates we compute

$$\int |\hat{a}_{\rho+1}^- \psi_n^\rho|^2 \mathrm{d}\xi = \int \psi_n^{\rho *} \hat{a}_{\rho+1}^+ \hat{a}_{\rho+1}^- \psi_n^\rho \mathrm{d}\xi = (\lambda_n - (\rho + \tfrac{1}{2})^2) \int |\psi_n^\rho|^2 \mathrm{d}\xi \tag{29}$$

and

$$\int |\hat{a}_\rho^+ \psi_n^\rho|^2 \mathrm{d}\xi = \int \psi_n^{\rho *} \hat{a}_\rho^- \hat{a}_\rho^+ \psi_n^\rho \mathrm{d}\xi = (\lambda_n - (\rho - \tfrac{1}{2})^2) \int |\psi_n^\rho|^2 \mathrm{d}\xi \,. \tag{30}$$

Since all eigenvalues of $\hat{H}_\rho$ are larger than the potential minimum $\rho^2 - \frac{1}{4} = (\rho + \frac{1}{2})(\rho - \frac{1}{2})$, we must have started with an eigenvalue $\lambda_n > (\rho - \frac{1}{2})^2$. When applying $\hat{a}_\rho^-$ several times, we get eigenstates to $\hat{H}_\rho$ with higher and higher values for ρ, and since λ_n stays the same in this process and the integrals in (29) are non-negative, (29) is consistent only if $\lambda_n = (\rho + n + \frac{1}{2})^2$ and $n \in \mathbb{N}_0$ a non-negative integer. The state with the highest allowed ρ at fixed λ_n, i.e. for $n = 0$, is then annihilated by $\hat{a}_{\rho+1}^-$ as a consequence of (29):

$$\hat{a}_{\rho+1}^- \psi_0^\rho = 0$$

which has normalized solution

$$\psi_0^\rho(\xi) = \frac{\sqrt{\frac{1}{2}\Gamma(2\rho + 2)}}{2^\rho \Gamma(\rho + 1)} \sin^{\rho+1/2} \xi \,. \tag{31}$$

Normalized eigenstates with values $n > 0$ are obtained by applying $\hat{a}^+_\rho$ and using (30) with $\lambda_n - (\rho - \frac{1}{2})^2 = (2\rho + n)(n + 1)$ such that $\psi^{\rho-1}_{n+1} = [(2\rho + n)(n + 1)]^{-1/2}\hat{a}^+_\rho \psi^\rho_n$. The eigenstate ψ^ρ_n can thus be obtained from the known $\psi^{\rho+n}_0$ as

$$\psi^\rho_n = \sqrt{\frac{\Gamma(2\rho + n + 1)}{n!\,\Gamma(2\rho + 2n + 1)}}\, \hat{a}^+_{\rho+1} \cdots \hat{a}^+_{\rho+n} \psi^{\rho+n}_0 \,.$$

Using

$$\hat{a}^+_\rho = \frac{\mathrm{d}}{\mathrm{d}\xi} + (\rho - \tfrac{1}{2}) \cot\xi = (\sin\xi)^{1/2-\rho} \frac{\mathrm{d}}{\mathrm{d}\xi} (\sin\xi)^{\rho-1/2} = -(\sin\xi)^{3/2-\rho} \frac{\mathrm{d}}{\mathrm{d}\cos\xi} (\sin\xi)^{\rho-1/2}$$

such that

$$\hat{a}^+_{\rho+1} \cdots \hat{a}^+_{\rho+n} = (-1)^n (\sin\xi)^{1/2-\rho} \frac{\mathrm{d}^n}{(\mathrm{d}\cos\xi)^n} (\sin\xi)^{\rho+n-1/2}$$

this yields all eigenstates as

$$\psi^\rho_n(\xi) = (-1)^n \frac{\sqrt{(\rho + n + \frac{1}{2})\Gamma(2\rho + n + 1)}}{2^{\rho+n}\Gamma(\rho + n + 1)\sqrt{n!}} \sin^{1/2-\rho}\xi \frac{\mathrm{d}^n}{(\mathrm{d}\cos\xi)^n} \sin^{2(\rho+n)}\xi \tag{32}$$

of $\hat{H}_\rho$ with eigenvalue $\lambda_n = (\rho + n + 1/2)^2$, $n \in \mathbb{N}_0$. If $\rho = 1/2$, i.e. $k = 0$, the potential vanishes and the poles at $x = 0, \pi$ disappear. In this case boundary conditions leave more freedom since states do not need to vanish at $x = 0, \pi$ which results in the additional eigenvalue $\lambda = 0$ for the case $\rho = 1/2$.

These eigenfunctions can be expressed in terms of Legendre functions[1] $P^\mu_\nu(z)$ (see Appendix)

$$\psi^\rho_n(\xi) = \sqrt{(\rho + n + \tfrac{1}{2})\Gamma(2\rho + n + 1)/n!}\, \sqrt{|\sin\xi|}\, P^{-\rho}_{\rho+n}(\cos\xi) \tag{33}$$

such that the sum of degree and order, $\nu + \mu = n$, is a non-negative integer and the order $\mu = -\rho$ is negative.

3.2 Volume spectrum

We now have to construct the volume operator from the basic building blocks $\hat{E}^x$ and $\hat{E}^\varphi = \frac{\gamma\ell_{\mathrm{P}}^2}{8\pi}\sqrt{\hat{H}}$. In particular, expressions like $(P^\varphi)^2$ and $(P^\beta)^2$ in H have to be regularized upon quantization since $\hat{P}^\varphi$ and $\hat{P}^\beta$ are operator valued distributions.

We do that by decomposing a given interval $\mathcal{I}$, whose volume $V = 4\pi \int_{\mathcal{I}} \mathrm{d}x \sqrt{|E^x|} E^\varphi$ we want to quantize, into subintervals $\mathcal{I}_i = [x_i - \epsilon/2, x_i + \epsilon/2]$ each of size ϵ: $\mathcal{I} = \bigcup_i \mathcal{I}_i$.

[1] It is amusing to note that such as in rotationally symmetric systems in quantum mechanics also in spherically symmetric quantum geometry Legendre functions play an important role (with the difference that here the labels are always non-integer).

The volume can then be written as

$$\begin{aligned} V(\mathcal{I}) &= 4\pi\int_{\mathcal{I}}\mathrm{d}x\sqrt{|E^x|}E^\varphi = 4\pi\sum_i\int_{\mathcal{I}_i}\mathrm{d}x\sqrt{|E^x|}E^\varphi \\ &\approx 4\pi\sum_i\sqrt{|E^x(x_i)|}E^\varphi(x_i)\epsilon = 2\pi\sum_i\sqrt{|E^x(x_i)|}\sqrt{P^\varphi(x_i)^2\epsilon^2+P^\beta(x_i)^2\epsilon^2/A_\varphi(x_i)^2} \\ &\approx 2\pi\sum_i\sqrt{|E^x(x_i)|}\sqrt{(\textstyle\int_{\mathcal{I}_i}P^\varphi)^2+(\int_{\mathcal{I}_i}P^\beta)^2/A_\varphi(x_i)^2} \end{aligned}$$

where now only squares of fluxes occur which can immediately be quantized. We now choose the subdivision of $\mathcal{I}$ fine enough such that each $\mathcal{I}_i$ contains at most one vertex of the spin network state we act on, and immediately obtain the volume operator

$$\hat{V}(\mathcal{I}) = \frac{\gamma\ell_{\mathrm{P}}^2}{2}\sum_v\sqrt{|\hat{E}^x(v)|\hat{H}(v)} \tag{34}$$

where only vertices of graphs appear since otherwise $\hat{P}^\varphi$ and $\hat{P}^\beta$, and thus $\hat{H}(v) = -\partial^2/\partial A_\varphi(v)^2 - (\sin A_\varphi(v))^{-2}\,\partial^2/\partial\beta(v)^2$, would annihilate the state.

Using the spectra of $\hat{E}^x$ and $\hat{H}$, we find the volume spectrum as

$$V_{k,n} = (\tfrac{1}{2}\gamma\ell_{\mathrm{P}}^2)^{3/2}(4\pi)^{-1/2}\sum_v\sqrt{\tfrac{1}{2}|k_{e^+(v)}+k_{e^-(v)}|}\left(n_v+\tfrac{1}{2}\left(1+\sqrt{1+(k_{e^+(v)}-k_{e^-(v)})^2}\right)\right) \tag{35}$$

where $k_{e^+(v)}$ is the label of the edge outgoing from v, $k_{e^-(v)}$ the label of the edge incoming to v, and n_v a label of the vertex with $n_v\in\mathbb{N}_0$ if $k_{e^+(v)}\neq k_{e^-(v)}$ and $n_v\in\mathbb{N}_0\cup\{-1\}$ if $k_{e^+(v)} = k_{e^-(v)}$. Moreover, for the spectrum on gauge invariant states the difference $k_{e^+(v)} - k_{e^-(v)}$ must be even [19], and thus also the sum $k_{e^+(v)} + k_{e^-(v)}$.

The A_φ-dependence of volume eigenstates is now via an $\hat{H}$-eigenstate $\psi^{\rho(v)}_{n(v)}(A_\varphi)$, where $\rho(v) = \frac{1}{2}\sqrt{1+(k_{e^+(v)}-k_{e^-(v)})^2}$ using the eigenvalue $\frac{1}{2}(k_{e^+(v)} - k_{e^-(v)})$ of $\hat{P}^\beta$, explicitly given by

$$S_{g,k,n} = \prod_{e\in g}\exp\left(\tfrac{1}{2}ik_e\int_e(A_x+\beta')\mathrm{d}x\right)\prod_{v\in V(g)}\psi_{n_v}^{\frac{1}{2}\sqrt{1+(k_{e^+(v)}-k_{e^-(v)})^2}}(A_{\varphi(v)})\,. \tag{36}$$

3.3 Level splitting

As in any loop quantization, the spherically symmetric volume operator has a discrete spectrum. Due to the symmetry, however, the set of eigenvalues (considering also non-zero values for ϵ) is the whole positive real line (see [23] for a discussion of this issue in the isotropic context; discussing observables may reduce the spectrum to a discrete subset [26]). It is nevertheless instructive to compare the main series of the volume spectra, which are obtained by restricting to a separable subspace corresponding to equidistant flux eigenvalues containing zero, for different symmetric models. Similarly to the spectroscopy

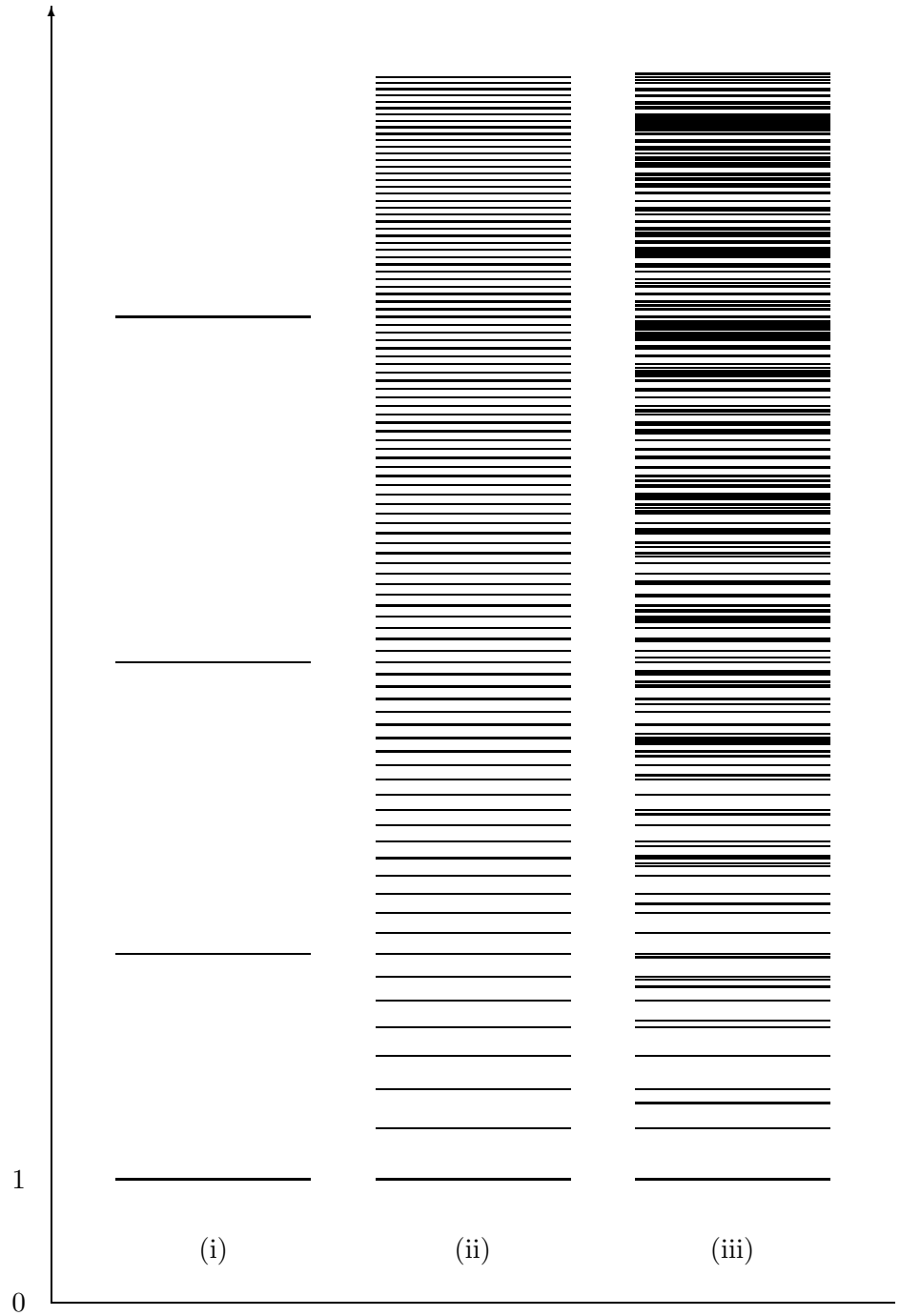

Figure 1: Comparison between the main series of the isotropic volume spectrum $n^{3/2}$, $n \in \mathbb{N}_0$ (i), the homogeneous LRS (single rotational axis) volume spectrum $m\sqrt{n}$, $m, n \in \mathbb{N}_0$ (ii) and one vertex contribution of the spherically symmetric volume spectrum (35) (iii). Constant factors resulting from the symplectic structure have been ignored.

of atoms, one then observes that levels split when the symmetry is weakened [20], as illustrated in Fig. 1.

Comparing in particular the homogeneous LRS spectrum with a single spherically symmetric vertex contribution shows that the spectra differ only in details such that the homogeneous levels split only slightly. Moreover, all the levels of the LRS spectrum are preserved and reappear in the spherically symmetric spectrum for $k_{e^+(v)} = k_{e^-(v)}$. Since the classical geometry in a single point of a spherically symmetric space is given by a homogeneous LRS metric we would in fact expect a close relationship between the quantum geometries in a given vertex. A spherically symmetric vertex is closest to a homogeneous state if $k_{e^+(v)} = k_{e^-(v)}$ (a homogeneous state [27, 11] would be built with point holonomies also of A_x, which can be seen as holonomies for closed edges leaving and entering in the vertex which requires equal spin numbers) in which case the spectra are identical. If $k_{e^+(v)} \neq k_{e^-(v)}$, inhomogeneity becomes noticeable and the levels split. One can view $k_{e^+(v)} - k_{e^-(v)} \in 2\mathbb{Z}$ as a new quantum number of spherically symmetric vertices characterizing the degree of inhomogeneity. The larger $k_{e^+(v)} - k_{e^-(v)}$, the more strongly the levels split. Even though there is no smooth transition between a homogeneous LRS state and a spherically symmetric one since a more symmetric state is distributional in the less symmetric model, one can visualize the splitting by introducing a continuous parameter $h \in [0, 1]$. With this parameter, we have a family of spectra

$$V^{(h)}_{k_+,k_-,n} = \sqrt{\tfrac{1}{2}|k_+ + k_-|}\left(n + \tfrac{1}{2}\left(1 + \sqrt{1 + h^2(k_+ - k_-)^2}\right)\right) .$$

For $h = 0$ we get the homogeneous spectrum, while $h = 1$ gives all levels of the spherically symmetric vertex contribution. Values in between do not correspond to physical models but are introduced for the sake of illustration. This clearly shows that levels with $k_+ - k_- = 0$ do not split, whereas all other levels split approximately linearly and the stronger the larger $k_+ - k_-$ is. Fig. 2 shows the splitting of all levels which will be smaller than five after the splitting is completed, from which one can recognize the lower part of Fig. 1.

Thus, even though the spherically symmetric volume operator at first sight looks very different from the homogeneous one (where fluxes and triad components are identical and holonomies are not used for the operator), its single vertex spectrum displays only slight traces of the inhomogeneity. As discussed, there are several ambiguities affecting the spherically symmetric operator, mainly coming from the replacement of A_φ. If we choose a value for δ different from $\delta = 1$ as used above, the spectrum of $\hat{H}$ is rescaled by δ^2 and that of $\hat{V}$ by δ. For this value, we would be considering a different series of states given by $e^{in\delta A_\varphi}$, which would similarly define a different series of states in the LRS Hilbert space. On those states, also the LRS volume operator is rescaled by exactly the same value δ such that the relationship between the spectra is preserved. If we choose a different function other than $\delta^{-1}\sin\delta A_\varphi$, on the other hand, the spherically symmetric spectrum would change without a corresponding change in the LRS spectrum. This would have an effect on the exact position of the new levels in the spherically symmetric spectrum compared to the LRS one.

Comparing all three cases of Fig. 1 shows that the difference between the isotropic

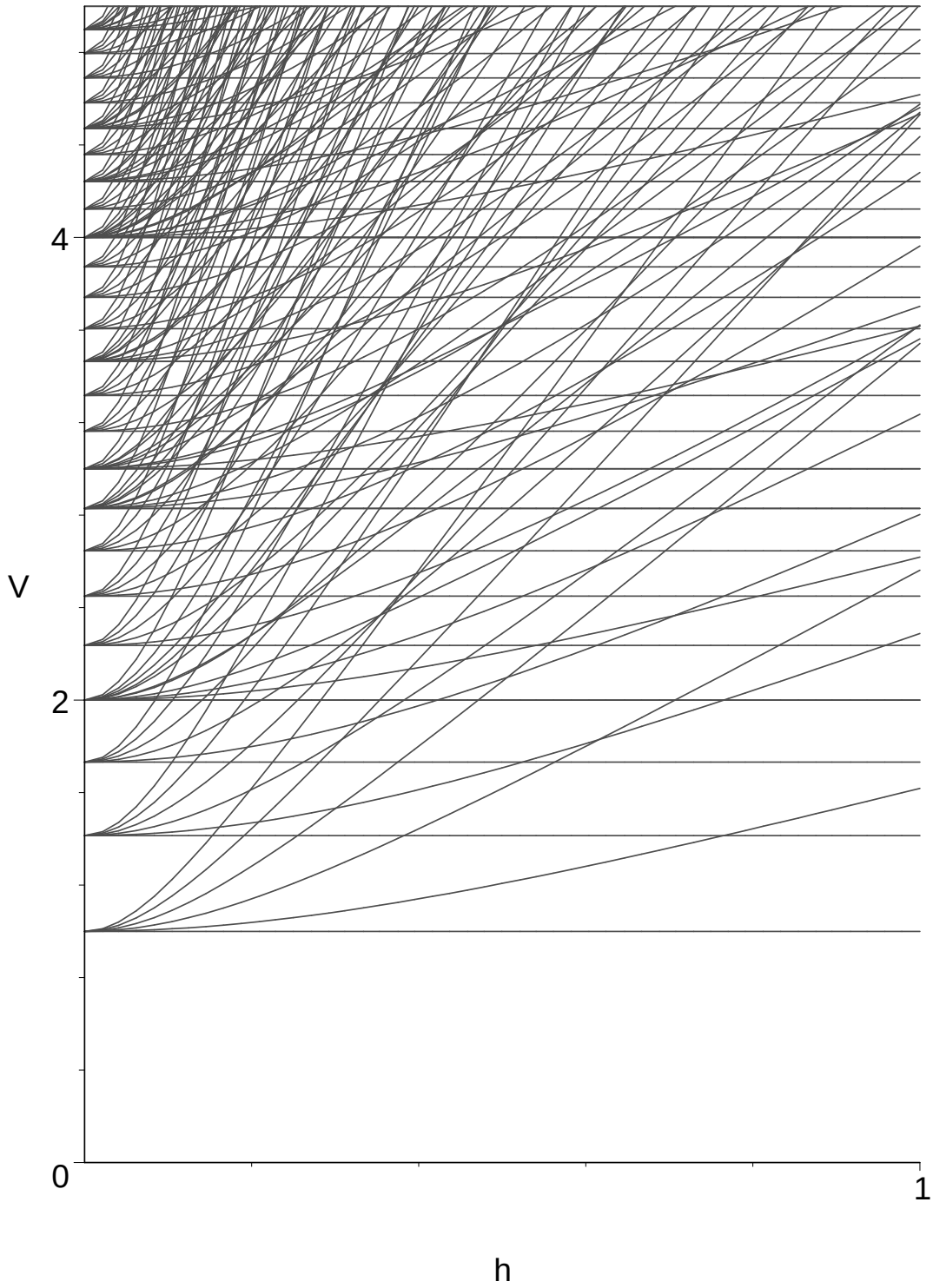

Figure 2: Explicit splitting of the LRS levels with an auxiliary parameter $0 < h < 1$.

spectrum and the homogeneous LRS spectrum is much more pronounced than that between the homogeneous LRS spectrum and the single vertex spherically symmetric spectrum. When breaking isotropy, many more lines emerge which cannot even be attributed to individual level splitting, and a spectrum with increasing level distance turns into one with decreasing distance. When breaking the homogeneity of the LRS model, on the other hand, the single vertex spectrum results from small level splitting, which can be considered as just fine structure of the overall spectrum. This may be taken as an indication that the transition from homogeneous to inhomogeneous models, though technically very involved, is less severe in geometrical and physical consequences than the transition from isotropy to anisotropy. Since physical results such as the non-singular behavior [9] have been extended from the isotropic context to the anisotropic one [11, 12], an extension to inhomogeneous models may be feasible, too.

Note, however, that we used only a single vertex contribution of the spherically symmetric volume spectrum for this comparison. Since the spectrum (35) is a vertex sum over non-integer contributions, it will become dense much more rapidly at large eigenvalues than just a single vertex contribution.

3.4 Triad spectrum

Eq. (24) shows that the eigenvalues of the triad component E^φ in a single vertex are given by the square root of those of the operator $\hat{H}$,

$$E^\varphi_{k,n} = \frac{\gamma\ell_{\mathrm{P}}^2}{8\pi}\sum_v \left(n_v + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+(k_{e^+(v)}-k_{e^-(v)})^2}\right) .$$

Unlike for the triad component E^x or flux operators, this spectrum is not equidistant since vertex contributions are in general irrational. Thus, for large eigenvalues the behavior is very different and the spectrum crowds rapidly if several vertices are considered. This observation may be of relevance for the semiclassical limit.

It is possible to choose a quantization resulting in an equidistant spectrum even for the triad component E^φ. To do so, we replace $(\hat{P}^\beta)^2$ in $\hat{E}^\varphi$ by $(\hat{P}^\beta)^2 - \gamma^2\ell_{\mathrm{P}}^4/64\pi^2$ such that the label ρ used in the derivation of the $\hat{H}$-spectrum would now be identical to the integer k labeling $\hat{P}^\beta$-eigenstates. Such a replacement would not spoil the classical limit since the added constant vanishes when $\hbar \to 0$. Vertex sums such as in the E^φ-spectrum would then be over half-integers, which would not crowd even for many vertices. Moreover, a vertex contribution to the spherically symmetric volume spectrum would be identical to the LRS spectrum such that all fine structure would disappear.

However, this replacement only works on states with non-zero $\hat{P}^\beta$-eigenvalue, for otherwise the potential would be negative and unbounded from below when $k = 0$. One can avoid this by choosing $(\hat{P}^\beta)^2 - \gamma^2\ell_{\mathrm{P}}^4/64\pi^2$ on $\hat{P}^\beta$-eigenstates with non-zero eigenvalue, and the previous prescription $(\hat{P}^\beta)^2$ on the zero eigenstate. Since this special treatment of the zero eigenstate seems artificial, we do not use the resulting operator here and regard the non-equidistant triad spectrum and the related fine structure of the volume spectrum as a robust consequence of the loop quantization.

4 Operators in the volume eigenbasis

We now have the full eigenbasis of the volume operator at our disposal which can be used for further calculations. As another consequence we have seen that the triad operators $\hat{E}^x$ and $\int_{\mathcal{I}} \hat{E}^\varphi$ commute with each other such that a triad eigenbasis exists (given by the volume eigenstates). This may be useful for investigations of the Hamiltonian constraint just as in the homogeneous case [10, 11, 12]. The difference to the homogeneous case is that the triad representation is not identical to the flux representation which was derived for spherical symmetry in [19]. For particular calculations one then has to see which representation is easier to use, and maybe has to transform between the triad and the flux representation. We will use the explicit form of the triad eigenstates together with identities for Legendre functions (see, e.g., [28]) in order to write down this transformation and to express some operators in both the flux and the triad representation.

4.1 Volume eigenstates in the flux representation

For the Legendre functions we have the trigonometric expansion (A.9) which gives the A_φ-dependence of volume eigenstates for fixed k_+, k_- and n, and thus fixed ρ and n as

$$\begin{aligned}\psi^\rho_n(A_\varphi) &= 2^{1-\rho}\pi^{-1/2}\sqrt{n!(\rho+n+\tfrac{1}{2})\Gamma(2\rho+n+1)}(\sin A_\varphi)^{-\rho+1/2} \\ &\quad \sum_{k=0}^{\infty}\left(\begin{array}{c} n+k \\ k \end{array}\right)\frac{\Gamma(k-\rho+\frac{1}{2})}{\Gamma(-\rho+\frac{1}{2})\Gamma(n+k+\rho+\frac{3}{2})}\sin((n+2k+1)A_\varphi)\,.\end{aligned}$$

The expansion in flux eigenstates is then obtained by using

$$(\sin A_\varphi)^{-\rho+1/2} = \sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c} \frac{1}{4}-\frac{1}{2}\rho \\ k \end{array}\right)\cos^{2k} A_\varphi \tag{37}$$

and

$$\cos^{2k} A_\varphi = \frac{1}{2^{2k}}\binom{2k}{k} + \sum_{j=1}^{2k}\frac{1+(-1)^j}{2^{2k}}\binom{2k}{k-\frac{1}{2}j}\cos(jA_\varphi)$$

as well as

$$\cos jA_\varphi \sin kA_\varphi = \tfrac{1}{2}(\sin(j+k)A_\varphi - \sin(j-k)A_\varphi)$$

after which collecting terms with equal frequency leads to the expansion of $\psi^\rho_n(A_\varphi)$ in terms of flux eigenstates.

4.2 Holonomies and fluxes in the triad representation

Holonomies $\cos A_\varphi$ and $\sin A_\varphi$ and the flux $\mathrm{d}/\mathrm{d}A_\varphi$ have simple actions (19), (18) and (21) in the flux representation, but are more complicated when acting on volume eigenstates. Still, in some cases one can make use of some identities satisfied by Legendre functions

(which for convenience are collected in the Appendix) in order to get explicit expressions for the action of holonomy and flux operators in the volume eigenbasis.

The simplest operator will be $\cos A_\varphi$ since we can use (A.1) and obtain a simple recursion where ρ is fixed:

$$(\cos A_\varphi)\psi^\rho_n(A_\varphi) = \frac{1}{\sqrt{2\rho+2n+1}}\left(\sqrt{\frac{(n+1)(2\rho+n+1)}{2\rho+2n+3}}\,\psi^\rho_{n+1}(A_\varphi) + \sqrt{\frac{n(2\rho+n)}{2\rho+2n-1}}\,\psi^\rho_{n-1}(A_\varphi)\right). \tag{38}$$

If $\rho=\frac{1}{2}$, for instance, both coefficients reduce to $1/2$ and we reobtain (19) since in this case a volume eigenstate is a flux eigenstate.

Already $\sin A_\varphi$ is more complicated since simple recursion relations like (A.2) would change the label ρ by an integer, which would not be a volume eigenstate (no difference between different allowed values of $\rho=\sqrt{k^2+1/4}$ with integer k is an integer). One can use these functions in intermediate steps of the calculation, but in the end all have to be expressed in terms of the allowed volume eigenstates. Even when the sine has been used in intermediate steps, there may be cancellations (either directly or after using identities like (A.5) for Legendre functions) in some cases such that in the end the result will be expressed as a linear combination of finitely many volume eigenstates. Instead of using a recursion relation, we can use the expansion (37) and use the action of $\cos A_\varphi$. This results in an infinite linear combination of volume eigenstates.

The square $\sin^2 A_\varphi$, on the other hand, does have a simple action since it can simply be written as $1-\cos^2 A_\varphi$, for which we can use the action derived before. We then obtain

$$\begin{aligned}(\sin^2 A_\varphi)\psi^\rho_n(A_\varphi) &= 2\frac{(\rho+n)(\rho+n+1)-1+\rho^2}{(2\rho+2n+3)(2\rho+2n-1)}\,\psi^\rho_n(A_\varphi)\\ &\quad -\frac{1}{2\rho+2n-1}\sqrt{\frac{n(n-1)(2\rho+n)(2\rho+n-1)}{(2\rho+2n+1)(2\rho+2n-3)}}\,\psi^\rho_{n-2}(A_\varphi)\\ &\quad -\frac{1}{2\rho+2n+3}\sqrt{\frac{(n+1)(n+2)(2\rho+n+1)(2\rho+n+2)}{(2\rho+2n+1)(2\rho+2n+5)}}\,\psi^\rho_{n+2}(A_\varphi)\end{aligned}$$

which again for $\rho=\frac{1}{2}$ reduces to the flux result since all coefficients collapse to $\frac{1}{4}$.

Flux operators, i.e. derivatives acting on the Legendre function, would in general also change ρ by an integer. When we multiply holonomies with a flux operator, however, there may be simpler expressions in some cases. For instance, (A.8) implies

$$\frac{\mathrm{d}\psi^\rho_n(A_\varphi)}{\mathrm{d}A_\varphi} = \frac{\sqrt{2\rho+2n+1}}{2\sqrt{1-\cos^2 A_\varphi}}\left(\sqrt{\frac{(n+1)(2\rho+n+1)}{2\rho+2n+3}}\psi^\rho_{n+1}(A_\varphi) - \sqrt{\frac{n(2\rho+n)}{2\rho+2n-1}}\psi^\rho_{n-1}(A_\varphi)\right)$$

such that $(\sin A_\varphi)\hat{P}^\varphi$ would have a simple action on volume eigenstates, even though the action of $\sin A_\varphi$ or $\hat{P}^\varphi$ separately is more complicated.

Those operators are necessary to consider when composite operators such as the Hamiltonian constraint are computed. One common example is that of commutators between holonomies and the volume operator, such that powers of sine and cosine act with the volume operator in between.

4.3 Euclidean observables

In [17] the spherically symmetric system has been reduced explicitly in complex Ashtekar variables and in particular a complete set of observables has been found. We can use the same expressions, translated to real variables, for the Euclidean model, which are then given by the ADM mass

$$m = (A_1^2 + A_2^2 - 1)\sqrt{|E^x|} = (A_\varphi^2 - 1)\sqrt{|E^x|} \tag{39}$$

evaluated at the boundary at infinity, and its conjugate

$$T = (\gamma\kappa)^{-1} \int_B \mathrm{d}x \frac{A_1 E^1 + A_2 E^2}{(A_1^2 + A_2^2)\sqrt{|E^x|}} = (\gamma\kappa)^{-1} \int_B \mathrm{d}x \frac{P^\varphi}{2A_\varphi \sqrt{|E^x|}}\,. \tag{40}$$

In the original variables A_I, E^I the expressions look rather complicated to quantize, but using the variables A_φ and P^φ adapted to the loop quantization, we can quantize the observables along the lines used before, after replacing A_φ suitably by holonomies.

We use $\sin A_\varphi$ instead of A_φ in (39) and obtain the operator

$$\hat{m} = -\cos^2 A_\varphi(\infty)\sqrt{|E^x(\infty)|} \tag{41}$$

without ordering ambiguities. The argument ∞ here refers to the boundary point of a state corresponding to spatial infinity. For (40) we choose the replacement $\sin A_\varphi \cos A_\varphi$ instead of a simple sine in order to preserve the conjugacy of the new expressions. Moreover, here we have to be careful with the ordering between A_φ and P^φ. If we again subdivide the integration over B into integrations over intervals $\mathcal{I}_i$ of size ϵ and with mid-points x_i, we obtain

$$T = (2\gamma\kappa)^{-1} \lim_{\epsilon\to 0} \sum_i E^x(x_i)^{-1/2} A_\varphi(x_i)^{-1} \int_{\mathcal{I}_i} \mathrm{d}x P^\varphi(x)\,.$$

After quantizing we will be able to remove the regulator only if there are only finitely many non-zero contributions to the sum. This is the case if we use the ordering as indicated above, with the flux to the right. For this ordering we obtain the operator

$$\hat{T} = (2\gamma\kappa)^{-1} \sum_v \widehat{(|E^x(v)|^{-1/2})}(\sin A_\varphi(v)\cos A_\varphi(v))^{-1}\hat{P}_v^\varphi \tag{42}$$

which is not symmetric in the kinematical inner product ($\hat{P}_v^\varphi$ refers to a single vertex contribution to (21)). If we reorder before we remove the regulator, the resulting operator

will not be well-defined since $\hat{P}^{\varphi}(\sin A_{\varphi}\cos A_{\varphi})^{-1}$ would have non-zero action even where the state has no vertex. One can, however, reorder after the regulator has been removed, such that we simply symmetrize the operator $\hat{T}$.

There are also several inverse expressions in T, which could make the operator ill-defined. We indicated in the notation above that we use a well-defined, even finite operator for $|E^x|^{-1/2}$ along the lines of [29]. The inverses of $\sin A_{\varphi}$ and $\cos A_{\varphi}$ are obtained as multiplication operators, which are unbounded but densely defined. Computing the operator $\hat{T}$ explicitly is nevertheless complicated since the flux eigenstates do not lie in the domain of definition of the inverse sine and cosine.

On volume eigenstates the mass $\hat{m}$ has a simple action since we only need to multiply with the cosine as in (38). For $\hat{T}$, on the other hand, we have the flux operator which has a complicated expression in the volume eigenbasis, and also inverse powers of sine and cosine. The inverse sine can be directly applied to a volume eigenstate using the relation (A.7), where the right hand side has changed orders of the Legendre functions such that it effectively corresponds to an infinite superposition of volume eigenstates, but the inverse cosine again does not have the eigenstates in its domain of definition.

Looking at the algebra of the quantum observables $\hat{m}$ and $\hat{T}$ with a quantized Hamiltonian constraint and the issue of self-adjointness with respect to possible physical inner products can provide additional tests for the quantization scheme. In particular the fact that operators like the observables and also the Hamiltonian constraint have to be rewritten in terms of holonomies makes it more non-trivial for the quantum algebra to reflect faithfully the classical algebra. This is in addition to factor ordering issues that play a role in any quantization with more complicated constraints.

5 Discussion

We have diagonalized the volume operator of spherically symmetric quantum geometry explicitly, which shows that spherical symmetry can supply a substantial amount of simplification compared to the full theory. This makes it possible to study also composite operators explicitly, where usually the volume operator plays a central role. Having an explicit diagonalization of the volume operator is one of the reasons why homogeneous models can be dealt with directly with by now many cosmological applications. However, while explicitly diagonalized, the spherically symmetric volume operator is much more complicated than the homogeneous or isotropic ones. Its eigenstates are not simply flux eigenstates but have a more complicated dependence on the connection components via Legendre functions. Since holonomies in general have a complicated action on volume eigenstates, composite operators like the Hamiltonian constraint, where commutators of the form $h[h^{-1},\hat{V}]$ between holonomies and the volume operator appear, become more difficult to handle. Unless subtle cancellations happen in the construction of those operators, for instance making use of relations like (A.5) between Legendre functions, the Hamiltonian constraint in the triad (or flux) representation will be a difference operator of infinite order in the vertex labels n_v (while the order in edge labels k_e will be finite as

in homogeneous models [30, 10, 11]). The physical significance of this fact depends on the interpretation of solutions to the constraint and the issue of time. Compared to the full theory the possibility of explicit, though rather cumbersome, expressions will substantially improve the outlook to understand the quantum system and in particular field theoretical issues (the constraint algebra, semiclassical states) that appear here. At least numerical computations should be possible to implement in a straightforward way, while even this is complicated in the full theory [31].

Since the spherically symmetric volume operator is, unlike the full one as well as homogeneous expressions, a function of the basic expressions involving also connection components, there are more quantization ambiguities. There are, however, no factor ordering ambiguities and so the consequences are rather minor compared to, e.g., quantizations for inverse powers of the volume in homogeneous models [29, 32] where the ordering is relevant, too. In particular the good agreement with the LRS homogeneous spectrum is reassuring, where quantization ambiguities just affect the fine structure of the spectrum.

Acknowledgements

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Appendix

A Properties of Legendre functions

Recursion relations [28]:

$$zP^\mu_\nu(z) = \frac{\nu-\mu+1}{2\nu+1}P^\mu_{\nu+1}(z) + \frac{\mu+\nu}{2\nu+1}P^\mu_{\nu-1}(z) \tag{A.1}$$

$$\sqrt{z^2-1}P^\mu_\nu(z) = \frac{1}{2\nu+1}(P^{\mu+1}_{\nu+1}(z) - P^{\mu+1}_{\nu-1}(z)) \tag{A.2}$$

$$P^{\mu+1}_\nu(z) = (z^2-1)^{-1/2}\left((\nu-\mu)zP^\mu_\nu(z) - (\nu+\mu)P^\mu_{\nu-1}(z)\right) \tag{A.3}$$

$$(z^2-1)\frac{\mathrm{d}P^\mu_\nu(z)}{\mathrm{d}z} = \nu z P^\mu_\nu(z) - (\nu+\mu)P^\mu_{\nu-1}(z) \tag{A.4}$$

Combining (A.1) with (A.3) and then with (A.2) shows that

$$P^{\mu+2}_{\nu+1}(z) - P^{\mu+2}_{\nu-1}(z) = (\nu-\mu)(\nu-\mu+1)P^\mu_{\nu+1}(z) - (\nu-\mu-1)(\nu+\mu)P^\mu_{\nu-1}(z)\,. \tag{A.5}$$

From (A.3) together with (A.1) we obtain

$$\sqrt{z^2-1}P_\nu^\mu(z) = \frac{1}{2\nu+1}\left((\nu-\mu+1)(\nu-\mu+2)P_{\nu+1}^{\mu-1}(z) - (\nu+\mu)(\nu+\mu-1)P_{\nu-1}^{\mu-1}(z)\right) \tag{A.6}$$

which is independent of the relation (A.2). We can thus combine both relations, shifting the label μ by two in (A.6), in order to eliminate $P_{\nu-1}^{\mu+1}(z)$ on the right hand side. After dividing by $\sqrt{z^2-1}$, we have

$$\frac{1}{\sqrt{z^2-1}}P_\nu^\mu(z) = \frac{1}{2\mu}\left((\nu+\mu)(\nu+\mu-1)P_{\nu-1}^{\mu-1}(z) - P_{\nu-1}^{\mu+1}(z)\right) . \tag{A.7}$$

Most of these identities involve Legendre functions with different orders μ, in particular those involving derivatives. One combination with fixed order can be obtained by computing

$$\begin{aligned}\frac{\mathrm{d}(1-z^2)^{1/4}P(z)}{\mathrm{d}\xi} &= -\frac{\cos\xi}{2\sqrt{\sin\xi}}P(\cos\xi) - \sin^{3/2}\xi\frac{\mathrm{d}P(\cos\xi)}{\mathrm{d}\cos\xi} \\ &= \frac{z}{2(1-z^2)^{1/4}}P(z) - (1-z^2)^{3/4}\frac{\mathrm{d}P(z)}{\mathrm{d}z}\end{aligned}$$

where $z=\cos\xi$. Using (A.4) and (A.1) we obtain

$$\begin{aligned}\frac{\mathrm{d}(1-z^2)^{1/4}P_\nu^\mu(z)}{\mathrm{d}\xi} &= (1-z^2)^{-1/4}\left((\nu+\tfrac{1}{2})zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)\right) \\ &= \tfrac{1}{2}(1-z^2)^{-1/4}\left((\nu-\mu+1)P_{\nu+1}^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)\right) . \end{aligned} \tag{A.8}$$

Trigonometric expansion [28]:

$$P_\nu^\mu(\cos(\xi)) = \frac{2^{\mu+1}}{\sqrt{\pi}}\sin^\mu\xi\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\sum_{l=0}^{\infty}\frac{(\mu+\frac{1}{2})_l(\nu+\mu+1)_l}{l!(\nu+\frac{3}{2})_l}\sin\left((\nu+\mu+2l+1)\xi\right) \tag{A.9}$$

where $(\nu)_l := \Gamma(\nu+l)/\Gamma(\nu)$.

B Legendre functions for negative order and integer sum of order and degree

The formula

$$\frac{(-1)^{n+m}}{2^n n!}(1-z^2)^{m/2}\frac{\mathrm{d}^{n+m}}{\mathrm{d}z^{n+m}}(1-z^2)^n = P_n^m(z)$$

can be found in the standard literature (e.g., [28, 33]) for positive integer labels n and m. This would not be applicable in our case since $\mu=-\rho$ and $\nu=\rho+n$ are non-integer. However, for all the volume eigenstates $\mu=-\rho$ is negative and $\mu+\nu=n$ is a non-negative integer. In this case, the same formula applies.

Lemma 1 *Let $\mu < 0$, and $\mu + \nu$ be a non-negative integer. Then*

$$P_\nu^\mu(z) = \frac{(-1)^{\nu+\mu}}{2^\nu \Gamma(\nu+1)} (1-z^2)^{\mu/2} \frac{\mathrm{d}^{\nu+\mu}}{\mathrm{d}z^{\nu+\mu}} (1-z^2)^\nu \tag{B.10}$$

Proof: We will prove (B.10) by induction over $n = \nu + \mu$. Since [28]

$$P_\rho^{-\rho}(z) = \frac{(z^2-1)^{\frac{1}{2}\rho}}{2^\rho \Gamma(\rho+1)} \tag{B.11}$$

the identity is true for $n = 0$. Let us now assume that it is true for a given n. Multiplying both sides of (B.10) by $(1-z^2)^{-\mu/2}$ and differentiating with respect to z we obtain

$$\begin{aligned}
\frac{(-1)^{\nu+\mu}}{2^\nu \Gamma(\nu+1)} \frac{\mathrm{d}^{\nu+\mu+1}}{\mathrm{d}z^{\nu+\mu+1}} (1-z^2)^\nu &= \frac{\mathrm{d}}{\mathrm{d}z} \left((1-z^2)^{-\mu/2} P_\nu^\mu(z) \right) \\
&= \mu z (1-z^2)^{-\mu/2-1} P_\nu^\mu(z) + (1-z^2)^{-\mu/2} \frac{\mathrm{d}P_\nu^\mu(z)}{\mathrm{d}z} \\
&= \mu z (1-z^2)^{-\mu/2-1} P_\nu^\mu(z) \\
&\quad + (1-z^2)^{-\mu/2-1} \left(-(\nu-\mu+1) P_{\nu+1}^\mu(z) + z(\nu+1) P_\nu^\mu(z) \right) \\
&= -(1-z^2)^{-(\mu+1)/2} P_\nu^{\mu+1}(z)
\end{aligned} \tag{B.12}$$

where in the last two steps use was made of

$$\frac{\mathrm{d}P_\nu^\mu(z)}{\mathrm{d}z} = \frac{1}{1-z^2} \left((\nu+1) z P_\nu^\mu(z) - (\nu-\mu+1) P_{\nu+1}^\mu(z) \right) \tag{B.13}$$

and

$$(\nu-\mu+1) P_{\nu+1}^\mu(z) - (\nu+\mu+1) z P_\nu^\mu(z) = \sqrt{1-z^2}\, P_\nu^{\mu+1}(z)\,, \tag{B.14}$$

respectively. Here, (B.13) follows directly from (A.4) together with (A.1) whereas (B.14) is identical to (A.3). Multiplying equation (B.12) by $-(1-z^2)^{(\mu+1)/2}$ we infer that our identity (B.10) is also true for $n+1$ thus completing the proof. □

Again for positive integer m and n the function $P_n^m(z)$ (a polynomial in this case) is normalizable with

$$\int_{-1}^{1} P_n^m(z)^2 \mathrm{d}z = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!}\,.$$

Also this formula holds true in our case:

Lemma 2 *Let $\mu < 0$, and $\mu + \nu$ be a non-negative integer. Then*

$$\int_{-1}^{1} P_\nu^\mu(z)^2 \mathrm{d}z = \frac{1}{\nu+\frac{1}{2}} \frac{(\nu+\mu)!}{\Gamma(\nu-\mu+1)}\,. \tag{B.15}$$

Proof: We use the representation (B.10) and integrate by parts where boundary terms vanish:

$$\begin{aligned}
\int P_\nu^\mu(z)^2 \mathrm{d}z &= \frac{1}{2^{2\nu}\Gamma(\nu+1)^2}\int (1-z^2)^\mu \left(\frac{\mathrm{d}^{\nu+\mu}}{\mathrm{d}z^{\nu+\mu}}(1-z^2)^\nu\right)^2 \mathrm{d}z \\
&= \frac{-1}{2^{2\nu}\Gamma(\nu+1)^2}\int \left(\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right)\left((1-z^2)^\mu \frac{\mathrm{d}^{\nu+\mu+1}}{\mathrm{d}z^{\nu+\mu+1}}(1-z^2)^\nu\right. \\
&\quad \left. -2\mu z(1-z^2)^{\mu-1}\frac{\mathrm{d}^{\nu+\mu}}{\mathrm{d}z^{\nu+\mu}}(1-z^2)^\nu\right)\mathrm{d}z \\
&= \frac{-1}{2^{2\nu}\Gamma(\nu+1)^2}\int (1-z^2)^{(\mu-1)/2}\left(\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right) \\
&\quad \times\left((1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left((1-z^2)^{(\mu-1)/2}\left(\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right)\right)\right. \\
&\quad -2z(1-z^2)^{(\mu-1)/2}\frac{\mathrm{d}^{\nu+\mu}}{\mathrm{d}z^{\nu+\mu}}(1-z^2)^\nu \\
&\quad \left.+(\mu-1)(1-(\mu-3)z^2/(1-z^2))(1-z^2)^{(\mu-1)/2}\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right)\mathrm{d}z \\
&= \frac{-1}{2^{2\nu}\Gamma(\nu+1)^2}\int (1-z^2)^{(\mu-1)/2}\left(\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right) \\
&\quad \times\left((1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2}-2z\frac{\mathrm{d}}{\mathrm{d}z}+\mu(\mu-1)-\frac{(\mu-1)^2}{1-z^2}\right) \\
&\quad \times\left((1-z^2)^{(\mu-1)/2}\frac{\mathrm{d}^{\nu+\mu-1}}{\mathrm{d}z^{\nu+\mu-1}}(1-z^2)^\nu\right)\mathrm{d}z \\
&= (\nu(\nu+1)-\mu(\mu-1))\int P_\nu^{\mu-1}(z)^2\mathrm{d}z
\end{aligned}$$

where we used the differential equation

$$(1-z^2)\frac{\mathrm{d}^2P_\nu^\mu(z)}{\mathrm{d}z^2}-2z\frac{\mathrm{d}P_\nu^\mu(z)}{\mathrm{d}z}+\left(\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right)P_\nu^\mu(z)=0 \tag{B.16}$$

obeyed by the Legendre functions. Thus,

$$\int P_\nu^\mu(z)^2\mathrm{d}z = \frac{1}{(\nu-\mu)(\nu+\mu+1)}\int P_\nu^{\mu+1}(z)^2\mathrm{d}z$$

and starting from

$$\int P_\nu^{-\nu}(z)^2\mathrm{d}z = \frac{2}{\Gamma(2\nu+2)}$$

which follows from (B.11) the proof by induction is complete. □

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