# Supersymmetry, Kähler Geometry and Beyond*

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**Abstract.** This lecture briefly describes E. Kähler's impact on the development of supersymmetric field theories, which play a central role in modern attempts to unify the fundamental laws of physics.

## 1 Introduction

As we review progress in theoretical physics over the past decades, the fundamental importance of symmetries is a striking and persistent feature of successful model building in physics. More than any other notion, the concept of symmetry has enabled us to make progress in our understanding of nature at the smallest and the largest scales. The work of E. Kähler, and especially the theory of Kähler manifolds have come to occupy a central place in recent efforts to unify gravity with the other fundamental forces. In this lecture I will try to explain briefly why this is so. The main message will be that the restrictions that come with introducing more symmetry (in particular, more *super*symmetry) are accompanied by similar restrictions on the geometry of the associated models. In this way physicists have been able to recover much of the terrain conquered by pure mathematics since E. Kähler's groundbreaking work on complex manifolds in the early 1930s of the 20th century [12].

In modern physics, symmetries come in different guises. On the one hand, we distinguish between space-time symmetries and internal symmetries. On the other hand, both types of symmetries can appear as global (rigid) or local (gauge) symmetries; in the former, the transformation parameters are constant, whereas for the latter they vary with the space-time coordinates. As every physicist knows, rigid symmetries are associated with conservation laws by Noether's theorem, which states that there is a conserved charge for every exact symmetry (e.g. linear momentum is conserved in translationally invariant theories). By contrast, the presence of *local* symmetries always indicates a redundancy in the parametrization of a physical system, such that two parametrizations that are related by a gauge transformations must be considered as physically equivalent.

The most fundamental local symmetry of space-time is general covariance or (in more mathematical parlance) invariance under space-time diffeomorphisms. According to Einstein, space-time is a pseudo-Riemannian manifold $(M, g)$ whose metric $g$ is determined by Einstein's equations (see e.g. [8])

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

where $T_{\mu\nu}$ is the matter energy momentum tensor, which acts as a source of the gravitational field. $\Lambda$ is the cosmological constant, which was considered an embarrassment by Einstein (but is different from zero according to the most recent cosmological measurements). This equation is generally covariant by construction: the laws of physics must not depend on the coordinate system that one chooses to formulate them. The diffeomorphism symmetry of Einstein's equations is broken when one considers special solutions, i.e. any specific space-time metric $g_{\mu\nu}$ satisfying (1). We are then left only with a rigid symmetry, namely the isometry group Isom($M$) of the manifold $M$ under consideration; of course, this group may be trivial (and will be trivial for most solutions). For instance, if the cosmological constant $\Lambda$ vanishes and there are no matter sources, the simplest solution to (1) is Minkowski space, and Isom($M$) is just the Poincaré group.[1]

*Internal local symmetries* play an essential role in the modern description of elementary particles. Namely, the so-called standard model of elementary particle physics is a (spontaneously broken) Yang Mills gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$ and certain fermionic matter multiplets (the quarks and leptons). The symmetry is called "internal" because in contrast to space-time diffeomorphisms these groups do not act on physical space-time, but rather on an abstract internal space. Apart from simplifying the construction and determining the possible interactions of elementary particles, the Yang Mills symmetries are absolutely crucial for the consistency of the standard model. The fact that quantum corrections can be consistently computed and unambiguous predictions be made with a finite number of parameters rests on the *renormalizability* of these models, which itself is ensured by the gauge symmetry. While renormalizability is a perturbative notion, non-perturbative effects have also been shown to arise in Yang Mills theories. Namely, there exist solitonic solutions of the classical equations of motion (instantons and magnetic monopoles) which are thought to play an essential role not only in explaining various non-perturbative phenomena in particle physics, and are also considered to provide crucial insights into non-perturbative aspects of modern superstring theory.

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[1] As is well known the simplest solution of Einstein's equations with positive cosmological constant and without matter sources is de Sitter space. This is nothing but the coset space SO(1, 4)/ SO(1, 3), whose isometry group is the de Sitter group SO(1, 4). Curiously, the de Sitter group coincides with the "new Poincaré group" considered by E. Kähler in connection with the coset space SO(1, 4)/ SO(4), which is a generalization of the Poincaré upper half plane, and at the same time may be regarded as a Euclideanization of anti-de Sitter space SO(2, 3)/ SO(1, 3).

## 2 Supersymmetry

Given the important role of space-time and internal symmetries, already in the late 1960s the question was asked whether it might not be possible to unify them in one simple group (to be sure, at that time physicists were mainly concerned with the rigid Poincaré symmetry and the so-called flavor symmetries of strong interaction physics). After several unsuccessful attempts, it was realized that this aim cannot be achieved within the framework of ordinary Lie algebra theory: the famous Coleman–Mandula No-Go Theorem [6] states that the most general symmetry of the S matrix compatible with the general postulates of relativistic quantum field theory is always a direct product of the space-time Poincaré symmetry and some internal symmetry. Of course, an essential ingredient in this proof was the assumption that the S matrix should be non-trivial, as free field theory can admit many more symmetries which are incompatible with non-trivial self-interactions.

As with most no-go theorems in physics there was a loophole. This was hidden in the assumption which seemed the most obvious of all, namely the requirement that the symmetry should be realized as an ordinary Lie algebra. At about the same time physicists started "experimenting" with new symmetry concepts (see [13] for the early history). The breakthrough occurred in 1973 when Wess and Zumino discovered the first example of a rigidly *supersymmetric* quantum field theory in four space-time dimensions [14]. The essential new idea was to admit besides the standard (bosonic) Lie algebra generators fermionic generators obeying anti-commutation relations. The concomitant transformation parameters $\varepsilon$ (still space-time independent) must then anti-commute, i.e. generate a Grassmann algebra. The associated symmetry "rotations" thus act in some abstract "superspace" with both bosonic (commuting) and fermionic (anti-commuting) coordinates.

The analysis of Coleman and Mandula was subsequently generalized and super-seded by the work of Haag, Lopuszanski and Sohnius, who were able to classify all possible supersymmetries of the S matrix [11]. This work is still the basis of all the work done nowadays in supersymmetric model building. According to [11], the most general supersymmetry in four space-time dimensions contains the Poincaré algebra, which is generated by the momentum operators $P^\mu$ and the Lorentz generators $M^{\mu\nu}$, and a number of so-called "central charges" $U^{IJ}$ and $V^{IJ}$, which commute with all the elements of the Lie algebra, as well as $N$ real (Majorana) fermionic charges $Q^I_\alpha$ (for $I, J = 1, \ldots, N$) transforming as spinors under the Lorentz group in accordance with the spin statistics theorem). In this case (i.e. in the presence of $N$ fermionic generators) one speaks of "$N$-extended supersymmetry". Referring for the full details to ref. [11], let us here record only the crucial relation

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\mathcal{C}\gamma_\mu)_{\alpha\beta} P^\mu + \mathcal{C}_{\alpha\beta}(U^{IJ} + \gamma^5 V^{IJ}) \tag{2}$$

showing how two supersymmetry transformations commute to give a translation in space-time plus an action of the central charge generators. Here $\gamma_\mu$ are the usual $\gamma$-matrices generating the Clifford algebra, and $\mathcal{C}$ is the charge conjugation matrix.

Having classified the possible superalgebras, the next task was to construct models realizing the associated supersymmetries. What is important here is that these models should allow non-trivial interactions, as a symmetry that can only be realized on free fields would not be of much interest for the description of the real world. The effort to construct all possible supersymmetric field theories kept theoretical physicists busy for several years. To make a long story short, it turns out that the construction of models becomes more and more difficult as $N$ is increased, and at the same time the possibilities become more scarce, such that for the maximum allowed values of $N$ there remains (essentially) only one theory. The possibilities are different according to whether one is dealing with rigid or local supersymmetry. In the first case, the maximum helicity appearing in a supermultiplet cannot be greater than one (corresponding to a vector particle), and we have the bound $N \leq 4$. The maximally extended theory in this case is the celebrated $N = 4$ supersymmetric Yang Mills theory [10], and this theory is still being studied by quantum field theorists because of its wondrous finiteness properties (most recently highlighted in the context of the so-called AdS/CFT correspondence).

For local supersymmetry, on the other hand, there are more possibilities. As one can immediately see from (2), the commutator of two local supersymmetry transformations gives a local translation, which is nothing but an infinitesimal coordinate transformation. In this way, it is almost obvious that local supersymmetry implies gravity, as this is the only way to accommodate the symmetry under local translations. The resulting locally supersymmetric extension of the Einstein's theory is supergravity [9]; in addition to the graviton field, it requires a fermionic gauge field of spin $\frac{3}{2}$, the gravitino. In order to avoid yet higher massless spin fields (for which we don't know how to construct consistent self-interactions), the helicities in a supermultiplet can only go up to $h = 2$, which yields the bound $N \leq 8$. The maximally extended theory is $N = 8$ supergravity [7] which could play some role in the ultimate unification of particle physics and gravitation.

A salient feature of locally supersymmetric theories is the presence of a differential geometric structure not only in the gravitational sector (which is still governed by the Einstein–Hilbert Lagrangian), but also in the sector containing the scalar and fermionic matter fields. More specifically, the scalar matter fields appearing in these theories are always governed by a non-linear $\sigma$-model based on some Riemannian manifold. This manifold is "internal" in the sense that it has nothing to do with space-time, rather it is attached to every space-time point. The unification of space-time and internal symmetries is thus beautifully realized at the level of differential geometry. (In the context of Kaluza–Klein theories one can view this differential geometric structure of the scalar sector as a remnant of pure gravity in higher dimensions.) Just like the associated supersymmetric models, the possible choices for these manifolds get more restricted as one increases the number $N$ of supersymmetries. It is here that E. Kähler's work enters the stage. Namely, as first noticed in [17], Kähler geometry automatically appears when one tries to supersymmetrize non-linear $\sigma$-models already in the context of simple $N = 1$ supersymmetry in four dimensions. As shown not

much later, going to higher $N$ implies further restrictions, so that for $N = 2$ one gets quaternionic manifolds (see [4], and [15] for more recent developments and many further references). For yet higher $N$, the manifolds are associated with exceptional geometries: the choices of the internal manifolds become restricted to coset spaces involving the exceptional groups.

The appearance of geometrical structures in the scalar sector is, of course, not restricted to four space-time dimensions. One of the earliest investigations on the connection between supersymmetry and the theory of Kähler and hyper-Kähler manifolds was in fact done in the context of two-dimensional supersymmetric non-linear $\sigma$-models [2]. One feature that sets these models apart from their (non-renormalizable) higher-dimensional analogs is their UV behavior, which is much better than that of the generic (non-supersymmetric) models, and for special examples leads to completely UV finite theories. However, in the remainder I will turn to another example illustrating the interplay between supersymmetry and differential geometry, namely the case of locally supersymmetric models in three space-time dimensions [16].

## 3 Supersymmetry and Differential Geometry in Three Dimensions

The representations of massless supermultiplets relevant for the construction of supersymmetric field theories in three dimensions are particularly simple to classify. The main reason for this is that there is no spin any more because the little group becomes trivial (recall that the little group is the rotation group in the transverse dimensions, which would be SO(1) in three dimensions). As a consequence, the supercharges carry only internal indices $I, J = 1, \ldots, N$, and the most general superalgebra in the Lorentz frame appropriate to a massless particle reduces to

$$\{Q^I, Q^J\} = 2\delta^{IJ} \tag{3}$$

where the supercharges have been rescaled by an irrelevant factor for convenience. In addition, we have the fermion number operator $F$ satisfying

$$F^2 = 1 \tag{4}$$

$$\{Q^I, F\} = 0. \tag{5}$$

Thus, the massless supermultiplets of $N$-extended supersymmetry in three dimensions are in one-to-one correspondence with the representations of *real* Clifford algebras in $N + 1$ dimensions. The latter have been given in [3], and obey the famous periodicity 8 property. The centralizer $Z$ for each Clifford algebra is one of the three division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions): we have $Z = \mathbb{R}$ for $N = 0, 1, 7 \bmod 8$, $Z = \mathbb{C}$ for $N = 2, 6 \bmod 8$, and $Z = \mathbb{H}$ for $N = 3, 4, 5 \bmod 8$. Accordingly, one finds that the geometries of the associated $\sigma$-models will be Riemannian, Kähler, or quaternionic, respectively. However, just like in higher dimensions, supersymmetric models cannot exist for arbitrary $N$. Rather one finds the bound $N \leq 16$, which is derived in [16]

by invoking a subtle theorem of [5] according to which the holonomy group of a Riemannian manifold uniquely determines the manifold if it does not act transitively on the unit sphere in tangent space. Of course, physicists had already guessed this bound beforehand because it is directly related to the corresponding bounds in higher dimensions based on the absence of massless higher spin particles.

The actual construction of the models is somewhat tedious, and I will therefore be sketchy. As already mentioned, in all cases the scalar sector is governed by a non-linear $\sigma$-model. There is thus some internal $n$-dimensional Riemannian manifold $(\mathcal{M}, G)$ with metric $G_{ij}$, which is locally coordinatized by the fields $\varphi^i$ (with $i, j = 1, \ldots, n$), which are themselves functions of the space-time coordinates. The space $\mathcal{M}$ is usually called the "target space" of the $\sigma$-model. Thus, we have a map

$$\varphi : M \longrightarrow \mathcal{M}. \tag{6}$$

The Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{2} G_{ij}(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j. \tag{7}$$

Remember that $g_{\mu\nu}$ is the metric of the space-time manifold $M$. The equation of motion following from this Lagrangian requires $\varphi^i$ to be harmonic.

In physics the above model first appeared as an effective description of pion self-interactions: here the internal manifolds is just the three-sphere $\mathcal{M} = S^3$, with $G$ the standard metric on $S^3$ (as a function of the three pion fields), and the symmetry of the system is

$$\mathrm{Isom}(S^3) = \mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R. \tag{8}$$

This model actually does work rather well, even though it is only an approximation. From the point of view of QCD (which is now generally accepted as the correct theory of strong interactions), this success is explained by the fact that the symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ survives as an (approximate) chiral symmetry acting on the quark doublets $q = (u, d)$ for approximately massless quarks.

We now want to make this model supersymmetric. For this purpose, we introduce a set of fermionic partners $\chi^i$ to the bosonic fields $\varphi^i$. This is always the first step in supersymmetric model building: we must ensure that the number of physical (propagating) bosonic and fermionic degrees of freedom is the same. The ($N = 1$) supersymmetric extension of the Lagrangian (7) is

$$\mathcal{L} = -\tfrac{1}{2} G_{ij}(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j - \tfrac{1}{2} G_{ij}(\varphi) \bar{\chi}^i \gamma^\mu D_\mu \chi^j$$
$$- \tfrac{1}{24} R_{ijkl}(G(\varphi)) \bar{\chi}^i \gamma^\mu \chi^j \bar{\chi}^k \gamma_\mu \chi^l. \tag{9}$$

Here $R_{ijkl}$ is the Riemann tensor of $(\mathcal{M}, G)$ and the covariant derivative is defined by

$$D_\mu \chi^i := \partial_\mu \chi^i + \Gamma^i_{jk}(G) \partial_\mu \varphi^j \chi^k \tag{10}$$

with the affine connection $\Gamma^i_{jk}$ computed from the internal metric $G_{ij}$ in the usual fashion. The Lagrangian (9) is invariant (i.e. varies into a total derivative) under the

$N = 1$ supersymmetry variations

$$\delta\varphi^i = \tfrac{1}{2}\,\bar\varepsilon\chi^i \tag{11}$$

$$\delta\chi^i = \tfrac{1}{2}\,\gamma^\mu\partial_\mu\varphi^i\varepsilon - \Gamma^i_{jk}(G)\delta\varphi^j\chi^k. \tag{12}$$

We conclude that in three dimensions a $\sigma$-model can be made $N = 1$ supersymmetric for *any* Riemannian manifold $(\mathcal{M}, G)$ in this way.

As we increase the number of supersymmetries to $N = 2$, the possibilities become more restricted. Let us denote by $\eta$ the second supersymmetry parameter, and proceed from the ansatz

$$\delta\varphi^i = \tfrac{1}{2}\,\bar\varepsilon\chi^i + \tfrac{1}{2}\,I^i{}_j\bar\eta\chi^j \tag{13}$$

$$\delta\chi^i = \tfrac{1}{2}\,\gamma^\mu\partial_\mu\varphi^j(\delta^i_j\varepsilon + I^i{}_j\eta) + \Gamma^i_{jk}(G)\delta\varphi^j\chi^k. \tag{14}$$

A calculation completely analogous to the one presented in [2] then shows that supersymmetry requires the tensor $I^i{}_j$ to satisfy the following relations:

$$G_{ik}I^k{}_j + G_{jk}I^k{}_i = 0 \tag{15}$$

$$I^i{}_k I^k{}_j = -\delta^i_j \tag{16}$$

$$D_i I^j{}_k = 0 \tag{17}$$

$$I^k{}_m R_{ij}{}^m{}_l = R_{ij}{}^k{}_m I^m{}_l. \tag{18}$$

But these are precisely the conditions stating that the tensor $I^i{}_j$ is a complex structure on $(\mathcal{M}, G)$, and therefore that $(\mathcal{M}, G)$ is a Kähler manifold! The conclusion is therefore that $N = 2$ supersymmetry is no longer compatible with any Riemannian manifold. Rather, only those models for which $(\mathcal{M}, G)$ is a Kähler manifold can be made $N = 2$ supersymmetric.

If one wants yet more supersymmetry, one gets further restrictions [2]. Replacing the second supersymmetry parameter $\eta$ by several parameters $\varepsilon^P$ and generalizing (13) to

$$\delta\varphi^i = \tfrac{1}{2}\,\bar\varepsilon\chi^i + \tfrac{1}{2}\,f^i_{Pj}\bar\varepsilon^P\chi^j \tag{19}$$

$$\delta\chi^i = \tfrac{1}{2}\,\gamma^\mu\partial_\mu\varphi^j(\delta^i_j\varepsilon + f^i_{Pj}\varepsilon^P) + \Gamma^i_{jk}(G)\delta\varphi^j\chi^k \tag{20}$$

with summation over the labels $P$, one finds that the tensors $f^i_{Pj}$ must satisfy analogous properties as the complex structure for $N = 2$. In fact, it turns out that $N = 4$ is the only possibility, such that $P = 1, 2, 3$, and the space $(\mathcal{M}, G)$ must now be hyper-Kähler [2]. For rigid supersymmetry this exhausts all the possibilities.

There are more possibilities for local supersymmetry, for which we can go up to $N = 16$, but also more restrictions. The associated models have more complicated Lagrangians since in addition to the matter fields $\varphi^i$ and $\chi^i$, we need also the dreibein field (the "square root" of the metric $g_{\mu\nu}$ introduced earlier), and as many gravitino fields $\psi^I_\mu$ as there are supersymmetries. The latter fields carry no local (propagating) degrees of freedom, but are nonetheless indispensable for formulating the theory.

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The locally supersymmetric Lagrangians then consist of a combination of the above $\sigma$-model Lagrangians, the Einstein and Rarita Schwinger Lagrangians in three dimensions and several quartic fermionic terms (usually the most tedious to calculate). The supersymmetry of the model is then entirely encoded into the restrictions that must be imposed on the internal manifold $(\mathcal{M}, G)$. The complete table of matter-coupled supergravities in three space-time dimensions is given below, with $N$ the number of local supersymmetries, and $k$ the number of matter supermultiplets that can be coupled to the basic supergravity multiplet.

Table 1. $\sigma$-model target manifolds in three space-time dimensions.

| $N$ | $k$ | $\mathcal{M}$ |
|-----|------|---------------|
| 1 | $k \in \mathbb{N}$ | Riemannian |
| 2 | $k \in \mathbb{N}$ | Kähler |
| 3 | $k \in \mathbb{N}$ | quaternionic |
| 4 | $k \in \mathbb{N}$ | (quaternionic)$^2$ |
| 5 | $k \in \mathbb{N}$ | $Sp(2,k)/Sp(2) \otimes Sp(k)$ |
| 6 | $k \in \mathbb{N}$ | $SU(4,k)/S(U(4) \otimes U(k))$ |
| 8 | $k \in \mathbb{N}$ | $SO(8,k)/SO(8) \otimes SO(k)$ |
| 9 | $k = 1$ | $F_{4(-20)}/SO(9)$ |
| 10 | $k = 1$ | $E_{6(-14)}/SO(10) \otimes SO(2)$ |
| 12 | $k = 1$ | $E_{7(-5)}/SO(12) \otimes SO(3)$ |
| 16 | $k = 1$ | $E_{8(8)}/SO(16)$ |

In conclusion the requirement of local supersymmetry goes hand in hand with the restrictions on the geometry of the internal manifold. For $N = 1, 2, 3$ the target manifolds $(\mathcal{M}, G)$ must be general Riemannian, Kähler, or quaternionic, respectively, but are otherwise arbitrary; in particular, the number of matter fields (or, more precisely, of matter supermultiplets) can be freely chosen. For $N = 4$ $\mathcal{M}$ must be a product of two quaternionic manifolds. Beyond $N = 4$ the target manifolds are completely determined by supersymmetry, with the number of matter supermultiplets still arbitrary for $N \leq 8$. For the values $N = 9, 10, 12, 16$ we obtain *unique* theories; remarkably, there are no matter coupled supergravities at all for the intermediate values $N = 7, 11, 13, 14, 15$! There have been several attempts to associate the higher $N$ theories with some kind of *octonionic* geometry (recall that the octonions $\mathbb{O}$ are the last of the division algebras and are both non-commutative as well as non-associative), but so far the only link that has emerged is the fact that the exceptional groups appearing in the coset spaces are themselves linked to the octonions in an as yet not completely understood way. For $N > 16$, only Chern Simons-type theories can be constructed [1], but no theories with propagating matter degrees of freedom (and hence nontrivial internal manifold $\mathcal{M}$) exist any more.

In view of the intimate links between supersymmetry and differential geometry it is perhaps not surprising that many results of complex or quaternionic differential geometry have neat, though sometimes only heuristic, derivations based on supersymmetry (this applies in particular to recent developments in quaternionic geometry, see e.g. [15] for further details). The reason is that many arguments that are somewhat involved when phrased in terms of the metric $G_{ij}$ simplify considerably when analyzed in terms of supersymmetry variations involving only first order derivatives.

Unfortunately, however, we have no indication so far from experimental physics that any of these beautiful structures are actually realized in nature. At least at our present level of understanding, the most sophisticated models with maximal supersymmetry are too restrictive to match real physics, whereas the models with "little" (in practise $N = 1$) supersymmetry may not be restrictive enough to make falsifiable predictions. Kähler geometry is associated with low supersymmetry, and thus leaves enough room for (semi-)realistic model building. In fact, a glance at any paper dealing with supersymmetric phenomenology (just have a look at the pertinent papers that you can find on the shelves this week) immediately reveals the ubiquity of Kähler potentials in modern elementary particle physics. Indeed, it is quite likely that the real world is not maximally supersymmetric at the energy scales accessible to present day experimental physics, but may still admit some residual supersymmetry at energy scales of $\mathcal{O}(1\ TeV)$ which could show up via the production of supersymmetric partners of the known elementary particles in upcoming accelerator experiments. If this were the case, Kähler geometry would have found a beautiful match in the world of elementary particle physics.

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