

Relativistic Elasticity

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Abstract

Relativistic elasticity on an arbitrary spacetime is formulated as a Lagrangian field theory which is covariant under spacetime diffeomorphisms. This theory is the relativistic version of classical elasticity in the hyperelastic, materially frame-indifferent case and, on Minkowski space, reduces to the latter in the limit $c \rightarrow \infty$. The field equations are cast into a first – order symmetric hyperbolic system. As a consequence one obtains local-in-time existence and uniqueness theorems under various circumstances.

Keyword: elasticity, general relativity

1 Introduction

By far the most popular relativistic matter model has been that of a perfect fluid. One reason for this lies in the relative simplicity of this model. Another reason is that astrophysical objects, which are the objects most likely to exhibit effects of Special or General Relativity, are usually described quite

well as bodies composed of perfect fluid. Elasticity, on the other hand, is relevant for describing the statics and dynamics of solids which we meet in everyday life, and for this relativistic effects can safely be ignored. But, whatever experimental relevance it might have (e.g. for describing neutron star crusts or bar detectors of gravitational waves), it is clearly of conceptual interest to have a consistent theory of bulk matter which is in accordance with the principles of relativistic physics.

A basic consistency check on such a theory is to see whether its equations have a well-posed initial value problem. Christodoulou, in the book [3], has announced without proof a general existence theorem, covering the case of elasticity, using a new notion of hyperbolicity. We show that the elasticity equations on an arbitrary spacetime can in a very natural manner be cast into the form of a quasilinear, symmetric hyperbolic system¹.

Relativistic elasticity has been treated by many authors. The earliest references we are aware of are Herglotz [7], in 1911(!) for Special Relativity and Nordström [14] in 1916, on General Relativity. A very influential paper has been that of Carter and Quintana [2], of which a clear presentation has been given by Ehlers [5]. A very recent application of the Carter formalism is by Karlovini and Samuelsson [10]. We also would like to mention the work of Maugin [13]. An excellent exposition can be found in the text book of Soper [15]. Recent useful references are Kijowski and Magli [12], which also contains an extensive bibliography, Christodoulou [3] and Tahvildar-Zadeh [16].

In the present work we treat elasticity as a field theory derived from a Lagrangian. Our basic fields are maps from spacetime into a 3-dimensional "material manifold" \mathcal{B} . Objects defined on \mathcal{B} are physically interpreted as properties of the material prior to the action of deformations or other fields.

In Section 2 we review general properties of Lagrangian field theories with Lagrangians depending on the map f and its first derivatives and which are covariant under spacetime diffeomorphisms. In Section 3 we show that the Euler-Lagrange equations of these theories, when written as a first-order system, can in a very natural manner be rewritten as a "symmetric system", i.e. as a quasilinear first - order system where the coefficient matrices of the derivative - terms are symmetric.

Now recall that, in the quasilinear case, the coefficient matrices are in general functions of the independent and dependent variables. The former

¹For nonrelativistic elasticity such a formulation has been found by John [9].

are points in spacetime. The latter variables, in our case, consist of possible values for the field and its first derivatives. All these together form what we call the "deformation bundle" \mathcal{D} over spacetime. According to the general theory the symmetric system found in Sect.3 is symmetric hyperbolic at a point of \mathcal{D} if there is a covector at the corresponding spacetime point so that a certain positivity requirement is fulfilled for the coefficient matrices. The existence of such "hyperbolic points" of \mathcal{D} requires further constitutive hypotheses to which we turn in Sect.4. Here we define a class of theories for which \mathcal{D} has a subbundle consisting of "unstrained" or "natural" points which are in addition stress – free and for which the elasticity tensor has the standard "Hookean" structure. We show that, for such materials, our equations are indeed symmetric hyperbolic near natural points in the deformation bundle.

In Section 5 we discuss various physical situations covered by our existence theorem and the interpretation of the initial data.

Finally (in Appendix B) we treat the non-relativistic limit $c \rightarrow \infty$ of elasticity in Minkowski space. As to be expected, the basic structure of the equations remains essentially unchanged and we obtain again local existence and uniqueness. These equations are equivalent to those of the standard nonrelativistic theory in the hyperelastic, materially frame-indifferent case (see Ciarlet [4] and Gurtin [6]).

2 Preliminaries

States of relativistic continua are best described by maps f from spacetime (M, g) to a 3-dimensional manifold \mathcal{B} , called "material manifold". In local coordinates we write $X^A = f^A(x^\mu)$. The space \mathcal{B} is the abstract collection of particles making up the continuous medium. By "abstract" we mean that we view \mathcal{B} as a space of its own right independent of "physical space". There are several reasons for doing so, the most important one being that there is, in a relativistic theory, no natural notion of "spatial location at given time".

The inverse images of points of \mathcal{B} are supposed to form a timelike congruence of M . Thus there is, given f , a timelike vector u^μ , unique up to scale, so that

$$u^\mu \partial_\mu f^A = 0, \tag{2.1}$$

Given a time orientation for (M, g) , u^μ is fixed by requiring u^μ to be future–

pointing and

$$g_{\mu\nu}u^\mu u^\nu = -1 . \quad (2.2)$$

The dynamics will be given by an action principle of the forme

$$S[f] = \int_M \rho(f, \partial f; g) \sqrt{-\det(g)} d^4x \quad (2.3)$$

with associated Euler – Lagrange equations

$$-\mathcal{E}_A := \frac{1}{\sqrt{-\det(g)}} \partial_\mu \left(\sqrt{-\det(g)} \frac{\partial \rho}{\partial (\partial_\mu f^A)} \right) - \frac{\partial \rho}{\partial f^A} = 0 . \quad (2.4)$$

The Lagrangian ρ is thus a function on the "deformation bundle" \mathcal{D} with base $M \times \mathcal{B}$, coordinatized by $(x^\mu, X^A, F^B{}_\nu)$ where $F^A{}_\mu$ is of rank three with null space timelike w.r. to g . The geometrical interpretation of $F^A{}_\mu$ at (x^μ, X^A) is that of a linear map from $T_x(M)$ to $T_X(\mathcal{B})$. While we have no desire to enter formal excesses, we do from now on use notation which emphasizes the difference between a general fibre point $(X^A, F^B{}_\mu)$ and a pair of values $(f^A(x), \partial_\mu f^B(x))$ of $(X^A, F^B{}_\nu)$ along a particular map f . The four velocity u^μ for example can be viewed as a function on the deformation bundle. A convenient way of expressing the rank-condition and explicitly writing down this function is as follows: Choose a volume form Ω_{ABC} on B and require

$$\omega_{\mu\nu\lambda} = F^A{}_\mu F^B{}_\nu F^C{}_\lambda \Omega_{ABC}(X) \quad (2.5)$$

to be non-zero and

$$\epsilon^{\mu\nu\lambda\rho} \omega_{\nu\lambda\rho} \quad (2.6)$$

to be future-pointing timelike, where $\epsilon^{\mu\nu\lambda\rho}$ is the volume form associated with the spacetime metric. This means that the linear maps $F^A{}_\mu$ preserve orientation. Given this orientation, there is a positive function n on the deformation bundle so that

$$u^\mu = \frac{1}{3!n} \epsilon^{\mu\nu\lambda\rho} \omega_{\nu\lambda\rho} \quad (2.7)$$

is normalized.

Along a map f , nu^μ is a vector field on M satisfying the continuity equation

$$\nabla_\mu (nu^\mu) = 0 \quad (2.8)$$

identically. The physical meaning of n is that of a density of number of particles in the state given by f .

We now come back to the Lagrangian ρ which is a function on the deformation bundle with x -dependence only via the spacetime metric g , i.e. $\rho = \rho(X, F; g)$. As a further ingredient we assume covariance of the Lagrangian under spacetime diffeomorphisms, i.e. that, along a map f , ρ behaves as a scalar under transformations of the coordinates x^μ . This is equivalent to the following condition on ρ :

$$\mathcal{L}_\xi \rho = \frac{\partial \rho}{\partial X^A} (\mathcal{L}_\xi f)^A + \frac{\partial \rho}{\partial F^A_\mu} \mathcal{L}_\xi (\partial_\mu f^A) + \frac{\partial \rho}{\partial g^{\mu\nu}} (\mathcal{L}_\xi g)^{\mu\nu} \quad (2.9)$$

for arbitrary vector fields $\xi^\mu(x)$. Diffeomorphism invariance has several important consequences. The first is that the energy momentum tensor $T_{\mu\nu}$ defined by

$$T_{\mu\nu} = 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho g_{\mu\nu} \quad (2.10)$$

obeys the identity

$$-\nabla_\nu T_\mu{}^\nu = \mathcal{E}_A F^A{}_\mu, \quad (2.11)$$

independently of the field equations. (Notice that both the left-hand side of (2.11) and \mathcal{E}_A should be viewed as functions on the "second jet space" with elements given by $(x^\mu, X^A, F^A{}_\nu, F^B{}_{\lambda\rho} = F^B{}_{(\lambda\rho)})$). From Equ. (2.11) and the regularity condition on $F^A{}_\mu$ it follows that the Euler-Lagrange equations are equivalent to the energy-momentum tensor being divergence-free.

Secondly, writing out the Lie-derivatives in (2.9) in terms of partial derivatives, we find the equivalent relation

$$F^A{}_\mu \frac{\partial \rho}{\partial F^A{}_\nu} = 2g^{\nu\lambda} \frac{\partial \rho}{\partial g^{\mu\lambda}}. \quad (2.12)$$

Equ.(2.12), in turn, has the corollaries

$$\frac{\partial \rho}{\partial g^{\mu\nu}} u^\nu = 0 \quad (2.13)$$

and

$$F^A{}_{[\mu} g_{\nu]\lambda} \frac{\partial \rho}{\partial F^A{}_\lambda} = 0, \quad (2.14)$$

Equ.(2.13) implies that

$$T_{\mu\nu} u^\nu = -\rho u_\mu. \quad (2.15)$$

Thus the energy momentum tensor has the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + S_{\mu\nu} , \quad (2.16)$$

where the stress tensor satisfies $S_{\mu\nu} u^\nu = 0$. It follows that there exists $\tau_{AB} = \tau_{(AB)}$ on the deformation bundle so that

$$T_{\mu\nu} = \rho u_\mu u_\nu + n \tau_{AB} F^A{}_\mu F^B{}_\nu . \quad (2.17)$$

(The factor n in front of τ_{AB} is inserted for later convenience.) In other words, in the rest frame of matter, $T_{\mu\nu}$ has no mixed space–time components and the Lagrangian ρ is the energy density of the matter. The quantity τ_{AB} is called "second Piola–Kirchhoff" stress tensor in the elasticity literature.

As a special case consider a Lagrangian which just depends on n , as defined above. We find that

$$\frac{\partial n}{\partial g^{\mu\nu}} = \frac{n}{2} h_{\mu\nu} \quad (2.18)$$

with

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu . \quad (2.19)$$

Thus

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \quad (2.20)$$

with

$$p = n \frac{\partial \rho}{\partial n} - \rho . \quad (2.21)$$

Since, by (2.11), the field equations on f are equivalent to $\nabla_\nu T_\mu{}^\nu = 0$, we have recovered the standard formulation of perfect fluids: one simply considers $\nabla_\nu T_\mu{}^\nu = 0$ as the equations of motion for u^μ as a vector field on M , supplements them by the continuity equation (2.8) and completely forgets about $f : M \rightarrow \mathcal{B}$.

3 The symmetric system

The equations $\mathcal{E}_A = 0$ can be written as (we suppress the $g_{\mu\nu}$ -dependence)

$$M^{\mu\nu}{}_{AB}(f, \partial f) \partial_\mu \partial_\nu f^B(x) = G_A(f, \partial f) , \quad (3.1)$$

where

$$M^{\mu\nu}{}_{AB} := \frac{\partial^2 \rho}{\partial F^B{}_\nu \partial F^A{}_\mu} = M^{\nu\mu}{}_{BA} \quad (3.2)$$

The quantities $M^{\mu\nu}{}_{AB}$ can be viewed as a quadratic form on the space of matrices $m^A{}_\mu$ at each point of the deformation bundle. Alternatively they can be viewed as a map sending matrices $m^A{}_\mu$ to matrices $n_B{}^\nu$. This map, the "Legendre map", is in what follows required to be non-degenerate, i.e. $M^{\mu\nu}{}_{AB} m^B{}_\nu = 0$ implies $m^A{}_\mu = 0$. Another restriction on $M^{\mu\nu}{}_{AB}$ follows from Equ.(2.14). Namely, differentiating (2.14) w.r. to $F^A{}_\mu$ and using the rank-condition one easily finds that

$$u_\mu u^{[\nu} M^{\lambda]\mu}{}_{AB} = 0 . \quad (3.3)$$

We now have the properties of $M^{\mu\nu}{}_{AB}$ necessary for rewriting (3.1) as an equivalent first-order system. Let us define

$$\begin{aligned} W^{\mu\nu}{}_{AB}{}^{(\lambda)} &:= u^\mu M^{\lambda\nu}{}_{AB} - 2u^{[\lambda} M^{\nu]\mu}{}_{BA} \\ &= u^\mu M^{\lambda\nu}{}_{AB} + u^\nu M^{\lambda\mu}{}_{BA} - u^\lambda M^{\nu\mu}{}_{BA} . \end{aligned} \quad (3.4)$$

By (3.2)

$$W^{\mu\nu}{}_{AB}{}^{(\lambda)} = W^{\nu\mu}{}_{BA}{}^{(\lambda)} \quad (3.5)$$

Furthermore, since the last two terms in (3.4) are antisymmetric in λ and ν , from (3.1) there follows that

$$W^{\mu\nu}{}_{AB}{}^{(\lambda)}(f, \partial f) \partial_\lambda \partial_\nu f^B = u^\mu(f, \partial f) G_A(f, \partial f) \quad (3.6)$$

We now replace (3.1) by the following first-order system:

$$W^{\mu\nu}{}_{AB}{}^{(\lambda)}(X, F) \partial_\lambda F^B{}_\nu = u^\mu(X, F) G_A(X, F) \quad (3.7)$$

$$- u^\lambda(X, F) \partial_\lambda X^A = 0 . \quad (3.8)$$

We can write (3.7,3.8) in matrix form as

$$\begin{pmatrix} W^{\mu\nu}{}_{AB}{}^{(\lambda)} & 0 \\ 0 & -\delta^A{}_B u^\lambda \end{pmatrix} \partial_\lambda \begin{pmatrix} F^B{}_\nu \\ X^B \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \quad (3.9)$$

and see that we have a symmetric system because of (3.5) ².

²The quantity $W^{\mu\nu}{}_{AB}{}^{(\lambda)}$ also appears in [3], but is used there for a different purpose.

We have just seen that, taking a solutions $X^A = f^A(x)$ of (3.1) and setting $F^A{}_\mu = \partial_\mu f^A(x)$ we obtain a solution of (3.7,3.8). The converse is given by the following

Theorem 1: Suppose we have a solution pair $(X^A(x), F^A{}_\mu(x))$ of Equ.'s (3.7,3.8) and a hypersurface Σ , on which

(i) $\partial_i X^A$ has rank three ³

(ii) $(\partial_i X^A - F^A{}_i)|_\Sigma = 0$

Then $f^A = X^A(x)$ satisfies $\partial_\mu f^A = F^A{}_\mu$ and Equ.'s (3.1).

Proof: Contraction of (3.7) with u_μ and (3.3) imply

$$M^{\lambda\nu}{}_{AB} \partial_\lambda F^B{}_\nu = G_A \quad (3.10)$$

It remains to show that $F^A{}_\mu = \partial_\mu f^A$. Inserting (3.10) back into (3.6) gives

$$u^\lambda M^{\nu\mu}{}_{BA} \partial_{[\lambda} F^B{}_{\nu]} = 0 . \quad (3.11)$$

Using the regularity of the Legendre map this leads to

$$u^\lambda \partial_{[\lambda} F^B{}_{\nu]} = 0 . \quad (3.12)$$

We now write $X = (X^A)$ as a triple of 0-forms and $F = F^A{}_\mu dx^\mu$ as a triple of 1-forms. Consider the expression $F - dX$. From the definition of u^μ and (3.8) it follows that

$$i_u(F - dX) = 0 \quad (3.13)$$

Thus

$$\mathcal{L}_u(F - dX) = i_u d(F - dX) = i_u dF = 0 , \quad (3.14)$$

where we use (3.12) in the last equality. The condition (ii) states that the pull-back of $F - dX$ to Σ is zero. Thus, since u^μ is transversal to Σ , $F - dX$ vanishes identically, and the proof is complete.

Before commenting on the relation between the first and second-order systems we introduce a further piece of notation. The rank-condition on $F^A{}_\mu$ means that it has an inverse on the space orthogonal to u^μ . Thus there exists $F_A{}^\mu$ such that

$$F_A{}^\mu F^B{}_\mu = \delta_A^B , \quad F^A{}_\mu F_A{}^\nu = h_\mu{}^\nu , \quad (3.15)$$

where $h_\mu{}^\nu$ is defined by (2.19).

³Condition (i) is the same as saying that $u^\mu(X, F; g)$ is transversal to Σ

Defining

$$H^{AB} = F^A{}_{\mu} F^B{}_{\nu} g^{\mu\nu} . \quad (3.16)$$

we can write F_A^μ down explicitly as

$$F_A^\mu = H_{AB} F^{B\mu} = H_{AB} g^{\mu\nu} F^B{}_{\nu} , \quad (3.17)$$

where H_{AB} is defined as the inverse of H^{AB} , i.e. $H_{AB} H^{BC} = \delta_A^C$.

With these definitions the relation (3.3) implies the existence of quantities $\mu_{AB} = \mu_{(AB)}$ and $U_{CADB} = U_{DBCA}$ such that

$$M^{\mu\nu}{}_{AB} = -\mu_{AB} u^\mu u^\nu + U_{ACBD} F^{C\mu} F^{D\nu} \quad (3.18)$$

Thus the regularity conditions is equivalent to both μ_{AB} and U_{CADB} being regular in that $\mu_{AB} \alpha^B = 0 \implies \alpha^A = 0$ and $U_{ABCD} \alpha^{CD} = 0 \implies \alpha^{AB} = 0$.

Next, using (3.3) and (3.4), it follows that

$$W^{\mu\nu}{}_{AB}{}^{(\lambda)} u_\lambda = \mu_{AB} u^\mu u^\nu + U_{ACBD} F^{C\mu} F^{D\nu} . \quad (3.19)$$

Note that, compared to $M^{\mu\nu}{}_{AB}$, only the sign of the first term has changed. The regularity of $M^{\mu\nu}{}_{AB}$ is thus equivalent to the requirement that u_μ be a non-characteristic covector for the first-order system (3.7,3.8).

Now recall the notion of characteristic covectors for the second-order system (3.1). Namely k_μ is characteristic iff

$$\Delta_{AB} = M^{\mu\nu}{}_{AB} k_\mu k_\nu \quad (3.20)$$

is singular, i.e. $\Delta = \det(\Delta_{AB}) = 0$. If k_μ is characteristic in this sense and α^A an associated eigenvector, i.e. $\Delta_{AB} \alpha^B = 0$, then $W^{\mu\nu}{}_{AB}{}^{(\lambda)} k_\lambda k_\nu \alpha^B = 0$ i.e. k_μ is also characteristic for the system (3.7,3.8).

The first-order system (3.7,3.8) is called symmetric hyperbolic if, in addition to the symmetry of the coefficients, there exists a "subcharacteristic" covector (hypersurface element), that-is-to-say a covector k_λ for which the matrix

$$\begin{pmatrix} W^{\mu\nu}{}_{AB}{}^{(\lambda)} k_\lambda & 0 \\ 0 & -H_{AB} u^\lambda k_\lambda \end{pmatrix} \quad (3.21)$$

is positive definite in the variables $(m^A{}_\mu, l^A)$. In fact, using the variables $m^A{}_\mu u^\mu = -\alpha^A$, $m^A{}_\mu F^{B\mu} = \alpha^{AB}$, l^A , the quadratic form corresponding to (3.21), for $u_\lambda = k_\lambda$, is given by

$$\mu_{AB} \alpha^A \alpha^B + U_{ACBD} \alpha^{AC} \alpha^{BD} + H_{AB} l^A l^B . \quad (3.22)$$

⁴ H_{AB} describes the distance of particles of \mathcal{B} in spacetime.

Therefore, u_λ is subcharacteristic for (3.7,3.8) iff the first two forms in (3.22) are positive definite. Notice that these conditions are also sufficient for regularity. The validity of these conditions is studied in the next section.

4 Hyperbolicity

We first return to the covariance condition Equ.(2.12) and claim that it is equivalent to the requirement that $\rho(X, F; g)$ is a function just of (X, H) with H^{AB} given by (3.16). To prove this assertion one first observes that Equ.(2.13) implies

$$\rho(X^A, F^B{}_\mu; g_{\nu\lambda}) = \rho(X^A, F^B{}_\mu; g_{\nu\lambda} + s u_\nu u_\lambda) \quad (4.1)$$

for all real numbers s . Setting $s = 1$ in Equ.(4.1) and using that $h_{\mu\nu} = F^A{}_\mu F^B{}_\nu H_{AB}$ we infer that there is a function σ so that

$$\rho(X^A, F^B{}_\mu; g_{\nu\lambda}) = \sigma(X^A, F^B{}_\mu, H^{CD}) . \quad (4.2)$$

Using (4.2) again in Equ.(2.12) we see that σ has to be independent of $F^A{}_\mu$, thus proving our assertion.

We now write ρ as

$$\rho = n\epsilon , \quad (4.3)$$

where ϵ is a positive function with

$$\epsilon = \epsilon(X^A, H^{BC}) . \quad (4.4)$$

The quantity ϵ is the relativistic version of the “stored-energy function” of standard elasticity. It is possible to factorize ρ in this way since the number density n is uniquely given in terms of (X^A, H^{BC}) by virtue of

$$6n^2 = H^{AA'} H^{BB'} H^{CC'} \Omega_{ABC} \Omega_{A'B'C'} , \quad n > 0 . \quad (4.5)$$

Just as a pair $(X^A, F^B{}_\nu)$, along a map $f : M \rightarrow \mathcal{B}$, measures deformation, the pairs (X^A, H^{BC}) measure ”strain”. Thus triples (x, X, H) might be considered as forming the ”strain bundle” over $M \times \mathcal{B}$ in which the deformation bundle \mathcal{D} is embedded via Equ.(3.16). Using the definition of $F_A{}^\mu$ we easily find that

$$\frac{\partial F_A{}^\mu}{\partial F^B{}_\nu} = -F_A{}^\nu F_B{}^\mu - H_{AB} u^\mu u^\nu \quad (4.6)$$

and, from the definition of u^μ , that

$$\frac{\partial u^\mu}{\partial F^A{}_\nu} = -F_A{}^\mu u^\nu . \quad (4.7)$$

From (4.5) we deduce that

$$\frac{\partial n}{\partial F^A{}_\mu} = n F_A{}^\mu \quad (4.8)$$

which, using (4.6,4.7), leads to

$$\frac{\partial^2 n}{\partial F^B{}_\nu \partial F^A{}_\mu} = 2n F_A{}^{[\mu} F_B{}^{\nu]} - n H_{AB} u^\mu u^\nu . \quad (4.9)$$

The quantities μ_{AB} and U_{ACBD} entering the expression (3.18) for $M^{\mu\nu}{}_{AB}$ using (4.8, 4.9) can now be written as

$$\mu_{AB} = n(\epsilon H_{AB} + \tau_{AB}) \quad (4.10)$$

$$U_{ACBD} = n(\tau_{AB} H_{CD} + \tau_{AC} H_{BD} + \tau_{BD} H_{AC} + 2 \frac{\partial \tau_{BD}}{\partial H_{AC}} + 2\epsilon H_{A[C} H_{D]B}) , \quad (4.11)$$

where the second Piola–Kirchhoff stress tensor τ_{AB} entering (2.17) is given by

$$\tau_{AB} = 2 \frac{\partial \epsilon}{\partial H_{AB}} . \quad (4.12)$$

Note that μ_{AB} and U_{ACBD} are functions solely on the strain bundle. Furthermore the dependence on the volume form Ω_{ABC} is only via the multiplicative factor n . It will follow that our results in this paper do not depend on the choice of Ω_{ABC} .

Of particular interest are points $(\overset{\circ}{x}, \overset{\circ}{X}, \overset{\circ}{H})$ of the strain bundle which are stress-free i.e. for which τ_{AB} vanishes. Then the regularity condition on μ_{AB} is clearly satisfied since $\epsilon > 0$, but that on U_{ACBD} becomes delicate: Since the fourth term in (4.11), in addition to $U_{ACBD} = U_{CADB}$, is symmetric in both (BD) and (AC) , this term annihilates elements α^{AB} with $\alpha^{AB} = \alpha^{(AB)}$ which the last term simply multiplies by a positive constant. On the other hand the second term annihilates elements α^{AB} with $\alpha^{AB} = \alpha^{[AB]}$. Thus regularity requires a balance between these two terms, which is not necessarily satisfied in the cases one wants to consider. But there is a possible cure: One can try to add further terms proportional to $H_{C[A} H_{B]D}$ to U_{CABD} so that

regularity is satisfied. By that same token one might hope to also satisfy part 2 of the hyperbolicity requirement, namely the existence of a timelike covector k_μ . We would, by this manouvre, change $M^{\mu\nu}{}_{AB}$ into $\bar{M}^{\mu\nu}{}_{AB}$ and accordingly $W^{\mu\nu}{}_{AB}^{(\lambda)}$ into $\bar{W}^{\mu\nu}{}_{AB}^{(\lambda)}$. The second-order system, however, would be unchanged. We would thus be able to use $\bar{W}^{\mu\nu}{}_{AB}^{(\lambda)}$ instead of $W^{\mu\nu}{}_{AB}^{(\lambda)}$ in Theorem 1. Consequently the statements at the end of Section 3 can be generalized by saying: The system (3.7,3.8) with $\bar{W}^{\mu\nu}{}_{AB}^{(\lambda)}$ is symmetric hyperbolic if there exist quantities Λ_{ACBD} on the strain bundle with $\Lambda_{ACBD} = \Lambda_{CADB}$, but also $\Lambda_{ACBD} = -\Lambda_{ADBC}$, in such a way that

$$\bar{U}_{ACBD} = U_{ACBD} + \Lambda_{ACBD} \quad (4.13)$$

is positive definite. A necessary condition for this to be the case is that U_{ACBD} is positive definite on rank-one elements, i.e. on elements α^{AC} of the form

$$\alpha^{AC} = \lambda^A \mu^C, \quad (4.14)$$

since that property remains unchanged under (4.13). Thus the question is whether rank-one positivity, called Legendre–Hadamard condition in the time independent theory, is sufficient for the existence of a suitable Λ_{ABCD} . The answer, in three or more space dimensions, is in general "no" (see Ball [1] and references therein). Luckily, in the situation we shall presently consider, the answer is affirmative ⁵.

We firstly imagine there to be given a positive definite metric $G^{AB}(X)$ (usually taken flat in standard elasticity) on \mathcal{B} playing the role of "zero strain". Points $(\overset{\circ}{x}, \overset{\circ}{X}, \overset{\circ}{F})$ on \mathcal{D} will be called "natural" or strain-free when $\overset{\circ}{F}{}^A{}_\mu \overset{\circ}{F}{}^B{}_\nu g^{\mu\nu}(\overset{\circ}{x}) = G^{AB}(\overset{\circ}{X})$ ⁶. Next suppose that the stored energy function ϵ depends on X only via G^{AB} and is covariant under diffeomorphisms of \mathcal{B} . It is not difficult to see that this implies that ϵ only depends on the principal invariants of the linear map \mathcal{H} with components given by $(\mathcal{H})^A{}_B = H^{AC}G_{CB}(X)$. In that case it is easy to see that $M^{\mu\nu}{}_{AB}$ transforms tensorially both under transformations of spacetime and body coordinates, and in that case the full field equations can be written $M^{\mu\nu}{}_{AB} \bar{\nabla}_\mu \nabla_\nu f^B = 0$,

⁵Christodoulou claims that his new hyperbolic theory just requires rank-one positivity (see [3]). The standard theorems for nonrelativistic elasticity in the second-order formulation also only require rank-one positivity, see [8].

⁶We note that there is for general spacetimes no cross section $X^A = f^A(x)$, $F^A{}_\mu(x) = (\partial_\mu f^A)(x)$ consisting of natural points. The existence of such a map $f : M \rightarrow \mathcal{B}$ requires the flow of the associated vector field u^μ on M to be Born-rigid (see [17]).

where $\bar{\nabla}_\mu$ is the appropriate "double-covariant" derivative. (We could, without altering our results, add some thermodynamics by allowing ϵ to depend on an additional scalar function on \mathcal{B} , the entropy density.)

Finally we assume that the stress τ_{AB} vanishes at natural points. We now expand ϵ at natural points or, equivalently, at $\mathcal{H} = \mathcal{I}$, where \mathcal{I} is the identity map. It follows that there are constants m, q, r such that

$$\epsilon = m + \frac{m}{8}[q \operatorname{tr}(\mathcal{H} - \mathcal{I})^2 + r(\operatorname{tr}(\mathcal{H} - \mathcal{I}))^2] + O((\mathcal{H} - \mathcal{I})^3) \quad (4.15)$$

where $m > 0$ is the rest mass per particle. (In the prestressed case there would, in Equ. (4.15), appear a linear term of the form $p(2\mathring{n})^{-1}(\operatorname{tr}(\mathcal{H} - \mathcal{I}))$. If the background pressure p is positive our results below continue to hold but the expressions for the phase velocities change.)

The connection between q, r and the Lamé "constants" is $\lambda = m \mathring{n} q$ and $\mu = m \mathring{n} r$.⁷

From (4.15) and (4.11) we deduce

$$\mathring{U}_{ACBD} = m \mathring{n} [q G_{AC} G_{BD} + 2r G_{D(C} G_{A)B} + 2G_{A[C} G_{D]B}] \quad (4.16)$$

One easily finds that \mathring{U} is rank-one positive iff $r > 0$ and $2r + q > 0$. We set

$$q = c_1^2 - 2c_2^2, \quad r = c_2^2. \quad (4.17)$$

Inserting (4.16) and $\mathring{\tau}_{AB} = 0$ into $M^{\mu\nu}{}_{AB}$ we find that $\Delta = \det(\Delta_{AB}) = M^{\mu\nu}{}_{AB} k_\mu k_\nu$, in a frame adapted to \mathring{u}^μ , is given by

$$(m \mathring{n})^3 (\mathring{g}^{\mu\nu} k_\mu k_\nu) (\mathring{g}^{\mu\nu} k_\mu k_\nu)^2 \quad (4.18)$$

where

$$\mathring{g}^{\mu\nu} = g^{\mu\nu} + \left(1 - \frac{1}{c_1^2}\right) \mathring{u}^\mu \mathring{u}^\nu \quad (4.19)$$

and

$$\mathring{g}^{\mu\nu} = g^{\mu\nu} + \left(1 - \frac{1}{c_2^2}\right) \mathring{u}^\mu \mathring{u}^\nu. \quad (4.20)$$

⁷Since in our formulation the volume element Ω is independent of the metric G^{AB} , there is in general no need for \mathring{n} to be constant. Thus the Lamé coefficients could depend on \mathring{X} , but not their quotient. The phase velocities of sound are constant, as we see below.

Thus k_μ is characteristic for the second order system at natural points if either k_μ is null w.r. to $\overset{1}{g}{}^{\mu\nu}$ or k_μ is null w.r. to $\overset{2}{g}{}^{\mu\nu}$. For the associated vectors α^A with $\Delta_{AB}\alpha^B = 0$ we get that $\overset{1}{\alpha}{}^A$ is proportional to $\overset{\circ}{F}{}^A{}_\mu k^\mu$ and $\overset{2}{\alpha}{}^A$ is determined by $\overset{1}{\alpha}{}^A \overset{2}{\alpha}{}^B G_{AB} = 0$. Thus we have a longitudinal mode propagating at phase velocity c_1 and two transversal modes propagating at phase velocity c_2 .

We now come to the question of symmetric hyperbolicity of (3.7,3.8), possibly after modifying $\overset{\circ}{U}{}_{ACBD}$ by changing the factor of 2 in front of $G_{A[C}G_{D]B}$ in (4.16) into a generic constant σ . Setting $\sigma = 4c_2^2 - 2\delta$ we find the following

Theorem 2: Let $0 < \delta < \frac{3c_1^2}{2}$ and $0 < \delta < 2c_2^2$. Then the system (3.7,3.8) is symmetric hyperbolic at natural points.

Proof: We show that $k_\mu = \overset{\circ}{u}_\mu$ is timelike. Keeping in mind (3.22), this is equivalent to the positivity of $S(\alpha, \alpha)$ given by

$$\begin{aligned} S(\alpha, \alpha) &= [(c_1^2 - 2c_2^2)G_{AC}G_{BD} + 2c_2^2G_{D(C}G_{A)B}] \\ &\quad + (4c_2^2 - 2\delta)G_{A[C}G_{D]B}] \alpha^{AC} \alpha^{BD} > 0 \end{aligned} \quad (4.21)$$

Decomposing

$$\alpha^{AB} = \omega^{AB} + \kappa^{AB} + \frac{\kappa}{3}G^{AB} \quad (4.22)$$

where $\omega^{AB} = \omega^{[AB]}$, $\kappa^{AB} = \kappa^{(AB)}$ and $\kappa^{AB}G_{AB} = 0$, we see that

$$S(\alpha, \alpha) = (c_1^2 - \frac{2\delta}{3})\kappa^2 + \delta\kappa_{AB}\kappa^{AB} + (2c_2^2 - \delta)\delta\omega_{AB}\omega^{AB}, \quad (4.23)$$

from which our assertion follows.

It follows from the above theorem that the system (3.7,3.8) has a well-posed initial value problem. But the allowable Cauchy data are quite restricted. Namely they have to be near ones for which the matter is initially static. The general allowable Cauchy data can be inferred from the following

Theorem 3: Suppose the equation of state $\epsilon(H^{AB})$ is of the form (4.15) with $0 < \frac{4}{3}c_2^2 < c_1^2$. Then the system (3.7,3.8) with $0 < \delta < 2c_2^2$ is symmetric hyperbolic at the natural points on the deformation bundle $(\overset{\circ}{x}, \overset{\circ}{X}, \overset{\circ}{F})$ i.e. ones where $\overset{\circ}{F}{}^A{}_\mu \overset{\circ}{F}{}^B{}_\nu g^{\mu\nu}(\overset{\circ}{x}) = G^{AB}(\overset{\circ}{X})$. Covectors k_μ at $\overset{\circ}{x}$ are timelike iff

$$\overset{1}{g}{}^{\mu\nu}k_\mu k_\nu < 0 \quad \text{and} \quad \overset{\circ}{u}{}^\mu k_\mu < 0 \quad (4.24)$$

where $\overset{1}{g}{}^{\mu\nu}$ is given by (4.19). We remark that the additional hypothesis, i.e. that $\frac{4}{3}c_2^2 < c_1^2$, is not necessary for hyperbolicity but convenient for a complete characterization of timelike covectors. This condition is however physically entirely reasonable since it is equivalent to that the 'bulk modulus' given by $k = \frac{3\lambda+2\mu}{3}$ be positive.

Proof: Recall the definition (3.4) of $W^{\mu\nu}{}_{AB}{}^{(\lambda)}$ in terms of $M^{\mu\nu}{}_{AB}$. In the case at hand $\bar{M}^{\mu\nu}{}_{AB}$ at natural points is given by

$$\begin{aligned} \frac{1}{m \overset{\circ}{n}} \bar{M}^{\mu\nu}{}_{AB} &= -G_{AB} \overset{\circ}{u}{}^\mu \overset{\circ}{u}{}^\nu + [(c_1^2 - 2c_2^2)G_{AC}G_{BD} + 2c_2^2 G_{D(C}G_{A)B}] \\ &\quad + (4c_2^2 - 2\delta)G_{A[C}G_{D]B}] \overset{\circ}{F}{}^{C\mu} \overset{\circ}{F}{}^{D\nu} \end{aligned} \quad (4.25)$$

We have to study $\frac{1}{m \overset{\circ}{n}} \bar{W}(k) = \frac{1}{m \overset{\circ}{n}} \bar{W}^{\mu\nu}{}_{AB}{}^{(\lambda)} k_\lambda m^B{}_\nu m^A{}_\nu$. Decomposing

$$m^A{}_\mu = \alpha^A \overset{\circ}{u}{}_\mu + \overset{\circ}{F}{}^A{}_\mu \alpha^B{}_A \quad (4.26)$$

and

$$k_\mu = \omega \overset{\circ}{u}{}_\mu + \overset{\circ}{F}{}^A{}_\mu k_A \quad (4.27)$$

there results

$$\begin{aligned} \frac{1}{m \overset{\circ}{n}} \bar{W}(k, k) &= \omega \alpha_A \alpha^A + (c_1^2 - \delta) \omega \kappa^2 + 2(\delta - c_1^2) \kappa \alpha^A k_A + 2c_2^2 \omega \alpha_{AB} \alpha^{[AB]} \\ &\quad + \delta \omega \alpha_{AB} \alpha^{BA} + 4c_2^2 \alpha^A k^B \alpha_{[AB]} - 2\delta \alpha^A k^B \alpha_{AB} . \end{aligned} \quad (4.28)$$

Here indices are raised and lowered with G_{AB} . In Appendix A it is shown that \bar{W} is positive definite iff

$$\frac{\omega^2}{c_1^2} - k^A k_A > 0 , \quad (4.29)$$

from which the Theorem follows.

We now take up the discussion of characteristic covectors of Sect.3. It is easy to see from (3.20) that the characteristic covectors, as defined there for the 2nd-order system, are the same for $M^{\mu\nu}{}_{AB}$ as for $\bar{M}^{\mu\nu}{}_{AB}$. It remains to find the characteristic covectors for the 1st-order system (3.7,3.8). Due to the block diagonal form of (3.7,3.8) the relevant determinant is given by $-(u^\mu k_\mu)^3 D$, where D is the (12×12) - determinant

$$D = \det\left(\frac{1}{m \overset{\circ}{n}} \bar{W}^{\mu\nu}{}_{AB}{}^{(\lambda)} k_\lambda\right) . \quad (4.30)$$

The determinant, in a frame adapted to \mathring{u}^μ , is ($a = c_1^2$, $b = c_2^2$):

$$D = \det \begin{pmatrix} \rho\omega, 0, 0, a k_1, (\delta - b) k_2, (\delta - b) k_3, b k_2, (a - \delta) k_1, 0, b k_3, 0, (a - \delta) k_1 \\ 0, \rho\omega, 0, (a - \delta) k_2, b k_1, 0, (\delta - b) k_1, a k_2, (\delta - b) k_3, 0, b k_3, (a - \delta) k_2 \\ 0, 0, \rho\omega, (a - \delta) k_3, 0, b k_1, 0, (a - \delta) k_3, b k_2, (\delta - b) k_1, (\delta - b) k_2, a k_3 \\ a k_1, (a - \delta) k_2, (a - \delta) k_3, a\omega, 0, 0, 0, a\omega - \delta\omega, 0, 0, 0, a\omega - \delta\omega \\ (\delta - b) k_2, b k_1, 0, 0, b\omega, 0, -b\omega + \delta\omega, 0, 0, 0, 0, 0 \\ (\delta - b) k_3, 0, b k_1, 0, 0, b\omega, 0, 0, 0, -b\omega + \delta\omega, 0, 0 \\ b k_2, (\delta - b) k_1, 0, 0, -b\omega + \delta\omega, 0, b\omega, 0, 0, 0, 0, 0 \\ (a - \delta) k_1, a k_2, (a - \delta) k_3, a\omega - \delta\omega, 0, 0, 0, a\omega, 0, 0, 0, a\omega - \delta\omega \\ 0, (\delta - b) k_3, b k_2, 0, 0, 0, 0, 0, b\omega, 0, -b\omega + \delta\omega, 0 \\ b k_3, 0, (\delta - b) k_1, 0, 0, -b\omega + \delta\omega, 0, 0, 0, b\omega, 0, 0 \\ 0, b k_3, (\delta - b) k_2, 0, 0, 0, 0, 0, -b\omega + \delta\omega, 0, b\omega, 0 \\ (a - \delta) k_1, (a - \delta) k_2, a k_3, a\omega - \delta\omega, 0, 0, 0, a\omega - \delta\omega, 0, 0, 0, a\omega \end{pmatrix}$$

which gives (using Maple)

$$D = -\omega^6 \delta^5 (2b - \delta)^3 (-2\delta + 3a) (-\omega^2 \rho + b k_1^2 + b k_2^2 + b k_3^2)^2 (-\omega^2 \rho + a k_1^2 + a k_2^2 + a k_3^2)$$

or

$$D = -\delta^5 (u^\mu k_\mu)^6 (2c_2^2 - \delta)^3 (3c_1^2 - 2\delta) (\mathring{g}^{\mu\nu} k_\mu k_\nu) (\mathring{g}^{\mu\nu} k_\mu k_\nu)^2. \quad (4.31)$$

Note that Δ appears as a factor in D , as it has to be.

Finally we have to find Cauchy data. Such are given by a smooth hypersurface $\Sigma \in M$ and on it data $X(x), F(x)$ satisfying conditions (i,ii) of Theorem 1 so that the conormal n_μ of Σ is everywhere timelike in the sense of the symmetric hyperbolic system (3.9). The easiest way to achieve this is as follows: Pick an arbitrary $\Sigma \subset M$ and \mathring{y} a point on Σ . Choose $(\mathring{X}, \mathring{F})$ so that $G^{AB}(\mathring{X}) = \mathring{F}^A{}_\mu \mathring{F}^B{}_\nu g^{\mu\nu}(\mathring{y})$ and, in addition, so that the conormal \mathring{n}_μ of Σ at \mathring{y} is timelike w.r. to $\mathring{g}^{\mu\nu} = g^{\mu\nu} + (1 - \frac{1}{c_1^2}) \mathring{u}^\mu \mathring{u}^\mu$, for \mathring{u}^μ given in terms of $\mathring{F}^A{}_\mu$ by $\mathring{u}^\mu = \frac{1}{3!n} \epsilon^{\mu\nu\lambda\rho} \mathring{F}^A{}_\nu \mathring{F}^B{}_\lambda \mathring{F}^C{}_\rho \mathring{\Omega}_{ABC}$.

Note that this condition is satisfied automatically when $c_1^2 \leq 1$ and Σ is spacelike w.r. to $g_{\mu\nu}$ and can not be satisfied when $c_1^2 > 1$ and Σ is timelike w.r. to $g_{\mu\nu}$. In the other cases it can be satisfied by a suitable choice of $\mathring{F}^A{}_\nu$.

We now choose a function $\bar{f} : \Sigma \rightarrow B$ so that $\bar{f}^A|_{\mathring{y}} = \mathring{X}^A$ and $\partial_i \bar{f}^A|_{\mathring{y}} = \mathring{F}^A{}_i$. Choose $\bar{F}^A{}_\mu$ on Σ so that $\bar{F}^A{}_\mu = \mathring{F}^A{}_\mu$ at \mathring{y} and $\bar{F}^A{}_i = \partial_i \bar{f}^A$ everywhere. Then,

in a sufficiently small neighbourhood O of $\overset{\circ}{y}$, the field equations (3.1) with initial data $f^A = \bar{f}^A$ and $\partial_\mu f^A = \bar{F}^A_\mu$ on Σ have a unique local solution.

5 Discussion

We want to discuss certain physical situations in which our results show existence and uniqueness of solutions of the elasticity equations. Let us stress that in the present paper we are concerned only with local questions. In our formulation of elasticity theory we have as yet no way to solve the boundary initial value problem corresponding to the motion of a finite elastic body, i.e. where the normal component of the stress tensor at the (free!) boundary of the body is required to be zero. Hence we have to consider just parts of the body or infinitely extended bodies.

First we consider elasticity in Special Relativity. Hence our spacetime metric is Minkowski space, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. Furthermore we assume that the coordinates are inertial, then $t = \text{const}$ is the natural Cauchy surface. We assume the body metric to be flat, i.e. $G^{AB} = \delta^{AB}$. The map $\overset{\circ}{f}: (t, x^1, x^2, x^3) \mapsto (X^1 = x^1, X^2 = x^2, X^3 = x^3)$ is a solution for the elasticity equation if the Lagrangian is given by (4.15). The interpretation of this solution is that of a relaxed body at rest at all times. The obvious physical question is to deform the body slightly at $t = 0$ and ask for solutions. Any map f sufficiently near to $\overset{\circ}{f}$ at $t=0$ defines data f^A, F^A_μ at $t = 0$ satisfying the positivity requirement via Theorem 3 in Section 4. We thus obtain existence locally in time for pieces of the body or for an infinitely extended body.

Consider next a general spacetime to be given and in it a spacelike hypersurface Σ . If the geometry on and near Σ is "close" to that of Minkowski space and the data close to the ones in the previous paragraph, we obtain again a solution local in time.

In an arbitrary spacetime we can, following the procedure at the end of Sect 4, construct solutions locally both in space and time.

Consider as a further example a spacetime of the form

$$ds^2 = -dt^2 + g_{ik}(t, x^j)dx^i dx^k \quad (5.1)$$

Let ψ be a diffeomorphism from $t = 0$ onto the body and define the body metric G^{AB} such that ψ becomes an isometry. Then the system becomes

symmetric hyperbolic for the data $f^A = \psi^A, \partial_\mu f^A = 0$ at $t = 0$, or data near-by. We obtain solutions globally in space, locally in time.

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A Appendix A

We start with the expression (4.25) and decompose α^{AB} in the form

$$\alpha^{AB} = \omega^{AB} + \kappa^{AB} + \frac{\kappa}{3}G^{AB} \quad (\text{A.1})$$

where $\omega^{AB} = \omega^{[AB]}$, $\kappa^{AB} = \kappa^{(AB)}$ and $\kappa^{AB}G_{AB} = 0$. There results

$$\begin{aligned} \frac{1}{m \dot{n}} \bar{W}(k)(m, m) &= \omega \alpha_A \alpha^A + 2(\delta - c_1^2) \alpha^A k_A \kappa \\ &+ 2\omega c_2^2 \omega_{AB} \omega^{AB} + 4c_2^2 \omega_{AB} \alpha^A k^B \\ &- 2\delta(\omega_{AB} + \kappa_{AB} + \frac{\kappa}{3}h_{AB}) \alpha^A k^B \\ &+ (c_1^2 - \delta) \omega \kappa^2 + \delta \kappa (-\omega_{AB} \omega^{AB} + \kappa_{AB} \kappa^{AB} + \frac{\kappa^2}{3}) \end{aligned} \quad (\text{A.2})$$

We now proceed as follows: we first eliminate all linear k_A - terms by substituting α_A with β_A given by

$$\beta_A = \alpha_A + \frac{1}{\omega} \left(\frac{2\delta}{3} - c_1^2 \right) \kappa k_A + \frac{1}{\omega} (2c_2^2 - \delta) \omega_{AB} k^B - \frac{\delta}{\omega} \kappa_{AB} k^B \quad (\text{A.3})$$

Next we eliminate terms linear in κ_{AB} by setting

$$\begin{aligned} \bar{\kappa}_{AB} &= A \kappa_{AB} - \frac{\delta \kappa}{A \omega} (c_1^2 - \frac{2\delta}{3}) k_A k_B \\ &+ \frac{\delta \kappa k_C k^C}{A \omega} (c_1^2 - \frac{2\delta}{3}) \frac{1}{3} h_{AB} \\ &+ \frac{\delta}{A \omega} (2c_2^2 - \delta) k_{(A} \omega_{B)C} k^C \end{aligned} \quad (\text{A.4})$$

for some constant $A \neq 0$. Thirdly, note the identity

$$2\bar{\kappa}_{ABC} \bar{\kappa}^{ABC} = \kappa_{AB} \kappa^{AB} k_C k^C - \frac{3}{2} \kappa_{AB} k^B \kappa^A_C k^C \quad (\text{A.5})$$

where $\bar{\kappa}_{ABC}$ is the trace-free part of $\kappa_{A[B}k_{C]}$, i.e.

$$\bar{\kappa}_{ABC} := \kappa_{A[B}k_{C]} + \frac{1}{2}h_{A[B}\kappa_{C]D}k^D \quad (\text{A.6})$$

We use (A.6) to eliminate $\kappa_{AB}k^B\kappa^A_Ck^C$ in favour of the other two quantities in (A.5). We do the analogous thing for ω_{AB} , using

$$3\omega_{[AB}k_{C]}\omega^{[AB}k^{C]} = \omega_{AB}\omega^{AB}k_Ck^C - \omega^A_Bk^B\omega_{AC}k^C \quad (\text{A.7})$$

The last two operations leave us with just having to worry about the signs of the terms proportional to κ^2 , $\kappa_{AB}\kappa^{AB}$ and $\omega_{AB}\omega^{AB}$. We have to choose A such that all the signs are positive and, for the "only if" -direction, have to make an optimal choice in terms of the allowed range for k_Ak^A . Our choice is $A = \frac{\delta\omega}{c_1^2}(c_1^2 - \frac{2\delta}{3})$. We finally obtain

$$\begin{aligned} \frac{1}{m \overset{\circ}{n}} \bar{W}(k)(m, m) &= \omega\beta_A\beta^A + \delta\omega\left(1 - \frac{2\delta}{3c_1^2}\right)\bar{\kappa}_{AB}\bar{\kappa}^{AB} \quad (\text{A.8}) \\ &+ \frac{2\delta^2\omega}{3}\left(\frac{1}{c_1^3} - \frac{k_Ck^C}{\omega^2}\right)\kappa_{AB}\kappa^{AB} \\ &+ \frac{2\delta\omega c_1^2}{3}\left(c_1^2 - \frac{2\delta}{3}\right)\left(\frac{1}{c_1^2} - \frac{k_Ck^C}{\omega^2}\right)\left(\frac{3}{2\delta} + \frac{k_Dk^D}{\omega^2}\right)\kappa^2 \\ &+ \omega(2c_2^2 - \delta)D\left(\frac{k^Ak_A}{\omega^2}\right)\omega_{AB}\omega^{AB} \\ &+ \frac{3(2c_2^2 - \delta)}{2\omega}\left[1 + \frac{\delta c_1^2}{2\omega^2(c_1^2 - \frac{2\delta}{3})}\right]\omega_{[AB}k_{C]}\omega^{[AB}k^{C]} \\ &+ \frac{4\delta^2}{3\omega}\bar{\kappa}_{ABC}\bar{\kappa}^{ABC} \end{aligned}$$

where

$$D(\eta) = 1 - \frac{2c_2^2 - \delta}{2\omega^2}\eta - \frac{(2c_2^2 - \delta)\delta c_1^2}{4\omega^2(c_1^2 - \frac{2\delta}{3})}\eta^2 \quad (\text{A.9})$$

Observe that $D(0) > 0$ and

$$D\left(\frac{1}{c_2^2}\right) = \frac{\delta^2}{4c_2^2}\frac{c_1^2 - \frac{4}{3}c_2^2}{c_1^2 - \frac{2\delta}{3}} > 0 \quad (\text{A.10})$$

Consequently $D(\eta)$ is positive for all $\eta \in [0, \frac{1}{c_1^2}]$, since $\frac{1}{c_1^2} < \frac{1}{c_2^2}$ and $D(\eta)$ is monotonically decreasing for $\eta > 0$. Thus for $\frac{1}{c_1^2} - \frac{k_Ak^A}{\omega^2} > 0$, $\bar{W}(k)(m, m)$

is positive, except possibly when $\omega_{AB}, \bar{\kappa}_{AB}, \kappa$ and β_A are all zero. But then $\kappa_{AB} = 0 = \alpha_A$.

Suppose, conversely, that $\frac{1}{c_1^2} - \frac{k_A k^A}{\omega^2} \leq 0$. Then choose $\omega_{AB} = 0$, κ non-zero, κ_{AB} such that $\bar{\kappa}_{AB} = 0$ and α_A such that $\beta_A = 0$. Clearly κ_{AB} is non-zero of the form "trace-free part of $k_A k_B$ ", whence the last line in (A.8) is zero, as are the first and fourth line. But the second and third line are non-positive. Thus, (A.2) is positive definite iff $\frac{\omega^2}{c_1^2} - k_A k^A > 0$.

B Appendix B

Here we outline how nonrelativistic elasticity is recovered by taking the limit $c \rightarrow \infty$ in our equations when (M, g) is the Minkowski space ⁸. We have

$$g_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + \delta_{ik} dx^i dx^k \quad (\text{B.1})$$

$$g^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{c^2} (\partial_t)^2 + \delta^{ik} \partial_i \partial_k \quad (\text{B.2})$$

and

$$u^\mu \partial_\mu = \frac{1}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (\partial_t + v^i \partial_i) \quad (\text{B.3})$$

$$u_\mu dx^\mu = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (-cdt + \frac{v_i}{c} dx^i) \quad (\text{B.4})$$

where $v_i = \delta_{ik} v^k$ and $v^2 = v_i v^i$. The vector v^i is determined from $F^A{}_\mu dx^\mu = F^A dt + f^A{}_i dx^i$ by

$$F^A + F^A{}_i v^i = 0 \quad (\text{B.5})$$

The tensor $h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$ goes in the limit $c \rightarrow \infty$ to $\delta^\mu{}_\nu - v^\mu \tau_\nu$ where

$$v^\mu \partial_\mu = \partial_t + v^i \partial_i \quad (\text{B.6})$$

and

$$\tau_\mu dx^\mu = dt \quad (\text{B.7})$$

The quantity $F_A{}^\mu \partial_\mu$ tends to $F_A{}^i \partial_i$ and

$$F_A{}^\mu F^A{}_\nu = \delta^\mu{}_\nu - v^\mu \tau_\nu, \quad F^A{}_\mu F_A{}^\mu = \delta_B^A \quad (\text{B.8})$$

⁸A related discussion can be found in [15].

The quantity $H^{AB} = F^A{}_{\mu} F^B{}_{\nu} g^{\mu\nu}$ tends to

$$K^{AB} = F^A{}_i F^B{}_j \delta^{ij} \quad (\text{B.9})$$

Finally, the equation of state $\epsilon(X^A, H^{BC})$ can not be expected to have a finite limit as $c \rightarrow \infty$, but $e(X^A, H^{BC})$ has a limit, where

$$\epsilon = mc^2 + e(H^{AB}) = mc^2 + e(K^{AB}) + O\left(\frac{1}{c^2}\right) \quad (\text{B.10})$$

We omit the term mc^2 (which diverges as $c \rightarrow \infty$) from the Lagrangian density (B.10) because it contributes nmc^2 to the Lagrangian density which has vanishing Lagrangian derivative and thus leaves the field equations unchanged. We insert (4.10,4.11) into (3.19) and take the limit $c \rightarrow \infty$ to finally obtain

$$M^{\mu\nu}{}_{AB} = -nmK_{AB} v^{\mu}v^{\nu} + n \left[\tau_{AB}K_{CD} + \tau_{AC}K_{BD} + \tau_{BD}K_{AC} + 2\frac{\partial\tau_{AC}}{\partial K^{BD}} \right] F^{C\mu} F^{D\nu}, \quad (\text{B.11})$$

where $F^{A\mu} = F^A{}_{\mu} h^{\mu\nu}$ with

$$h^{\mu\nu} \partial_{\mu} \partial_{\nu} = \delta^{ik} \partial_i \partial_k. \quad (\text{B.12})$$

Furthermore $\tau_{AB} = 2\frac{\partial e}{\partial K^{AB}}$ and n is defined by

$$6n^2 = K^{AA'} K^{BB'} K^{CC'} \Omega_{ABC} \Omega_{A'B'C'}, n > 0 \quad (\text{B.13})$$

Thus the structure of the nonrelativistic equations is very similar to that of the relativistic ones. In particular, we can obtain existence (locally in time) by essentially the method described in the body of the paper.

In nonrelativistic elasticity it is common to use as the basic field variable $\phi^i(t, X)$ defined by

$$f^A(t, \phi(t, X)) = X^A \quad (\text{B.14})$$

We state without proof that the equations derived above, when rewritten in terms of ϕ , are equivalent to the ones usually considered in the frame-indifferent, hyperelastic case (see Gurtin [6]).

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