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Splitting spinning strings in AdS/CFT

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Abstract: We study the semiclassical decay of macroscopic spinning strings in $\mathrm{AdS}_5 \times S^5$ through spontaneous splitting of the folded string worldsheet. Based on similar considerations in flat space this decay channel is expected to dominate the full quantum computation. The outgoing strings are uniquely specified by an infinite set of conserved (local) charges with a regular expansion in inverse powers of the initial angular momentum. We compute these charges and determine functional relations between them. Finally, a preliminary discussion of the corresponding calculation in the non-planar sector of the dual gauge theory is presented.

Keywords: AdS/CFT, spinning strings.

Contents

1. Introduction and summary

Within the framework of the AdS/CFT correspondence it has proved fruitful to explore sectors of both theories which are characterized by large charges. In the Berenstein, Maldacena and Nastase limit [1] one considers operators of $\mathrm{U}(1)_R$ charge J such that $J \sim N^2$ as $N \to \infty$ with g_{YM} finite. This leads to a subsector of the theory for which the quantum corrections are under control, despite the fact that the 't Hooft coupling $g_{\mathrm{YM}}^2 N$ becomes large. A remarkable feature of this limit is that it maintains a full genus expansion [2, 3]. The effective genus parameter turns out to be $g_2 = J^2/N$, which is nonvanishing despite the large-N limit. This allows one to compare the process of string splitting with a computation in the dual gauge theory, by determining decay widths of the corresponding BMN operators [4]. At leading order in $\lambda' = g_{\mathrm{YM}}^2 N/J^2$ and g_2 these computations have been shown to agree [5] with light-cone string field theory [6].

Following the BMN idea, one can consider subsectors of the gauge theory with several large charges [7, 8, 9, 10]. As before, the operators in these sectors have controlled quantum corrections, allowing one to make a direct comparison with string theory. The objects dual to such operators have been identified as large, macroscopic spinning strings in $\mathrm{AdS}_5 \times S^5$ [8, 9, 11]. The energies of these strings and the anomalous dimensions of the gauge theory operators are in agreement up to two-loop order in λ' [12, 10, 13, 14] [1]. Clearly however, these computations only probe zeroth-order effects in the string coupling constant g_s.

[1] They actually start to disagree at the three loop level [15]. This mismatch has been argued to arise from the non-commutativity of taking $J \to \infty$ and expanding in λ' [16].

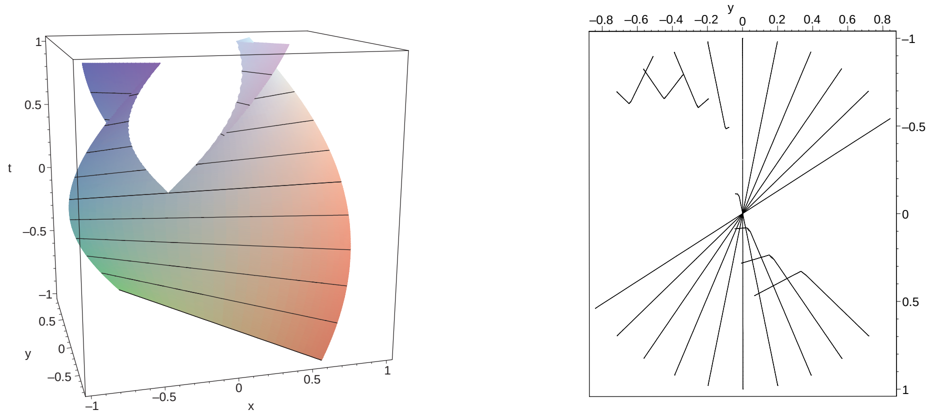


Figure 1: Semi-classical decay of a folded, rotating string in flat space-time, following [17]. The plot on the right shows snapshots at various values of τ. The outgoing pieces exhibit kinks, which propagate outward along the strings. New momenta $P_x^I = -P_x^{II}$ are generated in the decay process.

The central question to be addressed in this work is what can be said about $g_s \neq 0$ effects for large spinning strings. In order to study string splitting, one would in principle need to compute the decay widths of long spinning strings and compare these to a dual computation of gauge theory operators with many impurities. Unfortunately, the determination of decay widths in quantum $\mathrm{AdS}_5 \times S^5$ string theory is at least for the time being out of reach.

It is, however, possible to analyze the decay semi-classically. In *flat* space-time, the semi-classical decay of macroscopic strings was analyzed in detail by Iengo and Russo [17]. They also compared the semi-classical results to those of a full quantum treatment. In the semi-classical approach, one starts with a classical, rotating closed string solution. At a given time $\tau = 0$, the string can spontaneously split if two points σ and σ' on the string coincide in target space, and if their velocities agree. The string described by these boundary conditions, $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma')$ and $\dot{X}^\mu(\tau, \sigma) = \dot{X}^\mu(\tau, \sigma')$, then forms a "figure eight". The splitting is realized by declaring that from $\tau = 0$ onward, each of the two string pieces ("left and right" from the overlapping point), *separately* satisfy periodic boundary conditions. The initial conditions on the positions and velocities of the outgoing pieces are simply taken to be those of the incoming string at the moment $\tau = 0$ of splitting. The effect of the splitting propagates with the speed of light along the outgoing pieces, leading to kink-like shapes (see figure 1).

The relations between the energies and angular momenta of the outgoing strings are determined completely by conservation laws, i.e. one does not need to derive the explicit string shapes in order to obtain these relations. From the relations between the charges one can then produce a curve in, for instance, the plane spanned by the masses M_I and M_{II} of the outgoing string pieces. In flat space-time, this curve can be compared with a *full quantum* string computation of the decay rate. It has been shown [18, 19] that the

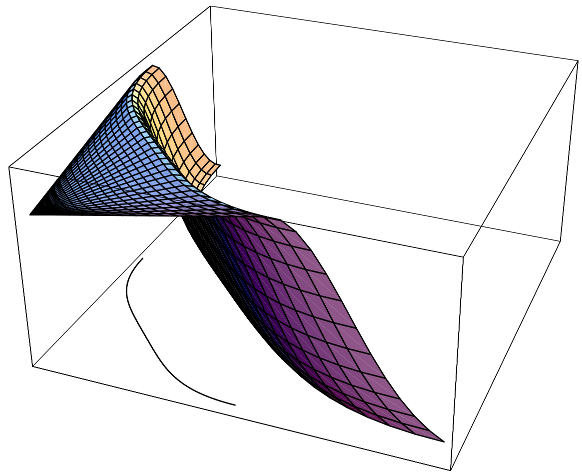


Figure 2: Sketch of the relation between the semi-classical and the full quantum calculations. The surface depicts the quantum decay amplitude over the (horizontal) plane spanned by the mass-square of the two outgoing strings, $(M_I)^2$ and $(M_{II})^2$. The amplitude reaches its maximum over the curve allowed by semi-classical decay.

quantum decay rate, as a function of the outgoing masses, reaches its maximum very close to the curve obtained from the classical analysis (see figure 2). In order to understand this relation between the semi-classical computation and the full quantum treatment, it is important to realize that the space of kinematically allowed decays of a string is much larger than the channel which is available using a semi-classical treatment (in the sense described above). Hence, the semi-classical analysis only describes one particular decay channel. Fortunately the quantum analysis of [18, 19] shows that this is the most probable channel (at least in flat space-time).

In the present paper we will analyze the decay of semi-classical strings on $\mathrm{AdS}_5 \times S^5$, with the goal of producing predictions which can in principle be verified on the gauge theory side. We will focus on the folded string which is rotating on the S^5 factor of the background. We first review the properties of this folded spinning string and all its charges. There are "global" charges (angular momenta and energy), associated to the isometries of the target space, as well as an infinite set of "local" (commuting) charges, related to the integrable structure of the string sigma model. Due to the fact that we consider a rigidly rotating string, all local charges are uniquely determined as functions of the global charges.

We then analyze the semi-classical splitting process. This splitting process introduces one new parameter a, the splitting parameter, and all the local and global charges of the outgoing strings are uniquely determined as functions of this parameter and the global charges of the incoming string. This is now the case, despite the fact that the outgoing strings are highly *non-rigid*, i.e. strongly *fluctuating* (as in the flat-space case depicted in figure 1). It is a consequence of the "continuous" way in which the worldsheets of in- and out- strings have been glued together. Since both local and global charges are conserved under the free ($g_s = 0$) evolution of the string, they can be computed by integrating the charge densities at the moment of splitting, i.e. at the moment when the charge densities of the in- and out-going strings coincide. In this way, one circumvents the need to explicitly construct the solutions for the outgoing strings.

In the decay process, several charges which were zero for the ingoing string now get turned on for the individual outgoing pieces. This is similar to the decay in flat space-time, where one generates new momenta $P_x^I = -P_x^{II}$ (the pieces move away from each other). For the string on S^5, new angular momentum components get turned on. These new angular momenta are neither Casimirs nor elements of the Cartan subalgebra, indicating that the outgoing strings are descendants rather than highest-weight states in the gauge theory dual. Having found these new charges, we in addition also construct the generating functional for all the higher charges as a function of the charges of the incoming string and the splitting parameter a.

The conservation of an infinite set of local charges is a consequence of our construction. It can be checked explicitly that the *quantum* decay of the string in flat space does not preserve the higher charges. However, for the most dominant, semiclassical decay processes, the conservation laws do hold for all charges. This suggests that the conjectured integrability in the planar sector of $\mathcal{N}=4$ super-Yang-Mills may be extended in a certain sense to the non-planar sector.

In order to make a comparison to the gauge theory, it is necessary to express the decay process on the string theory side purely in terms of relations between charges. In principle there is an infinite set of relations, for all the local charges, but we focus on the relations between the global charges. We present these relations in section 3.2. The gauge theory quantum amplitude is expected to attain a maximum over the semi-classical string decay curves, just as in figure 2.

On the gauge theory side, the decay of long strings can be analyzed in the spin chain picture of [20]. The splitting operator, which when acting on a single-trace operator produces a double-trace operator, is represented by the non-planar part of the dilatation operator. In section 4 we discuss these gauge theory ingredients. We argue that, despite the fact that we are not in the BMN regime, an effective genus counting parameter $\mathcal{J}^2/N$ is still present. We also discuss the higher conserved charges and their role in reducing the possible spin chain decay channels. A detailed study of this decay is, as we will argue, hampered by the complexity of the spin chain wave functions, and we leave this for future work.

Finally, let us note that our construction can easily be extended to the study of strings in less symmetric backgrounds (and in particular to backgrounds dual to confining gauge theories). This could provide one with information about meson and glue-ball decays. A study of these processes is under investigation.

2. Review of the folded string solution

In order to set up our notation and introduce the new charges that will be generated in the process of the decay, let us briefly review the construction of the folded two spin string rotating in S^5, as first presented by Frolov and Tseytlin [9]. Using the parameterization

$$X_1 + iX_2 = \sin\gamma\,\cos\psi\,e^{i\varphi_1}\,,\quad X_3 + iX_4 = \sin\gamma\,\sin\psi\,e^{i\varphi_2}\,,\quad X_5 + iX_6 = \cos\gamma\,e^{i\varphi_3}\,, \tag{2.1}$$

the metric on a five-sphere $X_1^2 + \cdots + X_6^2 = 1$ can be written as

$$\mathrm{d}s^2_{S^5} = \mathrm{d}\gamma^2 + \cos^2\gamma\,\mathrm{d}\varphi_3^2 + \sin^2\gamma(\mathrm{d}\psi^2 + \cos^2\psi\,\mathrm{d}\varphi_1^2 + \sin^2\psi\,\mathrm{d}\varphi_2^2)\,, \tag{2.2}$$

whereas the metric on AdS_5 reads $\mathrm{d}s^2_{\mathrm{AdS}_5} = \mathrm{d}\rho^2 - \cosh^2\rho\,\mathrm{d}t^2 + \sinh^2\rho\,\mathrm{d}\Omega_3^2$. The two-spin string solution is given by the equations

$$t = \kappa\tau\,,\quad \rho = 0\,,\quad \gamma = \frac{\pi}{2}\,,\quad \varphi_3 = 0\,,\quad \varphi_1 = w_1\tau\,,\quad \varphi_2 = w_2\tau\,,\quad \psi = \psi(\sigma)\,, \tag{2.3}$$

where κ, w_1 and w_2 are constants. The equation which determines the profile of $\psi(\sigma)$ is

$$\psi'' + \frac{1}{2}w_{21}^2\sin(2\psi) = 0\,,\quad w_{21}^2 \equiv w_2^2 - w_1^2 \geq 0\,. \tag{2.4}$$

By integrating this equation once, we obtain the following equation,

$$\psi'^2 = w_{21}^2(\sin^2\psi_0 - \sin^2\psi)\,. \tag{2.5}$$

Here the constant ψ_0 corresponds to the target-space length of the folded string; the point at which the first derivative of ψ vanishes is the point at which the world-sheet of the string turns back onto itself. The conformal gauge constraints imply

$$\kappa^2 = w_2^2\sin^2\psi_0 + w_1^2\cos^2\psi_0\,. \tag{2.6}$$

The motion of the string is confined to a three-sphere embedded in the five sphere, which will remain true also for the two outgoing strings after the decay process to be considered in the next section. As the isometry group of the three sphere is $SO(4)$, there are 6 conserved angular momenta J_{ij}, associated to the 6 Killing vectors,

$$J_{ij} = \sqrt{\lambda}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}(X_i\dot{X}_j - X_j\dot{X}_i) \equiv \sqrt{\lambda}\mathcal{J}_{ij}\,,\quad (i,j = 1\cdots 4)\,. \tag{2.7}$$

Explicitly, using the parameterization (2.1) these can be rewritten as (using $\gamma = \frac{\pi}{2}$)

$$\mathcal{J}_{12} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \cos^2\psi \dot{\varphi}_1\,, \tag{2.8}$$

$$\mathcal{J}_{34} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \sin^2\psi \dot{\varphi}_2\,, \tag{2.9}$$

$$\mathcal{J}_{13} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\big(\cos\varphi_1\cos\varphi_2\dot{\psi} + \sin\psi\cos\psi(-\sin\varphi_2\cos\varphi_2\dot{\varphi}_2 + \sin\varphi_1\cos\varphi_2\dot{\varphi}_1)\big)\,, \tag{2.10}$$

$$\mathcal{J}_{24} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\big(\sin\varphi_1\sin\varphi_2\dot{\psi} + \sin\psi\cos\psi(\sin\varphi_1\cos\varphi_2\dot{\varphi}_2 - \sin\varphi_2\cos\varphi_1\dot{\varphi}_1)\big)\,, \tag{2.11}$$

$$\mathcal{J}_{14} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\big(\cos\varphi_1\sin\varphi_2\dot{\psi} + \sin\psi\cos\psi(\cos\varphi_1\cos\varphi_2\dot{\varphi}_2 + \sin\varphi_2\sin\varphi_1\dot{\varphi}_1)\big)\,, \tag{2.12}$$

$$\mathcal{J}_{23} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\big(\sin\varphi_1\cos\varphi_2\dot{\psi} - \sin\psi\cos\psi(\sin\varphi_1\sin\varphi_2\dot{\varphi}_2 + \cos\varphi_2\cos\varphi_1\dot{\varphi}_1)\big)\,. \tag{2.13}$$

An additional charge is the energy E, which is associated to the translation invariance with respect to global time. Using the constraint (2.6) one finds that the energy is given by

$$E = \sqrt{\lambda}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\dot{X}^0 = \sqrt{\lambda}\,\kappa = \sqrt{\lambda}\sqrt{w_2^2\sin^2\psi_0 + w_1^2\cos^2\psi_0}\,. \tag{2.14}$$

Before the decay, the string (2.3) carries two (mutually commuting) angular momenta, J_{12} and J_{34}

$$\mathcal{J}_{12} = \frac{2w_1}{\pi w_{21}}\int_0^{\psi_0}\frac{\cos^2\psi\,\mathrm{d}\psi}{\sqrt{\sin^2\psi_0 - \sin^2\psi}} = \frac{2\omega_1}{\pi\,\omega_{21}}E(q)\,, \tag{2.15}$$

$$\mathcal{J}_{34} = \frac{2w_2}{\pi w_{21}}\int_0^{\psi_0}\frac{\sin^2\psi\,\mathrm{d}\psi}{\sqrt{\sin^2\psi_0 - \sin^2\psi}} = \frac{2\omega_2}{\pi\,\omega_{21}}(K(q) - E(q))\,, \qquad q \equiv \sin^2\psi_0\,. \tag{2.16}$$

The expressions on the right-hand sides have been obtained by making a change of variables to ψ', defined by $\sin\psi/\sin\psi_0 = \sin\psi'$. [2]

Although the worldsheet densities for the other angular momenta are non-zero before the split, they vanish when integrated over the world-sheet. The two non-vanishing angular momenta are related by

$$1 = \frac{\mathcal{J}_{12}}{\omega_1} + \frac{\mathcal{J}_{34}}{\omega_2}\,. \tag{2.18}$$

as a consequence of (2.8) and (2.9). Moreover one derives from (2.15), (2.16) and (2.14)

$$\sqrt{\omega_2^2 - \omega_1^2} = \frac{2}{\pi}K(q)\,, \qquad q = \frac{\kappa^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\,. \tag{2.19}$$

[2] Our conventions for the elliptic integrals are

$$E(x;q) = \int_0^x \mathrm{d}\varphi\sqrt{1 - q\sin^2\varphi}\,, \qquad F(x;q) = \int_0^x \mathrm{d}\varphi\frac{1}{\sqrt{1 - q\sin^2\varphi}}\,. \tag{2.17}$$

with $E(q) := E(\pi/2;q)$ and $K(q) := F(\pi/2;q)$.

It then follows that all information to determine the energy $\mathrm{E}=\sqrt{\lambda}\,\mathcal{E}(\mathcal{J}_{12},\mathcal{J}_{34})$ as a function of the angular momenta lies within the two equations

$$\frac{4}{\pi^2}\,q = \frac{\mathcal{E}^2}{K(q)^2} - \frac{\mathcal{J}_{12}^2}{E(q)^2}\,, \tag{2.20}$$

$$\frac{4}{\pi^2} = \frac{\mathcal{J}_{34}^2}{(K(q)-E(q))^2} - \frac{\mathcal{J}_{12}^2}{E(q)^2}\,. \tag{2.21}$$

upon elimination of q. This may be achieved iteratively in an expansion for large total angular momentum $\mathcal{J} = \mathcal{J}_{12} + \mathcal{J}_{34}$ via the ansatz

$$q = q_0 + \frac{q_1}{\mathcal{J}^2} + \frac{q_2}{\mathcal{J}^4} + \dots\,, \tag{2.22}$$

$$\mathcal{E} = \mathcal{J}\,\mathcal{E}_0 + \frac{\mathcal{E}_1}{\mathcal{J}} + \frac{\mathcal{E}_2}{\mathcal{J}^3} + \dots\,. \tag{2.23}$$

One finds that q_0 is determined by the equation

$$\frac{E(q_0)}{K(q_0)} = 1-\alpha\,, \qquad\qquad \alpha := \frac{\mathcal{J}_{34}}{\mathcal{J}}\,. \tag{2.24}$$

All the higher terms q_i with $i > 0$ are then given algebraically as functions of q_0, and similarly for $\mathcal{E}_i$. The first non-trivial terms which one finds are

$$q_1 = -\frac{4\,E(q_0)\,K(q_0)^2\,(K(q_0)-E(q_0))\,(1-q_0)\,q_0}{\pi^2\,(E(q_0)^2 - 2\,E(q_0)\,K(q_0)\,(1-q_0) + (1-q_0)\,K(q_0)^2)}\,, \tag{2.25}$$

as well as

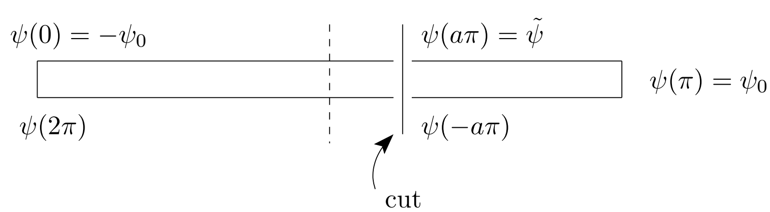
$$\mathcal{E}_0 = 1\,, \quad \mathcal{E}_1 = \frac{2}{\pi^2}\,K(q_0)\,(E(q_0) - (1-q_0)\,K(q_0))\,. \tag{2.26}$$

3. Semiclassical decay of the folded string

3.1 The splitting

Let us now consider the spontaneous splitting of the solution (2.3). As mentioned in the introduction, we will focus on the relations satisfied by the charges of the outgoing strings. In order to obtain those relations, we fortunately do not need to derive the precise form of the solutions $X^I_\mu(\tau,\sigma)$ for the outgoing strings. While those solutions could be obtained in flat space-time [17], it would be much more difficult to do so in the $\mathrm{AdS}_5\times S^5$ background.

We choose a parametrization on the world-sheet such that at the "end"-point of the folded, unsplit string we have $\psi(\sigma=0) = -\psi_0$ on one end of the string and $\psi(\sigma=\pi)=\psi_0$ on the other end. The splitting occurs on the worldsheet at points $\sigma=\pi a$ and $\sigma=-\pi a$, which are both mapped to the same target space point $\psi(-\pi a)=\psi(\pi a)=\tilde{\psi}$. The setup is thus



The charges carried by the outgoing strings can be calculated by evaluating the expressions (2.8)-(2.14), using the solution of the string *before* the decay, but integrating them over the lengths of each piece of string separately (i.e. $\sigma \in [-\pi a, \pi a]$ for the first piece and $\sigma \in [-\pi, -\pi a] \cup [\pi a, \pi]$ for the second one). In these calculations the string solution is evaluated at the moment of the decay, which without loss of generality we will take to be $\tau_d = 0$. This is consistent, as the initial conditions generated from the unsplit solution for the outgoing two string pieces are consistent, i.e. obey the Virasoro constraint.

The splitting parameter a is then related to the splitting point $\tilde{\psi}$ via (2.5) by

$$2\pi a = \frac{2}{\omega_{21}} \int_{-\psi_0}^{\tilde{\psi}} \frac{\mathrm{d}\psi}{\sqrt{\sin^2\psi_0 - \sin^2\psi}} = \frac{2}{\omega_{21}} \left(K(q) + F(x;q)\right) \tag{3.1}$$

where $x := \arcsin(\frac{\sin\tilde{\psi}}{\sin\psi_0})$. The "mirror" equation to this is

$$\pi(1-a)\sqrt{\omega_2^2 - \omega_1^2} = K(q) - F(x;q)\,. \tag{3.2}$$

The angular momenta J_{12} and J_{34}, which were non-zero before the split, get distributed between the outgoing string pieces I and II according to

$$\mathcal{J}_{12}^{I} = \frac{\omega_1}{\pi\omega_{21}} \left(E(q) + E(x;q)\right)\,, \tag{3.3}$$

$$\mathcal{J}_{34}^{I} = \frac{\omega_2}{\pi\omega_{21}} \left(K(q) - E(q) + F(x;q) - E(x;q)\right)\,, \tag{3.4}$$

$$\mathcal{J}_{12}^{II} = \frac{\omega_1}{\pi\omega_{21}} \left(E(q) - E(x;q)\right)\,, \tag{3.5}$$

$$\mathcal{J}_{34}^{II} = \frac{\omega_2}{\pi\omega_{21}} \left(K(q) - E(q) - F(x;q) + E(x;q)\right)\,. \tag{3.6}$$

Moreover one has as a consequence of the above

$$a = \frac{\mathcal{J}_{12}^{I}}{\omega_1} + \frac{\mathcal{J}_{34}^{I}}{\omega_2}\,, \qquad 1 - a = \frac{\mathcal{J}_{12}^{II}}{\omega_1} + \frac{\mathcal{J}_{34}^{II}}{\omega_2}\,. \tag{3.7}$$

The remaining angular momenta (2.10)–(2.13) vanish before the split, but they become non-zero for the outgoing strings,

$$\mathcal{J}_{13}^{I} = \mathcal{J}_{13}^{II} = 0\,, \qquad \mathcal{J}_{24}^{I} = \mathcal{J}_{24}^{II} = 0\,, \tag{3.8}$$

$$\mathcal{J}_{14}^{I} = -\mathcal{J}_{14}^{II} = -\frac{w_2}{\pi w_{21}}\sqrt{\sin^2\psi_0 - \sin^2\tilde{\psi}}\,, \tag{3.9}$$

$$\mathcal{J}_{23}^{I} = -\mathcal{J}_{23}^{II} = \frac{w_1}{\pi w_{21}}\sqrt{\sin^2\psi_0 - \sin^2\tilde{\psi}}\,. \tag{3.10}$$

The sum of each of these momenta is zero in accordance with the conservation laws. The conformal gauge constraint (2.6) remains untouched,

$$q = \frac{\kappa^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\,, \tag{3.11}$$

and the energies of two outgoing string pieces are given by

$$\mathcal{E}^I = \kappa\, a\,, \qquad \mathcal{E}^{II} = \kappa\,(1-a)\,. \tag{3.12}$$

A further relation is satisfied by the newly generated angular momenta and the splitting parameter x,

$$(\mathcal{J}^I_{14})^2 - (\mathcal{J}^I_{23})^2 = \frac{1}{\pi^2}\, q^2 \cos^2 x\,. \tag{3.13}$$

This equation determines x as a function of q and $\Delta := (\mathcal{J}^I_{14})^2 - (\mathcal{J}^I_{23})^2$.

3.2 Relations between outgoing charges

The goal now is to eliminate the parameters x and q related to the splitting point and initial string length, and express all conserved charges in terms of a minimal set of independent ones. Recall that before the split, the input data which determine all the string charges are the total momentum $\mathcal{J}$ and the filling fraction $\alpha = \mathcal{J}_{34}/\mathcal{J}$. The energy $\mathcal{E}(\alpha, \mathcal{J})$ is expressed in terms of these two parameters through (2.23), which can be compared with the gauge theory order by order in the $1/\mathcal{J}$ expansion.

The split introduces only one extra free parameter, namely the point x at which the string splits, while the number of measurable charges doubles: α^I,α^{II}, $\mathcal{J}^I$ and $\mathcal{J}^{II}$. Hence after the split, the number of dependent quantities, as well as the number of functional relations between them (which should be compared to the gauge theory) is larger. Depending on which quantities we want to relate, the choice for the set of independent parameters might be different.

The first functional relation we want to establish is the relation between the two angular momenta carried by the first part of the string,

$$\beta_{12} := \frac{\mathcal{J}^I_{12}}{\mathcal{J}_{12}}\,, \qquad \beta_{34} := \frac{\mathcal{J}^I_{34}}{\mathcal{J}_{34}}\,. \tag{3.14}$$

where the total momentum $\mathcal{J}$ is expressed as

$$\mathcal{J} := \underbrace{\mathcal{J}^I_{12} + \mathcal{J}^{II}_{12}}_{=:\mathcal{J}_{12}} + \underbrace{\mathcal{J}^I_{34} + \mathcal{J}^{II}_{34}}_{=:\mathcal{J}_{34}}\,. \tag{3.15}$$

Combining the equations (3.3) and (3.4) with equations (2.15) and (2.16) one deduces that

$$\beta_{12} = \frac{1}{2}\left(1 + \frac{E(x;q)}{E(q)}\right), \tag{3.16}$$

$$\beta_{34} = \frac{1}{2}\left(1 + \frac{F(x;q) - E(x,q)}{K(q) - E(q)}\right). \tag{3.17}$$

The parameter q appearing in these equations is given as a series in $1/\mathcal{J}$ with coefficients fixed by the data of the unsplit string (see equations (2.22), (2.24) and (2.25)). The splitting point x should now be eliminated by a combination of global charges of the outgoing strings, which is at our disposal. We could choose either β_{12}, β_{34} or something like the string length fraction $\frac{\mathcal{J}_{12}^I+\mathcal{J}_{34}^I}{\mathcal{J}}$ as the free parameter of the splitting process. Any such choice will lead to an expansion of x in $1/\mathcal{J}^2$,

$$x = x_0 + \frac{x_1}{\mathcal{J}^2} + \frac{x_2}{\mathcal{J}^4} + \dots\,. \tag{3.18}$$

In the remainder of this section we shall choose β_{12} as the new parameter. This yields the first two coefficients of (3.18) as

$$\beta_{12} = \frac{1}{2}\left(1 + \frac{E(x_0;q_0)}{E(q_0)}\right), \tag{3.19}$$

$$x_1 = \frac{q_1}{2q_0}\frac{E(q_0)F(x_0;q_0) - K(q_0)E(x_0;q_0)}{E(q_0)\sqrt{1-q_0\,\sin^2 x_0}}, \tag{3.20}$$

where q_1 is given in (2.25) and induces a $1/\mathcal{J}^2$ expansion for β_{34}. Substituting the expansion for q and x in the second equation (3.17), one is left with the desired first functional relation, namely $\beta_{34} = \beta_{34}(\beta_{12}, \alpha, \mathcal{J})$, given as a series in $1/\mathcal{J}$

$$\beta_{34} = \beta_{34}^0 + \frac{\beta_{34}^1}{\mathcal{J}^2} + \dots\,, \tag{3.21}$$

$$\beta_{34}^0 = \frac{1}{2}\left(1 + \frac{F(x_0;q_0) - E(x_0,q_0)}{K(q_0) - E(q_0)}\right), \tag{3.22}$$

$$\beta_{34}^1 = \frac{2\,q_0\,x_1\left(1-\sin^2 x_0\,[q_0+(1-q_0)\,(K(q_0)-E(q_0))]\right)+(K(q_0)-E(q_0))\,q_1\,\sin x_0\,\cos x_0}{4\,(E(q_0)-K(q_0))^2\,(q_0-1)\sqrt{1-q_0\,\sin^2 x_0}}\,. \tag{3.23}$$

One might wonder whether from the gauge-theory perspective it makes sense for the splitting parameter x and the outgoing angular momentum fraction β_{34} to be dependent on $\mathcal{J}$. After all, the splitting Hamiltonian commutes with the R-charge operators $\mathcal{J}_{12}$ and $\mathcal{J}_{34}$. Hence, going up higher in perturbation theory should not induce coupling-constant dependent modifications to the R-charges of the outgoing strings. However, the reason why (3.21) is a sensible result is that the semi-classical string calculation captures only a part (namely the maximum) of the full quantum surface of the decay process. The position of the maximum varies as we go higher up in perturbation theory. At each order in perturbation theory, the most probable outgoing string with fixed $\mathcal{J}_{12}^I$ is carrying a different $\mathcal{J}_{34}^I$. This effectively means that the maximal probability varies with $\mathcal{J}$.

In figure 3 we plot a collection of functions $\beta_{34}^0(\beta_{12})$ for values of the filling fraction before the split ranging from 0.05 to 0.5. These are all computed using the leading values for the parameter q_0. We choose to restrict to this region of the filling fraction since the folded string solution (2.3) does not posses a $\mathcal{J}_{12} \leftrightarrow \mathcal{J}_{34}$ symmetry[3] and only the solutions with $\alpha < 0.5$ have been identified on the gauge side [13]. The solutions with $\alpha > 0.5$ are conjectured to correspond to operators of higher bare dimension, which do not have a BMN limit. Note that figure 3 possesses a symmetry with respect to the point $(0.5, 0.5)$ as a consequence of the geometry of the folded string ($\psi(\sigma) = \psi(\pi - \sigma)$). Note also that the point $(1, 1)$ corresponds to the unsplit string.

[3] On the other hand the sigma model is, as expected, invariant under this symmetry. The implementation of this symmetry requires the simultaneous transformations $\phi_1 \leftrightarrow \phi_2$, and $\psi \leftrightarrow \frac{\pi}{2} - \psi$.

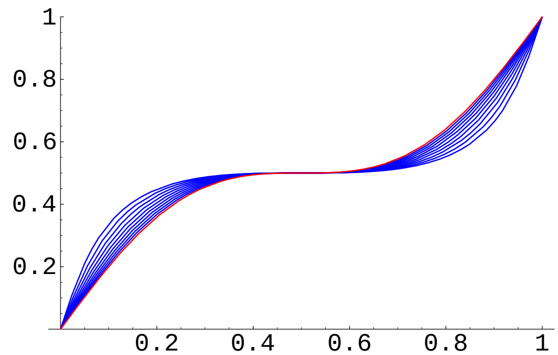


Figure 3: Plot of the relation between β_{12} (horizontal) and β_{34}^0 (vertical) as defined in (3.14). The various curves correspond to various values for the filling fraction $\alpha \in [0.05, \ldots 0.5]$.

The second functional relation we want to obtain is a relation between the energy of the first outgoing piece $\mathcal{E}^I$ and the parameters $(\mathcal{J}, \alpha, \beta_{12})$. Eliminating κ, a, ω_1 and ω_2 from equation (3.11) using (3.12), (3.1) and (3.3) leaves us with

$$\frac{q}{\pi^2} = \frac{(\mathcal{E}^I)^2}{(K(q) + F(x;q))^2} - \frac{(\mathcal{J}_{12}^I)^2}{(E(q) + E(x;q))^2}\,, \tag{3.24}$$

which is the split analogue of (2.20). The "mirror" equation for the second half of the string is obtained from (3.24) by replacing the indices $I \to II$ and $x \to -x$. Using equations (3.16) and (3.17), this can be simplified to

$$\frac{4}{\pi^2}\,q = \frac{(\mathcal{E}^I)^2}{[\beta_{34}\,K(q) + (\beta_{12} - \beta_{34})\,E(q)]^2} - \frac{(\mathcal{J}_{12})^2}{E(q)^2}\,. \tag{3.25}$$

Combining this with (2.20) we learn that

$$\mathcal{E}^I = \Big(\beta_{34} + (\beta_{12} - \beta_{34})\,\frac{E(q)}{K(q)}\Big)\,\mathcal{E}\,, \qquad \mathcal{E}^{II} = \mathcal{E} - \mathcal{E}^I\,. \tag{3.26}$$

This equation, together with equation (3.17) for β_{34} and equation (2.22) for q, defines $\mathcal{E}^{I/II}(\mathcal{J}, \alpha, \beta_{12})$ as a series in $1/\mathcal{J}$,

$$\mathcal{E}^{I/II} = \mathcal{J}\,\mathcal{E}_0^{I/II} + \mathcal{E}_1^{I/II}\,\frac{1}{\mathcal{J}} + \ldots\,. \tag{3.27}$$

The first coefficient in the expansion is given by

$$\mathcal{E}_0^I = \frac{\mathcal{J}_{12}^I}{\mathcal{J}} + \frac{\mathcal{J}_{34}^{I,0}}{\mathcal{J}} = (1 - \alpha)\,\beta_{12} + \alpha\,\beta_{34}^0\,, \tag{3.28}$$

and is in agreement with the (trivial) gauge theory prediction: the two decay products (single trace operators) have engineering dimensions J_{12}^I and $J_{34}^{I,0}$. In figure 4 we plot the energy of the first string piece as a function of β_{12}, for various filling fractions. The coefficient at order $1/\mathcal{J}$ of (3.27) reads

$$\mathcal{E}_1^I = \tfrac{2}{\pi^2}\,K(q_0)\,\Big[(K(q_0) - E(q_0))\,\beta_{34}^0\,(q_0 - 1) + \beta_{12}\,E(q_0)\,q_0\Big] + \beta_{34}^1\,(1 - \tfrac{E(q_0)}{K(q_0)})\,, \tag{3.29}$$

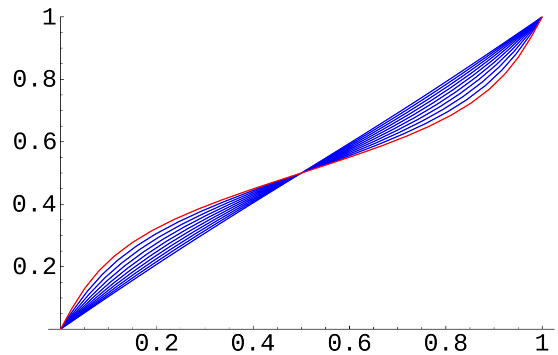


Figure 4: The energy $\mathcal{E}_0^I$ of the first outgoing string as a function of β_{12}, for various filling fractions $\alpha \in [0.05, \ldots, 0.5]$. The straight line corresponds to $\alpha = 0.5$.

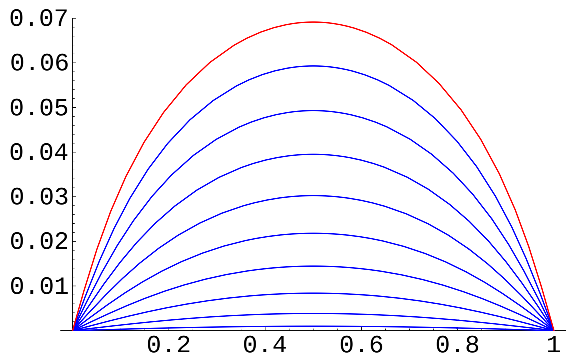


Figure 5: The combination of new angular charges $\Delta := (\mathcal{J}_{14}^I)^2 - (\mathcal{J}_{23}^I)^2$ plotted as a function of β_{12}, again for α in the range $[0.05, \ldots, 0.5]$. The upper curve corresponds to $\alpha = 0.5$.

which yields a prediction of the anomalous dimension at one loop of the first decay product (single trace operator) in the dual gauge theory.

The third functional relation is obtained by eliminating x from (3.16) and (3.13), after which one can express $\Delta := (\mathcal{J}_{14}^I)^2 - (\mathcal{J}_{23}^I)^2$ as a function of β_{12} in an expansion in $1/\mathcal{J}$. The corresponding plot is given in figure 5.

3.3 Higher charges and traces of integrability in the splitting process

Thus far we have only discussed the behavior of the string energy and angular momenta under the decay process. However, the classical string sigma model is known to possess an infinite number of local, conserved and commuting charges Q_n due to its integrability [21, 22, 23], the first non-vanishing of which is the Hamiltonian $Q_2 = \mathcal{H}$. These were written down explicitly in the work of [24] for the folded string solution in terms of a generating functional. On the other hand, one does not expect the string sigma model to remain integrable once string interactions are included (i.e. when $g_s \neq 0$). This may be seen explicitly from the dual gauge theory side: nonplanar graphs break the integrability of the planar theory[4]. Nevertheless it is obvious that, for the semi-classical decay process we are studying here, the higher charges Q_n *are* conserved. This conservation follows from the same logic that was used for the calculation of the energy and angular momenta. If the initial charges are given via a charge density as $Q_n = \int \mathrm{d}\sigma\, q_n(\sigma,\tau)$, then the charges of the outgoing strings after the split are simply

$$Q_n^I = \int_0^{2\pi a} \mathrm{d}\sigma\, q_n(\sigma,\tau)\,, \quad Q_n^{II} = Q_n - Q^I\,. \tag{3.30}$$

Here one uses the charge densities $q_n(\sigma,\tau)$ *before* the split. In appendix A we explicitly derive the generating functional for the commuting charges of the outgoing strings by generalizing the work of [24]. As a side remark let us note that this knowledge could in principle be used to construct the explicit form of the outgoing string solutions $X^I(\tau,\sigma)$ and $X^{II}(\tau,\sigma)$.

How is this result to be reconciled with the breakdown of integrability at $g_s \neq 0$? Again we need to remember that the quantum string decay leads to a full surface of possible decay channels, which generically will not preserve the charges beyond Q_2. A subset of channels will, however, preserve all Q_n. It is precisely this subsector which should capture the semiclassical string decay analyzed in the previous subsections and is expected to dominate the decay amplitude.

4. The decay from the gauge theory side

Let us now turn to the discussion of the splitting process in the dual gauge theory. Our exposition will not be complete, but we will set the scene for the full calculation and point out the technical difficulties that one will have to face.

In the large-N limit, the dilatation operator of $\mathcal{N}=4$ super-Yang-Mills factorizes as the product of a universal space-time dependent factor times a combinatorial factor acting on the fields inside composite operators. The string splitting vertex is encoded in the non-planar piece of this dilatation operator. In the relevant SU(2) sector of two chiral complex scalar adjoint fields Z and W the (space-time independent part of the) dilatation operator is known to be [25]

$$D_2 = -\frac{g_{\mathrm{YM}}^2}{8\pi^2}\ \mathrm{Tr}[Z,W][\check{Z},\check{W}]\,, \tag{4.1}$$

where $\check{Z}_{ab} := \delta/\delta Z_{ba}$ is the matrix derivative (for a pedagogical derivation see [26]). The action of this operator can be expressed in the language of spin chains, by considering the action of D_2 on two fields in an arbitrary single trace operator $\mathrm{Tr}(WAZB)$. One finds

$$D_2 \circ \mathrm{Tr}(WAZB) = \\ \frac{g_{\mathrm{YM}}^2}{8\pi^2}\ \mathrm{Tr}\,A\,\Big(\mathrm{Tr}(WZB) - \mathrm{Tr}(ZWB)\Big) + \frac{g_{\mathrm{YM}}^2}{8\pi^2}\ \mathrm{Tr}\,B\,\Big(\mathrm{Tr}(ZWA) - \mathrm{Tr}(WZA)\Big)\,. \tag{4.2}$$

[4]Concretely, one observes degeneracies in the spectrum of planar $\mathcal{N}=4$ super-Yang-Mills (the so-called "planar parity pairs") which may be attributed to the existence of a higher conserved charge [12]. These degeneracies are lifted, however, by $1/N$ corrections: a clear signal of the breakdown of integrability.

The planar (nearest neighbor) contribution is obtained when A is the identity operator:

$$D_2^{\text{planar}} = \frac{g_{\text{YM}}^2 N}{8\pi^2} \sum_{i=1}^{L} (\delta_{i,i+1} - P_{i,i+1}) \,, \tag{4.3}$$

with $P_{i,j}$ the permutation operator permuting the fields (spins) at sites i and j. The system described by D_2^{planar} is nothing but the Heisenberg $\text{XXX}_{1/2}$ model [20]. The non-planar contribution may be written as

$$D_2^{\text{splitting}} = \frac{g_{\text{YM}}^2}{8\pi^2} \sum_{i,j} (\delta_{i,j} - P_{i,j}) \, \mathcal{S}_{ij} \,, \tag{4.4}$$

where the sum is over non-nearest neighbors and the splitting operator acts on sites i and j as ($\phi_k \in Z, W$)

$$\begin{aligned} &\mathcal{S}_{ij} \circ \text{Tr}(\phi_1 \ldots \phi_{i-1}\phi_i\phi_{i+1} \ldots \phi_{j-1}\phi_j\phi_{j+1} \ldots \phi_L) = \\ &\quad \text{Tr}(\phi_1 \ldots \phi_i\phi_j \ldots \phi_L) \; \text{Tr}(\phi_{i+1} \ldots \phi_{j-1}) + \text{Tr}(\phi_1 \ldots \phi_{i-1}\phi_{j+1} \ldots \phi_L) \; \text{Tr}(\phi_i \ldots \phi_j) \,. \end{aligned} \tag{4.5}$$

That is, we have a Heisenberg exchange interaction multiplied by a chain splitting operation (see also [27] for a related discussion).

While the dilatation operator is thus under control, the initial gauge theory operator dual to the single folded string solution with angular momenta $\mathcal{J}_{12}$ and $\mathcal{J}_{34}$ is less understood. The dual gauge operator may be written as

$$\text{Tr}(Z^{\mathcal{J}_{12}} W^{\mathcal{J}_{34}}) + \ldots \tag{4.6}$$

where the dots stand for suitable permutations of the Z and W's – which are of essential importance for the evaluation of decay amplitudes! The spin chain picture has proved to be very efficient for the task of diagonalizing D_2^{planar} for long operators ($\mathcal{J} \to \infty$) with the technology of the Bethe ansatz. There every eigenstate of the "free" Hamiltonian D_2^{planar} is parametrized by a set of Bethe roots λ_i with $i = 1, \ldots, \mathcal{J}_{34}$, which are determined through the Bethe equations

$$\left(\frac{\lambda_i + i/2}{\lambda_i - i/2} \right)^L = \prod_{k \neq i} \left(\frac{\lambda_i - \lambda_k + i}{\lambda_i - \lambda_k - i} \right) , \qquad \prod_{i=1}^{J_{34}} \frac{\lambda_i + i/2}{\lambda_i - i/2} = 1 \,, \tag{4.7}$$

where $L := \mathcal{J}_{12} + \mathcal{J}_{34} = \mathcal{J}$. The corresponding eigenstate may then be written down explicitly as follows. Denote by $|\{m_1, m_2, \ldots, m_{J_{34}}\}\rangle_L$ the single trace operator of length L with W's appearing at positions m_i, e.g.

$$|\{1, 3, 4\}\rangle_{L=7} = \text{Tr}(WZWWZZZ) \,.$$

Introduce the quasi-momenta p_i and the scattering phases φ_{ij}

$$p_i := -i \, \ln \left(\frac{\lambda_i + i/2}{\lambda_i - i/2} \right) , \qquad \varphi_{i,j} := -i \, \ln \left(\frac{\lambda_i - \lambda_k + i}{\lambda_i - \lambda_k - i} \right) , \tag{4.8}$$

then the eigenstate of $D_2^{\rm planar}$ may be written down explicitly [28][5]. It is the rather formidable object

$$|\psi\rangle = \tag{4.9}$$
$$\sum_{\substack{1\leq m_1<m_2<\ldots\\ \ldots<m_{\mathcal{J}_{34}}\leq L}} \sum_{\mathcal{P}\in{\rm Perm}_{\mathcal{J}_{34}}} \exp\Big[i\sum_{i=1}^{\mathcal{J}_{34}} p_{\mathcal{P}(i)}\cdot m_i + \frac{i}{2}\sum_{i<j}^{\mathcal{J}_{34}} \varphi_{\mathcal{P}(i),\mathcal{P}(j)}\Big]\,\Big|\{m_1,m_2,\ldots,m_{\mathcal{J}_{34}}\}\Big\rangle_L$$

where the second sum is over all $\mathcal{J}_{34}!$ permutations of the labels $\{1,2,3,\ldots,\mathcal{J}_{34}\}$. As Bethe showed in 1931 this is an eigenstate of the free Hamiltonian

$$D_2^{\rm planar}\,|\psi\rangle = \frac{g_{\rm YM}^2\,N}{2\pi^2}\sum_{i=1}^{\mathcal{J}_{34}}\sin^2\left(\frac{p_i}{2}\right)\,|\psi\rangle\,. \tag{4.10}$$

In order to make contact to our semiclassical string considerations we need to take the thermodynamic limit $L,\mathcal{J}_{34}\to\infty$ with $\mathcal{J}_{34}/L=\alpha$ fixed. Due to the unknown structure of the continuum limit of the permutation group the Bethe wave function $|\psi\rangle$ becomes a monstrous object in this limit[6]. This is in stark contrast to the Bethe equations, which actually simplify in the same limit. Even worse, we would now want to act with the splitting Hamiltonian $D_2^{\rm splitting}$ of (4.4) on $|\psi\rangle$ in the thermodynamic limit, in order to describe the quantum decay of the semiclassical folded string solution. This is the core of the problem which hampers a direct analytic computation of the splitting in the gauge theory. In principle one could attempt to address this problem numerically. Here however, one faces technical limitations, as the minimal length of the spin chain for which distinguishable structures limiting to the continuum folded string configuration start to emerge is 26 (with half filling fraction) [10]. This implies that the wave function $|\psi\rangle$ contains 2^{26} terms, most of which have coefficients of the same order.

An alert reader might wonder whether the gauge theory again develops an effective genus counting parameter $\mathcal{J}^2/N$ in the thermodynamic limit ($\mathcal{J}_{12}$, $\mathcal{J}_{34}$, $N\to\infty$) as it does in the BMN limit where $\mathcal{J}_{34}$ remains finite [3, 2]. It is very plausible that this is the case. Indeed a simple pilot computation of the free theory two point function of two operators of type (4.6) in the this limit confirms the expectation. One finds using the method of "highways" [3] up to genus one

$$\begin{aligned}\langle{\rm Tr}(\bar Z^J\,\bar W^J)\;{\rm Tr}(Z^J\,W^J)\rangle &= N^{2J}\left(1+\frac{1}{N^2}\left(2\left[\binom{J+1}{3}+\binom{J+1}{4}\right]+(J-1)^2\right)+\ldots\right)\\ &\to N^{2J}\left(1+12\,\frac{J^4}{N^2}+\ldots\right)\end{aligned} \tag{4.11}$$

which displays the expected J^4/N^2 scaling behavior.

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[5] For a nice hands-on review of this topic see [29].

[6] If one stays with a small number of impurities $\mathcal{J}_{34}$ and takes $L\to\infty$ the state remains manageable and is dual to excited states of the plane wave superstring in the BMN correspondence [1]. In this limit one finds $p_i=n_i/\mathcal{J}$ with integer n_i and $\varphi_{i,j}\to 0$.

It is instructive to look at the decay of a number of lower eigenstates of $D_2^{\rm planar}$ as toy calculations exemplifying the general logic of the quantum decay. All eigenstates $D_2^{\rm planar}$ may be classified by Bethe roots, or equivalently by the values of the local, conserved charges Q_i. At one loop, these are given by the moments of the resolvent, and the first nonvanishing charge corresponds to the one loop anomalous dimension of the state $E = Q_2$. On top of this, all states carry a representation of the global R-symmetry group. In the SU(2) sector, these are realized through the operators

$$J_z \equiv J_{12} - J_{34} = {\rm Tr}(W\check{W} - Z\check{Z})\,, \qquad J_+ = {\rm Tr}(W\check{Z})\,, \qquad J_- = {\rm Tr}(Z\check{W})\,, \tag{4.12}$$

where, geometrically, the operators J_{12} and J_{34} correspond to the rotations in the two two-planes $W, \bar{W}$ and $Z, \bar{Z}$. It is easy to check that the operators (4.12) obey an su(2) algebra: $[J_+, J_-] = J_z$ and $[J_z, J_\pm] = \pm 2\, J_\pm$. The full (planar and non-planar) dilatation operator D_2 indeed commutes with these operators: $[D_2, J_z] = 0 = [D_2, J_\pm]$. Hence, the total spin and the J_z charge of a given initial state is conserved in the decay process. Highest-weight single trace states obey $J_- \,|{\rm HWS}\rangle = 0$ and correspond to ensembles of Bethe roots at finite values. Acting with J_+ on $|{\rm HWS}\rangle$ increases the number of impurities but leaves the energy invariant. This corresponds to adding Bethe roots at infinity. Note also that expectation values of ("non-Cartan") operators J_x and J_y in the $|{\rm HWS}\rangle$ obviously vanish.

All the local conserved charges of the Heisenberg $\text{XXX}_{1/2}$ are known explicitly [30]. As we illustrate now, these are generically not conserved in the decay process. Explicitly, the first three charges are given by [12]

$$\begin{aligned} Q_2 &= 2\sum_{i=1}^{L}(1 - P_{i,i+1})\,, \\ Q_3 &= 4\sum_{i=1}^{L}(P_{i,i+1}\,P_{i+1,i+2} - P_{i+1,i+2}\,P_{i,i+1})\,, \\ Q_4 &= \sum_{i=1}^{L}\big(-2P_{i,i+1} + P_{i,i+1}\,P_{i+1,i+2} + P_{i+1,i+2}\,P_{i,i+1} \\ &\qquad + P_{i,i+1}\,P_{i+2,i+3}\,P_{i+1,i+2} + P_{i+1,i+2}\,P_{i,i+1}\,P_{i+2,i+3} \\ &\qquad - P_{i,i+1}\,P_{i+1,i+2}\,P_{i+2,i+3} - P_{i+2,i+3}\,P_{i+1,i+2}\,P_{i,i+1}\big)\,. \end{aligned} \tag{4.13}$$

The odd charges have negative parity and either annihilate a highest-weight state, or pair them to a partner of opposite parity, which is degenerate in energy. Let us for example, look at the decay of two highest-weight states of length 9,

$$\begin{aligned} \mathcal{O}_-^{9,4} &= -\,{\rm Tr}(Z^4WZW^3) + {\rm Tr}(Z^4W^3ZW) + {\rm Tr}(Z^3WZ^2W^3) - {\rm Tr}(Z^3W^3Z^2W) \\ &\quad + {\rm Tr}(Z^3WZWZW^2) - {\rm Tr}(Z^3W^2ZWZW) \qquad q_2 = 5,\ q_4 = 1 \\ \mathcal{O}_+^{9,2} &= {\rm Tr}(Z^7W^2) - {\rm Tr}(Z^6WZW) \qquad\qquad\qquad\qquad q_2 = 4,\ q_4 = -16\,, \end{aligned} \tag{4.14}$$

where we have also spelled out their charges with respect to the Q_2 and Q_4 operators. The first state decays into Q_4 non-conserving constituents:

$$D_2^{\rm splitting} \circ \mathcal{O}_-^{9,4} = 12\Big({\rm Tr}(Z^2)\, \mathcal{O}_-^{7,4} + {\rm Tr}(ZW)\, \mathcal{O}_-^{7,3} \Big)\,, \tag{4.15}$$

where

$$\begin{aligned} \mathcal{O}_-^{7,4} &= \mathrm{Tr}(Z^2W^3ZW) - \mathrm{Tr}(Z^2WZW^3)\,, \qquad q_2 = 5, q_4 = -5\,, \\ \mathcal{O}_-^{7,3} &= \mathrm{Tr}(Z^3WZW^2) - \mathrm{Tr}(Z^3W^2ZW)\,, \qquad q_2 = 5, q_4 = -5\,. \end{aligned} \tag{4.16}$$

The protected states $\mathrm{Tr}(Z^2)$ and $\mathrm{Tr}(ZW)$ have vanishing charges. Therefore Q_4 is not conserved in both decay channels. Note also that

$$J_- \circ \mathcal{O}_-^{7,4} = -\mathcal{O}_-^{7,3} \quad J_- \circ \mathrm{Tr}(ZW) = \mathrm{Tr}(Z^2) \quad J_- \circ \{\, \mathcal{O}_-^{9,4}, \mathcal{O}_-^{7,3}, \mathrm{Tr}(Z^2)\,\} = 0\,. \tag{4.17}$$

Hence a highest-weight state does not necessarily decay into products of highest-weight states.

The highest-weight state $\mathcal{O}_+^{9,2}$ of (4.14) on the other hand has only a single decay channel

$$D_2^{\text{splitting}} \circ \mathcal{O}_+^{9,2} = 8\ \mathrm{Tr}(Z^2)\, \mathcal{O}_+^{5,2} + (\text{non } Q_2 \text{ preserving channels})\,, \tag{4.18}$$

where

$$\mathcal{O}_+^{5,2} = \mathrm{Tr}(Z^3W^2) - \mathrm{Tr}(Z^2WZW)\,, \qquad q_2 = 4,\, q_4 = -16 \tag{4.19}$$

Hence here the higher local charge Q_4 is conserved.

In summary, by considering the decays of short operators, we see that the highest-weight states do not need to decay into a product of highest-weight states, and that higher charges are not preserved in this decay process. However, in the thermodynamic limit we expect the decay to be dominated by the channels which *do* preserve all higher charges. The outgoing states are not highest-weight states. This can be seen from the fact that in the semiclassical calculations, equations (3.9) and (3.10) indicate that the expectation values for the "non-Cartan" angular momenta, $\langle J_{13}\rangle$ and $\langle J_{24}\rangle$, are non-vanishing after the decay.

5. Outlook

The folded spinning string for which we have calculated the classical decay process has been identified on the gauge theory side [10] by solving the Bethe equations (4.7). In the sector in which the number of impurities is odd, it appears as the first highest-weight state above the vacuum. Using (4.10) it is possible to compute its energy directly from the Bethe roots. For half-filling, an analysis of various spin chain lengths (up to length 46) has shown that this energy is approximated by

$$\mathcal{E} - \mathcal{J}\mathcal{E}_0 = \frac{0.356}{\mathcal{J}} + \dots\,. \tag{5.1}$$

This matches the result computed from (2.24) and (2.26) for $\alpha = 1/2$.

In principle, one can apply the machinery of section 4 in order to study the decay of this state. The charges which enter in the classical decay relations on the string side, as depicted in figure 3–5, have direct analogues in the spin chain. As an example, the parameters β_{12} and β_{34} are related to filling fractions α and α^I and chain lengths L and L^I according to

$$\beta_{34} \leftrightarrow \frac{\alpha^I L^I}{\alpha L}\,, \quad \beta_{12} \leftrightarrow \frac{(1-\alpha^I)L^I}{(1-\alpha)L}\,. \tag{5.2}$$

As explained in the introduction, the splitting amplitude of the spin chain is expected to attain its maximum over the semi-classical curves found in section 3.2.

However, as we have already alluded to in the previous section, the main obstacle is the complexity of the Bethe wave function (4.9). Even for moderately large spin chains, the Bethe state is a monstrous object, which complicates a brute force analysis through a numerical treatment. One possible simplification can perhaps be obtained by using the coherent state wave function of [31]. However, a potential problem in this approach seems to arise from the inability to write down wave functions for the outgoing strings.

An additional guideline for a better analytic understanding is the existence of the higher local charges. The decay channels in which these charges are conserved are expected to correspond to semi-classical decay, and form only a small subsector of all possible channels. We will return to this spin chain analysis in future work.

Finally an interesting question which deserves investigation concerns the circular string solution of Frolov and Tseytlin [8]. This solution has also been successfully matched to gauge theory [10]. Clearly the circular string is semiclasssically stable, as it does not self intersect. How does this property reflect itself in the dual gauge or spin chain description?

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A. Generating function for commuting charges of the outgoing string

In the case of the $O(6)$ model, an infinite set of commuting charges can be constructed using the so-called Bäcklund transformations. Namely, given a particular solution X_0^μ, by solving a set of equations [22] one can derive a "dressed" solution $X^\mu(\gamma)$ (with $X^\mu(0) = X_0^\mu$) as a power series in the spectral parameter γ. The coefficients in the series are determined though certain recursive relations. The generating function for the charges is then obtained from the dressed solution as

$$\Gamma(\gamma) = \int \frac{\mathrm{d}\sigma}{4\pi} \left(\gamma(X(\gamma)\cdot X_\xi) + \gamma^3(X(\gamma)\cdot X_\eta)\right) , \tag{A.1}$$

where $\xi = \frac{1}{2}(\tau+\sigma)$ and $\eta = \frac{1}{2}(\tau-\sigma)$, while subscripts denote partial derivatives. In the case of the folded string, a solution to the Bäcklund transformation has been constructed exactly (to all orders in γ) in [24] and is given by

$$Z_i(\gamma) \equiv X_i + iX_{i+3} = r_i(\sigma,\gamma)e^{i\alpha_i(\sigma,\gamma)}e^{i\omega_i\tau} , \quad (i=1,2,3) . \tag{A.2}$$

The r_i are defined as

$$r_1(\sigma,\gamma) = \mathrm{dn}(\sqrt{\omega_{21}^2}\sigma+\nu,t) , \quad r_2(\sigma,\gamma) = \sqrt{t}\,\mathrm{sn}(\sqrt{\omega_{21}^2}\sigma+\nu,t) , \quad r_3(\sigma,\gamma) = 0 . \tag{A.3}$$

The constant phases α_i are given by

$$\cos\alpha_1 = \frac{1-\gamma^2}{1+\gamma^2}\frac{1}{\mathrm{dn}\,\nu} , \quad \cos\alpha_2 = \frac{1-\gamma^2}{1+\gamma^2}\frac{\mathrm{cn}\,\nu}{\mathrm{dn}\,\nu} . \tag{A.4}$$

The functional dependence of the parameter ν on the spectral parameter γ is given by the equation

$$1 - \frac{\omega_1^2}{\omega_{21}^2}\frac{\mathrm{sn}^2\,\nu}{\mathrm{cn}^2\,\nu} - \left(\frac{1-\gamma^2}{1+\gamma^2}\right)^2 \frac{1}{1-t^2\,\mathrm{sn}^2\,\nu} = 0 . \tag{A.5}$$

In addition, the periodicity condition for the folded string implies the relation

$$\frac{\pi}{2}\sqrt{\omega_{21}^2} = K(t) , \quad \omega_{21}^2 \equiv \omega_2^2 - \omega_1^2 , \tag{A.6}$$

which is (2.19) of the main text.

Inserting (A.2) into the expression for the generating function of the charges (A.1), one obtains

$$\Gamma(\gamma;a) = \gamma \int_0^{2\pi a} \frac{\mathrm{d}\sigma}{2\pi} r_i(\sigma,\gamma)\left[(1-\gamma^2)\cos\alpha_i\, r_i'(\sigma,0) + (1+\gamma^2)\omega_i\,\sin\alpha_i\, r_i(\sigma,0)\right] , \tag{A.7}$$

where here the constant a is the splitting parameter. For $a=1$, this integral determines the charges of the incoming string, and it has been computed in [24]. To evaluate this integral for a generic value of the parameter a, and hence determine the generating functions for commuting charges of the outgoing strings, one needs the following four integrals:

$$\int_0^{2\pi a} \frac{\mathrm{d}\sigma}{4\pi} r_1(\sigma,\gamma)\, r_1(\sigma,0) = \frac{\mathrm{dn}\,\nu}{2K(t)\,\mathrm{sn}^2\,\nu}\left[aK(t) - \mathrm{cn}\,\nu\,\Pi(a,t\,\mathrm{sn}^2\,\nu,t)\right] + \frac{1}{16K(t)}\frac{\mathrm{cn}\,\nu}{\mathrm{sn}\,\nu}\ln|A(\nu,t)| , \tag{A.8}$$

$$\int_0^{2\pi a} \frac{d\sigma}{4\pi} r_2(\sigma,\gamma)\, r_2(\sigma,0) = -\frac{\operatorname{cn}\nu\,\operatorname{dn}\nu}{2K(t)\operatorname{sn}^2\nu}\Big[aK(t) - \Pi(a, t\operatorname{sn}^2\nu, t)\Big] - \frac{1}{16K(t)}\frac{1}{\operatorname{sn}\nu}\ln|A(\nu,t)|\,, \quad (A.9)$$

$$\int_0^{2\pi a} \frac{d\sigma}{4\pi} r_1(\sigma,\gamma)\, r_1'(\sigma,0) = \frac{\operatorname{cn}\nu}{\pi\operatorname{sn}^3\nu}\Big[\operatorname{dn}^2\nu(K(t)a - \Pi(a, t\operatorname{sn}^2\nu, t)) + \operatorname{sn}^2\nu\,(K(t)a - \tfrac{1}{4}E(\tilde{a},t))\Big] - \frac{1}{8\pi}\frac{\operatorname{dn}\nu}{\operatorname{sn}^2\nu}\ln|A(\nu,t)|\,, \quad (A.10)$$

$$\int_0^{2\pi a} \frac{d\sigma}{4\pi} r_2(\sigma,\gamma)\, r_2'(\sigma,0) = \frac{1}{\pi\operatorname{sn}^3\nu}\Big[\operatorname{cn}^2\nu\operatorname{dn}^2\nu\Pi(a, t\operatorname{sn}^2, t) + \operatorname{sn}^2\nu\,\tfrac{1}{4}E(\tilde{a},t) - \operatorname{dn}^2\nu K(t)a\Big] - \frac{1}{8\pi}\frac{\operatorname{dn}\nu\operatorname{cn}\nu}{\operatorname{sn}^2\nu}\ln|A(\nu,t)|\,, \quad (A.11)$$

where

$$A(\nu,t) = t\operatorname{sn}^2\nu\operatorname{sn}^2\big(4K(t)a\big) - 1 \qquad \text{and} \qquad \tilde{a} = \operatorname{am}\big(4aK(t),t\big)\,. \quad (A.12)$$

Here $\Pi(a,m^2,t)$ is the incomplete elliptic integral of the third kind, defined as

$$\Pi(a,m^2,t) = \frac{K(t)}{2\pi}\int_0^{2\pi a}\frac{d\sigma}{1 - m^2\operatorname{sn}^2\big(\frac{2}{\pi}K(t)\sigma, t\big)}\,. \quad (A.13)$$

The expression for the generating function for the charges of the outgoing string of "length"-a can be now be written as

$$\begin{aligned}
\Gamma(\gamma;a) &= \Gamma(\gamma)^I + \Gamma^{II}(\gamma)\,,\\
\Gamma^I(\gamma;a) &= -\frac{\gamma\operatorname{cn}\gamma}{(1+\gamma^2)\pi\operatorname{dn}\nu\operatorname{sn}^3\nu}\left(B(\nu,\gamma)K(t) + C(\nu,\gamma)\Pi(a,t^2\operatorname{sn}^2\nu,t)\right)\,,\\
\Gamma^{II}(\gamma;a) &= \frac{1}{4\pi}\frac{\gamma}{1+\gamma^2}\frac{1}{\operatorname{sn}^2\nu}\left((1+\gamma^4) - 2\operatorname{cn}^2\nu\gamma^2\right)\ln|A(\nu,t)|\,,
\end{aligned} \quad (A.14)$$

where the functions $B(\nu,\gamma)$ and $C(\nu,\gamma)$ are given by

$$\begin{aligned}
B(\nu,\gamma) &= (1-\gamma^2)^2\left(\left(\operatorname{sn}^2\nu - \operatorname{dn}^2\nu\right)a + \frac{\operatorname{cn}^2\nu}{a} - 1\right) - (1+\gamma^2)^2\frac{\operatorname{dn}^2\nu}{a} + 2(1+\gamma^4)\operatorname{dn}^2\nu\,,\\
C(\nu,\gamma) &= (1-\gamma^2)^2\left(\operatorname{cn}^2\nu\left(\frac{1}{a} - 1\right) + \frac{\operatorname{dn}^2\nu}{a} - \operatorname{sn}^2\nu\operatorname{dn}^2\nu\right) - (1+\gamma^2)^2\operatorname{cn}^2\nu\operatorname{dn}^2\nu\,,
\end{aligned} \quad (A.15)$$

and the function $A(\nu,t)$ is given in (A.12).

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